RESOLUTIONS AND SEMIDUALIZING MODULES

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ABSTRACT

Projective and injective modules are of key importance in algebra, in part because of their useful homological properties. The notion of C-projective and C-injective modules generalizes these constructions. In particular, these modules may be used to construct resolutions and define related homological dimensions in a natural way. When C is a semidualizing module, the C-projective and C-injective modules have particularly useful homological properties. Further, one may combine projective and C-projective resolutions to construct complete PC-resolutions (and, dually, complete IC-resolutions) that yield other modules with nice homological properties. This paper surveys some of the literature on these constructions.

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TABLE OF CONTENTS

1. INTRODUCTION

Projective and injective modules play key roles in the study of rings. In particular, their homological definitions have deep consequences that come into play when studying complexes of modules and associated resolutions. Throughout our study, R is a commutative noetherian ring with identity. Recall that an R-module P is *projective* if the functor $\mathrm{Hom}_R(P, -)$ is exact; that is, it respects exactness of sequences of R-module homomorphisms. An R-module I is *injective* if the functor $\mathrm{Hom}_R(-, I)$ is exact.

Given any R-module M, we may construct resolutions using projective or injective modules that terminate with M. An *augmented projective resolution* of M is an exact sequence of R-module homomorphisms of the form

$$P^+ := \cdots \to P_1 \to P_0 \to M \to 0$$

where each P_i is projective. Dually, an *augmented injective coresolution* of M is an exact sequence of the form

$$^+I := \quad 0 \to M \to I^0 \to I^1 \to \cdots$$

where each I^j is injective. It can be shown that every module admits both a projective resolution and an injective coresolution.

We use these resolutions to define dimensions of modules. Informally, the *projective dimension* $\mathrm{pd}_R(M)$ of M is the length of the shortest possible projective resolution of M. The *injective dimension* $\mathrm{id}_R(M)$ of M is the length of the shortest possible injective coresolution of M. As an example, if P is projective, we have $\mathrm{pd}_R(P) = 0$ since $0 \to P \to P \to 0$ is an augmented projective resolution of P. Similarly, if I is injective, we have $\mathrm{id}_R(I) = 0$ since $0 \to I \to I \to 0$ is an augmented injective coresolution of I.

We say a noetherian ring R is *local* if it has a unique maximal ideal, and that a local ring is *regular* if every R-module has finite projective dimension. One of the first major applications of homological techniques to commutative algebra is the following characterization of regular local rings by Auslander, Buchsbaum, and Serre. There is also an identical version for modules of finite injective dimension.

Theorem 1.1 ([2, 1, 10])**.** *Let R be a local noetherian ring with residue field k. The following conditions are equivalent:*

(i) *R is regular;*

(ii) $\mathrm{pd}_R(M) < \infty$ *for every finitely generated R-module M; and*

(iii) $\mathrm{pd}_R(k) < \infty$.

A particularly useful characterization of modules of finite projective or injective dimension is the following standard result.

Theorem 1.2. *Let M be an R-module.*

(a) *We have* $\mathrm{pd}_R(M) \leq n$ *if and only if* $\mathrm{Ext}^i_R(M, -) = 0$ *for all* $i > n$.

(b) *We have* $\mathrm{id}_R(M) \leq n$ *if and only if* $\mathrm{Ext}^i_R(-, M) = 0$ *for all* $i > n$.

Here $\mathrm{Ext}^i_R(M, N)$ is the cohomology module $H^i(\mathrm{Hom}_R(P, N)) \cong H^i(\mathrm{Hom}_R(M, J))$, where P is any truncated projective resolution of M and J is any truncated injective coresolution of N; we may also use the homological notation $H^i \equiv H_{-i}$. Notice that the exactness of the functors $\mathrm{Hom}_R(P, -)$ (where P is projective) and $\mathrm{Hom}_R(-, I)$ (where I is injective) follows immediately from this result.

There is a generalization of projective and injective modules that extends these homological constructions in a natural way. If C is a finitely generated R-module, a *C-projective* module is an R-module isomorphic to a module of the form $C \otimes_R P$, where P is projective. A *C-injective* module is isomorphic to a module of the form $\mathrm{Hom}_R(C, I)$, where I is injective. These modules are particularly nice when C is a semidualizing R-module: a finitely generated R-module C is *semidualizing* if the homothety map $R \to \mathrm{Hom}_R(C, C)$ is an isomorphism and we have the vanishing $\mathrm{Ext}^i_R(C, C) = 0$ for $i > 0$.

Given any R-module M and finitely generated R-module C, we may construct resolutions using C-projective or C-injective modules that terminate with M, the so-called *augmented proper $\mathcal{P}_C$-projective resolutions* and *augmented proper $\mathcal{I}_C$-injective coresolutions*; see Section 2 for definitions. In Proposition 3.4 below we show that every module admits both when C is semidualizing.

Such resolutions give rise to related dimensions. Informally, the *$\mathcal{P}_C$-projective dimension* of M, denoted $\mathcal{P}_C\text{-}\mathrm{pd}_R(M)$, is the length of the shortest possible proper $\mathcal{P}_C$-projective resolution of M. The *$\mathcal{I}_C$-injective dimension* $\mathcal{I}_C\text{-}\mathrm{id}_R(M)$ of M is the length of the shortest possible proper $\mathcal{I}_C$-injective coresolution of M. Similar to the projective and injective cases, we have $\mathcal{P}_C\text{-}\mathrm{pd}_R(C \otimes_R P) = 0$ if P is projective, and $\mathcal{I}_C\text{-}\mathrm{id}_R(\mathrm{Hom}_R(C, I)) = 0$ if I is injective.

If C is semidualizing, we obtain the following relationships between projective and $\mathcal{P}_C$-projective dimensions and, dually, between injective and $\mathcal{I}_C$-injective dimensions of modules. See Theorem 3.15, which is a main result of Section 3.

Theorem 1.3 ([11, Theorem 2.11])**.** *Let C be a semidualizing R-module and M any R-module. The following equalities hold:*

(a) $\mathrm{pd}_R(M) = \mathcal{P}_C\text{-}\mathrm{pd}_R(C \otimes_R M)$*;*

(b) $\mathcal{I}_C\text{-id}_R(M) = \text{id}_R(C \otimes_R M)$;

(c) $\mathcal{P}_C\text{-pd}_R(M) = \text{pd}_R(\text{Hom}_R(C, M))$; *and*

(d) $\text{id}_R(M) = \mathcal{I}_C\text{-id}_R(\text{Hom}_R(C, M))$.

The next result is a version of the Auslander-Buchsbaum-Serre theorem that uses finite $\mathcal{P}_C$-projective dimension to detect the regularity of a local ring; it is Theorem 3.17.

Theorem 1.4 ([11, Proposition 5.1])**.** *Let R be a local noetherian ring with residue field k, and C a semidualizing R-module. The following conditions are equivalent:*

(i) *R is regular;*

(ii) *$\mathcal{P}_C\text{-pd}_R(M) < \infty$ for every finitely generated R-module M; and*

(iii) *$\mathcal{P}_C\text{-pd}_R(k) < \infty$.*

Homological properties of projective and injective modules often extend to the C-projective and C-injective settings. As a matter of notation, we denote Ext-modules constructed using proper $\mathcal{P}_C$-projective resolutions by $\text{Ext}_{\mathcal{P}_C}$, and Ext-modules constructed using proper $\mathcal{I}_C$-injective coresolutions by $\text{Ext}_{\mathcal{I}_C}$; see Definition 3.6. This yields a version of Theorem 1.2 that uses the C-projective and C-injective constructions; see Theorem 4.3.

Theorem 1.5 ([11, Theorem 3.2])**.** *Let C be a semidualizing R-module and M any R-module.*

(a) *We have $\mathcal{P}_C\text{-pd}_R(M) \leq n$ if and only if $\text{Ext}^i_{\mathcal{P}_C}(M, -) = 0$ for all $i > n$.*

(b) *We have $\mathcal{I}_C\text{-id}_R(M) \leq n$ if and only if $\text{Ext}^i_{\mathcal{I}_C}(-, M) = 0$ for all $i > n$.*

A useful application of these constructions is to relate $\text{Ext}_{\mathcal{P}_C}$- and $\text{Ext}_{\mathcal{I}_C}$-modules to the standard Ext-modules from Theorem 1.2. This uses the *Foxby classes* $\mathcal{B}_C(R)$ and $\mathcal{A}_C(R)$ from Definition 2.7; see Theorem 4.11, a main result of Section 4.

Theorem 1.6 ([11, Corollary 4.2])**.** *Let C be a semidualizing R-module.*

(a) *If $M, N \in \mathcal{B}_C(R)$, then $\text{Ext}^i_{\mathcal{P}_C}(M, N) \cong \text{Ext}^i_R(M, N)$ for all $i \geq 0$.*

(b) *If $M, N \in \mathcal{A}_C(R)$, then $\text{Ext}^i_{\mathcal{I}_C}(M, N) \cong \text{Ext}^i_R(M, N)$ for all $i \geq 0$.*

It is useful to combine projective and C-projective resolutions to form *complete PC-resolutions* and, dually, to combine injective and C-injective resolutions to form *complete IC-resolutions* of modules; modules admitting such resolutions are *G_C-projective* or, dually, *G_C-injective*. These resolutions have many properties in common with other constructions discussed above. For example, we introduce the notions of *resolving*, *quasi-resolving*, *coresolving*, and *quasi-coresolving* classes of modules; see Definition 5.16. These classes model the standard behavior of projective and injective modules in short exact sequences. The next result is contained in Theorem 5.17 and Theorem 5.21, the main results of Section 5.

Theorem 1.7 ([12, Theorem 2.8]). *Let C be a semidualizing R-module.*

(a) *The class of C-projective R-modules is quasi-resolving.*

(b) *The class of C-injective R-modules is quasi-coresolving.*

(c) *The class of G_C-projective R-modules is resolving.*

(d) *The class of G_C-injective R-modules is coresolving.*

2. SEMIDUALIZING MODULES AND FOXBY CLASSES

In this section we introduce several useful constructions and definitions.

Definition 2.1. An R-module C is *semidualizing* if it satisfies the following conditions:

1. C is finitely generated;
2. the homothety map $\chi_C^R : R \to \mathrm{Hom}_R(C, C)$ is an isomorphism, where $\chi_C^R(r)(c) := rc$; and
3. $\mathrm{Ext}_R^i(C, C) = 0$ for $i > 0$.

In particular, R itself is semidualizing over R, so a semidualizing module over a noetherian ring always exists. The following lemma says that the classes $\mathcal{P}_C$ and $\mathcal{I}_C$ from Definition 3.1 are "precovering" and "preenveloping", respectively.

Lemma 2.2 ([5, Proposition 5.10])**.** *Let M be an R-module and C a semidualizing R-module.*

(a) *There exists a projective module P and a homomorphism $\phi : C \otimes_R P \to M$ such that for every projective module Q, the induced homomorphism*

$$\mathrm{Hom}_R(C \otimes_R Q, C \otimes_R P) \xrightarrow{\mathrm{Hom}(C \otimes Q, \phi)} \mathrm{Hom}_R(C \otimes_R Q, M)$$

is surjective.

(b) *There exists an injective module I and a homomorphism $\psi : M \to \mathrm{Hom}_R(C, I)$ such that for every injective module J, the induced homomorphism*

$$\mathrm{Hom}_R(\mathrm{Hom}_R(C, I), \mathrm{Hom}_R(C, J)) \xrightarrow{\mathrm{Hom}(\psi, \mathrm{Hom}(C,J))} \mathrm{Hom}_R(M, \mathrm{Hom}_R(C, J))$$

is surjective.

Definition 2.3. We say an injective R-module E is *faithfully injective* if for any sequence X of R-module homomorphisms such that $\mathrm{Hom}_R(X, E)$ is exact, X is exact.

Remark 2.4 ([6, Corollary 3.2])**.** Every commutative ring admits a faithfully injective module. In particular, let $E := \bigoplus_{\mathfrak{m}} E_R(R/\mathfrak{m})$, where E_R denotes the injective hull over R and the sum is taken over all maximal ideals $\mathfrak{m}$ of R; then E is faithfully injective.

Definition 2.5. We say a projective R-module Q is *faithfully projective* if for any sequence X of R-module homomorphisms such that $\mathrm{Hom}_R(Q, X)$ is exact, X is exact.

Remark 2.6. Every nonzero free module is faithfully projective, so every commutative ring admits a faithfully projective module.

The next two classes of R-modules are called the *Foxby classes*. We use them extensively.

Definition 2.7. Let C be a finitely generated R-module. The *Bass class* $\mathcal{B}_C(R)$ (or simply $\mathcal{B}_C$ if there is no confusion) is the class of all R-modules M such that

1. the natural map $\xi_M^C : C \otimes_R \operatorname{Hom}_R(C, M) \to M$ is an isomorphism, where $\xi_M^C(c \otimes \phi) := \phi(c)$; and
2. $\operatorname{Ext}_R^i(C, M) = 0 = \operatorname{Tor}_i^R(C, \operatorname{Hom}_R(C, M))$ for $i > 0$.

The *Auslander class* $\mathcal{A}_C(R)$ (or simply $\mathcal{A}_C$ if there is no confusion) is the class of all R-modules M such that

1. the natural map $\gamma_M^C : M \to \operatorname{Hom}_R(C, C \otimes_R M)$ is an isomorphism, where $\gamma_M^C(m)(c) := c \otimes m$; and
2. $\operatorname{Tor}_i^R(C, M) = 0 = \operatorname{Ext}_R^i(C, C \otimes_R M)$ for $i > 0$.

Here $\operatorname{Tor}_i^R(M, N)$ is the homology module $H_i(P, N) \cong H_i(M, Q)$, where P is any truncated projective resolution of M and Q is any truncated projective resolution of N. When C is semidualizing, the Foxby classes have several nice properties, especially when we are dealing with modules of finite projective or injective dimension.

Proposition 2.8 ([7, Propositions 3.1.7, 3.1.9, 3.1.10])**.** *Let C be a semidualizing R-module.*

(a) *If any two modules in a short exact sequence are in $\mathcal{A}_C$, then so is the third.*

(b) *If M is an R-module of finite flat dimension, then $M \in \mathcal{A}_C$.*

(c) *If any two modules in a short exact sequence are in $\mathcal{B}_C$, then so is the third.*

(d) *If M is an R-module of finite injective dimension, then $M \in \mathcal{B}_C$.*

The next theorem provides an elegant connection between the Foxby classes when C is semidualizing. The proof is straightforward but quite involved, so we omit it here.

Theorem 2.9 (Foxby equivalence [7, Theorem 3.2.1])**.** *Let C be a semidualizing R-module and M any R-module.*

(a) *We have $M \in \mathcal{B}_C$ if and only if $\operatorname{Hom}_R(C, M) \in \mathcal{A}_C$.*

(b) *We have $M \in \mathcal{A}_C$ if and only if $C \otimes_R M \in \mathcal{B}_C$.*

The previous two results immediately imply the following corollary.

Corollary 2.10. *Let C be a semidualizing R-module.*

(a) *If P is a projective R-module, then $C \otimes_R P \in \mathcal{B}_C$.*

(b) *If I is an injective R-module, then* $\operatorname{Hom}_R(C, I) \in \mathcal{A}_C$.

3. C-PROJECTIVE AND C-INJECTIVE RESOLUTIONS

We now construct our first generalization of projective and injective modules.

Definition 3.1. Let C be a finitely generated R-module. We say an R-module M is *C-projective* if it is isomorphic to a module of the form $C \otimes_R P$, where P is projective. The class of C-projective modules is denoted $\mathcal{P}_C$. We say M is *C-injective* if it is isomorphic to a module of the form $\mathrm{Hom}_R(C, I)$, where I is injective. The class of C-injective modules is denoted $\mathcal{I}_C$.

Proposition 3.2. *The class of C-projective modules is closed under direct sums. The class of C-injective modules is closed under direct products and sums.*

Proof. If $\{C \otimes_R P_\lambda\} \subset \mathcal{P}_C$, then $\bigoplus_\lambda (C \otimes_R P_\lambda) \cong C \otimes_R (\bigoplus_\lambda P_\lambda) \in \mathcal{P}_C$ since the class of projective R-modules is closed under direct sums. Similarly, if $\{\mathrm{Hom}_R(C, I_\lambda)\} \subset \mathcal{I}_C$, then $\prod_\lambda \mathrm{Hom}_R(C, I_\lambda) \cong \mathrm{Hom}_R\,(C, \prod_\lambda I_\lambda) \in \mathcal{I}_C$ and $\bigoplus_\lambda \mathrm{Hom}_R(C, I_\lambda) \cong \mathrm{Hom}_R\,(C, \prod_\lambda I_\lambda) \in \mathcal{I}_C$ since the class of injective R-modules is closed under direct products and sums and C is finitely generated over a noetherian ring.

Just as every module admits both a projective resolution and an injective coresolution, every module admits an augmented proper $\mathcal{P}_C$-projective resolution and an augmented proper $\mathcal{I}_C$-injective coresolution, defined next; see Proposition 3.4.

Definition 3.3. Let C be a finitely generated R-module and M any R-module.

(1) An *augmented proper $\mathcal{P}_C$-projective resolution* of M is an R-complex

$$X^+ := \cdots \xrightarrow{\partial_2} C \otimes_R P_1 \xrightarrow{\partial_1} C \otimes_R P_0 \xrightarrow{\tau} M \longrightarrow 0$$

of R-module homomorphisms such that each P_i is projective and the complex $\mathrm{Hom}_R(C \otimes_R Q, X^+)$ is exact for every projective module Q. The truncated complex

$$X := \cdots \xrightarrow{\partial_2} C \otimes_R P_1 \xrightarrow{\partial_1} C \otimes_R P_0 \longrightarrow 0$$

is the *proper $\mathcal{P}_C$-projective resolution* of M corresponding to X^+.

(2) An *augmented coproper $\mathcal{P}_C$-projective coresolution* of M is an R-complex

$$^+X := \quad 0 \longrightarrow M \xrightarrow{\epsilon} C \otimes_R Q^0 \xrightarrow{\partial^0} C \otimes_R Q^1 \xrightarrow{\partial^1} \cdots$$

of R-module homomorphisms such that each Q^i is projective and the complex $\operatorname{Hom}_R({}^+X, C\otimes_R Q)$ is exact for every projective module Q. The truncated complex

$$X := \; 0 \longrightarrow C\otimes_R Q^0 \xrightarrow{\partial^0} C\otimes_R Q^1 \xrightarrow{\partial^1} \cdots$$

is the *coproper $\mathcal{P}_C$-projective coresolution* of M corresponding to ${}^+X$.

(3) An *augmented proper $\mathcal{I}_C$-injective coresolution* of M is an R-complex

$${}^+Y := \quad 0 \longrightarrow M \xrightarrow{\epsilon} \operatorname{Hom}_R(C, I^0) \xrightarrow{\partial^0} \operatorname{Hom}_R(C, I^1) \xrightarrow{\partial^1} \cdots$$

of R-module homomorphisms such that each I^i is injective and the complex $\operatorname{Hom}_R({}^+Y, \operatorname{Hom}_R(C, J))$ is exact for every injective module J. The truncated complex

$$Y := \quad 0 \longrightarrow \operatorname{Hom}_R(C, I^0) \xrightarrow{\partial^0} \operatorname{Hom}_R(C, I^1) \xrightarrow{\partial^1} \cdots$$

is the *proper $\mathcal{I}_C$-injective coresolution* of M corresponding to ${}^+Y$.

(4) An *augmented coproper $\mathcal{I}_C$-injective resolution* of M is an R-complex

$$Y^+ := \; \cdots \xrightarrow{\partial^2} \operatorname{Hom}_R(C, I_1) \xrightarrow{\partial^1} \operatorname{Hom}_R(C, I_0) \xrightarrow{\tau} M \longrightarrow 0$$

of R-module homomorphisms such that each I_i is injective and the complex $\operatorname{Hom}_R(\operatorname{Hom}_R(C, J), Y^+)$ is exact for every injective module J. The truncated complex

$$Y := \; \cdots \xrightarrow{\partial^2} \operatorname{Hom}_R(C, I_1) \xrightarrow{\partial^1} \operatorname{Hom}_R(C, I_0) \longrightarrow 0$$

is the *coproper $\mathcal{I}_C$-injective resolution* of M corresponding to Y^+.

Proposition 3.4. *Let M and C be R-modules.*

(a) *An augmented proper $\mathcal{P}_C$-projective resolution of M exists.*

(b) *An augmented proper $\mathcal{I}_C$-injective coresolution of M exists.*

Proof. (a) Lemma 2.2(a) gives a projective module P_0 and homomorphism $\tau : C\otimes_R P_0 \to M$ such that $\operatorname{Hom}_R(C\otimes_R Q, \tau)$ is surjective for any projective module Q. Let $K_0 := \ker\tau$. Applying Lemma 2.2(a) again, we obtain a projective module P_1 and homomorphism $\phi_1 : C\otimes_R P_1 \to K_0$ such that $\operatorname{Hom}_R(C\otimes_R Q, \phi_1)$ is surjective. Define the homomorphism ∂_1 to make the following diagram commute:

$$\begin{tikzcd}
C \otimes_R P_1 \arrow[rr, "\partial_1"] \arrow[dr, "\phi_1"'] & & C \otimes_R P_0 \arrow[r, "\tau"] & M \arrow[r] & 0 \\
& K_0 \arrow[ur, hook] & & &
\end{tikzcd}$$

Continue inductively. For each $i > 0$, let $K_i := \ker \partial_i$ and apply Lemma 2.2(a) to obtain the following commutative diagram:

$$\begin{tikzcd}
C \otimes_R P_{i+1} \arrow[rr, "\partial_{i+1}"] \arrow[dr, "\phi_{i+1}"'] & & C \otimes_R P_i \\
& K_i \arrow[ur, hook] &
\end{tikzcd}$$

Define the sequence

$$X^+ := \cdots \xrightarrow{\partial^2} C \otimes_R P_1 \xrightarrow{\partial_1} C \otimes_R P_0 \xrightarrow{\tau} M \to 0$$

using these maps. Note that by our construction, X is an R-complex. It remains to show that the complex $\operatorname{Hom}_R(C \otimes_R Q, X^+)$ is exact for any projective module Q. For $i \geq 0$, consider the sequence $0 \to K_i \to C \otimes_R P_i \xrightarrow{\phi_i} K_{i-1}$, which is exact by construction if we define $K_{-1} := M$ and $\phi_0 := \tau$. Since the map $\operatorname{Hom}_R(C \otimes_R Q, \phi_i)$ is surjective, the sequence

$$0 \to \operatorname{Hom}_R(C \otimes_R Q, K_i) \to \operatorname{Hom}_R(C \otimes_R Q, C \otimes_R P_i) \to \operatorname{Hom}_R(C \otimes_R Q, K_{i-1}) \to 0$$

is exact by left-exactness of the functor $\operatorname{Hom}_R(C \otimes_R Q, -)$. A standard sequence-splicing argument implies that $\operatorname{Hom}_R(C \otimes_R Q, X^+)$ is exact.

(b) Lemma 2.2(b) gives an injective module I^0 and homomorphism $\epsilon : M \to \operatorname{Hom}_R(C, I^0)$ such that $\operatorname{Hom}_R(\epsilon, \operatorname{Hom}_R(C, J))$ is surjective for any injective module J. Let $L^0 := \operatorname{coker} \epsilon$. Applying Lemma 2.2(b) again, we obtain an injective module I^1 and homomorphism $\psi^0 : L^0 \to \operatorname{Hom}_R(C, I^1)$ such that the map $\operatorname{Hom}_R(\psi^0, \operatorname{Hom}_R(C, J))$ is surjective. Define the homomorphism ∂^0 to make the following diagram commute:

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r, "\epsilon"] & \operatorname{Hom}_R(C, I^0) \arrow[rr, "\partial^0"] \arrow[dr] & & \operatorname{Hom}_R(C, I^1) \\
& & & L^0 \arrow[ur, "\psi^0"'] &
\end{tikzcd}$$

Continue inductively. For each $i > 0$, let $L^i := \operatorname{coker} \partial^{i-1}$ and apply Lemma 2.2(b) to obtain the following commutative diagram:

$$\begin{tikzcd} \mathrm{Hom}_R(C,I^i) \arrow[rr, "\partial^i"] \arrow[dr] & & \mathrm{Hom}_R(C,I^{i+1}) \\ & L^i \arrow[ur, "\psi^i"'] & \end{tikzcd}$$

Define the sequence

$$^+Y := \quad 0 \to M \xrightarrow{\epsilon} \mathrm{Hom}_R(C,I^0) \xrightarrow{\partial^0} \mathrm{Hom}_R(C,I^1) \xrightarrow{\partial^1} \cdots$$

using these maps. Note that by our construction, Y is an R-complex. It remains to show that the complex $\mathrm{Hom}_R(^+Y, \mathrm{Hom}_R(C,J))$ is exact for any injective module J. For $i \geq -1$, consider the sequence $L^i \xrightarrow{\psi^i} \mathrm{Hom}_R(C,I^{i+1}) \to L^{i+1} \to 0$, which is exact by construction if we define $L^{-1} := M$ and $\psi^{-1} := \epsilon$. Since the map $\mathrm{Hom}_R(\psi^i, \mathrm{Hom}_R(C,J))$ is surjective, the sequence

$$0 \to \mathrm{Hom}_R(L^{i+1}, \mathrm{Hom}_R(C,J)) \to \mathrm{Hom}_R(\mathrm{Hom}_R(C,I^{i+1}), \mathrm{Hom}_R(C,J)) \to \mathrm{Hom}_R(L^i, \mathrm{Hom}_R(C,J)) \to 0$$

is exact by left-exactness of the functor $\mathrm{Hom}_R(-, \mathrm{Hom}_R(C,J))$. A standard sequence-splicing argument implies that $\mathrm{Hom}_R(^+Y, \mathrm{Hom}_R(C,J))$ is exact.

We use these resolutions to define associated homological dimensions.

Definition 3.5. If C is a finitely generated R-module, the $\mathcal{P}_C$*-projective dimension* of an R-module M is

$$\mathcal{P}_C\text{-pd}(M) := \inf \{\sup\{n : X_n \neq 0\} : X \text{ is a proper } \mathcal{P}_C\text{-projective resolution of } M\}$$

and the $\mathcal{I}_C$*-injective dimension* of M is

$$\mathcal{I}_C\text{-id}(M) := \inf \{\sup\{n : Y^n \neq 0\} : Y \text{ is a proper } \mathcal{I}_C\text{-injective coresolution of } M\}.$$

Definition 3.6. Let M, N, C be R-modules and $i \geq 0$.

(1) Let X be a proper $\mathcal{P}_C$-projective resolution of M. We define $\mathrm{Ext}^i_{\mathcal{P}_C}(M,N) := H^i(\mathrm{Hom}_R(X,N))$.

(2) Let Y be a proper $\mathcal{I}_C$-injective coresolution of N. We define $\mathrm{Ext}^i_{\mathcal{I}_C}(M,N) := H^i(\mathrm{Hom}_R(M,Y))$.

Remark 3.7 ([9, Remark 1.12])**.** Both "relative Ext-modules" above are well-defined and independent of the choice of resolution.

Notice that $\mathcal{P}_C$-projective resolutions and $\mathcal{I}_C$-injective coresolutions are not exact in general. However, the next few results establish Foxby class conditions under which they are exact.

Lemma 3.8 ([11, Lemma 2.1])**.** *Let C be a semidualizing R-module and M any R-module.*

(a) *If X^+ is an augmented proper $\mathcal{P}_C$-projective resolution of M, then* $\operatorname{Hom}_R(C, X^+)$ *is an augmented projective resolution of* $\operatorname{Hom}_R(C, M)$.

(b) *If ${}^+Y$ is an augmented proper $\mathcal{I}_C$-injective coresolution of M, then $C \otimes_R {}^+Y$ is an augmented injective coresolution of $C \otimes_R M$.*

Proof. (a) Let

$$X^+ := \cdots \to C \otimes_R P_1 \to C \otimes_R P_0 \to M \to 0$$

be an augmented proper $\mathcal{P}_C$-projective resolution of M, and consider

$$\operatorname{Hom}_R(C, X^+) = \cdots \to \operatorname{Hom}_R(C, C \otimes_R P_1) \to \operatorname{Hom}_R(C, C \otimes_R P_0) \to \operatorname{Hom}_R(C, M) \to 0$$

where, for each $i \geq 0$, the module P_i is flat since it is projective, so it has finite flat dimension and thus $P_i \in \mathcal{A}_C$ by Proposition 2.8(b). Definition 2.7 implies $\operatorname{Hom}_R(C, C \otimes_R P_i) \cong P_i$. Then

$$\operatorname{Hom}_R(C, X^+) \cong \cdots \to P_1 \to P_0 \to \operatorname{Hom}_R(C, M) \to 0$$

is exact by Definition 3.3(1) since $C \cong C \otimes_R R$ is C-projective since R is projective as an R-module.

(b) Let

$${}^+Y := \quad 0 \to M \to \operatorname{Hom}_R(C, I^0) \to \operatorname{Hom}_R(C, I^1) \to \cdots$$

be an augmented proper $\mathcal{I}_C$-injective coresolution of M, and consider

$$C \otimes_R {}^+Y = \quad 0 \to C \otimes_R M \to C \otimes_R \operatorname{Hom}_R(C, I^0) \to C \otimes_R \operatorname{Hom}_R(C, I^1) \to \cdots$$

where, for each $i \geq 0$, the module I^i has finite injective dimension and thus $I^i \in \mathcal{B}_C$ by Proposition 2.8(d). We have

$$C \otimes_R {}^+Y \cong \quad 0 \to C \otimes_R M \to I^0 \to I^1 \to \cdots$$

since, by Definition 2.7, $C \otimes_R \operatorname{Hom}_R(C, I^i) \cong I^i$ for each i. To show this complex is exact, choose E to be any faithfully injective R-module. Since ${}^+Y$ is an augmented proper $\mathcal{I}_C$-injective coresolution, by Definition 3.3(3) the complex $\operatorname{Hom}_R({}^+Y, \operatorname{Hom}_R(C, E)) \cong \operatorname{Hom}_R(C \otimes_R {}^+Y, E)$ is exact. Since E is faithfully injective, the complex $C \otimes_R {}^+Y$ is exact.

Proposition 3.9 ([11, Proposition 2.2]). *Let C be a semidualizing R-module, M any R-module, and $n > 0$ an integer.*

(a) *The following conditions are equivalent:*

 (i) *There exists an augmented proper $\mathcal{P}_C$-projective resolution of M that is exact in homological degree less than n.*

 (ii) *Each augmented proper $\mathcal{P}_C$-projective resolution of M is exact in homological degree less than n.*

 (iii) *The natural homomorphism $\xi_M^C : C \otimes_R \operatorname{Hom}_R(C, M) \to M$ is an isomorphism and for $0 < i < n$ we have $\operatorname{Tor}_i^R(C, \operatorname{Hom}_R(C, M)) = 0$.*

(b) *The following conditions are equivalent:*

 (i) *There exists an augmented proper $\mathcal{I}_C$-injective coresolution of M that is exact in cohomological degree less than n.*

 (ii) *Each augmented proper $\mathcal{I}_C$-injective coresolution of M is exact in cohomological degree less than n.*

 (iii) *The natural homomorphism $\gamma_M^C : M \to \operatorname{Hom}_R(C, C \otimes_R M)$ is an isomorphism and for $0 < i < n$ we have $\operatorname{Ext}_R^i(C, C \otimes_R M) = 0$.*

Proof. (a) (ii) $\Rightarrow$ (i). This follows immediately from Proposition 3.4(a).

(i) $\Rightarrow$ (iii). Let X^+ be an augmented proper $\mathcal{P}_C$-projective resolution of M that is exact in degree less than n. We have shown in Lemma 3.8(a) that $\operatorname{Hom}_R(C, X)$ is a projective resolution of $\operatorname{Hom}_R(C, M)$. Since each X_i is in $\mathcal{P}_C \subset \mathcal{B}_C$ by Corollary 2.10(a), we have $C \otimes_R \operatorname{Hom}_R(C, X) \cong X$, and this complex is exact in positive homological degree less than n. This gives the desired vanishing since $\operatorname{Tor}_i^R(C, \operatorname{Hom}_R(C, M)) = H_i(C \otimes_R \operatorname{Hom}_R(C, X))$. To show the given isomorphism, consider the commutative diagram

$$\begin{tikzcd}
C \otimes_R \operatorname{Hom}_R(C, X_1) \arrow[r] \arrow[d, "\cong"', "\xi_{X_1}^C"] & C \otimes_R \operatorname{Hom}_R(C, X_0) \arrow[r] \arrow[d, "\cong"', "\xi_{X_0}^C"] & C \otimes_R \operatorname{Hom}_R(C, M) \arrow[r] \arrow[d, "\xi_M^C"] & 0 \arrow[r] \arrow[d, "\cong"] & 0 \arrow[d, "\cong"] \\
X_1 \arrow[r] & X_0 \arrow[r] & M \arrow[r] & 0 \arrow[r] & 0
\end{tikzcd}$$

where the bottom row is exact by assumption and the top row is exact by Lemma 3.8(a) and right-exactness of the functor $C \otimes_R -$. An application of the Five Lemma ensures that ξ_M^C is an isomorphism.

(iii) $\Rightarrow$ (ii). Let X^+ be an augmented $\mathcal{P}_C$-projective resolution of M. Since ξ_M^C is an isomorphism and each $X_i \in \mathcal{B}_C$ by Corollary 2.10(a), we have $X^+ \cong C \otimes_R \operatorname{Hom}_R(C, X^+)$. Since $\operatorname{Tor}_i^R(C, \operatorname{Hom}_R(C, M)) = 0$ for $0 < i < n$ and the associated tensor product is right-exact, the complex X^+ is exact in homological degree less than n.

(b) (ii) $\Rightarrow$ (i). This follows immediately from Proposition 3.4(a).

(i) $\Rightarrow$ (iii). Let Y^+ be an augmented proper $\mathcal{I}_C$-injective coresolution of M that is exact in cohomological degree less than n. We have shown in Lemma 3.8(b) that $C \otimes_R Y$ is an injective coresolution of $C \otimes_R M$. Since each Y^i is in $\mathcal{I}_C \subset \mathcal{A}_C$ by Corollary 2.10(b), we have $\mathrm{Hom}_R(C, C \otimes_R Y) \cong Y$, and this complex is exact in positive cohomological degree less than n. This gives the desired vanishing since $\mathrm{Ext}^i_R(C, C \otimes_R M) = H^i(\mathrm{Hom}_R(C, C \otimes_R Y))$. To show the given isomorphism, let $Y^i := \mathrm{Hom}_R(C, I^i)$ for $i \geq 0$ and consider the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] \arrow[d, "\cong"'] & 0 \arrow[r] \arrow[d, "\cong"'] & M \arrow[r, "\epsilon"] \arrow[d, "\gamma^C_M"] & Y^0 \arrow[r] \arrow[d, "\cong"', "\gamma^C_{Y^0}"] & Y^1 \arrow[d, "\cong"', "\gamma^C_{Y^1}"] \\
0 \arrow[r] & 0 \arrow[r] & \mathrm{Hom}_R(C, C \otimes_R M) \arrow[r, "\overline{\epsilon}"] & \mathrm{Hom}_R(C, C \otimes_R Y^0) \arrow[r] & \mathrm{Hom}_R(C, C \otimes_R Y^1)
\end{tikzcd}$$

where the top row is exact by assumption and the bottom row is exact by Lemma 3.8(b) and left-exactness of the functor $\mathrm{Hom}_R(C, -)$. An application of the Five Lemma ensures that γ^C_M is an isomorphism.

(iii) $\Rightarrow$ (ii). Let ${}^+Y$ be an augmented proper $\mathcal{I}_C$-coinjective resolution of M. Since γ^C_M is an isomorphism and each $Y^i \in \mathcal{A}_C$ by Corollary 2.10(b), we have ${}^+Y \cong \mathrm{Hom}_R(C, C \otimes_R {}^+Y)$. Since we have $\mathrm{Ext}^i_R(C, C \otimes_R M) = 0$ for $0 < i < n$ and the associated Hom functor is left-exact, the complex ${}^+Y$ is exact in cohomological degree less than n.

Remark 3.10. Note that the case $n = 0$ in the previous result is false. To see this, suppose by way of contradiction that (b) $\Rightarrow$ (c). Consider the case when R is a local ring with maximal ideal $\mathfrak{m}$ and residue field $k := R/\mathfrak{m}$. If we choose $C \ncong R$, then C is not cyclic (as in [7, Corollary 2.1.14]), so Nakayama's lemma implies that $\mathrm{Hom}_R(C, k) \cong k^{\beta_0} \cong C \otimes_R k$ for some integer $\beta_0 \geq 2$. Then

$$C \otimes_R \mathrm{Hom}_R(C, k) \cong C \otimes_R k^{\beta_0} \cong (C \otimes_R k)^{\beta_0} \cong (k^{\beta_0})^{\beta_0} = k^{\beta_0^2}.$$

Since $C \otimes_R k \cong k^{\beta_0}$ and $\beta_0 \neq 0$, there is a surjection $C \otimes_R k \twoheadrightarrow k$. There is a surjection $R \twoheadrightarrow k$, so right-exactness of the tensor product implies that $C \otimes_R R \twoheadrightarrow C \otimes_R k \twoheadrightarrow k$. For any augmented proper $\mathcal{P}_C$-projective resolution X^+ of k we have the following commutative diagram by properness:

$$\begin{tikzcd}
 & C \otimes_R R \arrow[dl] \arrow[d] \\
X_0 \arrow[r, "\tau"'] & k
\end{tikzcd}$$

It follows that τ is a surjection, so every augmented proper $\mathcal{P}_C$-projective resolution of k is exact in homological degree -1. By assumption and our previous work, we have the isomorphism $k^{\beta_0^2} \cong C \otimes_R \operatorname{Hom}_R(C,k) \cong k$, which is a contradiction since $\beta_0 \geq 2$.

Corollary 3.11 ([11, Corollary 2.4]). *Let C be a semidualizing R-module and M an R–module.*

(a) *If $M \in \mathcal{B}_C$, then every augmented proper $\mathcal{P}_C$-projective resolution of M is exact.*

(b) *If $M \in \mathcal{A}_C$, then every augmented proper $\mathcal{I}_C$-injective coresolution of M is exact.*

Proof. Both results follow immediately from Proposition 3.9 and Definition 2.7.

We can use Foxby equivalence to relate membership in the Foxby classes to $\mathcal{P}_C$-projective and $\mathcal{I}_C$-injective dimensions.

Corollary 3.12 ([11, Corollary 2.9]). *Let C be a semidualizing R-module and M an R-module.*

(a) *If $\mathcal{P}_C\text{-}\mathrm{pd}_R(M)$ is finite, then $M \in \mathcal{B}_C$.*

(b) *If $\mathcal{I}_C\text{-}\mathrm{id}_R(M)$ is finite, then $M \in \mathcal{A}_C$.*

Proof. (a) Suppose $\mathcal{P}_C\text{-}\mathrm{pd}_R(M) = n$. By Definition 3.5, there exists an augmented proper $\mathcal{P}_C$-projective resolution X^+ such that $X_i = 0$ for all $i > n$. By Lemma 3.8(a), this means that $\operatorname{Hom}_R(C, X^+)$ is a bounded augmented projective resolution of $\operatorname{Hom}_R(C,M)$, so $\operatorname{Hom}_R(C,M)$ has finite flat dimension. It follows by Proposition 2.8(b) that $\operatorname{Hom}_R(C,M) \in \mathcal{A}_C$, and Theorem 2.9(a) implies that $M \in \mathcal{B}_C$.

(b) Suppose $\mathcal{I}_C\text{-}\mathrm{id}_R(M) = n$. By Definition 3.5, there exists an augmented proper $\mathcal{I}_C$-injective coresolution ${}^+Y$ such that $Y^i = 0$ for all $i > n$. By Lemma 3.8(b), this means that $C \otimes_R {}^+Y$ is a bounded augmented injective coresolution of $C \otimes_R M$, so $C \otimes_R M$ has finite injective dimension. It follows by Proposition 2.8(d) that $C \otimes_R M \in \mathcal{B}_C$, and Theorem 2.9(b) implies that $M \in \mathcal{A}_C$.

Corollary 3.13 ([11, Corollary 2.10]). *Let C be a semidualizing R-module and M any R-module.*

(a) *The inequality $\mathcal{P}_C\text{-}\mathrm{pd}_R(M) \leq n$ holds if and only if there is an exact sequence*

$$0 \to C \otimes_R P_n \to \cdots C \otimes_R P_0 \to M \to 0$$

with each P_i a projective R-module. If such a sequence exists, then it is proper and $M \in \mathcal{B}_C$.

(b) *The inequality $\mathcal{I}_C\text{-}\mathrm{id}_R(M) \leq n$ holds if and only if there is an exact sequence*

$$0 \to M \to \operatorname{Hom}_R(C, I^0) \to \cdots \to \operatorname{Hom}_R(C, I^n) \to 0$$

with each I^i an injective R-module. If such a sequence exists, then it is proper and $M \in \mathcal{A}_C$.

Proof. (a) Suppose first that $\mathcal{P}_C\text{-pd}_R(M) \leq n$ for some n. Then $M \in \mathcal{B}_C$ by Corollary 3.12(a), so Corollary 3.11(a) implies that every augmented proper $\mathcal{P}_C$-projective resolution of M is exact. Since M has an augmented proper $\mathcal{P}_C$-projective resolution X^+ such that $X_i = 0$ for $i > n$, we are done. To show the converse, let

$$X^+ := \quad 0 \longrightarrow C \otimes_R P_n \xrightarrow{\partial_n} \cdots \xrightarrow{\partial_1} C \otimes_R P_0 \xrightarrow{\tau} M \longrightarrow 0$$

be exact, where each P_i is projective. Since each $C \otimes_R P_i \in \mathcal{B}_C$ by Corollary 2.10(a), we have that $M \in \mathcal{B}_C$ by Proposition 2.8(c). Consider the associated short exact sequences

$$0 \to K_i \to C \otimes_R P_i \to K_{i-1} \to 0$$

for $K_i := \operatorname{Im} \partial_{i+1}$, where $K_{-1} := \operatorname{Im} \tau = M$. We have the long exact sequence in $\operatorname{Ext}^i_R(C, -)$

$$0 \to \operatorname{Hom}_R(C, K_i) \to \operatorname{Hom}_R(C, C \otimes_R P_i) \to \operatorname{Hom}_R(C, K_{i-1}) \to \operatorname{Ext}^1_R(C, K_i) \to \cdots$$

with $\operatorname{Ext}^1_R(C, K_i) = 0$ since $K_i \in \mathcal{B}_C$ by Proposition 2.8(c). A standard sequence-splicing argument implies exactness of $\operatorname{Hom}_R(C, X^+)$. If Q is any projective R-module, Hom-tensor adjointness implies that $\operatorname{Hom}_R(C \otimes_R Q, X^+) \cong \operatorname{Hom}_R(Q, \operatorname{Hom}_R(C, X^+))$ is exact, so X^+ is proper and the desired inequality holds.

(b) Suppose first that $\mathcal{I}_C\text{-id}_R(M) \leq n$ for some n. Then $M \in \mathcal{A}_C$ by Corollary 2.8(b), so Corollary 3.11(b) implies that every augmented proper $\mathcal{I}_C$-injective coresolution of M is exact. Since M has an augmented proper $\mathcal{I}_C$-injective coresolution ${}^+Y$ such that $Y^i = 0$ for $i > n$, we are done. To show the converse, let

$${}^+Y := \quad 0 \longrightarrow M \xrightarrow{\epsilon} \operatorname{Hom}_R(C, I^0) \xrightarrow{\partial^0} \cdots \xrightarrow{\partial^{n-1}} \operatorname{Hom}_R(C, I^n) \longrightarrow 0$$

be exact, where each I^i is injective. Since each $\operatorname{Hom}_R(C, I^i) \in \mathcal{A}_C$ by Corollary 3.12(b), we have that $M \in \mathcal{B}_C$ by Proposition 2.8(a). Consider the associated short exact sequences

$$0 \to L^{i-1} \to \operatorname{Hom}_R(C, I^i) \to L^i \to 0$$

for $L^i := \operatorname{Im} \partial^i \cong M$, where $L^{-1} := \operatorname{Im} \epsilon$. We have the long exact sequence in $\operatorname{Tor}^R_i(C, -)$

$$\cdots \to \operatorname{Tor}^R_1(C, L^i) \to C \otimes_R L^{i-1} \to C \otimes_R \operatorname{Hom}_R(C, I^i) \to C \otimes_R L^i \to 0$$

with $\operatorname{Tor}^R_1(C, L^i) = 0$ since $L^i \in \mathcal{A}_C$ by Definition 2.7. A standard sequence-splicing argument implies exactness of $C \otimes_R {}^+Y$. If J is any injective R-module, Hom-tensor adjointness implies that the complex $\mathrm{Hom}_R({}^+Y \otimes_R C, J) \cong \mathrm{Hom}_R({}^+Y, \mathrm{Hom}_R(C,J))$ is exact, so ${}^+Y$ is proper; hence the desired inequality.

The following corollary shows that modules of $\mathcal{P}_C$-projective dimension zero are in $\mathcal{P}_C$ and, dually, that modules of $\mathcal{I}_C$-injective dimension zero are in $\mathcal{I}_C$. Because augmented proper $\mathcal{P}_C$-projective resolutions and $\mathcal{I}_C$-injective coresolutions are not exact in general, this result requires a proof.

Corollary 3.14. *Let C be a semidualizing R-module and M an R-module.*

(a) *We have $\mathcal{P}_C\text{-pd}_R(M) = 0$ if and only if $M \in \mathcal{P}_C$.*

(b) *We have $\mathcal{I}_C\text{-id}_R(M) = 0$ if and only if $M \in \mathcal{I}_C$.*

Proof. (a) Suppose first that $M \in \mathcal{P}_C \subset \mathcal{B}_C$. Then $0 \to C \otimes_R P \xrightarrow{\cong} M \to 0$ is an augmented proper $\mathcal{P}_C$-projective resolution of M, so $\mathcal{P}_C\text{-pd}_R(M) = 0$. Conversely, if $\mathcal{P}_C\text{-pd}_R(M) = 0$, then $M \in \mathcal{B}_C$ by Corollary 3.12(a). Corollary 3.11(a) implies that the augmented proper $\mathcal{P}_C$-projective resolution $0 \to C \otimes_R P \to M \to 0$ of M is exact, so $M \cong C \otimes_R P$ and hence $M \in \mathcal{P}_C$.

(b) Suppose first that $M \in \mathcal{I}_C \subset \mathcal{A}_C$. Then $0 \to M \xrightarrow{\cong} \mathrm{Hom}_R(C,I) \to 0$ is an augmented proper $\mathcal{I}_C$-injective coresolution of M, so $\mathcal{I}_C\text{-id}_R(M) = 0$. Conversely, if $\mathcal{I}_C\text{-id}_R(M) = 0$, then $M \in \mathcal{A}_C$ by Corollary 3.12(b). Corollary 3.11(b) implies that the augmented proper $\mathcal{I}_C$-injective coresolution $0 \to M \to \mathrm{Hom}_R(C,I) \to 0$ of M is exact, so $M \cong \mathrm{Hom}_R(C,I)$ and hence $M \in \mathcal{I}_C$.

The next result establishes a connection between projective and C-projective dimension, and between injective and C-injective dimension, when C is semidualizing. It is Theorem 1.3 from the introduction.

Theorem 3.15 ([11, Theorem 2.11])**.** *Let C be a semidualizing R-module and M any R-module. The following equations hold:*

(a) $\mathrm{pd}_R(M) = \mathcal{P}_C\text{-pd}_R(C \otimes_R M)$*;*

(b) $\mathcal{I}_C\text{-id}_R(M) = \mathrm{id}_R(C \otimes_R M)$*;*

(c) $\mathcal{P}_C\text{-pd}_R(M) = \mathrm{pd}_R(\mathrm{Hom}_R(C,M))$*; and*

(d) $\mathrm{id}_R(M) = \mathcal{I}_C\text{-id}_R(\mathrm{Hom}_R(C,M))$.

Proof. (a) Let $\mathrm{pd}_R(M) = n < \infty$, and consider an augmented projective resolution X^+ of M of length n. Since M has finite flat dimension, $M \in \mathcal{A}_C$ by Proposition 2.8(b) and by Definition 2.7 we have $\mathrm{Tor}_i^R(C,M) = 0$ for $i \geq 1$. This and right-exactness of the tensor product imply the complex $C \otimes_R X^+$ is exact of length n, and hence an augmented proper $\mathcal{P}_C$-projective resolution of $C \otimes_R M$ by Corollary 3.13(a). This implies the inequality $\mathcal{P}_C\text{-pd}_R(C \otimes_R M) \leq \text{pd}_R(M)$. To show the reverse inequality, suppose we have $\mathcal{P}_C\text{-pd}_R(C \otimes_R M) = m < \infty$. We have shown in Corollary 3.13(a) that there exists an exact augmented proper $\mathcal{P}_C$-projective resolution Z^+ of $C \otimes_R M$ of length m and $C \otimes_R M \in \mathcal{B}_C$, so Theorem 2.9(b) implies that $M \in \mathcal{A}_C$. Hence $\text{Hom}_R(C, C \otimes_R M) \cong M$. By Lemma 3.8(a), we have that $\text{Hom}_R(C, Z^+)$ is an augmented projective resolution of $\text{Hom}_R(C, C \otimes_R M) \cong M$ of length m, so $\text{pd}_R(M) \leq \mathcal{P}_C\text{-pd}_R(C \otimes_R M)$ and we are done.

(b) Let $\mathcal{I}_C\text{-id}_R(M) = n < \infty$. We have shown in Corollary 3.13(b) that there exists an exact augmented proper $\mathcal{I}_C$-injective coresolution ${}^+Y$ of M of length n and $M \in \mathcal{A}_C$. By Lemma 3.8(b), we have that $C \otimes_R {}^+Y$ is an augmented injective coresolution of $C \otimes_R M$ of length n, so $\text{id}_R(C \otimes_R M) \leq \mathcal{I}_C\text{-id}_R(M)$. To show the reverse inequality, suppose we have $\text{id}_R(C \otimes_R M) = m < \infty$, and consider an augmented injective coresolution ${}^+Z$ of $C \otimes_R M$ of length m. Since $C \otimes_R M$ has finite injective dimension, we have $C \otimes_R M \in \mathcal{B}_C$ by Proposition 2.8(d), and by Definition 2.7 we have $\text{Ext}^i_R(C, C \otimes_R M) = 0$ for $i \geq 1$. This and left-exactness of the Hom functor imply the complex $\text{Hom}_R(C, {}^+Z)$ is exact of length m, and hence an augmented proper $\mathcal{I}_C$-injective coresolution of $\text{Hom}_R(C, C \otimes_R M) \cong M$ since $M \in \mathcal{A}_C$, so $\mathcal{I}_C\text{-id}_R(M) \leq \text{id}_R(C \otimes_R M)$ and we are done.

(c) Let $\mathcal{P}_C\text{-pd}_R(M) = n < \infty$. We have shown in Corollary 3.13(a) that there exists an exact augmented proper $\mathcal{P}_C$-projective resolution X^+ of M of length n and $M \in \mathcal{B}_C$. By Lemma 3.8(a), we have that $\text{Hom}_R(C, X^+)$ is an augmented projective resolution of $\text{Hom}_R(C, M)$ of length n, so $\text{pd}_R(\text{Hom}_R(C, M)) \leq \mathcal{P}_C\text{-pd}_R(M)$. To show the reverse inequality, suppose we have $\text{pd}_R(\text{Hom}_R(C, M)) = m < \infty$, and consider an augmented projective resolution Z^+ of $\text{Hom}_R(C, M)$ of length m. Since $\text{Hom}_R(C, M)$ has finite projective dimension, we have $\text{Hom}_R(C, M) \in \mathcal{A}_C$ by Proposition 2.8(b) and by Definition 2.7, we have $\text{Tor}^R_i(C, \text{Hom}_R(C, M)) = 0$ for $i \geq 1$. This and right-exactness of the tensor product imply the complex $C \otimes_R \text{Hom}_R(C, M)$ is exact of length m, and hence an augmented proper $\mathcal{P}_C$-projective resolution of $C \otimes_R \text{Hom}_R(C, M) \cong M$ since $M \in \mathcal{B}_C$, so $\mathcal{P}_C\text{-pd}_R(M) \leq \text{pd}_R(\text{Hom}_R(C, M))$ and we are done.

(d) Let $\text{id}_R(M) = n < \infty$, and consider an augmented injective coresolution ${}^+Y$ of M of length n. Since M has finite injective dimension, $M \in \mathcal{B}_C$ by Proposition 2.8(d) and by Definition 2.7 we have $\text{Ext}^i_R(C, M) = 0$ for $i \geq 1$. This and left-exactness of the Hom functor imply the complex $\text{Hom}_R(C, {}^+Y)$ is exact of length n, and hence an augmented proper $\mathcal{I}_C$-injective coresolution of $\text{Hom}_R(C, M)$ by Corollary 3.13(b). This implies the inequality $\mathcal{I}_C\text{-id}_R(\text{Hom}_R(C, M)) \leq \text{id}_R(M)$. To show the reverse inequality, suppose we have $\mathcal{I}_C\text{-id}_R(\text{Hom}_R(C, M)) = m < \infty$. We have shown in Corollary 3.13(b) that there exists an exact augmented proper $\mathcal{I}_C$-injective coresolution ${}^+Z$ of $\text{Hom}_R(C, M)$ of length m and $\text{Hom}_R(C, M) \in \mathcal{A}_C$, so Theorem 2.9(b) implies that $M \in \mathcal{B}_C$. Hence $C \otimes_R \text{Hom}_R(C, M) \cong M$. By Lemma 3.8(b), we have that $C \otimes_R {}^+Z$ is an augmented injective coresolution of $C \otimes_R \text{Hom}_R(C, M) \cong M$ of length m, so

$\mathrm{id}_R(M) \leq \mathcal{I}_C\text{-}\mathrm{id}_R(\mathrm{Hom}_R(C,M))$ and we are done.

Armed with these results, we can extend Foxby equivalence to include the modules of finite C-projective and C-injective dimensions.

Theorem 3.16 (Extended Foxby equivalence [11, Theorem 2.12])**.** *Let C be a semidualizing R-module, and let $n \geq 0$ be an integer. Let $\widehat{\mathcal{P}_C}(R)_{\leq n}, \widehat{\mathcal{P}}(R)_{\leq n}, \widehat{\mathcal{I}_C}(R)_{\leq n}, \widehat{\mathcal{I}}_{\leq n}$ be the classes of R-modules of C-projective, projective, C-injective, and injective dimension of at most n, respectively. Let $\mathcal{P}(R)$ and $\mathcal{I}(R)$ be the classes of projective and injective R-modules, respectively. We have the following equivalences of classes:*

$$\begin{tikzcd}
\mathcal{P}(R) \arrow[r, shift left, "C\otimes_R -"] \arrow[d, hook] & \mathcal{P}_C(R) \arrow[l, shift left, "\mathrm{Hom}_R(C,-)"] \arrow[d, hook] \\
\widehat{\mathcal{P}}(R)_{\leq n} \arrow[r, shift left, "C\otimes_R -"] \arrow[d, hook] & \widehat{\mathcal{P}_C}(R)_{\leq n} \arrow[l, shift left, "\mathrm{Hom}_R(C,-)"] \arrow[d, hook] \\
\mathcal{A}_C(R) \arrow[r, shift left, "C\otimes_R -"] & \mathcal{B}_C(R) \arrow[l, shift left, "\mathrm{Hom}_R(C,-)"] \\
\widehat{\mathcal{I}_C}(R)_{\leq n} \arrow[r, shift left, "C\otimes_R -"] \arrow[u, hook] & \widehat{\mathcal{I}}(R)_{\leq n} \arrow[l, shift left, "\mathrm{Hom}_R(C,-)"] \arrow[u, hook] \\
\mathcal{I}_C(R) \arrow[r, shift left, "C\otimes_R -"] \arrow[u, hook] & \mathcal{I}(R) \arrow[l, shift left, "\mathrm{Hom}_R(C,-)"] \arrow[u, hook]
\end{tikzcd}$$

Proof. We prove the claim only for the top half of the diagram; the bottom half follows similarly. The equivalences $\mathcal{P}(R) \longleftrightarrow \mathcal{P}_C(R)$ and $\widehat{\mathcal{P}}(R)_{\leq n} \longleftrightarrow \widehat{\mathcal{P}_C}(R)_{\leq n}$ follow from Corollary 3.14 and Theorem 3.15(a). The equivalence $\mathcal{A}_C(R) \longleftrightarrow \mathcal{B}_C(R)$ follows from Theorem 2.9. The inclusions $\mathcal{P}(R) \hookrightarrow \widehat{\mathcal{P}}(R)_{\leq n}$ and $\mathcal{P}_C(R) \hookrightarrow \widehat{\mathcal{P}_C}(R)_{\leq n}$ are immediate. The inclusion $\widehat{\mathcal{P}}(R)_{\leq n} \hookrightarrow \mathcal{A}_C(R)$ follows from Proposition 2.8(b), and the inclusion $\widehat{\mathcal{P}_C}(R)_{\leq n} \hookrightarrow \mathcal{B}_C(R)$ follows from Corollary 3.12(a).

The next result is Theorem 1.4 from the introduction, and is a version of the Auslander-Buchsbaum-Serre result in Theorem 1.1.

Theorem 3.17 ([11, Proposition 5.1])**.** *Let R be a local noetherian ring with residue field k, and C a semidualizing R-module. The following conditions are equivalent:*

(i) *R is regular;*

(ii) *$\mathcal{P}_C\text{-}\mathrm{pd}_R(M) < \infty$ for every finitely generated R-module M; and*

(iii) *$\mathcal{P}_C\text{-}\mathrm{pd}_R(k) < \infty$.*

Proof. (ii) $\Rightarrow$ (iii). This is immediate.

(iii) $\Rightarrow$ (i). Suppose that $\mathcal{P}_C\text{-pd}_R(k)$ is finite. By Theorem 3.15(c), $\text{pd}_R(\text{Hom}_R(C,k)) = \mathcal{P}_C\text{-pd}_R(k)$ is also finite. Since $\text{Hom}_R(C,k)$ is a nonzero vector space over k by Nakayama's lemma, we have that $\text{pd}_R(k)$ is finite. Theorem 1.1 implies that R is regular.

(i) $\Rightarrow$ (ii). Suppose that R is a regular local ring; then R is Gorenstein. It can be shown, as in [7, Corollary 4.1.9], that in this case the only semidualizing R-module is R itself. Theorem 1.1 implies that $\text{pd}_R(M) = \mathcal{P}_C\text{-pd}_R(M) < \infty$ for any R-module M.

4. RELATIVE COHOMOLOGY AND Ext-VANISHING

Recall that in Section 1 we characterized projective and injective modules by Ext-vanishing. When we compute Ext-modules using $\mathcal{P}_C$-resolutions or $\mathcal{I}_C$-coresolutions, we obtain an analagous result.

Theorem 4.1 ([11, Theorem 3.1]). *Let C be a semidualizing R-module and M any R-module.*

(a) *The following conditions are equivalent:*

(i) $\operatorname{Ext}^1_{\mathcal{P}_C}(M,-) = 0$;

(ii) $\operatorname{Ext}^i_{\mathcal{P}_C}(M,-) = 0$ *for* $i \geq 1$; *and*

(iii) $M \in \mathcal{P}_C$.

(b) *The following conditions are equivalent:*

(i) $\operatorname{Ext}^1_{\mathcal{I}_C}(-,M) = 0$;

(ii) $\operatorname{Ext}^i_{\mathcal{I}_C}(-,M) = 0$ *for* $i \geq 1$; *and*

(iii) $M \in \mathcal{I}_C$.

Proof. (a) (iii) $\Rightarrow$ (ii). Suppose that $M \in \mathcal{P}_C$; then the complex $\cdots \to 0 \to M \to M \to 0$ is an exact augmented proper $\mathcal{P}_C$-projective resolution of M so we have the desired vanishing.

(ii) $\Rightarrow$ (i). This is immediate.

(i) $\Rightarrow$ (iii). Consider the commutative diagram

$$\begin{tikzcd} X^+ := \cdots \arrow[r] & C\otimes_R P_2 \arrow[r, "\partial_2"] & C\otimes_R P_1 \arrow[rr, "\partial_1"] \arrow[dr, "\alpha"'] & & C\otimes_R P_0 \arrow[r, "\tau"] & M \arrow[r] & 0 \\ & & & K_0 \arrow[ur, hook, "\beta"'] & & & \end{tikzcd}$$

where X^+ is an augmented proper $\mathcal{P}_C$-projective resolution of M. Here β is the inclusion map and $K_0 := \ker\tau$. Since X^+ is an R-complex, we have $\beta\alpha\partial_2 = \partial_1\partial_2 = 0$; since β is injective, we have $\alpha\partial_2 = 0$. Since $\operatorname{Ext}^1_{\mathcal{P}_C}(M,K_0) = 0$ by assumption, the sequence

$$\operatorname{Hom}_R(C\otimes_R P_0, K_0) \xrightarrow{\partial_1^*} \operatorname{Hom}_R(C\otimes_R P_1, K_0) \xrightarrow{\partial_2^*} \operatorname{Hom}_R(C\otimes_R P_2, K_0)$$

is exact. Since $0 = \alpha\partial_2 \in \operatorname{Hom}_R(C\otimes_R P_2, K_0)$, we have $\partial_2^*(\alpha) = 0$, so $\alpha \in \operatorname{Im}\partial_1^*$. Hence there exists $\gamma \in \operatorname{Hom}_R(C\otimes_R P_0, K_0)$ such that $\alpha = \partial_1^*(\gamma) = \gamma\partial_1 = \gamma\beta\alpha$ by commutativity of the diagram. The last equation implies that $\operatorname{Hom}_R(C,\alpha) = \operatorname{Hom}_R(C,\gamma\beta\alpha) = \operatorname{Hom}_R(C,\gamma)\operatorname{Hom}_R(C,\beta)\operatorname{Hom}_R(C,\alpha)$. Observe that the sequence

$$0 \longrightarrow K_1 \longrightarrow C \otimes_R P_1 \xrightarrow{\alpha} K_0 \longrightarrow 0$$

is $\operatorname{Hom}(C,-)$-exact by properness, where $K_1 := \ker \partial_1$. This means that $\operatorname{Hom}_R(C,\alpha)$ is surjective, so we have the composition $\operatorname{Hom}_R(C,\gamma)\operatorname{Hom}_R(C,\beta) = 1_{\operatorname{Hom}(C,K_0)}$. This and properness of X imply that the sequence

$$\begin{tikzcd} 0 \arrow[r] & \operatorname{Hom}_R(C,K_0) \arrow[r, shift left, "\operatorname{Hom}(C,\beta)"] & \operatorname{Hom}_R(C, C\otimes_R P_0) \arrow[l, shift left, "\operatorname{Hom}(C,\gamma)"] \arrow[r] & \operatorname{Hom}_R(C,M) \arrow[r] & 0 \end{tikzcd}$$

is split exact. Now $\operatorname{Hom}_R(C, C\otimes_R P_0) \cong P_0$ is projective, so $\operatorname{Hom}_R(C,M)$ is also projective and hence $\mathcal{P}_C\text{-}\operatorname{pd}_R(M) = \operatorname{pd}_R(\operatorname{Hom}_R(C,M)) = 0$ by Theorem 3.15(c). Corollary 3.13(a) implies that there is an exact sequence $0 \to C \otimes_R Q_0 \to M \to 0$, so $M \cong C \otimes_R Q_0$ for some projective Q_0 and $M \in \mathcal{P}_C$.

(b) (iii) $\Rightarrow$ (ii). Suppose that $M \in \mathcal{I}_C$; then the complex $0 \to M \to M \to 0 \to \cdots$ is an exact augmented proper $\mathcal{I}_C$-injective coresolution of M so we have the desired vanishing.

(ii) $\Rightarrow$ (i). This is immediate.

(i) $\Rightarrow$ (iii). Consider the commutative diagram

$$\begin{tikzcd} {}^{+}Y := & 0 \arrow[r] & M \arrow[r, "\epsilon"] & \operatorname{Hom}_R(C,I^0) \arrow[rr, "\partial^0"] \arrow[dr, "\beta"'] & & \operatorname{Hom}_R(C,I^1) \arrow[r, "\partial^1"] & \operatorname{Hom}_R(C,I^2) \arrow[r] & \cdots \\ & & & & L^0 \arrow[ur, "\alpha"'] & & & \end{tikzcd}$$

where ${}^{+}Y$ is an augmented proper $\mathcal{I}_C$-injective coresolution of M. Here $L^0 := \operatorname{Im}\partial^0$. Since ${}^{+}Y$ is an R-complex, we have $\partial^1\alpha\beta = \partial^1\partial^0 = 0$; since β is surjective, we have $\partial^1\alpha = 0$. Since $\operatorname{Ext}^1_{\mathcal{I}_C}(L^0, M) = 0$ by assumption, the sequence

$$\operatorname{Hom}_R(L^0, \operatorname{Hom}_R(C,I^0)) \xrightarrow{\partial^0_*} \operatorname{Hom}_R(L^0, \operatorname{Hom}_R(C,I^1)) \xrightarrow{\partial^1_*} \operatorname{Hom}_R(L^0, \operatorname{Hom}_R(C,I^2))$$

is exact. Since $0 = \partial^1\alpha \in \operatorname{Hom}_R(L^0, \operatorname{Hom}_R(C,I^2))$, we have $\partial^1_*(\alpha) = 0$, so $\alpha \in \operatorname{Im}\partial^0_*$. Hence there exists $\gamma \in \operatorname{Hom}_R(L^0, \operatorname{Hom}_R(C,I^0))$ such that $\alpha = \partial^0_*(\gamma) = \partial^0\gamma = \alpha\beta\gamma$ by commutativity of the diagram. The last equation implies that $C\otimes\alpha = C\otimes(\gamma\beta\alpha) = (C\otimes\gamma)(C\otimes\beta)(C\otimes\alpha)$. Observe that the sequence

$$0 \longrightarrow L^0 \xrightarrow{\alpha} \operatorname{Hom}_R(C,I^0) \longrightarrow L^1 \longrightarrow 0$$

is $(C\otimes_R -)$-exact, where $L^1 := \operatorname{Im}\partial^1$. This means that $C\otimes\alpha$ is injective, so we have the composition

$(C \otimes \gamma)(C \otimes \beta) = 1_{C \otimes_R \operatorname{Hom}_R(C,I^0)}$. This and properness of Y imply that the sequence

$$0 \longrightarrow C \otimes_R M \longrightarrow C \otimes_R \operatorname{Hom}_R(C, I^0) \underset{C\otimes\gamma}{\overset{C\otimes\beta}{\rightleftarrows}} C \otimes_R L^0 \longrightarrow 0$$

is split exact. Now $C \otimes_R \operatorname{Hom}_R(C, I^0) \cong I^0$ is injective, so $C \otimes_R M$ is also injective and hence $\mathcal{I}_C\text{-id}_R(M) = \operatorname{id}_R(C \otimes_R M) = 0$ by Theorem 3.15(b). Corollary 3.13(b) implies that there is an exact sequence $0 \to M \to \operatorname{Hom}_R(C, J^0) \to 0$, so $M \cong \operatorname{Hom}_R(C, J^0)$ for some injective J^0 and $M \in \mathcal{I}_C$.

The next lemma is sometimes called "dimension shifting" or "degree shifting".

Lemma 4.2 ([11, 1.7]). *Let M, N, and C be R-modules, and $n \geq 1$.*

(a) *If X^+ is an augmented proper $\mathcal{P}_C$-projective resolution of M with the usual notation, then there is an isomorphism $\operatorname{Ext}^{n+1}_{\mathcal{P}_C}(M, N) \cong \operatorname{Ext}^1_{\mathcal{P}_C}(\operatorname{coker} \partial_{n+1}, N)$ for all $n \geq 0$.*

(b) *If ${}^+Y$ is an augmented proper $\mathcal{I}_C$-injective coresolution of N with the usual notation, then there is an isomorphism $\operatorname{Ext}^{n+1}_{\mathcal{I}_C}(M, N) \cong \operatorname{Ext}^1_{\mathcal{I}_C}(M, \ker \partial^n)$ for all $n \geq 0$.*

The next result extends Theorem 4.1 by dimension shifting to investigate modules of finite C-projective and C-injective dimension. It contains Theorem 1.5 from the introduction.

Theorem 4.3 ([11, Theorem 3.2]). *Let C be a semidualizing R-module, M any R-module, and $n \geq 0$.*

(a) *The following conditions are equivalent:*

   (i) $\operatorname{Ext}^{n+1}_{\mathcal{P}_C}(M, -) = 0$;

   (ii) $\operatorname{Ext}^{n+i}_{\mathcal{P}_C}(M, -) = 0$ *for $i \geq 1$; and*

   (iii) $\mathcal{P}_C\text{-pd}_R(M) \leq n$.

(b) *The following conditions are equivalent:*

   (i) $\operatorname{Ext}^{n+1}_{\mathcal{I}_C}(-, M) = 0$;

   (ii) $\operatorname{Ext}^{n+i}_{\mathcal{I}_C}(-, M) = 0$ *for $i \geq 1$; and*

   (iii) $\mathcal{I}_C\text{-id}_R(M) \leq n$.

Proof. (a) (iii) $\Rightarrow$ (ii). This follows by definition.

(ii) $\Rightarrow$ (i). This is immediate.

(i) $\Rightarrow$ (iii). Let

$$X^+ := \cdots \xrightarrow{\partial_{n+1}} C \otimes_R P_n \to C \otimes_R P_{n-1} \to \cdots \to M \to 0$$

be an augmented proper $\mathcal{P}_C$-projective resolution of M. There is a corresponding sequence

$$(X^+)' := \quad 0 \to \operatorname{coker} \partial_{n+1} \to C \otimes_R P_{n-1} \to \cdots \to M \to 0$$

such that $\operatorname{Hom}_R(C \otimes_R Q, (X^+)')$ is exact for every projective module Q. Lemma 4.2(a) implies that $\operatorname{Ext}^1_{\mathcal{P}_C}(\operatorname{coker} \partial_{n+1}, -) \cong \operatorname{Ext}^{n+1}_{\mathcal{P}_C}(M, -) = 0$ by assumption. By Theorem 4.1(a), we have that $\operatorname{coker} \partial_{n+1} \in \mathcal{P}_C$, so $(X^+)'$ is an augmented proper $\mathcal{P}_C$-projective resolution of M. The inequality $\mathcal{P}_C\text{-}\mathrm{pd}_R(M) \leq n$ follows by definition.

(b) (iii) $\Rightarrow$ (ii). This follows by definition.

(ii) $\Rightarrow$ (i). This is immediate.

(i) $\Rightarrow$ (iii). Let

$$^+Y := \quad 0 \to M \to \cdots \to \operatorname{Hom}_R(C, I^{n-1}) \to \operatorname{Hom}_R(C, I^n) \xrightarrow{\partial^n} \cdots$$

be an augmented proper $\mathcal{I}_C$-injective coresolution of M. There is a corresponding sequence

$$^+Y' := \quad 0 \to M \to \cdots \to \operatorname{Hom}_R(C, I^{n-1}) \to \ker \partial^n \to 0$$

such that $\operatorname{Hom}_R(^+Y', \operatorname{Hom}_R(C, J))$ is exact for every injective module J. Lemma 4.2(b) implies that $\operatorname{Ext}^1_{\mathcal{I}_C}(-, \ker \partial^n) \cong \operatorname{Ext}^{n+1}_{\mathcal{I}_C}(-, M) = 0$ by assumption. By Theorem 4.1(b), we have that $\ker \partial^n \in \mathcal{I}_C$, so $^+Y'$ is an augmented proper $\mathcal{I}_C$-injective coresolution of M. The inequality $\mathcal{I}_C\text{-}\mathrm{id}_R(M) \leq n$ follows by definition.

Recall that modules of finite projective and injective dimensions satisfy the standard "two of three" condition; that is, if any two modules in a short exact sequence have finite projective (resp. injective) dimension, then so does the third module. The next result shows that modules of finite $\mathcal{P}_C$-projective dimension and of finite $\mathcal{I}_C$-injective dimension satisfy an analagous condition.

Proposition 4.4 ([11, Proposition 3.4]). *Let C be a semidualizing R-module. Consider an exact sequence*

$$X = \quad 0 \to M' \to M \to M'' \to 0$$

of R-module homomorphisms.

(a) *If any two of the modules have finite $\mathcal{P}_C$-projective dimension, then so does the third.*

(b) *If any two of the modules have finite $\mathcal{I}_C$-injective dimension, then so does the third.*

Proof. (a) Suppose that any two of the (possibly nonzero) modules in X have finite $\mathcal{P}_C$-projective dimension. Since Corollary 3.12(a) implies that these two modules are in $\mathcal{B}_C$, Proposition 2.8(c) implies that the remaining module must also be in $\mathcal{B}_C$. Applying Definition 2.7, we have that $\operatorname{Ext}^1_R(C, M') = 0$ and hence the sequence

$$0 \to \operatorname{Hom}_R(C, M') \to \operatorname{Hom}_R(C, M) \to \operatorname{Hom}_R(C, M'') \to 0$$

is exact.

Since two of the (possible nonzero) modules in X have finite $\mathcal{P}_C$-projective dimension, the corresponding Hom-modules have finite projective dimension by Theorem 3.15(c). It can be shown (as in [8], Corollary VII.3.9) that this implies the remaining Hom-module must also have finite projective dimension, so Theorem 3.15(c) ensures that its corresponding module in X has finite $\mathcal{P}_C$-projective dimension.

(b) Suppose that any two of the (possibly nonzero) modules in X have finite $\mathcal{I}_C$-injective dimension. Since Corollary 3.12(b) implies that these two modules are in $\mathcal{A}_C$, Proposition 2.8(a) implies that the remaining module must also be in $\mathcal{A}_C$. Applying Definition 2.7, we have that $\operatorname{Tor}^R_1(C, M'') = 0$ and hence the sequence

$$0 \to C \otimes_R M' \to C \otimes_R M \to C \otimes_R M'' \to 0$$

is exact.

Since two of the (possible nonzero) modules in X have finite $\mathcal{I}_C$-injective dimension, the corresponding tensor products have finite injective dimension by Theorem 3.15(b). It can be shown (as in [8], Corollary VII.5.7) that this implies the remaining tensor product must also have finite injective dimension, so Theorem 3.15(b) ensures that its corresponding module in X has finite $\mathcal{I}_C$-injective dimension.

Extended Foxby equivalence is used to prove the next result, which relates relative and absolute Ext-modules.

Theorem 4.5 ([11, Theorem 4.1]). *Let C be a semidualizing R-module, and let M and N be any R-modules. Then for $i \geq 0$ we have the following isomorphisms:*

(a) $\operatorname{Ext}^i_{\mathcal{P}_C}(M, N) \cong \operatorname{Ext}^i_R(\operatorname{Hom}_R(C, M), \operatorname{Hom}_R(C, N))$*; and*

(b) $\operatorname{Ext}^i_{\mathcal{I}_C}(M, N) \cong \operatorname{Ext}^i_R(C \otimes_R M, C \otimes_R N)$.

Proof. (a) Let X^+ be an augmented proper $\mathcal{P}_C$-projective resolution of M, and choose $i \geq 0$. Lemma 3.8(a) implies that $\operatorname{Hom}_R(C, X)$ is a projective resolution of $\operatorname{Hom}_R(C, M)$. Hence, we have

$$\begin{aligned}
\operatorname{Ext}^i_{\mathcal{P}_C}(M, N) &= H^i(\operatorname{Hom}_R(X, N)) \\
&\cong H^i(\operatorname{Hom}_R(C \otimes_R \operatorname{Hom}_R(C, X), N)) \\
&\cong H^i(\operatorname{Hom}_R(\operatorname{Hom}_R(C, X), \operatorname{Hom}_R(C, N))) \\
&= \operatorname{Ext}^i_R(\operatorname{Hom}_R(C, M), \operatorname{Hom}_R(C, N))
\end{aligned}$$

where the first isomorphism follows since $\mathcal{P}_C \subset \mathcal{B}_C$ from Theorem 3.16 and each module in X is C-projective, and the second follows from Hom-tensor adjointness.

(b) Let ${}^+Y$ be an augmented proper $\mathcal{I}_C$-injective coresolution of N, and choose $i \geq 0$. Lemma 3.8(b) implies that $C \otimes_R {}^+Y$ is an injective coresolution of $C \otimes_R N$. Hence, we have

$$\begin{aligned}
\operatorname{Ext}^i_{\mathcal{I}_C}(M, N) &= H^i(\operatorname{Hom}_R(M, Y)) \\
&\cong H^i(\operatorname{Hom}_R(M, \operatorname{Hom}_R(C, C \otimes_R Y))) \\
&\cong H^i(\operatorname{Hom}_R(C \otimes_R M, C \otimes_R Y)) \\
&= \operatorname{Ext}^i_R(C \otimes_R M, C \otimes_R N)
\end{aligned}$$

where the first isomorphism follows since $\mathcal{I}_C \subset \mathcal{A}_C$ from Theorem 3.16 and each module in Y is C-injective, and the second follows from Hom-tensor adjointness.

The next three definitions are for use in Lemma 4.10 and Theorem 4.11.

Definition 4.6. Let X and Y be R-complexes and $f : X \to Y$ a chain map. We say f is a *quasiisomorphism* if the induced map $H_i(f) : H_i(X) \to H_i(Y)$ is an isomorphism for all i. In this case, we write $f : X \xrightarrow{\simeq} Y$.

Definition 4.7. A *double complex* of R-module homomorphisms is a commutative diagram

$$\begin{tikzcd}
 & \vdots \arrow[d] & \vdots \arrow[d] & \vdots \arrow[d] & \\
\cdots & P_{0,2} \arrow[l] \arrow[d, "d_{0,2}"] & P_{1,2} \arrow[l, "\partial_{1,2}"'] \arrow[d, "d_{1,2}"] & P_{2,2} \arrow[l, "\partial_{2,2}"'] \arrow[d, "d_{2,2}"] & \cdots \arrow[l] \\
\cdots & P_{0,1} \arrow[l] \arrow[d, "d_{0,1}"] & P_{1,1} \arrow[l, "\partial_{1,1}"'] \arrow[d, "d_{1,1}"] & P_{2,1} \arrow[l, "\partial_{2,1}"'] \arrow[d, "d_{2,1}"] & \cdots \arrow[l] \\
\cdots & P_{0,0} \arrow[l] \arrow[d] & P_{1,0} \arrow[l, "\partial_{1,0}"'] \arrow[d] & P_{2,0} \arrow[l, "\partial_{2,0}"'] \arrow[d] & \cdots \arrow[l] \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

where each row and column is an R-complex.

Definition 4.8. Let P be a double complex over R, with notation as above. Define $P_m := \bigoplus_{i+j=m} P_{i,j}$ for each m and the maps $D_m : P_m \to P_{m-1}$ via $D_m(c_{i,j}) := \partial_{i,j}(c_{i,j}) + (-1)^i d_{i,j}(c_{i,j})$. Then

$$\cdots \to P_1 \xrightarrow{D_1} P_0 \xrightarrow{D_0} P_{-1} \to \cdots$$

is the *total complex* of P, denoted $\operatorname{Tot} P$.

Remark 4.9. If P is a double complex over R, then $\operatorname{Tot} P$ is an R-complex.

Lemma 4.10 ([3, 6.14 and 6.12])**.** *Let $X \xrightarrow{\simeq} Y$ be a quasiisomorphism of R-complexes.*

(a) *If I is a bounded above complex of injective R-modules, then* $\operatorname{Tot}\operatorname{Hom}_R(X, I) \xleftarrow{\simeq} \operatorname{Tot}\operatorname{Hom}_R(Y, I)$.

(b) *If P is a bounded below complex of projective R-modules, then* $\operatorname{Tot}\operatorname{Hom}_R(P, X) \xrightarrow{\simeq} \operatorname{Tot}\operatorname{Hom}_R(P, Y)$.

The following result is Theorem 1.6 from the introduction.

Theorem 4.11 ([11, Corollary 4.2])**.** *Let C be a semidualizing R-module, and let M and N be any R-modules.*

(a) *If $M, N \in \mathcal{B}_C$, then* $\operatorname{Ext}^i_{\mathcal{P}_C}(M, N) \cong \operatorname{Ext}^i_R(M, N)$ *for $i \geq 0$.*

(b) *If $M, N \in \mathcal{A}_C$, then* $\operatorname{Ext}^i_{\mathcal{I}_C}(M, N) \cong \operatorname{Ext}^i_R(M, N)$ *for $i \geq 0$.*

Proof. (a) Choose $i \geq 0$. Let P^+ be an augmented projective resolution of $\operatorname{Hom}_R(C, M)$ and ${}^+I$ an augmented injective coresolution of N. Since $M, N \in \mathcal{B}_C$, the complexes $C \otimes_R P^+$ and $\operatorname{Hom}_R(C, {}^+I)$ are exact;

in particular, $C \otimes_R P \xrightarrow{\simeq} C \otimes_R \operatorname{Hom}_R(C, M) \cong M$ and $\operatorname{Hom}_R(C, I) \xleftarrow{\simeq} \operatorname{Hom}_R(C, N)$. Lemma 4.10 gives the quasiisomorphisms

$$\begin{aligned} \operatorname{Tot}\operatorname{Hom}_R(C \otimes_R P, I) &\xleftarrow{\simeq} \operatorname{Tot}\operatorname{Hom}_R(M, I) \\ \operatorname{Tot}\operatorname{Hom}_R(P, \operatorname{Hom}_R(C, I)) &\xleftarrow{\simeq} \operatorname{Tot}\operatorname{Hom}_R(P, \operatorname{Hom}_R(C, N)) \end{aligned}$$

that each give rise to isomorphisms of homology modules. Since there is an isomorphism

$$\operatorname{Hom}_R(C \otimes_R P, I) \cong \operatorname{Hom}_R(P, \operatorname{Hom}_R(C, I))$$

of double complexes, we have

$$\begin{aligned} \operatorname{Ext}^i_R(M, N) &= H^i(\operatorname{Hom}_R(M, I)) \\ &= H^i(\operatorname{Tot}\operatorname{Hom}_R(M, I)) \\ &\cong H^i(\operatorname{Tot}\operatorname{Hom}_R(C \otimes_R P, I)) \\ &\cong H^i(\operatorname{Tot}\operatorname{Hom}_R(P, \operatorname{Hom}_R(C, I))) \\ &\cong H^i(\operatorname{Tot}\operatorname{Hom}_R(P, \operatorname{Hom}_R(C, N))) \\ &= H^i(\operatorname{Hom}_R(P, \operatorname{Hom}_R(C, N))) \\ &= \operatorname{Ext}^i_R(\operatorname{Hom}_R(C, M), \operatorname{Hom}_R(C, N)) \\ &\cong \operatorname{Ext}^i_{\mathcal{P}_C}(M, N) \end{aligned}$$

where the last isomorphism is by Theorem 4.5(a).

(b) Choose $i \geq 0$. Let P^+ be an augmented projective resolution of M and ^+I an augmented injective coresolution of $C \otimes_R N$. Since $M, N \in \mathcal{A}_C$, the complexes $C \otimes_R P^+$ and $\operatorname{Hom}_R(C, {}^+I)$ are exact; in particular, $\operatorname{Hom}_R(C, I) \xleftarrow{\simeq} \operatorname{Hom}_R(C, C \otimes_R N) \cong N$ and $C \otimes_R P \xrightarrow{\simeq} C \otimes_R M$. Lemma 4.10 gives the quasiisomorphisms

$$\begin{aligned} \operatorname{Tot}\operatorname{Hom}_R(P, \operatorname{Hom}_R(C, I)) &\xleftarrow{\simeq} \operatorname{Tot}\operatorname{Hom}_R(P, N) \\ \operatorname{Tot}\operatorname{Hom}_R(C \otimes_R P, I) &\xleftarrow{\simeq} \operatorname{Tot}\operatorname{Hom}_R(C \otimes_R M, I) \end{aligned}$$

that each give rise to isomorphisms of homology modules. Since there is an isomorphism

$$\operatorname{Hom}_R(P, \operatorname{Hom}_R(C, I)) \cong \operatorname{Hom}_R(C \otimes_R P, I)$$

of double complexes, we have

$$\begin{aligned}
\mathrm{Ext}^i_R(M,N) &= H^i(\mathrm{Hom}_R(P,N)) \\
&= H^i(\mathrm{Tot}\,\mathrm{Hom}_R(P,N)) \\
&\cong H^i(\mathrm{Tot}\,\mathrm{Hom}_R(P,\mathrm{Hom}_R(C,I))) \\
&\cong H^i(\mathrm{Tot}\,\mathrm{Hom}_R(C\otimes_R P, I)) \\
&\cong H^i(\mathrm{Tot}\,\mathrm{Hom}_R(C\otimes_R M, I)) \\
&= H^i(\mathrm{Hom}_R(C\otimes_R M, I)) \\
&= \mathrm{Ext}^i_R(C\otimes_R M, C\otimes_R N) \\
&= \mathrm{Ext}^i_{\mathcal{I}_C}(M,N)
\end{aligned}$$

where the last isomorphism is by Theorem 4.5(b).

5. COMPLETE PC- AND IC-RESOLUTIONS

In this section we combine projective and C-projective resolutions and, dually, injective and C-injective resolutions in a useful way.

Definition 5.1. Let C be an R-module. A *complete PC-resolution* is an exact sequence of R-module homomorphisms

$$X := \cdots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} C \otimes_R Q^0 \xrightarrow{\partial^0} C \otimes_R Q^1 \xrightarrow{\partial^1} \cdots$$

such that each P_i and Q^j is projective and the complex $\operatorname{Hom}_R(X, C \otimes_R Q)$ is exact for every projective module Q.

A *complete IC-resolution* is an exact sequence of R-module homomorphisms

$$Y := \cdots \xrightarrow{\partial_2} \operatorname{Hom}_R(C, I_1) \xrightarrow{\partial_1} \operatorname{Hom}_R(C, I_0) \xrightarrow{\partial_0} J^0 \xrightarrow{\partial^0} J^1 \xrightarrow{\partial^1} \cdots$$

such that each I_i and J^j is injective and the complex $\operatorname{Hom}_R(\operatorname{Hom}_R(C, J), Y)$ is exact for every injective module J.

Definition 5.2. Let C be an R-module. An R-module M is said to be *G_C-projective* if there exists a complete PC-resolution (using the notation above) with $M \cong \operatorname{Im} \partial_0 = \ker \partial^0$. The module M is said to be *G_C-injective* if there exists a complete IC-resolution (using the notation above) with $M \cong \ker \partial^0 = \operatorname{Im} \partial_0$.

The next result shows that complete PC- and IC-resolutions are built from $\mathcal{P}_C$-projective resolutions and $\mathcal{I}_C$-injective coresolutions, respectively.

Proposition 5.3 ([12, Proposition 2.2])**.** *Let M and C be R-modules.*

(a) *The module M is G_C-projective if and only if M admits an exact augmented coproper $\mathcal{P}_C$-projective coresolution and $\operatorname{Ext}^i_R(M, C \otimes_R Q) = 0$ for $i > 0$ and for all projective modules Q.*

(b) *The module M is G_C-injective if and only if M admits an exact augmented coproper $\mathcal{I}_C$-injective resolution and $\operatorname{Ext}^i_R(\operatorname{Hom}_R(C, J), M) = 0$ for $i > 0$ and for all injective modules J.*

Proof. (a) Suppose first that M is a G_C-projective R-module. Then there exists a complete PC-resolution

$$X := \cdots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} C \otimes_R Q^0 \xrightarrow{\partial^0} C \otimes_R Q^1 \xrightarrow{\partial^1} \cdots$$

such that $M \cong \operatorname{Im} \partial_0$ by Definition 5.2. Here we adopt the convention that $X_i := P_i$ for $i \geq 0$ and

$X_i := C \otimes_R Q^{-i-1}$ otherwise. Form the augmented projective resolution

$$P^+ := \cdots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\tau} M \to 0$$

of M by truncation of X. Since the complex $\mathrm{Hom}_R(X, C \otimes_R Q)$ is exact for every projective module Q by Definition 5.1 and the complexes P^+ and X agree in positive degree, we obtain that

$$\begin{aligned} \mathrm{Ext}_R^i(M, C \otimes_R Q) &= H^i(\mathrm{Hom}_R(P, C \otimes_R Q)) \\ &= H^i(\mathrm{Hom}_R(X, C \otimes_R Q)) \\ &= 0 \end{aligned}$$

for $i > 0$; exactness in degree $i \in \{-1, 0\}$ follows from left-exactness of $\mathrm{Hom}_R(-, C \otimes_R Q)$. Consider the complex

$$^+Y := \quad 0 \to M \xrightarrow{\epsilon} C \otimes_R Q^0 \xrightarrow{\partial^0} C \otimes_R Q^1 \xrightarrow{\partial^1} \cdots$$

obtained by truncation, where ϵ is chosen such that $\epsilon\tau = \partial_0$. Since $Y_i = X_{i-1}$ for $i \leq 0$ and the maps ∂^i are equal, we obtain that $H_i(\mathrm{Hom}_R(^+Y, C \otimes_R Q)) = H_{i+1}(\mathrm{Hom}_R(X, C \otimes_R Q)) = 0$ for $i \geq 1$ by Definition 5.1 and exactness in degree $i \in \{-1, 0\}$ follows from left-exactness of $\mathrm{Hom}_R(-, C \otimes_R Q)$. Hence ^+Y is an augmented coproper $\mathcal{P}_C$-projective coresolution of M.

To show the converse, let

$$P^+ := \cdots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\tau} M \to 0$$

be an augmented projective resolution of M and

$$^+Y := \quad 0 \to M \xrightarrow{\epsilon} C \otimes_R Q^0 \xrightarrow{\partial^0} C \otimes_R Q^1 \xrightarrow{\partial^1} \cdots$$

an exact augmented coproper $\mathcal{P}_C$-projective coresolution of M. Form the complex

$$X := \quad \cdots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\epsilon\tau} C \otimes_R Q^0 \xrightarrow{\partial^0} C \otimes_R Q^1 \xrightarrow{\partial^1} \cdots$$

that we claim is a complete PC-resolution of M. Here we adopt the same degree convention as above. Since X is exact by construction, it suffices to show that $\mathrm{Hom}_R(X, C \otimes_R Q)$ is exact for every projective R-module Q.

For $i > 0$, we have

$$\begin{aligned} H^i(\operatorname{Hom}_R(X, C \otimes_R Q)) &= H^i(\operatorname{Hom}_R(P, C \otimes_R Q)) \\ &= \operatorname{Ext}^i_R(M, C \otimes_R Q) \\ &= 0 \end{aligned}$$

by assumption. For $i > 1$, we have $H_i(\operatorname{Hom}_R(X, C \otimes_R Q)) = H_{i-1}(\operatorname{Hom}_R({}^+Y, C \otimes_R Q)) = 0$ by coproperness of ${}^+Y$. Left-exactness of the functor $\operatorname{Hom}_R(-, C \otimes_R Q)$ gives exactness of $\operatorname{Hom}_R(X, C \otimes_R Q)$ in degree $i \in \{0, 1\}$. Hence $\operatorname{Hom}_R(X, C \otimes_R Q)$ is exact and M is G_C-projective by Definition 5.2.

(b) Suppose first that M is a G_C-injective R-module. Then there exists a complete IC-resolution

$$Y := \cdots \xrightarrow{\partial_2} \operatorname{Hom}_R(C, I_1) \xrightarrow{\partial_1} \operatorname{Hom}_R(C, I_0) \xrightarrow{\partial_0} J^0 \xrightarrow{\partial^0} J^1 \xrightarrow{\partial^1} \cdots$$

such that $M \cong \ker \partial^0$ by Definition 5.2. Here we adopt the convention that $Y_i := J^{-i}$ for $i \leq 0$ and $Y_i := \operatorname{Hom}_R(C, I_{i-1})$ otherwise. Form the augmented injective coresolution

$${}^+J := \quad 0 \to M \xrightarrow{\epsilon} J^0 \xrightarrow{\partial^0} J^1 \xrightarrow{\partial^1} \cdots$$

of M by truncation of Y. Since the complex $\operatorname{Hom}_R(\operatorname{Hom}_R(C, I), Y)$ is exact for every injective module I by Definition 5.1 and the complexes ${}^+J$ and Y agree in negative degree, we obtain that

$$\begin{aligned} \operatorname{Ext}^i_R(\operatorname{Hom}_R(C, I), M) &= H^i(\operatorname{Hom}_R(\operatorname{Hom}_R(C, I), J)) \\ &= H^i(\operatorname{Hom}_R(\operatorname{Hom}_R(C, I), Y)) \\ &= 0 \end{aligned}$$

for $i > 0$; exactness in degree $i \in \{-1, 0\}$ follows from left-exactness of $\operatorname{Hom}_R(\operatorname{Hom}_R(C, I), -)$. Consider the complex

$$Z^+ := \cdots \xrightarrow{\partial^2} \operatorname{Hom}_R(C, I_1) \xrightarrow{\partial^1} \operatorname{Hom}_R(C, I_0) \xrightarrow{\tau} M \to 0$$

obtained by truncation, where τ is chosen such that $\epsilon\tau = \partial$. Since $Z_{i-1} = Y_i$ for $i \geq 1$, we obtain that $H_i(\operatorname{Hom}_R(\operatorname{Hom}_R(C, I), Z^+)) = H_{i+1}(\operatorname{Hom}_R(\operatorname{Hom}_R(C, I), Y)) = 0$ for $i > 0$ by Definition 5.1 and exactness in degree $i \in \{-1, 0\}$ follows from left-exactness. Hence Z^+ is an augmented coproper $\mathcal{I}_C$-injective resolution of M.

To show the converse, let

$$^{+}J := \quad 0 \to M \xrightarrow{\epsilon} J^0 \xrightarrow{\partial^0} J^1 \xrightarrow{\partial^1} \cdots$$

be an augmented injective coresolution of M and

$$Z^{+} := \cdots \xrightarrow{\partial_2} \operatorname{Hom}_R(C, I_1) \xrightarrow{\partial_1} \operatorname{Hom}_R(C, I_0) \xrightarrow{\tau} M \to 0$$

an exact augmented coproper $\mathcal{I}_C$-injective resolution of M. Form the complex

$$Y := \ \cdots \xrightarrow{\partial_2} \operatorname{Hom}_R(C, I_1) \xrightarrow{\partial_1} \operatorname{Hom}_R(C, I_0) \xrightarrow{\epsilon\tau} J^0 \xrightarrow{\partial^0} J^1 \xrightarrow{\partial^1} \cdots$$

that we claim is a complete IC-resolution of M. Here we adopt the same degree convention as above. To show this, it suffices to show that $\operatorname{Hom}_R(\operatorname{Hom}_R(C, I), Y)$ is exact for every injective module I.

For $i > 0$, we have

$$\begin{aligned} H^i(\operatorname{Hom}_R(\operatorname{Hom}_R(C, I), Y)) &= H^i(\operatorname{Hom}_R(\operatorname{Hom}_R(C, I), {}^{+}J)) \\ &= \operatorname{Ext}^i_R(\operatorname{Hom}_R(C, I), M) \\ &= 0 \end{aligned}$$

by assumption. For $i > 1$, we have $H_i(\operatorname{Hom}_R(\operatorname{Hom}_R(C, I), Y)) = H_{i-1}(\operatorname{Hom}_R(\operatorname{Hom}_R(C, I), Z^{+})) = 0$ by coproperness of Z^{+}. Left-exactness of the functor $\operatorname{Hom}_R(\operatorname{Hom}_R(C, I), -)$ gives exactness of the complex $\operatorname{Hom}_R(\operatorname{Hom}_R(C, I), Y)$ in degree $i \in \{0, 1\}$. Hence $\operatorname{Hom}_R(\operatorname{Hom}_R(C, I), Y)$ is exact and M is G_C-injective by Definition 5.2.

Corollary 5.4 ([4, Proposition 2.4])**.** *Let C be an R-module. If M is G_C-projective, then M admits a complete PC-resolution of the form*

$$\cdots \to F_1 \to F_0 \to C \otimes_R G^0 \to C \otimes_R G^1 \to \cdots$$

where each F_i and G^j is free.

Proof. Let Q be a projective R-module. Since every R-module admits a free resolution, there exists an augmented free resolution

$$\cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\tau} M \to 0$$

of M that is $\operatorname{Hom}_R(-, C \otimes_R Q)$-exact by left-exactness. By Proposition 5.3(a), there is a complete PC-resolution of M yielding the exact sequence

$$0 \to M \xrightarrow{\epsilon} C \otimes_R Q^0 \xrightarrow{\beta^0} C \otimes_R Q^1 \xrightarrow{\beta^1} \cdots$$

by truncation; this sequence is $\operatorname{Hom}_R(-, C \otimes_R Q)$-exact since it is an augmented coproper $\mathcal{P}_C$-projective coresolution of M. Since each Q^j is projective, it is the direct summand of a free module. Choose P^0 such that $G^0 = Q^0 \oplus P^0$ is free. For $j > 0$, inductively choose P^j such that $G^j = Q^j \oplus P^{j-1} \oplus P^j$ is free.

Define the sequence

$$X := \cdots \xrightarrow{\partial_2} F_1 \xrightarrow{\partial_1} F_0 \xrightarrow{\partial_0} C \otimes_R G^0 \xrightarrow{\partial^0} C \otimes_R G^1 \xrightarrow{\partial^1} \cdots$$

combining the above resolution and coresolution. Define the map $\partial^0 : C \otimes_R G^0 \to C \otimes_R G^1$ via $\partial^0(x, y) := (\beta^0(x), y, 0)$. For $j > 0$, define the map $\partial^j : C \otimes_R G^j \to C \otimes_R G^{j+1}$ via $\partial^j(x, y, z) := (\beta^j(x), z, 0)$. Define the map $\partial_0 : F_0 \to C \otimes_R G^0$ via $\partial_0(x) := (\epsilon\tau(x), 0)$. Note that since X is constructed in negative homological degree by summing the above $\mathcal{P}_C$-projective coresolution with exact sequences of the form $0 \to C \otimes_R P^j \to C \otimes_R P^j \to 0$, it is still $\operatorname{Hom}_R(-, C \otimes_R Q)$-exact in these degrees.

We show that X is exact. Exactness at F_i (for $i > 0$) follows immediately from the given free resolution. To show exactness at F_0, observe that $\partial_0\partial_1 = (\epsilon\tau\partial_1, 0) = (0, 0)$ since the given free resolution is an R-complex. Further, suppose $y \in \ker \partial_0$; then $y \in \ker(\epsilon\tau)$ and $\tau(y) \in \ker \epsilon$. Since ϵ is injective, we have $\tau(y) = 0$ and hence $y \in \ker \tau = \operatorname{Im} \partial_1$.

To show exactness at $C \otimes_R G^0$, observe that $\partial^0\partial_0 = (\beta^0\epsilon\tau, 0, 0) = (0, 0)$. Further, suppose $(x, y) \in \ker \partial^0$; then $y = 0$ and $x \in \ker \beta^0 = \operatorname{Im} \epsilon$. Hence there is some $m \in M$ such that $\epsilon(m) = x$. Surjectivity of τ gives $m' \in F_0$ such that $\tau(m') = m$. It follows that

$$\partial_0(m') = (\epsilon\tau(m'), 0) = (\epsilon(m), 0) = (x, 0) = (x, y)$$

as desired. To show exactness at $C \otimes_R Q^i$ for $i > 0$, observe that $\partial^i\partial^{i-1} = (\beta^i\beta^{i-1}, 0, 0) = (0, 0, 0)$. Further, suppose $(x, y, z) \in \ker \partial^i$; then $z = 0$ and $x \in \ker \beta^i = \operatorname{Im} \beta^{i-1}$. Hence there is some x' such that $\beta^{i-1}(x') = x$. It follows that

$$\partial^{i-1}(x', 0, y) = (\beta^{i-1}(x'), y, 0) = (x, y, 0) = (x, y, z)$$

and hence X is exact. It can be shown that $\operatorname{Hom}_R(X, C \otimes_R Q)$ is exact for every projective R-module Q.

Hence X is a complete PC-resolution of M. ◆

The next result shows that complete PC- and IC-injective resolutions play nicely with products and coproducts.

Proposition 5.5 ([12, Proposition 2.4]). *Let C be an R-module.*

(a) *If $\{X_\lambda\}$ is a set of complete PC-resolutions, then the complex $\coprod_\lambda X_\lambda$ is a complete PC-resolution.*

(b) *If $\{Y_\lambda\}$ is a set of complete IC-resolutions, then the complex $\prod_\lambda Y_\lambda$ is a complete IC-resolution.*

Proof. (a) Suppose that $\{X_\lambda\}$ is such a set. Then X_λ and $\mathrm{Hom}_R(X_\lambda, C \otimes_R Q)$ are exact for any projective module Q; hence $\coprod_\lambda X_\lambda$ and $\prod_\lambda \mathrm{Hom}_R(X_\lambda, C \otimes_R Q)$ are also exact. Recall that there is an isomorphism

$$\prod_\lambda \mathrm{Hom}_R(X_\lambda, C \otimes_R Q) \cong \mathrm{Hom}_R\left(\coprod_\lambda X_\lambda, C \otimes_R Q\right)$$

of R-modules. Thus $\mathrm{Hom}_R\left(\coprod_\lambda, C \otimes_R Q\right)$ is exact and $\coprod_\lambda X_\lambda$ is a complete PC-resolution.

(b) Suppose that $\{Y_\lambda\}$ is such a set. Then Y_λ and $\mathrm{Hom}_R(\mathrm{Hom}_R(C, J), Y_\lambda)$ are exact for any injective module J; hence $\prod_\lambda Y_\lambda$ and $\prod_\lambda \mathrm{Hom}_R(\mathrm{Hom}_R(C, J), Y_\lambda)$ are also exact. Recall that there is an isomorphism

$$\prod_\lambda \mathrm{Hom}_R(\mathrm{Hom}_R(C, J), Y_\lambda) \cong \mathrm{Hom}_R\left(\mathrm{Hom}_R(C, J), \prod_\lambda Y_\lambda\right)$$

of R-modules. Thus $\mathrm{Hom}_R\left(\mathrm{Hom}_R(C, J), \prod_\lambda Y_\lambda\right)$ is exact and $\prod_\lambda Y_\lambda$ is a complete IC-resolution. ◆

Corollary 5.6. *The class of G_C-projective modules is closed under direct sums and the class of G_C-injective modules is closed under direct products.*

Proof. Since taking cokernels commutes with coproducts and taking kernels commutes with products, the results follows immediately from Proposition 5.5. ◆

We use the following six lemmata below.

Lemma 5.7. *Let F be a flat R-module and I an injective R-module. Then $\mathrm{Hom}_R(F, I)$ is also injective.*

Proof. Let X be an exact sequence of R-module homomorphisms. By definition, $X \otimes_R F$ is exact since F is flat. Since I injective, the sequence $\mathrm{Hom}_R(X \otimes_R F, I)$ is exact by definition. Applying Hom-tensor adjointness, we find that $\mathrm{Hom}_R(X \otimes_R F, I) \cong \mathrm{Hom}_R(X, \mathrm{Hom}_R(F, I))$ is exact, so $\mathrm{Hom}_R(F, I)$ is injective. ◆

Lemma 5.8. *Let P and Q be projective R-modules. Then $P \otimes_R Q$ is projective.*

Proof. Since P and Q are projective, the functors $\mathrm{Hom}_R(P,-)$ and $\mathrm{Hom}_R(Q,-)$ are exact. Hom-tensor adjointness gives that $\mathrm{Hom}_R(P \otimes_R Q, -) \cong \mathrm{Hom}_R(P, \mathrm{Hom}_R(Q,-))$ is also exact, so $P \otimes_R Q$ is projective.

Lemma 5.9 ([12, Lemma 2.5]). *Let C be an R-module.*

(a) *Let P and Q be projective R-modules and X an R-complex. If $\mathrm{Hom}_R(X, C \otimes_R Q)$ is exact, then the complex $\mathrm{Hom}_R(P \otimes_R X, C \otimes_R Q)$ is exact. In particular, if X is a complete PC-resolution of a module M, then $P \otimes_R X$ is a complete PC-resolution of $P \otimes_R M$.*

(b) *Let P be a projective R-module, J an injective R-module, and Y an R-complex. If $\mathrm{Hom}_R(\mathrm{Hom}_R(C,J),Y)$ is exact, then the complex $\mathrm{Hom}_R(\mathrm{Hom}_R(C,J), \mathrm{Hom}_R(P,Y))$ is exact. In particular, if Y is a complete IC-resolution of a module N, then $\mathrm{Hom}_R(P,Y)$ is a complete IC-resolution of $\mathrm{Hom}_R(P,N)$.*

Proof. (a) Suppose that $\mathrm{Hom}_R(X, C \otimes_R Q)$ is exact, where Q is projective. Since P is projective, the complex

$$\mathrm{Hom}_R(P \otimes_R X, C \otimes_R Q) \cong \mathrm{Hom}_R(P, \mathrm{Hom}_R(X, C \otimes_R Q))$$

is exact. Now suppose

$$X = \cdots \to P_1 \to P_0 \to C \otimes_R Q^0 \to C \otimes_R Q^1 \to \cdots$$

is a complete PC-resolution of M. Since P is projective and X is exact, the complex

$$\begin{aligned} P \otimes_R X &= \cdots \to P \otimes_R P_1 \to P \otimes_R P_0 \to P \otimes_R (C \otimes_R Q^0) \to P \otimes_R (C \otimes_R Q^1) \to \cdots \\ &\cong \cdots \to P \otimes_R P_1 \to P \otimes_R P_0 \to C \otimes_R (P \otimes_R Q^0) \to C \otimes_R (P \otimes_R Q^1) \to \cdots \end{aligned}$$

is exact; further, it is a complete PC-resolution of $P \otimes_R M$ since each $P \otimes_R P_i$ and $P \otimes_R Q^j$ is projective by Lemma 5.8 and since taking cokernels commutes with $P \otimes_R -$ by right-exactness.

(b) Suppose that $\mathrm{Hom}_R(\mathrm{Hom}_R(C,J),Y)$ is exact, where J is injective. The complex

$$\begin{aligned} \mathrm{Hom}_R(\mathrm{Hom}_R(C,J), \mathrm{Hom}_R(P,Y)) &\cong \mathrm{Hom}_R(P \otimes_R \mathrm{Hom}_R(C,J), Y) \\ &\cong \mathrm{Hom}_R(P, \mathrm{Hom}_R(\mathrm{Hom}_R(C,J),Y)) \end{aligned}$$

is exact since P is projective. Now suppose

$$Y = \cdots \to \mathrm{Hom}_R(C, I_1) \to \mathrm{Hom}_R(C, I_0) \to J^0 \to J^1 \to \cdots$$

is a complete IC-resolution of N. Since P is projective and X is exact, the complex

$$\begin{aligned}\mathrm{Hom}_R(P,Y) &= \cdots \to \mathrm{Hom}_R(P,\mathrm{Hom}_R(C,I_0)) \to \mathrm{Hom}_R(P,J^0) \to \cdots \\ &\cong \cdots \to \mathrm{Hom}_R(C,\mathrm{Hom}_R(P,I_0)) \to \mathrm{Hom}_R(P,J^0) \to \cdots\end{aligned}$$

is exact; further, it is a complete IC-resolution of $\mathrm{Hom}_R(P,N)$ since each $\mathrm{Hom}_R(P,I_i)$ and $\mathrm{Hom}_R(P,J^j)$ is injective by Lemma 5.7 and since taking kernels commutes with $\mathrm{Hom}_R(P,-)$ by left-exactness.

Lemma 5.10 ([7, Lemmata A.1.2, A.1.3])**.** *Let L,M,N be R-modules.*

(a) *If L is finitely generated and N is flat, then* $\mathrm{Hom}_R(L,M)\otimes_R N \cong \mathrm{Hom}_R(L,M\otimes_R N)$.

(b) *If L is finitely generated and N is injective, then* $L\otimes_R \mathrm{Hom}_R(M,N) \cong \mathrm{Hom}_R(\mathrm{Hom}_R(L,M),N)$.

Lemma 5.11 ([7, Proposition 5.4.1(a)])**.** *Let C be a semidualizing R-module and M a finitely generated R-module. If there is an isomorphism $M\cong \mathrm{Hom}_R(\mathrm{Hom}_R(M,C),C)$, then the natural biduality map $M\to \mathrm{Hom}_R(\mathrm{Hom}_R(M,C),C)$ is an isomorphism.*

Lemma 5.12. *Let C be a semidualizing R-module and β a positive integer. The biduality maps $R^\beta \to \mathrm{Hom}_R(\mathrm{Hom}_R(R^\beta,C),C)$ and $C^\beta \to \mathrm{Hom}_R(\mathrm{Hom}_R(C^\beta,C),C)$ are isomorphisms.*

Proof. We have

$$\begin{aligned}\mathrm{Hom}_R(\mathrm{Hom}_R(R^\beta,C),C) &\cong \mathrm{Hom}_R(\mathrm{Hom}_R(R,C)^\beta,C)\\ &\cong \mathrm{Hom}_R(C^\beta,C)\\ &\cong \mathrm{Hom}_R(C,C)^\beta\\ &\cong R^\beta\end{aligned}$$

where the last isomorphism follows since C is semidualizing. We have

$$\begin{aligned}\mathrm{Hom}_R(\mathrm{Hom}_R(C^\beta,C),C) &\cong \mathrm{Hom}_R(\mathrm{Hom}_R(C,C)^\beta,C)\\ &\cong \mathrm{Hom}_R(R^\beta,C)\\ &\cong \mathrm{Hom}_R(R,C)^\beta\\ &\cong C^\beta\end{aligned}$$

similarly. By Lemma 5.11, the corresponding biduality maps are isomorphisms.

The next proposition shows how the construction of G_C-projective modules generalizes that of projective and C-projective modules. The construction of G_C-injective modules similarly generalizes that of injective and C-injective modules.

Proposition 5.13 ([12, Proposition 2.6]). *Let C be a semidualizing R-module.*

(a) *If P is a projective R-module, then P and $C \otimes_R P$ are G_C-projective.*

(b) *If I is an injective R-module, then I and $\operatorname{Hom}_R(C, I)$ are G_C-injective.*

Proof. (a) From Lemma 5.9(a), it suffices to show that C and R admit complete PC-resolutions. We first show that C admits such a resolution. Since C is semidualizing, there is an augmented projective resolution

$$X^+ := \cdots \to R^{\beta_1} \to R^{\beta_0} \to C \to 0$$

that is exact; in particular, $C \cong \operatorname{coker}(R^{\beta_1} \to R^{\beta_0})$. It remains to show that $\operatorname{Hom}_R(X^+, C \otimes_R Q)$ is exact for any projective R-module Q. Exactness in nonnegative degrees follows from left-exactness of the functor $\operatorname{Hom}_R(-, C \otimes_R Q)$. For $i > 1$ we have

$$\begin{aligned} H^i(\operatorname{Hom}_R(X^+, C \otimes_R Q)) &\cong H^i(\operatorname{Hom}_R(X^+, C) \otimes_R Q) \\ &\cong H^i(\operatorname{Hom}_R(X^+, C)) \otimes_R Q \\ &= \operatorname{Ext}^i_R(C, C) \otimes_R Q \\ &= 0 \end{aligned}$$

where the first isomorphism follows from Lemma 5.10(a) since each X_i^+ is finitely generated and Q is flat, the second follows from [8, Theorem IV.1.10(a)] since Q is flat, and the final equality follows from Definition 2.1. We now claim that $\operatorname{Hom}_R(X^+, C)$ is a complete PC-resolution of R. We have

$$\begin{aligned} \operatorname{Hom}_R(X^+, C) &= 0 \to \operatorname{Hom}_R(C, C) \to \operatorname{Hom}_R(R^{\beta_0}, C) \to \operatorname{Hom}_R(R^{\beta_1}, C) \to \cdots \\ &\cong 0 \to R \to \operatorname{Hom}_R(R, C)^{\beta_0} \to \operatorname{Hom}_R(R, C)^{\beta_1} \to \cdots \\ &\cong 0 \to R \to C^{\beta_0} \to C^{\beta_1} \to \cdots \end{aligned}$$

since $\operatorname{Hom}_R(C, C) \cong R$ by Definition 2.1. Thus $\operatorname{Hom}_R(X^+, C)$ has the correct form for a complete PC-resolution of R. The complex $\operatorname{Hom}_R(X^+, C)$ is exact by setting $Q = R$ in our previous work. Further, the biduality map $X^+ \to \operatorname{Hom}_R(\operatorname{Hom}_R(X^+, C), C)$ is an isomorphism by Lemma 5.12. Hence for any projective module Q we have

$$\begin{aligned}\mathrm{Hom}_R(\mathrm{Hom}_R(X^+, C), C \otimes_R Q) &\cong \mathrm{Hom}_R(\mathrm{Hom}_R(X^+, C), C) \otimes_R Q \\ &\cong X^+ \otimes_R Q\end{aligned}$$

as before. Since X^+ is exact and Q is projective, the complex $X^+ \otimes_R Q$ is exact, so the complex $\mathrm{Hom}_R(X^+, C)$ is a complete PC-resolution of R.

(b) Let I be an injective R-module. There exists an augmented injective resolution

$$^+Y = \quad 0 \to \mathrm{Hom}_R(C, I) \to I^0 \to I^1 \to \cdots$$

of $\mathrm{Hom}_R(C, I)$. We claim that ^+Y is a complete IC-resolution of $\mathrm{Hom}_R(C, I)$, giving that $\mathrm{Hom}_R(C, I)$ is G_C-injective. It suffices to show that

$$\mathrm{Hom}_R(\mathrm{Hom}_R(C, J), {}^+Y) = \quad 0 \to \mathrm{Hom}_R(\mathrm{Hom}_R(C, J), \mathrm{Hom}_R(C, I)) \to \mathrm{Hom}_R(\mathrm{Hom}_R(C, J), I^0) \to \cdots$$

is exact for every injective R-module J. We obtain exactness at $\mathrm{Hom}_R(\mathrm{Hom}_R(C, J), \mathrm{Hom}_R(C, I))$ and $\mathrm{Hom}_R(\mathrm{Hom}_R(C, J), I^0)$ by left-exactness of the functor $\mathrm{Hom}_R(\mathrm{Hom}_R(C, J), -)$. For the remaining terms, it suffices to show that $\mathrm{Ext}^i_R(\mathrm{Hom}_R(C, J), \mathrm{Hom}_R(C, I)) = 0$ for $i \geq 1$. Let P^+ be an augmented projective resolution of C. Then $\mathrm{Hom}_R(P, I)$ is an augmented injective resolution of $\mathrm{Hom}_R(C, I)$, so

$$\begin{aligned}\mathrm{Ext}^i_R(\mathrm{Hom}_R(C, J), \mathrm{Hom}_R(C, I)) &\cong H^i(\mathrm{Hom}_R(\mathrm{Hom}_R(C, J), \mathrm{Hom}_R(P, I))) \\ &\cong H^i(\mathrm{Hom}_R(P \otimes_R \mathrm{Hom}_R(C, J), I)) \\ &\cong \mathrm{Hom}_R(H_i(P \otimes_R \mathrm{Hom}_R(C, J)), I) \\ &\cong \mathrm{Hom}_R(\mathrm{Tor}^R_i(C, \mathrm{Hom}_R(C, J)), I) \\ &= 0\end{aligned}$$

since $J \in \mathcal{B}_C$. Hence $\mathrm{Hom}_R(C, I)$ is G_C-injective. It remains to show that I is G_C-injective. Let

$$X := 0 \to R \to C^{\beta_0} \to C^{\beta^1} \to \cdots$$

be the complete PC-resolution of R from part (a). We claim that $\mathrm{Hom}_R(X, I)$ is a complete IC-resolution of I, and hence that I is G_C-injective. Indeed,

$$\begin{aligned}\operatorname{Hom}_R(X,I) &= \cdots \to \operatorname{Hom}_R(C^{\beta_1},I) \to \operatorname{Hom}_R(C^{\beta_0},I) \to \operatorname{Hom}_R(R,I) \to 0\\ &\cong \cdots \to \operatorname{Hom}_R(C,I^{\beta_1}) \to \operatorname{Hom}_R(C,I^{\beta_0}) \to I \to 0\end{aligned}$$

where each $\operatorname{Hom}_R(C,I^\beta) \in \mathcal{I}_C$. Thus $\operatorname{Hom}_R(X,I)$ has the correct form for a complete IC-resolution of I. We must also show that $\operatorname{Hom}_R(\operatorname{Hom}_R(C,J),\operatorname{Hom}_R(X,I))$ is exact for any injective module J. We have

$$\begin{aligned}\operatorname{Hom}_R(\operatorname{Hom}_R(C,J),\operatorname{Hom}_R(X,I)) &\cong \operatorname{Hom}_R(X\otimes_R \operatorname{Hom}_R(C,J),I)\\ &\cong \operatorname{Hom}_R(X,\operatorname{Hom}_R(\operatorname{Hom}_R(C,J),I))\\ &\cong \operatorname{Hom}_R(X,C\otimes_R \operatorname{Hom}_R(J,I))\\ &\cong \operatorname{Hom}_R(X,C)\otimes_R \operatorname{Hom}_R(J,I)\end{aligned}$$

where the third isomorphism follows from Lemma 5.10(b) since I is injective and C is finitely generated, and the last isomorphism follows from Lemma 5.10(a) since $\operatorname{Hom}_R(J,I)$ is flat and $\operatorname{Hom}_R(X,C)_i$ is finitely generated for all i. By the proof of part (a), the complex $\operatorname{Hom}_R(X,C)$ is an augmented free resolution of C, so it is exact; the flatness of $\operatorname{Hom}_R(J,I)$ gives that the complex $\operatorname{Hom}_R(X,C)\otimes_R \operatorname{Hom}_R(J,I)$ is also exact. Hence I is G_C-injective.

Corollary 5.14. *Let C be a semidualizing R-module. Every R-module admits a G_C-projective resolution and a G_C-injective coresolution.*

Proof. Every module admits a projective resolution and an injective coresolution. The result follows since every projective module is G_C-projective and every injective module is G_C-injective by Proposition 5.13.

Next, we obtain some useful Ext-vanishing.

Proposition 5.15 ([12, Proposition 2.7])**.** *Let C be a semidualizing R-module.*

(a) *If X is a complete PC-resolution and L has finite $\mathcal{P}_C$-projective dimension, then* $\operatorname{Hom}_R(X,L)$ *is exact. In particular, if M is G_C-projective, then* $\operatorname{Ext}^i_R(M,L)=0$ *for* $i>0$.

(b) *If Y is a complete IC-resolution and L has finite $\mathcal{I}_C$-injective dimension, then* $\operatorname{Hom}_R(L,Y)$ *is exact. In particular, if M is G_C-injective, then* $\operatorname{Ext}^i_R(L,M)=0$ *for* $i>0$.

Proof. (a) We proceed by induction on $n := \mathcal{P}_C\text{-}\mathrm{pd}_R(L)$. In the case where $n=0$, we have $L\cong C\otimes_R P_0$ for some projective R-module P_0, so $\operatorname{Hom}_R(X,L)\cong \operatorname{Hom}_R(X,C\otimes_R P_0)$ is exact by definition. In the case where $n > 0$, consider an augmented proper $\mathcal{P}_C$-projective resolution

$$0 \to C \otimes_R P_n \to \cdots \to C \otimes_R P_1 \to C \otimes_R P_0 \to L \to 0$$

of L. By Corollary 3.13(a), this resolution is exact. Form the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & C \otimes_R P_n \arrow[r] & \cdots \arrow[r] & C \otimes_R P_1 \arrow[rr, "\partial_1"] \arrow[dr] & & C \otimes_R P_0 \arrow[r] & L \arrow[r] & 0 \\
& & & & \operatorname{Im}\partial_1 \arrow[ur] \arrow[dr] & & & \\
& & & 0 \arrow[ur] & & 0 & &
\end{tikzcd}$$

where the two crossing subsequences are exact by construction. The complex $\operatorname{Hom}_R(X, C \otimes P_0)$ is exact by the base case previously shown. Observe that $\mathcal{P}_C\text{-pd}_R(\operatorname{Im}\partial_1) < n$ by our diagram, so by induction the complex $\operatorname{Hom}_R(X, \operatorname{Im}\partial_1)$ is exact. The sequence

$$0 \to \operatorname{Hom}_R(X, \operatorname{Im}\partial_1) \to \operatorname{Hom}_R(X, C \otimes_R P_0) \to \operatorname{Hom}_R(X, L)$$

is exact by left-exactness of the functor $\operatorname{Hom}_R(X, -)$.

We show that the map $\operatorname{Hom}_R(X, C \otimes_R P_0) \to \operatorname{Hom}_R(X, L)$ is surjective. By the long exact sequence in $\operatorname{Ext}_R(X_i, -)$, it suffices to show that $\operatorname{Ext}^1_R(X_i, \operatorname{Im}\partial_1) = 0$ for all i. By definition, each X_i is either projective or C-projective. If X_i is projective, then $\operatorname{Ext}^1_R(X_i, \operatorname{Im}\partial_1) = 0$. If instead $X_i \in \mathcal{P}_C$, then $\operatorname{Ext}^1_R(X_i, \operatorname{Im}\partial_1) \cong \operatorname{Ext}^1_{\mathcal{P}_C}(X_i, \operatorname{Im}\partial_1) = 0$, where the isomorphism follows from Corollary 3.13(a) and Theorem 4.11(a), and the vanishing is from Theorem 4.1(a).

The long exact sequence in homology implies that $\operatorname{Hom}_R(X, L)$ is exact. If M is G_C-projective, then the Ext-vanishing follows by using a truncation of X as a projective resolution of M.

(b) We proceed by induction on $n := \mathcal{I}_C\text{-id}_R(L)$. In the case where $n = 0$, we have $L \cong \operatorname{Hom}_R(C, I^0)$ for some injective R-module I^0, so $\operatorname{Hom}_R(L, Y) \cong \operatorname{Hom}_R(\operatorname{Hom}_R(C, I^0), Y)$ is exact by definition. In the case where $n > 0$, consider an augmented proper $\mathcal{I}_C$-injective coresolution

$$0 \to L \to \operatorname{Hom}_R(C, I^0) \to \operatorname{Hom}_R(C, I^1) \to \cdots \to \operatorname{Hom}_R(C, I^n) \to 0$$

of L. By Corollary 3.13(b), this coresolution is exact. Form the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & L \arrow[r] & \mathrm{Hom}_R(C,I^0) \arrow[rr, "\partial^0"] \arrow[dr] & & \mathrm{Hom}_R(C,I^1) \arrow[r] & \cdots \arrow[r] & \mathrm{Hom}_R(C,I^n) \arrow[r] & 0 \\
& & & \mathrm{Im}\,\partial^0 \arrow[ur] \arrow[dr] & & & & \\
& & 0 \arrow[ur] & & 0 & & &
\end{tikzcd}$$

where the two crossing subsequences are exact by construction. The complex $\mathrm{Hom}_R(Y, \mathrm{Hom}_R(C, I^0))$ is exact by the base case previously shown. Observe that $\mathcal{I}_C\text{-id}_R(\mathrm{Im}\,\partial^0) < n$ by our diagram, so by induction the complex $\mathrm{Hom}_R(Y, \mathrm{Im}\,\partial^0)$ is exact. The sequence

$$0 \to \mathrm{Hom}_R(\mathrm{Im}\,\partial^0, Y) \to \mathrm{Hom}_R(\mathrm{Hom}_R(C, I^0), Y) \to \mathrm{Hom}_R(L, Y)$$

is exact by left-exactness of the functor $\mathrm{Hom}_R(-, Y)$.

We show that the map $\mathrm{Hom}_R(\mathrm{Hom}_R(C, I^0), Y) \to \mathrm{Hom}_R(L, Y)$ is surjective. By the long exact sequence in $\mathrm{Ext}_R(-, Y^i)$, it suffices to show that $\mathrm{Ext}^1_R(\mathrm{Im}\,\partial^0, Y^i) = 0$ for all i. By definition, each Y^i is either injective or C-injective. If Y^i is injective, then $\mathrm{Ext}^1_R(\mathrm{Im}\,\partial^0, Y^i) = 0$. If instead $Y^i \in \mathcal{I}_C$, then $\mathrm{Ext}^1_R(\mathrm{Im}\,\partial^0, Y^i) \cong \mathrm{Ext}^1_{\mathcal{I}_C}(\mathrm{Im}\,\partial^0, Y^i) = 0$, where the isomorphism follows from Corollary 3.13(b) and Theorem 4.11(b), and the vanishing is from Theorem 4.1(b).

The long exact sequence in homology implies that $\mathrm{Hom}_R(L, Y)$ is exact. If M is G_C-injective, then the Ext-vanishing follows by using a truncation of Y as an injective coresolution of M.

The next definition leads to a main result. It is a slightly weaker version of the usual "two of three" condition on modules in short exact sequences.

Definition 5.16. Let χ be a class of R-modules.

1. We say χ is *resolving* if it satisfies the following conditions:

   (a) $\mathcal{P}(R) \subset \chi$;

   (b) for every short exact sequence $0 \to M' \to M \to M'' \to 0$ of R-module homomorphisms with $M'' \in \chi$, we have $M \in \chi$ if and only if $M' \in \chi$; and

   (c) χ is closed under direct summands.

   If χ satisfies only conditions (b) and (c), we say it is *quasi-resolving*.

2. We say χ is *coresolving* if it satisfies the following conditions:

(a) $\mathcal{I}(R) \subset \chi$;

(b) for every short exact sequence $0 \to M' \to M \to M'' \to 0$ of R-module homomorphisms with $M' \in \chi$, we have $M \in \chi$ if and only if $M'' \in \chi$; and

(c) χ is closed under direct summands.

If χ satisfies only conditions (b) and (c), we say it is *quasi-coresolving.*

The next result is part of Theorem 1.7 from the introduction.

Theorem 5.17. *Let C be a semidualizing R-module.*

(a) *The class of C-projective modules is quasi-resolving.*

(b) *The class of C-injective modules is quasi-coresolving.*

Proof. (a) Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of R-module homomorphisms such that $M'' \in \mathcal{P}_C$. Assume that either $M'' \in \mathcal{P}_C$ or $M \in \mathcal{P}_C$. Then $M'' \in \mathcal{B}_C$ by Theorem 3.16 and either $M' \in \mathcal{B}_C$ or $M \in \mathcal{B}_C$. By Proposition 2.8(c), we conclude that $M' \in \mathcal{B}_C$ and $M \in \mathcal{B}_C$. In either of our cases, the sequence

$$0 \to \mathrm{Hom}_R(C, M') \to \mathrm{Hom}_R(C, M) \to \mathrm{Hom}_R(C, M'') \to 0$$

is exact since $\mathrm{Ext}^1_R(C, M') = 0$ by Definition 2.7. Since $\mathrm{Hom}_R(C, M'')$ is projective by Theorem 3.15(c) and Corollary 3.14(a), we have that $\mathrm{Hom}_R(C, M')$ is projective if and only if $\mathrm{Hom}_R(C, M)$ is projective. It follows by the same corollary that $\mathcal{P}_C\text{-pd}_R(M') = 0$ if and only if $\mathcal{P}_C\text{-pd}_R(M) = 0$.

It remains to show that $\mathcal{P}_C$ is closed under direct summands. Let $M = L \oplus N \in \mathcal{P}_C$. We must show that $L \in \mathcal{P}_C$. Theorem 3.16 implies that $\mathrm{Hom}_R(C, M) \cong \mathrm{Hom}_R(C, L) \oplus \mathrm{Hom}_R(C, N)$ is projective, so the summands are also projective. Another application of Theorem 3.16 implies that $L \in \mathcal{P}_C$. Hence $\mathcal{P}_C$ is quasi-resolving.

(b) Let $0 \to M' \to M \to M'' \to 0$ be a short exact sequence of R-module homomorphisms such that $M' \in \mathcal{I}_C$. Assume that either $M'' \in \mathcal{I}_C$ or $M \in \mathcal{I}_C$. Then $M' \in \mathcal{A}_C$ by Theorem 3.16 and either $M'' \in \mathcal{A}_C$ or $M \in \mathcal{A}_C$. By Proposition 2.8(a), we conclude that $M \in \mathcal{A}_C$ and $M'' \in \mathcal{A}_C$. In either case, the sequence

$$0 \to C \otimes_R M' \to C \otimes_R M \to C \otimes_R M'' \to 0$$

is exact since $\mathrm{Tor}_1^R(C, M'') = 0$ by Definition 2.7. Since $C \otimes_R M'$ is injective by Theorem 3.15(b) and Corollary 3.14(b), we have that $C \otimes_R M$ is injective if and only if $C \otimes_R M''$ is injective. It follows by the same corollary that $\mathcal{I}_C\text{-id}_R(M) = 0$ if and only if $\mathcal{I}_C\text{-id}_R(M'') = 0$.

It remains to show that $\mathcal{I}_C$ is closed under direct summands. Let $M = L \oplus N \in \mathcal{I}_C$. We must show that $L \in \mathcal{I}_C$. Theorem 3.16 implies that $C \otimes_R M \cong (C \otimes_R L) \oplus (C \otimes_R N)$ is injective, so the summands are also injective. Another application of Theorem 3.16 implies that $L \in \mathcal{I}_C$. Hence $\mathcal{I}_C$ is quasi-coresolving.

To show that the classes of G_C-projective and G_C-injective modules are resolving and coresolving, respectively, we need modified versions of some well-known results.

Lemma 5.18 (Horseshoe lemma). *Let $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ be a short exact sequence of R-module homomorphisms.*

(a) *If X' and X'' are complete PC-resolutions of M' and M'' (respectively), then there is a complete PC-resolution X of M such that there is a degreewise split exact sequence $0 \to X' \to X \to X'' \to 0$ of R-complexes. In particular, if M' and M'' are G_C-projective, then so is M.*

(b) *If Y' and Y'' are complete IC-resolutions of M' and M'' (respectively), then there is a complete IC-resolution Y of M such that there is a degreewise split exact sequence $0 \to Y' \to Y \to Y'' \to 0$ of R-complexes. In particular, if M' and M'' are G_C-injective, then so is M.*

Proof. (a) Suppose that M' and M'' are G_C-projective with complete PC-resolutions X' and X'', respectively. From X' and X'' we may extract augmented projective resolutions $\cdots \to P_1' \to P_0' \to M' \to 0$ and $\cdots \to P_1'' \to P_0'' \to M'' \to 0$ and, applying the projective horseshoe lemma (as in [8, Lemma VIII.3.2]), we may construct an augmented projective resolution $\cdots \to P_1' \oplus P_1'' \to P_0' \oplus P_0'' \to M \to 0$ of M such that the diagram

$$\begin{tikzcd}
 & \vdots \arrow[d, "\partial_2'"'] & \vdots \arrow[d, "\partial_2"] & \vdots \arrow[d, "\partial_2''"] & \\
0 \arrow[r] & P_1' \arrow[r] \arrow[d, "\partial_1'"'] & P_1' \oplus P_1'' \arrow[r] \arrow[d, "\partial_1"] & P_1'' \arrow[r] \arrow[d, "\partial_1''"] & 0 \\
0 \arrow[r] & P_0' \arrow[r] \arrow[d, "\tau'"'] & P_0' \oplus P_0'' \arrow[r] \arrow[d, "\tau"] & P_0'' \arrow[r] \arrow[d, "\tau''"] & 0 \\
0 \arrow[r] & M' \arrow[r, "f"] \arrow[d] & M \arrow[r, "g"] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}$$

with exact rows commutes. We may extract augmented coproper $\mathcal{P}_C$-projective coresolutions $0 \to M' \xrightarrow{\epsilon'} {}^0X' \to {}^1X' \to \cdots$ and $0 \to M'' \xrightarrow{\epsilon''} {}^0X'' \to {}^1X'' \to \cdots$, where we define ${}^iX' := C \otimes_R {}^iQ'$ for some projective module ${}^iQ'$ (and similarly for ${}^iX''$). Applying the functor $\mathrm{Hom}_R(-, {}^0X')$ to the original short exact sequence, we obtain the exact sequence

$$0 \to \mathrm{Hom}_R(M'', {}^0X') \to \mathrm{Hom}_R(M, {}^0X') \to \mathrm{Hom}_R(M', {}^0X') \to \mathrm{Ext}^1_R(M'', {}^0X')$$

and the vanishing $\mathrm{Ext}^1_R(M'', {}^0X') = 0$ by Proposition 5.3(a) since ${}^0X' \in \mathcal{P}_C$. This vanishing implies the map $\mathrm{Hom}_R(M, {}^0X') \to \mathrm{Hom}_R(M', {}^0X')$ is surjective, so we obtain the following commutative diagram, where the existence of the map $\epsilon \in \mathrm{Hom}_R(M', {}^0X')$ implies the existence of the map $h \in \mathrm{Hom}_R(M, {}^0X')$ such that $\epsilon = hf$ and we define $\epsilon := (h, e''g)$:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M' \arrow[r, "f"] \arrow[d, "\epsilon'"'] & M \arrow[r, "g"] \arrow[d, "\epsilon"] \arrow[dl, dashed, "h"'] & M'' \arrow[r] \arrow[d, "\epsilon''"] & 0 \\
0 \arrow[r] & {}^0X' \arrow[r] & {}^0X' \oplus {}^0X'' \arrow[r] & {}^0X'' \arrow[r] & 0
\end{tikzcd}$$

We then apply the snake lemma and obtain the next commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M' \arrow[r, "f"] \arrow[d, "\epsilon'"'] & M \arrow[r, "g"] \arrow[d, "\epsilon"] \arrow[dl, dashed, "h"'] & M'' \arrow[r] \arrow[d, "\epsilon''"] & 0 \\
0 \arrow[r] & {}^0X' \arrow[r] \arrow[d] & {}^0X' \oplus {}^0X'' \arrow[r] \arrow[d] & {}^0X'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{coker} \epsilon' \arrow[r] & \operatorname{coker} \epsilon \arrow[r] & \operatorname{coker} \epsilon'' \arrow[r] & 0
\end{tikzcd}$$

with exact rows and columns. Since $\operatorname{coker} \epsilon'$ and $\operatorname{coker} \epsilon''$ have coproper $\mathcal{P}_C$-projective coresolutions, we may apply the same procedure to the bottom row of the diagram. We obtain the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & \operatorname{coker} \epsilon' \arrow[r] \arrow[d, "{}^0\partial'"'] & \operatorname{coker} \epsilon \arrow[r] \arrow[d, "{}^0\partial"] \arrow[dl, dashed, "h^0"'] & \operatorname{coker} \epsilon'' \arrow[r] \arrow[d, "{}^0\partial''"] & 0 \\
0 \arrow[r] & {}^1X' \arrow[r] \arrow[d] & {}^1X' \oplus {}^1X'' \arrow[r] \arrow[d] & {}^1X'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \operatorname{coker} {}^0\partial' \arrow[r] & \operatorname{coker} {}^0\partial \arrow[r] & \operatorname{coker} {}^0\partial'' \arrow[r] & 0
\end{tikzcd}$$

with exact rows and columns, where we define ${}^0\partial$ as above. Splicing these diagrams together and continuing this process inductively, we obtain the commutative diagram

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M' \arrow[r] \arrow[d, "\epsilon'"'] & M \arrow[r] \arrow[d, "\epsilon"] & M'' \arrow[r] \arrow[d, "\epsilon''"] & 0 \\
0 \arrow[r] & {}^0X' \arrow[r] \arrow[d, "{}^0\partial'"'] & {}^0X' \oplus {}^0X'' \arrow[r] \arrow[d, "\partial^0"] & {}^0X'' \arrow[r] \arrow[d, "{}^0\partial''"] & 0 \\
0 \arrow[r] & {}^1X' \arrow[r] \arrow[d, "{}^1\partial'"'] & {}^1X' \oplus {}^1X'' \arrow[r] \arrow[d, "\partial^1"] & {}^1X'' \arrow[r] \arrow[d, "{}^1\partial''"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}$$

with exact rows and columns. Splicing this diagram with the diagram obtained from the projective horseshoe lemma yields the exact sequence

$$0 \to X' \to X \to X'' \to 0.$$

We claim that that X is a complete PC-resolution of M. To show this, fix any projective R-module Q. We have the exact sequence

$$0 \to \mathrm{Hom}_R(X'', C \otimes_R Q) \to \mathrm{Hom}_R(X, C \otimes_R Q) \to \mathrm{Hom}_R(X', C \otimes_R Q) \to 0$$

by Ext-vanishing, so exactness of $\mathrm{Hom}_R(X', C \otimes_R Q)$ and $\mathrm{Hom}_R(X'', C \otimes_R Q)$ imply exactness of the complex $\mathrm{Hom}_R(X, C \otimes_R Q)$ using the long exact sequence in homology. Further, we have

$$M \cong \mathrm{coker}(P_1' \oplus P_1'' \to P_0' \oplus P_0'')$$

by construction.

(b) Suppose that M' and M'' are G_C-injective with complete IC-resolutions Y' and Y'' respectively. From Y' and Y'' we may extract augmented injective coresolutions $0 \to M' \to {}^0J' \to {}^1J' \to \cdots$ and $0 \to M'' \to {}^0J'' \to {}^1J'' \to \cdots$ and, appying the injective horseshoe lemma (as in [8, Lemma VIII.3.4]), we may construct an augmented injective coresolution $0 \to M \to {}^0J' \oplus {}^0J'' \to {}^1J' \oplus {}^1J'' \to \cdots$ of M such that the diagram

$$
\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
0 \arrow[r] & M' \arrow[r, "f"] \arrow[d, "\epsilon'"'] & M \arrow[r, "g"] \arrow[d, "\epsilon"] & M'' \arrow[r] \arrow[d, "\epsilon''"] & 0 \\
0 \arrow[r] & {}^0J' \arrow[r] \arrow[d, "{}^0\partial'"'] & {}^0J' \oplus {}^0J'' \arrow[r] \arrow[d, "{}^0\partial"] & {}^0J'' \arrow[r] \arrow[d, "{}^0\partial''"] & 0 \\
0 \arrow[r] & {}^1J' \arrow[r] \arrow[d, "{}^1\partial'"'] & {}^1J' \oplus {}^1J'' \arrow[r] \arrow[d, "{}^1\partial"] & {}^1J'' \arrow[r] \arrow[d, "{}^1\partial''"] & 0 \\
 & \vdots & \vdots & \vdots &
\end{tikzcd}
$$

with exact rows commutes. We may extract augmented coproper $\mathcal{I}_C$-injective resolutions $\cdots \to Y_1' \to Y_0' \xrightarrow{\tau'} M' \to 0$ and $\cdots \to Y_1'' \to Y_0'' \xrightarrow{\tau''} M'' \to 0$ where we define $Y_i' := \operatorname{Hom}_R(C, I_i')$ for some injective module I_i' (and similarly for Y_i''). Applying the functor $\operatorname{Hom}_R(Y_0'', -)$ to the original short exact sequence, we obtain the exact sequence

$$0 \to \operatorname{Hom}_R(Y_0'', M') \to \operatorname{Hom}_R(Y_0'', M) \to \operatorname{Hom}_R(Y_0'', M'') \to \operatorname{Ext}_R^1(Y_0'', M')$$

and the vanishing $\operatorname{Ext}_R^i(Y_0'', M') = 0$ by Proposition 5.3(b) since $Y_0'' \in \mathcal{I}_C$. Since the map $\operatorname{Hom}_R(Y_0'', M) \to \operatorname{Hom}_R(Y_0'', M'')$ is surjective, we obtain the following commutative diagram, where the existence of the map $\tau'' \in \operatorname{Hom}_R(Y_0'', M'')$ implies the existence of the map $h \in \operatorname{Hom}_R(Y_0'', M)$ such that $\tau'' = gh$ and we define τ such that $(x, y) \mapsto f\tau'(x) + h(y)$:

$$
\begin{tikzcd}
0 \arrow[r] & Y_0' \arrow[r] \arrow[d, "\tau'"'] & Y_0' \oplus Y_0'' \arrow[r] \arrow[d, "\tau"] & Y_0'' \arrow[r] \arrow[d, "\tau''"] \arrow[dl, dashed, "h"'] & 0 \\
0 \arrow[r] & M' \arrow[r, "f"] \arrow[d] & M \arrow[r, "g"] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

We apply the snake lemma and obtain the commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & \ker\tau' \arrow[r] \arrow[d] & \ker\tau \arrow[r] \arrow[d] & \ker\tau'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Y_0' \arrow[r] \arrow[d, "\tau'"'] & Y_0' \oplus Y_0'' \arrow[r] \arrow[d, "\tau"] & Y_0'' \arrow[r] \arrow[d, "\tau''"] \arrow[dl, dashed, "h"'] & 0 \\
0 \arrow[r] & M' \arrow[r, "f"] \arrow[d] & M \arrow[r, "g"] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

with exact rows and columns. Since $\ker\tau''$ and $\ker\tau$ have coproper $\mathcal{I}_C$-injective coresolutions, we may apply the same procedure to the top row of the diagram. We obtain the commutative diagram

$$
\begin{tikzcd}
0 \arrow[r] & \ker\partial_1' \arrow[r] \arrow[d] & \ker d_1 \arrow[r] \arrow[d] & \ker\partial_1'' \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & Y_1' \arrow[r] \arrow[d, "\partial_1'"'] & Y_1' \oplus Y_1'' \arrow[r] \arrow[d, "d_1"] & Y_1'' \arrow[r] \arrow[d, "\partial_1''"] \arrow[dl, dashed, "h_1"'] & 0 \\
0 \arrow[r] & \ker\tau' \arrow[r] & \ker\tau \arrow[r] & \ker\tau'' \arrow[r] & 0
\end{tikzcd}
$$

with exact rows and columns, where we define d_1 as above. Splicing these diagrams together and continuing this process inductively, we obtain the commutative diagram

$$
\begin{tikzcd}
 & \vdots \arrow[d, "\partial_2'"'] & \vdots \arrow[d, "\partial_2"] & \vdots \arrow[d, "\partial_2''"] & \\
0 \arrow[r] & Y_1' \arrow[r] \arrow[d, "\partial_1'"'] & Y_1' \oplus Y_1'' \arrow[r] \arrow[d, "\partial_1"] & Y_1'' \arrow[r] \arrow[d, "\partial_2''"] & 0 \\
0 \arrow[r] & Y_0' \arrow[r] \arrow[d, "\tau'"'] & Y_0' \oplus Y_0'' \arrow[r] \arrow[d, "\tau"] & Y_0'' \arrow[r] \arrow[d, "\tau''"] & 0 \\
0 \arrow[r] & M' \arrow[r] \arrow[d] & M \arrow[r] \arrow[d] & M'' \arrow[r] \arrow[d] & 0 \\
 & 0 & 0 & 0 &
\end{tikzcd}
$$

with exact rows and columns. Splicing this diagram with the diagram obtained from the injective horseshoe lemma yields the exact sequence

$$0 \to Y' \to Y \to Y'' \to 0.$$

We claim that Y is a complete IC-resolution of M. To show this, fix any injective R-module J. We have the exact sequence

$$0 \to \mathrm{Hom}_R(\mathrm{Hom}_R(C,J),Y') \to \mathrm{Hom}_R(\mathrm{Hom}_R(C,J),Y) \to \mathrm{Hom}_R(\mathrm{Hom}_R(C,J),Y'') \to 0$$

by Ext-vanishing, so exactness of $\mathrm{Hom}_R(\mathrm{Hom}_R(C,J),Y')$ and $\mathrm{Hom}_R(\mathrm{Hom}_R(C,J),Y'')$ imply exactness of $\mathrm{Hom}_R(\mathrm{Hom}_R(C,J),Y)$ using the long exact sequence in homology. Further, we have

$$M \cong \ker({}^0J' \oplus {}^0J'' \to {}^1J' \oplus {}^1J'')$$

by construction.

Lemma 5.19 (Lifting lemma)**.** *Let C be a semidualizing R-module.*

(a) *Let X and X' be complete PC-resolutions of R-modules M and M', respectively. Let $f: M \to M'$ be a homomorphism. Then f induces a chain map $F: X \to X'$ such that the diagram*

$$\begin{tikzcd} X_0 \arrow[r] \arrow[d, "F_0"'] & M \arrow[r] \arrow[d, "f"] & {}^0X \arrow[d, "F^0"] \\ X_0' \arrow[r] & M' \arrow[r] & {}^0X' \end{tikzcd}$$

commutes.

(b) *Let Y and Y' be complete IC-resolutions of R-modules M and M', respectively. Let $g: M \to M'$ be a homomorphism. Then g induces a chain map $G: Y \to Y'$ such that the diagram*

$$\begin{tikzcd} Y_0 \arrow[r] \arrow[d, "G_0"'] & M \arrow[r] \arrow[d, "g"] & {}^0Y \arrow[d, "G^0"] \\ Y_0' \arrow[r] & M' \arrow[r] & {}^0Y' \end{tikzcd}$$

commutes.

Proof. (a) Let

$$\begin{aligned} X &:= \cdots \to P_1 \to P_0 \to C \otimes_R Q^0 \to C \otimes_R Q^1 \to \cdots \\ X' &:= \cdots \to P_1' \to P_0' \to C \otimes_R (Q^0)' \to C \otimes_R (Q^1)' \to \cdots \end{aligned}$$

be complete PC-resolutions of M and M', respectively. By the construction in Proposition 5.3(a), there are augmented projective resolutions $P := \cdots \to P_1 \to P_0 \to M \to 0$ and $P' := \cdots \to P_1' \to P_0' \to M' \to 0$ of M and M', respectively. The projective lifting lemma (as in [8, Proposition VI.3.2]) gives maps such that the diagram

$$\begin{tikzcd}
\cdots \arrow[r] & P_1 \arrow[r] \arrow[d, "f_1"] & P_0 \arrow[r] \arrow[d, "f_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r] & P_1' \arrow[r] & P_0' \arrow[r] & M' \arrow[r] & 0
\end{tikzcd}$$

defines a chain map $P \to P'$. Proposition 5.3(a) gives augmented coproper $\mathcal{P}_C$-projective coresolutions

$$0 \to M \to C \otimes_R Q^0 \xrightarrow{\partial^0} C \otimes_R Q^1 \xrightarrow{\partial^1} \cdots$$

$$0 \to M' \to C \otimes_R (Q^0)' \xrightarrow{(\partial^0)'} C \otimes_R (Q^1)' \xrightarrow{(\partial^1)'} \cdots$$

of M and M', respectively. Consider the exact sequence

$$0 \to M \to C \otimes_R Q^0 \to \operatorname{Im} \partial^0 \to 0$$

and the associated sequence

$$0 \to \operatorname{Hom}_R(\operatorname{Im} \partial^0, C \otimes_R (Q^0)') \to \operatorname{Hom}_R(C \otimes_R Q^0, C \otimes_R (Q^0)') \to \operatorname{Hom}_R(M, C \otimes_R (Q^0)') \to 0$$

that is also exact since $\operatorname{Ext}^1_R(\operatorname{Im} \partial^0, C \otimes_R (Q^0)') = 0$ by Proposition 5.3(a). Hence we have a surjection

$$\operatorname{Hom}_R(C \otimes_R Q^0) \twoheadrightarrow \operatorname{Hom}_R(M, C \otimes_R (Q^0)')$$

that gives a map f^0 such that the diagram

$$\begin{tikzcd}
0 \arrow[r] & M \arrow[r] \arrow[d, "f"] & C \otimes_R Q^0 \arrow[r, "\partial^0"] \arrow[d, "f^0"] & C \otimes_R Q^1 \arrow[r, "\partial^1"] & \cdots \\
0 \arrow[r] & M' \arrow[r] & C \otimes_R (Q^0)' \arrow[r, "(\partial^0)'"] & C \otimes_R (Q^1)' \arrow[r, "(\partial^1)'"] & \cdots
\end{tikzcd}$$

commutes. Continue this process inductively to obtain a chain map between these coresolutions induced by f. A routine diagram chase gives the resulting chain map $F : X \to X'$.

(b) Let

$$\begin{aligned} Y &:= \cdots \to \operatorname{Hom}_R(C, I_1) \to \operatorname{Hom}_R(C, I_0) \to J^0 \to J^1 \to \cdots \\ Y' &:= \cdots \to \operatorname{Hom}_R(C, I_1') \to \operatorname{Hom}_R(C, I_0') \to (J^0)' \to (J^1)' \to \cdots \end{aligned}$$

be complete IC-resolutions of M and M', respectively. By the construction in Proposition 5.3(b), there are augmented injective coresolutions $J := 0 \to M \to J^0 \to J^1 \to \cdots$ and $J' := 0 \to M' \to (J^0)' \to (J^1)' \to \cdots$ of M and M', respectively. The injective lifting lemma (as in [8, Proposition VI.3.6]) gives maps such that the diagram

$$\begin{tikzcd} 0 \arrow[r] & M \arrow[r] \arrow[d, "g"] & J^0 \arrow[r] \arrow[d, "g^0"] & J^1 \arrow[r] \arrow[d, "g^1"] & \cdots \\ 0 \arrow[r] & M' \arrow[r] & (J^0)' \arrow[r] & (J^1)' \arrow[r] & \cdots \end{tikzcd}$$

defines a chain map $J \to J'$. Proposition 5.3(b) gives augmented coproper $\mathcal{I}_C$-injective resolutions

$$\cdots \xrightarrow{\partial^2} \operatorname{Hom}_R(C, I_1) \xrightarrow{\partial^1} \operatorname{Hom}_R(C, I_0) \to M \to 0$$

$$\cdots \xrightarrow{(\partial^2)'} \operatorname{Hom}_R(C, I_1') \xrightarrow{(\partial^1)'} \operatorname{Hom}_R(C, I_0') \to M' \to 0$$

of N and N', respectively. Consider the exact sequence

$$0 \to \operatorname{Im}(\partial^1)' \to \operatorname{Hom}_R(C, I_0') \to M' \to 0$$

and the associated sequence

$$0 \to \operatorname{Hom}_R(\operatorname{Hom}_R(C, I_0), \operatorname{Im}(\partial^1)') \to \operatorname{Hom}_R(\operatorname{Hom}_R(C, I_0), \operatorname{Hom}_R(C, I_0')) \to \operatorname{Hom}_R(\operatorname{Hom}_R(C, I_0), M') \to 0$$

that is also exact since $\operatorname{Ext}^1_R(\operatorname{Hom}_R(C, I_0), \operatorname{Im}(\partial^1)') = 0$ by Proposition 5.3(b). Hence we have a surjection

$$\operatorname{Hom}_R(\operatorname{Hom}_R(C, I_0), \operatorname{Hom}_R(C, I_0')) \twoheadrightarrow \operatorname{Hom}_R(\operatorname{Hom}_R(C, I_0), M')$$

that gives a map g_0 such that the diagram

$$\begin{tikzcd} \cdots \arrow[r, "\partial^2"] & \operatorname{Hom}_R(C, I_0) \arrow[r, "\partial^1"] & \operatorname{Hom}_R(C, I_0) \arrow[r] \arrow[d, "g^0"] & M \arrow[r] \arrow[d, "g"] & 0 \\ \cdots \arrow[r, "(\partial^2)'"] & \operatorname{Hom}_R(C, I_1') \arrow[r, "(\partial^1)'"] & \operatorname{Hom}_R(C, I_0') \arrow[r] & M' \arrow[r] & 0 \end{tikzcd}$$

commutes. Continue this process inductively to obtain a chain map between these resolutions induced by g. A routine diagram chase gives the resulting chain map $G : Y \to Y'$. ■

Lemma 5.20. *Let C be a semidualizing R-module.*

(a) *Let X and X' be complete $\mathcal{P}C$-resolutions of R-modules M and M', respectively. Let $f : M \twoheadrightarrow M'$ be an epimorphism. Then there exists a complete $\mathcal{P}C$-resolution $\overline{X}$ of M such that f induces a surjective chain map $F : \overline{X} \to X'$.*

(b) *Let Y and Y' be complete $\mathcal{I}C$-resolutions of R-modules N and N', respectively. Let $g : N \hookrightarrow N'$ be a monomorphism. Then there exists a complete $\mathcal{I}C$-resolution $\overline{Y}'$ of N' such that g induces an injective chain map $G : Y \to \overline{Y}'$.*

Proof. (a) We construct the resolution and lifting by induction, constructing at each step a new resolution and a lifting that is surjective in increasing homological degree. The proof of Lemma 5.19(a) gives augmented projective resolutions of M and M' and a lifting $\{\ldots, f_1, f_0, f\}$ making the diagram

$$\begin{tikzcd}
\cdots \arrow[r, "\partial_2"] & P_1 \arrow[r, "\partial_1"] \arrow[d, "f_1"'] & P_0 \arrow[r, "\tau"] \arrow[d, "f_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r, "\partial_2'"] & P_1' \arrow[r, "\partial_1'"] & P_0' \arrow[r, "\tau'"] & M' \arrow[r] & 0
\end{tikzcd}$$

commute. Construct the commutative digram

$$\begin{tikzcd}
0 \arrow[r] & \ker\tau \arrow[r] \arrow[d, "g_0"'] & P_0 \arrow[r] \arrow[d, "f_0"] & M \arrow[r] \arrow[d, "f"] & 0 \\
0 \arrow[r] & \ker\tau' \arrow[r] & P_0' \arrow[r] & M' \arrow[r] & 0 \\
0 \arrow[r] & P_1' \oplus \ker\tau \arrow[r, "\begin{pmatrix} 1 & 0 \\ 0 & \epsilon_0 \end{pmatrix}"'] \arrow[u, "\begin{pmatrix} \tau_1' & g_0 \end{pmatrix}"] & P_1' \oplus P_0 \arrow[r, "\begin{pmatrix} 0 & \tau \end{pmatrix}"'] \arrow[u, "\begin{pmatrix} \partial_1' & f_0 \end{pmatrix}"'] & M \arrow[r] \arrow[u, "f"'] & 0
\end{tikzcd}$$

where the maps are defined by the first diagram and the associated natural maps $\tau_1' : P_1' \to \ker\tau'$ and $\epsilon_0 : \ker\tau \to P_0$, and g^0 is the induced map on kernels. Here the maps f and $\begin{pmatrix} \tau_1' & g_0 \end{pmatrix}$ are surjective by construction, so $\begin{pmatrix} \partial_1' & f_0 \end{pmatrix}$ must also be surjective by the snake lemma. Use this map to construct the augmented projective resolution of M and lifting

$$\begin{tikzcd}
\cdots \arrow[r, "\left(\begin{smallmatrix}0\\ \partial_2\end{smallmatrix}\right)"] & P_1' \oplus P_1 \arrow[r, "\left(\begin{smallmatrix}1 & 0\\ 0 & \partial_1\end{smallmatrix}\right)"] \arrow[d, "(1\ f_1)"'] & P_1' \oplus P_0 \arrow[r, "(0\ \tau)"] \arrow[d, "(\partial_1'\ f_0)"] & M \arrow[r] \arrow[d, "f"] & 0 \\
\cdots \arrow[r, "\partial_2'"] & P_1' \arrow[r, "\partial_1'"] & P_0' \arrow[r, "\tau'"] & M' \arrow[r] & 0
\end{tikzcd}$$

that is surjective at the two rightmost degrees and exact since it is of the form $P^+ \oplus Z'$ for $Z' := 0 \to P_1' \xrightarrow{1} P_1' \to 0$. Continue this process inductively using this diagram and its associated natural maps to obtain a projective resolution of M and surjective lifting. The construction of an appropriate augmented coproper $\mathcal{P}_C$-projective coresolution of M and projective lifting follow similarly; we splice them together to obtain $\overline{X}$.

(b) We construct the resolution and lifting by induction, constructing at each step a new resolution and a lifting that is surjective in increasing cohomological degree. The proof of Lemma 5.19(b) gives augmented injective coresolutions of N and N' and a lifting $\{g, g^0, g^1, \ldots\}$ making the digram

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "\epsilon"] \arrow[d, "g"'] & J^0 \arrow[r, "\partial^0"] \arrow[d, "g^0"] & J^1 \arrow[r, "\partial^1"] \arrow[d, "g^1"] & \cdots \\
0 \arrow[r] & N' \arrow[r, "\epsilon'"] & (J^0)' \arrow[r, "(\partial^0)'"] & (J^1)' \arrow[r, "(\partial^1)'"] & \cdots
\end{tikzcd}$$

commute. Construct the commutative diagram

$$\begin{tikzcd}
0 \arrow[r] & N' \arrow[r, "(0\ \epsilon')"] & J^1 \oplus (J^0)' \arrow[r, "\left(\begin{smallmatrix}1 & 0\\ 0 & (\tau^0)'\end{smallmatrix}\right)"] & J^1 \oplus \operatorname{coker}\epsilon' \arrow[r] & 0 \\
0 \arrow[r] & N \arrow[r] \arrow[u, "g"] \arrow[d, "g"'] & J^0 \arrow[r] \arrow[u, "(\partial^0\ g^0)"'] \arrow[d, "g^0"] & \operatorname{coker}\epsilon \arrow[r] \arrow[u, "(\epsilon^0\ h^0)"'] \arrow[d, "h^0"] & 0 \\
0 \arrow[r] & N' \arrow[r] & (J^0)' \arrow[r] & \operatorname{coker}\epsilon' \arrow[r] & 0
\end{tikzcd}$$

where the maps are defined by the first diagram and the associated natural maps $(\tau^0)' : (J^0)' \to \operatorname{coker}\epsilon'$ and $\epsilon^0 : \operatorname{coker}\epsilon \to J^1$, and h^0 is the induced map on cokernels. Here the maps g and $\begin{pmatrix}\epsilon^0 & h^0\end{pmatrix}$ are injective by construction, so $\begin{pmatrix}\partial^0 & g^0\end{pmatrix}$ must also be injective by the snake lemma. Use this map to construct the augmented injective coresolution of N' and lifting

$$\begin{tikzcd}
0 \arrow[r] & N \arrow[r, "\epsilon"] \arrow[d, "g"'] & J^0 \arrow[r, "\partial^0"] \arrow[d, "\begin{pmatrix} \partial^0 & g^0 \end{pmatrix}"] & J^1 \arrow[r, "\partial^1"] \arrow[d, "\begin{pmatrix} 1 & g^1 \end{pmatrix}"] & \cdots \\
0 \arrow[r] & N' \arrow[r, "\begin{pmatrix} 0 & \epsilon' \end{pmatrix}"'] & J^1 \oplus (J^0)' \arrow[r, "\begin{pmatrix} 1 & 0 \\ 0 & (\partial^0)' \end{pmatrix}"'] & J^1 \oplus (J^1)' \arrow[r, "\begin{pmatrix} 0 \\ (\partial^1)' \end{pmatrix}"'] & \cdots
\end{tikzcd}$$

that is injective at the two leftmost degrees since it is of the form ${}^{+}I \oplus Z$ for $Z := 0 \to J^1 \xrightarrow{1} J^1 \to 0$. Continue this process inductively using this diagram and its associated natural maps to obtain an injective coresolution of N' and injective lifting. The construction of an appropriate augmented coproper $\mathcal{I}_C$-injective resolution of N' and injective lifting follow similarly; we splice them together to obtain $\overline{Y}'$.

Here is the main result of this section. It is part of Theorem 1.7 from the introduction.

Theorem 5.21 ([12, Theorem 2.8] and [4, Proposition 1.4])**.** *Let C be a semidualizing R-module.*

(a) *The class of G_C-projective modules is resolving.*

(b) *The class of G_C-injective modules is coresolving.*

Proof. (a) Let $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ be an exact sequence of R-modules. Suppose that M' and M'' are G_C-projective. Lemma 5.18(a) implies that M is also G_C-projective. If M and M'' are G_C-projective, there are corresponding complete PC-resolutions X and X'', respectively, and a chain map $G : X \to X''$ induced by g. By Lemma 5.20(a), we may assume without loss of generality that G is surjective. There is the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & \ker G_1 \arrow[r] \arrow[d] & \ker G_0 \arrow[r] \arrow[d] & \ker G^0 \arrow[r] \arrow[d] & \ker G^1 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & P_1 \arrow[r] \arrow[d, "G_1"] & P_0 \arrow[r] \arrow[d, "G_0"] & C \otimes_R Q^0 \arrow[r] \arrow[d, "G^0"] & C \otimes_R Q^1 \arrow[r] \arrow[d, "G^1"] & \cdots \\
\cdots \arrow[r] & P_1'' \arrow[r] \arrow[d] & P_0'' \arrow[r] \arrow[d] & C \otimes_R (Q^0)'' \arrow[r] \arrow[d] & C \otimes_R (Q^1)'' \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

Denote the top row of the above diagram by X'; this complex is exact by the long exact sequence in homology. Since the bottom two rows are complete PC-resolutions, long exact sequences in $\operatorname{Ext}_R(-, C \otimes_R Q)$ give that each column is $\operatorname{Hom}_R(-, C \otimes_R Q)$-exact for every projective R-module Q by Propositions 5.3(a) and 5.13(a). Hence $\operatorname{Hom}_R(X, C \otimes_R Q)$ is exact by the associated long exact sequence in homology. Since the classes of projective and C-projective modules are quasi-resolving, each $\ker G_i$ is projective and each $\ker G^j$ is C-projective. Truncation of the above diagram after the second column gives, by the snake lemma, the sequence

$$0 \to \operatorname{coker}(\ker G_1 \to \ker G_0) \to \underbrace{\operatorname{coker} \partial_1}_{\cong M} \xrightarrow{g} \underbrace{\operatorname{coker} \partial_1''}_{\cong M''} \to 0$$

and implies that $\operatorname{coker}(\ker G_1 \to \ker G_0) \cong \ker g \cong M'$; hence M' is G_C-projective.

It remains to show that the class of G_C-projective modules is closed under direct summands. Let $Y \oplus Z$ be G_C-projective. If we define $W = Y \oplus Z \oplus Y \oplus Z \oplus \cdots$, then $W \in \mathcal{P}_C$ by Proposition 5.5(a). Since $W \cong Y \oplus (Y \oplus Z) \oplus (Y \oplus Z) \cdots = Y \oplus W$, there is an exact sequence $0 \to Y \to Y \oplus W \to W \to 0$. Since we have shown that $Y \oplus W \cong W$ is G_C-projective, our previous work shows that Y is also G_C-projective and we are done.

(b) Let $0 \to M' \xrightarrow{f} M \xrightarrow{g} M'' \to 0$ be an exact sequence of R-modules. Suppose that M' and M'' are G_C-injective. Lemma 5.18(b) implies that M is also G_C-injective. If M' and M are G_C-injective, there are corresponding complete IC-resolutions Y' and Y, respectively, and a chain map $F : Y' \to Y$ induced by f. By Lemma 5.20(b), we may assume without loss of generality that F is injective. There is the following commutative diagram:

$$\begin{tikzcd}
 & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & 0 \arrow[d] & \\
\cdots \arrow[r] & \operatorname{Hom}_R(C, I_1') \arrow[r] \arrow[d, "F_1"] & \operatorname{Hom}_R(C, I_0') \arrow[r] \arrow[d, "F_0"] & (J^0)' \arrow[r] \arrow[d, "F^0"] & (J^1)' \arrow[r] \arrow[d, "F^1"] & \cdots \\
\cdots \arrow[r] & \operatorname{Hom}_R(C, I_1) \arrow[r] \arrow[d] & \operatorname{Hom}_R(C, I_0) \arrow[r] \arrow[d] & J^0 \arrow[r] \arrow[d] & J^1 \arrow[r] \arrow[d] & \cdots \\
\cdots \arrow[r] & \operatorname{coker} F_1 \arrow[r] \arrow[d] & \operatorname{coker} F_0 \arrow[r] \arrow[d] & \operatorname{coker} F^0 \arrow[r] \arrow[d] & \operatorname{coker} F^1 \arrow[r] \arrow[d] & \cdots \\
 & 0 & 0 & 0 & 0 &
\end{tikzcd}$$

Denote the bottom row of the above diagram by Y''; this complex is exact by the long exact sequence in homology. Since the top two rows are complete IC-resolutions, long exact sequences in $\operatorname{Ext}_R(\operatorname{Hom}_R(C, J), -)$ give that each column is $\operatorname{Hom}_R(\operatorname{Hom}_R(C, J), -)$-exact for every injective R-module J by Propositions 5.3(b) and 5.13(b). Hence $\operatorname{Hom}_R(\operatorname{Hom}_R(C, J), Y)$ is exact by the associated long exact sequence in homology. Since the classes of injective and C-injective modules are quasi-coresolving, each $\operatorname{coker} F^i$ is injective and each $\operatorname{coker} F_i$ is C-injective. Truncation of the above diagram after the second column gives, by the snake lemma, the sequence

$$0 \to \underbrace{\ker(\partial^0)'}_{\cong M'} \xrightarrow{f} \underbrace{\ker \partial^0}_{\cong M} \to \ker(\operatorname{coker} F^0 \to \operatorname{coker} F^1) \to 0$$

and implies that $\ker(\operatorname{coker} F^0 \to \operatorname{coker} F^1) \cong \operatorname{Im} f \cong M''$; hence M'' is G_C-injective.

It remains to show that the class of G_C-injective modules is closed under direct summands. Let $Y \oplus Z$ be G_C-projective. If we define $W = Y \times Z \times Y \times Z \times \cdots$, then $W \in \mathcal{I}_C$ by Proposition 5.5(b). Since $W \cong Y \times (Y \times Z) \times (Y \times Z) \times = Y \times W \cong W \times Y$, there is an exact sequence $0 \to W \to W \oplus Y \to Y \to 0$. Since we have shown that $W \oplus Y \cong W$ is G_C-injective, our previous work shows that Y is also G_C-injective and we are done.

Corollary 5.22 ([12, Proposition 2.9]). *Let C be a semidualizing R-module.*

(a) *Every cokernel in a complete PC-resolution is G_C-projective.*

(b) *Every kernel in a complete IC-resolution is G_C-injective.*

Proof. (a) Consider a complete PC-resolution

$$X := \cdots \xrightarrow{\partial_2} P_1 \xrightarrow{\partial_1} P_0 \xrightarrow{\partial_0} C \otimes_R Q^0 \xrightarrow{\partial^0} C \otimes_R Q^1 \xrightarrow{\partial^1} \cdots$$

of an R-module M. There is a short exact sequence $0 \to \operatorname{coker} \partial_2 \to P_0 \to M \to 0$ since $M \cong \operatorname{coker} \partial_1$ by definition. Since both P_0 and M are G_C-projective, Theorem 5.21(a) implies that $\operatorname{coker} \partial_2$ is also G_C-projective. Inductively, we form short exact sequences of the form $0 \to \operatorname{coker} \partial_{i+1} \to P_{i-1} \to \operatorname{coker} \partial_i \to 0$ and, since both P_{i-1} and $\operatorname{coker} \partial_i$ are G_C-projective, we have that $\operatorname{coker} \partial_{i+1}$ is also G_C-projective.

By Proposition 5.3(a), it suffices to show that $\operatorname{Ext}^i_R(\operatorname{coker} \partial^j, C \otimes_R Q) = 0$ for $i > 0$, $j \geq 0$, and all projective R-modules Q. Consider the short exact sequence $0 \to M \to C \otimes_R Q^0 \to \operatorname{coker} \partial_0 \to 0$. We have that M is G_C-projective by assumption, and Proposition 5.13(a) implies that $C \otimes_R Q^0$ is also G_C-projective. Hence, we have $\operatorname{Ext}^i_R(M, C \otimes_R Q) = 0 = \operatorname{Ext}^i_R(C \otimes_R Q^0, C \otimes_R Q)$ for all $i \geq 1$. This gives $\operatorname{Ext}^i_R(\operatorname{coker} \partial_0, C \otimes_R Q) = 0$ for $i > 1$. Since $\operatorname{Hom}_R(X, C \otimes_R Q)$ is exact, the sequence

$$0 \to \operatorname{Hom}_R(\operatorname{coker} \partial_0, C \otimes_R Q) \to \operatorname{Hom}_R(C \otimes_R Q^0, C \otimes_R Q) \to \operatorname{Hom}_R(M, C \otimes_R Q) \to 0$$

is exact as well, so $\operatorname{Ext}^1_R(\operatorname{coker} \partial_0, C \otimes_R Q) = 0$. Inductively, we form short exact sequences of the form $0 \to \operatorname{coker} \partial^{j-1} \to C \otimes_R Q^{j+1} \to \operatorname{coker} \partial^j \to 0$ and apply the same reasoning to conclude that each $\operatorname{coker} \partial^j$ is G_C-projective.

(b) Consider a complete IC-resolution

$$Y := \cdots \xrightarrow{\partial_2} \mathrm{Hom}_R(C, I_1) \xrightarrow{\partial_1} \mathrm{Hom}_R(C, I_0) \xrightarrow{\partial_0} J^0 \xrightarrow{\partial^0} J^1 \xrightarrow{\partial^1}\to \cdots$$

of an R-module M. There is a short exact sequence $0 \to M \to J^0 \to \ker \partial^1 \to 0$ since $M \cong \ker \partial^0$ by definition. Since both J^0 and M are G_C-injective, Theorem 5.21(a) implies that $\ker \partial^1$ is also G_C-injective. Inductively, we form short exact sequences of the form $0 \to \ker \partial^i \to J^i \to \ker \partial^{i+1} \to 0$ and, since both J^i and $\ker \partial^i$ are G_C-injective, we have that $\ker \partial^{i+1}$ is also G_C-injective.

By Proposition 5.3(b), it suffices to show that $\mathrm{Ext}^i_R(\mathrm{Hom}_R(C, J), \ker \partial_j) = 0$ for all $i > 0$, $j \geq 0$, and all injective R-modules J. Consider the short exact sequence $0 \to \ker \partial_0 \to \mathrm{Hom}_R(C, I_0) \to M \to 0$. We have that M is G_C-injective by assumption, and Proposition 5.13(b) implies that $\mathrm{Hom}_R(C, I_0)$ is also G_C-injective. Hence, we have $\mathrm{Ext}^i_R(\mathrm{Hom}_R(C, J), M) = 0 = \mathrm{Ext}^i_R(\mathrm{Hom}_R(C, J), \mathrm{Hom}_R(C, I_0))$ for all $i \geq 1$. This gives $\mathrm{Ext}^i_R(\mathrm{Hom}_R(C, J), \ker \partial_0) = 0$ for $i > 1$. Since $\mathrm{Hom}_R(\mathrm{Hom}_R(C, J), Y)$ is exact, the sequence

$$0 \to \mathrm{Hom}_R(\mathrm{Hom}_R(C, J), M) \to \mathrm{Hom}_R(\mathrm{Hom}_R(C, J), \mathrm{Hom}_R(C, I_0)) \to \mathrm{Hom}_R(\mathrm{Hom}_R(C, J), \ker \partial_0) \to 0$$

is exact as well, so $\mathrm{Ext}^1_R(\mathrm{Hom}_R(C, J), \ker \partial_0) = 0$. Inductively, we form short exact sequences of the form $0 \to \ker \partial^j \to \mathrm{Hom}_R(C, I^j) \to \ker \partial^{j-1} \to 0$ and apply the same reasoning to conclude that each $\ker \partial^j$ is G_C-injective.

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