Simple Cell Response Properties Imply Receptive Field Structure: Balanced Gabor and/or Bandlimited Field Functions. SUPPLEMENT. Appendices A, B, C and Figures 13-16.

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Abstract

Complement. The classical receptive fields of simple cells in mammalian primary visual cortex demonstrate three cardinal response properties: 1) they do not respond to stimuli which are spatially homogeneous; 2) they respond best to stimuli in a preferred orientation (direction); and 3) they do not respond to stimuli in other, non-preferred orientations (directions). We refer to these as the Balanced Field Property, the Maximum Response Direction Property, and the Zero Response Direction Property, respectively. These empirically-determined response properties are used to derive a complete characterization of elementary receptive field functions defined as products of a circularly symmetric weight function and a simple periodic carrier. Two disjoint classes of elementary receptive field functions result: the balanced Gabor class, a generalization of the traditional Gabor filter, and a bandlimited class whose Fourier transforms have compact support (i.e., are zero-valued outside of a bounded range). The detailed specification of these two classes of receptive field functions from empirically-based postulates may prove useful to neurophysiologists seeking to test alternative theories of simple cell receptive field structure, and to computational neuroscientists seeking basis functions with which to model human vision.

Supplement. This Supplement provides detailed proofs of the main results of the complementary paper. Lemmas 1, 2, 3 and Theorems A.1,

A.2, and B.1 are stated in the paper. Appendix A proves Theorems A.1 and A.2. Appendix B proves Theorem B.1. Appendix C proves several miscellaneous results mentioned in the paper. Equation numbering continues the numbering of the paper.

Keywords. Bandlimited, Bessel function, Fourier transform, Gabor filter, Hankel transform, receptive fields, simple cells, visual cortex

We denote the following frequently occurring expressions by:

$$\begin{aligned} s_{\pm}(\omega, x) &:= \sqrt{(\sin(\omega))^2 + (x \pm \cos(\omega))^2} \\ &= \sqrt{1 \pm 2\cos(\omega)x + x^2} \end{aligned} \tag{73}$$

Appendix A. Proofs for Lemma A and Theorems A.1 and A.2.

This Appendix states and proves a key result, Lemma A, and applies it to prove Theorems A.1 and A.2 (Section 4).

It will be useful to extend the domain of $H_q(\rho)$ to the whole real line and rephrase Lemma 2. Since $q(r)$ satisfies (12), the Hankel transform $H_q(\rho)$, defined by (14), exists for all real ρ and is even because $J_0(x)$ is even. Properties (15) become

$$H_q(\rho) \text{ is } C(-\infty, +\infty), H_q(0) = 1, \text{ and } H_q(\pm\infty) = 0 \tag{74}$$

With respect to Lemma 2, we can assume $\alpha_{\text{ZR}} - \alpha_R$ is restricted to $[0, +\pi/2]$ because conditions (25, 26) are unchanged when $\alpha_{\text{ZR}} - \alpha_R$ is translated by $\pm\pi$ and the cosine is even. Conditions (25, 26) are also unchanged when $x = \lambda_R/\lambda_P$ is replaced by $-x$ (and hold at $x = 0$ by continuity). Lemma 2 can therefore be restated as:

Lemma 2′. Let the receptive field function $R\left(\vec{x}\right)$ be given by (11). Let $q(r)$ satisfy (12) (in which case the Hankel transform $H_q(\rho)$ exists and satisfies (74)). Then $R\left(\vec{x}\right)$ has the ZRD Property if and only if there exists a value β_R, $0 \le \beta_R \le \pi/2$, such that the following relations hold for $-\infty < x < +\infty$:

$$\cos(\phi_R)\left[H_q\left(s_-(\beta_R, x)\right) + H_q\left(s_+(\beta_R, x)\right)\right] = 2b_R H_q(x) \tag{75}$$

$$\sin(\phi_R)\left[H_q\left(s_-(\beta_R, x)\right) - H_q\left(s_+(\beta_R, x)\right)\right] = 0 \tag{76}$$

$$b_R = \cos(\phi_R)\, H_q(1) \tag{77}$$

In such a case, $\alpha_{\mathrm{ZR}} = \alpha_R + \beta_R$ is a zero response direction.

We now state the main lemma and apply it to prove Theorems A.1 and A.2. The remainder of the Appendix is the proof of the lemma.

Lemma A. Let ϕ_0, β_0 be given with $0 \le \beta_0 \le \frac{\pi}{2}$. Consider the following three conditions on a function $h(x)$:

(A1) $h(x)$ is even and continuous on the real line with $h(0) = 1$ and $h(\pm\infty) = 0$.

(A2) $\cos(\phi_0)\left[h\left(s_-(\beta_0, x)\right) + h\left(s_+(\beta_0, x)\right) - 2h(1)h(x)\right] = 0$ for all x.

(A3) $\sin(\phi_0)\left[h\left(s_-(\beta_0, x)\right) - h\left(s_+(\beta_0, x)\right)\right] = 0$ for all x.

Then:

(1a) Assume $\beta_0 = \frac{\pi}{2}$ and $\cos(\phi_0) = 0$. Then $h(x)$ satisfies all three conditions iff it satisfies the first, that is, (A1).

(1b) Assume $\beta_0 = \frac{\pi}{2}$ and $\cos(\phi_0) \ne 0$. Then:

1. $h(x)$ satisfies the three conditions with $h(1) > 0$ iff $h(x) = h(1)^{x^2} F\left(x^2\right)$ where $F(y)$ is $C[0, +\infty)$, $F(0) = 1$, $F(y+1) = F(y)$ for $y \ge 0$, and $0 < h(1) < 1$.
2. $h(x)$ satisfies the three conditions with $h(1) = 0$ iff $h(x) = 0$ for $|x| \ge 1$ with $h(x)$ even and continuous on the real line and $h(0) = 1$.
3. $h(x)$ satisfies the three conditions with $h(1) < 0$ iff $h(x) = (-h(1))^{x^2} F\left(x^2\right)$ where $F(y)$ is $C[0, +\infty)$, $F(0) = 1$, $F(y+1) = -F(y)$ for $y \ge 0$, and $-1 < h(1) < 0$.

(2a) Assume $0 < \beta_0 < \frac{\pi}{2}$ and $\cos(\phi_0) \ne \pm 1$. Then $h(x)$ satisfies the three conditions iff $h(x) = 0$ for $|x| \ge \sin(\beta_0)$ with $h(x)$ even and continuous on the real line and $h(0) = 1$.

(2b) Assume $0 < \beta_0 < \frac{\pi}{2}$ and $\cos(\phi_0) = \pm 1$. Then:

1. If $h(1) = 0$, then $h(x)$ satisfies the three conditions iff $h(x) = 0$ for $|x| \ge \sin(\beta_0)$ with $h(x)$ even and continuous on the real line and $h(0) = 1$.
2. If $0 < |h(1)| < 1$, then $h(x)$ satisfies the three conditions iff $h(x)$ satisfies (A1) and (A2).
3. If $|h(1)| \ge 1$, then no $h(x)$ satisfies the three conditions.

(3) Assume $\beta_0 = 0$. Then no $h(x)$ satisfies the three conditions.

Proof of Theorem A.1. Assume $R\left(\vec{x}\right)$ has the ZRD Property with a ZRD α_{ZR}. Apply Lemma 2′ with $\beta_R = \alpha_{\mathrm{ZR}} - \alpha_R$. Then the Hankel transform, extended to the real line as an even function $H_q(x)$, satisfies hypothesis (A1) of Lemma A and satisfies the following relations for all x, corresponding to hypotheses (A2) and (A3) of Lemma A:

$$\cos\left(\phi_R\right)\left[H_q\left(s_-(\beta_R,x)\right) + H_q\left(s_+(\beta_R,x)\right) - 2H_q(1)H_q(x)\right] = 0 \tag{78}$$

$$\sin\left(\phi_R\right)\left[H_q\left(s_-(\beta_R,x)\right) - H_q\left(s_+(\beta_R,x)\right)\right] = 0 \tag{79}$$

Apply Lemma A. Since $\cos\left(\phi_R\right) \neq 0$, conclusion (1a) of Lemma A does not occur. By conclusion (3) of Lemma A, $\beta_R = 0$ does not occur, so $0 < \beta_R \leq \pi/2$.

Assume $0 < \beta_R < \pi/2$ occurs. Then either conclusion (2a) or (2b) of Lemma A occurs. Assume conclusion (2a) occurs. Then $H_q(x)$ satisfies $H_q(x) = 0$ for $|x| \geq \sin\left(\beta_R\right)$, where we already know that $\alpha_{\mathrm{ZR}} = \alpha_R + \beta_R$ is a ZRD. Notice that, for each β_R' with $\beta_R \leq \beta_R' \leq \pi/2$, (78) and (79) are still satisfied for $H_q(x)$, hence $\alpha_{\mathrm{ZR}}' = \alpha_R + \beta_R'$ is also a ZRD by Lemma 2′, giving a sector of ZRDs. Result (3) of Theorem A.1 follows by extending the interval where $H_q(x)$ vanishes to its maximum extent. Now assume conclusion (2b) of Lemma A occurs. By (2b.3), $|H_q(1)| \geq 1$ does not occur. Then either $H_q(1) \neq 0$, in which case (2b.2) occurs and gives result (4) of Theorem A.1, or $H_q(1) = 0$ occurs, in which case (2b.1) occurs and gives result (3) of Theorem A.1 (after repeating the analysis for conclusion (2a)).

We can now assume (78) and (79) hold for $\beta_R = \pi/2$ and for no other value of β_R (otherwise, we return to the previous case). Since (78) and (79) hold only for this value, Lemma 2′ implies the ZRD is the unique value $\alpha_{\mathrm{ZR}} = \alpha_R + \pi/2$. Only conclusion (1b) of Lemma A applies, and its subcases (1b.1,2,3) give results (1), (2), (3) of Theorem A.1.

Conversely, assume a receptive field function $R\left(\vec{x}\right)$ is given such that (11) and (12) hold and let the resulting Hankel transform $H_q(\rho)$ be extended to the real line as an even function. We wish to show that the field function has the ZRD Property.

Assume result (1) of Theorem A.1 holds, so that $\alpha_{\mathrm{ZR}} = \alpha_R + \pi/2$ is a ZRD. Apply Lemma 2′ with $\beta_R = \alpha_{\mathrm{ZR}} - \alpha_R$ and b_R defined by (77). Then direct substitution shows that (75) and (76) are satisfied, so the ZRD Property holds. Similarly, by direct substitution into the conditions of Lemma 2′, results (2), (3), (4) of Theorem A.1 also imply the ZRD Property, noting that results (3) and (4), although stated for $\rho \geq 0$, are preserved for even extensions of $H_q(\rho)$ to the real line. Theorem A.1 is proved.

Proof of Theorem A.2. A receptive field function $R\left(\vec{x}\right)$ is given such that (11) and (12) hold. Let the Hankel transform be extended to the real line as an even function $H_q(x)$. We wish to determine the ZRDs (if any). By Lemma 2′, each ZRD α_{ZR} determines a corresponding $\beta_R = \alpha_{\mathrm{ZR}} - \alpha_R$ for which equations (75,76,77) are satisfied. That is, $H_q(x)$ necessarily satisfies hypothesis (A1) of Lemma A and equations (75,76) correspond to hypotheses (A2) and (A3) of

Lemma A. We can now apply Lemma A to determine the values β_R. Since $\cos(\phi_R) = 0$, only conclusions (1a), (2a), and (3) of Lemma A can occur. By conclusion (3), $\beta_R = 0$ does not occur, so $0 < \beta_R \leq \pi/2$.

Assume $\beta_R = \pi/2$. Then conclusion (1a) of Lemma A occurs, and equations (75,76) are automatically satisfied because they are vacuous. That is, there is always a ZRD, namely, $\alpha_{\mathrm{ZR}} = \alpha_R + \pi/2$, proving result (1) of Theorem A.2.

Assume $0 < \beta_R < \pi/2$. Then conclusion (2a) of Lemma A occurs, and $H_q(x)$ satisfies $H_q(x) = 0$ for $|x| \geq \sin(\beta_R)$. We can therefore form equations (75,76) not only for β_R but for each β'_R with $\beta_R \leq \beta'_R \leq \pi/2$ as well. By Lemma $2'$, there is a sector of ZRDs, namely, $\alpha'_{\mathrm{ZR}} = \alpha_R + \beta'_R$.

We have now shown that there is always at least one ZRD, corresponding to $\beta_R = \pi/2$, and that if there is more than one ZRD, the ZRDs must occur as a sector, corresponding to $\beta_R \leq \beta'_R \leq \pi/2$ (and thus including $\pi/2$). Furthermore, we have shown that if more than one ZRD occurs, then $H_q(x)$ must satisfy $H_q(x) = 0$ for $|x| \geq s_0$ for some $0 < s_0 < 1$ and that, when $H_q(x)$ satisfies such a condition, a sector of ZRDs is implied. Altogether, this proves result (2) of Theorem A.2 and completes the proof.

Proof of Lemma A.

Case (1a). Assume $\beta_0 = \frac{\pi}{2}$ and $\cos(\phi_0) = 0$. Then (A2) and (A3) are vacuous. Hence $h(x)$ satisfies the three conditions iff it satisfies (A1).

Case (1b). Assume $\beta_0 = \frac{\pi}{2}$ and $\cos(\phi_0) \neq 0$. Notice (A3) is vacuous, and (A2) reduces to $h\left(\sqrt{1+x^2}\right) = h(1)h(x)$ for all x.

If $h(1) = 0$, it follows that $h(x) = 0$ for $|x| \geq 1$. That is, (A1,2,3) and $h(1) = 0$ give the conclusion of (1b.2), and the converse follows directly.

If $h(1) > 0$, define $F(y) := h(1)^{-y} h\left(\sqrt{y}\right)$ for $y \geq 0$. Then $F(y)$ is $C[0, +\infty)$, $F(0) = 1$, and the reduced form of (A2) gives $F(y+1) = F(y)$. Then $h(x) = h(1)^{x^2} F\left(x^2\right)$. Using $h(\pm\infty) = 0$ from (A1) gives $0 < h(1) < 1$. This proves the representation (1b.1) for $h(x)$, and the converse follows directly.

If $h(1) < 0$, define $F(y) := |h(1)|^{-y} h\left(\sqrt{y}\right)$ for $y \geq 0$. Proceeding as for $h(1) > 0$ proves the representation (1b.3), and the converse follows directly.

Case (2a). Assume $0 < \beta_0 < \frac{\pi}{2}$ and $\cos(\phi_0) \neq \pm 1$. Then, since $\sin(\phi_0) \neq 0$, (A3) implies

$$h\left(s_+(\beta_0, x)\right) = h\left(s_-(\beta_0, x)\right) \text{ for all } x$$

The function $h\left(\sqrt{\sin(\beta_0)^2 + x^2}\right)$ is then periodic with period $2\cos(\beta_0)$ and, by $h(\pm\infty) = 0$ from (A1), must be identically zero. Thus, $h(x) = 0$ for $|x| \geq \sin(\beta_0)$. In particular, $h(1) = 0$, so (A2) is necessarily satisfied whatever the value of $\cos(\phi_0)$. Representation (2a) follows. The converse is direct.

Case (2b). Assume $0 < \beta_0 < \frac{\pi}{2}$ and $\cos(\phi_0) = \pm 1$. Then $\sin(\phi_0) = 0$ and (A3) is vacuous.

If $h(1) = 0$, then (A2) reduces to

$$h\left(s_+(\beta_0, x)\right) = -h\left(s_-(\beta_0, x)\right) \text{ for all } x$$

The function $h\left(\sqrt{\sin(\beta_0)^2 + x^2}\right)$ is therefore periodic with period $4\cos(\beta_0)$

and, by $h(\pm\infty) = 0$ from (A1), must be identically zero. Thus, $h(x) = 0$ for $|x| \geq \sin(\beta_0)$. Representation (2b.1) follows, and the converse is direct.

If $0 < |h(1)| < 1$, then the stated result (2b.2) is immediate.

For $|h(1)| \geq 1$, first consider $h(1) \geq 1$. Notice (A2) becomes

$$\frac{1}{2}\left[h\left(s_-(\beta_0, x)\right) + h\left(s_+(\beta_0, x)\right)\right] = h(1)h(x) \text{ for all } x$$

By (A1), $h(x)$ must attain an absolute maximum $M = h(x_0) \geq h(1) \geq 1$. Applying (A2) with $x = x_0$ forces

$$h\left(s_-(\beta_0, x_0)\right) = h\left(s_+(\beta_0, x_0)\right) = M \text{ and } h(1) = 1$$

Thus, $h(x)$ also attains the absolute maximum M at $x_1 = s_+(\beta_0, x_0) > x_0 + \cos(\beta_0)$. Repeating the argument at x_1 gives an absolute maximum at $x_2 = s_+(\beta_0, x_1) > x_0 + 2\cos(\beta_0)$, and so on, giving a sequence of points $x_n \to +\infty$ with $h(x_n) = M \geq 1$, contradicting $h(\pm\infty) = 0$ in (A1).

Now consider $h(1) \leq -1$. By (A1), $h(x)$ must attain an absolute maximum $M \geq h(0) = 1$, and $h(x)$ must have an absolute minimum $m \leq h(1) \leq -1$. Assume $M \geq -m$. Let $h(x_0) = M$. Applying (A2) with $x = x_0$ gives

$$\frac{1}{2}\left[h\left(s_-(\beta_0, x_0)\right) + h\left(s_+(\beta_0, x_0)\right)\right] = h(1)h(x_0) \leq -M \leq m$$

This forces $m = -M$ and $h(1) = -1$ and

$$h\left(s_-(\beta_0, x_0)\right) = h\left(s_+(\beta_0, x_0)\right) = m$$

Thus, $h(x)$ attains the absolute minimum m at $x_{1/2} = s_+(\beta_0, x_0) > x_0 + \cos(\beta_0)$. Repeating the argument at $x = x_{1/2}$ gives

$$\frac{1}{2}\left[h\left(s_-(\beta_0, x_{1/2})\right) + h\left(s_+(\beta_0, x_{1/2})\right)\right] = h(1)h(x_{1/2}) = -m = M$$

forcing $h(x_1) = M$ at $x_1 = s_+(\beta_0, x_{1/2}) > x_0 + 2\cos(\beta_0)$. Repeat to construct a sequence $x_n \to +\infty$ such that $h(x_n) = M \geq 1$, contradicting $h(\pm\infty) = 0$ in (A1). Now assume $M < -m$. Let $h(x_0) = m$. Applying (A2) with $x = x_0$ gives

$$\frac{1}{2}\left[h\left(s_-(\beta_0, x_0)\right) + h\left(s_+(\beta_0, x_0)\right)\right] = h(1)h(x_0) \geq -m > M$$

But $h\left(s_\pm(\beta_0, x_0)\right) \leq M$, contradiction. This proves (2b.3).

Case (3). Assume $\beta_0 = 0$. Then $s_\pm(\beta_0, x) = |1 \pm x|$. If $\sin(\phi_0) \neq 0$, then (A3) implies $h(x)$ is periodic, forcing a contradiction by the argument of case (2a). If $\sin(\phi_0) = 0$, then, since $h(x)$ is even, (A2) becomes an ordinary difference equation, namely, $h(x-1) + h(x+1) = 2h(1)h(x)$. If $|h(1)| > 1$, there are two real characteristic roots with one root greater than one. If $|h(1)| = 1$, there is a double root, either $+1$ or -1. If $|h(1)| < 1$, there are two complex conjugate roots with modulus one. Picking arbitrary starting points $x_0, x_0 + 1$, it can be shown by explicit solution that, for all these cases, $h(\pm\infty) = 0$ holds iff $h(x_0) = h(x_0 + 1) = 0$, forcing $h(x) = 0$ for all x and thus contradicting (A1). This completes the proof of Lemma A.

Appendix B. Proof for Theorem B.1.

This appendix proves Theorems B.1, the characterization theorem for elementary receptive field functions with the MRD Property. The following lemma is used in the proof of Theorem B.1.

Lemma B. The following are equivalent:

(1) $h(x)$ is $C[0, +\infty)$ and satisfies $h\left(\sqrt{1 + x^2}\right) = h(1)h(x)$ for $x \geq 0$ with $h(1) \neq 0$.

(2) $h(x) = e^{-cx^2} f\left(x^2\right)$ where $f(y)$ is $C[0, +\infty)$, $f(1) = \pm 1$, and $f(y + 1) = f(1)f(y)$ for $y \geq 0$.

In case (1), $h(0) = 1$ necessarily holds. In case (2), $f(0) = 1$ necessarily holds.

Proof of Lemma B. Assume (1) holds. Define $f(y){:=}e^{cy}h\left(\sqrt{y}\right)$ for $y \geq 0$, where $e^{-c} = |h(1)| > 0$. Then $f(y)$ is $C[0, +\infty)$, $f(1) = h(1)/|h(1)| = \pm 1$, and $f(y+1) = f(1)f(y)$ follows using the recursion relation in (1). Assume (2) holds. Then $h(x)$ is $C[0, +\infty)$, $h(1) = e^{-c}f(1) \neq 0$, and $h\left(\sqrt{1 + x^2}\right) = h(1)h(x)$ follows by substitution. The lemma is proved.

Proof of Theorem B.1. It may be helpful to outline the structure of the proof. We first assume $R\left(\vec{x}\right)$ has the MRD Property. Then, as noted in Section 2, it has the ZRD Property and, since $\cos\left(\phi_R\right) \neq 0$, Theorem A.1 applies. We consider the four cases of Theorem A.1 in reverse order:

- case (4), which will be shown inconsistent with the MRD Property;
- case (3), which will imply result (2) of Theorem B.1;
- cases (1) and (2) together, where case (2) will be shown inconsistent with the MRD Property and case (1) will imply result (1) of Theorem B.1.

Having derived cases (1) and (2) of Theorem B.1, we will then prove the converse, that each of these cases implies the MRD Property.

Assume $R\left(\vec{x}\right)$ has the MRD Property. Then, as noted at the end of Section 3, $R\left(\vec{x}\right)$ has the ZRD Property, Theorem A.1 applies, and we will work through its four cases.

Assume case (4) of Theorem A.1 holds. Then $\cos\left(\phi_R\right) = \pm 1$ and the Hankel transform $H_q(\rho)$ satisfies

$$H_q\left(s_-(\zeta_0, \rho)\right) + H_q\left(s_+(\zeta_0, \rho)\right) = 2H_q(1)H_q(\rho) \text{ for } \rho \geq 0$$

with $0 < |H_q(1)| < 1$, where ζ_0 is some fixed value with $0 < \zeta_0 < \frac{\pi}{2}$, and where ZRDs are given by $\alpha_{\mathrm{ZR}} - \alpha_R = \pm\zeta_0$. The MRD Property then implies that there must be a sector of ZRDs, $|\alpha_{\mathrm{ZR}} - \alpha_R| \geq \zeta_0$. Consequently, applying Lemma 2 with $\cos(\phi_R) = \pm 1$ yields, for $\rho > 0$,

$$H_q(s_-(\alpha,\rho)) + H_q(s_+(\alpha,\rho)) = 2H_q(1)H_q(\rho) \text{ for } \zeta_0 \leq \alpha \leq \frac{\pi}{2} \tag{80}$$

Taking $\alpha = \frac{\pi}{2}$ and using the fact that $H_q(\rho)$ is $C[0,\infty)$ gives

$$H_q\left(\sqrt{1+\rho^2}\right) = H_q(1)H_q(\rho) \text{ for } \rho \geq 0 \tag{81}$$

Since $H_q(1) \neq 0$, Lemma B applies to give $H_q(\rho) = e^{-c\rho^2} f\left(\rho^2\right)$ for $\rho \geq 0$ with corresponding conditions on $f(y)$. Since $f(y)$ is already periodic (with period 2) on the half-line, it can be extended to the whole line and the extended function will be useful below. So the representation can be summarized as

$$\begin{aligned} H_q(\rho) = e^{-c\rho^2} f\left(\rho^2\right) &\text{ for } \rho \geq 0 \\ &\text{where } f(y) \text{ is } C(-\infty,+\infty), f(1) = \pm 1, \\ &\text{and } f(y+1) = f(1)f(y) \text{ for all } y. \end{aligned} \tag{82}$$

Notice $c > 0$ because $H_q(+\infty) = 0$. Combining this representation with (B.1) gives, for each $\rho > 0$,

$$\begin{aligned} e^{+2c\cos(\alpha)\rho} f\left(1 - 2\rho\cos(\alpha) + \rho^2\right) + e^{-2c\cos(\alpha)\rho} f\left(1 + 2\rho\cos(\alpha) + \rho^2\right) \\ = 2f(1)f\left(\rho^2\right) \text{ for } \zeta_0 \leq \alpha \leq \frac{\pi}{2} \end{aligned} \tag{83}$$

Setting $x = 2\cos(\alpha)\rho$ and using the symmetry of the expression gives, for each $\rho > 0$,

$$e^{+cx} f\left(1 - x + \rho^2\right) + e^{-cx} f\left(1 + x + \rho^2\right) = 2f(1)f\left(\rho^2\right) \text{ for } |x| \leq 2\cos(\zeta_0)\rho \tag{84}$$

Setting $\rho = 2N$, where N is a positive integer, and using the properties of f gives

$$e^{+cx} f(-x) + e^{-cx} f(x) = 2 \text{ for } |x| \leq 4\cos(\zeta_0) N \tag{85}$$

Letting $N \to \infty$ gives

$$e^{+cx} f(-x) + e^{-cx} f(x) = 2 \text{ for all } x \tag{86}$$

However, dividing by e^{+cx} and taking a limit, implies $f(-\infty) = 0$, a contradiction since f is periodic and $f(0) = 1$. Consequently, case (4) of Theorem A.1 cannot occur.

Assume case (3) of Theorem A.1 holds. The Hankel transform $H_q(\rho)$ is then zero on an interval of the form $0 < \sin(\zeta_R) \leq \rho < +\infty$, the largest such interval on which $H_q(\rho)$ vanishes. In particular, $H_q(\rho) = 0$ for $\rho \geq 1$, which implies, writing $\alpha = \alpha_P - \alpha_R$,

$$\begin{aligned} b_R = \cos(\phi_R) H_q(1) &= 0 \\ H_q(s_+(\alpha, \lambda_R/\lambda_P)) &= 0 \text{ for } 0 \leq \alpha \leq \frac{\pi}{2} \end{aligned} \tag{87}$$

Consequently, (3.9a) simplifies to

$$N(\alpha_P, \lambda_P) = |H_q(s_-(\alpha, \lambda_R/\lambda_P))| \text{ for } 0 \le \alpha \le \frac{\pi}{2} \tag{88}$$

which becomes, for $\lambda_P = \lambda_R$,

$$N(\alpha_P, \lambda_R) = \left| H_q\left(\sqrt{2 - 2\cos(\alpha)}\right)\right| \text{ for } 0 \le \alpha \le \frac{\pi}{2} \tag{89}$$

Then the MRD Property says that N must strictly decrease from its maximum $|H_q(0)|$ (at $\alpha = 0$) until it reaches zero (at $2 - 2\cos(\alpha) = \sin(\zeta_R)^2$) and then remains zero (as α increases to $\pi/2$). Thus, $H_q(\rho)$ must strictly decrease from its maximum of one at $\rho = 0$ to zero at $\rho = \sin(\zeta_R)$, in particular, $H_q(\rho) \ge 0$. This proves that the condition of result (2) of Theorem B.1 is necessary for the MRD Property. Observation (a), that the MRD is given by $\alpha_{\text{MR}} = \alpha_R$, is a general result noted in the discussion of Lemma 3, and observation (b) on the sector of ZRDs follows from Theorem A.1.

Assume case (1) or case (2) of Theorem A.1 holds. Then in both cases the Hankel transform can be written as $H_q(\rho) = e^{-c\rho^2} F(\rho^2)$, where $e^{-c} = |H_q(1)|$ with $c > 0$ and $F(y)$ is periodic with period 2 on the half-line. The periodic extension of $F(y)$ to the whole line will be useful, and the representation can be summarized as

$$\begin{aligned} &H_q(\rho) = e^{-c\rho^2} F(\rho^2) \text{ for } \rho \ge 0 \\ &\qquad \text{where } F(y) \text{ is } C(-\infty, +\infty), F(0) = 1, F(1) = \pm 1, \\ &\qquad \text{and } F(y+1) = F(1)F(y) \end{aligned} \tag{90}$$

where $F(1) = +1$ corresponds to case (1) and $F(1) = -1$ to case (2) of Theorem A.1. Combining this representation with (23) and writing $\alpha = \alpha_P - \alpha_R$ and $r = \lambda_R/\lambda_P$ gives

$$N(\alpha_P, \lambda_P)^2 = e^{-c\left(1+r^2\right)} \cdot N_1(\alpha, r)^2 \tag{91}$$

where $N_1(\alpha, r) > 0$ is given by

$$\begin{aligned} &N_1(\alpha, r)^2 := \\ &\cos(\phi_R)^2 \ \big[\, e^{-2c\cos(\alpha)r} F\left(1 + 2\cos(\alpha)r + r^2\right) + \\ &\qquad e^{+2c\cos(\alpha)r} F\left(1 - 2\cos(\alpha)r + r^2\right) - 2F(1)F\left(r^2\right)\big]^2 + \\ &\sin(\phi_R)^2 \ \big[\, e^{-2c\cos(\alpha)r} F\left(1 + 2\cos(\alpha)r + r^2\right) - \\ &\qquad e^{+2c\cos(\alpha)r} F\left(1 - 2\cos(\alpha)r + r^2\right)\,\big]^2 \end{aligned} \tag{92}$$

The MRD Property implies that, for each $r > 0$, $N_1(\alpha, r)$ is either identically zero or is initially positive and then strictly decreasing to zero (and remains zero) as α increases from 0 to $\pi/2$. Set $y = 2\cos(\alpha)r$ and notice the resulting expression for N_1 is even in y. Then, for each $r > 0$ and for $|y| \le 2r$,

$$\begin{aligned} N_1^2 \ = \ &\cos(\phi_R)^2 \left(e^{-cy} F\left(1 + y + r^2\right) + e^{+cy} F\left(1 - y + r^2\right) - 2F(1)F\left(r^2\right)\right)^2 + \\ &\sin(\phi_R)^2 \left(e^{-cy} F\left(1 + y + r^2\right) - e^{+cy} F\left(1 - y + r^2\right)\right)^2 \end{aligned} \tag{93}$$

where N_1 is either zero on the whole interval $|y| \le 2r$ or is initially zero and then strictly increasing as $|y|$ increases to $2r$. Setting $r^2 = a + 2N$, where a is an arbitrary real value and N is a sufficiently large positive integer, and using the properties of F from (90) gives, for $|y| \le 2\sqrt{a+2N}$,

$$\begin{aligned} N_1^2 &= \cos(\phi_R)^2 \left(e^{-cy}F(a+y) + e^{+cy}F(a-y) - 2F(a)\right)^2 + \\ &\quad \sin(\phi_R)^2 \left(e^{-cy}F(a+y) - e^{+cy}F(a-y)\right)^2 \end{aligned} \tag{94}$$

Let $N \to +\infty$. Then, for each real a, the expression T_F, defined by

$$\begin{aligned} &T_F(\phi_R, c, a; y) := \\ &\quad \cos(\phi_R)^2 \left(e^{cy}F(a-y) + e^{-cy}F(a+y) - 2F(a)\right)^2 + \\ &\quad \sin(\phi_R)^2 \left(e^{cy}F(a-y) - e^{-cy}F(a+y)\right)^2 \end{aligned} \tag{95}$$

is either zero for all $|y|$ or is initially zero and then strictly increasing as $|y|$ increases indefinitely. In particular, $T_F(y)$ is nondecreasing on $y \ge 0$.
CLAIM 1. $e^{2cy}F(-y)^2$ is nondecreasing on $-\infty < y < +\infty$.
PROOF. Let $a = 0$ and $y = x+2N$ for positive integer N in (95) with $0 \le x \le 2$. Then

$$\begin{aligned} &e^{-4Nc}T_F(x+2N) = \\ &\quad (\cos(\phi_R))^2 \left(e^{cx}F(-x) + e^{-cx-4Nc}F(x) - 2e^{-2Nc}F(a)\right)^2 + \\ &\quad (\sin(\phi_R))^2 \left(e^{cx}F(-x) - e^{-cx-4Nc}F(x)\right)^2 \end{aligned} \tag{96}$$

is a sequence of nondecreasing functions converging uniformly on $0 \le x \le 2$ to $(e^{cx}F(-x))^2$. The limit function is then nondecreasing on $0 \le x \le 2$ and, since shifting by 2 simply multiplies the function by a constant, that is, $\left(e^{c(x+2)}F(-(x+2))\right)^2 = e^{4c}\left(e^{cx}F(-x)\right)^2$, the function must be nondecreasing on the entire line.
CLAIM 2. $e^{cy}F(-y)$ is positive and nondecreasing on $-\infty < y < +\infty$.
PROOF. If this function was zero at some value $y = y_0$, then $e^{2cy}F(-y)^2$ would, by Claim 1, be zero on $y \le y_0$ and, by periodicity of F, be identically zero. But it has the value 1 at $y = 0$. Consequently, $e^{cy}F(-y)$ is nonzero and thus does not change sign and is necessarily positive. It is then nondecreasing since its square is nondecreasing by Claim 1.
CLAIM 3. Case (2) of Theorem A.1 does not hold.
PROOF. In case (2), $F(y)$ satisfies $F(y+1) = -F(y)$. Thus, $F(0) = 1$ and $F(1) = -1$, contradicting Claim 2.

As a result of Claim 3, only case (1) of Theorem A.1 holds, which implies that $F(y)$ is periodic with period 1. This proves the initial part of result (1) of Theorem B.1.
CLAIM 4. For each real a, $T_F(\phi_R, c, a; y)$ is nondecreasing on $0 \le y \le \delta_a$ and is strictly increasing on $\delta_a \le y < +\infty$, where $0 \le \delta_a < 1$.

PROOF. As noted in connection with (95), $T_F(y)$ is either (a) zero for all $y \geq 0$ or (b) initially zero and then strictly increasing. Since $F(y)$ has period 1 and is nonzero, we have

$$T_F(1) = \left(\cos(\phi_R)^2 \left(e^c + e^{-c} - 2\right)^2 + \sin(\phi_R)^2 \left(e^c - e^{-c}\right)^2\right) F(a)^2 > 0 \quad (97)$$

Alternative (b) must hold, and $T_F(y)$ is in the strictly increasing regime at $y = 1$, so $\delta_a < 1$.

CLAIM 5. $e^{cy}F(-y)$ is positive and strictly increasing on $-\infty < y < +\infty$.

PROOF. By Claim 2, the function is positive and nondecreasing. Assume it is not strictly increasing. Then it must be constant on some interval:

$$e^{+cx}F(-x) = e^{+cb}F(-b) \text{ on } |x - b| < \delta \quad (98)$$

which implies

$$e^{-cx}F(+x) = e^{+cb}F(-b) \text{ on } |x + b| < \delta \quad (99)$$

Consider Claim 4 with $a = -b$ and $y = z + N$ for integers $N >> 1$ and $|z| < \delta$:

$$\begin{aligned} &T_F(\phi_R, c, -b; y) \\ &\quad = \cos(\phi_R)^2 \left(e^{c(z+N)}F(-b-z) + e^{-c(z+N)}F(-b+z) - 2F(-b)\right)^2 + \\ &\qquad \sin(\phi_R)^2 \left(e^{c(z+N)}F(-b-z) - e^{-c(z+N)}F(-b+z)\right)^2 \\ &\quad = \left(\cos(\phi_R)^2 \left(e^{cN} + e^{-cN} - 2\right)^2 + \sin(\phi_R)^2 \left(e^{cN} - e^{-cN}\right)^2\right) F(-b)^2 \end{aligned}$$

That is, $T_F(y)$ is constant on subintervals $|y - N| < \delta$ for sufficiently large N, contradicting Claim 4.

Note that Claim 5 implies the functions $e^{cy}F(a-y) = e^{ca}e^{c(y-a)}F(a-y)$ are strictly increasing and the functions $e^{-cy}F(a+y)$ are strictly decreasing. In particular, Claim 5 gives (c) under result (1) of Theorem B.1.

CLAIM 6. Assume $\cos(\phi_R) \neq \pm 1$. Then, for each real a, $T_F(\phi_R, c, a; y)$ is strictly increasing on $0 \leq y < +\infty$.

PROOF. By Claim 4, for each a, $T_F(\phi_R, c, a; y)$ must be initially zero, then strictly increasing. We show that the initially zero interval must always reduce to a single point. Assume it does not. Then, for some a_0, $T_F(a_0; y) = 0$ on some interval $|y| \leq \delta_0$ with $\delta_0 > 0$. Since $\sin(\phi_R) \neq 0$, $T_F(a_0; y) = 0$ gives two equations

$$\begin{aligned} e^{-cy}F(a_0 + y) + e^{+cy}F(a_0 - y) &= 2F(a_0) \text{ and} \\ e^{-cy}F(a_0 + y) - e^{+cy}F(a_0 - y) &= 0 \text{ on } |y| \leq \delta_0 \end{aligned}$$

which can be solved to give $e^{-cy}F(a_0 + y) = F(a_0)$ and $e^{+cy}F(a_0 - y) = F(a_0)$ on $|y| \leq \delta_0$. Set $x = y + N$ for positive integers N with $|y| \leq \delta_0$ and combine these expressions with the periodicity of F to obtain

$$\begin{aligned} &T_F(a_0; x) = \\ &F(a_0)^2 \left(\cos(\phi_R)^2 \left(e^{+cN} + e^{-cN} - 2\right)^2 + \sin(\phi_R)^2 \left(e^{+cN} - e^{-cN}\right)^2\right) \end{aligned} \quad (100)$$

This is a contradiction because it shows $T_F(a_0; x)$ to be constant on the intervals $|x - N| = |y| \le \delta_0$, but $T_F(a_0; x)$ must be strictly increasing on such intervals for sufficiently large N by Claim 4.

Claim 6 proves part (**) of result (1) for Theorem B.1.

CLAIM 7. Assume $\cos(\phi_R) = \pm 1$. Then:

(7a) For each a, the function $f(a; y) = e^{cy} F(a - y) + e^{-cy} F(a + y) - 2F(a)$ is initially zero on $0 \le y \le \delta_a$ and strictly increasing on $\delta_a \le y < +\infty$. (The value δ_a satisfies the bound $0 \le \delta_a < 1$.)

(7b) In (7a), $\delta_a > 0$ if and only if the value a satisfies $e^{-cy} F(y) = e^{-ca} F(a) - K(a)(y - a)$ on $|y - a| \le \delta_a$ for some constant $K(a) > 0$.

PROOF. Since $\sin(\phi_R) = 0$, the even function $T_F(a; y)$ in (95) reduces to $T_F(a; y) = f(a; y)^2$, which is nondecreasing for $y \ge 0$ by Claim 4. Since $f(a; 0) = 0$ and $f(a; y)$ is eventually positive due to the growth of the term $e^{+cy} F(a - y)$, we have $f(a; y) \ge 0$ and nondecreasing. Combining this behavior with the general behavior of $T_F(a; y)$ given by Claim 4, which also gives $\delta_a < 1$, proves Claim (7a).

For Claim (7b), it will be convenient to use the fact that $f(a; y)$ is an even function of y. Let a_0 be a value such that $f(a_0; y) = 0$ on $|y| \le \delta_0$ for some positive δ_0, that is,

$$e^{+cy} F(a_0 - y) + e^{-cy} F(a_0 + y) - 2F(a_0) = 0 \text{ on } |y| \le \delta_0 \tag{101}$$

This equation can be solved for e^{+cy}:

$$e^{+cy} F(a_0 - y) = F(a_0) \left(1 \pm \sqrt{1 - K_0(y)}\right) \text{ for } |y| \le \delta_0 \tag{102}$$

where $K_0(y) = F(a_0 + y) F(a_0 - y) / F(a_0)^2$ is even, $K_0(0) = 1$, and $K_0(y) \le 1$ for $|y| \le \delta_0$. By Claim 5, $e^{+cy} F(a_0 - y)$ is strictly increasing, which forces

$$\begin{aligned} & e^{+cy} F(a_0 - y) \\ & = F(a_0) \left(1 + \sqrt{1 - K_0(y)}\right) \text{ for } 0 \le y \le \delta_0 \\ & = F(a_0) \left(1 - \sqrt{1 - K_0(y)}\right) \text{ for } -\delta_0 \le y \le 0 \end{aligned} \tag{103}$$

with the even function $K_0(y)$ strictly decreasing as $|y|$ increases. Now consider $f(a; y)$ with $a = a_0 + \epsilon$, where $|\epsilon| \le \delta_0 / 2$, which is an even function of y. Thus, for $|\epsilon| \le \delta_0 / 2$ and $|y| \le \delta_0 / 2$,

$$\begin{aligned} & f(a_0 + \epsilon; y) = e^{+cy} F(a_0 + \epsilon - y) + e^{-cy} F(a_0 + \epsilon + y) - 2F(a_0 + \epsilon) \\ & = e^{+c\epsilon} F(a_0) \Big[\operatorname{sgn}(y - \epsilon) \sqrt{1 - K_0(y - \epsilon)} - \operatorname{sgn}(y + \epsilon) \sqrt{1 - K_0(y + \epsilon)} + \\ & \quad 2 \operatorname{sgn}(\epsilon) \sqrt{1 - K_0(\epsilon)}\Big] \end{aligned} \tag{104}$$

Consider the two cases $\epsilon = \pm\epsilon_0$ with $\epsilon_0 > 0$. In each case set $y = \epsilon_0$ and use $f(a;y) \geq 0$ to obtain

$$f\left(a_0+\epsilon_0;\epsilon_0\right) = e^{+c\epsilon_0}F\left(a_0\right)\left[0-\sqrt{1-K_0\left(2\epsilon_0\right)}+2\sqrt{1-K_0\left(\epsilon_0\right)}\right] \geq 0 \quad (105)$$

$$f\left(a_0-\epsilon_0;\epsilon_0\right) = e^{-c\epsilon_0}F\left(a_0\right)\left[+\sqrt{1-K_0\left(2\epsilon_0\right)}-0-2\sqrt{1-K_0\left(\epsilon_0\right)}\right] \geq 0 \quad (106)$$

Consequently,

$$\sqrt{1-K_0(2\epsilon)} = 2\sqrt{1-K_0(\epsilon)} \text{ for } |\epsilon| \leq \delta_0/2 \quad (107)$$

Setting $z(\epsilon) = \sqrt{1-K_0(\epsilon)}$, note that continuous solutions to the functional equation $z(2\epsilon) = 2z(\epsilon)$ on intervals $0 \leq \epsilon < \epsilon_0$ must be given by $z(\epsilon) = k_0\epsilon$ for constants k_0. (Use the functional equation to derive this formula for a dense set and note a continuous function is determined by its values on a dense set.) Consequently, since $\sqrt{1-K_0(\epsilon)}$ is a strictly increasing function of $|\epsilon|$,

$$\sqrt{1-K_0(x)} = k_0|x| \text{ for some } k_0 > 0 \text{ and for } |x| \leq \delta_0 \quad (108)$$

Equation (103) now becomes

$$e^{+cy}F\left(a_0-y\right) = F\left(a_0\right)\left(1+k_0 y\right) \text{ for some } k_0 > 0 \text{ and } |y| \leq \delta_0 \quad (109)$$

that is,

$$\begin{gathered} e^{-cy}F(y) = e^{-ca_0}F\left(a_0\right) - K_1\left(y-a_0\right) \\ \text{for some constant } K_1 > 0 \text{ and } \left|y-a_0\right| \leq \delta_0 \end{gathered} \quad (110)$$

which is precisely the "only if" part of Claim (7b).

To obtain the "if" part of Claim (7b), start with (B.18b), obtain expressions for $e^{-cy}F\left(a_0+y\right)$ and $e^{+cy}F\left(a_0-y\right)$, and observe that $f\left(a_0;y\right) = 0$ for $|y| \leq \delta_0$.

Claim 7 proves part (*) of result (1) of Theorem B.1, including the stated condition on the existence of nontrivial initially zero intervals. Altogether, Claims 1-7 complete the proof that the conditions in result (1) of Theorem B.1 are necessary for the MRD Property. Observation (a), that the MRD is given by $\alpha_{\mathrm{MR}} = \alpha_R$, is a general result noted in the discussion of Lemma 3; observation (b) on the unique ZRD follows from Theorem A.1; and observation (c) was established by Claim 5.

To prove the converse: We wish to show that results (1) and (2) of Theorem B.1 are sufficient conditions, that is, that they imply the MRD Property. For each result, we calculate $N\left(\alpha_P,\lambda_P\right)$, given by (23), and show that it satisfies the criteria of Lemma 3. Notice that the ZRD Property holds for each case, so the Balanced Field Property also holds and $b_R = \cos\left(\phi_R\right)H_q(1)$ in (23) for both cases. Similarly, $\alpha_{\mathrm{MR}} = \alpha_R$ in both cases.

Assume result (1) of Theorem B.1 holds. Then $H_q(\rho) = e^{-c\rho^2}F\left(\rho^2\right)$ with $c > 0$ and the stated conditions on $F(y)$ hold. Substituting into (23) gives

$$N\left(\alpha_P,\lambda_P\right)^2 = \exp\left(-c\left(1+\frac{\lambda_R^2}{\lambda_P^2}\right)\right)\cdot T_F\left(\phi_R, c, \frac{\lambda_R^2}{\lambda_P^2}; 2\cos\left(\alpha_P-\alpha_R\right)\frac{\lambda_R}{\lambda_P}\right) \quad (111)$$

where $T_F(\phi_R, c, a; y)$ is given by (95) and we note the particular case $T_F(\pm\pi, c, a; y) = f(a; y)^2$. Since $T_F(\phi_R, c, a; y)$ is strictly increasing on $y \geq 0$ for arbitrary real a or (in the particular case $\phi_R = \pm\pi$) may be initially zero and then strictly increasing, it immediately follows that, for each fixed λ_P, N is strictly decreasing to zero as $\cos(\alpha_P - \alpha_R)$ decreases, that is, as $|\alpha_P - \alpha_R|$ increases, and either reaches zero at the ZRD $|\alpha_P - \alpha_R| = \pi/2$ or (in the particular case $\phi_R = \pm\pi$) may reach zero before then and remains zero until $|\alpha_P - \alpha_R| = \pi/2$. The conditions of Lemma 3 are satisfied.

Assume result (2) of Theorem B.1 holds. Then $H_q(\rho) = 0$ on $\sin(\zeta_R) \leq \rho < +\infty$ and is positive and strictly decreasing on $0 \leq \rho < \sin(\zeta_R)$, so $b_R = 0$ and the expression for N reduces to the single term

$$\begin{array}{c} N(\alpha_P, \lambda_P) = H_q\left(s_-(\alpha_P - \alpha_R, \lambda_R/\lambda_P)\right) \\ \text{for } -\frac{\pi}{2} \leq \alpha_P - \alpha_R \leq +\frac{\pi}{2} \end{array} \tag{112}$$

It immediately follows that, for each fixed λ_P, N is strictly decreasing to zero, and remains zero, as $\cos(\alpha_P - \alpha_R)$ decreases, that is, as $|\alpha_P - \alpha_R|$ increases. The conditions of Lemma 3 are satisfied. The proof of Theorem B.1 is now complete.

Appendix C. Miscellaneous Derivations.

Section 3. Derivation of eqn. (18): Fourier transform of elementary receptive field function in terms of Hankel transforms.

We wish to derive eqn. (18), the representation of the Fourier transform of an elementary receptive field function (11) in terms of the Hankel transform $H_q(\rho)$ of the weight function $q(r)$. By elementary properties of the Fourier transform (as defined by eqn. (4)), the Fourier transform of the elementary receptive field function $R(x_1, x_2)$ can be reduced to the Fourier transform of the weight function $p(x_1, x_2)$:

$$\begin{aligned} R\left(\overrightarrow{x}\right) &= \frac{2\pi}{\lambda_R^2} q\left(\frac{2\pi}{\lambda_R}\left|\overrightarrow{x}\right|\right)\left(\cos\left(\frac{2\pi}{\lambda_R}\overrightarrow{d}(\alpha_R)\cdot\overrightarrow{x} - \phi_R\right) - b_R\right) \\ &= \frac{2\pi}{\lambda_R^2} p\left(\frac{2\pi}{\lambda_R}x_1, \frac{2\pi}{\lambda_R}x_2\right)\left(\cos\left(\frac{2\pi}{\lambda_R}\overrightarrow{d}(\alpha_R)\cdot\overrightarrow{x} - \phi_R\right) - b_R\right) \end{aligned} \tag{113}$$

which has the Fourier transform

$$\begin{aligned} F_R(s_1, s_2) = \ & \frac{1}{4\pi}\left(e^{+i\phi_R} F_p\left(\frac{\lambda_R}{2\pi}\left(s_1 + \frac{1}{\lambda_R}\cos(\alpha_R)\right), \frac{\lambda_R}{2\pi}\left(s_2 + \frac{1}{\lambda_R}\sin(\alpha_R)\right)\right)\right. \\ & \left. + e^{-i\phi_R} F_p\left(\frac{\lambda_R}{2\pi}\left(s_1 - \frac{1}{\lambda_R}\cos(\alpha_R)\right), \frac{\lambda_R}{2\pi}\left(s_2 - \frac{1}{\lambda_R}\sin(\alpha_R)\right)\right)\right) \\ & - \frac{b_R}{2\pi} F_p\left(\frac{\lambda_R}{2\pi}s_1, \frac{\lambda_R}{2\pi}s_2\right) \end{aligned} \tag{114}$$

The 2D-Fourier transform of a circularly symmetric function can be expressed as the Hankel transform (defined by eqn. (14)) of the radial form:

$$p(x_1, x_2) = q(r) \tag{115}$$

which has the Hankel transform

$$F_p\left(s_1, s_2\right) = 2\pi H_q(2\pi\rho) \text{ where } \rho = \sqrt{s_1^2 + s_2^2} \tag{116}$$

Combining (114) and (116) gives (18).

Section 5a. Derivation of part (2) of the Closure Lemma for balanced Gabor weights.

We wish to prove part (2) of the Closure Lemma for balanced Gabor weights. It is sufficient to prove the following:
CLAIM. Given $0 < \gamma_1 < \gamma_2$ and $G(y)$ continuous on the real line with $e^{-\gamma_1 y}G(y)$ strictly decreasing and positive and an arbitrary real a. If

$$e^{\gamma_1 y}G(a-y) + e^{-\gamma_1 y}G(a+y) \text{ is increasing for } y \geq 0, \tag{117}$$

then

$$e^{\gamma_2 y}G(a-y) + e^{-\gamma_2 y}G(a+y) \text{ is strictly increasing for } y > 0. \tag{118}$$

PROOF. Note $e^{-\gamma_1 y}G(y)$ is strictly decreasing and positive implies

$$e^{-\gamma_1 y}G(a+y) \text{ and } e^{-\gamma_2 y}G(a+y) \text{ are strictly decreasing and positive} \tag{119}$$

$$e^{+\gamma_1 y}G(a-y) \text{ and } e^{+\gamma_2 y}G(a-y) \text{ are strictly increasing and positive} \tag{120}$$

By hypothesis, for $0 < y_1 < y_2$,

$$\begin{array}{c} e^{\gamma_1 y_2}G\left(a-y_2\right) + e^{-\gamma_1 y_2}G\left(a+y_2\right) \geq \\ e^{\gamma_1 y_1}G\left(a-y_1\right) + e^{-\gamma_1 y_1}G\left(a+y_1\right) \end{array} \tag{121}$$

We will show

$$\begin{array}{c} e^{\gamma_2 y_2}G\left(a-y_2\right) + e^{-\gamma_2 y_2}G\left(a+y_2\right) > \\ e^{\gamma_2 y_1}G\left(a-y_1\right) + e^{-\gamma_2 y_1}G\left(a+y_1\right) \end{array} \tag{122}$$

Multiply (121) by $e^{(\gamma_2-\gamma_1)y_2}$ to obtain

$$\begin{array}{c} e^{\gamma_2 y_2}G\left(a-y_2\right) + e^{(\gamma_2-2\gamma_1)y_2}G\left(a+y_2\right) \geq \\ e^{\gamma_2 y_2}e^{\gamma_1(-y_2+y_1)}G\left(a-y_1\right) + e^{\gamma_2 y_2}e^{-\gamma_1(y_2+y_1)}G\left(a+y_1\right) \end{array} \tag{123}$$

$$\begin{array}{l} e^{\gamma_2 y_2}G\left(a-y_2\right) + e^{-\gamma_2 y_2}G\left(a+y_2\right) \\ -e^{\gamma_2 y_1}G\left(a-y_1\right) - e^{-\gamma_2 y_1}G\left(a+y_1\right) \geq \\ \;+e^{-\gamma_2 y_2}G\left(a+y_2\right) - e^{(\gamma_2-2\gamma_1)y_2}G\left(a+y_2\right) \\ \;+e^{\gamma_2 y_2}e^{\gamma_1(-y_2+y_1)}G\left(a-y_1\right) + e^{\gamma_2 y_2}e^{-\gamma_1(y_2+y_1)}G\left(a+y_1\right) \\ \;-e^{\gamma_2 y_1}G\left(a-y_1\right) - e^{-\gamma_2 y_1}G\left(a+y_1\right) \end{array} \tag{124}$$

The desired result (122) follows if we can show the right side R is positive. We have

$$\begin{array}{rl} R = & -\left(e^{2(\gamma_2-\gamma_1)y_2} - 1\right)e^{-\gamma_2 y_2}G\left(a+y_2\right) \\ & +\left(e^{\gamma_2 y_2}e^{\gamma_1(-y_2+y_1)} - e^{\gamma_2 y_1}\right)G\left(a-y_1\right) \\ & +\left(e^{\gamma_2 y_2}e^{-\gamma_1(y_2+y_1)} - e^{-\gamma_2 y_1}\right)G\left(a+y_1\right) \end{array} \tag{125}$$

As noted, $0 < e^{-\gamma_2 y_2} G(a+y_2) < e^{-\gamma_2 y_1} G(a+y_1)$, so

$$
\begin{aligned}
R > \quad & -\left(e^{2(\gamma_2-\gamma_1)y_2}-1\right)e^{-\gamma_2 y_1}G(a+y_1) \\
& +\left(e^{\gamma_2 y_2}e^{\gamma_1(-y_2+y_1)}-e^{\gamma_2 y_1}\right)G(a-y_1) \\
& +\left(e^{\gamma_2 y_2}e^{-\gamma_1(y_2+y_1)}-e^{-\gamma_2 y_1}\right)G(a+y_1) \\
= \quad & \left(e^{(\gamma_2-\gamma_1)(y_2+y_1)}-e^{2(\gamma_2-\gamma_1)y_2}\right)e^{-\gamma_2 y_1}G(a+y_1) \\
& +\left(e^{\gamma_2 y_2}e^{\gamma_1(-y_2+y_1)}-e^{\gamma_2 y_1}\right)G(a-y_1) \\
= \quad & \left(e^{(\gamma_2-\gamma_1)(y_2-y_1)}-1\right)e^{\gamma_2 y_1}G(a-y_1) \\
& +e^{(\gamma_2-\gamma_1)y_2}\left(e^{(\gamma_2-\gamma_1)y_1}-e^{(\gamma_2-\gamma_1)y_2}\right)e^{-\gamma_2 y_1}G(a+y_1) > 0
\end{aligned}
\tag{126}
$$

The claim is proved.

Section 5a. Derivation of eqn. (55): Condition determining balanced Gabor weights of order 1.

Balanced Gabor weights of order 1 have Hankel transforms $H_g(\rho) = e^{-\gamma_R \rho^2} G\left(\rho^2\right)$ where $G(y){:=}\frac{1+c_R\cos(2\pi y-\psi_R)}{1+c_R\cos(\psi_R)}$ and $G(y)$ must satisfy the conditions (a), (b), (c), (d) of the balanced Gabor definition at the beginning of Section 5a. We will derive eqn. (55), a characterization of these conditions in terms of the parameters γ_R, c_R, ψ_R for the order 1 case. As noted in the discussion following (55), condition (d) is vacuous here.

Condition (a): $H_q(\rho) = e^{-c\rho^2} F\left(\rho^2\right)$ where $F(y)$ is continuous and positive on the real line, $F(0) = 1$, $F(y+1) = F(y)$. Clearly $G(y)$ satisfies this requirement iff $|c_R| < 1$.

Condition (b): $e^{-cy}F(y)$ is strictly decreasing on the real line, that is, $\frac{d}{dy}\left(e^{-cy}F(y)\right) \leq 0$ (with equality at most at isolated points).

CLAIM. $G(y)$ satisfies condition (b) iff ${c_R}^2 \leq \frac{{\gamma_R}^2}{4\pi^2+{\gamma_R}^2}$.
PROOF. The condition can be restated as $\frac{G'(y)}{G(y)} \leq c$. Then $\frac{G'(y)}{G(y)} = \frac{-2\pi c_R \sin(2\pi y-\psi_R)}{1+c_R\cos(2\pi y-\psi_R)}$ has min-max values $\frac{\pm 2\pi c_R}{\sqrt{1-{c_R}^2}}$ (occurring at $\cos(2\pi y-\psi_R) = -c_R$), that is, max-value $\frac{2\pi|c_R|}{\sqrt{1-{c_R}^2}}$. Take $\frac{2\pi|c_R|}{\sqrt{1-{c_R}^2}} = \gamma_R$, or ${c_R}^2 \leq \frac{{\gamma_R}^2}{4\pi^2+{\gamma_R}^2}$.

Condition (c): For each real a, $T_F(c,a;y){:=}e^{cy}F(a-y)+e^{-cy}F(a+y)$ is strictly increasing for $y \geq 0$. That is, taking a derivative and expanding: for each real a, $(F'(a-y)-cF(a-y)) - e^{-2cy}(F'(a+y)-cF(a+y)) \leq 0$ for $y \geq 0$ (with equality at most at isolated points).

CLAIM. $G(y)$ satisfies condition (c) iff $|c_R| \leq \frac{{\gamma_R}^2}{{\gamma_R}^2+4\pi^2}$.
PROOF. Set $G_1(y){:=}1+c_R\cos(2\pi y-\psi_R)$. The problem is to determine the range of c for which the inequality

$$(G_1'(a-y)-cG_1(a-y)) - e^{-2cy}\left(G_1'(a+y)-cG_1(a+y)\right) \leq 0$$

holds on $y \geq 0$ for all real a. Notice

$$G_1'(y) - cG_1(y) = -2\pi c_R \sin(2\pi y - \psi_R) - c\,(1 + c_R \cos(2\pi y - \psi_R))$$

The problem is now to determine the range of c such that, for $y \geq 0$,

$$\begin{array}{l}[-2\pi c_R \sin(2\pi(a-y) - \psi_R) - c\,(1 + c_R \cos(2\pi(a-y) - \psi_R))] \\ -e^{-2cy}\,[-2\pi c_R \sin(2\pi(a+y) - \psi_R) - c\,(1 + c_R \cos(2\pi(a+y) - \psi_R))] \leq 0\end{array}$$

Writing $C_1 = \cos(2\pi a - \psi_R)$, $S_1 = \sin(2\pi a - \psi_R)$, this inequality is equivalent to

$$\begin{array}{l} c_R C_1 \left[\left(1 + e^{-2cy}\right) 2\pi \sin(2\pi y) - \left(1 - e^{-2cy}\right) c \cos(2\pi y)\right] \\ \;+ c_R S_1 \left[-\left(1 - e^{-2cy}\right) 2\pi \cos(2\pi y) - \left(1 + e^{-2cy}\right) c \sin(2\pi y)\right] \\ \;\leq c\left(1 - e^{-2cy}\right)\end{array}$$

for $y \geq 0$. Setting

$$\begin{array}{rcl} A_1(c,y) & := & \coth(cy) 2\pi \sin(2\pi y) - c\cos(2\pi y) \\ B_1(c,y) & := & 2\pi \cos(2\pi y) + \coth(cy) c \sin(2\pi y)\end{array}$$

gives, for $y > 0$,

$$c_R\,(C_1 A_1(c,y) - S_1 B_1(c,y)) \leq c$$

The expression on the left (given $C_1, S_1 = \cos(2\pi a - \psi_R), \sin(2\pi a - \psi_R)$) has, for $y > 0$, the precise upper bound over all a:

$$|c_R| \sqrt{A_1(c,y)^2 + B_1(c,y)^2} \leq c$$

That is, for each $y > 0$, the left side is simply the amplitude at y as a varies of the preceding expression and will be attained for infinitely many values of a. Hence the inequality is necessary and sufficient to insure that the preceding inequalities are maintained at each $y > 0$. It can be rewritten as

$$\left(\cos(2\pi y)^2 + \coth(cy)^2 \sin(2\pi y)^2\right)\left(4\pi^2 + c^2\right) {c_R}^2 \leq c^2 \tag{127}$$

for $y > 0$.

CLAIM. $f(y) := \cos(2\pi y)^2 + \coth(cy)^2 \sin(2\pi y)^2 \leq 1 + \frac{4\pi^2}{c^2}$ for $y \geq 0$.

PROOF. Note $f(0) = 1 + \frac{4\pi^2}{c^2}$ and $f(y) = 1 + (\operatorname{csch}(cy)\sin(2\pi y))^2$. Consider $g(y) := \operatorname{csch}(cy)\sin(2\pi y)$ for $y > 0$ and notice

$$\begin{array}{rl} g'(y) & = \frac{\sinh(cy)(2\pi\cos(2\pi y)) - \sin(2\pi y) c \cosh(cy)}{\sinh(cy)^2} \\ & = (\text{positive term})(\tanh(cy)(2\pi\cos(2\pi y)) - \sin(2\pi y) c\end{array}$$

Hence critical points satisfy $\tanh(cy) = \tan(2\pi y)\frac{c}{2\pi}$. That is, at critical points,

$$\operatorname{csch}(cy)^2 = \frac{1 - \tanh(cy)^2}{\tanh(cy)^2} = \frac{1 - \tan(2\pi y)^2 \left(\frac{c}{2\pi}\right)^2}{\tan(2\pi y)^2 \left(\frac{c}{2\pi}\right)^2}$$

in which case

$$f(y) = 1 + \operatorname{csch}(cy)^2 \sin(2\pi y)^2 = \cos(2\pi y)^2 \left(1 + \frac{4\pi^2}{c^2}\right)$$

Hence $f(y) \leq 1 + \frac{4\pi^2}{c^2}$ for $y \geq 0$. The claim is proved.

Now observe that (127) implies the bound

$$c_R^2 \leq \left(\frac{c^2}{4\pi^2 + c^2}\right)^2$$

which is (55), and conversely, the claim shows that this bound implies that (127) holds. Thus (55) characterizes the parameter range for balanced Gabor weights of order 1.

Figure Captions for Figures 13 - 16.

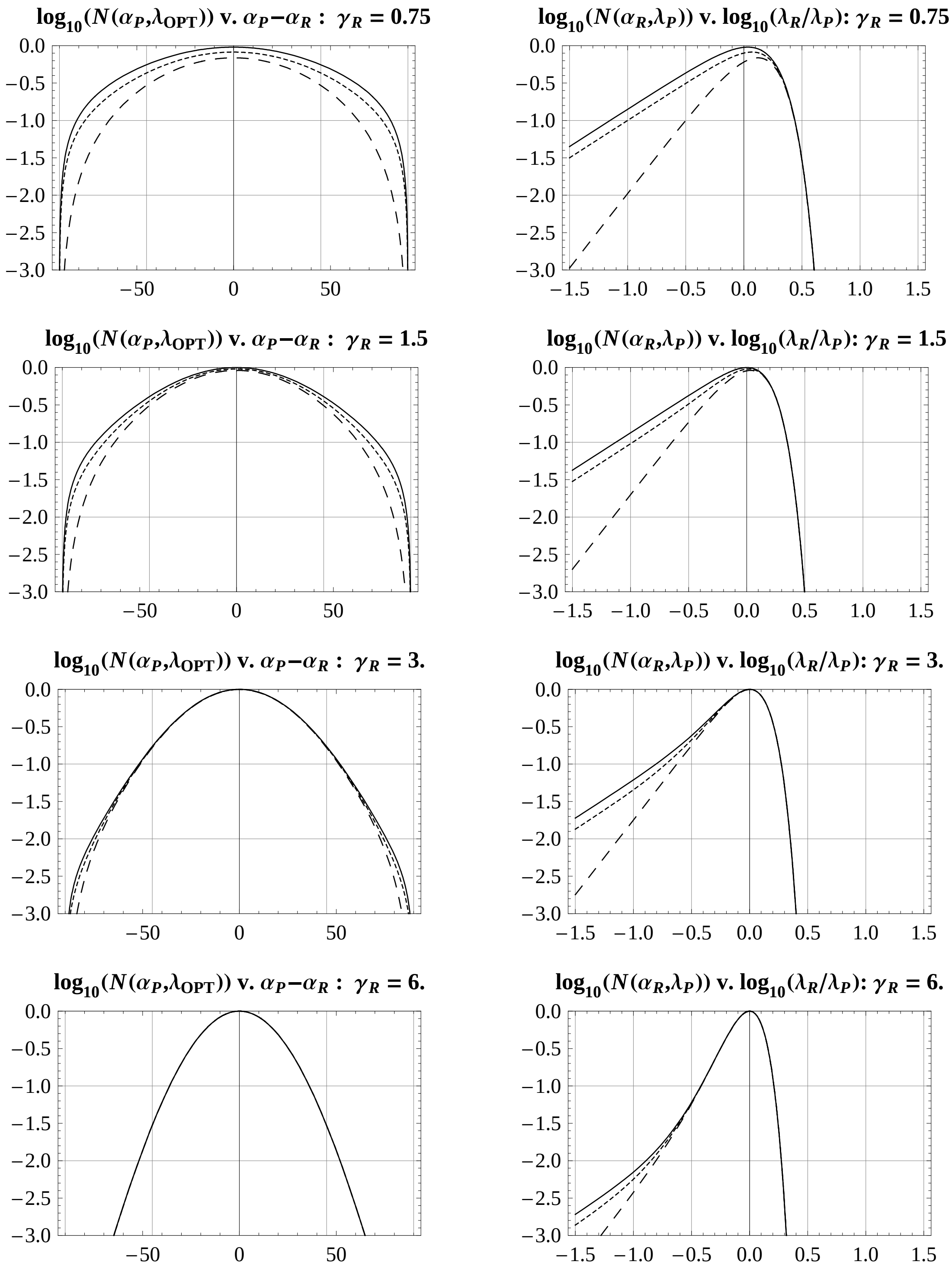
Figure 13. Log-log plot of Fig. 3. Left panels illustrate orientation max-response functions (in degrees) and right panels plot spatial frequency max-response functions for simple balanced Gabor receptive field functions as they vary with carrier spatial phase: 0 deg (cosine-type, dashed), 45 deg (mixed-type, dotted) and 90 deg (sine-type, solid). Response curves for $\gamma_R = 0.75$, 1.5, 3.0, 6.0 are illustrated. Note that as exponent γ_R increases max-response curves become independent of carrier spatial phase ϕ_R and that sine-type fields always give larger responses than cosine-type fields with corresponding parameters. The cosine-type spatial frequency max-response matches the Fourier transform of Fig. 2 (differing only by a scale factor).

Figure 14. Log-log plot of Fig. 6. Four pairs of plots showing orientation max-response functions in degrees (left) and spatial frequency max-response functions (right) for nonsimple balanced Gabor receptive field functions as they vary with c_R for four exponents, $\gamma_R = 0.75, 1.5, 3.0, 6.0$ (with $\psi_R = 0$), where $c_R = 0$ (simple balanced Gabor, dashed) and $c_R = \pm\gamma_R^2/\left(\gamma_R^2 + 4\pi^2\right)$, corresponding to the max (solid) and min (dotted) boundary values of this coefficient. For each value of c_R there is plotted a triplet of curves corresponding to receptive field phase, $\phi_R = 0$ deg (cosine-type), 45 deg (mixed-type) and 90 deg (sine-type). Note that as γ_R increases the dependence of both orientation and spatial frequency response on receptive field phase ϕ_R, (holding c_R fixed) steadily decreases, becoming virtually independent of receptive field phase ϕ_R at $\gamma_R = 6.0$. For each value of γ_R, the cosine-type spatial frequency max-response function matches the Fourier transform of Fig. 5 (differing only by a scale factor). In particular, the plot for $\gamma_R = 6.0$ matches the corresponding plot for Fig. 5 because virtually no variation with carrier phase occurs here.

Figure 15. Log-log plot of Fig. 9. Four pairs of plots showing orientation max-response functions in degrees (left) and spatial frequency max-response functions (right) for nonsimple balanced Gabor receptive field functions as they vary with c_R for four exponents, $\gamma_R = 0.75, 1.5, 3.0, 6.0$ (with $\psi_R = \pi/2$), where $c_R = 0$ (simple balanced Gabor, dashed) and $c_R = \pm\gamma_R^2/\left(\gamma_R^2 + 4\pi^2\right)$, corresponding to the max (solid) and min (dotted) boundary values of this coefficient. For each value of c_R there is plotted a triplet of curves corresponding to receptive field phase, $\phi_R = 0$ deg (cosine-type), 45 deg (mixed-type) and 90 deg (sine-type). Note that as γ_R increases the dependence of both orientation and spatial frequency response on receptive field phase ϕ_R (holding c_R fixed) steadily decreases, becoming virtually independent of receptive field phase ϕ_R at $\gamma_R = 6.0$. For each value of γ_R, the cosine-type spatial frequency max-response function matches the Fourier transform of Fig. 7 (differing only by a scale factor). In particular, the plot for $\gamma_R = 6.0$ matches the corresponding plot for Fig. 7 because virtually no variation with carrier phase occurs here.

Figure 16. Log-log plot of Fig. 12. Four pairs of plots showing orientation

max-response functions in degrees (left) and spatial frequency max-response functions (right) for bandlimited Bessel receptive field functions as they vary with $s_R = 1.0, 0.85, 0.7, 0.5$, and $\nu_R = 2.0$ (dotted), 3.5 (dashed), and 5.0 (solid). Unlike balanced Gabor receptive field functions, the max-response of bandlimited receptive field functions is independent of the receptive field phase ϕ_R. As support parameter s_R values decrease the receptive field function narrows in effective bandwidth. For a particular support parameter value, as the order of the Bessel weight ν_R increases the receptive field function narrows in effective bandwidth.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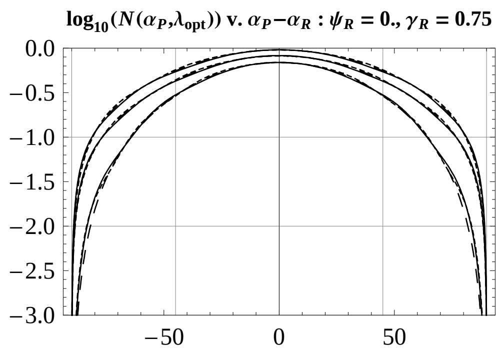
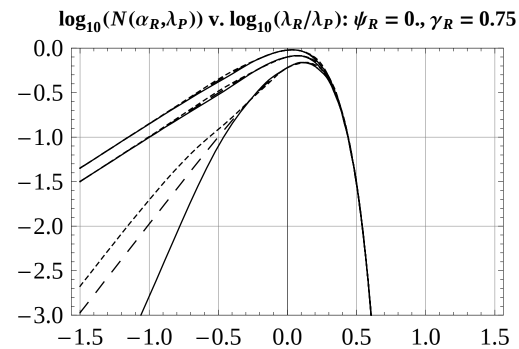
log10(N(αR,λP)) v. log10(λR/λP): γR = 0.75
log10(N(αP,λOPT)) v. αP−αR : γR = 1.5
log10(N(αR,λP)) v. log10(λR/λP): γR = 1.5
log10(N(αP,λOPT)) v. αP−αR : γR = 3.
log10(N(αR,λP)) v. log10(λR/λP): γR = 3.
log10(N(αP,λOPT)) v. αP−αR : γR = 6.
log10(N(αR,λP)) v. log10(λR/λP): γR = 6.


log10(N(αP,λopt)) v. αP−αR : ψR = 0., γR = 0.75
0.0 −0.5 −1.0 −1.5 −2.0 −2.5 −3.0
−50 0 50


log10(N(αR,λP)) v. log10(λR/λP): ψR = 0., γR = 0.75
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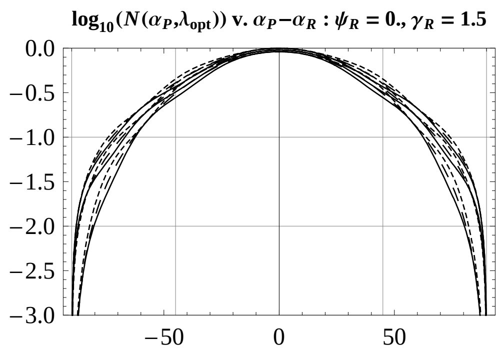
−1.5 −1.0 −0.5 0.0 0.5 1.0 1.5


log10(N(αP,λopt)) v. αP−αR : ψR = 0., γR = 1.5
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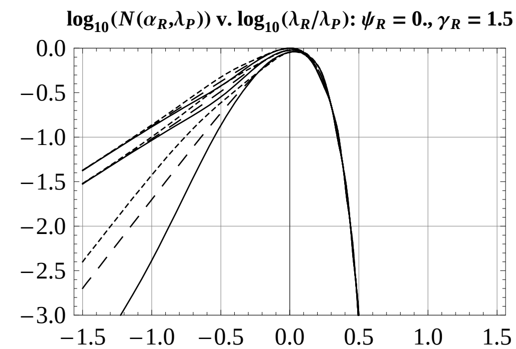
−50 0 50


log10(N(αR,λP)) v. log10(λR/λP): ψR = 0., γR = 1.5
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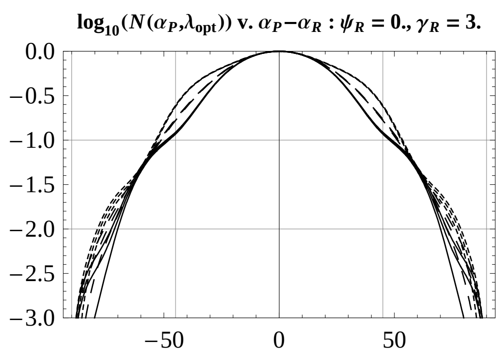
−1.5 −1.0 −0.5 0.0 0.5 1.0 1.5


log10(N(αP,λopt)) v. αP−αR : ψR = 0., γR = 3.
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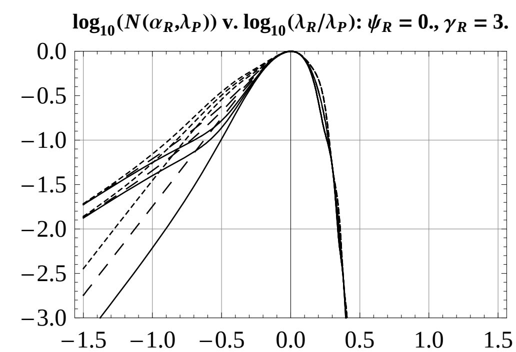
−50 0 50


log10(N(αR,λP)) v. log10(λR/λP): ψR = 0., γR = 3.
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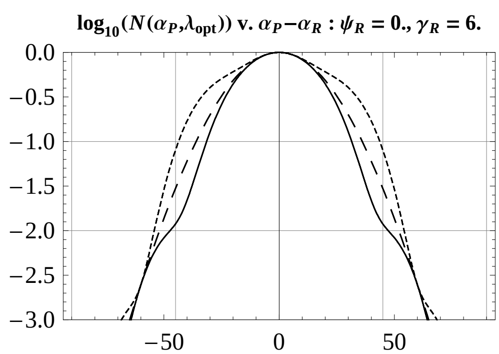
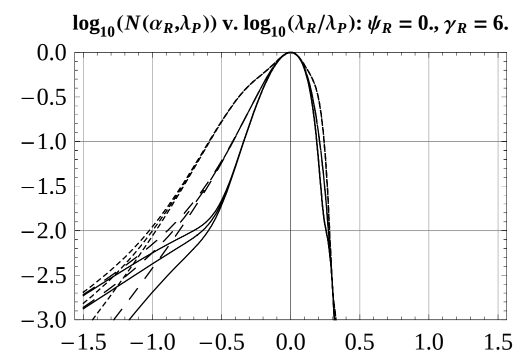
−1.5 −1.0 −0.5 0.0 0.5 1.0 1.5


log10(N(αP,λopt)) v. αP−αR : ψR = 0., γR = 6.
0.0 −0.5 −1.0 −1.5 −2.0 −2.5 −3.0
−50 0 50


log10(N(αR,λP)) v. log10(λR/λP): ψR = 0., γR = 6.
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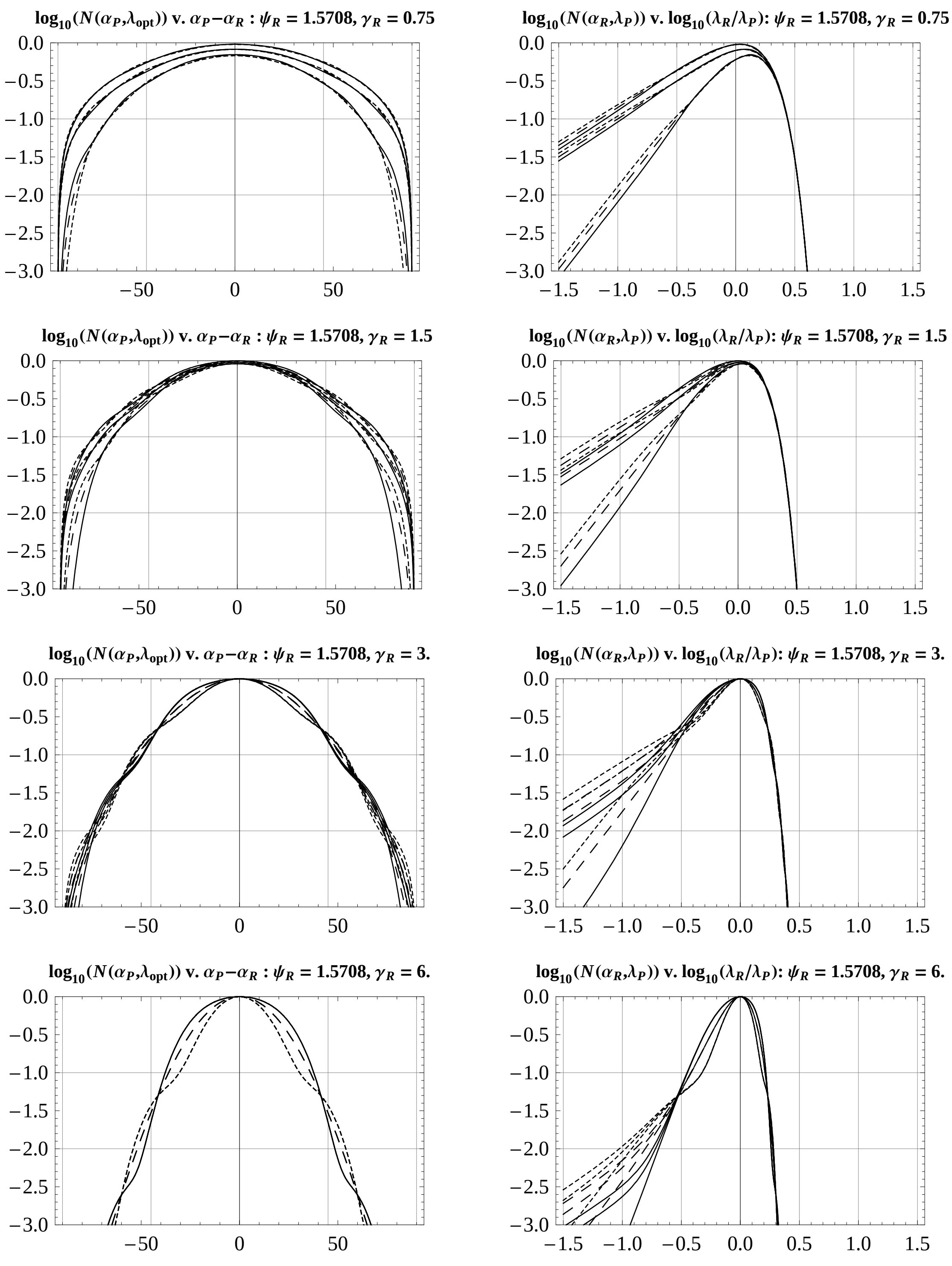
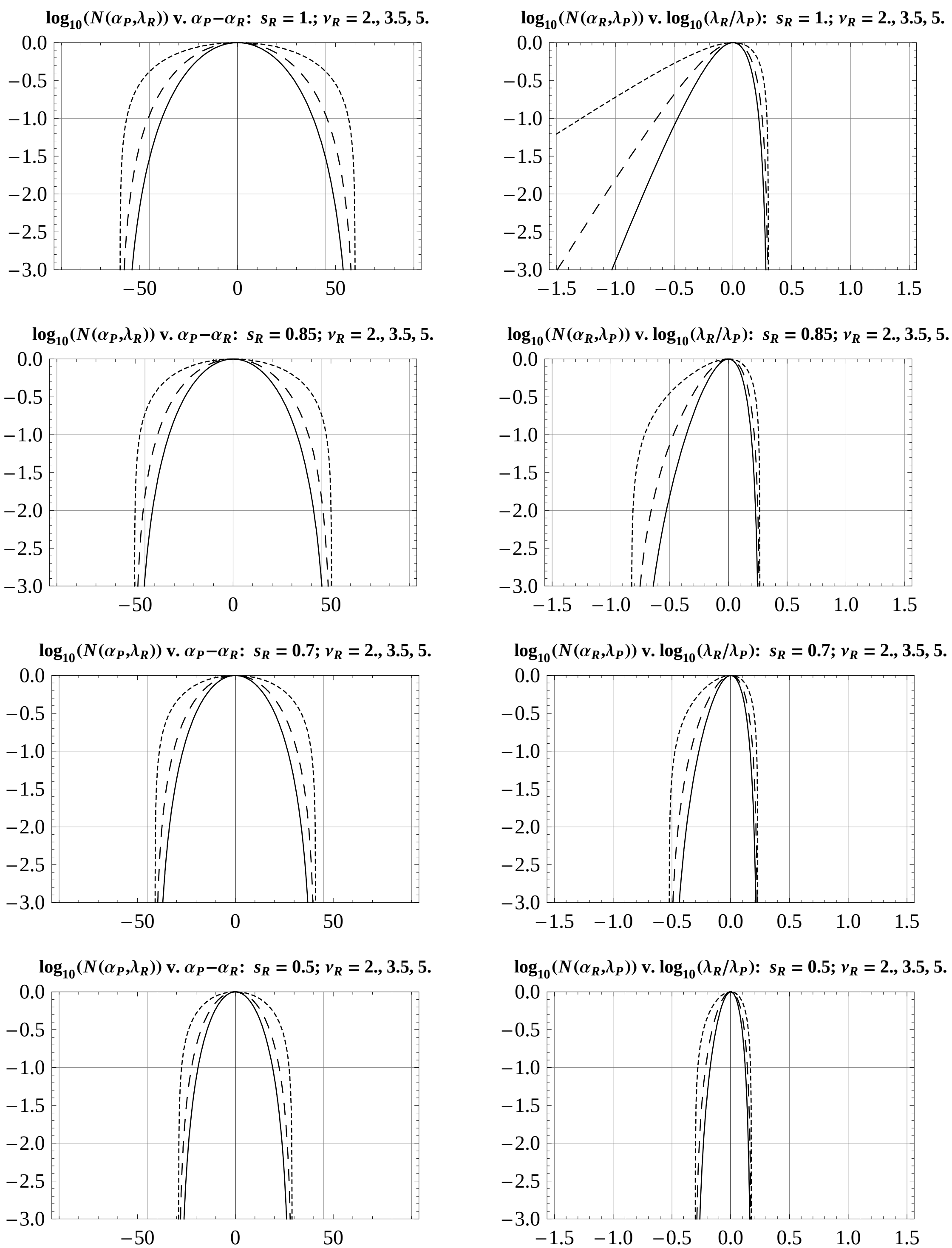
−1.5 −1.0 −0.5 0.0 0.5 1.0 1.5


$\log_{10}(N(\alpha_P,\lambda_{\text{opt}}))$ v. $\alpha_P-\alpha_R$: $\psi_R = 1.5708$, $\gamma_R = 0.75$
$\log_{10}(N(\alpha_R,\lambda_P))$ v. $\log_{10}(\lambda_R/\lambda_P)$: $\psi_R = 1.5708$, $\gamma_R = 0.75$
$\log_{10}(N(\alpha_P,\lambda_{\text{opt}}))$ v. $\alpha_P-\alpha_R$: $\psi_R = 1.5708$, $\gamma_R = 1.5$
$\log_{10}(N(\alpha_R,\lambda_P))$ v. $\log_{10}(\lambda_R/\lambda_P)$: $\psi_R = 1.5708$, $\gamma_R = 1.5$
$\log_{10}(N(\alpha_P,\lambda_{\text{opt}}))$ v. $\alpha_P-\alpha_R$: $\psi_R = 1.5708$, $\gamma_R = 3.$
$\log_{10}(N(\alpha_R,\lambda_P))$ v. $\log_{10}(\lambda_R/\lambda_P)$: $\psi_R = 1.5708$, $\gamma_R = 3.$
$\log_{10}(N(\alpha_P,\lambda_{\text{opt}}))$ v. $\alpha_P-\alpha_R$: $\psi_R = 1.5708$, $\gamma_R = 6.$
$\log_{10}(N(\alpha_R,\lambda_P))$ v. $\log_{10}(\lambda_R/\lambda_P)$: $\psi_R = 1.5708$, $\gamma_R = 6.$



log10(N(αP,λR)) v. αP−αR: sR = 1.; νR = 2., 3.5, 5.
log10(N(αR,λP)) v. log10(λR/λP): sR = 1.; νR = 2., 3.5, 5.
log10(N(αP,λR)) v. αP−αR: sR = 0.85; νR = 2., 3.5, 5.
log10(N(αR,λP)) v. log10(λR/λP): sR = 0.85; νR = 2., 3.5, 5.
log10(N(αP,λR)) v. αP−αR: sR = 0.7; νR = 2., 3.5, 5.
log10(N(αR,λP)) v. log10(λR/λP): sR = 0.7; νR = 2., 3.5, 5.
log10(N(αP,λR)) v. αP−αR: sR = 0.5; νR = 2., 3.5, 5.
log10(N(αR,λP)) v. log10(λR/λP): sR = 0.5; νR = 2., 3.5, 5.