HOMOLOGICAL DIMENSIONS WITH RESPECT TO A SEMIDUALIZING COMPLEX

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ABSTRACT

A fintitely generated R-module C is semidualizing if $R \cong \mathrm{Hom}_R(C, C)$ and $\mathrm{Ext}_R^{\geqslant 1}(C, C) = 0$. In this dissertation we build off of Takahashi and White's $\mathcal{P}_C$-projective dimension and $\mathcal{I}_C$-injective dimension to define these dimensions for when C is a semidualizing complex. In particular, for an R-complex X and a semidualizing R-complex C, the $\mathcal{P}_C$-projecitve dimensions is $\mathcal{P}_C\text{-}\operatorname{pd}_R(X) = \sup(C)+\operatorname{pd}_R(\mathbf{R}\mathrm{Hom}_R(C, X)$. The $\mathcal{F}_C$-projective dimension is defined similarly and the $\mathcal{I}_C$-injective dimension is defined dually. We develop the framework for these homological dimensions by establishing base change results and local-global behavior. Furthermore, we investigate how these dimensions interact with other invariants. In addition we answer a question from Takahashi and White [27] that generalizes a Theorem of Foxby [9]. His result states that if there exists a module with finite depth, finite flat dimension, and finite injective dimension over a local ring R, then R is Gorenstein. We use our generalized definitions of $\mathcal{F}_C$-projective dimension and $\mathcal{I}_C-$injective dimension involving a semidualizing R-complex to improve on Foxby's result by answering Takahashi and White's question. In particular, we prove for a semidualizing module C, if there exists a module with finite depth, finite $\mathcal{F}_C$-projective dimension, and finite $\mathcal{I}_C$-injective dimension over a local ring R, then R is Gorenstein.

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DEDICATION

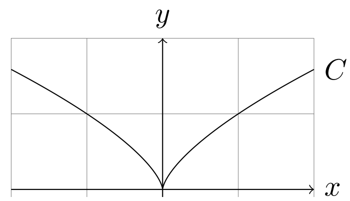
To Carrie and Adelaide.

TABLE OF CONTENTS

1. INTRODUCTION

1.1. Motivation

Algebra and geometry are two central areas of mathematics. We use geometry to help model the world around us. Some geometric objects are nicer than others. For example, lines and planes are flat and smooth. A more interesting example of a smooth shape is a circle of radius one $S : x^2 + y^2 = 1$. However, many shapes have singularities (e.g., sharp corners), so are not smooth. For example, the following curve $C : x^2 = y^3$ has a singularity at $x = 0$.



How can we understand these curves and other shapes, especially in high dimensions where we cannot visualize them? Algebra holds the key via functions. Polynomials of the form

$$a_0 + a_1X + \cdots + a_nX^n$$

give us functions from the real numbers to the real numbers ($\mathbb{R} \to \mathbb{R}$). In a similar way we have notions of polynomials in two variables ($\mathbb{R}^2 \to \mathbb{R}$) and polynomials $C \to \mathbb{R}$ and $S \to \mathbb{R}$. Ring theory uses these sets of functions to distinguish between different geometric objects such as $\mathbb{R}$, $\mathbb{R}^2$, C, and S.

Consider the set of all polynomial functions. You may notice that any two polynomials added together produce another polynomial. Also any two polynomials multiplied together produce another polynomial. Sets that have the properties of being closed under multiplication and addition are called *rings*. These are the major objects of study in my research. Note that different geometric objects have different polynomial rings, so one can use these algebraic constructions to differentiate between geometric objects.

A useful algebraic structure for studying rings is a module. In many circumstances we can characterize types of rings by describing the associated modules. In my research, I study modules over a ring R via homological constructions (this too has its roots in geometry). For instance, a powerful technique for studying R-modules A and B is to combine them as $\operatorname{Ext}_R^i(A,B)$, $\operatorname{Tor}_i^R(A,B)$, $\operatorname{Hom}_R(A,B)$, and $A \otimes_R B$.

One important type of module is a projective module. If P is a projective module, then P has the property $\operatorname{Ext}_R^i(P,M) = 0$ for all $i \geqslant 1$ and for all R-modules M. Not all modules are projective. Therefore we may want to know how close a module M is to being projective. The *projective dimension* of M (denoted $\operatorname{pd}_R(M)$) is the invariant we use to do this. If $\operatorname{pd}_R(M)$ is a large number, then M is far from being projective. See Section 1.2 for a more precise definition of the projective dimension.

From the geometric point of view the regular local rings are quite important. Oscar Zariski proved that regular local rings correspond to smooth points on algebraic varieties, e.g., on curves. Auslander, Buchsbaum, and Serre combined these algebraic, geometric, and homological notions in the following celebrated result.

Theorem 1.1.1 ([3, 26])**.** *Let R be a local ring with maximal ideal $\mathfrak{m}$ and residue field k. If R is a regular local ring, then $\operatorname{pd}_R(M) < \infty$ for all R-modules M; conversely, if $\operatorname{pd}_R(k) < \infty$, then R is a regular local ring.*

1.2. Homological Dimensions

Let R be a commutative noetherian ring with identity and M be an R-module. The *projective, flat, and injective dimensions* of an R-module M are now classical invariants that are important for studying M and R. The projective dimension is defined to be the length of the shortest projective resolution of M. A projective resolution of M is a sequence of R-modules and R-module homomorphisms

$$\cdots \xrightarrow{\partial_3^P} P_2 \xrightarrow{\partial_2^P} P_1 \xrightarrow{\partial_1^P} P_0 \xrightarrow{\pi} M \to 0$$

that is exact and each P_i is projective. The flat dimension of M (denoted $\operatorname{fd}_R(M)$) is defined in the same way as projective dimension, only we use flat R-modules in place of projective R-modules. The injective dimension of M (denoted $\operatorname{id}_R(M)$) is defined to be the length of the shortest injective resolution of M. An injective resolution of M is a sequence of R-modules and

R-module homomorphisms

$$0 \to M \xrightarrow{\varepsilon} I^0 \xrightarrow{\partial_I^1} I^1 \xrightarrow{\partial_I^2} I^2 \xrightarrow{\partial_I^3} \cdots$$

that is exact and each I^i is an injective module. These dimensions were later generalized for R-complexes by Foxby [9] and many useful results about dimensions for modules also hold true for complexes.

A finitely generated R-module C is *semidualizing* if $R \cong \mathrm{Hom}_R(C, C)$ and $\mathrm{Ext}_R^{\geqslant 1}(C, C) = 0$. For example, R is a semidualizing R-module. Semidualizing modules were first introduced by Foxby [8]. They were later studied by Vasconcelos [29], and Golod [14]. Semidualizing modules are useful, e.g., for proving results about Bass numbers [5, 22] and compositions of local ring homomorphisms [5, 21].

Takahashi and White [27] defined, for a semidualizing R-module C, the $\mathcal{P}_C$-projective and $\mathcal{I}_C$-injective dimensions. The *$\mathcal{P}_C$-projective dimension* of an R-module M (denoted $\mathcal{P}_C\text{-}\mathrm{pd}_R(M)$) is the length of the shortest resolution of M by modules of the form $C \otimes_R P$ where P is a projective module. They define *$\mathcal{I}_C$-injective dimension* (denoted $\mathcal{I}_C\text{-}\mathrm{id}_R(M)$) dually, and one defines the *$\mathcal{F}_C$-projective dimension* (denoted $\mathcal{F}_C\text{-}\mathrm{pd}_R(M)$) similarly. In Chapter 3 we extend these constructions to the realm of R-complexes. We work in the derived category $\mathcal{D}(R)$. See Chapter 2 for background information.

To understand the $\mathcal{P}_C$-projective, $\mathcal{F}_C$-projective, and $\mathcal{I}_C$-injective dimensions in the context when C is a semidualizing R-complex, we use the following result; see Theorem 3.1.9 below.

Theorem 1.2.1. *Let $X \in \mathcal{D}_\mathrm{b}(R)$, and let C be a semidualizing R-complex.*

(a) *We have $\mathrm{pd}_R(\mathbf{R}\mathrm{Hom}_R(C, X)) < \infty$ if and only if there exists $Y \in \mathcal{D}_\mathrm{b}(R)$ such that $\mathrm{pd}_R(Y) < \infty$ and $X \simeq C \otimes_R^{\mathbf{L}} Y$ in $\mathcal{D}(R)$. When these conditions are satisfied, one has $Y \simeq \mathbf{R}\mathrm{Hom}_R(C, X)$ and $X \in \mathcal{B}_C(R)$.*

(b) *We have $\mathrm{fd}_R(\mathbf{R}\mathrm{Hom}_R(C, X)) < \infty$ if and only if there exists $Y \in \mathcal{D}_\mathrm{b}(R)$ such that $\mathrm{fd}_R(Y) < \infty$ and $X \simeq C \otimes_R^{\mathbf{L}} Y$ in $\mathcal{D}(R)$. When these conditions are satisfied, one has $Y \simeq \mathbf{R}\mathrm{Hom}_R(C, X)$ and $X \in \mathcal{B}_C(R)$.*

(c) *We have $\mathrm{id}_R(C \otimes_R^{\mathbf{L}} X) < \infty$ if and only if there exists $Y \in \mathcal{D}_\mathrm{b}(R)$ such that $\mathrm{id}_R(Y) < \infty$ and $X \simeq \mathbf{R}\mathrm{Hom}_R(C, Y)$ in $\mathcal{D}(R)$. When these conditions are satisfied, one has $Y \simeq C \otimes_R^{\mathbf{L}} X$ and*

$X \in \mathcal{A}_C(R)$.

With this in mind, we define e.g., $\mathcal{P}_C\text{-}\operatorname{pd}_R(X) := \sup(C) + \operatorname{pd}_R(\mathbf{R}\mathrm{Hom}_R(C, X))$; thus $\mathcal{P}_C\text{-}\operatorname{pd}_R(X) < \infty$ if and only if X satisfies the equivalent conditions of Theorem 1.2.1(a). We define $\mathcal{F}_C\text{-}\operatorname{pd}_R(X)$ and $\mathcal{I}_C\text{-}\operatorname{id}_R(X)$ similarly.

In Section 3.1 we develop the foundations of these homological dimensions. For instance, we establish finite flat dimension base change (3.1.11) and local-global principles (3.1.16-3.1.18). Also in Theorem 3.1.10 we show how these notions naturally augment Foxby Equivalence. In Section 3.2 we establish some stability results and the following; see Theorem 3.2.9.

Theorem 1.2.2. *Assume R has a dualizing complex D and let $X \in \mathcal{D}_{\mathrm{b}}(R)$. Then $\mathcal{F}_C\text{-}\operatorname{pd}_R(X) < \infty$ if and only if $\mathcal{I}_{C^\dagger}\text{-}\operatorname{id}_R(X) < \infty$ where $C^\dagger = \mathbf{R}\mathrm{Hom}_R(C, D)$.*

This result is key for the work in Section 3.3 where we generalize a result of Foxby that states, if there exists an R-complex X that has finite flat dimension and finite injective dimension, then $R_{\mathfrak{p}}$ is a Gorenstein ring for all $\mathfrak{p} \in \operatorname{supp}_R(X)$; see Chapter 2 for definitions of small support and Gorenstein rings.

In Takahashi and White's investigation in [27] they posed the following question: When R is a local Cohen-Macaulay ring admitting a dualizing module and C is a semidualizing R-module, if there exists an R-module M such that $\mathcal{P}_C\text{-}\operatorname{pd}_R(M) < \infty$ and $\mathcal{I}_C\text{-}\operatorname{id}_R(M) < \infty$, must R be Gorenstein? If M has infinite depth, then the answer is false. However, if we additionally assume that M has finite depth, then an affirmative answer to this question would yield a generalization of Foxby's theorem.

Partial answers to Takahashi and White's question are given by Araya and Takahashi [1] and Sather-Wagstaff and Yassemi [24]. We give a complete answer to this question in the following result; see Chapter 2 for background on complexes and the derived category, and Theorem 3.3.2 for the proof.

Theorem 1.2.3. *Let C be a semidualizing R-complex. If there exists an R-complex $X \in \mathcal{D}_{\mathrm{b}}(R)$ such that $\mathcal{F}_C\text{-}\operatorname{pd}_R(X) < \infty$ and $\mathcal{I}_C\text{-}\operatorname{id}_R(X) < \infty$, then $R_{\mathfrak{p}}$ is Gorenstein for all $\mathfrak{p} \in \operatorname{supp}_R(X)$.*

The new results of this dissertation can be found in two peer-reviewed research papers. The results of Sections 3.1 and 3.2 can be found in [28], which has been accepted for publication, and the results from Section 3.3 can be found in [25], which has appeared.

2. BACKGROUND

2.1. Homological Constructions and the Category of R-Complexes

Throughout this chapter let R be a commutative noetherian ring with identity, and let all R-modules be unital.

In this section we define the objects and look at some of the basic properties from which we build the derived category. Most of these constructions and results can be found in [6, 20].

Definition 2.1.1. Let M, N, L be R-modules. A *commutative diagram* is a diagram of R-module homomorphisms

$$\begin{tikzcd} M \arrow[r, "\alpha"] \arrow[dr, "\beta"'] & N \arrow[d, "\gamma"] \\ & L \end{tikzcd}$$

such that $\gamma \circ \alpha = \beta$.

Definition 2.1.2. A sequence of R-module homomorphisms

$$X = \cdots \xrightarrow{\partial^X_{i+1}} X_i \xrightarrow{\partial^X_i} X_{i-1} \xrightarrow{\partial^X_{i-1}} \cdots$$

is a *chain complex* or an *R-complex* if $\partial^X_{i-1} \circ \partial^X_i = 0$ for all i. The module in the i^{th} *degree* of the R-complex X is X_i and we set $|x| = i$ when $x \in X_i$. The i^{th} *homology module* of X is

$$\mathrm{H}_i(X) := \mathrm{Ker}(\partial^X_i)/\,\mathrm{Im}(\partial^X_{i+1}).$$

We say that the complex X is *exact* if $\mathrm{H}_i(X) = 0$ for all i. The n^{th} *suspension* or *shift* of X is the R-complex $\Sigma^n X$ defined by $(\Sigma^n X)_i := X_{i-n}$ and $\partial^{\Sigma^n X}_i := (-1)^n \partial^X_{i-n}$. Furthermore, set $\Sigma X := \Sigma^1 X$.

Remark 2.1.3. An R-module M is an R-complex concentrated in degree 0. It is the complex

$$\cdots \to 0 \to 0 \to M \to 0 \to 0 \to \cdots .$$

Definition 2.1.4. Let X and Y be R-complexes. A *chain map* $F : X \to Y$ is a sequence of R-module homomorphisms $\{F_i : X_i \to Y_i\}_{i\in\mathbb{Z}}$ making the following diagram commute

$$\begin{array}{ccc} X & & \cdots \xrightarrow{\partial^X_{i+1}} X_i \xrightarrow{\partial^X_i} X_{i-1} \xrightarrow{\partial^X_{i-1}} \cdots \\ {\scriptstyle F}\downarrow & & \qquad\quad {\scriptstyle F_i}\downarrow \qquad\ {\scriptstyle F_{i-1}}\downarrow \\ Y & & \cdots \xrightarrow{\partial^Y_{i+1}} Y_i \xrightarrow{\partial^Y_i} Y_{i-1} \xrightarrow{\partial^Y_{i-1}} \cdots \end{array}$$

An isomorphism from X to Y, denoted $X \cong Y$, is a chain map $F : X \to Y$ such that each map $F_i : X_i \to Y_i$ is an isomorphism. If $F : X \to Y$ is a chain map, then the induced map $\mathrm{H}_i(F) : \mathrm{H}_i(X) \to \mathrm{H}_i(Y)$ given by $\mathrm{H}_i(F)(\overline{x}) := \overline{F_i(x)}$ is a well-defined R-module homomorphism. A *quasiisomorphism* from X to Y, denoted $X \xrightarrow{\simeq} Y$, is a chain map $F : X \to Y$ such that the induced map $\mathrm{H}_i(F) : \mathrm{H}_i(X) \to \mathrm{H}_i(Y)$ is an isomorphism for all i.

Example 2.1.5. Let X be an R-complex and let $r \in R$. The *homothety* map $\mu^X_r : X \to X$ given by $\mu^X_r(x) := rx$ is a chain map.

Fact 2.1.6. Let X, Y be R-complexes. If $F : X \to Y$ is an isomorphism, then the induced map on homology $\mathrm{H}_i(F)$ is an isomorphism for all i. Hence F is a quasiisomorphism.

Remark 2.1.7. The converse to Fact 2.1.6 fails in general. This is the motivation for the derived category in which we "force" all quasiisomorphisms to be isomorphisms.

The following boundedness conditions are important for many results and constructions.

Definition 2.1.8. An R-complex X is *bounded above*, *bounded below*, or *bounded* if $X_i = 0$ for all $i \gg 0$, $i \ll 0$, or $|i| \gg 0$, respectively. An R-complex X is *homologically bounded above*, *homologically bounded below*, or *homologically bounded* if $\mathrm{H}_i(X) = 0$ for all $i \gg 0$, $i \ll 0$, or $|i| \gg 0$, respectively. An R-complex X is *homologically finite* if X is homologically bounded and $\mathrm{H}_i(X)$ is finitely generated for all i.

Definition 2.1.9. The *supremum*, *infimum*, and *amplitude* of an R-complex X are

$$\sup(X) := \sup\{i \in \mathbb{Z} \mid \mathrm{H}_i(X) \neq 0\}$$

$$\inf(X) := \inf\{i \in \mathbb{Z} \mid \mathrm{H}_i(X) \neq 0\}$$

$$\mathrm{amp}(X) := \sup(X) - \inf(X).$$

Fact 2.1.10. The category of R-complexes $\mathcal{C}(R)$ is the category where the objects are the R-complexes and the morphisms are the chain maps.

The next constructions are central to our main result.

Definition 2.1.11. Let X and Y be R-complexes.

(1) The *homomorphism complex* $\operatorname{Hom}_R(X,Y) \in \mathcal{C}(R)$ in degree n is defined as

$$\operatorname{Hom}_R(X,Y)_n := \prod_{p\in\mathbb{Z}} \operatorname{Hom}_R(X_p, Y_{p+n})$$

with $\partial_n^{\operatorname{Hom}_R(X,Y)}(\{f_p\}) := \{\partial_{p+n}^Y f_p - (-1)^n f_{p-1}\partial_p^X\}$.

(2) The *tensor product complex* $X \otimes_R Y \in \mathcal{C}(R)$ in degree n is defined as

$$(X \otimes_R Y)_n := \coprod_{p\in\mathbb{Z}} X_p \otimes_R Y_{n-p}$$

where $\partial_n^{X\otimes_R Y}$ is given on a generator $x_p \otimes y_{n-p} \in (X \otimes_R Y)_n$ by

$$\partial_n^{X\otimes_R Y}(x_p \otimes y_{n-p}) := \partial_p^X(x_p) \otimes y_{n-p} + (-1)^p x_p \otimes \partial_{n-p}^Y(y_{n-p}).$$

Example 2.1.12. Let X be an R-complex. The *homothety morphism* $\chi_X^R : R \to \operatorname{Hom}_R(X,X)$, given in degree i by $\chi_X^R(r)_i(x) = rx$, is a morphism in $\mathcal{C}(R)$.

Fact 2.1.13. Let X, Y, and Z be R-complexes, and let $F : X \to Y$ be a chain map. Then there are well-defined induced chain maps

(a) $\operatorname{Hom}_R(Z,F) : \operatorname{Hom}_R(Z,X) \to \operatorname{Hom}_R(Z,Y)$ defined by $\{f_p\} \mapsto \{F_{p+|f|}f_p\}$,

(b) $\operatorname{Hom}_R(F,X) : \operatorname{Hom}_R(Y,Z) \to \operatorname{Hom}_R(X,Z)$ defined by $\{f_p\} \mapsto \{f_pF_p\}$,

(c) $Z \otimes_R F : Z \otimes_R X \to Z \otimes_R Y$ defined by $z_p \otimes x_q \mapsto z_p \otimes F_q(x_q)$, and

(d) $F \otimes_R Z : X \otimes_R Z \to Y \otimes_R Z$ defined by $x_p \otimes z_q \mapsto F_p(x_p) \otimes z_q$.

The following isomorphisms are very useful for our work.

Fact 2.1.14. Let X, Y and Z be R-complexes. There are natural isomorphisms

(a) *Hom cancellation*: $\operatorname{Hom}_R(R, X) \cong X$,

(b) *Tensor cancellation*: $R \otimes_R X \cong X$,

(c) *Commutativity of tensor product*: $X \otimes_R Y \cong Y \otimes_R X$,

(d) *Associativity*: $X \otimes_R (Y \otimes_R Z) \cong (X \otimes_R Y) \otimes_R Z$, and

(e) *Adjointness*: $\operatorname{Hom}_R(X \otimes_R Y, Z) \cong \operatorname{Hom}_R(X, \operatorname{Hom}_R(Y, Z))$.

Resolutions and associated homological dimensions are central to this dissertation.

Definition 2.1.15. Let X be an R-complex.

(1) A *projective resolution* of X is a quasiisomorphism $P \xrightarrow{\simeq} X$, such that P is a bounded below R-complex of projective R-modules.

(2) A *flat resolution* of X is a quasiisomorphism $F \xrightarrow{\simeq} X$ such that F is a bounded below R-complex of flat R-modules.

(3) A *injective resolution* of X is a quasiisomorphism $X \xrightarrow{\simeq} I$ such that I is a bounded above R-complex of injective R-modules.

Remark 2.1.16. Let M be an R-module with projective resolution

$$P^+ = \left(\cdots \xrightarrow{\partial_3^P} P_2 \xrightarrow{\partial_2^P} P_1 \xrightarrow{\partial_1^P} P_0 \xrightarrow{\pi} M \to 0 \right).$$

Set P to be the *truncated projective resolution*

$$P = \left(\cdots \xrightarrow{\partial_3^P} P_2 \xrightarrow{\partial_2^P} P_1 \xrightarrow{\partial_1^P} P_0 \to 0 \right).$$

It is straightforward to show that the following diagram

$$\begin{tikzcd}
P \arrow[d, "\simeq"'] & \cdots \arrow[r, "\partial_3^P"] \arrow[d] & P_2 \arrow[r, "\partial_2^P"] \arrow[d] & P_1 \arrow[r, "\partial_1^P"] \arrow[d] & P_0 \arrow[r] \arrow[d, "\pi"'] & 0 \arrow[d] \\
M & \cdots \arrow[r] & 0 \arrow[r] & 0 \arrow[r] & M \arrow[r] & 0
\end{tikzcd}$$

is a projective resolution of M, as in Definition 2.1.15.

This shows that projective resolutions of an R-module, as defined in the introduction, give projective resolutions as in Definition 2.1.15. Similarly, it can be shown that flat and injective resolutions, as defined in the introduction, give flat and injective resolutions as in Definition 2.1.15.

Fact 2.1.17. Let X and Y be R-complexes.

(a) The complex X is homologically bounded below if and only if X has a projective resolution (equivalently, a flat resolution).

(b) The complex Y is homologically bounded above if and only if Y has an injective resolution.

Definition 2.1.18. Let X be a homologically bounded below complex and let Y be a homologically bounded above complex.

(1) The *projective dimension* of X is defined as

$$\mathrm{pd}_R(X) := \inf\{\sup\{i \in \mathbb{Z} \mid P_i \neq 0\} \mid P \xrightarrow{\simeq} X \text{ is a projective resolution}\}.$$

(2) The *flat dimension* of X is defined as

$$\mathrm{fd}_R(X) := \inf\{\sup\{i \in \mathbb{Z} \mid F_i \neq 0\} \mid F \xrightarrow{\simeq} X \text{ is a flat resolution}\}.$$

(3) The *injective dimension* of Y is defined as

$$\mathrm{id}_R(X) := \inf\{\sup\{i \in \mathbb{Z} \mid I_{-i} \neq 0\} \mid X \xrightarrow{\simeq} I \text{ is an injective resolution}\}.$$

The following definition is important to understand "triangles" in the derived category. It is also used to define the Koszul complex which is an important object in Section 3.3.

Definition 2.1.19. Let X, Y be R-complexes and let $f : X \to Y$ be a chain map. The *mapping* *cone* of f is the sequence $\mathrm{Cone}(f)$ defined as

$$\mathrm{Cone}(f) := \cdots \to \begin{array}{c} Y_i \\ \oplus \\ X_{i-1} \end{array} \xrightarrow{\begin{pmatrix} \partial_i^Y & f_{i-1} \\ 0 & -\partial_{i-1}^X \end{pmatrix}} \begin{array}{c} Y_{i-1} \\ \oplus \\ X_{i-2} \end{array} \xrightarrow{\begin{pmatrix} \partial_{i-1}^Y & f_{i-2} \\ 0 & -\partial_{i-2}^X \end{pmatrix}} \begin{array}{c} Y_{i-2} \\ \oplus \\ X_{i-3} \end{array} \to \cdots$$

Fact 2.1.20. Let X, Y be R-complexes and let $f : X \to Y$ be a chain map.

(a) The sequence $\mathrm{Cone}(f)$ is an R-complex.

(b) There exists an exact sequence

$$0 \to Y \xrightarrow{\iota} \mathrm{Cone}(f) \xrightarrow{\pi} \Sigma X \to 0$$

where ι and π are the natural injection and surjection morphisms.

(c) The chain map f is a quasiisomorphism if and only if $\mathrm{Cone}(f)$ is exact.

Definition 2.1.21. Let M be an R-module and let $\mathbf{x} = x_1, \ldots, x_n \in R$. The *Koszul complex* $K(\mathbf{x}; M)$ is built inductively on n.

Base case: $n = 1$.

$$K(x_1; M) := \left(0 \to M \xrightarrow{\mu_{x_1}^M} M \to 0\right) \cong \mathrm{Cone}(\mu_{x_1}^M)$$

where $\mu_{x_1}^M(m) = x_1 m$ is the homothety map.

Inductive step: Assume $n \geqslant 2$ and that $K^{n-1} := K(x_1, \ldots, x_{n-1})$ has been constructed. Let $\mu_{x_n}^{K^{n-1}} : K^{n-1} \to K^{n-1}$ be the homothety map and set

$$K(\mathbf{x}; M) := K(x_1, \ldots, x_{n-1}, x_n) := \mathrm{Cone}(\mu_{x_n}^{K^{n-1}}).$$

Also, set $K(\mathbf{x}) := K(\mathbf{x}; R)$.

Fact 2.1.22. Let $\mathbf{x} = x_1, \ldots, x_n \in R$. The Koszul complex $K(\mathbf{x}; M)$ is an R-complex for any R-module M.

Fact 2.1.23. Let $\mathbf{x} = x_1, \ldots, x_n \in R$. Then $\mathrm{pd}_R(K(\mathbf{x})) < \infty$.

The Koszul complex is an important tool that we use in the proof of Theorem 1.2.3 from the introduction.

2.2. The Homotopy Category

The homotopy category is built from the category of R-complexes, and is a tool to build the derived category.

Definition 2.2.1. Let X, Y be R-complexes and let $f : X \to Y$ be a chain map.

(1) f is *null homotopic* or *homotopic to 0* (denoted $f \sim 0$) if for all i there exists $s_i : X_i \to Y_{i+1}$ such that $f_i = \partial^Y_{i+1} s_i + s_{i-1} \partial^X_i$. A *homotopy* between f and 0 is $s = \{s_i\}$.

$$
\begin{tikzcd}
X \arrow[d, "f"'] & \cdots \arrow[r, "\partial^X_{i+2}"] & X_{i+1} \arrow[r, "\partial^X_{i+1}"] \arrow[d, "f_{i+1}"'] & X_i \arrow[r, "\partial^X_i"] \arrow[d, "f_i"] \arrow[dl, dotted, "s_i"] & X_{i-1} \arrow[r, "\partial^X_{i-1}"] \arrow[d, "f_{i-1}"] \arrow[dl, dotted, "s_{i-1}"] & \cdots \\
Y & \cdots \arrow[r, "\partial^Y_{i+2}"'] & Y_{i+1} \arrow[r, "\partial^Y_{i+1}"'] & Y_i \arrow[r, "\partial^Y_i"'] & Y_{i-1} \arrow[r, "\partial^Y_{i-1}"'] & \cdots
\end{tikzcd}
$$

(2) Let $g : X \to Y$ be a chain map. We say that f and g are *homotopic* (denoted $f \sim g$) if $f - g \sim 0$.

(3) We say that f is a *homotopy equivalence* if there exists a chain map $h : Y \to X$ such that $fh \sim \mathrm{id}_Y$ and $hf \sim \mathrm{id}_X$.

Fact 2.2.2. Let X, Y be R-complexes and let $f, g : X \to Y$ be chain maps such that $f \sim g$.

(a) For all $i \in \mathbb{Z}$, one has $\mathrm{H}_i(f) = \mathrm{H}_i(g)$.

(b) f is a quasiisomorphism if and only if g is a quasiisomorphism.

(c) If f is a homotopy equivalence, then $\mathrm{H}_i(f)$ is an isomorphism for all i, i.e., f is a quasiisomorphism.

(d) "Homotopy equivalent" is an equivalence relation on the class of all R-complexes.

Notation 2.2.3. Let $\mathcal{X}$ be a category. Let $\mathcal{M}_{\mathcal{X}}(X, Y)$ denote the set of all morphisms $X \to Y$ in the category $\mathcal{X}$.

Definition 2.2.4. The *homotopy category* $\mathcal{K}(R)$ is the category with objects the R-complexes and morphism sets $\mathcal{M}_{\mathcal{K}(R)}(X,Y) := \mathcal{M}_{\mathcal{C}(R)}(X,Y)/\,\mathrm{I}(X,Y) = \mathrm{H}_0(\mathrm{Hom}_R(X,Y))$ where

$$\mathrm{I}(X,Y) := \{\text{chain maps } f : X \to Y \mid f \sim 0\}.$$

Remark 2.2.5. The *homologically bounded above homotopy category* $\mathcal{K}_{(\sqsubset)}(R)$ and *homologically bounded below homotopy category* $\mathcal{K}_{(\sqsupset)}(R)$ are defined similarly with homologically bounded above and homologically bounded below R-complexes, respectively, as the objects.

Fact 2.2.6. Let $f, g \in \mathcal{M}_{\mathcal{C}(R)}(X,Y)$ with image $\overline{f} \in \mathcal{M}_{\mathcal{K}(R)}(X,Y)$.

(a) $\overline{f}$ is an isomorphism in $\mathcal{K}(R)$ if and only if f is a homotopy equivalence in $\mathcal{C}(R)$.

(b) $\overline{f} = 0$ if and only if $f \sim 0$ in $\mathcal{C}(R)$.

(c) $\overline{f} = \overline{g}$ in $\mathcal{K}(R)$ if and only if $f \sim g$ in $\mathcal{C}(R)$.

Unlike the category of R-complexes, the homotopy category is *not* abelian. However, it does have an important structure, which we describe next.

Definition 2.2.7. A category $\mathcal{X}$ is *additive* if $\mathcal{M}_{\mathcal{X}}(X,Y)$ is an additive abelian group for all objects X, Y such that composition respects addition.

Definition 2.2.8. Let $\mathcal{X}$ be an additive category, and assume that $\mathcal{X}$ has an equivalence $\Sigma : \mathcal{X} \to \mathcal{X}$ with a quasi-inverse Σ^{-1}. A *triangle* in $\mathcal{X}$ is a sequence of morphisms

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$$

in $\mathcal{X}$. A *morphism of triangles* in $\mathcal{X}$ is a commutative diagram

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "g"'] & Y \arrow[r, "\beta"] \arrow[d, "f"'] & Z \arrow[r, "\gamma"] \arrow[d, "h"'] & \Sigma X \arrow[d, "\Sigma f"'] \\
X' \arrow[r, "\alpha'"] & Y' \arrow[r, "\beta'"] & Z' \arrow[r, "\gamma'"] & \Sigma X'
\end{tikzcd}$$

where X, Y, Z, X', Y' and Z' are all objects in $\mathcal{X}$.

Definition 2.2.9. An additive category $\mathcal{X}$ is *triangulated* if there exists a distinguished class of triangles (*distinguished triangles* or *exact triangles*) satisfying the following axioms:

TR 1. If two triangles are isomorphic triangles in $\mathcal{X}$ such that one of the triangles is distinguished, then the other triangle is distinguished as well. For any object X in $\mathcal{X}$, the following triangle is distinguished

$$0 \to X \xrightarrow{\mathrm{id}_X} X \to 0.$$

For any morphisms $\alpha : X \to Y$ in $\mathcal{X}$, there exists a distinguished triangle

$$X \xrightarrow{\alpha} Y \to Z \to \Sigma X.$$

TR 2. The triangle $X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$ is distinguished if and only if the *shifted* triangle

$$Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X \xrightarrow{\Sigma\alpha} \Sigma Y$$

is distinguished.

TR 3. Given two distinguished triangles in $\mathcal{X}$ such that the leftmost square in the following diagram commutes in $\mathcal{X}$

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "g"'] & Y \arrow[r, "\beta"] \arrow[d, "f"'] & Z \arrow[r, "\gamma"] & \Sigma X \arrow[d, "\Sigma f"'] \\
X' \arrow[r, "\alpha'"] & Y' \arrow[r, "\beta'"] & Z' \arrow[r, "\gamma'"] & \Sigma X'
\end{tikzcd}$$

there is a morphism $h : Z \to Z'$ making all squares in the following diagram commute in $\mathcal{X}$

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "g"'] & Y \arrow[r, "\beta"] \arrow[d, "f"'] & Z \arrow[r, "\gamma"] \arrow[d, dashed, "\exists\, h"'] & \Sigma X \arrow[d, "\Sigma f"'] \\
X' \arrow[r, "\alpha'"] & Y' \arrow[r, "\beta'"] & Z' \arrow[r, "\gamma'"] & \Sigma X'
\end{tikzcd}$$

TR 4. *Octahedral Axiom.* If the following triangles are distinguished in $\mathcal{X}$

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma} \Sigma X$$

$$X \xrightarrow{a} U \xrightarrow{b} V \xrightarrow{c} \Sigma Z$$

$$Y \xrightarrow{f} U \xrightarrow{g} W \xrightarrow{h} \Sigma Y$$

there exists a distinguished triangle $Z \xrightarrow{r} V \xrightarrow{s} W \xrightarrow{t} \Sigma Z$ making the following "octahedron" diagram commutes in $\mathcal{X}$.

$$\begin{tikzcd}
X \arrow[r, "\alpha"] \arrow[d, "="'] & Y \arrow[r, "\beta"] \arrow[d, "f"'] & Z \arrow[r, "\gamma"] \arrow[d, dashed, "r"'] & \Sigma X \arrow[d, "="'] \\
X \arrow[r, "a"] & U \arrow[r, "b"] \arrow[d, "g"'] & V \arrow[r, "c"] \arrow[d, dashed, "s"'] & \Sigma X \arrow[d, "\Sigma\alpha"'] \\
 & W \arrow[r, "="] \arrow[d, "h"'] & W \arrow[r, "h"] \arrow[d, dashed, "t"'] & \Sigma Y \\
 & \Sigma Y \arrow[r, "\Sigma b"] & \Sigma Z &
\end{tikzcd}$$

Remark 2.2.10. The commutative diagram in the octahedral axiom is called an octahedron because the four distinguished triangles can be arranged in such a way to create an octahedron.

Fact 2.2.11. The category $\mathcal{K}(R)$ is triangulated where the distinguished triangles are triangles in $\mathcal{K}(R)$ that are isomorphic to triangles of the form

$$X \xrightarrow{\overline{\alpha}} Y \xrightarrow{\overline{\iota}} \text{Cone}(\alpha) \xrightarrow{\overline{\pi}} \Sigma X$$

where α is a morphism in the category of R-complexes, i.e., a chain map, and ι and π are the chain maps from Definition 2.1.19.

Definition 2.2.12. Let $X, Y \in \mathcal{K}_{(\sqsubset)}(R)$. An *inj-diagram* from X to Y is a sequence $X \xrightarrow{\overline{\alpha}} U \xleftarrow{\overline{\beta}} Y$ where $\overline{\beta}$ is a quasiisomorphism.

Remark 2.2.13. The term *inj-diagram* is not a standard term in the literature. We use it here for convenience of notation.

Definition 2.2.14. Let $X, Y \in \mathcal{K}_{(\sqsubset)}(R)$. Two inj-diagrams $X \to U \xleftarrow{\simeq} Y$ and $X \to U' \xleftarrow{\simeq} Y$ are equivalent if there exists an R-complex U'' such that the following diagram commutes in $\mathcal{K}(R)$

$$
\begin{tikzcd}
 & U \arrow[d, "\simeq"'] & \\
X \arrow[ur] \arrow[r] \arrow[dr] & U'' & Y \arrow[ul, "\simeq"'] \arrow[l, "\simeq"'] \arrow[dl, "\simeq"] \\
 & U' . \arrow[u, "\simeq"] &
\end{tikzcd}
$$

Fact 2.2.15. The equivalence defined in Definition 2.2.14 is an equivalence relation on the class of all inj-diagrams from X to Y.

2.3. The Derived Category

We are now in a position to define the derived category. The derived category was first constructed by Verdier and Grothendieck [30]. Loosely, the derived category is defined by formally localizing the homotopy category by formally inverting all the quasiisomorphisms.

Definition 2.3.1. The *derived category* of R is the category $\mathcal{D}(R)$ in which the objects are the R-complexes and the morphisms $X \to Y$ are defined to be equivalence classes of inj-diagrams from X to Y. Let $X \to Y$ be a morphism in $\mathcal{D}(R)$ represented by the inj-diagram $X \xrightarrow{\overline{\alpha}} U \underset{\simeq}{\xleftarrow{\overline{\beta}}} Y$. We denote this morphism $X \to Y$ as $\overline{\alpha}/\overline{\beta}$. The *bounded above derived category* of R is the full subcategory $\mathcal{D}_{-}(R)$ of $\mathcal{D}(R)$ whose objects are homologically bounded above R-complexes. The *bounded derived category* of R is the full subcategory $\mathcal{D}_{\mathrm{b}}(R)$ of $\mathcal{D}(R)$ whose objects are the homologically bounded R-complexes.

Let $X \xrightarrow{\overline{\alpha}/\overline{\beta}} Y \xrightarrow{\overline{\gamma}/\overline{\delta}} Z$ be morphisms in $\mathcal{D}(R)$, represented by inj-diagrams $X \xrightarrow{\overline{\alpha}} V \underset{\simeq}{\xleftarrow{\overline{\beta}}} Y$ and $Y \xrightarrow{\overline{\gamma}} U \underset{\simeq}{\xleftarrow{\overline{\delta}}} Z$. If there is a commutative diagram

$$
\begin{tikzcd}
X \arrow[r, "\overline{\alpha}"] & V \arrow[dr, "\overline{\varpi}"'] & Y \arrow[l, "\overline{\beta}"', "\simeq"] \arrow[r, "\overline{\gamma}"] & U \arrow[dl, "\overline{\sigma}", "\simeq"'] & Z \arrow[l, "\overline{\delta}"', "\simeq"] \\
 & & W & &
\end{tikzcd}
$$

in $\mathcal{K}(R)$, then the composition $X \xrightarrow{(\overline{\gamma}/\overline{\delta})\circ(\overline{\alpha}/\overline{\beta})} Z$ is defined to be $(\overline{\varpi\alpha})/(\overline{\sigma\delta})$, the morphism represented by the inj-diagram $X \xrightarrow{\overline{\varpi\alpha}} W \underset{\simeq}{\xleftarrow{\overline{\sigma\delta}}} Z$.

The existence of compositions of morphisms in $\mathcal{D}(R)$ is a subtle point. This is explained in Fact 2.3.3, which uses the following "Lifting Lemma". It is standard knowledge in the area, but we do not of know a good reference, so we include a proof here.

Fact 2.3.2. Let I be a bounded above complex of injective R-modules. For every quasiisomorphism $\alpha : X \to Y$ and for every chain map $\beta : X \to Y$, there exists a chain map $\gamma : Y \to I$ such that $\gamma\alpha \sim \beta$. Moreover, γ is unique up to homotopy.

Proof: Since I is a bounded above complex of injective R-modules and α is a quasiisomorphism, the induced chain map $\mathrm{Hom}_R(\alpha, I) : \mathrm{Hom}_R(Y, I) \to \mathrm{Hom}_R(X, I)$ is also a quasiisomorphism. In particular, the induced map

$$\mathrm{H}_0(\mathrm{Hom}_R(\alpha, I)) : \mathrm{H}_0(\mathrm{Hom}_R(Y, I)) \to \mathrm{H}_0(\mathrm{Hom}_R(X, I))$$

is an isomorphism. Given the description of $\mathrm{H}_0(\mathrm{Hom}_R(Y, I))$ and $\mathrm{H}_0(\mathrm{Hom}_R(X, I))$ from Definition 2.2.4, the surjectivity of this map explains the existence of γ, and the injectivity of the map explains the uniqueness. $\square$

Fact 2.3.3. Let $X \xrightarrow{\overline{\alpha}/\overline{\beta}} Y \xrightarrow{\overline{\gamma}/\overline{\delta}} Z$ be morphisms in $\mathcal{D}(R)$, represented by inj-diagrams $X \xrightarrow{\overline{\alpha}} V \underset{\simeq}{\xleftarrow{\overline{\beta}}} Y$ and $Y \xrightarrow{\overline{\gamma}} U \underset{\simeq}{\xleftarrow{\overline{\delta}}} Z$. Assume that U has an injective resolution $U \underset{\simeq}{\xrightarrow{\sigma}} J$. Fact 2.3.2 implies that there is a chain map $\omega : V \to J$ making the following diagram commute in $\mathcal{K}(R)$.

$$\begin{tikzcd} X \arrow[r, "\overline{\alpha}"] & V \arrow[dr, "\overline{\omega}"'] & Y \arrow[l, "\overline{\beta}", "\simeq"'] \arrow[r, "\overline{\gamma}"] & U \arrow[dl, "\simeq"', "\overline{\sigma}"] & Z \arrow[l, "\overline{\delta}", "\simeq"'] \\ & & J & & \end{tikzcd}$$

It follows that the composition $X \xrightarrow{(\overline{\gamma}/\overline{\delta})\circ(\overline{\alpha}/\overline{\beta})} Z$ is defined in this situation. The composition also exists in general, though this requires a more general notion of injective resolutions.

The minus sign in $\mathcal{D}_-(R)$ is meant to suggest that the complexes in this category live mostly in negative degrees.

Remark 2.3.4. The *bounded below derived category* of R, denoted $\mathcal{D}_+(R)$, can be constructed in a similar way using bounded below R-complexes as the objects. However, the morphisms from $X \to Y$ are defined to be equivalence classes of diagrams of chain maps of the form $X \xleftarrow{\simeq} U \to Y$.

Notation 2.3.5. The full subcategory of $\mathcal{D}(R)$ of homologically degreewise finite R-complexes is denoted $\mathcal{D}^{\mathrm{f}}(R)$, i.e., $X \in \mathcal{D}^{\mathrm{f}}(R)$ if and only if $\mathrm{H}_i(X)$ is finitely generated for all i. Let $* \in \{-,+,\mathrm{b}\}$. Then we write $\mathcal{D}^{\mathrm{f}}_*(R) = \mathcal{D}^{\mathrm{f}}(R) \cap \mathcal{D}_*(R)$.

Notation 2.3.6. Let $\mathcal{P}(R)$, $\mathcal{F}(R)$, and $\mathcal{I}(R)$ denote the full subcategories of $\mathcal{D}_{\mathrm{b}}(R)$ consisting of complexes of finite projective, flat, and injective dimensions, respectively.

Fact 2.3.7. There is a natural functor $\mathcal{F} : \mathcal{K}(R) \to \mathcal{D}(R)$ given on objects by $\mathcal{F}(X) = X$ and morphisms by $\mathcal{F}(\overline{g}) = \overline{g}/\overline{1}$.

Definition 2.3.8. Let $X \in \mathcal{D}_+(R)$, $Y \in \mathcal{D}(R)$, and let $P \xrightarrow{\simeq} X$ be a projective resolution of X.

(1) The *right derived functor* $\mathbf{R}\mathrm{Hom}_R(X,Y)$ is defined as $\mathbf{R}\mathrm{Hom}_R(X,Y) := \mathrm{Hom}_R(P,Y)$.

(2) The *left derived functor* $X \otimes^{\mathbf{L}}_R Y$ is defined as $X \otimes^{\mathbf{L}}_R Y := P \otimes_R Y$.

Remark 2.3.9. Let M and N be R-modules. Then $\mathrm{H}_{-i}(\mathbf{R}\mathrm{Hom}_R(M,N)) \cong \mathrm{Ext}^i_R(M,N)$ and $\mathrm{H}_i(M \otimes^{\mathbf{L}}_R N) \cong \mathrm{Tor}^R_i(M,N)$.

Remark 2.3.10. The functors $\mathbf{R}\mathrm{Hom}_R(-,-)$ and $- \otimes^{\mathbf{L}}_R -$ can be defined using complexes from $\mathcal{D}(R)$. However, projective resolutions of homologically unbounded complexes are more technical to define and construct.

Fact 2.3.11 (Balance)**.** Let $X \in \mathcal{D}_+(R)$, $Y \in \mathcal{D}(R)$. Let $P \xrightarrow{\simeq} X$ a projective resolution.

(a) If Y has an injective resolution $Y \xrightarrow{\simeq} J$, then

$$\mathbf{R}\mathrm{Hom}_R(X,Y) = \mathrm{Hom}_R(P,Y) \simeq \mathrm{Hom}_R(P,J) \simeq \mathrm{Hom}_R(X,J).$$

(b) If Y has a flat resolution $F \xrightarrow{\simeq} Y$, then

$$X \otimes^{\mathbf{L}}_R Y = P \otimes_R Y \simeq P \otimes_R F \simeq X \otimes_R F.$$

The next few facts describe how these functors restrict to important subcategories.

Fact 2.3.12 (Boundedness)**.** (a) We have that the bi-functor $- \otimes^{\mathbf{L}}_R -$ restricts to a bi-functor $\mathcal{D}_+(R) \times \mathcal{D}_+(R) \to \mathcal{D}_+(R)$.

(b) We have that the bi-functor $\mathbf{R}\mathrm{Hom}_R(-,-)$ restricts to a bi-functor $\mathcal{D}_+(R) \times \mathcal{D}_-(R) \to \mathcal{D}_-(R)$.

Fact 2.3.13 (Finiteness)**.** (a) We have that the bi-functor $\mathbf{R}\mathrm{Hom}_R(-,-)$ restricts to a bi-functor $\mathcal{D}^{\mathrm{f}}_+ \times \mathcal{D}^{\mathrm{f}}_-(R) \to \mathcal{D}^{\mathrm{f}}_-(R)$.

(b) We have that the bi-functor $- \otimes^{\mathbf{L}}_R -$ restricts to a bi-functor $\mathcal{D}^{\mathrm{f}}_+(R) \times \mathcal{D}^{\mathrm{f}}_+(R) \to \mathcal{D}^{\mathrm{f}}_+(R)$.

Fact 2.3.14. Let $X \in \mathcal{D}_{\mathrm{b}}(R)$ such that $\mathrm{pd}_R(X) < \infty$.

(a) The functor $X \otimes^{\mathbf{L}}_R -$ restricts to a functor $\mathcal{D}_{\mathrm{b}}(R) \to \mathcal{D}_{\mathrm{b}}(R)$.

(b) The functor $X \otimes^{\mathbf{L}}_R -$ restricts to a functor $\mathcal{D}_-(R) \to \mathcal{D}_-(R)$.

(c) The functor $\mathbf{R}\mathrm{Hom}_R(X,-)$ restricts to a functor $\mathcal{D}_{\mathrm{b}}(R) \to \mathcal{D}_{\mathrm{b}}(R)$.

(d) The functor $\mathbf{R}\mathrm{Hom}_R(X,-)$ restricts to a functor $\mathcal{D}_+(R) \to \mathcal{D}_+(R)$.

We use the next isomorphsims frequently in this work.

Fact 2.3.15. Let $X, Y, Z \in \mathcal{D}(R)$. There are natural isomorphisms in $\mathcal{D}(R)$:

(a) *Hom-cancellation*: $\mathbf{R}\mathrm{Hom}_R(R, X) \simeq X$,

(b) *Tensor-cancellation*: $R \otimes^{\mathbf{L}}_R X \simeq X$,

(c) *Commutativity of tensor product*: $X \otimes^{\mathbf{L}}_R Y \simeq Y \otimes^{\mathbf{L}}_R X$,

(d) *Associativity of tensor product*: $X \otimes^{\mathbf{L}}_R (Y \otimes^{\mathbf{L}}_R Z) \simeq (X \otimes^{\mathbf{L}}_R Y) \otimes^{\mathbf{L}}_R Z$, and

(e) *Adjointness*: $\mathbf{R}\mathrm{Hom}_R(X \otimes^{\mathbf{L}}_R Y, Z) \simeq \mathbf{R}\mathrm{Hom}_R(X, \mathbf{R}\mathrm{Hom}_R(Y, Z))$.

Fact 2.3.16 ([4, Lemma 4.4])**.** Let $L, M, N \in \mathcal{D}(R)$. Assume that $L \in \mathcal{D}^{\mathrm{f}}_+(R)$.

The natural *tensor-evaluation* morphism

$$\omega_{LMN} : \mathbf{R}\mathrm{Hom}_R(L, M) \otimes^{\mathbf{L}}_R N \to \mathbf{R}\mathrm{Hom}_R(L, M \otimes^{\mathbf{L}}_R N)$$

is an isomorphism when $M \in \mathcal{D}_-(R)$ and either $L \in \mathcal{P}(R)$ or $N \in \mathcal{F}(R)$.

The natural *Hom-evaluation* morphism

$$\theta_{LMN} : L \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(M, N) \to \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(L, M), N)$$

is an isomorphism when $M \in \mathcal{D}_{\mathrm{b}}(R)$ and either $L \in \mathcal{P}(R)$ or $N \in \mathcal{I}(R)$.

The next few results are tools for detecting (finiteness of) homological dimensions.

Fact 2.3.17 ([4, Theorem 2.4])**.** Let $X \in \mathcal{D}_{\mathrm{b}}(R)$.

(a) If $n = \mathrm{pd}_R(X) < \infty$ and $P \xrightarrow{\simeq} X$ is any projective resolution of X, then $n \geqslant \sup(X)$, and for all $j \geqslant n$, the module $\mathrm{Coker}(\partial^P_{j+1})$ is projective.

(b) If $n = \mathrm{fd}_R(X) < \infty$ and $F \xrightarrow{\simeq} X$ is any flat resolution of X, then $n \geqslant \sup(X)$, and for all $j \geqslant n$, the module $\mathrm{Coker}(\partial^F_{j+1})$ is flat.

(c) If $m = \mathrm{id}_R(X) < \infty$ and $X \xrightarrow{\simeq} I$ is any injective resolution of X, then $m \geqslant -\inf(X)$, and for all $j \geqslant -n$, the module $\mathrm{Ker}(\partial^I_j)$ is injective.

Fact 2.3.18 ([4, Proposition 4.5])**.** Let $X, Y \in \mathcal{D}(R)$.

(a) If $\mathrm{id}_R(Y) < \infty$, then $\mathrm{fd}_R(\mathbf{R}\mathrm{Hom}_R(X,Y)) \leqslant \mathrm{id}_R(X) + \sup(Y)$.

(b) If $\mathrm{fd}_R(Y) < \infty$, then $\mathrm{id}_R(X \otimes^{\mathbf{L}}_R Y) \leqslant \mathrm{id}_R(X) - \inf(Y)$.

The following result is for use in Section 3.2.

Lemma 2.3.19. *Let* $X \in \mathcal{D}_{\mathrm{b}}(R)$*.*

(a) *If E is a faithfully injective R-module, then* $\mathrm{id}_R(\mathbf{R}\mathrm{Hom}_R(X,E)) = \mathrm{fd}_R(X)$*.*

(b) *If F is a faithfully flat R-module, then* $\mathrm{fd}_R(X \otimes^{\mathbf{L}}_R F) = \mathrm{fd}_R(X)$*.*

(c) *If E is a faithfully injective R-module, then* $\mathrm{id}_R(X) = \mathrm{fd}_R(\mathbf{R}\mathrm{Hom}_R(X,E))$*.*

(d) *If F is a faithfully flat R-module, then* $\mathrm{id}_R(X) = \mathrm{id}_R(X \otimes^{\mathbf{L}}_R E)$*.*

Proof: (a) The inequality ($\leqslant$) follows from [4, Theorem 4.1(I)]. For the other inequality assume that $\mathrm{id}_R(\mathbf{R}\mathrm{Hom}_R(X,E)) = n < \infty$ and let $F \xrightarrow{\simeq} X$ be a flat resolution. Note that $n = \mathrm{id}_R(\mathbf{R}\mathrm{Hom}_R(X,E) \geqslant -\inf(\mathbf{R}\mathrm{Hom}_R(X,E)) = \sup(X)$. Since E is injective and F_i is flat, we must have that $\mathrm{Hom}_R(F_i,E)$ is injective for all i. Therefore $\mathrm{Hom}_R(F,E)$ is a complex of injective modules such that $\mathbf{R}\mathrm{Hom}_R(X,E) \simeq \mathrm{Hom}_R(F,E)$.

$$\mathrm{Hom}_R(F,E) = F^* = \left(0 \to F^*_j \xrightarrow{\partial^{F^*}_j} \cdots \xrightarrow{\partial^{F^*}_{-(n-2)}} F^*_{-(n-1)} \xrightarrow{\partial^{F^*}_{-(n-1)}} F^*_{-n} \xrightarrow{\partial^{F^*}_{-n}} F^*_{-(n+1)} \to \cdots\right)$$

where $\partial_{-i}^{F^*} = \mathrm{Hom}_R(\partial_{i+1}^F, E)$ and $F_i^* = \mathrm{Hom}_R(F_i, E)$. It now follows from Fact 2.3.17(c) that $\mathrm{Ker}(\partial_{-n}^{F^*})$ is injective. Observe that there is an isomorphism

$$\mathrm{Ker}(\partial_{-n}^{F^*}) = \mathrm{Ker}(\mathrm{Hom}_R(\partial_{n+1}^F, E)) \cong \mathrm{Hom}_R(\mathrm{Coker}(\partial_{n+1}^F), E).$$

Therefore $\mathrm{Hom}_R(\mathrm{Coker}(\partial_{n+1}^F), E)$ is injective.

Claim: $\mathrm{Coker}(\partial_{n+1}^F)$ is flat. Indeed, let $\mathcal{S}$ be an exact sequence of R-modules and let $M := \mathrm{Coker}(\partial_{n+1}^F)$. Since $\mathrm{Hom}_R(M, E)$ is injective, we have that $\mathrm{Hom}_R(\mathcal{S}, \mathrm{Hom}_R(M, E))$ is exact. By Hom-tensor adjointness we have

$$\mathrm{Hom}_R(\mathcal{S}, \mathrm{Hom}_R(M, E)) \cong \mathrm{Hom}_R(\mathcal{S} \otimes_R M, E).$$

Since E is faithfully injective, it follows that $\mathcal{S} \otimes_R M$ is exact. Therefore M is flat, as claimed.

We conclude that the truncated complex

$$F' = (0 \to M \to F_{n-1} \to \cdots \to F_j \to 0$$

is a flat resolution of X since $n \geqslant \sup(X)$. Thus we have $\mathrm{fd}_R(X) \leqslant n = \mathrm{id}_R(\mathbf{R}\mathrm{Hom}_R(X, E))$, as desired.

The proofs of (b), (c), and (d) are similar. $\square$

Definition 2.3.20. Let $(R, \mathfrak{m}, k)$ be a local ring, the *depth* and *width* of an R-complex $X \in \mathcal{D}(R)$ are defined by Foxby [9] and Yassemi [31] as

$$\mathrm{depth}_R(X) := -\sup(\mathbf{R}\mathrm{Hom}_R(k, X))$$
$$\mathrm{width}_R(X) := \inf(k \otimes_R^{\mathbf{L}} X).$$

Remark 2.3.21. Let $(R, \mathfrak{m}, k)$ be local and let M be an R-module. Then we have $\mathrm{depth}_R(M) := \min\{i \mid \mathrm{Ext}_R^i(k, M) \neq 0\}$. Recall that for all i we have

$$\mathrm{Ext}_R^i(k, M) \cong \mathrm{H}_{-i}(\mathbf{R}\mathrm{Hom}_R(k, M)).$$

It follows readily that $\operatorname{depth}_R(M) = \min\{i \mid \operatorname{Ext}^i_R(k, M) \neq 0\} = -\sup(\mathbf{R}\mathrm{Hom}_R(k, X))$.

In a similar way one can show that

$$\operatorname{width}_R(M) := \min\{i \mid \operatorname{Tor}^R_i(k, M) \neq 0\} = \inf(k \otimes^{\mathbf{L}}_R M).$$

Definition 2.3.22. The *small support* of an R-complex $X \in \mathcal{D}(R)$ is defined as follows:

$$\operatorname{supp}_R(X) := \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_R X \not\simeq 0\}.$$

An important property of the small support is given in the following.

Fact 2.3.23 ([9, Proposition 2.7])**.** If $X, Y \in \mathcal{D}(R)$, then

$$\operatorname{supp}_R(X \otimes^{\mathbf{L}}_R Y) = \operatorname{supp}_R(X) \cap \operatorname{supp}_R(Y).$$

The following facts are used in the proof of Theorem 1.2.3.

Fact 2.3.24 (Künneth Formula [20, Corollary 10.84])**.** Let $(R, \mathfrak{m}, k)$ be a local ring and $X, Y \in \mathcal{D}(R)$. Then

$$\mathrm{H}_n(X \otimes^{\mathbf{L}}_k Y) \cong \bigoplus_{p+q=n} \mathrm{H}_p(X) \otimes_k \mathrm{H}_q(Y).$$

Fact 2.3.25 ([23, Fact 3.4])**.** Let $(R, \mathfrak{m}, k)$ be a local ring and let $\mathbf{x} = x_1, \ldots, x_n$ be a generating sequence for $\mathfrak{m}$. Then $\operatorname{supp}_R(K(\mathbf{x})) = \{\mathfrak{m}\}$.

2.4. Semidualizing Complexes and Gorenstein Rings

Semidualizing complexes originate in work of Grothendieck and Hartshorne [15, 16], Foxby [8], Avramov and Foxby [5], and Christensen [7]. For the non-commutative case, see, e.g., Araya, Takahashi, and Yoshino [2] and Holm and White [17].

Definition 2.4.1. Let $Y \in \mathcal{D}_{\mathrm{b}}(R)$. The *homothety morphism* $\chi^R_Y : R \to \mathbf{R}\mathrm{Hom}_R(Y, Y)$ in $\mathcal{D}(R)$ is the morphism $\overline{\chi^R_J}/\overline{\mathrm{id}}$ represented by the following inj-diagram

$$R \xrightarrow{\overline{\chi^R_J}} \operatorname{Hom}_R(J, J) \xleftarrow{\overline{\mathrm{id}}} \operatorname{Hom}_R(J, J)$$

where $Y \xrightarrow{\simeq} J$ is an injective resolution of Y, and χ^R_J is from Example 2.1.12.

Definition 2.4.2. Let $C \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$. Then C is a *semidualizing complex* if $\chi^R_C : R \to \mathbf{R}\mathrm{Hom}_R(C, C)$ is an isomorphism in $\mathcal{D}(R)$. A complex $D \in \mathcal{D}(R)$ is a *dualizing complex* if D is semidualizing and has finite injective dimension.

Dualizing complexes were introduced by Grothendieck and Hartshorne [15].

Example 2.4.3. The ring R is a semidualizing R-complex. Because R is finitely generated over itself and it is a complex concentrated in degree 0, we have $R \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$. Therefore the only thing to show is that $\chi^R_R : R \to \mathbf{R}\mathrm{Hom}_R(R, R)$ is an isomorphism in $\mathcal{D}(R)$. It is straightforward to show that χ^R_R is the Hom-cancellation isomorphism $R \xrightarrow{\simeq} \mathbf{R}\mathrm{Hom}_R(R, R)$ from Fact 2.3.15. We conclude that χ^R_R is an isomorphism in $\mathcal{D}(R)$.

The following definitions are important for use in Chapter 3. They originate in work of Foxby [8], Avramov and Foxby [5] and Christensen [7].

Definition 2.4.4 (Foxby Classes)**.**

(1) The *Auslander Class* with respect to C is the full subcategory $\mathcal{A}_C(R) \subseteq \mathcal{D}_{\mathrm{b}}(R)$ such that a complex X is in $\mathcal{A}_C(R)$ if and only if $C \otimes^{\mathbf{L}}_R X \in \mathcal{D}_{\mathrm{b}}(R)$ and the natural morphism

$$\gamma^C_X : X \to \mathbf{R}\mathrm{Hom}_R(C, C \otimes^{\mathbf{L}}_R X)$$

is an isomorphism in $\mathcal{D}(R)$.

(2) The *Bass Class* with respect to C is the full subcategory $\mathcal{B}_C(R) \subseteq \mathcal{D}_{\mathrm{b}}(R)$ such that a complex Y is in $\mathcal{B}_C(R)$ if and only if $\mathbf{R}\mathrm{Hom}_R(C, Y) \in \mathcal{D}_{\mathrm{b}}(R)$ and the natural morphism

$$\xi^C_Y : C \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C, Y) \to Y$$

is an isomorphism in $\mathcal{D}(R)$.

For a generalized diagramatic version of the next result, see Theorem 3.1.10. It was first proved for modules by Foxby [8].

Fact 2.4.5 (Foxby Equivalence [7, Theorem 4.6])**.** Let $X, Y \in \mathcal{D}_{\mathrm{b}}(R)$.

(a) One has $X \in \mathcal{A}_C(R)$ if and only if $C \otimes^{\mathbf{L}}_R X \in \mathcal{B}_C(R)$.

(b) One has $Y \in \mathcal{B}_C(R)$ if and only if $\mathbf{R}\mathrm{Hom}_R(C, Y) \in \mathcal{A}_C(R)$.

Note that Theorem 3.1.9 is a generalization of the following result.

Fact 2.4.6 ([7, Proposition 4.4])**.** Let $X \in \mathcal{D}_{\mathrm{b}}(R)$.

(a) If $\mathrm{fd}_R(X) < \infty$ (e.g., $\mathrm{pd}_R(X) < \infty$), then $X \in \mathcal{A}_C(R)$.

(b) If $\mathrm{id}_R(X) < \infty$, then $X \in \mathcal{B}_C(R)$.

Definition 2.4.7. A local ring R is *Gorenstein* if $\mathrm{id}_R(R) < \infty$. A (not necessarily local) ring R is *Gorenstein* if $R_{\mathfrak{p}}$ is a Gorenstein local ring for all $\mathfrak{p} \in \mathrm{Spec}(R)$.

Example 2.4.8. Let k be a field. The ring $k[\![X]\!]$ is a Gorenstein local ring.

Fact 2.4.9 ([7, Corollary 8.6])**.** A local ring is Gorenstein if and only if it has a dualizing complex and the only semidualizing module is the ring itself.

3. HOMOLOGICAL DIMENSIONS AND SEMIDUALIZING COMPLEXES

Throughout this chapter R and S are commutative noetherian rings with identity and C is a semidualizing R-complex.

3.1. C-Homological Dimensions for Complexes

In this section we define the $\mathcal{P}_C$-projective, $\mathcal{F}_C$-projective, and $\mathcal{I}_C$-injective dimensions and build their foundations.

Definition 3.1.1. Let $X \in \mathcal{D}_{\mathrm{b}}(R)$.

(1) The *$\mathcal{P}_C$-projective dimension* of X is defined as

$$\mathcal{P}_C\text{-}\operatorname{pd}_R(X) = \sup(C) + \operatorname{pd}_R(\mathbf{R}\mathrm{Hom}_R(C, X)).$$

(2) The *$\mathcal{F}_C$-projective dimension* of X is defined as

$$\mathcal{F}_C\text{-}\operatorname{pd}_R(X) = \sup(C) + \operatorname{fd}_R(\mathbf{R}\mathrm{Hom}_R(C, X)).$$

(3) The *$\mathcal{I}_C$-injective dimension* of X is defined as

$$\mathcal{I}_C\text{-}\operatorname{id}_R(X) = \sup(C) + \operatorname{id}_R(C \otimes^{\mathbf{L}}_R X).$$

Let $\mathcal{P}_C(R)$, $\mathcal{F}_C(R)$, and $\mathcal{I}_C(R)$ denote the full subcategories of $\mathcal{D}_{\mathrm{b}}(R)$ of all complexes of finite C-projective, C-flat, and C-injective dimension, respectively.

Remark 3.1.2. Let $X \in \mathcal{D}_{\mathrm{b}}(R)$. Observe that $\sup(C) < \infty$. Hence $\mathcal{P}_C\text{-}\operatorname{pd}_R(X) < \infty$ if and only if $\operatorname{pd}_R(\mathbf{R}\mathrm{Hom}_R(C, X)) < \infty$. If $\mathcal{P}_C\text{-}\operatorname{pd}_R(X) < \infty$, then Fact 2.4.6(a) implies that $\mathbf{R}\mathrm{Hom}_R(C, X) \in \mathcal{A}_C(R)$ and Foxby Equivalence (2.4.5) implies that $X \in \mathcal{B}_C(R)$. Similarly, $\mathcal{F}_C\text{-}\operatorname{pd}_R(X) < \infty$ if and only if $\operatorname{fd}_R(\mathbf{R}\mathrm{Hom}_R(C, X)) < \infty$. If $\mathcal{F}_C\text{-}\operatorname{pd}_R(X) < \infty$, then $X \in \mathcal{B}_C(R)$. Also we have $\mathcal{I}_C\text{-}\operatorname{id}_R(X) < \infty$ if and only if $\operatorname{id}_R(C \otimes^{\mathbf{L}}_R X) < \infty$. Hence, if $\mathcal{I}_C\text{-}\operatorname{id}_R(X) < \infty$, then $X \in \mathcal{A}_C(R)$.

Remark 3.1.3. Let $X \in \mathcal{D}_{\mathrm{b}}(R)$. Note that when $C = R$ we have that $\mathcal{P}_C\text{-}\operatorname{pd}_R(X) = \sup(R) + \operatorname{pd}_R(\mathbf{R}\mathrm{Hom}_R(R, X)) = \operatorname{pd}_R(X)$. Similarly in this case $\mathcal{F}_C\text{-}\operatorname{pd}_R(X) = \operatorname{fd}_R(X)$ and $\mathcal{I}_C\text{-}\operatorname{id}_R(X) = \operatorname{id}_R(X)$.

Remark 3.1.4. Let M be an R-module. When C is a semidualizing R-module, Takahashi and White [27, Theorem 2.11], using the definition described in Section 1.2, showed that $\mathcal{P}_C\text{-}\operatorname{pd}_R(X) = \operatorname{pd}_R(\mathbf{R}\mathrm{Hom}_R(C, X))$. Since $\sup(C) = 0$ in this case, Definition 3.1.1(1) shows that our definition is consistent with the one from [27]. In a similar way, it can be shown that $\mathcal{I}_C\text{-}\operatorname{id}$ recovers Takahashi and White's definition in this case.

The next result compares $\mathcal{F}_C\text{-}\operatorname{pd}$ with $\mathcal{P}_C\text{-}\operatorname{pd}$.

Proposition 3.1.5. *Let $X \in \mathcal{D}_{\mathrm{b}}(R)$. Then*

$$\mathcal{F}_C\text{-}\operatorname{pd}_R(X) \leqslant \mathcal{P}_C\text{-}\operatorname{pd}_R(X) \leqslant \mathcal{F}_C\text{-}\operatorname{pd}_R(X) + \dim(R).$$

In particular if $\dim(R) < \infty$, then we have $\mathcal{P}_C\text{-}\operatorname{pd}_R(X) < \infty$ if and only if $\mathcal{F}_C\text{-}\operatorname{pd}_R(X) < \infty$.

Proof: Assume that $\mathcal{P}_C\text{-}\operatorname{pd}_R(X) = n < \infty$. Then

$$\operatorname{fd}_R(\mathbf{R}\mathrm{Hom}_R(C, X)) \leqslant \operatorname{pd}_R(\mathbf{R}\mathrm{Hom}_R(C, X)) = n - \sup(C) < \infty.$$

It now follows that $\mathcal{F}_C\text{-}\operatorname{pd}_R(X) \leqslant n$.

Next assume that $\dim(R) < \infty$ and $\mathcal{F}_C\text{-}\operatorname{pd}_R(X) = n < \infty$. By [19] we have

$$\operatorname{pd}_R(\mathbf{R}\mathrm{Hom}_R(C, X)) \leqslant \operatorname{fd}_R(\mathbf{R}\mathrm{Hom}_R(C, X)) + \dim(R) = n - \sup(C) + \dim(R).$$

Therefore $\mathcal{P}_C\text{-}\operatorname{pd}_R(X) \leqslant \dim(R) + n$.

$\square$

The following three results are versions of [27, Theorem 2.11] involving a semidualizing complex.

Proposition 3.1.6. *Let $X \in \mathcal{D}_{\mathrm{b}}(R)$. Then we have*

$$\mathcal{P}_C\text{-}\operatorname{pd}_R(C \otimes^{\mathbf{L}}_R X) = \sup(C) + \operatorname{pd}_R(X).$$

In particular, $\mathcal{P}_C\text{-}\operatorname{pd}_R(C \otimes^{\mathbf{L}}_R X) < \infty$ *if and only if* $\operatorname{pd}_R(X) < \infty$.

Proof: Let $n \in \mathbb{Z}$. We prove that $\mathcal{P}_C\text{-}\operatorname{pd}_R(C \otimes^{\mathbf{L}}_R X) \leqslant n$ if and only if $\sup(C) + \operatorname{pd}_R(X) \leqslant n$.

For the forward implication assume that $\mathcal{P}_C\text{-}\operatorname{pd}_R(C \otimes^{\mathbf{L}}_R X) \leqslant n$. Then by Definition 3.1.1(1) we have

$$\sup(C) + \operatorname{pd}_R(\mathbf{R}\mathrm{Hom}_R(C, C \otimes^{\mathbf{L}}_R X)) = \mathcal{P}_C\text{-}\operatorname{pd}_R(C \otimes^{\mathbf{L}}_R X) \leqslant n.$$

Thus $\operatorname{pd}_R(\mathbf{R}\mathrm{Hom}_R(C, C \otimes^{\mathbf{L}}_R X)) < \infty$. Fact 2.4.6(a) implies $\mathbf{R}\mathrm{Hom}_R(C, C \otimes^{\mathbf{L}}_R X) \in \mathcal{A}_C(R)$. By Foxby Equivalence (2.4.5) we have $C \otimes^{\mathbf{L}}_R X \in \mathcal{B}_C(R)$ and $X \in \mathcal{A}_C(R)$. Therefore we have $X \simeq \mathbf{R}\mathrm{Hom}_R(C, C \otimes^{\mathbf{L}}_R X)$ and $\operatorname{pd}_R(\mathbf{R}\mathrm{Hom}_R(C, C \otimes^{\mathbf{L}}_R X)) = \operatorname{pd}_R(X)$. Thus $\sup(C) + \operatorname{pd}_R(X) \leqslant n$.

For the reverse implication assume that $\sup(C) + \operatorname{pd}_R(X) \leqslant n$. In particular, we have that $\operatorname{pd}_R(X) < \infty$. Therefore $X \in \mathcal{A}_C(R)$ and $X \simeq \mathbf{R}\mathrm{Hom}_R(C, C \otimes^{\mathbf{L}}_R X)$. It follows that $\operatorname{pd}_R(X) = \operatorname{pd}_R(\mathbf{R}\mathrm{Hom}_R(C, C \otimes^{\mathbf{L}}_R X)$. By Definition 3.1.1(1) we have

$$\mathcal{P}_C\text{-}\operatorname{pd}_R(C \otimes^{\mathbf{L}}_R X) = \sup(C) + \operatorname{pd}_R(\mathbf{R}\mathrm{Hom}_R(C, C \otimes^{\mathbf{L}}_R X)) = \sup(C) + \operatorname{pd}_R(X) \leqslant n.$$

$\square$

The next result is proven like Proposition 3.1.6.

Proposition 3.1.7. *Let* $X \in \mathcal{D}_{\mathrm{b}}(R)$. *Then we have*

$$\mathcal{F}_C\text{-}\operatorname{pd}_R(C \otimes^{\mathbf{L}}_R X) = \sup(C) + \operatorname{fd}_R(X).$$

In particular, $\mathcal{F}_C\text{-}\operatorname{pd}_R(C \otimes^{\mathbf{L}}_R X) < \infty$ *if and only if* $\operatorname{fd}_R(X) < \infty$.

The following result is proven dually to Proposition 3.1.6. We include this proof to show that the proofs of the $\mathcal{I}_C$- id results are dual arguments to the $\mathcal{P}_C$- pd results. Throughout the rest of this chapter we will prove the $\mathcal{P}_C$- pd results and leave the $\mathcal{F}_C$- pd and $\mathcal{I}_C$- id results left to the interested reader.

Proposition 3.1.8. *Let* $X \in \mathcal{D}_{\mathrm{b}}(R)$. *Then we have*

$$\mathcal{I}_C\text{-}\operatorname{id}_R(\mathbf{R}\mathrm{Hom}_R(C, X)) = \sup(C) + \operatorname{id}_R(X).$$

In particular, $\mathcal{I}_C\text{-}\operatorname{id}_R(\mathbf{R}\mathrm{Hom}_R(C,X)) < \infty$ *if and only if* $\operatorname{id}_R(X) < \infty$.

Proof: Let $n \in \mathbb{Z}$. We prove that $\mathcal{I}_C\text{-}\operatorname{id}_R(C \otimes^{\mathbf{L}}_R X) \leqslant n$ if and only if $\sup(C) + \operatorname{id}_R(X) \leqslant n$.

For the forward implication assume that $\mathcal{I}_C\text{-}\operatorname{id}_R(\mathbf{R}\mathrm{Hom}_R(C,X)) \leqslant n$. Then by Definition 3.1.1(3) we have

$$\sup(C) + \operatorname{id}_R(C \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C,X)) = \mathcal{I}_C\text{-}\operatorname{id}_R(\mathbf{R}\mathrm{Hom}_R(C,X)) \leqslant n.$$

Thus $\operatorname{id}_R(C \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C,X)) < \infty$. Fact 2.4.6(b) implies $C \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C,X) \in \mathcal{B}_C(R)$. By Foxby Equivalence (2.4.5) we have $\mathbf{R}\mathrm{Hom}_R(C,X) \in \mathcal{A}_C(R)$ and $X \in \mathcal{B}_C(R)$. Therefore we have $X \simeq C \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C,X)$ and $\operatorname{id}_R(C \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C,X)) = \operatorname{id}_R(X)$. Thus $\sup(C) + \operatorname{id}_R(X) \leqslant n$.

For the reverse implication assume that $\sup(C) + \operatorname{id}_R(X) \leqslant n$. In particular, we have that $\operatorname{id}_R(X) < \infty$. Therefore $X \in \mathcal{B}_C(R)$ and $X \simeq C \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C,X)$. It follows that $\operatorname{id}_R(X) = \operatorname{id}_R(C \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C,X))$. By Definition 3.1.1(3) we have

$$\mathcal{I}_C\text{-}\operatorname{id}_R(\mathbf{R}\mathrm{Hom}_R(C,X)) = \sup(C) + \operatorname{id}_R(C \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C,X)) = \sup(C) + \operatorname{id}_R(X) \leqslant n.$$

$\square$

Next, we have Theorem 1.2.1 from the introduction.

Theorem 3.1.9. *Let* $X \in \mathcal{D}_{\mathrm{b}}(R)$.

(a) *We have* $\mathcal{P}_C\text{-}\operatorname{pd}_R(X) < \infty$ *if and only if there exists* $Y \in \mathcal{D}_{\mathrm{b}}(R)$ *such that* $\operatorname{pd}_R(Y) < \infty$ *and* $X \simeq C \otimes^{\mathbf{L}}_R Y$. *When these conditions are satisfied, one has* $Y \simeq \mathbf{R}\mathrm{Hom}_R(C,X)$ *and* $X \in \mathcal{B}_C(R)$.

(b) *We have* $\mathcal{F}_C\text{-}\operatorname{pd}_R(X) < \infty$ *if and only if there exists* $Y \in \mathcal{D}_{\mathrm{b}}(R)$ *such that* $\operatorname{fd}_R(Y) < \infty$ *and* $X \simeq C \otimes^{\mathbf{L}}_R Y$. *When these conditions are satisfied, one has* $Y \simeq \mathbf{R}\mathrm{Hom}_R(C,X)$ *and* $X \in \mathcal{B}_C(R)$.

(c) *We have* $\mathcal{I}_C\text{-}\operatorname{id}_R(X) < \infty$ *if and only if there exists* $Y \in \mathcal{D}_{\mathrm{b}}(R)$ *such that* $\operatorname{id}_R(Y) < \infty$ *and* $X \simeq \mathbf{R}\mathrm{Hom}_R(C,Y)$. *When these conditions are satisfied, one has* $Y \simeq C \otimes^{\mathbf{L}}_R X$ *and* $X \in \mathcal{A}_C(R)$.

Proof: (a) For the forward implication assume that $\mathcal{P}_C\text{-}\operatorname{pd}_R(X) < \infty$. Then by Definition 3.1.1(1) we have $\operatorname{pd}_R(\mathbf{R}\mathrm{Hom}_R(C,X)) = \mathcal{P}_C\text{-}\operatorname{pd}_R(X) - \sup(C) < \infty$. Fact 2.4.6(a) implies that

$\mathbf{R}\mathrm{Hom}_R(C,X) \in \mathcal{A}_C(R)$ and Foxby Equivalence implies that $X \in \mathcal{B}_C(R)$. Thus $X \simeq C \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C,X) \simeq C \otimes^{\mathbf{L}}_R Y$ with $Y = \mathbf{R}\mathrm{Hom}_R(C,X)$.

For the reverse implication assume that there exists a $Y \in \mathcal{D}_{\mathrm{b}}(R)$ such that $\mathrm{pd}_R(Y) < \infty$ and $X \simeq C \otimes^{\mathbf{L}}_R Y$. Then Fact 2.4.6(a) implies that $Y \in \mathcal{A}_C(R)$ and hence we have

$$Y \simeq \mathbf{R}\mathrm{Hom}_R(C, C \otimes^{\mathbf{L}}_R Y) \simeq \mathbf{R}\mathrm{Hom}_R(C,X).$$

It now follows by Definition 3.1.1(1) that $\mathcal{P}_C\text{-}\mathrm{pd}_R(X) < \infty$.

Parts (b) and (c) are proven similarly. $\square$

The previous results give rise to a generalized Foxby Equivalence.

Theorem 3.1.10 (Foxby Equivalence)**.** *There is a commutative diagram*

$$\begin{tikzcd}
\mathcal{I}_C(R) \arrow[r, shift left] \arrow[d, hook] & \mathcal{I}(R) \arrow[l, shift left] \arrow[d, hook] \\
\mathcal{A}_C(R) \arrow[r, shift left, "C \otimes^{\mathbf{L}}_R -"] & \mathcal{B}_C(R) \arrow[l, shift left, "\mathbf{R}\mathrm{Hom}_R(C,-)"] \\
\mathcal{F}(R) \arrow[r, shift left] \arrow[u, hook] & \mathcal{F}_C(R) \arrow[l, shift left] \arrow[u, hook] \\
\mathcal{P}(R) \arrow[r, shift left] \arrow[u, hook] & \mathcal{P}_C(R) \arrow[l, shift left] \arrow[u, hook]
\end{tikzcd}$$

where the vertical arrows are full embeddings, and the unlabeled horizontal arrows are quasi-inverse equivalences of categories.

The next result shows how $\mathcal{P}_C$-pd and $\mathcal{F}_C$-pd transfer along a ring homomorphism of finite flat dimension. Note that if $\varphi : R \to S$ is a ring homomorphism of finite flat dimension, then $C \otimes^{\mathbf{L}}_R S$ is a semidualizing S-complex by [7, Theorem 5.6] and [11, Theorem II(a)].

Proposition 3.1.11. *Let $\varphi : R \to S$ be a ring homomorphism of finite flat dimension and $X \in \mathcal{D}_{\mathrm{b}}(R)$. Then one has*

(a) $\mathcal{P}_{C\otimes^{\mathbf{L}}_R S}\text{-}\mathrm{pd}_S(X \otimes^{\mathbf{L}}_R S) - \sup(C \otimes^{\mathbf{L}}_R S) \leqslant \mathcal{P}_C\text{-}\mathrm{pd}_R(X) - \sup(C)$,

(b) $\mathcal{F}_{C\otimes^{\mathbf{L}}_R S}\text{-}\mathrm{pd}_S(X \otimes^{\mathbf{L}}_R S) - \sup(C \otimes^{\mathbf{L}}_R S) \leqslant \mathcal{F}_C\text{-}\mathrm{pd}_R(X) - \sup(C)$,

(c) $\mathcal{P}_{C\otimes^{\mathbf{L}}_R S}\text{-}\mathrm{pd}_S(X \otimes^{\mathbf{L}}_R S) \leqslant \mathcal{P}_C\text{-}\mathrm{pd}_R(X)$, *and*

(d) $\mathcal{F}_{C\otimes^{\mathbf{L}}_R S}\text{-}\operatorname{pd}_S(X\otimes^{\mathbf{L}}_R S)\leqslant \mathcal{F}_C\text{-}\operatorname{pd}_R(X)$.

Equality holds when φ is faithfully flat.

Proof: (a) and (c): Assume that $\mathcal{P}_C\text{-}\operatorname{pd}_R(X)-\sup(C)=n<\infty$. Then $\operatorname{pd}_R(\mathbf{R}\mathrm{Hom}_R(C,X))=n$ and hence by base change we have

$$\operatorname{pd}_S(\mathbf{R}\mathrm{Hom}_R(C,X)\otimes^{\mathbf{L}}_R S)\leqslant \operatorname{pd}_R(\mathbf{R}\mathrm{Hom}_R(C,X))=n.$$

Observe by tensor-evaluation (2.3.16) and Hom-tensor adjointness, there are isomorphisms

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_R(C,X)\otimes^{\mathbf{L}}_R S &\simeq \mathbf{R}\mathrm{Hom}_R(C,X\otimes^{\mathbf{L}}_R S)\\ &\simeq \mathbf{R}\mathrm{Hom}_R(C,\mathbf{R}\mathrm{Hom}_S(S,X\otimes^{\mathbf{L}}_R S))\\ &\simeq \mathbf{R}\mathrm{Hom}_S(C\otimes^{\mathbf{L}}_R S,X\otimes^{\mathbf{L}}_R S).\end{aligned}$$

Therefore $\operatorname{pd}_S(\mathbf{R}\mathrm{Hom}_S(C\otimes^{\mathbf{L}}_R S,X\otimes^{\mathbf{L}}_R S))\leqslant n$. Thus we have

$$\mathcal{P}_{C\otimes^{\mathbf{L}}_R S}\text{-}\operatorname{pd}_S(X\otimes^{\mathbf{L}}_R S)-\sup(C\otimes^{\mathbf{L}}_R S)\leqslant n=\mathcal{P}_C\text{-}\operatorname{pd}_R(X)-\sup(C)$$

that is, the inequality in (a) holds.

Observe that since $\operatorname{fd}_R(S)<\infty$, we have $S\in\mathcal{A}_C(R)$ and hence $\sup(C\otimes^{\mathbf{L}}_R S)\leqslant\sup(C)$ by [7, Proposition 4.8(a)]. Hence the inequality in (c) follows from part (a).

Now assume that φ is faithfully flat. Therefore one has that $\sup(C\otimes^{\mathbf{L}}_R S)=\sup(C)$. Hence it suffices to show that $\mathcal{P}_{C\otimes^{\mathbf{L}}_R S}\text{-}\operatorname{pd}_R(X\otimes^{\mathbf{L}}_R S)\geqslant\mathcal{P}_C\text{-}\operatorname{pd}_R(X)$. Assume that $\mathcal{P}_{C\otimes^{\mathbf{L}}_R S}\text{-}\operatorname{pd}_R(X\otimes^{\mathbf{L}}_R S)=n<\infty$. Then

$$\operatorname{pd}_S(\mathbf{R}\mathrm{Hom}_R(C,X)\otimes^{\mathbf{L}}_R S)=\operatorname{pd}_S(\mathbf{R}\mathrm{Hom}_S(C\otimes^{\mathbf{L}}_R S,X\otimes^{\mathbf{L}}_R S))=n-\sup(C\otimes^{\mathbf{L}}_R S).$$

Therefore we have $\operatorname{pd}_S(\mathbf{R}\mathrm{Hom}_R(C,X)\otimes^{\mathbf{L}}_R S)\leqslant n-\sup(C)$. Observe that if P is an R-module such that $P\otimes_R S$ is projective over S, then P is projective over R by [18, Theorem 9.6] and [19]. A standard truncation argument thus shows that

$$\operatorname{pd}_R(\mathbf{R}\mathrm{Hom}_R(C,X)) \leqslant \operatorname{pd}_S(\mathbf{R}\mathrm{Hom}_R(C,X) \otimes^{\mathbf{L}}_R S) = n - \sup(C)$$

as desired.

Parts (d) and (b) are proven similarly. $\square$

Corollary 3.1.12. *Let $X \in \mathcal{D}_{\mathrm{b}}(R)$, and let $U \subset R$ be a multiplicatively closed subset. Then there are inequalities*

(a) $\mathcal{P}_{U^{-1}C}\text{-}\operatorname{pd}_{U^{-1}R}(U^{-1}X) \leqslant \mathcal{P}_C\text{-}\operatorname{pd}_R(X)$,

(b) $\mathcal{F}_{U^{-1}C}\text{-}\operatorname{pd}_{U^{-1}R}(U^{-1}X) \leqslant \mathcal{F}_C\text{-}\operatorname{pd}_R(X)$,

(c) $\mathcal{I}_{U^{-1}C}\text{-}\operatorname{id}_{U^{-1}R}(U^{-1}X) \leqslant \mathcal{I}_C\text{-}\operatorname{id}_R(X)$,

(d) $\mathcal{P}_{U^{-1}C}\text{-}\operatorname{pd}_{U^{-1}R}(U^{-1}X) - \sup(U^{-1}C) \leqslant \mathcal{P}_C\text{-}\operatorname{pd}_R(X) - \sup(C)$,

(e) $\mathcal{F}_{U^{-1}C}\text{-}\operatorname{pd}_{U^{-1}R}(U^{-1}X) - \sup(U^{-1}C) \leqslant \mathcal{F}_C\text{-}\operatorname{pd}_R(X) - \sup(C)$, *and*

(f) $\mathcal{I}_{U^{-1}C}\text{-}\operatorname{id}_{U^{-1}R}(U^{-1}X) - \sup(U^{-1}C) \leqslant \mathcal{I}_C\text{-}\operatorname{id}_R(X) - \sup(C)$.

Proof: The map $\varphi : R \to U^{-1}R$ is flat. Hence (a), (b), (d), and (e) follow from Proposition 3.1.11. Parts (c) and (f) are proven similarly to Proposition 3.1.11. $\square$

Remark 3.1.13. Observe that to obtain the inequality in Corollary 3.1.12 we need the inequality $\sup(U^{-1}C) \leqslant \sup(C)$ to hold. If we had defined $\mathcal{P}_C\text{-}\operatorname{pd}_R(X)$ as $\inf(C) + \operatorname{pd}_R(\mathbf{R}\mathrm{Hom}_R(C,X))$, then Corollarly 3.1.12 would not hold because $\inf(U^{-1}C) \nleqslant \inf(C)$. This is why we choose $\sup(C)$ instead of $\inf(C)$ in the definition of $\mathcal{P}_C\text{-}\operatorname{pd}$.

The next result is a local-global principal for Bass classes.

Lemma 3.1.14. *Let $X \in \mathcal{D}_{\mathrm{b}}(R)$. The following conditions are equivalent:*

(i) $X \in \mathcal{B}_C(R)$*;*

(ii) $U^{-1}X \in \mathcal{B}_{U^{-1}C}(U^{-1}R)$ *for all multiplicatively closed subsets $U \subset R$;*

(iii) $X_\mathfrak{p} \in \mathcal{B}_{C_\mathfrak{p}}(R_\mathfrak{p})$ *for all* $\mathfrak{p} \in \mathrm{Spec}(R)$*;*

(iv) $X_\mathfrak{p} \in \mathcal{B}_{C_\mathfrak{p}}(R_\mathfrak{p})$ *for all* $\mathfrak{p} \in \mathrm{Supp}(R)$*;*

(v) $X_\mathfrak{m} \in \mathcal{B}_{C_\mathfrak{m}}(R_\mathfrak{m})$ *for all* $\mathfrak{m} \in \mathrm{Max}(R)$*; and*

(vi) $X_\mathfrak{m} \in \mathcal{B}_{C_\mathfrak{m}}(R_\mathfrak{m})$ *for all* $\mathfrak{m} \in \mathrm{Supp}(R) \cap \mathrm{Max}(R)$*.*

Proof: The implications (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) $\Rightarrow$ (vi) and (iii) $\Rightarrow$ (v) $\Rightarrow$ (vi) follow from definitions. We prove (v) $\Rightarrow$ (i) and (vi) $\Rightarrow$ (v).

For the implication (v) $\Rightarrow$ (i), assume $X_\mathfrak{m} \in \mathcal{B}_{C_\mathfrak{m}}(R_\mathfrak{m})$ for all $\mathfrak{m} \in \mathrm{Max}(R)$. We use the following commutative diagram in $\mathcal{D}(R)$:

$$\begin{tikzcd}
C_\mathfrak{m} \otimes^{\mathbf{L}}_{R_\mathfrak{m}} \mathbf{R}\mathrm{Hom}_R(C,X)_\mathfrak{m} \arrow[r, "\simeq"] \arrow[d, "\simeq"] & \left[C \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(C,X)\right]_\mathfrak{m} \arrow[d, "(\xi^C_X)_\mathfrak{m}"] \\
C_\mathfrak{m} \otimes^{\mathbf{L}}_{R_\mathfrak{m}} \mathbf{R}\mathrm{Hom}_{R_\mathfrak{m}}(C_\mathfrak{m},X_\mathfrak{m}) \arrow[r, "\xi^{C_\mathfrak{m}}_{X_\mathfrak{m}}"] & X_\mathfrak{m}.
\end{tikzcd}$$

As $X_\mathfrak{m} \in \mathcal{B}_{C_\mathfrak{m}}(R_\mathfrak{m})$ for all $\mathfrak{m} \in \mathrm{Max}(R)$, the morphism $\xi^{C_\mathfrak{m}}_{X_\mathfrak{m}}$ is an isomorphism for all $\mathfrak{m} \in \mathrm{Max}(R)$. Commutativity of the above diagram now forces $(\xi^C_X)_\mathfrak{m}$ to be an isomrophism for all $\mathfrak{m} \in \mathrm{Max}(R)$. Therefore ξ^C_X is an isomorphism.

It remains to show that $\mathbf{R}\mathrm{Hom}_R(C,X) \in \mathcal{D}_\mathrm{b}(R)$. As $\mathbf{R}\mathrm{Hom}_R(C,X) \in \mathcal{D}_-(R)$, it suffices to show that $\mathbf{R}\mathrm{Hom}_R(C,X) \in \mathcal{D}_+(R)$. By assumption $X_\mathfrak{m} \in \mathcal{B}_{C_\mathfrak{m}}(R_\mathfrak{m})$. Then for all $\mathfrak{m} \in \mathrm{Max}(R)$ we have

$$\begin{aligned}
\inf(\mathbf{R}\mathrm{Hom}_R(C,X)_\mathfrak{m}) &= \inf(\mathbf{R}\mathrm{Hom}_{R_\mathfrak{m}}(C_\mathfrak{m},X_\mathfrak{m})) \\
&\geqslant \inf(X_\mathfrak{m}) - \sup(C_\mathfrak{m}) \\
&\geqslant \inf(X) - \sup(C)
\end{aligned}$$

where the equality is by the isomorphism $\mathbf{R}\mathrm{Hom}_R(C,X)_\mathfrak{m} \simeq \mathbf{R}\mathrm{Hom}_{R_\mathfrak{m}}(C_\mathfrak{m},X_\mathfrak{m})$, the first inequality is by [7, Proposition 4.8(c)], and the second inequality is by properties of localization. Thus $\inf(\mathbf{R}\mathrm{Hom}_R(C,X)) \geqslant \inf(X) - \sup(C) > -\infty$.

For the implication (vi) $\Rightarrow$ (v), assume $X_\mathfrak{m} \in \mathcal{B}_{C_\mathfrak{m}}(R_\mathfrak{m})$ for all $\mathfrak{m} \in \mathrm{Supp}_R(X) \cap \mathrm{Max}(R)$. Then for all $\mathfrak{m} \in \mathrm{Max}(R) \setminus \mathrm{Supp}_R(X)$ we have $X_\mathfrak{m} \simeq 0 \in \mathcal{B}_{C_\mathfrak{m}}(R_\mathfrak{m})$, as desired. $\square$

The following is proven similarly to Lemma 3.1.14

Lemma 3.1.15. *Let $X \in \mathcal{D}_{\mathrm{b}}(R)$. The following conditions are equivalent:*

(i) $X \in \mathcal{A}_C(R)$;

(ii) $U^{-1}X \in \mathcal{A}_{U^{-1}C}(U^{-1}R)$ *for all multiplicatively closed subsets* $U \subset R$;

(iii) $X_\mathfrak{p} \in \mathcal{A}_{C_\mathfrak{p}}(R_\mathfrak{p})$ *for all* $\mathfrak{p} \in \operatorname{Spec}(R)$;

(iv) $X_\mathfrak{p} \in \mathcal{A}_{C_\mathfrak{p}}(R_\mathfrak{p})$ *for all* $\mathfrak{p} \in \operatorname{Supp}(R)$;

(v) $X_\mathfrak{m} \in \mathcal{A}_{C_\mathfrak{m}}(R_\mathfrak{m})$ *for all* $\mathfrak{m} \in \operatorname{Max}(R)$; *and*

(vi) $X_\mathfrak{m} \in \mathcal{A}_{C_\mathfrak{m}}(R_\mathfrak{m})$ *for all* $\mathfrak{m} \in \operatorname{Supp}(R) \cap \operatorname{Max}(R)$.

Proposition 3.1.16. *Let $X \in \mathcal{D}_{\mathrm{b}}(R)$ and let $n \in \mathbb{Z}$. Consider the following conditions:*

(i) $\mathcal{P}_C\text{-}\mathrm{pd}_R(X) - \sup(C) \leqslant n$;

(ii) $\mathcal{P}_{U^{-1}C}\text{-}\mathrm{pd}_{U^{-1}R}(U^{-1}X) - \sup(U^{-1}C) \leqslant n$ *for each multiplicatively closed subset* $U \subset R$;

(iii) $\mathcal{P}_{C_\mathfrak{p}}\text{-}\mathrm{pd}_{R_\mathfrak{p}}(X_\mathfrak{p}) - \sup(C_\mathfrak{p}) \leqslant n$ *for each* $\mathfrak{p} \in \operatorname{Spec}(R)$; *and*

(iv) $\mathcal{P}_{C_\mathfrak{m}}\text{-}\mathrm{pd}_{R_\mathfrak{m}}(X_\mathfrak{m}) - \sup(C_\mathfrak{m}) \leqslant n$ *for each* $\mathfrak{m} \in \operatorname{Max}(R)$.

Then (i) $\Rightarrow$ (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv). *Furthermore, if $X \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$, then* (iv) $\Rightarrow$ (i) *and*

$$\begin{aligned}
\mathcal{P}_C\text{-}\mathrm{pd}_R(X) - c &= \sup\left\{ \begin{array}{l} \mathcal{P}_{U^{-1}C}\text{-}\mathrm{pd}_{U^{-1}R}(U^{-1}X) \\ -\sup(U^{-1}C) \end{array} \,\middle|\, U \subset R \text{ is multiplicatively closed} \right\} \\
&= \sup\{\mathcal{P}_{C_\mathfrak{p}}\text{-}\mathrm{pd}_{R_\mathfrak{p}}(X_\mathfrak{p}) - \sup(C_\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec}(R)\} \\
&= \sup\{\mathcal{P}_{C_\mathfrak{m}}\text{-}\mathrm{pd}_{R_\mathfrak{m}}(X_\mathfrak{m}) - \sup(C_\mathfrak{m}) \mid \mathfrak{m} \in \operatorname{Max}(R)\}
\end{aligned}$$

where $c = \sup(C)$.

Proof: Observe that (i) $\Rightarrow$ (ii) follows from Proposition 3.1.11. The implications (ii) $\Rightarrow$ (iii) $\Rightarrow$ (iv) follow from properties of localization. For the rest of the proof assume that $X \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$.

For the implication (iv) $\Rightarrow$ (i) assume that $\mathcal{P}_{C_\mathfrak{m}}\text{-}\operatorname{pd}_{R_\mathfrak{m}}(X_\mathfrak{m}) - \sup(C_\mathfrak{m}) \leqslant n < \infty$ for all $\mathfrak{m} \in \operatorname{Max}(R)$. Then by Remark 3.1.2 we have $X_\mathfrak{m} \in \mathcal{B}_{C_\mathfrak{m}}(R_\mathfrak{m})$ for all $\mathfrak{m} \in \operatorname{Max}(R)$. Therefore Lemma 3.1.14 implies that $X \in \mathcal{B}_C(R)$ and hence $\mathbf{R}\operatorname{Hom}_R(C, X) \in \mathcal{D}_\mathrm{b}(R)$. Now

$$\begin{aligned}
\mathcal{P}_C\text{-}\operatorname{pd}_R(X) - \sup(C) &= \operatorname{pd}_R(\mathbf{R}\operatorname{Hom}_R(C, X)) \\
&= \sup_{\mathfrak{m} \in \operatorname{Max}(R)} (\operatorname{pd}_{R_\mathfrak{m}}(\mathbf{R}\operatorname{Hom}_{R_\mathfrak{m}}(C_\mathfrak{m}, X_\mathfrak{m}))) \\
&\leqslant n
\end{aligned}$$

where the second equality is by [4, Proposition 5.3P].

For the equalities, assume first that $\mathcal{P}_C\text{-}\operatorname{pd}_R(X) - \sup(C) = n < \infty$. Then each displayed supremum in the statement is at most n. If any of the supremums are strictly less than n, then the above equivalence will force $\mathcal{P}_C\text{-}\operatorname{pd}_R(X) - \sup(C) < n$, contradicting our assumption. A similar argument establishes the desired equalities if we assume any of the supremums equal n.

Finally if any of the displayed values in the statement are infinite, then the above equivalences forces the other values to be infinite as well. $\square$

To prove the implication (iv) $\Rightarrow$ (i) in Proposition 3.1.16, the condition $X \in \mathcal{D}_\mathrm{b}^\mathrm{f}(R)$ is required; see [4, Proposition 5.3P]. However the flat and injective versions only require $X \in \mathcal{D}_\mathrm{b}(R)$; see [4, Propositions 5.3F, and 5.3I]. Thus the next two results are proven similarly to Proposition 3.1.16.

Proposition 3.1.17. *Let $X \in \mathcal{D}_\mathrm{b}(R)$ and let $n \in \mathbb{Z}$. The following conditions are equivalent:*

(i) $\mathcal{F}_C\text{-}\operatorname{pd}_R(X) - \sup(C) \leqslant n$;

(ii) $\mathcal{F}_{U^{-1}C}\text{-}\operatorname{pd}_{U^{-1}R}(U^{-1}X) - \sup(U^{-1}C) \leqslant n$ *for each multiplicatively closed subset* $U \subset R$;

(iii) $\mathcal{F}_{C_\mathfrak{p}}\text{-}\operatorname{pd}_{R_\mathfrak{p}}(X_\mathfrak{p}) - \sup(C_\mathfrak{p}) \leqslant n$ *for each prime ideal* $\mathfrak{p} \subset R$; *and*

(iv) $\mathcal{F}_{C_\mathfrak{m}}\text{-}\operatorname{pd}_{R_\mathfrak{m}}(X_\mathfrak{m}) - \sup(C_\mathfrak{m}) \leqslant n$ *for each maximal ideal* $\mathfrak{m} \subset R$.

Furthermore

$$\mathcal{F}_C\text{-}\operatorname{pd}_R(X) - c = \sup\left\{ \begin{array}{l} \mathcal{F}_{U^{-1}C}\text{-}\operatorname{pd}_{U^{-1}R}(U^{-1}X) \\ -\sup(U^{-1}C) \end{array} \;\middle|\; U \subset R \textit{ is multiplicatively closed} \right\}$$
$$= \sup\{\mathcal{F}_{C_\mathfrak{p}}\text{-}\operatorname{pd}_{R_\mathfrak{p}}(X_\mathfrak{p}) - \sup(C_\mathfrak{p}) \mid \mathfrak{p} \in \operatorname{Spec}(R)\}$$
$$= \sup\{\mathcal{F}_{C_\mathfrak{m}}\text{-}\operatorname{pd}_{R_\mathfrak{m}}(X_\mathfrak{m}) - \sup(C_\mathfrak{m}) \mid \mathfrak{m} \in \operatorname{Max}(R)\}$$

where $c = \sup(C)$.

Proposition 3.1.18. *Let* $X \in \mathcal{D}_\mathrm{b}(R)$ *and let* $n \in \mathbb{Z}$. *The following conditions are equqivalent:*

(i) $\mathcal{I}_C\text{-}\operatorname{id}_R(X) - \sup(C) \leqslant n$*;*

(ii) $\mathcal{I}_{U^{-1}C}\text{-}\operatorname{id}_{U^{-1}R}(U^{-1}X) - \sup(U^{-1}C) \leqslant n$ *for each multiplicatively closed subset* $U \subset R$*;*

(iii) $\mathcal{I}_{C_\mathfrak{p}}\text{-}\operatorname{id}_{R_\mathfrak{p}}(X_\mathfrak{p}) - \sup(C_\mathfrak{p}) \leqslant n$ *for each prime ideal* $\mathfrak{p} \subset R$*; and*

(iv) $\mathcal{I}_{C_\mathfrak{m}}\text{-}\operatorname{id}_{R_\mathfrak{m}}(X_\mathfrak{m}) - \sup(C_\mathfrak{m}) \leqslant n$ *for each maximal ideal* $\mathfrak{m} \subset R$.

Furthermore

$$\mathcal{I}_C\text{-}\operatorname{id}_R(X) - c = \sup\left\{ \begin{array}{l} \operatorname{id}_{U^{-1}R}(U^{-1}C \otimes^{\mathbf{L}}_{U^{-1}R} U^{-1}X) \\ -\sup(U^{-1}C) \end{array} \;\middle|\; U \subset R \textit{ is multiplicatively closed} \right\}$$
$$= \sup\{\operatorname{id}_{R_\mathfrak{p}}\text{-}\operatorname{id}_{R_\mathfrak{p}}(C_\mathfrak{p} \otimes^{\mathbf{L}}_{R_\mathfrak{p}} X_\mathfrak{p}) - \sup(C_\mathfrak{m}) \mid \mathfrak{p} \in \operatorname{Spec}(R)\}$$
$$= \sup\{\operatorname{id}_{R_\mathfrak{m}}\text{-}\operatorname{id}_{R_\mathfrak{m}}(C_\mathfrak{m} \otimes^{\mathbf{L}}_{R_\mathfrak{m}} X_\mathfrak{m}) - \sup(C_\mathfrak{m}) \mid \mathfrak{m} \in \operatorname{Max}(R)\}$$

where $c = \sup(C)$.

Remark 3.1.19. When C is a semidualizing R-module, e.g., $C = R$, we recover the known local-global conditions for $\mathcal{P}_C$- pd, $\mathcal{F}_C$- pd, $\mathcal{I}_C$- id, pd, fd, and id.

3.2. Stability Results

In this section we investigate the behaviour of $\mathcal{P}_C$- pd, $\mathcal{F}_C$- pd, and $\mathcal{I}_C$- id after applying the functors $\otimes^{\mathbf{L}}$ and $\mathbf{R}\mathrm{Hom}$.

Proposition 3.2.1. *Let* $X, Y \in \mathcal{D}_\mathrm{b}(R)$. *The following inequalities hold:*

(a) $\mathcal{P}_C\text{-}\operatorname{pd}_R(X \otimes^{\mathbf{L}}_R Y) \leqslant \mathcal{P}_C\text{-}\operatorname{pd}_R(X) + \operatorname{pd}_R(Y)$;

(b) $\mathcal{I}_C\text{-}\operatorname{id}_R(\mathbf{R}\operatorname{Hom}_R(X,Y)) \leqslant \mathcal{F}_C\text{-}\operatorname{pd}_R(X) + \operatorname{id}_R(Y)$; *and*

(c) $\mathcal{F}_C\text{-}\operatorname{pd}_R(X \otimes^{\mathbf{L}}_R Y) \leqslant \mathcal{F}_C\text{-}\operatorname{pd}_R(X) + \operatorname{fd}_R(Y)$.

Proof: (a) Without loss of generality we assume that $\mathcal{P}_C\text{-}\operatorname{pd}_R(X) < \infty$ and $\operatorname{pd}_R(Y) < \infty$. It now follows that $\mathcal{P}_C\text{-}\operatorname{pd}_R(X) = \sup(C) + \operatorname{pd}_R(\mathbf{R}\operatorname{Hom}_R(C,X))$. By [4, Theorem 4.1 (P)] we have that

$$\operatorname{pd}_R(\mathbf{R}\operatorname{Hom}_R(C,X) \otimes^{\mathbf{L}}_R Y) \leqslant \operatorname{pd}_R(\mathbf{R}\operatorname{Hom}_R(C,X)) + \operatorname{pd}_R(Y).$$

Since $\operatorname{pd}_R(Y) < \infty$ (hence $\operatorname{fd}_R(Y) < \infty$) we get tensor-evaluation (2.3.16) is an isomorphism in $\mathcal{D}(R)$. That is, $\mathbf{R}\operatorname{Hom}_R(C, X \otimes^{\mathbf{L}}_R Y) \simeq \mathbf{R}\operatorname{Hom}_R(C,X) \otimes^{\mathbf{L}}_R Y$. Hence we have

$$\operatorname{pd}_R(\mathbf{R}\operatorname{Hom}_R(C, X \otimes^{\mathbf{L}}_R Y)) \leqslant \operatorname{pd}_R(\mathbf{R}\operatorname{Hom}_R(C,X)) + \operatorname{pd}_R(Y).$$

By adding a $\sup(C)$ to each side we see that $\mathcal{P}_C\text{-}\operatorname{pd}_R(X \otimes^{\mathbf{L}}_R Y) \leqslant \mathcal{P}_C\text{-}\operatorname{pd}_R(X) + \operatorname{pd}_R(Y)$.

(b) and (c) are proven similarly to (a). □

Corollary 3.2.2. *Let $X \in \mathcal{D}_b(R)$. The following inequalities hold:*

(a) $\mathcal{P}_C\text{-}\operatorname{pd}_R(X \otimes^{\mathbf{L}}_R \mathbf{R}\operatorname{Hom}_R(C,Y)) \leqslant \mathcal{P}_C\text{-}\operatorname{pd}_R(X) + \mathcal{P}_C\text{-}\operatorname{pd}_R(Y) - \sup(C)$;

(b) $\mathcal{I}_C\text{-}\operatorname{id}_R(\mathbf{R}\operatorname{Hom}_R(X, C \otimes^{\mathbf{L}}_R Y)) \leqslant \mathcal{F}_C\text{-}\operatorname{pd}_R(X) + \mathcal{I}_C\text{-}\operatorname{id}_R(Y) - \sup(C)$; *and*

(c) $\mathcal{F}_C\text{-}\operatorname{pd}_R(X \otimes^{\mathbf{L}}_R \mathbf{R}\operatorname{Hom}_R(C,Y)) \leqslant \mathcal{F}_C\text{-}\operatorname{pd}(RX) + \mathcal{F}_C\text{-}\operatorname{pd}_R(Y) - \sup(C)$.

Proof: (a) By Proposition 3.2.1(a) we have that $\mathcal{P}_C\text{-}\operatorname{pd}_R(X \otimes^{\mathbf{L}}_R \mathbf{R}\operatorname{Hom}_R(C,Y)) \leqslant \mathcal{P}_C\text{-}\operatorname{pd}_R(X) + \operatorname{pd}_R(\mathbf{R}\operatorname{Hom}_R(C,Y))$. Add and subtract $\sup(C)$ to the right hand side to obtain the result.

(b) and (c) are proven similarly. □

The next result is a version of Fact 2.3.18 involving a semidualizing complex.

Proposition 3.2.3. *Let $X, Y \in \mathcal{D}_\mathrm{b}(R)$.*

(a) *If* $\operatorname{id}_R(Y) < \infty$, *then* $\mathcal{F}_C\text{-}\operatorname{pd}_R(\mathbf{R}\operatorname{Hom}_R(X,Y)) \leqslant \mathcal{I}_C\text{-}\operatorname{id}_R(X) + \sup(Y)$.

(b) *If* $\operatorname{fd}_R(Y) < \infty$, *then* $\mathcal{I}_C\text{-}\operatorname{id}_R(X \otimes^{\mathbf{L}}_R Y) \leqslant \mathcal{I}_C\text{-}\operatorname{id}_R(X) - \inf(Y)$.

Proof: (a) Assume that $\operatorname{id}_R(Y) < \infty$. By Defintion 3.1.1 we get that $\mathcal{F}_C\text{-}\operatorname{pd}_R(\mathbf{R}\mathrm{Hom}_R(X,Y)) = \sup(C) + \operatorname{fd}_R(\mathbf{R}\mathrm{Hom}_R(C, \mathbf{R}\mathrm{Hom}_R(X,Y)))$. Hom-Tensor adjointness implies there is an isomorphism

$$\mathbf{R}\mathrm{Hom}_R(C, \mathbf{R}\mathrm{Hom}_R(X,Y)) \simeq \mathbf{R}\mathrm{Hom}_R(C \otimes^{\mathbf{L}}_R X, Y).$$

Therefore $\operatorname{fd}_R(\mathbf{R}\mathrm{Hom}_R(C, \mathbf{R}\mathrm{Hom}_R(X,Y)) = \operatorname{fd}_R(\mathbf{R}\mathrm{Hom}_R(C \otimes^{\mathbf{L}}_R X, Y))$. Hence by Fact 2.3.18(a) we have that

$$\operatorname{fd}_R(\mathbf{R}\mathrm{Hom}_R(C \otimes^{\mathbf{L}}_R X, Y) \leqslant \operatorname{id}_R(C \otimes^{\mathbf{L}}_R X) + \sup(Y).$$

By adding $\sup(C)$ to each side of the above inequality we obtain the desired result.

(b) is proven similarly. □

Proposition 3.2.4. *Let $X \in \mathcal{D}_{\mathrm{b}}(R)$. The following conditions are equivalent:*

(i) $\mathcal{F}_C\text{-}\operatorname{pd}_R(X) < \infty$*;*

(ii) $\mathcal{I}_C\text{-}\operatorname{id}_R(\mathbf{R}\mathrm{Hom}_R(X,Y)) < \infty$ *for all* $Y \in \mathcal{D}_{\mathrm{b}}(R)$ *such that* $\operatorname{id}_R(Y) < \infty$*; and*

(iii) $\mathcal{I}_C\text{-}\operatorname{id}_R(\mathbf{R}\mathrm{Hom}_R(X,E)) < \infty$ *for some faithfully injective R-module E.*

Proof: (i)⇒(ii) This follows from Proposition 3.2.1(b).

(ii)⇒(iii) Since E is a faithfully injective module it has $\operatorname{id}_R(E) = 0 < \infty$. Therefore (ii) implies that $\mathcal{I}_C\text{-}\operatorname{id}_R(\mathbf{R}\mathrm{Hom}_R(X,E)) < \infty$.

(iii)⇒(i) Assume there is a faithfully injective module E so that $\mathcal{I}_C\text{-}\operatorname{id}_R(\mathbf{R}\mathrm{Hom}_R(X,E)) < \infty$. Then by Definition 3.1.1(3) $\mathcal{I}_C\text{-}\operatorname{id}_R(\mathbf{R}\mathrm{Hom}_R(X,E)) = \sup(C) + \operatorname{id}_R(C \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(X,E))$. By Hom-evaluation (2.3.16) there is an isomorphism

$$\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(C,X),E) \simeq C \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(X,E).$$

It follows that $\operatorname{id}_R(C \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(X,E)) = \operatorname{id}_R(\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(C,X),E)) < \infty$. Therefore by Lemma 2.3.19(a) $\operatorname{fd}_R(\mathbf{R}\mathrm{Hom}_R(C,X)) < \infty$. It now follows that $\mathcal{F}_C\text{-}\operatorname{pd}_R(X) < \infty$. □

The following three propositions are proven similarly to Proposition 3.2.4.

Proposition 3.2.5. *Let $X \in \mathcal{D}_{\mathrm{b}}(R)$. The following conditions are equivalent:*

(i) $\mathcal{F}_C\text{-}\operatorname{pd}_R(X) < \infty$;

(ii) $\mathcal{F}_C\text{-}\operatorname{pd}_R(X \otimes^{\mathbf{L}}_R Y) < \infty$ *for all* $Y \in \mathcal{D}_{\mathrm{b}}(R)$ *such that* $\operatorname{fd}_R(Y) < \infty$;

(iii) $\mathcal{F}_C\text{-}\operatorname{pd}_R(X \otimes^{\mathbf{L}}_R F) < \infty$ *for some faithfully flat R-module F.*

Proposition 3.2.6. *Let $X \in \mathcal{D}_{\mathrm{b}}(R)$. The following conditions are equivalent:*

(i) $\mathcal{I}_C\text{-}\operatorname{id}_R(X) < \infty$;

(ii) $\mathcal{F}_C\text{-}\operatorname{pd}_R(\mathbf{R}\mathrm{Hom}_R(X, Y)) < \infty$ *for all* $Y \in \mathcal{D}_{\mathrm{b}}(R)$ *such that* $\operatorname{id}_R(Y) < \infty$;

(iii) $\mathcal{F}_C\text{-}\operatorname{pd}_R(\mathbf{R}\mathrm{Hom}_R(X, E)) < \infty$ *for some faithfully injective R-module E.*

Proposition 3.2.7. *Let $X \in \mathcal{D}_{\mathrm{b}}(R)$. The following conditions are equivalent:*

(i) $\mathcal{I}_C\text{-}\operatorname{id}_R(X) < \infty$;

(ii) $\mathcal{I}_C\text{-}\operatorname{id}_R(X \otimes^{\mathbf{L}}_R Y) < \infty$ *for all* $Y \in \mathcal{D}_{\mathrm{b}}(R)$ *such that* $\operatorname{fd}_R(Y) < \infty$;

(iii) $\mathcal{I}_C\text{-}\operatorname{id}_R(X \otimes^{\mathbf{L}}_R F) < \infty$ *for some faithfully flat R-module F.*

Corollary 3.2.8. *Let $X \in \mathcal{D}_{\mathrm{b}}(R)$. If there exists a dualizing complex D and $\mathcal{F}_C\text{-}\operatorname{pd}_R(X) < \infty$, then $\mathcal{I}_C\text{-}\operatorname{id}_R(X^\dagger) < \infty$ where $X^\dagger = \mathbf{R}\mathrm{Hom}_R(X, D)$.*

Proof: Since D is a dualizing complex, it has finite injective dimension. Therefore the result follows from Proposition 3.2.4. $\square$

The last result of this section establishes Theorem 1.2.2 from the introduction.

Theorem 3.2.9. *Assume R has a dualizing complex D and let $X \in \mathcal{D}_{\mathrm{b}}(R)$. Then $\mathcal{I}_C\text{-}\operatorname{id}_R(X) < \infty$ if and only if $\mathcal{F}_{C^\dagger}\text{-}\operatorname{pd}_R(X) < \infty$ where $C^\dagger = \mathbf{R}\mathrm{Hom}_R(C, D)$.*

Proof: For the forward implication assume that $\mathcal{I}_C\text{-}\operatorname{id}_R(X) < \infty$. Then set $J = C \otimes^{\mathbf{L}}_R X$. Since $\mathcal{I}_C\text{-}\operatorname{id}_R(X) < \infty$ we have that J has finite injective dimension. By Remark 3.1.2 we have $X \in \mathcal{A}_C(R)$. This explains the first isomorphism in the following display:

$$X \simeq \mathbf{R}\mathrm{Hom}_R(C, J) \simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\mathrm{Hom}_R(C^\dagger, D), J) \simeq C^\dagger \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(D, J).$$

The second isomorphism is from the isomorphism $C \simeq C^{\dagger\dagger}$, and the third is by Hom-evaluation (2.3.16). Observe that since $\mathrm{id}_R(D) < \infty$ and $\mathrm{id}_R(J) < \infty$ we have that $\mathrm{fd}_R(\mathbf{R}\mathrm{Hom}_R(D, J)) < \infty$ by Fact 2.3.18(a). Thus, it follows that $\mathcal{F}_{C^\dagger}\text{-}\mathrm{pd}_R(X) < \infty$ by the displayed isomorphisms.

For the reverse implication assume that $\mathcal{F}_{C^\dagger}\text{-}\mathrm{pd}_R(X) < \infty$. Then we can write $X \simeq C^\dagger \otimes^{\mathbf{L}}_R F$ where $F = \mathbf{R}\mathrm{Hom}_R(C^\dagger, X)$ and $\mathrm{fd}_R(F) < \infty$. We then have the following isomorphisms:

$$X \simeq C^\dagger \otimes^{\mathbf{L}}_R F = \mathbf{R}\mathrm{Hom}_R(C, D) \otimes^{\mathbf{L}}_R F \simeq \mathbf{R}\mathrm{Hom}_R(C, D \otimes^{\mathbf{L}}_R F)$$

where the second isomorphism is by tensor-evaluation (2.3.16). Since $\mathrm{id}_R(D) < \infty$ and $\mathrm{fd}_R(F) < \infty$ we have that $\mathrm{id}_R(D \otimes^{\mathbf{L}}_R F) < \infty$ by Fact 2.3.18(b). By Theorem 3.1.9(c) we conclude that $\mathcal{I}_C\text{-}\mathrm{id}_R(X) < \infty$ as desired. $\square$

3.3. Using Semidualizing Complexes to Detect Gorenstein Rings

The next result fully answers the question of Takahashi and White discussed in the introduction.

Theorem 3.3.1. *Let $(R, \mathfrak{m}, k)$ be a local ring. If there is an R-complex $X \in \mathcal{D}_{\mathrm{b}}(R)$ with finite depth, $\mathcal{F}_C\text{-}\mathrm{pd}_R(X) < \infty$ and $\mathcal{I}_C\text{-}\mathrm{id}_R(X) < \infty$, then R is Gorenstein.*

Proof: Case 1: $\mathrm{depth}_R(X) < \infty$ *and R has a dualizing complex D.*

We first observe that by [10, Theorem 4.6] we have that the following holds:

$$\mathrm{depth}_R(\mathbf{R}\mathrm{Hom}_R(C, X)) = \mathrm{width}_R(C) + \mathrm{depth}_R(X) < \infty. \qquad (\star)$$

Note that $\mathrm{depth}_R(X)$ is finite by assumption, and $\mathrm{width}_R(C)$ is finite by Nakayama's Lemma, as C is homologically finite: see [9, Lemma 2.1].

Set $C^\dagger := \mathbf{R}\mathrm{Hom}_R(C, D)$. The assumption $\mathcal{I}_C\text{-}\mathrm{id}_R(X) < \infty$ with Theorem 3.2.9 implies $\mathcal{F}_{C^\dagger}\text{-}\mathrm{pd}_R(X) < \infty$. Hence by Theorem 3.1.9(b) there exist R-complexes F, G of finite flat dimension such that $C \otimes^{\mathbf{L}}_R F \simeq X \simeq C^\dagger \otimes^{\mathbf{L}}_R G$. Since G has finite flat dimension, [7, Proposotion 4.4] implies $G \in \mathcal{A}_{C^\dagger}(R)$, which explains the first isomorphism in the following display:

$$G \simeq \mathbf{R}\mathrm{Hom}_R(C^\dagger, C^\dagger \otimes^{\mathbf{L}}_R G) \simeq \mathbf{R}\mathrm{Hom}_R(C^\dagger, C \otimes^{\mathbf{L}}_R F) \simeq \mathbf{R}\mathrm{Hom}_R(C^\dagger, C) \otimes^{\mathbf{L}}_R F.$$

The last isomorphism is by tensor evaluation [4, Lemma 4.4(F)].

Theorem 3.1.9(b) implies $F \simeq \mathbf{R}\mathrm{Hom}_R(C, X)$. By ($\star$) we have $\operatorname{depth}_R(F) < \infty$. It follows from [9, Proposition 2.8] that $k \otimes^{\mathbf{L}}_R F \not\simeq 0$. Set $U := \mathbf{R}\mathrm{Hom}_R(C^\dagger, C)$. Since C and $C^\dagger$ are in $\mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$, we have $U \in \mathcal{D}^{\mathrm{f}}_{-}(R)$.

Claim A: We have $U \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$.

To prove this claim it suffices to show that $U \in \mathcal{D}_+(R)$. Assume by way of contradiction that $\inf(U) = -\infty$. Then by [10, 4.5] we know that $\inf(k \otimes^{\mathbf{L}}_R U) = -\infty$. By tensor cancellation and the Künneth formula we have isomorphisms

$$\begin{aligned}
\mathrm{H}_n\left(k \otimes^{\mathbf{L}}_R (F \otimes^{\mathbf{L}}_R U)\right) &\cong \mathrm{H}_n\left((k \otimes^{\mathbf{L}}_R F) \otimes^{\mathbf{L}}_k (k \otimes^{\mathbf{L}}_R U)\right) \\
&\cong \bigoplus_{p+q=n} \mathrm{H}_p(k \otimes^{\mathbf{L}}_R F) \otimes_k \mathrm{H}_q(k \otimes^{\mathbf{L}}_R U).
\end{aligned}$$

Since $k \otimes^{\mathbf{L}}_R F \not\simeq 0$ and $\inf(k \otimes^{\mathbf{L}}_R U) = -\infty$ it follows that $\inf(k \otimes^{\mathbf{L}}_R (F \otimes^{\mathbf{L}}_R U)) = -\infty$. On the other hand, since $F \otimes^{\mathbf{L}}_R U \simeq G \in \mathcal{D}_{\mathrm{b}}(R)$ we have $k \otimes^{\mathbf{L}}_R (F \otimes^{\mathbf{L}}_R U) \simeq k \otimes^{\mathbf{L}}_R G \in \mathcal{D}_+(R)$, so $\inf(k \otimes^{\mathbf{L}}_R (F \otimes^{\mathbf{L}}_R U)) > -\infty$, a contradiction. This establishes Claim A.

Claim B: The complex U has finite projective dimension.

To show this claim assume by way of contradiction that $\operatorname{pd}_R(U) = \infty$. Then because $U \in \mathcal{D}^{\mathrm{f}}_{\mathrm{b}}(R)$ we have $\sup(k \otimes^{\mathbf{L}}_R U) = \infty$ by [4, Proposition 5.5]. As in the proof of Claim A, we conclude that $\sup(k \otimes^{\mathbf{L}}_R (F \otimes^{\mathbf{L}}_R U)) = \infty$. On the other hand, we have $k \otimes^{\mathbf{L}}_R (F \otimes^{\mathbf{L}}_R U) \simeq k \otimes^{\mathbf{L}}_R G$. Since G has finite flat dimension, this implies that $\sup(k \otimes^{\mathbf{L}}_R (F \otimes^{\mathbf{L}}_R U)) < \infty$, a contradiction. This concludes the proof of Claim B.

Now [13, Theorem 1.4] implies that $\mathbf{\Sigma}^n C \simeq C^\dagger = \mathbf{R}\mathrm{Hom}_R(C, D)$ for some $n \in \mathbb{Z}$. Hence by [12, Corollary 3.4] we deduce that R is Gorenstein. This concludes the proof of Case 1.

Case 2: $\operatorname{supp}_R(X) = \{\mathfrak{m}\}$.

For the proof of Case 2, first observe that R is Gorenstein if and only if $\widehat{R}$ is Gorenstein where $\widehat{R}$ is the $\mathfrak{m}$-adic completion of R. Since $\widehat{R}$ has a dualizing complex, by Case 1 it suffices to show that

(1) $\widehat{R} \otimes^{\mathbf{L}}_R X \in \mathcal{D}_{\mathrm{b}}(\widehat{R})$,

(2) $\widehat{R} \otimes^{\mathbf{L}}_R C$ is a semidualizing $\widehat{R}$-complex,

(3) $\mathcal{F}_{\widehat{R}\otimes^{\mathbf{L}}_R C}\text{-}\operatorname{pd}_{\widehat{R}}(\widehat{R}\otimes^{\mathbf{L}}_R X) < \infty$,

(4) $\mathcal{I}_{\widehat{R}\otimes^{\mathbf{L}}_R C}\text{-}\operatorname{id}_{\widehat{R}}(\widehat{R}\otimes^{\mathbf{L}}_R X) < \infty$, and

(5) $\operatorname{depth}_{\widehat{R}}(\widehat{R}\otimes^{\mathbf{L}}_R X) < \infty$.

Observe that (1) follows from the fact that $\widehat{R}$ is flat over R. Items (2) and (3) follow from [7, Lemma 2.6] and Proposition 3.1.11(d), respectively.

To prove (4) note that the first equality in the next sequence is by definition:

$$\begin{aligned}
\mathcal{I}_{\widehat{R}\otimes^{\mathbf{L}}_R C}\text{-}\operatorname{id}_{\widehat{R}}(\widehat{R}\otimes^{\mathbf{L}}_R X) &= \operatorname{id}_{\widehat{R}}\left((\widehat{R}\otimes^{\mathbf{L}}_R C)\otimes^{\mathbf{L}}_{\widehat{R}}(\widehat{R}\otimes^{\mathbf{L}}_R X)\right) + \sup(\widehat{R}\otimes^{\mathbf{L}}_R C) \\
&= \operatorname{id}_{\widehat{R}}\left(\widehat{R}\otimes^{\mathbf{L}}_R (C\otimes^{\mathbf{L}}_R X)\right) + \sup(\widehat{R}\otimes^{\mathbf{L}}_R C).
\end{aligned}$$

The second equality is by tensor cancellation. From the condition $\mathcal{I}_C\text{-}\operatorname{id}_R(X) < \infty$, we have $\operatorname{id}_R(C\otimes^{\mathbf{L}}_R X) < \infty$ by definition. Note that $\mathfrak{m} \in \operatorname{Spec}(R) = \operatorname{supp}_R(C)$ by [23, Proposition 6.6]. Therefore Fact 2.3.23 implies

$$\operatorname{supp}_R(C\otimes^{\mathbf{L}}_R X) = \operatorname{supp}_R(C)\cap\operatorname{supp}_R(X) = \{\mathfrak{m}\}.$$

Hence by [21, Lemma 3.4] the complex $\widehat{R}\otimes^{\mathbf{L}}_R (C\otimes^{\mathbf{L}}_R X)$ has finite injective dimension over $\widehat{R}$, so (4) holds.

For the proof of (5) consider the following sequence:

$$\begin{aligned}
\operatorname{depth}_{\widehat{R}}(\widehat{R}\otimes^{\mathbf{L}}_R X) &= -\sup\left(\mathbf{R}\operatorname{Hom}_{\widehat{R}}(k, \widehat{R}\otimes^{\mathbf{L}}_R X)\right) \\
&= -\sup\left(\mathbf{R}\operatorname{Hom}_{\widehat{R}}(\widehat{R}\otimes^{\mathbf{L}}_R k, \widehat{R}\otimes^{\mathbf{L}}_R X)\right) \\
&= -\sup\left(\widehat{R}\otimes^{\mathbf{L}}_R \mathbf{R}\operatorname{Hom}_R(k, X)\right) \\
&= -\sup\left(\mathbf{R}\operatorname{Hom}_R(k, X)\right) \\
&= \operatorname{depth}_R(X) \\
&< \infty.
\end{aligned}$$

The second equality is because $k \cong \widehat{R}\otimes^{\mathbf{L}}_R k$, and the fourth equality is because $\widehat{R}$ is faithfully flat over R. This establishes (5) and concludes the proof of Case 2.

Case 3: general case.

Let $\mathbf{x}$ be a generating sequence for $\mathfrak{m}$, and let $K = K^R(\mathbf{x})$ be the Koszul complex. Then $\operatorname{supp}_R(K) = \{\mathfrak{m}\}$. Since $\operatorname{depth}_R(X) < \infty$, we have that $\mathfrak{m} \in \operatorname{supp}_R(X)$ by [9, Proposition 2.8]. Hence, we conclude from Fact 2.3.23 that

$$\operatorname{supp}_R(K \otimes^{\mathbf{L}}_R X) = \operatorname{supp}_R(K) \cap \operatorname{supp}_R(X) = \{\mathfrak{m}\}.$$

By Case 2 it suffices to show that

(a) $\operatorname{depth}_R(K \otimes^{\mathbf{L}}_R X) < \infty$,

(b) $K \otimes^{\mathbf{L}}_R X \in \mathcal{D}_{\mathrm{b}}(R)$,

(c) $\mathcal{F}_C\text{-}\operatorname{pd}_R(K \otimes^{\mathbf{L}}_R X) < \infty$, and

(d) $\mathcal{I}_C\text{-}\operatorname{id}_R(K \otimes^{\mathbf{L}}_R X) < \infty$.

Item (a) follows from [9, Proposition 2.8]. For (b), use the conditions $\operatorname{pd}_R(K) < \infty$ and $X \in \mathcal{D}_{\mathrm{b}}(R)$. Items (c) and (d) follow from Proposition 3.2.5 and Proposition 3.2.7. This concludes the proof of Case 3. $\square$

The following result is Theorem 1.2.3 from the introduction.

Theorem 3.3.2. *If there exists an R-complex $X \in \mathcal{D}_{\mathrm{b}}(R)$ such that $\mathcal{F}_C\text{-}\operatorname{pd}_R(X) < \infty$ and $\mathcal{I}_C\text{-}\operatorname{id}_R(X) < \infty$, then $R_{\mathfrak{p}}$ is Gorenstein for all $\mathfrak{p} \in \operatorname{supp}_R(X)$.*

Proof: By Theorem 3.3.1 it suffices to show the following:

(i) $X_{\mathfrak{p}} \in \mathcal{D}_{\mathrm{b}}(R_{\mathfrak{p}})$,

(ii) $C_{\mathfrak{p}}$ is a semidualizing $R_{\mathfrak{p}}$-complex,

(iii) $\mathcal{F}_{C_{\mathfrak{p}}}\text{-}\operatorname{pd}_{R_{\mathfrak{p}}}(X_{\mathfrak{p}}) < \infty$,

(iv) $\mathcal{I}_{C_{\mathfrak{p}}}\text{-}\operatorname{id}_{R_{\mathfrak{p}}}(X_{\mathfrak{p}}) < \infty$, and

(v) $\operatorname{depth}_{R_{\mathfrak{p}}}(X_{\mathfrak{p}}) < \infty$.

Item (i) follows from the fact that $R_{\mathfrak{p}}$ is a flat over R. Item (ii) follows from [7, Lemma 2.5]. Items (iii) and (iv) are by Corollary 3.1.12.

(5) As $\mathfrak{p} \in \mathrm{supp}_R(X)$, we have $\mathfrak{p}R_{\mathfrak{p}} \in \mathrm{supp}_{R_{\mathfrak{p}}}(X_{\mathfrak{p}})$ by [23, Proposition 3.6]. Since $\mathfrak{p}R_{\mathfrak{p}}$ is the maximal ideal of the local ring $R_{\mathfrak{p}}$, we deduce from [9, Proposition 2.8] that $\mathrm{depth}_{R_{\mathfrak{p}}}(X_{\mathfrak{p}}) < \infty$. $\square$

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