COLORINGS OF ZERO-DIVISOR GRAPHS OF COMMUTATIVE RINGS

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ABSTRACT

We will focus on Beck's conjecture that the chromatic number of a zero-divisor graph of a ring R is equal to the clique number of the ring R. We begin by calculating the chromatic number of the zero-divisor graphs for some finite rings and characterizing rings whose zero-divisor graphs have finite chromatic number, known as colorings. We will discuss some properties of colorings and elements called separating elements, which will allow us to determine that Beck's conjecture holds for rings that are principal ideal rings and rings that are reduced. Then we will characterize the finite rings whose zero-divisor graphs have chromatic number less than or equal to four. In the general case, we will discuss a local ring that serves as a counterexample to Beck's conjecture.

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1. INTRODUCTION

We will investigate the connections between commutative ring theory and graph theory, focusing on the results of Istvan Beck in [1], and the counterexample given by Anderson and Naseer in [6]. Beck's results are centered around the following conjecture, and the counterexample in [6] shows this conjecture to be false.

Conjecture 1.1. *The chromatic number of a zero-divisor graph of a ring R is equal to the clique number of the ring R. That is, $\chi(\Gamma_0(R)) = cl(R)$.*

We will begin by discussing the chromatic number of some finite rings in Chapter 2. In Chapter 3, we will present several important results about the chromatic number of a zero-divisor graph of a ring R and the clique number of a ring R, and their relationship. Chapter 4 will consist of a discussion about the properties colorings, which are rings whose zero-divisor graphs have finite chromatic number. In Chapter 5, we will discuss the idea of separating elements which will help us address Beck's conjecture for reduced rings and principal ideal rings. Then we can finally describe the finite rings whose zero-divisor graphs have finite chromatic number in Chapter 6. We will use Chapter 7 to present and discuss a counterexample to Beck's conjecture that $\chi(\Gamma_0(R)) = cl(R)$ provided by Anderson and Naseer in [6].

Before we begin our investigation, it is necessary to state some ring theoretic and graph theoretic properties that will be important and useful.

1.1. Ring Theoretic Properties

Some good sources for our results in commutative algebra are [4], [5], and [2].

Definition 1.2. *A ring is a set together with two binary operations, $+$ and $\cdot$, satisfying the following:*

(a) $(R, +)$ is an abelian group,

(b) $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in R$,

(c) $a \cdot (b + c) = a \cdot b + a \cdot c$ and $(a + b) \cdot c = a \cdot c + b \cdot c$ for all $a, b, c \in R$.

The ring R is commutative if $ab = ba$ for all $a, b \in R$. If there is an element $1 = 1_R$ such that $1(a) = a(1) = a$ for all $a \in R$ then R is said to have a multiplicative identity. All rings discussed will be commutative with identity.

Definition 1.3. *An element $x \in R$ is said to be a zero-divisor if there exists some element $0 \neq y \in R$ such that $xy = 0$. The set of all zero-divisors is denoted $Z(R)$.*

Definition 1.4. *A subset $C = \{x_1, \ldots, x_n\}$ of R is called a clique if $x_i x_j = 0$ for all $i \neq j$. The clique number of R is denoted $cl(R)$. If R contains a clique with n elements and every clique contained in R has at most n elements, we say that the clique number of R is n and denote this as $cl(R) = n$. If the sizes of the cliques of R are without bound, the clique number is defined to be $cl(R) = \infty$.*

Notice that an integral domain is a ring with identity that contains no non-trivial zero-divisors.

Definition 1.5. *An element $x \in R$ is nilpotent if $x^n = 0$ for some $n \geq 1$.*

The set of all nilpotent elements, also known as the nilradical, will be denoted $\mathfrak{N}$. R is said to be reduced if $\mathfrak{N} = (0)$.

Definition 1.6. *An element $x \in R$ is a unit if there is an element $y \in R$ such that $xy = 1$.*

A commutative ring in which every nonzero element is a unit is called a field.

Fact 1.7. *Any field is an integral domain. In addition, any finite integral domain is a field.*

Definition 1.8. *The characteristic of a ring R is the smallest integer $n > 0$ such that $nx = 0$ for all $x \in R$. If no such n exists then $char(R) = 0$*

Definition 1.9. *A nonempty subset I of a ring R is called an ideal of R if it is closed under the operations of addition, and multiplication, i.e., for all $r \in R$ and $x \in I$, $rx \in I$.*

There are different types of ideals.

Definition 1.10. *An ideal $\mathfrak{P}$ is a prime ideal if $ab \in \mathfrak{P}$ implies $a \in \mathfrak{P}$ or $b \in \mathfrak{P}$.*

Definition 1.11. *An ideal $\mathfrak{M} \subset R$ is a maximal ideal if for any other ideal I in R such that $\mathfrak{M} \subseteq I \subseteq R$ then either $\mathfrak{M} = I$ or $I = R$.*

Definition 1.12. *An ideal I is principal if it is generated by one element, say $I = (a)$.*

Theorem 1.13. *The nilradical is the intersection of all prime ideals. Furthermore, the nilradical is the intersection of all minimal prime ideals.*

Proof. Suppose that R is a commutative ring with identity. Let $\mathfrak{N}$ be the ideal containing all nilpotent elements of R. Let $\alpha \in \mathfrak{N}$. Then $\alpha^n = 0$ for some $n \in \mathbb{N}$. Therefore $\alpha^n = 0 \in \mathfrak{P}$ for every prime ideal $\mathfrak{P}$. Since $\mathfrak{P}$ is a prime ideal, $\alpha^n \in \mathfrak{P}$ implies that either $\alpha \in \mathfrak{P}$ or $\alpha^{n-1} \in \mathfrak{P}$. If $\alpha \in \mathfrak{P}$, then we are done. If $\alpha^{n-1} \in \mathfrak{P}$, then either $\alpha \in \mathfrak{P}$ or $\alpha^{n-2} \in \mathfrak{P}$. Inductively, we continue this process until we have that if $\alpha^2 \in \mathfrak{P}$, then either $\alpha \in \mathfrak{P}$ or $\alpha \in \mathfrak{P}$. Therefore $\alpha \in \mathfrak{P}$, for every prime ideal. Hence $\alpha \in \bigcap_{\mathfrak{P}:\text{prime}} \mathfrak{P}$.

For the reverse containment, suppose that $\alpha \notin \mathfrak{N}$. We will show that $\alpha \notin \bigcap_{\mathfrak{P}:\text{prime}} \mathfrak{P}$. Since $\alpha \notin \mathfrak{N}$, we have that $\alpha^n \neq 0$ for any $n \in \mathbb{N}$. Let $\sum = \{I \subset R \mid \alpha^n \notin I \text{ for any } n \in \mathbb{N}\}$. Since $\alpha^n \neq 0$ for any $n \in \mathbb{N}$, we have that $(0) \in \sum$ and hence $\sum$ is nonempty. Now take a chain $\mathcal{C} = \{I_\alpha\}_{\alpha \in \Lambda} \subset \sum$. Let $A = \bigcup_{\alpha \in \Lambda} I_\alpha$. Notice that A is an ideal that does not contain α^n for any $n \in \mathbb{N}$ and A is an upper bound of $\mathcal{C}$ since $A \in \sum$. Since every chain in $\sum$ has an upper bound in $\sum$, then by Zorn's Lemma, $\sum$ has a maximal element, call it B. We now need to show that B is a prime ideal that does not contain a power of α. Suppose that $xy \in B$ but $x, y \notin B$. Then since B is a maximal element of $\sum$, neither of the ideals $(x) + B$ and $(y) + B$ are elements of $\sum$. Therefore there exist $m, n \in \mathbb{N}$ such that $\alpha^n \in (x) + B$ and $\alpha^m \in (y) + B$. Now let $\alpha^n = r_1 x + t_1$ and $\alpha^m = r_2 y + t_2$, where $r_1, r_2, \in R$ and $t_1, t_2 \in B$. Then

$$\begin{aligned}
\alpha^{n+m} &= \alpha^n \alpha^m \\
&= (r_1 x + t_1)(r_2 y + t_2) \\
&= r_1 x r_2 y + r_1 x t_2 + t_1 r_2 y + t_1 t_2 \\
&= r_1 r_2 x y + r_1 x t_2 + t_1 r_2 y + t_1 t_2.
\end{aligned}$$

Since $r_1 r_2 xy \in (xy) \subset B$ and $r_1 x t_2 + t_1 r_2 y + t_1 t_2 \in B$, we have that $\alpha^{n+m} \in (xy) + B = B$. As a result, $B \notin \sum$, which is a contradiction. Therefore either $x \in B$ or $y \in B$ and hence B is a prime ideal that does not contain α^n for all $n \in \mathfrak{N}$. In particular, B does not contain α and hence $\alpha \notin \bigcap_{\mathfrak{P}:\text{prime}} \mathfrak{P}$. Hence $\bigcap_{\mathfrak{P}:\text{prime}} \mathfrak{P} \subseteq \mathfrak{N}$.

Thus $\mathfrak{N} = \bigcap_{\mathfrak{P}:\text{prime}} \mathfrak{P}$. □

There is a relationship between ideals and the quotient group R/I, which can be seen in the next two theorems.

Theorem 1.14. *Let R be a commutative ring with identity and $I \subseteq R$ an ideal. Then the quotient group R/I is a ring with multiplication given by $(a+I)(b+I) = ab+I$ for all $a, b \in R$, and addition given by $(a+I)+(b+I) = (a+b)+I$ for all $a, b \in R$. Moreover R/I is commutative with identity.*

Theorem 1.15. *Let R be a commutative ring with identity and let $I \subsetneq R$ be an ideal.*

(a) *I is maximal if and only if R/I is a field.*

(b) *I is prime if and only if R/I is an integral domain.*

Proof. First we prove (a). Suppose R/I is a field. Then every nonzero element is a unit. Suppose J is an ideal that properly contains I. Consider the element $x \in J \setminus I$. Since $x \notin I$, the element $x + I$ is a nonzero element in R/I and every nonzero element in R/I is a unit, there is an element $y + I \in R/I$ such that $xy + I = (x + I)(y + I) = 1 + I$. Hence there is an element $\alpha \in I$ such that $xy + \alpha = 1$. Hence $(x, I) = R$. Since $(x, I) \subseteq J$, we must have that $J = R$, and as a result I is maximal.

For the converse, suppose that I is a maximal ideal. Consider the nonzero element $x + I \in R/I$. Since $x + I$ is a nonzero element, $x \notin I$. Therefore $(I, x) = R$. Hence there is an element $r \in R$ and $\alpha \in I$ such that $rx + \alpha = 1$. Therefore $(r + I)(x + I) = rx + I = 1 - \alpha + I = 1 + I$, which implies that $x + I$ is a unit. As a result, R/I is a field.

Now we prove (b). Suppose that R/I is an integral domain. Then R/I has no nonzero zero-divisors. Suppose $xy \in I$. Then $(x + I)(y + I) = xy + I = 0 + I$. Since R/I is an integral domain, either $x + I = 0 + I$ or $y + I = 0 + I$. Therefore either $x \in I$ or $y \in I$. Hence I is prime.

For the converse, suppose that I is prime. That is, if $xy \in I$, then $x \in I$ or $y \in I$. Suppose that $x + I \in R/I$ is a nonzero zero-divisor. This means that $x \notin I$. Then there is an element $y + I \in R/I$ such that $(x + I)(y + I) = xy + I = 0 + I$. Since I is prime and $x \notin I$, we must have that $y \in I$. Therefore $y + I = 0 + I$ and R/I is an integral domain. $\square$

Definition 1.16. *Let R and S be rings. A function $f : R \to S$ is said to be a ring homomorphism if*

(a) $f(a + b) = f(a) + f(b)$ for all $a, b \in R$.

(b) $f(ab) = f(a)f(b)$ for all $a, b \in R$.

Theorem 1.17. *Two Isomorphism Theorems*

(a) If $f : R \to S$ is a homomorphism, then f induces an isomorphism $\overline{f} : R/ker(f) \to im(f)$.

(b) If $I \subseteq J$ then J/I is an ideal of R/I and there is an isomorphism of rings $(R/I)/(J/I) \cong R/J$.

Definition 1.18. *(Localization) Let R is an integral domain. A nonempty subset $S \subseteq R \setminus \{0\}$ is said to be multiplicatively closed if $s, t \in S$ implies that $st \in S$. If $S \subsetneq R$ is a multiplicatively closed subset of $R \setminus \{0\}$, the localization of R at S is given by $R_S = \{\frac{r}{s} \mid r \in R, s \in S\}$ with the usual addition and multiplication of fractions $\frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{r_1 s_2 + r_2 s_1}{s_1 s_2}$ and $\left(\frac{r_1}{s_1}\right)\left(\frac{r_2}{s_2}\right) = \left(\frac{r_1 r_2}{s_1 s_2}\right)$. In addition, two elements $\frac{a}{b}$ and $\frac{c}{d}$ in $R_{\mathfrak{P}}$ are equivalent if and only if there exists an element $t \in S$ such that $t(ad - bc) = 0$.*

The localization that we will encounter is when $S = R \setminus \mathfrak{P}$, where $\mathfrak{P}$ is a prime ideal. We write $R_{\mathfrak{P}} = \{\frac{r}{s} \mid r \in R, s \in R \setminus \mathfrak{P}\}$. An important result of localizations is that $R_{\mathfrak{P}}$ is a local ring with maximal ideal $\mathfrak{P}R_{\mathfrak{P}}$.

Definition 1.19. *The Krull dimension of R is the supremum of the lengths of all chains of prime ideals in R. If $dim(R) = 0$, every prime ideal is maximal and minimal.*

Theorem 1.20. *If R is a finite ring then $dim(R) = 0$.*

Proof. Suppose that R is a finite ring and let $\mathfrak{P} \subset R$ a prime ideal. Since $\mathfrak{P}$ is prime, $R/\mathfrak{P}$ is an integral domain. Also, since R is finite, $R/\mathfrak{P}$ is finite. Therefore $R/\mathfrak{P}$ is a finite integral domain, which is a field. Hence $\mathfrak{P}$ is a maximal ideal, giving us that $dim(R) = 0$. $\square$

Definition 1.21. *For subsets I and K of R, $(I : K) = \{r \in R \mid rK \subset I\}$.*

Definition 1.22. *The annihilator, denoted $Ann(I) = (0 : I)$, of $I \subseteq R$ is the set of all elements $r \in R$ such that for each $s \in I$, $rs = 0$. If $I = \{x\}$ then $Ann(x) = (0 : x)$, which is the set of elements $r \in R$ such that $rx = 0$.*

Definition 1.23. *A prime ideal $\mathfrak{P}$ of R is called an associated prime ideal if $\mathfrak{P} = Ann(x)$ for some element $x \in R$. The set of all associated prime ideals is denoted $Ass(R)$.*

Definition 1.24. *The proper ideals A and B of the ring R are said to be comaximal if $A + B = R$.*

Theorem 1.25. *(Chinese Remainder Theorem) Let $A_1, \ldots, A_k$ be ideals in R. The map*

$$R \to R/A_1 \times \cdots \times R/A_k \text{ defined by } r \mapsto (r + A_1, \ldots, r + A_k)$$

is a ring homomorphism with kernel $A_1 \cap \cdots \cap A_k$. If for each $i, j \in \{1, 2, \ldots, k\}$ with $i \neq j$ the ideals A_i and A_j are comaximal, then this map is surjective and $A_1 \cap \cdots \cap A_k = A_1 \cdots A_k$, so

$$R/(A_1 \cdots A_k) \cong R/(A_1 \cap \cdots \cap A_k) \cong R/A_1 \times \cdots \times R/A_k.$$

The class of von Neumann regular rings will be of interest later. Here are a couple of basic results on von Neumann regular rings.

Definition 1.26. *R is von Neumann regular if for every $x \in R$, there exists $y \in R$ such that $x = xyx$.*

Theorem 1.27. *Any direct product or direct sum of fields is von Neumann regular.*

Theorem 1.28. *Suppose R is a commutative ring with identity. If R is von Neumann regular and $\mathfrak{P} \subset R$ be a prime ideal, then $R_{\mathfrak{P}} \cong R/\mathfrak{P}$.*

Now that we have stated useful ring theoretic properties, we can define some graph theoretic properties to understand the connection between commutative algebra and graph theory.

1.2. Graph Theoretic Properties

We define some graph theoretic properties that will be vital in our investigation of Beck's conjecture. Some useful sources for graph theory are [8] and [3].

Definition 1.29. *A graph is a pair $G = (V, E)$ of sets such that $E \subseteq V \times V$. It is always assumed that $V \cap E = \emptyset$. The elements of V are the vertices of the graph G, and the elements of E are the edges of the graph G. The picture of a graph is drawn by denoting each vertex as a dot, and joining two dots by drawing a line between them if two vertices form an edge.*

Definition 1.30. *A simple graph is an undirected graph that has no loops, which are edges connected at both ends to the same vertex, and has no more than one edge between any two vertices. The edges of the graph form a set, and each edge is a pair of distinct vertices. If a simple graph has n vertices, then each vertex can be connected to, or adjacent to, at most $n-1$ other vertices.*

Definition 1.31. *A complete graph is a simple graph in which every vertex is adjacent to every other vertex. A complete graph on n vertices is denoted K^n.*

Definition 1.32. *A zero-divisor graph of R, denoted $\Gamma_0(R)$, is a simple graph whose vertex set consists of elements of R including 0, and whose edge set is the set of elements (x, y) where $xy = 0$ for elements $x, y \in R$. If $xy = 0$, we say that x and y are adjacent.*

Definition 1.33. *The chromatic number of a zero-divisor graph of a ring R, denoted by $\chi(\Gamma_0(R))$, is the minimal number of colors required to assign each vertex in a zero-divisor graph a color so that no two adjacent vertices are assigned the same color.*

Definition 1.34. *In relation to the zero-divisor graph $\Gamma_0(R)$, a clique is a complete subgraph of the zero-divisor graph. We will denote the clique number of $\Gamma_0(R)$ as $cl(R)$, which is defined to be the greatest integer $r \geq 1$ such that $K^r \subset \Gamma_0(R)$. Also, if $K^r \subset \Gamma_0(R)$ for all $r \geq 1$, we write $cl(R) = \infty$.*

For simplicity, we will refer to $cl(\Gamma_0(R))$ as $cl(R)$.

Definition 1.35. *A graph $\Gamma_0(R)$ is said to be $k-$colorable if $\Gamma_0(R)$ can be colored with less than or equal to k colors.*

Theorem 1.36. *A graph is $2-$colorable if and only if the graph does not contain any odd cycle.*

The following result comes from [7].

Theorem 1.37. *Let k be a positive integer, and let the graph G have the property that any finite subgraph is $k-$colorable. Then G is $k-$colorable itself.*

2. THE CHROMATIC NUMBER OF SOME ZERO-DIVISOR GRAPHS

Now that we have established some basic properties of zero divisor graphs, we will present some of the rings from [1] and demonstrate how to calculate the chromatic number of their zero-divisor graphs.

Our first proposition is about zero-divisor graphs with chromatic number one.

Proposition 2.1. $\chi(\Gamma_0(R)) = 1$ *if and only if* $R = \{0\}$.

Proof. Suppose that $\chi(\Gamma_0(R)) = 1$. Then the zero-divisor graph must not contain any adjacencies and hence can only consist of the zero ring.

If $R = \{0\}$, then the zero-divisor graph can only consist of a vertex 0, which must have chromatic number one. □

The next proposition is about zero-divisor graphs with chromatic number two. We provide a diagram in Figure 2.1 of each of the zero-divisor graphs for each of the rings mentioned in this proposition.

Proposition 2.2. $\chi(\Gamma_0(R)) = 2$ *if and only if* R *is an integral domain,* $R \cong \mathbb{Z}_4$*, or* $R \cong \mathbb{Z}_2[X]/(X^2)$.

Proof. For the forward direction, suppose that $\chi(\Gamma_0(R)) = 2$. In the case that R is an integral domain, $\chi(\Gamma_0(R)) = 2$ because there are no nontrivial zero-divisors. Now, suppose that R is not an integral domain. We will verify that $R \cong \mathbb{Z}_4$ or $R \cong \mathbb{Z}_2[X]/(X^2)$. Let $xy = 0$ for nonzero elements x and y in R. Then the set $\{0, x, y\}$ must form a clique. We know that $\chi(\Gamma_0(R)) \geq cl(R)$ is always true, and we have that $\chi(\Gamma_0(R)) = 2$. Therefore $cl(R) \leq 2$. As a result, we must have $x = y$. Hence for a nonzero element x we have that $x^2 = 0$. We know that Rx is a clique and we can conclude that $|Rx| = 2$. Now let $a \in Ann(x)$. Then $ax = 0$ and the set $\{0, x, a\}$ is a clique in R. Since $cl(R) \leq 2$, either $a = 0$ or $a = x$. Therefore a is an element belonging $Rx = \{0, x\}$, giving us that $Ann(x) = Rx$. Hence $|Ann(x)| = |Rx| = 2$.

Now consider the exact sequence $0 \longrightarrow Ann(x) \xrightarrow{f} R \xrightarrow{g} Rx \longrightarrow 0$, where $g(r) = rx$. Certainly f is one-to-one as it is an inclusion map, and g is onto. Since g is onto, we have that

$im(g) = Rx$. It can be shown that $im(f) = ker(g)$ and that $Ann(x) = ker(g)$. Then by the first isomorphism theorem, $R/ker(g) \cong im(g)$. Therefore $Rx \cong im(g) \cong R/ker(g) \cong R/im(f) \cong R/Ann(x)$. By Lagrange's Theorem $|Rx| = |R|/|Ann(x)|$, which implies that $|R| = |Rx||Ann(x)| = 2 \cdot 2 = 4$. We know that the characteristic of R must divide the order of R, so the characteristic of R is either 2 or 4.

If $char(R) = 4$, we have $R \cong \mathbb{Z}_4$.

If $char(R) = 2$, we can derive that $R \cong \mathbb{Z}_2[X]/(X^2)$. Recall that $cl(R) = 2$, so $R \ncong \mathbb{Z}_2 \bigoplus \mathbb{Z}_2$ since it has a clique of size 3. Since $\chi(\Gamma_0(R)) = 2$, then for all nonzero elements $a, b \in R$ where $a \neq b$, we have that $ab \neq 0$. Since R must additively behave like $\mathbb{Z}_2 \bigoplus \mathbb{Z}_2$, we can write R as $\{0, 1, x, x+1\}$. Since we are under the assumption that R is not an integral domain and $ab \neq 0$ for $a \neq b$, we have that $x^2 = 0$. Therefore, $R \cong \mathbb{Z}_2[X]/(X^2)$.

For the reverse direction, suppose R is an integral domain. In an integral domain there are no nontrivial zero-divisors. Therefore in the zero-divisor graph, 0 is adjacent to every nonzero element while no two nonzero elements x and y can be adjacent. Therefore we can assign one color to the 0 vertex and we can color the rest of the vertices with a second color. Hence $\chi(\Gamma_0(R)) = 2$.

In $R \cong \mathbb{Z}_4$, there are four elements $\{0, 1, 2, 3\}$. The element 0 is adjacent to $1, 2$, and 3, but no two nonzero elements are adjacent. Therefore we can color 0 with one color and $1, 2$, and 3 with a second color. Hence $\chi(\Gamma_0(\mathbb{Z}_4)) = 2$.

Consider the ring $R \cong \mathbb{Z}_2[X]/(X^2)$. This ring has elements of the form $a + bx$, where $a \in \{0, 1\}$ and $b \in \{0, 1\}$. Therefore R consists of the elements 0, 1, x, and $1 + x$. Since no two nonzero elements multiply to zero, they cannot be adjacent in the zero-divisor graph. Therefore we can assign one color to 0 and a second color to the rest of the vertices. Hence $\chi(\Gamma_0(\mathbb{Z}_2[X]/(X^2))) = 2$. □

The next proposition gives us a clique number and chromatic number for rings of the form $R \cong \mathbb{Z}_n$.

Proposition 2.3. *Let $p_1, p_2, \ldots, p_k, q_1, q_2, \ldots, q_r$ be distinct prime numbers and $N = p_1^{2n_1} \ldots p_k^{2n_k} q_1^{2m_1+1} \ldots q_r^{2m_r+1}$. Then $\chi(\Gamma_0(\mathbb{Z}_N)) = cl(\mathbb{Z}_N) = p_1^{n_1} \ldots p_k^{n_k} q_1^{m_1} \ldots q_r^{m_r} + r$.*

Proof. Let $y_0 = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} q_1^{m_1+1} q_2^{m_2+1} \cdots q_r^{m_r+1}$. Then we have that $y_0^2 = p_1^{2n_1} p_2^{2n_2} \cdots p_k^{2n_k} q_1^{2m_1+2} q_2^{2m_2+2} \cdots q_r^{2m_r+2} = N q_1 q_2 \cdots q_r = 0$ in $\mathbb{Z}_N$. Notice that $\mathbb{Z}_N = \{0, 1, 2, \ldots, p_1^{n_1} \ldots p_k^{n_k} q_1^{m_1} \ldots q_r^{m_r} - 1, \ldots, p_1^{2n_1} \ldots p_k^{2n_k} q_1^{2m_1+1} \ldots q_r^{2m_r+1} - 1\}$. Consider $y_0\mathbb{Z}_N$, which

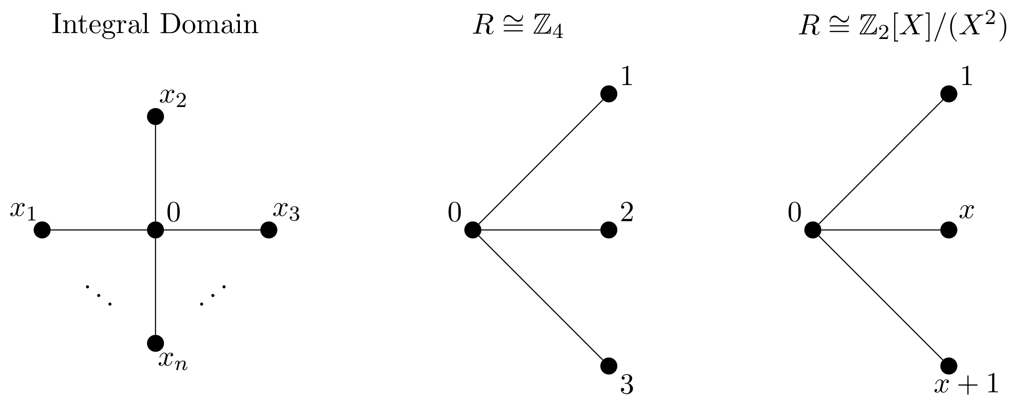

Figure 2.1: Zero-divisor graphs of rings with chromatic number 2.

equals the set:

$\{0, y_0, 2y_0, \ldots, y_0(p_1^{n_1} \ldots p_k^{n_k} q_1^{m_1} \ldots q_r^{m_r} - 1), \ldots, y_0(p_1^{2n_1} \ldots p_k^{2n_k} q_1^{2m_1+1} \ldots q_r^{2m_r+1} - 1)\}$. Every element beyond $y_0(p_1^{n_1} \ldots p_k^{n_k} q_1^{m_1} \ldots q_r^{m_r} - 1) = N - y_0$ is a repeat of some element before it. Consider $y_0(p_1^{n_1} \ldots p_k^{n_k} q_1^{m_1} \ldots q_r^{m_r}) = N$, which is 0 in $\mathbb{Z}_N$, and $y_0(p_1^{n_1} \ldots p_k^{n_k} q_1^{m_1} \ldots q_r^{m_r} + 1) = N + y_0$, which is y_0 in $\mathbb{Z}_N$. Therefore it must be that $y_0\mathbb{Z}_N = \{0, y_0, 2y_0, \ldots, y_0(p_1^{n_1} \ldots p_k^{n_k} q_1^{m_1} \ldots q_r^{m_r} - 1)\}$. In fact, $y_0\mathbb{Z}_N$ forms a clique since every element is a multiple of y_0 and $y_0^2 = 0$ in $\mathbb{Z}_N$. This clique has $\frac{N}{y_0} = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} q_1^{m_1} q_2^{m_2} \cdots q_r^{m_r}$ elements.

Let $y_i = \frac{y_0}{q_i}$ for $1 \le i \le r$. We claim that the set $C = y_0\mathbb{Z}_N \cup \{y_1, y_2, \ldots, y_r\}$ is a clique containing $t = p_1^{n_1} p_2^{n_2} \cdots p_k^{n_k} q_1^{m_1} q_2^{m_2} \cdots q_r^{m_r} + r$ elements and that $cl(R) \ge t$. Notice that $y_i \notin y_0\mathbb{Z}_N$ for $1 \le i \le r$ since $y_i < y_0$ for $1 \le i \le r$, which is because $y_i = \frac{y_0}{q_i} = p_1^{n_1} p_2^{n_2} \ldots p_k^{n_k} q_1^{m_1+1} \ldots q_{i-1}^{m_{i-1}+1} q_i^{m_i} q_{i+1}^{m_{i+1}+1} \ldots q_r^{m_r+1}$, and for every $0 \ne x \in y_0\mathbb{Z}_N$, we have that $x \ge y_0$. Furthermore, if we take $y_0 x \in y_0\mathbb{Z}_N$ and $y_i \in \{y_1, y_2, \ldots, y_r\}$ we have that $y_0 x y_i = y_0 x \cdot \frac{y_0}{q_i} = \frac{y_0^2 x}{q_i} = \frac{p_1^{2n_1} \cdots p_k^{2n_k} q_1^{2m_1+2} \cdots q_r^{2m_r+2} x}{q_i} = N q_1 \cdots q_{i-1} q_{i+1} \cdots q_r$, which is equal to 0 in $\mathbb{Z}_N$. If we take distinct elements y_i and y_j in $\{y_1, y_2, \ldots, y_r\}$, we get that $y_i y_j = \frac{y_0}{q_i} \cdot \frac{y_0}{q_j} = \frac{y_0^2}{q_i q_j} = \frac{N q_1 \cdots q_r}{q_i q_j} = N q_1 \cdots q_{i-1} q_{i+1} \cdots q_{j-1} q_{j+1} \cdots q_r$, which also equals 0 in $\mathbb{Z}_N$. Therefore C is a clique consisting of t elements, giving us that $cl(R) \ge t$. Hence $\chi(\Gamma_0(\mathbb{Z}_N)) \ge t$.

To show that $cl(R) \le t$, we begin by assigning a distinct color to each element in C. Let $x_i = N/p_i^{n_i}$ for $1 \le i \le k$. Notice that $x_1, \ldots x_k$ belong to C and have therefore been assigned a color. We will let $f(y)$ denote the color of element y and assign colors to the remaining elements in $\mathbb{Z}_N$. Take $x \notin y_0\mathbb{Z}_N$. If $p_1^{n_1} \cdots p_k^{n_k}$ divides x, we will define $f(x) = f(y_j)$, where $j = min\{i \mid q_i^{m_1+1}$

does not divide $x\}$. On the other hand, if $p_1^{n_1} \cdots p_k^{n_k}$ does not divide x, we will define $f(x) = f(x_j)$, where $j = min\{i \mid p_i^{n_i}$ does not divide $x\}$.

We will ensure that adjacent vertices are assigned distinct colors and that vertices assigned the same color are not adjacent. If $p_1^{n_1} \cdots p_k^{n_k}$ divides x, then x is assigned the same color as y_j. Since $q_j^{m_j+1}$ does not divide x, the exponent on q_j in x must be strictly less that $m_j + 1$. Furthermore, xy_j must contain a power of q_j that is strictly less than $(m_j+1)(m_j) = 2m_j+1$ since $y_j = \frac{y_0}{q_j} = p_1^{n_1} p_2^{n_2} \ldots p_k^{n_k} q_1^{m_1+1} \ldots q_{j-1}^{m_{j-1}+1} q_j^{m_j} q_{j+1}^{m_{j+1}+1} \ldots q_r^{m_r+1}$. Therefore xy_j cannot be 0 in $\mathbb{Z}_N$, and x and y_j cannot be adjacent. Hence x and y_j can be assigned the same color.

Now suppose that $p_1^{n_1} \cdots p_k^{n_k}$ does not divide x. In this case, x is assigned the same color as x_j. We will verify that x and x_j are not adjacent. Since $p_j^{n_j}$ does not divide x, the exponent on p_j in x must be strictly less than n_j. We also know that $x_j = N/p_j^{n_j} = p_1^{2n_1} \cdots p_{j-1}^{2n_{j-1}} p_j^{n_j} p_{j+1}^{2n_{j+1}} \cdots p_k^{2n_k} q_1^{2m_1+1} \cdots q_r^{2m_r+1}$. Therefore xx_j must contain a power of p_j that is strictly less than $(n_j)(n_j) = 2n_j$ Hence xx_j cannot be 0 in $\mathbb{Z}_N$, which implies that x and x_j cannot be adjacent. Therefore we can conclude that $\chi(\Gamma_0(\mathbb{Z}_N)) \leq t$. Since $\chi(\Gamma_0(\mathbb{Z}_N)) \geq cl(\mathbb{Z}_N)$, we also have that $cl(\mathbb{Z}_N) \leq t$. $\square$

3. RINGS WITH $\chi(\Gamma_0(R)) < \infty$

We will begin to characterize the rings whose zero-divisor graphs have finite chromatic number as in [1]. These results will help us prove Beck's first conjecture in the case of reduced rings. We will start off by stating a definition and an important lemma which will be useful in proving several other results.

Definition 3.1. *An element $x \in R$ is said to be finite if the ideal Rx is a finite set.*

Lemma 3.2. *If R has an infinite number of finite elements, then R contains an infinite clique.*

Proof. Let $x_1, x_2, \ldots, x_n, \ldots$ be a list of distinct finite elements contained in R. By definition, the ideals $(x_1)R, (x_2)R, \ldots (x_n)R, \ldots$ are finite sets. Consider the elements $x_1x_2, x_1x_3, \ldots, x_1x_n, \ldots$, which belong to the finite set $(x_1)R$. Since $(x_1)R$ is a finite set, there must be an infinite list of elements in the set $\{x_1x_2, x_1x_3, \ldots, x_1x_n, \ldots\}$ that are equal. That is, for an infinite subsequence $\{a_1, a_2, \ldots, a_k, \ldots\}$ of $\{2, 3, \ldots n, \ldots\}$, we have that $x_1x_{a_1} = x_1x_{a_2} = \cdots = x_1x_{a_k} = \ldots$. So we will now consider the sequence $\{x_{a_2}, x_{a_3}, \ldots, x_{a_k}, \ldots\}$. Since this sequence of elements is a subset of the list of distinct finite elements of R, the ideals $(x_{a_1})R, (x_{a_2})R, \ldots (x_{a_k})R, \ldots$ are finite sets.

Consider the elements $x_{a_1}x_{a_2}, x_{a_1}x_{a_3}, \ldots, x_{a_1}x_{a_k}, \ldots$, which belong to the finite set $(x_{a_1})R$. Since $(x_{a_1})R$ is a finite set, there must be an infinite list of elements belonging to the set of elements $\{x_{a_1}x_{a_2}, x_{a_1}x_{a_3}, \ldots, x_{a_1}x_{a_k}, \ldots\}$ that are equal to each other. Therefore for an infinite subsequence $\{a_{1,1}, a_{1,2}, \ldots, a_{1,k}, \ldots\}$ of $\{a_2, \ldots a_k, \ldots\}$, we have that $x_{a_1}x_{a_{1,1}} = x_{a_1}x_{a_{1,2}} = \cdots = x_{a_1}x_{a_{1,k}} = \ldots$.

If we continue this process, we can create a subsequence $y_1, y_2, \ldots y_k, \ldots$ of the sequence $x_1, x_2, \ldots, x_n, \ldots$ such that $y_iy_j = y_iy_m$ when $j, m > i$, where $y_1 = x_1, y_2 = x_{a_1}, y_3 = x_{a_{1,1}}$, and so on.

Now, we let $z_{ij} = y_i - y_j$. Suppose that $z_{ij}z_{kr} = 0$. We will show that this equality holds for $i < j < k < r$. Notice that $z_{ij}z_{kr} = (y_i - y_j)(y_k - y_r) = y_iy_k - y_iy_r - y_jy_k + y_jy_r = 0$ when either $y_iy_k = y_iy_r$ and $y_jy_k = y_jy_r$, or $y_iy_k = y_jy_k$ and $y_iy_r = y_jy_r$. Without loss of generality, we will assume that $y_iy_k = y_iy_r$ and $y_jy_k = y_jy_r$. By definition, $y_iy_k = y_iy_r$ when $k, r > i$ and $y_jy_k = y_jy_r$ when $k, r > i$. Since it does not matter which of i and j is larger and which of k and r are larger, we can assume $i < j$ and $k < r$. Hence $z_{ij}z_{kr} = 0$ if $i < j < k < r$.

We will construct an infinite clique. Consider $z_{1,2}z_{3,4} = 0 = z_{1,2}z_{3,5}$. Certainly $z_{3,4} \neq z_{3,5}$, which implies that at least one of $z_{3,4}$ and $z_{3,5}$ will be different from $z_{1,2}$. If it happens to be the case that $z_{3,4}$ is not equal to $z_{1,2}$, then $\{z_{1,2}, z_{3,4}\}$ is a clique containing two elements. Notice that $z_{6,7}, z_{6,8}, z_{6,9}$ are distinct elements and if one of them, say $z_{6,9}$, is not contained in the clique $\{z_{1,2}, z_{3,4}\}$, then we can conclude that $\{z_{1,2}, z_{3,4}, z_{6,9}\}$ is a clique consisting of three elements. We can continue in this way to construct an infinite clique. $\square$

Now we will state and prove a lemma from [1] that relates the size of a clique in R to that of a clique in R/I when I is a finite ideal.

Lemma 3.3. *Suppose that I is a finite ideal of R. R contains an infinite clique if and only if R/I has an infinite clique.*

Proof. Suppose that R has an infinite clique C and consider the homomorphism $\phi : R \to R/I$. Notice that the homomorphic image of C in R/I is $\overline{C}$, where $\overline{C} = \{c + I \mid c \in C\}$. Also, the homomorphic image $\overline{C}$ of C is a clique in $\overline{R} = R/I$ since for any two elements $x_1 + I$ and $x_2 + I$ in $\overline{C}$, we have that $(x_1+I)(x_2+I) = x_1x_2+I = 0+I$. We will verify that since I is a finite ideal in R, $\overline{C}$ is an infinite clique. Suppose that $\overline{C}$ is finite and consider $\phi(C) = \overline{C}$, where $C = \{x_1, x_2, \ldots, x_n, \ldots\}$ and $\overline{C} = \{\overline{x_1}, \overline{x_2}, \ldots, \overline{x_n}, \ldots\}$. Since $\overline{C}$ is finite, there are infinitely many $x_{i_1}, x_{i_2}, \cdots \in C$ such that $\overline{x_{i_j}} = \overline{x_{i_k}}$. Therefore for all $j \neq k$, $x_{i_j} - x_{i_k} \in I$. Hence I must be infinite, which is a contradiction. Hence $\overline{C}$ must be an infinite clique.

For the converse, suppose that $\overline{R} = R/I$ has an infinite clique $\{\overline{x_i}\}_{i=1}^{\infty}$, where $\overline{x_i} = x_i + I$. Therefore $\overline{x_i}\,\overline{x_j} = (x_i + I)(x_j + I) = x_ix_j + I = 0 + I = I$, which implies that $x_ix_j \in I$ for $i \neq j$. Since I is a finite ideal, the set $\{x_ix_j\}_{i \neq j}$ is a finite set. We can apply the technique from the proof of the previous lemma to construct an infinite clique in R. $\square$

The next two lemmas relate nilpotent elements in R and the size of the clique of R.

Lemma 3.4. *If the ring R contains a nilpotent element which is not finite, then R contains an infinite clique.*

Proof. Suppose R contains a nilpotent element that is not finite; that is, Rx is an infinite set and $x^n = 0$ for some n. We will proceed by induction on n to show that R must contain an infinite clique. If $x^2 = 0$ and Rx is infinite, then R contains an infinite clique Rx. Suppose that this lemma holds for nilpotent elements of degree $n-1$. Consider $x^n = 0$ for $n \geq 3$ and suppose that Rx is infinite. Let $y = x^2$. Then $y^{n-1} = (x^2)^{n-1} = x^{2n-2} = 0$. We have two cases, either Ry is infinite or Ry is finite. If Ry is infinite, then by the induction hypothesis, R contains an infinite clique. Suppose now that Ry is finite. Then we have that $\overline{Rx} = Rx/Ry$ is an infinite clique in $\overline{R/Ry}$ using the same arguments as in the proof of the previous lemma. By Lemma 3.3 since Ry is a finite ideal and R/Ry has an infinite clique, we can conclude that R has an infinite clique. $\square$

Lemma 3.5. *If the nilradical of R is infinite, then R has an infinite clique.*

Proof. Suppose that the nilradical $\mathfrak{N}$ of R is infinite. If every element in $\mathfrak{N}$ is finite, then by Lemma 3.2, R contains an infinite clique. Now, we suppose that $\mathfrak{N}$ contains an element that is not finite. Then by Lemma 3.4, R contains an infinite clique. $\square$

The following two lemmas establish some properties of annihilators in a ring.

Lemma 3.6. *If R be a reduced ring which does not contain an infinite clique, then R has the ascending chain condition on ideals of the form $Ann(x)$.*

Proof. Suppose R is a reduced ring that does not contain an infinite clique and suppose that we have an infinite chain $Ann(a_1) \subset Ann(a_2) \subset Ann(a_3) \subset \cdots$. Let $x_i \in Ann(a_i) \setminus Ann(a_{i-1})$ for $i = 2, 3, \cdots$. Consider $y_n = x_n a_{n-1}$ for $n = 2, 3, \cdots$, which are nonzero elements of R. We will verify that these elements form a clique. Without loss of generality, suppose that $k < j$ and consider $y_i y_j$. Since $x_i \in Ann(a_i)$ and $i < j$, we have that $x_i \in Ann(a_i) \subset Ann(a_{j-1}) \subset Ann(a_j)$. Therefore we have that $y_i y_j = (x_i a_{i-1})(x_j a_{j-1}) = (a_{i-1} x_j)(x_i a_{j-1}) = (a_{i-1} x_j)(0) = 0$. Hence the nonzero elements $y_n = x_n a_{n-1}$ for $n = 2, 3, \cdots$ form a clique.

We will verify that $y_i \neq y_j$ when $i \neq j$. Since $y_i y_j = 0$, if we were to have $y_i = y_j$ then we would have $y_i^2 = y_j^2 = 0$. This is a contradiction of the fact that R is a reduced ring. Hence $y_i \neq y_j$ when $i \neq j$. Since R does not contain an infinite clique, the nonzero elements $y_n = x_n a_{n-1}$ for $n = 2, 3, \cdots$ form a finite clique $\{y_1, y_2, \ldots, y_n\}$. Therefore the chain $Ann(a_1) \subset Ann(a_2) \subset Ann(a_3) \subset \cdots$ must stabilize. $\square$

Lemma 3.7. *If x and y are elements in R such that $Ann(x)$ and $Ann(y)$ are different prime ideals, then $xy = 0$.*

Proof. Suppose that x and y are elements in R such that $Ann(x)$ and $Ann(y)$ are different prime ideals and assume that $xy \neq 0$. Then $x \notin Ann(y)$ and $y \notin Ann(x)$. Since $Ann(x)$ and $Ann(y)$ are prime ideals, we can show that $(Ann(x) : y) = Ann(x)$ and $(Ann(y) : x) = Ann(y)$. By definition, $Ann(x) \subseteq (Ann(x) : y)$ and $Ann(y) \subseteq (Ann(y) : x)$. To show that $(Ann(x) : y) \subseteq Ann(x)$, we begin by letting $z \in (Ann(x) : y)$. Then $zy \in Ann(x)$, and since $Ann(x)$ is prime and $y \notin Ann(x)$, we have that $z \in Ann(x)$. The containment that $(Ann(y) : x) \subseteq Ann(y)$ can be proven similarly. Hence $(Ann(x) : y) = Ann(x)$ and $(Ann(y) : x) = Ann(y)$.

We will now prove that $(Ann(x) : y) = Ann(xy)$ and $(Ann(y) : x) = Ann(xy)$ so that we can conclude that $Ann(x) = Ann(y)$, which will give us that $xy = 0$. For $(Ann(x) : y) \subseteq Ann(xy)$, let $z \in (Ann(x) : y)$. Then $zy \in Ann(x)$ and therefore $zyx = 0$. Therefore $z \in Ann(xy)$. For the reverse containment, let $z \in Ann(xy)$. Then $zxy = 0$ and therefore $zy \in Ann(x)$. Therefore $z \in (Ann(x) : y)$. Hence $(Ann(x) : y) = Ann(xy)$. Notice that $(Ann(y) : x) = Ann(xy)$ can be proven similarly. Hence we have $Ann(x) = (Ann(x) : y) = (Ann(y) : x) = Ann(y)$, which is a contradiction since $Ann(x)$ and $Ann(y)$ were assumed to be distinct prime ideals. Hence it must be that $xy = 0$. $\square$

Now that we have proved these lemmas, we can prove the following theorem which relates the chromatic number of a zero-divisor graph, the clique number of a ring, and the nilradical of a ring.

Theorem 3.8. *If R is a reduced ring, the following statements are equivalent.*

(a) $\chi(\Gamma_0(R))$ is finite.

(b) $cl(R)$ is finite.

(c) The zero ideal in R is a finite intersection of prime ideals.

(d) The zero-divisor graph of R does not contain an infinite clique.

Proof. For $(a) \Rightarrow (b)$, if the clique number of the zero-divisor graph of R is infinite, then the zero-divisor graph of R has an infinite complete subgraph, which cannot possibly be colored with a finite number of colors.

For $(a) \Rightarrow (d)$, if the zero-divisor graph of R contained an infinite clique, it would be necessary to have infinitely many colors to color infinitely many vertices.

For $(b) \Rightarrow (d)$, if the clique of the zero-divisor graph of R is finite, the zero-divisor graph of R cannot have an infinite clique.

Now we will consider the implication $(c) \Rightarrow (a)$. Suppose that the zero ideal in R is a finite intersection of prime ideals. That is, assume $(0) = \bigcap_{i=1}^{n} \mathfrak{P}_i$, where each $\mathfrak{P}_i$ is a prime ideal for $i = 1, \ldots, n$. We define f to be a coloring on the zero-divisor graph of R by assigning color 0 to the element 0 by $f(0) = 0$ and assigning to each nonzero $x \in R$ contained in the zero-divisor graph the color pertaining to the minimum number i such that x is not in $\mathfrak{P}_i$. That is, $f(x) = \min\{i \mid x \notin \mathfrak{P}_i\}$.

We will verify that no two adjacent vertices can be assigned the same color. Notice that 0 is assigned its own color since it is adjacent to every element in a zero-divisor graph. Suppose that two nonzero elements x and y are adjacent. Then $xy = 0$ and therefore $xy \in \mathfrak{P}_i$ for all $1 \leq i \leq n$. Since $\mathfrak{P}_i$ is prime for all $1 \leq i \leq n$, we have that for each i, either $x \in \mathfrak{P}_i$ or $y \in \mathfrak{P}_i$. Suppose that x and y are assigned the same color. Let $j = min\{i \mid x \notin \mathfrak{P}_i\}$ and $j = min\{i \mid y \notin \mathfrak{P}_i\}$. Then $f(x) = j = f(y)$ and therefore $x \notin \mathfrak{P}_j$ and $y \notin \mathfrak{P}_j$. Hence $xy \notin \mathfrak{P}_j$, which is a contradiction. Therefore we require the use of at most $n+1$ colors and Hence $\chi(\Gamma_0(R)) \leq n+1$. Therefore the chromatic number of the zero-divisor graph of R must be finite.

Now, we must prove that $(d) \Rightarrow (c)$. Suppose that R is a reduced ring whose zero-divisor graph does not contain an infinite clique. Therefore by Theorem 3.6, R satisfies an ascending chain condition on ideals of the form $Ann(a)$. Let $Ann(x_i)$ with $i \in I$ be the distinct maximal members of the family $\{Ann(a) \mid a \neq 0\}$. We will verify that each $Ann(x_i)$ is a prime ideal. Suppose that $ab \in Ann(x_i)$ and $b \notin Ann(x_i)$. Then $bx_i \neq 0$ and $a \in Ann(bx_i)$. Now, $Ann(x_i) \subsetneq Ann(bx_i)$ and by maximality of $Ann(x_i)$, we have that $Ann(x_i) = Ann(bx_i)$. Therefore $a \in Ann(x_i)$. Hence each $Ann(x_i)$ is a prime ideal.

We claim that index set I is finite by Lemma 3.7. Suppose that the index set I is infinite. By Lemma 3.6, we know that if x_i and x_j in R are elements such that $Ann(x_i)$ and $Ann(x_j)$ are distinct prime ideals, then x and y must be adjacent. Since we have an infinite index set I, this means that we have an infinite clique but we know that there are no infinite cliques in the zero-divisor graph of R. Hence I must be a finite index set, say of order n.

We now claim that $\bigcap_{i=1}^n Ann(x_i) = 0$. Let $0 \neq x \in \bigcap_{i=1}^n Ann(x_i)$. Then $x \in Ann(x_i)$ and $xx_i = 0$ for all $i \in I$. Therefore $x_i \in Ann(x)$. We will verify that $Ann(x) \subseteq Ann(x_i)$ for some $i \in I$. Suppose that $Ann(x) \nsubseteq Ann(x_i)$ for all $i \in I$. Then $Ann(x)$ is maximal and Hence $Ann(x) = Ann(x_i)$ for some $i \in I$. This is a contradiction. Therefore $x_i \in Ann(x) \subseteq Ann(x_i)$. Hence $x_i^2 = 0$, which implies that x_i is nilpotent but since R is reduced $x_i = 0$. This is a contradiction to the fact that $x_i \neq 0$. Therefore $\bigcap_{i=1}^n Ann(x_i) = 0$ and the zero ideal is a finite intersection of prime ideals. $\square$

Now, we can prove Beck's conjecture for reduced rings as in [1].

Theorem 3.9. *Suppose that R is a nonzero reduced ring. If $\chi(\Gamma_0(R)) < \infty$, then R has finitely many minimal prime ideals. If R has $n < \infty$ minimal prime ideals, then $\chi(\Gamma_0(R)) = cl(R) = n+1$.*

Proof. Suppose that $\chi(\Gamma_0(R)) < \infty$. Since R is reduced, the nilradical $\mathfrak{N}$ of R is equal to the zero ideal. Then by Theorem 3.8, the zero ideal in R is a finite intersection of prime ideals, i.e. $\mathfrak{N} = (0) = \mathfrak{P}_1 \cap \cdots \cap \mathfrak{P}_n$. Since every prime ideal $\mathfrak{P}_i$ contains a minimal prime ideal Q_i, we have that the zero ideal is equal to the intersection of minimal prime ideals. That is, $\mathfrak{N} = (0) = Q_1 \cap \cdots \cap Q_n$. We will assume that each of these minimal ideals are distinct.

We need to show that R has finitely many minimal prime ideals. Suppose that R has infinitely many minimal prime ideals. Since the nilradical is equal to the intersection of all minimal prime ideals, we write $(0) = \mathfrak{N} = \bigcap_{i \in I} Q_i$. Therefore $Q_1 \cap \cdots \cap Q_n = (0) = \bigcap_{i \in I} Q_i$. As a result, $Q_1 \cap \cdots \cap Q_n \subseteq \bigcap_{i \in I} Q_{n+i} \subseteq Q_{n+i}$ for each $i \in I$. Since $Q_1 \cdots Q_n \subseteq Q_1 \cap \cdots \cap Q_n$, we also have that $Q_1 \cdots Q_n \subseteq Q_{n+i}$ for each $i \in I$. Since each Q_{n+i} is prime, we must have $Q_1 \subseteq Q_{n+i}$ or $Q_2 \subseteq Q_{n+i}$ or $\cdots$ or $Q_n \subseteq Q_{n+i}$ for each $i \in I$. Also, since each Q_{n+i} is minimal, we must have $Q_1 = Q_{n+i}$ or $Q_2 = Q_{n+i}$ or $\cdots$ or $Q_n = Q_{n+i}$ for each $i \in I$. Therefore the list of all minimal primes in R, $\{Q_1, \ldots, Q_n\}$, is exhaustive. Therefore R must have n many minimal prime ideals.

We can now show that $\chi(\Gamma_0(R)) = cl(R) = n+1$. By the proof of Theorem 3.7, we have that $\chi(\Gamma_0(R)) \leq n+1$. We know that $\chi(\Gamma_0(R)) \geq cl(R)$ is always true. The only thing left to prove is that $cl(R) \geq n+1$. We will prove this inequality by constructing a clique. Suppose that $Q_1 \cap \cdots \cap Q_{k-1} \cap Q_{k+1} \cap \cdots \cap Q_n \neq (0)$ for each $k \in \{1, \ldots, n\}$. We will select a nonzero $x_k \in Q_1 \cap \cdots \cap Q_{k-1} \cap Q_{k+1} \cap \cdots \cap Q_n$ where $x_k \notin Q_k$ for each $k \in \{1, \ldots, n\}$. Then $x_k \in Q_i$ for every $i \neq k$ and $x_k \notin Q_k$. Notice that $x_k x_i \in Q_i$ for all $i \neq k$ since $x_k \in Q_i$ for all $i \neq k$, and $x_k x_i \in Q_k$ for $k \neq i$ since $x_i \in Q_k$. As a result, we have that $x_k x_i \in Q_i$ for $i \in \{1, \ldots, n\}$. Therefore $x_k x_i = 0$ for all $i \neq k$. Hence $\{0, x_1, \ldots, x_n\}$ forms a clique consisting of $n+1$ elements and $cl(R) \geq n+1$. Finally, we have established that $cl(R) = \chi(\Gamma_0(R)) = n+1$. $\square$

Now we can state and prove Theorem 3.9 for a ring R that is not necessarily reduced, using some of the lemmas we stated previously.

Theorem 3.10. *The following conditions are equivalent for a ring R.*

(a) $\chi(\Gamma_0(R))$ *is finite.*

(b) $cl(R)$ *is finite.*

(c) The nilradical in R is finite and equals a finite intersection of prime ideals.

(d) The zero-divisor graph of R does not contain an infinite clique.

Proof. For $(a) \Rightarrow (b)$, if the clique number of the zero-divisor graph of R is infinite, then the zero-divisor graph of R has an infinite complete subgraph, which cannot possibly be colored with a finite number of colors.

For $(a) \Rightarrow (d)$, if the zero-divisor graph of R contained an infinite clique, it would be necessary to have infinitely many colors to color infinitely many vertices.

For $(b) \Rightarrow (d)$, if the clique of the zero-divisor graph of R is finite, the zero-divisor graph of R cannot have an infinite clique.

Now we will consider the implication $(c) \Rightarrow (a)$. Suppose that the nilradical in R is finite and equals a finite intersection of prime ideals. That is, assume $\mathfrak{N} = \bigcap_{i=1}^{n} \mathfrak{P}_i$, where each $\mathfrak{P}_i$ is a prime ideal for $i = 1, \ldots, n$. We will define a coloring f on the elements pertaining to the zero-divisor graph of R which are outside of $\mathfrak{N}$ by $f(x) = min\{i \mid x \notin \mathfrak{P}_i\}$.

We will verify that no two adjacent vertices can be assigned the same color. Notice that 0 is assigned its own color since it is adjacent to every element in a zero-divisor graph. Suppose that two nonzero elements x and y are adjacent. Then $xy = 0$ and therefore $xy \in \mathfrak{P}_i$ for all $1 \leq i \leq n$. Since $\mathfrak{P}_i$ is prime for all $1 \leq i \leq n$, we have that for each i, either $x \in \mathfrak{P}_i$ or $y \in \mathfrak{P}_i$. Suppose that x and y are assigned the same color. Let $j = min\{i \mid x \notin \mathfrak{P}_i\}$ and $j = min\{i \mid y \notin \mathfrak{P}_i\}$. Then $f(x) = j = f(y)$ and therefore $x \notin \mathfrak{P}_j$ and $y \notin \mathfrak{P}_j$. Hence $xy \notin \mathfrak{P}_j$, which is a contradiction. Therefore we require the use of at most $n + 1$ colors.

Since $\mathfrak{N}$ is finite, we need a finite number of additional colors to color the elements in $\mathfrak{N}$. We can use one of the colors used on the nonzero zero-divisors to color the elements that are not zero-divisors since the elements that are not zero-divisors cannot be adjacent to nonzero zero-divisors. Therefore the chromatic number of the zero-divisor graph of R must be finite.

Now, we must prove that $(d) \Rightarrow (c)$. Suppose that the zero-divisor graph of R does not have an infinite clique. Then by Lemma 3.5, the nilradical $\mathfrak{N}$ of R is finite, and by Lemma 3.3, $R/\mathfrak{N}$ does not have an infinite clique. Consider $R/\mathfrak{N}$. We can prove that $R/\mathfrak{N}$ is reduced by showing that $R/\mathfrak{N}$ has no nilpotent elements, which would give us that the nilradical of $R/\mathfrak{N}$ is $\overline{0}$. Suppose that $\overline{x} \in R/\mathfrak{N}$ and that $\overline{x}$ is nilpotent with $\overline{x}^n = 0$ for some n in $R/\mathfrak{N}$. Then $\overline{x}^n = 0$ implies that $x^n \in \mathfrak{N}$, which implies that $(x^n)^k = 0$ for some k. Therefore $x \in \mathfrak{N}$ and Hence $\overline{x} = 0$. Therefore the nilradical in $R/\mathfrak{N}$ must be $(\overline{0})$, which means that $R/\mathfrak{N}$ is a reduced ring. By Theorem 3.8, since $R/\mathfrak{N}$ is reduced, $(\overline{0})$ is a finite intersection of prime ideals, so we write $(\overline{0}) = \overline{\mathfrak{P}_1} \cap \overline{\mathfrak{P}_2} \cap \cdots \cap \overline{\mathfrak{P}_n}$. Since there is a one-to-one correspondence between the prime ideals of R containing $\mathfrak{N}$ and prime ideals in $R/\mathfrak{N}$, we can write $\mathfrak{N}$ as a finite intersection of prime ideals. That is, $\mathfrak{N} = \mathfrak{P}_1 \cap \mathfrak{P}_2 \cap \cdots \cap \mathfrak{P}_n$. $\square$

4. COLORINGS AND THEIR PROPERTIES

Since we have characterized the rings whose zero-divisor graphs have finite chromatic number, it is necessary to discuss some of the properties of these rings as in [1]. We will start by defining a coloring.

Definition 4.1. *A ring R is called a coloring if $\chi(\Gamma_0(R))$ is finite.*

This next lemma from [1] will help us prove a theorem that generalizes Lemma 3.7.

Lemma 4.2. *If I is a finite ideal in a ring R, then $(I : x)/Ann(x)$ is a finite $R-$module.*

Proof. Consider the exact sequence $0 \longrightarrow Ann(x) \xrightarrow{f} (I : x) \xrightarrow{g} (I : x)x \longrightarrow 0$ where $g(t) = tx$. Notice that $(I : x)x \subset I$ and since I is finite so is $(I : x)x$. We will verify that $(I : x)x \cong (I : x)/Ann(x)$ because then we can conclude that $(I : x)/Ann(x)$ is finite. Consider the homomorphism $g : (I : x) \to (I : x)x$. By the First Isomorphism Theorem, we have that $im(g) \cong (I : x)/ker(g)$. Since g is onto, $(I : x)x = im(g)$. We will verify that $ker(g) = Ann(x)$. Let $t \in ker(g)$. Then $0 = g(t)$. By definition, we know that $g(t) = tx$. Therefore $tx = 0$ and Hence $t \in Ann(x)$. Now let $t \in Ann(x)$. Then $tx = 0$. Since g is onto, $tx = g(t)$ and, Hence $g(t) = 0$ which implies that $t \in ker(g)$. Therefore $ker(g) = Ann(x)$. Hence $(I : x)x \cong (I : x)/Ann(x)$ and $(I : x)/Ann(x)$ is finite as an $R-$module. $\square$

We will now generalize Lemma 3.7 as in [1].

Theorem 4.3. *A coloring has the ascending chain condition on ideals of the form $Ann(a)$.*

Proof. Let R be a coloring and assume that $Ann(x_1) \subset Ann(x_2) \subset \cdots$. Since R is a coloring, we know that the chromatic number of the zero-divisor graph is finite and Hence by Theorem 3.10, the nilradical $\mathfrak{N}$ of R is finite and $\mathfrak{N} = \mathfrak{P}_1 \cap \mathfrak{P}_2 \cap \cdots \cap \mathfrak{P}_n$, where $\mathfrak{P}_i$ is a prime ideal for $q \leq i \leq n$. We will assume that $x_i \notin \mathfrak{N}$ for $i = 1, 2, \ldots$. Let $x \in R$. Then we claim that $(\mathfrak{N} : x) = (\mathfrak{P}_1 : x) \cap (\mathfrak{P}_2 : x) \cap \cdots \cap (\mathfrak{P}_n : x)$. We know that $(\mathfrak{N} : x) = ((\mathfrak{P}_1 \cap \mathfrak{P}_2 \cap \cdots \cap \mathfrak{P}_n) : x)$.

We will verify that $((\mathfrak{P}_1 \cap \mathfrak{P}_2 \cap \cdots \cap \mathfrak{P}_n) : x) = (\mathfrak{P}_1 : x) \cap (\mathfrak{P}_2 : x) \cap \cdots \cap (\mathfrak{P}_n : x)$. Let $t \in ((\mathfrak{P}_1 \cap \mathfrak{P}_2 \cap \cdots \cap \mathfrak{P}_n) : x)$. Then $tx \in \mathfrak{P}_1 \cap \mathfrak{P}_2 \cap \cdots \cap \mathfrak{P}_n$, which implies that $tx \in \mathfrak{P}_i$ for $1 \leq i \leq n$. Therefore $t \in (\mathfrak{P}_i : x)$ for $1 \leq i \leq n$ and as a result $t \in (\mathfrak{P}_1 : x) \cap (\mathfrak{P}_2 : x) \cap \cdots \cap (\mathfrak{P}_n : x)$. Hence the containment $((\mathfrak{P}_1 \cap \mathfrak{P}_2 \cap \cdots \cap \mathfrak{P}_n) : x) \subseteq (\mathfrak{P}_1 : x) \cap (\mathfrak{P}_2 : x) \cap \cdots \cap (\mathfrak{P}_n : x)$ holds.

For the reverse containment, let $t \in (\mathfrak{P}_1 : x) \cap (\mathfrak{P}_2 : x) \cap \cdots \cap (\mathfrak{P}_n : x)$. Then $t \in (\mathfrak{P}_i : x)$ for $1 \leq i \leq n$, which gives us that $tx \in \mathfrak{P}_i$ for $1 \leq i \leq n$. Therefore $tx \in (\mathfrak{P}_1 \cap \mathfrak{P}_2 \cap \cdots \cap \mathfrak{P}_n)$, which implies that $t \in ((\mathfrak{P}_1 \cap \mathfrak{P}_2 \cap \cdots \cap \mathfrak{P}_n) : x)$. Hence $(\mathfrak{P}_1 : x) \cap (\mathfrak{P}_2 : x) \cap \cdots \cap (\mathfrak{P}_n : x) \subseteq ((\mathfrak{P}_1 \cap \mathfrak{P}_2 \cap \cdots \cap \mathfrak{P}_n) : x)$.

Therefore we have established that $((\mathfrak{P}_1 \cap \mathfrak{P}_2 \cap \cdots \cap \mathfrak{P}_n) : x) = (\mathfrak{P}_1 : x) \cap (\mathfrak{P}_2 : x) \cap \cdots \cap (\mathfrak{P}_n : x)$. Therefore we write $(\mathfrak{N} : x) = (\mathfrak{P}_1 : x) \cap (\mathfrak{P}_2 : x) \cap \cdots \cap (\mathfrak{P}_n : x)$. We claim that the family $\{(\mathfrak{N} : x) \mid x \in R\}$ is finite. We will prove that for each $(\mathfrak{P}_i : x)$, either $(\mathfrak{P}_i : x) = R$ or $(\mathfrak{P}_i : x) = P_i$. Suppose that $x \in \mathfrak{P}_{\mathrm{i}}$. Then $(\mathfrak{P}_i : x) = R$ because the product between and $a \in R$ and x will be in $\mathfrak{P}_i$ since $\mathfrak{P}_i$ is an ideal. On the other hand, suppose that $x \in R \setminus \mathfrak{P}_i$. Let $a \in \mathfrak{P}_i$. Then since $\mathfrak{P}_i$ is an ideal, we have that $ax \in \mathfrak{P}_i$. Therefore $a \in (\mathfrak{P}_i : x)$. For the reverse containment, take $a \in (\mathfrak{P}_i : x)$. Then $ax \in \mathfrak{P}_i$. Since $x \notin \mathfrak{P}_i$ and $\mathfrak{P}_i$ is a prime ideal, we have that $a \in \mathfrak{P}_i$. As a result, the family $\{(\mathfrak{N} : x) \mid x \in R\}$ is finite. Therefore there must exist a subsequence y_j of x_i for which $(\mathfrak{N} : y_1) = (\mathfrak{N} : y_2) = \cdots$.

Consider $Ann(y_1) \subset Ann(y_2) \subset \cdots$. Then $Ann(y_1) \subset Ann(y_2) \subset \cdots \subset (\mathfrak{N} : y_1)$. Now, we take $a_1 \in Ann(y_1)$, $a_2 \in Ann(y_2) \setminus Ann(y_1)$, $a_3 \in Ann(y_3) \setminus Ann(y_2), \cdots$. We will show that $a_i + Ann(y_1) \neq a_j + Ann(y_1)$ for $i < j$. Suppose that $a_i + Ann(y_1) = a_j + Ann(y_1)$. Then $a_i - a_j \in Ann(y_1)$. Let $a_i - a_j = b_1'$ where $b_1' \in Ann(y_1)$. Since $Ann(y_1) \subset Ann(y_i)$, then $b_1' \in Ann(y_i)$ which implies that $0 = b_1' y_i = (a_i - a_j)y_i = a_i y_i - a_j y_i = 0 - a_j y_i$. Hence $a_j y_i = 0$, implying that $a_j \in Ann(y_i)$. This contradicts the choice of a_j. Therefore $a_i + Ann(y_1) \neq a_j + Ann(y_1)$ for $i < j$. Therefore $(\mathfrak{N} : y_1)/Ann(y_1)$ is infinite. This contradicts Lemma 4.2. Hence the ascending chain condition holds. $\square$

This theorem leads us to the next theorem, as in [1], which allows us to prove that given a Coloring, every minimal prime ideal is an associated prime ideal.

Theorem 4.4. *Suppose that R is a coloring. Then $Ass(R)$ is finite and $Z(R) = \bigcup_{\mathfrak{P} \in Ass(R)} \mathfrak{P}$. Furthermore, any minimal prime ideal $\mathfrak{P}$ is an associated prime ideal and $R_{\mathfrak{P}}$ is a field or a finite ring.*

Proof. Suppose that R is a coloring. Then by definition, $\chi(\Gamma_0(R)) < \infty$. By Lemma 3.10, if the chromatic number of a zero-divisor graph is finite, then so is the clique number. Hence $cl(R) < \infty$. Our first claim is that $Ass(R)$ is finite. Suppose that $Ass(R)$ is infinite. Let $\mathfrak{P} \in Ass(R)$. Then $\mathfrak{P} = Ann(x)$ for some $x \in R$. Since $Ass(R)$ is infinite, there are infinitely many prime ideals

$\mathfrak{P}_1, \mathfrak{P}_2, \ldots, \mathfrak{P}_n \ldots$ such that $\mathfrak{P}_i = Ann(x_i)$ for some $x_i \in R$. By Lemma 3.7, $x_i x_j = 0$ for all distinct prime ideals $Ann(x_i)$ and $Ann(x_j)$. Therefore the zero-divisor graph of R contains an infinite clique. This is a contradiction to the fact that the clique number is finite. Hence $Ass(R)$ is finite.

Now, we will prove that $Z(R) = \bigcup_{\mathfrak{P} \in Ass(R)} \mathfrak{P}$. Let $x \in Z(R)$. Then $x \in Ann(r)$ for some nonzero $r \in R$ and by the previous theorem we have that $Ann(r) \subset Ann(y)$ for some maximal $Ann(y)$, which implies that $x \in Ann(y)$. In the proof of Theorem 3.8, we proved that maximal members of the family $\{Ann(z) \mid z \neq 0\}$ are prime and therefore $Ann(y) = \mathfrak{P}$. This allows us to conclude that $Ann(y)$ is an associated prime ideal. For the reverse containment, suppose that $x \in \bigcup_{\mathfrak{P} \in Ass(R)} \mathfrak{P}$. Then $x \in \mathfrak{P}$ for some prime ideal $\mathfrak{P} \in Ass(R)$. Since $\mathfrak{P}$ is an associated prime ideal, we have that $\mathfrak{P} = Ann(s)$ for some $s \in R$. Therefore $x \in Ann(s)$, which in turn implies that $x \in Z(R)$. Hence $Z(R) = \bigcup_{\mathfrak{P} \in Ass(R)} \mathfrak{P}$.

Our next claim is that every minimal prime ideal $\mathfrak{P}$ is an associated prime ideal. We will begin by letting $\mathfrak{P}$ be a minimal prime ideal and supposing that $x \notin \mathfrak{P}$. Now, we will prove that $Ann(x) \subset \mathfrak{P}$. Let $z \in Ann(x)$. Then $zx = 0 \in \mathfrak{P}$ and since $\mathfrak{P}$ is prime and $x \notin \mathfrak{P}$, we have that $z \in \mathfrak{P}$ and we have established that $Ann(x) \subset \mathfrak{P}$. We will now choose $Ann(t)$ to be maximal in the family $\{Ann(y) \mid Ann(y) \subset \mathfrak{P}\}$. This family is nonempty since $Ann(x) \subset \mathfrak{P}$. We claim that $Ann(t)$ is a prime ideal and that $Ann(t) = \mathfrak{P}$. Let $ab \in Ann(t)$, $a \notin Ann(t)$, and $b \notin Ann(t)$. Let $a \notin \mathfrak{P}$ and consider $Ann(ta)$. We will show that $Ann(t) \subset Ann(ta) \subset \mathfrak{P}$. Let $\beta \in Ann(ta)$. Then $\beta(ta) = (\beta a)t = 0$ and therefore $\beta a \in Ann(t) \subset \mathfrak{P}$. Since $\mathfrak{P}$ is prime and $a \notin \mathfrak{P}$, we must have $\beta \in \mathfrak{P}$. Therefore $Ann(ta) \subset \mathfrak{P}$. We will show that $Ann(t) \subset Ann(ta)$. We know that $b \in Ann(ta)$ because $b(ta) = (ab)t = 0$ holds since $ab \in Ann(t)$ and $b \notin Ann(t)$. Therefore $Ann(t) \subset Ann(ta)$, which is a contradiction to the maximality of $Ann(t)$. Next, we will suppose that $a \in \mathfrak{P}$ and consider $Ann(ta)$. If $Ann(ta) \subset \mathfrak{P}$, we will arrive at the same contradiction as before. However, if we suppose that $a \in \mathfrak{P}$ and $Ann(ta) \not\subset \mathfrak{P}$, then we have $c \in Ann(ta)$ and $c \notin \mathfrak{P}$. Therefore we will consider $Ann(tc)$. Similarly to the case when $a \notin \mathfrak{P}$, it can be shown that $Ann(t) \subset Ann(tc) \subset \mathfrak{P}$ to arrive at the same contradiction. Hence $Ann(t)$ must be prime. Since $\mathfrak{P}$ is a minimal prime, we have that $Ann(t) = \mathfrak{P}$, which allows us to conclude that every minimal prime ideal is an associated prime ideal.

We must show that for a minimal prime $\mathfrak{P}$, either $R_{\mathfrak{P}}$ is a field or a finite ring. Let $\mathfrak{P}$ be a minimal prime ideal. Then we know that $\mathfrak{P} = Ann(x)$ for some $x \in R$. Suppose that $x \notin \mathfrak{P}$ and notice that $\mathfrak{P}R_{\mathfrak{P}}$ is defined as $\mathfrak{P}R_{\mathfrak{P}} = \{p \cdot \frac{r}{s} \mid p \in \mathfrak{P}, r \in R, s \in R \setminus \mathfrak{P}\}$. Let $p \cdot \frac{r}{s} = \frac{pr}{s} \in \mathfrak{P}R_{\mathfrak{P}}$. Notice that $prx = 0$ since $p \in \mathfrak{P} = Ann(x)$. Therefore $\frac{pr}{s} = \frac{0}{1}$ since for $x \notin \mathfrak{P}$, we have that $x(pr \cdot 1 - 0 \cdot s) = prx = 0$. Since $\frac{0}{1}$ is the zero element in $\mathfrak{P}R_{\mathfrak{P}}$, we can conclude that $\mathfrak{P}R_{\mathfrak{P}} = (0)$. Since $\mathfrak{P}R_{\mathfrak{P}} = (0)$ is the unique maximal ideal of $R_{\mathfrak{P}}$, the only ideals of $R_{\mathfrak{P}}$ are (0) and $R_{\mathfrak{P}}$. Suppose that $0 \neq x \in R_{\mathfrak{P}}$. Since $xR_{\mathfrak{P}}$ is a nonzero ideal in $R_{\mathfrak{P}}$, it must be that $xR_{\mathfrak{P}} = R_{\mathfrak{P}}$. Therefore $1 \in xR_{\mathfrak{P}}$, which implies that there exists an element $y \in R_{\mathfrak{P}}$ such that $xy = 1$. Since every nonzero element has an inverse, we must have that $R_{\mathfrak{P}}$ is a field.

Now, suppose that $x \in \mathfrak{P}$. We will show that $R_{\mathfrak{P}}$ must be finite. Since the nilradical, $\mathfrak{N}$, is the intersection of all minimal prime ideals, let $\mathfrak{N} = \mathfrak{P} \cap \mathfrak{P}_1 \cap \cdots \cap \mathfrak{P}_k$ where $\mathfrak{P}_1, \ldots, \mathfrak{P}_k$ are the rest of the minimal prime ideals. Take $y \in \mathfrak{P}_1 \cap \cdots \cap \mathfrak{P}_k \setminus \mathfrak{P}$. Then $y\mathfrak{P} \subseteq \mathfrak{N}$ since $\mathfrak{P}, \mathfrak{P}_1, \ldots, \mathfrak{P}_k$ are ideals. We claim that $\mathfrak{P}R_{\mathfrak{P}} = \mathfrak{N}R_{\mathfrak{P}}$. Let $\frac{a}{b} \in \mathfrak{P}R_{\mathfrak{P}}$. Then $\frac{a}{b} = \frac{ca}{cb}$ since $1 \in R \setminus \mathfrak{P}$ is such that $1(acb - bca) = 0$. Notice that $ca \in \mathfrak{N}$ and $cb \in R \setminus \mathfrak{P}$. Therefore $\mathfrak{P}R_{\mathfrak{P}} \subseteq \mathfrak{N}R_{\mathfrak{P}}$. The reverse containment comes from the fact that $\mathfrak{N} \subseteq \mathfrak{P}$. Hence $\mathfrak{P}R_{\mathfrak{P}} = \mathfrak{N}R_{\mathfrak{P}}$. Since $\mathfrak{N}$ is finite, we have that $\mathfrak{P}R_{\mathfrak{P}}$ is finite.

Certainly, $R/\mathfrak{P} \cong Rx$. Since $x \in \mathfrak{P} = Ann(x)$, we have $x^2 = 0$. Therefore Rx forms a clique. Notice that $cl(R)$ is finite since R is a coloring. Therefore Rx is finite. Now $R_{\mathfrak{P}}x \cong Rx \otimes_R R_{\mathfrak{P}} \cong (R/\mathfrak{P}) \otimes_R R_{\mathfrak{P}} \cong (R/\mathfrak{P})_{\mathfrak{P}} \cong R_{\mathfrak{P}}/\mathfrak{P}R_{\mathfrak{P}}$. Notice that $R_{\mathfrak{P}}x \subset \mathfrak{P}R_{\mathfrak{P}}$. Since $\mathfrak{P}R_{\mathfrak{P}}$ is finite, so is $R_{\mathfrak{P}}x$. Therefore $R_{\mathfrak{P}}/\mathfrak{P}R_{\mathfrak{P}}$ is finite. Since $|R_{\mathfrak{P}}| = |\mathfrak{P}R_{\mathfrak{P}}||R_{\mathfrak{P}}/\mathfrak{P}R_{\mathfrak{P}}|$ and each of $\mathfrak{P}R_{\mathfrak{P}}$ and $R_{\mathfrak{P}}/\mathfrak{P}R_{\mathfrak{P}}$ are finite, we have that $R_{\mathfrak{P}}$ is finite. $\square$

The next theorem gives us a characterization of associated primes within colorings.

Theorem 4.5. *Let $\mathfrak{P}$ be an associated prime ideal in a coloring. Then either $R_{\mathfrak{P}}$ is a field or $\mathfrak{P}$ is a maximal ideal.*

Proof. Since $\mathfrak{P}$ is an associated prime ideal, we let $\mathfrak{P} = Ann(x)$ for some $x \in R$. Suppose that $x \in \mathfrak{P}$. Then $x \in Ann(x)$ and Hence $x^2 = 0$ which implies that Rx forms a clique. We will prove that Rx must be finite. Suppose that Rx is not finite. Then by Lemma 3.4, R contains an infinite clique. This a contradiction to R being a coloring. Therefore Rx must be finite. Since $\mathfrak{P}$ is prime, $R/\mathfrak{P}$ is an integral domain. Also, $R/\mathfrak{P} \cong Rx$. To see this congruence, consider $\phi : R \to Rx$. By the First Isomorphism Theorem, $R/ker(\phi) \cong im(\phi)$. It can be shown that $im(\phi) = Rx$ and

$ker(\phi) = \mathfrak{P}$, where $\mathfrak{P} = Ann(x)$. Therefore using these equalities and isomorphism, $R/\mathfrak{P} \cong R/ker(\phi) \cong im(\phi) \cong Rx$. Since $R/\mathfrak{P} \cong Rx$ and Rx is finite, $R/\mathfrak{P}$ is also finite. Since $R/\mathfrak{P}$ is a finite integral domain, it is a field. Hence $\mathfrak{P}$ is a maximal ideal.

Now suppose that $x \notin \mathfrak{P}$. Using the same argument as in the proof of the previous theorem, since $x \notin \mathfrak{P}$ and $\mathfrak{P} = Ann(x)$, we can conclude that $\mathfrak{P}R_{\mathfrak{P}} = (0)$ and that $R_{\mathfrak{P}}$ is a field. $\square$

As a corollary to this theorem, we have the following.

Corollary 4.6. *An associated prime ideal in a coloring is either a maximal ideal or a minimal prime ideal.*

Proof. From the previous theorem, we have that an associated prime ideal in a coloring is a maximal ideal or $R_{\mathfrak{P}}$ is a field, so we are partially finished.

Now assume that $R_{\mathfrak{P}}$ is a field. Suppose that $\mathfrak{P}$ is not a minimal prime ideal. Then there is a prime ideal $\mathfrak{Q}$ contained in $\mathfrak{P}$; that is, $\mathfrak{Q} \subset \mathfrak{P}$. Since there is a one-to-one correspondence between the prime ideals in $R_{\mathfrak{P}}$ and the prime ideals in R contained in $\mathfrak{P}$, $S^{-1}\mathfrak{Q}$ is a prime ideal in $R_{\mathfrak{P}}$. Certainly, $S^{-1}\mathfrak{Q} \subset S^{-1}\mathfrak{P} = \mathfrak{P}R_{\mathfrak{P}}$, where $\mathfrak{P}R_{\mathfrak{P}} = (0)$ since $R_{\mathfrak{P}}$ is a field and $\mathfrak{P}R_{\mathfrak{P}}$ is the unique maximal ideal in $R_{\mathfrak{P}}$. This is a contradiction. Hence $\mathfrak{P}$ must be a minimal prime ideal. $\square$

Now we will discuss some of the properties of a family of colorings discussed in [1]. This first theorem is a fairly obvious property.

Theorem 4.7. *A subring of a coloring is itself a coloring.*

Proof. Suppose that the ring R is a coloring. Then $\chi(\Gamma_0(R)) < \infty$. Let $S \subseteq R$ be a subring and consider the zero-divisor graph on R. Since S is a subset of R, the zero-divisor graph of S is a subgraph of the zero-divisor graph of R. Therefore $\chi(\Gamma_0(S)) \leq \chi(\Gamma_0(R))$. Hence S must be a coloring. $\square$

The next few results prove that a quotient ring of a coloring is also a coloring.

Theorem 4.8. *Let I be a finite ideal in a coloring R. Then R/I is a coloring.*

Proof. Suppose that I is a finite ideal in a coloring R. Since R is a coloring, we know that $\chi(\Gamma_0(R)) < \infty$. By Theorem 3.10, $\chi(\Gamma_0(R)) < \infty$ is equivalent to $cl(R) < \infty$. Since R cannot contain an infinite clique, neither can R/I by Lemma 3.3. Therefore $cl(R/I) < \infty$ and equivalently by Theorem 3.10, $\chi(\Gamma_0(R)) < \infty$. Therefore R/I is a coloring. $\square$

Lemma 4.9. *Let x be an element in a coloring R. Then $R/Ann(x)$ is a coloring.*

Proof. Since R is a coloring, $\chi(\Gamma_0(R)) < \infty$ and by Theorem 3.10, $cl(R) < \infty$. Suppose that $\{\overline{r}_1, \overline{r}_2, \ldots, \overline{r}_n\}$ is a clique in $\overline{R} = R/Ann(x)$. Then $r_i \neq r_j$ for $i \neq j$ and $\overline{r}_i\overline{r}_j = 0$ for $i \neq j$. Therefore $(r_i + Ann(x))(r_j + Ann(x)) = r_i r_j + Ann(x) = Ann(x)$, which implies that $r_i r_j \in Ann(x)$. Hence $r_i r_j x = 0$. We will verify that $r_1 x, r_2 x, \ldots, r_n x$ are distinct elements in R. Suppose that $r_i x = r_j x$ for $i \neq j$. Then $r_i x - r_j x = (r_i - r_j)x = 0$, which implies that $r_i - r_j \in Ann(x)$. Let $z = r_i - r_j$, where $z \in Ann(x)$. Then $r_i + Ann(x) = (r_j + z + Ann(x)) = (r_j + Ann(x))$. Therefore $\overline{r}_i = \overline{r}_j$. This is a contradiction our assumption. Hence $r_1 x, r_2 x, \ldots, r_n x$ forms a clique in R. This demonstrates that, given a clique in $\overline{R}$, we can find a corresponding clique in R of the same size. In addition, since $cl(R) < \infty$, every clique in R is finite in size and is bounded by some value, say n. The size of any clique in $\overline{R}$ will also be bounded by n. As a result, $cl(\overline{R}) < \infty$ and by Theorem 3.10, $\chi(\Gamma_0(\overline{R})) < \infty$, which implies that $\overline{R}$ is a coloring. □

Theorem 4.10. *Let I be a finite ideal in a coloring R and assume $x \in R$. Then $R/(I : x)$ is a coloring.*

Proof. By Lemma 4.9, we know that $R/Ann(x)$ is a coloring. Also, by Lemma 4.2, we have that $(I : x)/Ann(x)$ is a finite ideal in $R/Ann(x)$. Therefore by Theorem 4.8, we conclude that $(R/Ann(x))/((I : x)/Ann(x))$ is a coloring. Then $(R/Ann(x))/((I : x)/Ann(x)) \cong R/(I : x)$ and therefore $R/(I : x)$ is a coloring. □

The next theorem is about the finite product of colorings.

Theorem 4.11. *A finite product of colorings is a coloring.*

Proof. Consider the product of two colorings R_1 and R_2 given by $R \cong R_1 \times R_2$. Since R_1 and R_2 are colorings, $\chi(\Gamma_0(R_1)) < \infty$ and $\chi(\Gamma_0(R_2)) < \infty$. By Lemma 3.10, this implies that $cl(R_1) < \infty$ and $cl(R_2) < \infty$. Suppose that $cl(R_1) = n$ and $cl(R_2) = m$. Let C be a maximal clique in $R = R_1 \times R_2$. If we restrict the first coordinate, there are at most n elements that form a clique, and if we restrict the second coordinate, there are at most m elements that form a clique. Hence $|C| \leq nm$. By Theorem 3.10, since C is a maximal clique, the clique number of R is finite and therefore $\chi(\Gamma_0(R_1 \times R_2)) < \infty$. Hence $R \cong R_1 \times R_2$ is a coloring. □

We can now generalize Lemma 4.9.

Theorem 4.12. *Let I be a finitely generated ideal in a coloring. Then $R/Ann(I)$ is a coloring.*

Proof. Let $I = (x_1, x_2, \ldots, x_n)$. We claim that $Ann(I) = Ann(x_1) \cap Ann(x_2) \cap \cdots \cap Ann(x_n)$. Let $z \in Ann(I)$. Then $zI = 0$. Therefore $zx_i = 0$ for $1 \leq i \leq n$. Hence $z \in Ann(x_i)$ for $1 \leq i \leq n$, which gives us that $z \in Ann(x_1) \cap \cdots \cap Ann(x_n)$. Therefore $Ann(I) \subseteq Ann(x_1) \cap \cdots \cap Ann(x_n)$. For the reverse containment, let $z \in Ann(x_1) \cap \cdots \cap Ann(x_n)$. Then $z \in Ann(x_i)$ for $1 \leq i \leq n$. Therefore $tx_i = 0$ for $1 \leq i \leq n$ and we have that $t \in Ann(I)$. Hence $Ann(x_1) \cap \cdots \cap Ann(x_n) \subseteq Ann(I)$, giving us that $Ann(I) = Ann(x_1) \cap Ann(x_2) \cap \cdots \cap Ann(x_n)$.

We claim that the map $\phi : R/Ann(I) \to R/Ann(x_1) \times \cdots \times R/Ann(x_n)$ defined by $\phi(r + Ann(I)) = (r + Ann(x_1), \ldots, r + Ann(x_n))$ is an injection. Suppose that $(r + Ann(x_1), \ldots, r + Ann(x_n)) = (s + Ann(x_1), \ldots, s + Ann(x_n))$. Then $r + Ann(x_i) = s + Ann(x_i)$ for all $i \in \{1, \ldots, n\}$. Therefore $r - s \in Ann(x_i)$ for all $i \in \{1, \ldots, n\}$. As a result, $r - s \in Ann(x_1) \cap Ann(x_2) \cap \cdots \cap Ann(x_n) = Ann(I)$. Hence $r + Ann(I) = s + Ann(I)$, and we have that ϕ is an injection.

By Lemma 4.9, $R/Ann(x_i)$ is a coloring for each $i \in \{1, \ldots, n\}$. Therefore $R/Ann(x_1) \times \cdots \times R/Ann(x_n)$ is a coloring by Theorem 4.11. Since ϕ is an injection, $R/Ann(I)$ is a subring of $R/Ann(x_1) \times \cdots \times R/Ann(x_n)$. Hence by Theorem 4.7, $R/Ann(I)$ is a coloring. □

The next theorem will allow us to consider the localization R_S of a Coloring R give us an idea of the chromatic number of the zero-divisor graph of R_S and clique number of R_S in relation to the chromatic number of the zero-divisor graph of R and clique number of R.

Theorem 4.13. *Let S be a multiplicatively closed set in a coloring R. Then R_S is a coloring. In addition, $\chi(\Gamma_0(R_S)) \leq \chi(\Gamma_0(R))$ and $cl(R_S) \leq cl(R)$.*

Proof. Since R is a coloring, we know that $\chi(\Gamma_0(R)) < \infty$. Suppose that $\chi(\Gamma_0(R)) = n$. As a result of [7], to show that the zero-divisor graph of R_S is $n-$colorable, it suffices to show that every finite subset of R_S is $n-$colorable. Let $x_1, x_2, \ldots, x_m$ be a finite subset of R_S, where $x_1 = \frac{r_1}{s}, x_2 = \frac{r_2}{s}, \cdots x_m = \frac{r_m}{s}$. We will show that this finite subset of R_S is $n-$colorable. To do this, we will associate each x_i with an element r_i' and show that $x_i x_j = 0$ if and only if $r_i' r_j' = 0$, so that we can assign the same colors to the $x_i's$ that are assigned to the r_i''s. To show this equivalence, we will establish some notations. By the properties of localizations, if $x_i x_j = 0$ for $i \neq j$ then $x_i x_j = \frac{r_i}{s} \cdot \frac{r_j}{s} = \frac{r_i r_j}{s^2} = \frac{0}{s^2}$, which implies that there exists for each pair i and j, where $i, j \in \{1, 2, \ldots m\}$, an element $u_{ij} \in S$ such that $u_{ij}(r_i r_j s^2 - 0 \cdot s^2) = 0$. Therefore $u_{ij} r_i r_j s^2 = 0$, where we define $s_{ij} = u_{ij} s^2$. Let $t = \Pi s_{lk}$, where $1 \leq l, k \leq m$ and define $r_i' = t r_i$. We now have the tools necessary to show that $x_i x_j = 0$ if and only if $r_i' r_j' = 0$.

For the forward containment, suppose that $x_i x_j = 0$ for $i \neq j$. Then using the properties of localization we discussed, for each pair i and j there exists an element $s_{ij} \in S$ such that $s_{ij} r_i r_j = 0$. From the discussion above, and since $s_{ij}^2 r_i r_j$ occurs in the product $(\Pi s_{lk})^2 r_i r_j$, we have that $r_i' r_j' = (tr_i)(tr_j) = (\Pi s_{lk}) r_i (\Pi s_{lk}) r_j = (\Pi s_{lk})^2 r_i r_j = 0$. Therefore $r_i' r_j' = 0$.

To show the reverse direction, suppose that $r_i' r_j' = 0$. Using the equalities defined in our discussion above, $0 = r_i' r_j' = (tr_i)(tr_j) = (\Pi s_{lk}) r_i (\Pi s_{lk}) r_j = (\Pi s_{lk})^2 r_i r_j$. We let $u = (\Pi s_{lk})^2$ and therefore $u r_i r_j = 0$. Hence $u(r_i r_j s^2 - 0 \cdot s^2) = 0$ and we have that $\frac{r_i r_j}{s^2} = \frac{0}{s^2}$. Therefore $x_i x_j = 0$.

We will now verify that the r_i''s are distinct. Suppose that $r_i' = r_j'$. Then $(\Pi s_{lk}) r_i = (\Pi s_{lk}) r_j$, which implies that $\Pi s_{lk}(r_i - r_j) = 0$ and therefore by letting $u = \Pi s_{lk}$ we have a $u \in S$ such that $u(r_i - r_j) = 0$. Therefore $u(r_i s - r_j s) = 0$ which implies $\frac{r_i}{s} = \frac{r_j}{s}$. This is a contradiction to the fact that the x_i's are distinct.

Now we have a correspondence between x_i and r_i', so x_i can be colored with the same color as r_i'. Therefore we have a valid coloring of $x_1, \ldots, x_m$. Since $\chi(\Gamma_0(R)) = n$, $\{x_1, \ldots, x_m\}$ is $n-$colorable. Therefore $\chi(\Gamma_0(R_S)) \leq n$ which gives us that R_S is a coloring and $\chi(\Gamma_0(R_S)) \leq \chi(\Gamma_0(R))$. This also tells us that $cl(R_S) \leq cl(R)$. □

5. SEPARATING ELEMENTS AND BECK'S CONJECTURE

Now that we have established some properties about colorings, we will discuss separating elements which will help us determine when $\chi(\Gamma_0(R)) = cl(R)$, as in [1]. We begin by defining separating elements.

Definition 5.1. *An element x in R is separating if $x \neq 0$ and $ab = 0$ imply that $xa = 0$ or $xb = 0$. To give a more local property, let I be an ideal. An element $x \in I$ is $I-$separating if $xI \neq (0)$ and whenever $ab = 0$ for some elements $a, b \in I$ then $xa = 0$ or $xb = 0$.*

As stated in [1], there are some remarks regarding this definition. Note that in this definition, it is not necessary for $a \neq b$. Also, an element x is separating if and only if x is $R-$separating. An $R-$separating element x contained in I is not $I-$separating if $xI = (0)$, but if it is the case that $xI \neq (0)$ then x is $I-$separating.

We are now ready to establish some theorems about separating elements.

Proposition 5.2. *If $Ann(x)$ is a prime ideal then x is separating.*

Proof. Suppose that $Ann(x)$ is a prime ideal. Assume $x \neq 0$ and let $y, z \in R$ be elements such that $yz = 0$. Then $yzx = 0$, which implies $yz \in Ann(x)$. Since $Ann(x)$ is prime, either $y \in Ann(x)$ or $z \in Ann(x)$. Therefore either $yx = 0$ or $zx = 0$. Thus x is separating. □

Proposition 5.3. *A nonzero ideal I in a coloring contains a separating element.*

Proof. Let R be a coloring and I be a nonzero ideal. Consider the family of annihilators $\{Ann(x_i) \mid 0 \neq x_i \in I\}$. Since $I \neq 0$, this family is nonempty. By Theorem 4.3, a coloring has an ascending chain condition on ideals of the form $Ann(a)$, so the family of annihilators $\{Ann(x_i) \mid 0 \neq x_i \in I\}$ must have a maximal element, say $Ann(x)$.

We claim that $Ann(x)$ is prime. Let $ab \in Ann(x)$ and $b \notin Ann(x)$. Then $abx = 0$ and $bx \neq 0$. Therefore $a \in Ann(bx)$. Notice that $Ann(x) \subsetneq Ann(bx)$. Since $Ann(x)$ is a maximal element of $\{Ann(x_i) \mid 0 \neq x_i \in I\}$, we must have that $Ann(x) = Ann(bx)$. As a result, $a \in Ann(x)$ and therefore $Ann(x)$ is prime.

By Proposition 5.3, since $Ann(x)$ is prime, x is a separating element. □

Theorem 5.4. *Let I be an ideal in a coloring not contained in the nilradical. Then I contains an $I-$separating element.*

Proof. Let I be an ideal in a coloring R not contained in the nilradical. Consider the family of annihilators $\{Ann(x_i) \mid 0 \neq x_i \in I\}$. Notice that $I \neq (0)$ since I is not contained in the nilradical $\mathfrak{N}$ of R. Therefore this family of annihilators is nonempty.

We claim that $I \nsubseteq Ann(x_i)$ for all $0 \neq x_i \in I$. Suppose that $I \subseteq Ann(x_i)$ for all nonzero $x_i \in I$. Then $x_i I = (0)$ for all nonzero $x_i \in I$. Therefore $x_i^2 = 0$ for all $x_i \in I$ and every element in I is nilpotent implying that $I \subseteq \mathfrak{N}$. This a contradiction to I not contained in the nilradical. Therefore $I \nsubseteq Ann(x_i)$ for all $0 \neq x_i \in I$.

Since R is a coloring, by Theorem 4.3, R has the ascending chain condition on ideals of the form $Ann(a)$. Therefore the family of annihilators $\{Ann(x_i) \mid 0 \neq x_i \in I\}$ must have a maximal element, say $Ann(x)$ where $I \nsubseteq Ann(x)$. Using the same argument as in the previous Proposition, $Ann(x)$ is prime. By Theorem 5.2, since $Ann(x)$ is a prime ideal, x is $R-$separating. We know that $xI \neq (0)$ because $I \nsubseteq Ann(x)$. Hence by definition, x is $I-$separating. □

Theorem 5.5. *Let I be a principal ideal in a coloring. If $I^2 \neq (0)$ then I contains an $I-$separating element.*

Proof. Let $I = Rx$ be a principal ideal in a coloring R and suppose that $x^2 \neq 0$. Consider the family of annihilators $\{Ann(x^2y) \mid y \in R \text{ and } x^2y \neq 0\}$. Notice that this set is nonempty since $Ann(x^2)$ is in this family. Since R has an ascending chain condition on annihilators, the family of annihilators $\{Ann(x^2y) \mid y \in R \text{ and } x^2y \neq 0\}$ has a maximal element, call it $Ann(x^2t) = (0 : x^2t)$. This element is prime by the same argument provided in Proposition 5.3.

We claim that xt is an $I-$separating element. Let $a, b \in I$ and assume $ab = 0$. Since I is principal, $a = rx$ and $b = sx$. Then $ab = (rx)(sx) = rsx^2 = 0$, which implies that $rs \in (0 : x^2t)$. Since $(0 : x^2t)$ is prime, either $r \in (0 : x^2t)$ or $s \in (0 : x^2t)$. If $r \in (0 : x^2t)$, then we have that $rx^2t = a(xt) = 0$. If $s \in (0 : x^2t)$, then $sx^2t = b(xt) = 0$. In addition, $(tx)x = tx^2 \neq 0$ implies that $(tx)I \neq (0)$.

Hence tx is an $I-$separating element. □

We will now begin to see the importance of separating elements in determining the relationship between the clique number of a ring and the chromatic number of the zero-divisor graph of a ring.

Lemma 5.6. *Let I be an ideal in a coloring and assume $x \in I$ is $I-$separating. Let $I' = Ann(x) \cap I$.*

(1) If $x^2 = 0$ then $cl(I') = cl(I)$ and $\chi(\Gamma_0(I')) = \chi(\Gamma_0(I))$.

(2) If $x^2 \neq 0$ then $cl(I') = cl(I) - 1$ and $\chi(\Gamma_0(I')) = \chi(\Gamma_0(I)) - 1$.

Proof. For (1), we assume that $x^2 = 0$. Then $x \in Ann(x)$. Therefore since $x \in I$ and $x \in Ann(x)$, we conclude that $x \in I'$.

We will first show that $cl(I') = cl(I)$. Since $I' \subseteq I$, certainly $cl(I') \leq cl(I)$. For the other inequality, let $cl(I) = n$ and choose a maximal clique $C = \{y_1, \ldots, y_n\}$ in I. We will consider two cases: $x \in C$ and $x \notin C$.

Case 1: Suppose $x \in C$. Then $x = y_i$ for some $i \in \{1, \ldots, n\}$, which implies that $xy_j = 0$ for all $j \in \{1, \ldots, i-1, i+1, \ldots, n\}$. Therefore $y_j \in Ann(x)$ for all $j \in \{1, \ldots, i-1, i+1, \ldots, n\}$. As a result, $y_j \in I'$ for all $i \in \{1, \ldots, i-1, i+1, \ldots, n\}$. Hence $C' = \{y_1, \ldots, y_{i-1}, x, y_{i+1}, \ldots, y_n\}$ is a clique in I' and therefore $cl(I') \geq n = cl(I)$. Therefore in this case, $cl(I') = cl(I)$.

Case 2: Now suppose that $x \notin C$. Then there exists an element in $y_i \in C$ such that $xy_i \neq 0$. Since C is a clique, $y_i y_j = 0$ for all $i \in \{1, 2, \ldots, i-1, i+1, \ldots n\}$. Since x is $I-$separating, either $y_i x = 0$ or $y_j x = 0$. But $y_i x \neq 0$, and so $y_j x = 0$ for all $j \in \{1, 2, \ldots, i-1, i+1, \ldots n\}$. Therefore $y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n \in Ann(x)$ implying that $y_1, \ldots, y_{i-1}, y_{i+1}, \ldots, y_n \subseteq I'$. Therefore $C' = \{y_1, y_2, \ldots, y_{i-1}, x, y_{i+1}, \ldots, y_n\}$ is a clique of size n in I'. Hence $cl(I') \geq n = cl(I)$. In this case, again $cl(I') = cl(I)$.

Now, we will show that $\chi(\Gamma_0(I')) = \chi(\Gamma_0(I))$. Since $I' \subseteq I$, certainly we have $\chi(\Gamma_0(I')) \leq \chi(\Gamma_0(I))$. For the reverse inequality, we begin by coloring the zero-divisor graph of I' first. Since $x \in \Gamma_0(I')$, it has been assigned a color. We claim that if $y \in \Gamma_0(I \setminus I')$ then y can be assigned the same color as x. Since $y \in I \setminus I'$, then $y \notin Ann(x)$ and therefore $xy \neq 0$. As a result, x and y are not adjacent and can be assigned the same color.

Our next claim is that any two elements in $a, b \in \Gamma_0(I \setminus I')$ are not adjacent and can be assigned the same color. If $ab = 0$, then since x is $I-$separating, we have that either $xa = 0$ or $xb = 0$. However, since $a, b \notin Ann(x)$, we must have $ab \neq 0$. Therefore a and b are not adjacent and can be assigned the same color.

We now claim that no other element $z \in \Gamma_0(I')$ that was assigned the same color as $x \in \Gamma_0(I')$ is adjacent to an element $y \in \Gamma_0(I \setminus I')$ which was also assigned the same color as x. Notice that $xz \neq 0$ since they were assigned the same color in I'. If $yz = 0$, then $xy = 0$ or $xz = 0$ since x is $I-$separating. However, neither of these is not possible since $y \notin Ann(x)$ and $xz \neq 0$. Therefore $yz \neq 0$ which tells us that y and z cannot be adjacent and can be assigned the same color.

Since we are able to color $\Gamma_0(I)$ using the same colors used to color $\Gamma_0(I')$, we have that $\chi(\Gamma_0(I)) \leq \chi(\Gamma_0(I'))$. Therefore $\chi(\Gamma_0(I')) = \chi(\Gamma_0(I))$.

For (2), assume that $x^2 \neq 0$. Then $x \notin Ann(x)$, which results in $x \notin I'$. We begin by proving that $cl(I') = cl(I) - 1$. Consider a maximal clique C in I' where $cl(I') = n$. All elements belonging to C are annihilators of x, so $C \cup \{x\}$ forms a clique in I of size $n+1$. Therefore $cl(I') + 1 \leq cl(I)$, which implies $cl(I') \leq cl(I) - 1$. For the reverse inequality, let $C = \{y_1, y_2, \ldots, y_n\}$ be a maximal clique in I. We have two cases: $x \notin C$ and $x \in C$.

Case 1: If $x \notin C$, there is a $y_i \in C$ such that $xy_i \neq 0$. Without loss of generality, assume $y_i = y_1$. Since C is a clique, we know that $y_1y_2 = y_1y_3 = \cdots = y_1y_n = 0$. Since x is $I-$separating, for all $k \in \{2, \ldots, n\}$ we have either $xy_1 = 0$ or $xy_k = 0$,but $xy_1 \neq 0$. This implies that $y_k \in Ann(x)$. Therefore $C' = \{x, y_2, \ldots, y_n\}$ is a clique in I' of size $cl(I)$, resulting in $cl(I') \geq cl(I)$.

Case 2: Suppose that $x \in C$. Without loss of generality, assume that $x = y_1$. Then since C is a clique, $xy_2 = xy_3 = \cdots = xy_n = 0$. Therefore $y_k \in Ann(x) \subseteq I'$ for all $k \in \{2, \ldots, n\}$. Therefore $C' = \{y_2, \ldots, y_n\}$ forms a clique in I' and has size $n - 1$. Hence $cl(I) - 1 \leq cl(I')$. As a result, we have proved that $cl(I') = cl(I) - 1$.

We will show that $\chi(\Gamma_0(I')) = \chi(\Gamma_0(I)) - 1$. We begin by coloring the zero-divisor graph of I'. We know that $x \notin I'$. Consider $y \in I' = Ann(x) \cap I$. Since $y \in Ann(x)$, we have that $yx = 0$. Therefore x must be assigned a color not already used in $\Gamma_0(I')$. Now consider $a, b \in I \setminus I'$. Certainly $a, b \notin Ann(x)$, so $xa \neq 0$ and $xb \neq 0$. If $ab = 0$, then $xa = 0$ or $xb = 0$ since x is $I-$separating. However, we know that $xa \neq 0$ and $xb \neq 0$. Therefore $ab \neq 0$. This tells us that the elements in $\Gamma_0(I \setminus I')$ are not adjacent and therefore can be assigned the same color. For this, we only need to include one additional color not used in $\Gamma_0(I')$. Therefore $\chi(\Gamma_0(I)) \leq \chi(\Gamma_0(I')) + 1$.

For the reverse inequality, color I first. Since $I \subseteq I'$, we can remove the vertices that belong to $\Gamma_0(I \setminus I')$ leaving us only with $\Gamma_0(I')$ which is already colored. For $y \in I'$, we know that $y \in Ann(x)$ and therefore $xy = 0$. This means that x must have been assigned a color that is different from those used on elements in $\Gamma_0(I')$. Since we removed the vertices belonging to $\Gamma_0(I \setminus I')$, the color used for x was removed and so we need one less color than was used to color $\Gamma_0(I)$. Therefore $\chi(\Gamma_0(I')) \leq \chi(\Gamma_0(I)) - 1$. Finally, we have that $\chi(\Gamma_0(I')) \leq \chi(\Gamma_0(I)) - 1$. $\square$

We are now ready to prove the following theorem, which is a generalization of the previous theorem.

Theorem 5.7. *Let I be an ideal in a coloring. Let $\{x_1, \ldots, x_n\}$ be a clique of $I-$separating elements. Define $k = |\{i \mid x_i^2 \neq 0\}|$ and $I' = I \cap Ann(x_1, \ldots, x_n)$. Then $cl(I') = cl(I) - k$ and $\chi(\Gamma_0(I')) = \chi(\Gamma_0(I)) - k$.*

Proof. We begin by showing that $cl(I') = cl(I) - k$. Let $x_1, \ldots, x_k \in \{x_i \mid x_i^2 \neq 0\}$ and $x_k, \ldots, x_n \in \{x_j \mid x_j^2 = 0\}$. Then $x_i \notin Ann(x_i)$ for $i \in \{1, \ldots, k\}$ and $x_j \in Ann(x_j)$ for $j \in \{k+1, \ldots, n\}$. Since $\{x_1, \ldots, x_n\}$ is a clique, we have that $x_i \in Ann(x_1, \ldots, x_{i-1}, x_{i+1}, \ldots, x_n)$ for $i \in \{1, \ldots, k\}$ and $x_j \in Ann(x_1, \ldots, x_n)$ for $j \in \{k+1, \ldots n\}$. As a result $x_1, \ldots, x_k \notin I'$ and $x_{k+1}, \ldots x_n \in I'$.

Suppose that C' is a clique in I' of size $cl(I')$. Notice that $I' \subseteq I$ and all elements in C' are annihilators of $x_1, \ldots, x_k$. Therefore since $x_1, \ldots, x_k \notin I'$, we have that $C' \cup \{x_1, \ldots, x_k\}$ forms a clique of size $cl(I') + k$ in I. Hence $cl(I) \geq cl(I') + k$, giving us that $cl(I') \leq cl(I) - k$.

Suppose that $C = \{y_1, \ldots, y_m\}$ is a clique in I. We will show that $cl(I') \geq cl(I) - k$. We will construct a clique in I' using C. For each $x_i \in \{x_1, \ldots, x_n\}$ either $x_i \in C$ or $x_i \notin C$. If $x_j \notin C$ for some $j \in \{1, \ldots, n\}$ then there exists an element, without loss of generality, $y_j \in C$ such that $x_j y_j \neq 0$. Since C is a clique, $y_j y_r = 0$ for all $r \in \{1, \ldots, j-1, j+1, \ldots, m\}$. In addition, since x_j is $I-$separating, we have that either $x_j y_j = 0$ or $x_j y_r = 0$ for all $r \in \{1, \ldots, j-1, j+1, \ldots, m\}$. However, we know that $x_j y_j \neq 0$ and therefore we must have $x_j y_r = 0$ for all $r \in \{1, \ldots, j-1, j+1, \ldots, m\}$. Hence we have a new clique $C_1 = \{y_1, \ldots, y_{j-1}, x_j, y_{j+1}, \ldots, y_m\}$. We can do this process for every $x_i \in \{x_1, \ldots, x_n\}$ that is not contained in C to get a new maximal clique of the same size. Therefore without loss of generality, we will assume that $\{x_1, \ldots, x_n\} \subseteq C$.

Since $x_i \in C$ for all $i \in \{1, \ldots, k\}$, we assume without loss of generality that $x_i = y_i$ for all $i \in \{1, \ldots, k\}$. Since $\{y_1, \ldots, y_m\}$ is a clique, $x_i y_i = 0$ for all $i, j \in \{1, \ldots, k\}$ where $i \neq j$, and for all $i \in \{k+1, \ldots n\}$ and $j \in \{k+1, \ldots, m\}$. Therefore $y_{n+1}, \ldots, y_m \in Ann(x_1, \ldots, x_n)$. Recall that $x_1, \ldots, x_k \notin I'$ and $x_{k+1}, \ldots, x_n \in I'$. Therefore if we restrict the clique C to I', we have that $C' = \{x_{k+1}, \ldots, x_n, y_{n+1}, \ldots, y_m\}$ is a clique in I'. Hence $cl(I') \geq cl(I) - k$.

Therefore we conclude that $cl(I') = cl(I) - k$.

We will now show that $\chi(\Gamma_0(I')) = \chi(\Gamma_0(I)) - k$. We begin by coloring the zero-divisor graph of I. Then $I' \subseteq I$ is colored. We will remove the vertices belonging to $I \setminus I'$. Recall that $x_1, \ldots, x_k \notin I'$. Notice that all elements in I' are in $Ann(x_1, \ldots, x_n)$ and therefore annihilate the elements $x_1, \ldots, x_k$. As a result, the elements $x_1, \ldots, x_k$ will require k colors and these colors do not appear in I'. Hence I' can be colored using $\chi(\Gamma_0(I)) - k$ colors. Therefore $\chi(\Gamma_0(I')) \geq \chi(\Gamma_0(I)) - k$.

Now we color the zero-divisor graph of I'. We know that $x_1, \ldots, x_k \notin I'$. Consider $x \in I' = I \cap Ann(x_1, \ldots, x_n)$. Since $x \in Ann(x_1, \ldots, x_n)$, we have that $xx_i = 0$ for all $i \in \{1, \ldots, n\}$. Therefore $x_1, \ldots, x_k$ must each be assigned colors not already used in $\Gamma_0(I')$. Now consider $y \in I \setminus I'$ where $y \notin \{x_1, \ldots, x_k\}$. Then $y \notin Ann(x_1, \ldots, x_n)$, and therefore there exists an element $x_i \in \{x_1, \ldots, x_n\}$ such that $yx_i \neq 0$. Hence y can be assigned the same color as x_i. Finally, take two elements $a, b \in I \setminus I'$ where $a, b \notin \{x_1, \ldots, x_k\}$. Certainly, $ax_i \neq 0$ and $bx_i \neq 0$ for all $i \in \{1, \ldots, n\}$. If $ab = 0$ then either $ax_i = 0$ or $bx_i = 0$ for all $i \in \{1, \ldots, n\}$ since x_i is $I-$separating for all $i \in \{1, \ldots, n\}$. However, we know that $ax_i \neq 0$ and $bx_i \neq 0$ for all $i \in \{1, \ldots, n\}$. Therefore $ab \neq 0$. This tells us that the elements in $\Gamma_0(I \setminus I')$ that are not in $\{x_1, \ldots, x_k\}$ are not adjacent, and therefore can be assigned the same color. Hence we only need k additional colors not used in coloring $\Gamma_0(I')$ to color $\Gamma_0(I)$. Thus $\chi(\Gamma_0(I)) \geq \chi(\Gamma_0(I')) + k$, giving us that $\chi(\Gamma_0(I')) \leq \chi(\Gamma_0(I)) - k$.

Therefore we can conclude that $\chi(\Gamma_0(I')) = \chi(\Gamma_0(I)) - k$ □

The next result is a corollary to the previous theorem.

Corollary 5.8. *Let $\mathfrak{P}_1, \ldots, \mathfrak{P}_n$ be the minimal prime ideals in a coloring R. Let $\varepsilon(R) = |\{i \mid R_{\mathfrak{P}_i}$ is a field $\}|$. Then $cl(R) = cl(\mathfrak{N}) + \varepsilon(R)$ and $\chi(\Gamma_0(R)) = \chi(\Gamma_0(\mathfrak{N})) + \varepsilon(R)$.*

Proof. We know that the nilradical is the intersection of all minimal prime ideals, so $\mathfrak{N} = \mathfrak{P}_1 \cap \cdots \cap \mathfrak{P}_n$. Certainly, $\mathfrak{N} = \mathfrak{P}_1 \cap \cdots \cap \mathfrak{P}_n \cap R$. Since $\mathfrak{P}_1, \ldots, \mathfrak{P}_n$ are minimal primes, by Theorem 4.4 they are also associated primes. Therefore $\mathfrak{P}_i = (0 : x_i) = Ann(x_i)$ for some $x_i \in R$ for all $i \in \{1, \ldots, n\}$. Then by Lemma 3.7, $x_i x_j = 0$ for all $i \neq j$. Therefore $\{x_1, \ldots, x_n\}$ forms a clique. To show that $\{x_1, \ldots, x_n\}$ is $R-$separating, assume that for $x, y \in R$ we have $xy = 0$. Then $xy \in Ann(x_i)$ for all $i \in \{1, \ldots, n\}$. Since $Ann(x_i)$ is prime, either $x \in Ann(x_i)$ or $y \in Ann(x_i)$ for all $i \in \{1, \ldots, n\}$. Therefore $xx_i = 0$ or $yx_i = 0$ for all $i \in \{1, \ldots, n\}$. As a result, we can conclude that $\{x_1, \ldots, x_n\}$ is a clique of $R-$separating elements.

Now we will verify that $R_{\mathfrak{P}_i}$ is a field if and only if $x_i^2 = 0$. Suppose that $R_{\mathfrak{P}_i}$ is a field. Then every nonzero element in $R_{\mathfrak{P}_i}$ is a unit. Consider the nonzero element $\frac{x_i}{1}$ with $x_i \neq 0$. Since $\frac{x_i}{1}$ is a unit, there exists an element $\frac{y_i}{1}$ such that $\frac{x_i}{1} \cdot \frac{y_i}{1} = \frac{x_i y_i}{1} = \frac{1}{1}$. Now $x_i \notin \mathfrak{P}_i = Ann(x_i)$ since $\mathfrak{P}_i$ cannot contain units. Therefore $x_i^2 \neq 0$. For the reverse containment, suppose that $x_i^2 \neq 0$. Then $x_i \notin Ann(x_i) = \mathfrak{P}_i$. Recall that the maximal ideal of $R_{\mathfrak{P}_i}$ is $\mathfrak{P}_i R_{\mathfrak{P}_i} = \{\frac{p}{s} \mid p \in \mathfrak{P}_i, s \in R \setminus \mathfrak{P}_i\}$. Let $\frac{p}{s} \in \mathfrak{P}_i R_{\mathfrak{P}_i}$. Notice that $x_i p = 0$ because $p \in \mathfrak{P}_i = Ann(x_i)$. Then we have that $\frac{p}{s} = \frac{0}{s}$

since $x_i(ps - 0 \cdot s) = 0$. Therefore $\mathfrak{P}_i R_{\mathfrak{P}_i} = (0)$, implying that $R_{\mathfrak{P}_i}$ is a field. This tells us that $k = |\{i \mid x_i^2 \neq 0\}| = |\{i \mid R_{\mathfrak{P}_i} \text{ is a field }\}| = \varepsilon(R)$.

Applying Theorem 5.7, $cl(\mathfrak{N}) = cl(R) - \varepsilon(R)$ and $\chi(\Gamma_0(\mathfrak{N})) = \chi(\Gamma_0(R) - \varepsilon(R)$. □

Theorem 5.8 allows us to see that the key to proving or disproving Beck's first conjecture may lie in the nilradical. The next result is stronger than Theorem 3.9.

Theorem 5.9. *Let R be a reduced coloring. Then $cl(I) = \chi(\Gamma_0(I)$ for any ideal $I \subset R$.*

Proof. Since R is a reduced coloring, the nilradical $\mathfrak{N}$ is the (0) ideal. Let I be an ideal in R. If $I = (0)$, then $cl(I) = \chi(\Gamma_0(I))$. Suppose that I is a nonzero ideal in R. Then I is not contained in the nilradical and by Theorem 5.4, I has an $I-$separating element, call it x. Notice that $x^2 \neq 0$ for otherwise x would be nilpotent and we would have $x = 0$. This would be a contradiction to the fact that x is $I-$separating. Now by Lemma 5.6, $cl(I') = cl(I) - 1$ and $\chi(\Gamma_0(I')) = \chi(\Gamma_0(I)) - 1$, where $I' = I \cap Ann(x)$.

We will prove that $cl(I) = \chi(\Gamma_0(I))$ by induction on $cl(I)$. If $cl(I) = 1$, the zero-divisor graph must consist of only one vertex and therefore $\chi(\Gamma_0(I)) = 1$. We will assume that if $cl(I) = n - 1$ then $cl(I) = \chi(\Gamma_0(I))$. Suppose that $cl(I) = n$. We will prove that $cl(I) = \chi(\Gamma_0(I))$. Since $cl(I) = n$ and $cl(I') = cl(I) - 1$, we have that $cl(I') = n - 1$. By induction hypothesis, we have $n - 1 = cl(I') = \chi(\Gamma_0(I'))$. Since $n - 1 = \chi(\Gamma_0(I'))$ and $\chi(\Gamma_0(I')) = \chi(\Gamma_0(I)) - 1$, we have $\chi(\Gamma_0(I)) - 1 = n - 1$. Therefore $\chi(\Gamma_0(I)) = n$ and finally, $\chi(\Gamma_0(I)) = n = cl(I)$. □

We will also prove Beck's conjecture in the case that R is a principal ideal ring.

Theorem 5.10. *Let R be a coloring which is a principal ideal ring. Then $\chi(\Gamma_0(I)) = cl(I)$ for any ideal I in R.*

Proof. We will show that we can make a reduction to the case when $I \subset \mathfrak{N}$. Suppose that $I \not\subset \mathfrak{N}$. Let $\mathfrak{P}_1, \ldots, \mathfrak{P}_n$ be the minimal prime ideals in R. Then $\mathfrak{N} = \mathfrak{P}_1 \cap \cdots \cap \mathfrak{P}_n$. Suppose that $I \not\subset \mathfrak{P}_1$. Then there exists an element $a \in I$ such that $a \notin \mathfrak{P}_1$. We claim that $Ann(a) \subseteq \mathfrak{P}_1$. Suppose that $Ann(a) \nsubseteq \mathfrak{P}_1$. Then there is an $x \in Ann(a)$ such that $x \notin \mathfrak{P}_1$. Therefore $xa = 0$ which implies that $xa \in \mathfrak{P}_1$ since 0 is in every prime ideal. Therefore either $x \in \mathfrak{P}_1$ or $a \in \mathfrak{P}_1$. However, we know that neither x nor a is in $\mathfrak{P}_1$. Therefore $Ann(a) \subseteq \mathfrak{P}_1$.

Consider the family of annihilators $\{Ann(y) \mid Ann(y) \subseteq \mathfrak{P}_1, z \in I, z \notin \mathfrak{P}_1\}$. By Theorem 4.3, R has an ascending chain condition on annihilators. Therefore the family of annihilators $\{Ann(y) \mid Ann(y) \subseteq \mathfrak{P}_1, z \in I, z \notin \mathfrak{P}_1\}$ has a maximal element, say $Ann(y_1)$. Notice that $Ann(y_1)$

is prime by the same argument used in Proposition 5.3. Since $Ann(y_1) \subseteq \mathfrak{P}_1$ and $\mathfrak{P}_1$ is a minimal prime, we must have that $Ann(y_1) = \mathfrak{P}_1$. By Proposition 5.2, since $Ann(y_1)$ is prime, y_1 is an $x-$separating element.

We claim that y_1 is $I-$separating. We know that $y_1 \in I$ and that y_1 is $R-$separating, so all that we must show is that $y_1 I \neq (0)$. If $y_1 I = (0)$, then $I \subseteq Ann(y_1) = \mathfrak{P}_1$ which is a contradiction. Therefore $y_1 I \neq (0)$, and y_1 is $I-$separating.

Define $I_1 = I \cap Ann(y_1) = I \cap \mathfrak{P}_1$. By Lemma 5.6, $\chi(\Gamma_0(I)) = cl(I)$ if and only if $\chi(\Gamma_0(I_1)) = cl(I_1)$. Therefore proving that $\chi(\Gamma_0(I)) = cl(I)$ is reduced to proving that $\chi(\Gamma_0(I_1)) = cl(I_1)$ for $I_1 \subseteq \mathfrak{P}_1$. As a result, we reduce to the case when $I \subset \mathfrak{N}$.

Suppose that $I = Rx \subset \mathfrak{N}$. We will consider I^2. If $I^2 = (0)$, then I is a clique and $cl(I) = |I| = \chi(\Gamma_0(I))$. If $I^2 \neq (0)$, by Theorem 5.5, I contains an $I-$separating element y_1. Let $I_1 = I \cap Ann(y_1)$. Then $I_1 \subseteq I$. If $I_1 = I$, we have that $I = I \cap Ann(y_1)$ and $I \subseteq Ann(y_1)$. Therefore $y_1 I = (0)$ which is not possible since y_1 is an I separating element. Hence $I_1 \subset I$. By Lemma 5.6, $\chi(\Gamma_0(I)) = cl(I)$ if and only if $\chi(\Gamma_0(I_1)) = cl(I_1)$.

Now we consider I_1^2. If $I_1^2 = (0)$, we know that I_1 is a clique and $cl(I_1) = |I_1| = \chi(\Gamma_0(I_1))$. By Lemma 5.6, $\chi(\Gamma_0(I)) = cl(I)$ if and only if $\chi(\Gamma_0(I_1)) = cl(I_1)$. In the case that $I_1^2 \neq (0)$, by Theorem 5.5, I_1 contains an I_1-separating element y_2. Let $I_2 = I_1 \cap Ann(y_2)$. Then $I_2 \subseteq I_1 \subseteq I$. If $I_2 = I_1$, we have that $I_1 = I_1 \cap Ann(y_1)$ and $I_1 \subseteq Ann(y_2)$. Therefore $y_2 I_1 = (0)$ which cannot be since y_2 is an I_1-separating element. Hence $I_2 \subset I_1$ and by Lemma 5.6, $\chi(\Gamma_0(I_1)) = cl(I_1)$ if and only if $\chi(\Gamma_0(I_2)) = cl(I_2)$. We repeat this process.

Since $\mathfrak{N}$ is finite, this process must terminate for otherwise we would have an infinite chain $I \supset I_1 \supset I_2 \supset \cdots$, which is a contradiction to $\mathfrak{N}$ being finite. Therefore we will have an $I_n = I_{n-1} \cap Ann(y_n) = I \cap Ann(y_1) \cap \cdots \cap Ann(y_n) = I \cap \mathfrak{P}_1 \cap \cdots \cap \mathfrak{P}_n$ where $I_n^2 = (0)$ and $\chi(\Gamma_0(I_n)) = |I_n| = cl(I_n)$. By Lemma 5.6, $\chi(\Gamma_0(I_{n-1})) = cl(I_{n-1})$ if and only if $\chi(\Gamma_0(I_n)) = cl(I_n)$. Hence $cl(I) = \chi(\Gamma_0(I))$. $\square$

The previous two theorems lead us to consider that the chromatic number equals the clique number for any ideal in a coloring, which we can see in the statement of the next theorem. The proof of the next theorem follows from the previous theorem's proof.

Theorem 5.11. *Let R be a coloring with the property that any ideal I for which $I^2 \neq (0)$ contains an $I-$separating element. Then $\chi(\Gamma_0(I)) = cl(I)$ for any ideal I in R.*

Now that we have proved that Beck's conjecture for principal ideal rings and reduced rings and their ideals, it seems natural to wonder about the clique number or chromatic number of the zero-divisor graph of an ideal I of a ring R, where R is a finite product of reduced rings or principal ideal rings.

Proposition 5.12. *Suppose that $R = R_1 \times \cdots \times R_n$. Let $I = I_1 \times \cdots \times I_n$ be an ideal of R. If $x_i \in I_i$ is I_i-separating, then $(0, \ldots, 0, x_i, 0, \ldots, 0) \in I$ is $I-$separating.*

Proof. Let $ab = 0$ for $a = (a_1, \ldots, a_n) \in I$ and $b = (b_1, \ldots, b_n) \in I$. Then $a_j b_j = 0$ for all $j \in \{1, \ldots, n\}$. Since $x_i \in I_i$ is I_i separating, we have that either $a_i x_i = 0$ or $b_i x_i = 0$, and $x_i I_i \neq (0)$. Then, either $(a_1, \ldots, a_{i-1}, a_i, a_{i+1}, \ldots a_n)(0, \ldots, 0, x_i, 0, \ldots, 0) = (0, \ldots, 0, 0, 0, \ldots, 0)$, or $(b_1, \ldots, b_{i-1}, b_i, b_{i+1}, \ldots b_n)(0, \ldots, 0, x_i, 0, \ldots, 0) = (0, \ldots, 0, 0, 0, \ldots, 0)$ for $a = (a_1, \ldots, a_n) \in I$ or $b = (b_1, \ldots, b_n) \in I$. Also, $x_i I \neq (0, \ldots, 0)$ since $x_i I_i \neq 0$. Therefore $(0, \ldots, 0, x_i, 0, \ldots, 0)$ is $I-$separating. ☐

Theorem 5.13. *Let R be a coloring that is a finite product of reduced rings and principal ideal rings. Then $\chi(\Gamma_0(I)) = cl(I)$ for any ideal $I \subset R$.*

Proof. Let R be a coloring that is a finite product of reduced rings and principal ideal rings. Then $R = R_1 \times \cdots \times R_n$, where each R_i is either a reduced ring or a principal ideal ring. Let $I \subsetneq R$ be an ideal where $I = I_1 \times \cdots \times I_n$, where I_i is an ideal of R_i for each $i \in \{1, \ldots, n\}$. If $I^2 = (0)$, then I is a clique and $cl(I) = |I| = \chi(\Gamma_0(I))$. If $I^2 \neq (0)$, then $I_i^2 \neq (0)$ for some $i \in \{1, \ldots, n\}$. Notice that $\underline{R_i} = (0, \ldots, 0, R_i, 0, \ldots, 0)$ is a subring of R, which is a coloring. If R_i is a reduced ring, we know that $\mathfrak{N} = 0$ and by Theorem 5.4, any ideal I_i not contained in the nilradical contains an I_i-separating element. If R_i is a principal ideal ring, then by Theorem 5.5, an ideal I_i in R_i contains an I_i-separating element. By the previous proposition, if we have an I_i-separating element x_i, then $(0, \ldots, 0, x_i, 0, \ldots, 0) \in I$ is an $I-$separating element. Now by Theorem 5.11, $\chi(\Gamma_0(I)) = cl(I)$ for any ideal I in R. ☐

6. CHARACTERIZATION OF FINITE RINGS R WITH $\chi(\Gamma_0(R)) \leq 4$

In this section, we will discuss the characterization of finite rings R with $\chi(\Gamma_0(R)) \leq 4$ as presented in [1] and [6]. We will start by discussing the work that Beck presented in [1] that verifies his conjecture that $\chi(\Gamma_0(R)) = cl(R)$ in the case that $\chi(\Gamma_0(R)) \leq 5$ or $cl(R) \leq 4$. Afterwards, we will proceed by discussing the work that Anderson and Naseer presented in [6].

We will begin by stating a proposition from [1] that proves Beck's conjecture for zero-divisor graphs whose chromatic number or clique number is less than or equal to two.

Proposition 6.1. *Given a coloring R, $\chi(\Gamma_0(R)) = cl(R)$ if $cl(R) \leq 2$ or $\chi(\Gamma_0(R)) \leq 2$.*

Proof. We will consider four cases: $\chi(\Gamma_0(R)) = 1, \chi(\Gamma_0(R)) = 2, cl(R) = 1$, and $cl(R) = 2$.

Assume that $\chi(\Gamma_0(R)) = 1$. By Proposition 2.1, R is the zero ring and therefore $cl(R) = 1$.

Suppose that $cl(R) = 1$. Then the zero-divisor graph of R cannot have any adjacencies. This is because 0 is always adjacent to all nonzero elements and if we have a nonzero element adjacent to 0, we would increase the clique number. Therefore $R = (0)$ and as a result, $\chi(R) = 1$.

Suppose $\chi(\Gamma_0(R)) = 2$. By Proposition 2.2, R is an integral domain, $R \cong \mathbb{Z}_4$, or $R \cong \mathbb{Z}_2[X]/(X^2)$. In the case that R is an integral domain, since there are no nontrivial zero-divisors, the largest clique can consist of exactly two elements which include 0 and a nonzero element $x \in R$.

If $R \cong \mathbb{Z}_4$, there are four elements which are $\{0, 1, 2, 3\}$. Since no two nonzero elements in this set multiply to zero, the largest clique in R is of size two and must consist of 0 and any one of the nonzero elements $1, 2$, or 3.

Finally, we consider the ring $R \cong \mathbb{Z}_2[X]/(X^2)$. The elements in this ring are $0, 1, x$, and $1+x$. Again, no two nonzero elements multiply to zero, so the largest clique in this ring can consist of two elements including only 0 and any one of the nonzero elements. In any case, we have that $cl(R) = 2$.

Now assume that $cl(R) = 2$. We know that $\chi(\Gamma_0(R)) \geq cl(R)$ is always true. Therefore $\chi(\Gamma_0(R)) \geq 2$. Assume that $\chi(\Gamma_0(R)) > 2$. Since 0 is always adjacent to every nonzero element in R, we must assign 0 its own color. Since the zero-divisor graph of R has chromatic number three, there must be nonzero elements x and y such that $xy = 0$. Therefore $\{0, x, y\}$ must form a clique giving us that $cl(R) \geq 3$, which is a contradiction. Therefore we must have $\chi(\Gamma_0(R)) = 2$. $\square$

The following proposition from [1] deals with Beck's conjecture in the case that the clique number and chromatic number are both three.

Proposition 6.2. *Let R be a coloring. Then $cl(R) = 3$ if and only if $\chi(\Gamma_0(R)) = 3$.*

Proof. We know that $\chi(\Gamma_0(R)) \geq cl(R)$ is always true and in Proposition 6.1 we proved that $cl(R) \leq 2$ if and only if $\chi(\Gamma_0(R)) \leq 2$; therefore it is enough to show that $\chi(\Gamma_0(R)) > 3$ implies that $cl(R) > 3$. We will begin by letting $R^* = R \setminus \{0\}$ and assuming that $\chi(\Gamma_0(R)) > 3$. Then we must have that $\chi(\Gamma_0(R^*)) \geq 3$ because 0 requires its own color and we are eliminating 0. Therefore the zero-divisor graph of R^* is not $2-$colorable and hence must contain an odd cycle. Let n be the minimal length of a cycle in R^* and assume that $n \geq 5$ since it is not $2-$colorable. We denote our cycle $C_n = x_n x_1 x_2 \cdots x_n x_1$. Therefore we have that $x_1x_2 = x_2x_3 = x_3x_4 = \cdots = x_{n-1}x_n = x_nx_1 = 0$. Now, suppose that $x_1x_k = 0$ for some $k \neq 1, 2, n$. Therefore we have two cycles $x_1x_2 \cdots x_kx_1$ and $x_kx_{k+1} \cdots x_nx_1x_k$ with length less than n, one of which must be odd. Since C_n is a cycle of minimal length, no smaller cycles can exist and therefore $x_ix_j = 0$ if and only if x_i and x_j are neighbors on C_n.

Let $y = x_1x_3$. Then $yx_2 = yx_4 = yx_n = 0$. Therefore y cannot belong to the cycle $x_1x_2 \cdots x_nx_1$ since it is adjacent to three elements contained in this cycle. Now, we have a cycle of length $n-2$ denoted by $yx_4 \cdots x_ny$. Hence R^* contains an odd cycle of length $n-2 \geq 3$, which implies that the clique number of R^* is greater or equal to 3 and hence R has a clique number greater than or equal to four. $\square$

Now, we will state a proposition from [1] that verifies Beck's conjecture in the case that the chromatic number and clique number are both four and an implication for when the chromatic number is five.

Proposition 6.3. *Let R be a coloring and let $k \leq 4$ be an integer. Then $\chi(\Gamma_0(R)) = k$ if and only if $cl(R) = k$. Moreover $\chi(\Gamma_0(R)) = 5$ implies $cl(R) = 5$.*

Proof. To prove that $\chi(\Gamma_0(R)) = k$ if and only if $cl(R) = k$, it suffices to show that $\chi(\Gamma_0(R)) \leq 4$ if and only if $cl(R) \leq 4$. The previous two propositions give us that $\chi(\Gamma_0(R)) \leq 3$ if and only if $cl(R) \leq 3$, and we know that $\chi(\Gamma_0(R)) \geq cl(R)$ is always true, so we only need to prove that $cl(R) \leq 4$ implies that $\chi(\Gamma_0(R)) \leq 4$. Suppose that $cl(R) \leq 4$. If R is reduced, by Theorem 5.9

$cl(R) = \chi(\Gamma_0(R))$. Therefore we assume that R is not reduced, i.e. the nilradical $\mathfrak{N}$ of R is nonzero. By Theorem 5.8, $cl(R) = cl(\mathfrak{N}) + \varepsilon(R)$ and $\chi(\Gamma_0(R)) = \chi(\Gamma_0(\mathfrak{N})) + \varepsilon(R)$.

We need to show that $cl(\mathfrak{N}) = \chi(\Gamma_0(\mathfrak{N}))$. Certainly, $\chi(\Gamma_0(R)) \geq cl(R)$. Therefore we only need to show that $\chi(\Gamma_0(R)) \leq cl(R)$ That is, we need to verify that $\chi(\Gamma_0(\mathfrak{N})) > 4$ implies $cl(\mathfrak{N}) > 4$. Let $I = \mathfrak{N} \cap Ann(\mathfrak{N})$. We claim that $\mathfrak{N}$ is nilpotent and $I \neq (0)$.

We will show that $\mathfrak{N}$ is nilpotent. Recall that $\mathfrak{N}$ is the set of nilpotent elements. Since R is a coloring, $\chi(\Gamma_0(R)) < \infty$ and by Theorem 3.10, $\mathfrak{N}$ is finite. Therefore let $\mathfrak{N} = \{j_1, \ldots, j_n\}$. We know that every element in $\mathfrak{N}$ is nilpotent and therefore $j_1^{a_1} = j_2^{a_2} = \cdots = j_n^{a_n} = 0$ for some $a_1, \ldots, a_n \in \mathbb{N}$. Let $a = max\{a_1, \ldots, a_n\}$ so that $j_i^a = 0$ for all $i \in \{1, \ldots, n\}$. Consider $\mathfrak{N}^{an} = \{\sum r_i j_{i_1} j_{i_2} \cdots j_{i_{an}} \mid r_i \in R, j_{i_k} \in \mathfrak{N}\}$. Since $j_{i_k} \in \mathfrak{N}$, we can write each product $j_{i_1} j_{i_2} \cdots j_{i_{an}}$ as $j_1^{b_1} j_2^{b_2} \cdots j_n^{b_n}$ where $b_1 + \cdots + b_n = an$ and at least one of the b_i's must be greater than or equal to a, say $b_r \geq a$. Therefore $j_r^{b_r} = 0$, which results in $\sum r_i j_1^{b_1} \cdots j_n^{b_n} = 0$. Hence $\mathfrak{N}^{an} = 0$ and $\mathfrak{N}$ is nilpotent.

Next, we will show that $I \neq (0)$. Suppose that $I = (0)$. Then for every nonzero element $x \in \mathfrak{N}$, we have that $x\mathfrak{N} \neq (0)$. Since $\mathfrak{N}$ is nilpotent, there is some $n \in \mathbb{N}$ such that $\mathfrak{N}^n = (0)$ but $\mathfrak{N}^{n-1} \neq (0)$. Let $0 \neq t \in \mathfrak{N}^{n-1}$. Since $\mathfrak{N}^{n-1} \subseteq \mathfrak{N}$, we have $t \in \mathfrak{N}$. Notice that $t\mathfrak{N} \neq 0$ but $t\mathfrak{N} \subseteq \mathfrak{N}^n = 0$. This is a contradiction. Therefore $I = \mathfrak{N} \cap Ann(\mathfrak{N}) \neq (0)$ and as a result, $|I| \geq 2$. Notice that I is a clique in $\mathfrak{N}$ because any element $x \in I$ is contained in both $\mathfrak{N}$ and $Ann(\mathfrak{N})$.

We have some cases to consider. If $I = \mathfrak{N}$, then certainly $\chi(\Gamma(\mathfrak{N})) = cl(\mathfrak{N})$. If $|I| > 4$, there is nothing to prove since $cl(\mathfrak{N}) > 4$. Therefore we assume that $|I| \leq 4$. Suppose $|I| = 4$. Let $x \in \mathfrak{N} \setminus I$. Then $I \cup \{x\}$ forms a clique in $\mathfrak{N}$ with five elements and therefore $cl(\mathfrak{N}) > 4$. If $|I| = 3$ and $\chi(\Gamma_0(\mathfrak{N})) > 4$ then there are at least two elements $x, y \in \mathfrak{N} \setminus I$ that are adjacent to each other and every element in I, for otherwise we would need less than five colors for $\Gamma_0(\mathfrak{N}))$. Therefore $I \cup \{x, y\}$ forms a clique in $\mathfrak{N}$ with five elements and so, $cl(\mathfrak{N}) > 4$. The last case is when $|I| = 2$ and $\chi(\Gamma_0(\mathfrak{N})) > 4$, which will require some work.

Let $I = (0, \gamma)$. Since I is an ideal $\gamma + \gamma = 0$. Also, since $\chi(\Gamma_0(\mathfrak{N})) \geq 5$, the set of elements belong to $\mathfrak{N} \backslash I$ in $\Gamma_0(\mathfrak{N}))$ will require three distinct colors. Since these elements are not $2-$colorable, the subgraph $\Gamma_0(\mathfrak{N} \setminus I)$ of $\Gamma_0(\mathfrak{N})$ must have an odd cycle. Suppose $C_n = x_1 x_2 \cdots x_n x_1$ is an odd cycle in $\Gamma_0(\mathfrak{N} \setminus I)$ of minimal length n where $n \geq 5$. Notice that a smaller odd cycle would result in $cl(\mathfrak{N}) > 4$. If $v_1 v_k = 0$ for some $k \neq 1, 2, n$ the cycle C_n would decompose into two smaller cycles, one of which would be an odd cycle. However, since C_n is a minimal odd cycle, there can be no smaller cycles. Therefore $x_ix_j = 0$ if and only if v_i and v_j are neighbors in C_n. Now let $y = x_ix_j$ where x_i and x_j are not neighbors in C_n. Then $yx_{i-1} = yx_{i+1} = yx_{j-1} = yx_{j+1} = 0$. Since y is adjacent to four elements in C_n, y cannot belong the the cycle C_n. Let $i \neq 1, 2, n$. Let $z = x_1x_i$. If i is even, $zx_2 \cdots x_{i-1}z$ is an odd cycle of length $i - 1 < n$. If i is odd, $zx_{i+1} \cdots x_nz$ is an odd cycle of length $n - (i - 1) < n$. Since C_n is minimal in $\Gamma_0(\mathfrak{N} \setminus I)$, we can conclude that $x_ix_j = 0$ if and only if x_i and x_j are neighbors and $x_ix_j = \gamma$ if and only if $i \neq j$, and x_i and x_j are not neighbors.

We now claim that $x_i^2 \neq 0$. Suppose that $x_i^2 = 0$ and $x_{i+1} \neq x_i + \gamma$. We can show that x_i, x_{i+1}, and $x_i + \gamma$ form a cycle in $\Gamma_0(\mathfrak{N} \setminus I)$. Certainly x_i and x_{i+1} are in $\Gamma_0(\mathfrak{N} \setminus I)$. Suppose that $x_i + \gamma \in I$. Then either $x_i + \gamma = 0$ or $x_i + \gamma = \gamma$. If $x_i + \gamma = 0$, we have that $x_i \in \gamma$. If $x_i + \gamma = \gamma$ then $x_i = \gamma \in I$. Both of these are contradictions since $x_i \notin I$. Hence $x_i + \gamma \in \mathfrak{N} \setminus I$. Now we consider the products between x_i, x_{i+1}, and $x_i + \gamma$. First, $x_ix_{i+1} = 0$ since x_i and x_{i+1} are neighbors in C_n. Also, $x_{i+1}(x_i + \gamma) = x_{i+1}x_i + x_{i+1}\gamma = 0$ since x_i and x_j are neighbors in C_n and $\gamma \in I = \mathfrak{N} \cap Ann(\mathfrak{N})$ annihilates all elements in $\mathfrak{N}$. Finally, $x_i(x_i + \gamma) = x_i^2 + x_i\gamma = 0$ since $x_i^2 = 0$ and $\gamma \in Ann(\mathfrak{N})$. Therefore we have have a cycle of length three in $\Gamma_0(\mathfrak{N} \setminus I)$. This is a contradiction since C_n is an odd cycle of minimal length at least five. Therefore we either have $x_i^2 \neq 0$ or $x_{i+1} = x_i + \gamma$. Suppose that $x_{i+1} = x_i + \gamma$. Then $x_{i+1}x_{i+2} = (x_i + \gamma)x_{i+2} = x_ix_{i+2} + \gamma x_{i+1} = x_ix_{i+2}$ since $\gamma \in Ann(\mathfrak{N})$. Also, since x_{i+1} and x_{i+2} are neighbors in C_n, we have that $0 = x_{i+1}x_{i+2} = x_ix_{i+2}$. This is a contradiction since x_i and x_{i+2} are not neighbors in C_n. Therefore we must have $x_i^2 \neq 0$ for $1 \leq i \leq n$. Now consider $t = x_1 + \cdots + x_{n-2}$. Then:

$$\begin{aligned} tx_{n-1} &= (x_1 + \cdots + x_{n-3} + x_{n-2})x_{n-1} \\ &= x_1x_{n-1} + \cdots + x_{n-3}x_{n-1} + x_{n-2}x_{n-1} \\ &= (n-3)\gamma \\ &= 0 \end{aligned}$$

since $x_ix_j \neq 0$ for $j \neq i + 1$, $\gamma + \gamma = 0$ and $n - 3$ an even number. Also, we have that

$$\begin{aligned} tx_n &= (x_1 + \cdots + x_{n-3} + x_{n-2})x_n \\ &= x_1x_n + x_2x_n \cdots + x_{n-3}x_n + x_{n-2}x_n \\ &= (n-3)\gamma \\ &= 0 \end{aligned}$$

since $x_i x_j \neq 0$ for $j \neq i+1$, $\gamma + \gamma = 0$ and $n-3$ an even number. We know that $x_n = i^2 \neq 0$ for $1 \leq i \leq n$, and so $t \neq x_n$ and $t \neq x_{n-1}$. Since n is odd, we write $n = 2k+1$ for some $k \in \mathbb{N}$. Then we have that

$$\begin{aligned}
tx_k &= (x_1 + \cdots + x_{k-2} + x_{k-1} + x_k + x_{k+1} + \cdots x_{2k-1})x_k \\
&= x_1x_k + \cdots + x_{k-2}x_k + x_{k-1}x_k + x_k^2 + x_{k+1}x_k + \cdots x_{2k-1}x_k \\
&= (k-3)\gamma + x_k^2 + ((2k-1) - (k+1))\gamma \\
&= (2k-4)\gamma + x_k^2 \\
&= x_k^2 \\
&\neq 0.
\end{aligned}$$

Therefore since $x_k \in \mathfrak{N}$, we have that $t \notin Ann(\mathfrak{N})$ and as a result $t \notin I$. Hence $t \neq 0$ and $t \neq \gamma$. This tells us that $\{0, x_{n-1}, x_n, t, \gamma\}$ forms a clique of size five in $\mathfrak{N}$. Therefore $cl(\mathfrak{N}) > 4$, implying that $cl(\mathfrak{N}) = \chi(\Gamma_0(\mathfrak{N}))$ for $\chi(\Gamma_0(\mathfrak{N})) \leq 4$ and $cl(\mathfrak{N}) \leq 4$. Thus $cl(R) = \chi(\Gamma_0(R))$ for $\chi(\Gamma_0(R)) \leq 4$ and $cl(R) \leq 4$.

To show that $\chi(\Gamma_0(R)) = 5$ implies $cl(R)$, assume $\chi(\Gamma_0(R)) = 5$. Certainly $\chi(\Gamma_0(\mathfrak{N})) \leq 5$ since $\mathfrak{N} \subseteq R$. If $\chi(\Gamma_0(\mathfrak{N})) \leq 4$, we have proved that $\chi(\Gamma_0(\mathfrak{N})) = cl(\mathfrak{N})$. Therefore we need only to consider the case when $\chi(\Gamma_0(\mathfrak{N})) = 5$. We know that $\chi(\Gamma_0(\mathfrak{N})) \geq cl(\mathfrak{N})$ is always true and so, $cl(\mathfrak{N}) \geq 4$. From the proof above, if $\chi(\Gamma_0(\mathfrak{N})) \leq 5$, then $cl(\mathfrak{N}) \geq 5$. Therefore $cl(\mathfrak{N}) = 5$ which implies that $\chi(\Gamma_0(\mathfrak{N})) = cl(\mathfrak{N})$. Hence $\chi(\Gamma_0(R)) = 5$ implies $cl(R) = 5$. □

Now we will find finite rings R for which $\chi(R) \leq 3$ as in [1].

By Proposition 2.1, we know that $\chi(\Gamma_0(R)) = 1$ if and only if $R = (0)$.

Proposition 2.2 tells us that $\chi(\Gamma_0(R)) = 2$ if and only if R is an integral domain, $R \cong \mathbb{Z}_4$, or $R \cong \mathbb{Z}_2[X]/(X^2)$. Notice that since R is a finite ring, we can replace integral domain with field.

We will now consider the case when $\chi(\Gamma_0(R)) = 3$. By Theorem 6.2, we know that $cl(R) = 3$ if and only if $\chi(\Gamma_0(R)) = 3$. Also, by Theorem 5.8 we know that $cl(R) = cl(\mathfrak{N}) + \varepsilon(R)$ and $\chi(\Gamma_0(R)) = \chi(\Gamma_0(\mathfrak{N})) + \varepsilon(R)$ where $\varepsilon(R)$ is the number of prime ideals such that $R_{\mathfrak{P}}$ is a field. Since $cl(R) \geq 1$, we have three cases: $\varepsilon(R) = 0$, $\varepsilon(R) = 1$, or $\varepsilon(R) = 2$.

Case 1: Consider $\varepsilon(R) = 2$. Then $cl(\mathfrak{N}) = 1$, which implies that $\mathfrak{N} = (0)$. Since $\chi(\Gamma_0(R)) = 3 < \infty$, by Theorem 3.9 R has two minimal prime ideals, say $\mathfrak{P}_1$ and $\mathfrak{P}_2$. Note that since R is finite, $dim(R) = 0$. Therefore every prime ideal in R is maximal and $(0) = \mathfrak{N} = \mathfrak{P}_1 \cap \mathfrak{P}_2$ where $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are maximal ideals. Then $\mathfrak{P}_1 + \mathfrak{P}_2 = R$ since $\mathfrak{P}_1$ and $\mathfrak{P}_2$ are maximal ideals and $\mathfrak{P}_1 \subsetneq \mathfrak{P}_1 + \mathfrak{P}_2$ and

$\mathfrak{P}_2 \subsetneq \mathfrak{P}_1 + \mathfrak{P}_2$. By the Chinese Remainder Theorem, $R \cong R/(0) \cong R/(\mathfrak{P}_1 \cap \mathfrak{P}_2) \cong R/\mathfrak{P}_1 \times R/\mathfrak{P}_2$. Therefore R is a product of two finite ideals.

Case 2: Suppose $\varepsilon(R) = 1$. Then $cl(\mathfrak{N}) = 2$ and R is not reduced. Let $\mathfrak{P}$ be a prime ideal for which $R_{\mathfrak{P}}$ is a field. Since R is a finite ring, $dim(R) = 0$ and every prime ideal is both maximal and minimal. Since $\mathfrak{P}$ is a minimal prime, by Theorem 4.4 $\mathfrak{P}$ is an associated prime ideal. Therefore $\mathfrak{P} = Ann(x)$ for some $x \in R$. Then $x\mathfrak{P} = (0)$, which implies $x \in Ann(\mathfrak{P})$. Since R_p is a field, every element $\frac{x}{1} \in R_{\mathfrak{P}}$ has an inverse. Notice that $x \notin \mathfrak{P}$. We will show that $\mathfrak{P} \cap (x) = (0)$. Take $z \in \mathfrak{P} \cap (x)$. Then $z \in \mathfrak{P}$ and $z = rx$. Since $\mathfrak{P}$ is a prime ideal and $rx \in \mathfrak{P}$, either $r \in \mathfrak{P}$ or $x \in \mathfrak{P}$. Since $x \notin \mathfrak{P}$, we must have $r \in \mathfrak{P}$. Therefore $rx = 0$ and we have established that $\mathfrak{P} \cap (x) = (0)$. Notice that $\mathfrak{P} \subsetneq \mathfrak{P} + (x)$. Therefore $\mathfrak{P} + (x) = R$. Since $x \in Ann(\mathfrak{P})$, we also know that $(x) \subseteq Ann(\mathfrak{P})$. Certainly we have that $\mathfrak{P} \subsetneq \mathfrak{P} + Ann(\mathfrak{P})$. Therefore $\mathfrak{P} + Ann(\mathfrak{P}) = R$. Next, we will verify that $\mathfrak{P} \cap Ann(\mathfrak{P}) = (0)$. Take $y \in \mathfrak{P} \cap Ann(\mathfrak{P})$. Then $y \in \mathfrak{P}$ and $yp = 0$ for all $p \in \mathfrak{P}$. Therefore $\mathfrak{P} \cap Ann(\mathfrak{P}) = (0)$. By the Chinese Remainder Theorem, $R \cong R/(0) \cong R/(\mathfrak{P} \cap Ann(\mathfrak{P})) \cong R/\mathfrak{P} \times R/Ann(\mathfrak{P})$.

Since $\mathfrak{P}$ is a maximal ideal, $R/\mathfrak{P}$ is a finite field. Let $R/\mathfrak{P} = k$ and $R/Ann(\mathfrak{P}) = S$, where k is a finite field. By assumption, we know $cl(R) = cl(k \times S) = 3$ and we always have that $cl(k \times S) \leq cl(k)cl(S)$. Since k is a finite field, $cl(k) = 2$ and therefore $cl(S) \geq 2$. If $cl(S) > 2$, there are at least three elements s_1, s_2, s_3 in S that are adjacent to one another. Notice that $(1, 0), (0, s_1), (0, s_2)$ and $(0, s_3)$ are all in $k \times S$ and form a clique of size 4. This is a contradiction. Therefore $cl(S) = 2$ which results in $\chi(\Gamma_0(S)) = 2$. By Proposition 2.2, $S \cong \mathbb{Z}_4$ or $S \cong \mathbb{Z}_2[X]/(X^2)$. Therefore $R \cong k \times \mathbb{Z}_4$ or $R \cong k \times \mathbb{Z}_2[X]/(X^2)$.

Case 3: Suppose that $\varepsilon(R) = 0$. Then there are no prime ideals $\mathfrak{P}$ such that $R_{\mathfrak{P}}$ is a field. In this case, $cl(\mathfrak{N}) = 3$. Since R is finite, we know that $dim(R) = 0$. Therefore every prime ideal is a minimal prime ideal and, hence an associated prime ideal. Suppose $\mathfrak{P}$ and $\mathfrak{Q}$ are two distinct prime ideals in R. Then $\mathfrak{P} = Ann(x)$ and $\mathfrak{Q} = Ann(y)$ for some elements x and y in R. By Lemma 3.7, $xy = 0$. Also, by Theorem 5.2, x and y are $R-$separating elements. Since $\varepsilon(R) = 0$, we have $x^2 = 0 = y^2$. Then $\{0, x, y, x + y\}$ forms a clique in $\mathfrak{N}$. However, since $cl(\mathfrak{N}) = 3$, it must be that $x + y = 0$ which leads to $\mathfrak{P} = \mathfrak{Q}$. Therefore R has a unique prime ideal $\mathfrak{P} = Ann(x)$ and $x^2 = 0$. As a result, $\mathfrak{N} = \mathfrak{P}$. This allows us to conclude that Rx is a clique and $2 \leq |Rx| \leq 3$. In addition, $Rx \subset Ann(\mathfrak{P}) = (0 : \mathfrak{P})$ and $(0 : \mathfrak{P})$ forms a clique. If $Rx \subsetneq (0 : \mathfrak{P})$ then $(0 : \mathfrak{P})$ must have at least 4 elements, which yields a contradiction. Hence $Rx = Ann(\mathfrak{P})$. Now we have some sub-cases to consider.

Sub-case 1: Suppose $|Rx| = 3$. Then $Rx = (0, x, y = -x)$. Notice that $\mathfrak{P} \cdot Rx = (0)$ because $\mathfrak{P} = Ann(x)$. We claim that $\mathfrak{P} = Rx$. Let $p \in R \setminus Rx$. Since $\mathfrak{P} = Ann(x)$, we would have a clique $\{0, x, y, p\}$ of 4 elements. This is a contradiction to $cl(\mathfrak{N}) = 3$. Therefore $\mathfrak{P} = Rx$. Consider the exact sequence $0 \longrightarrow Ann(x) \xrightarrow{f} R \xrightarrow{g} Rx \longrightarrow 0$. Since $R/Ann(x) \cong Rx$ and $|\mathfrak{P}| = |Rx| = 3$, by Lagrange's Theorem $|R| = |Rx||Ann(x)| = 9$. If $char(R) = 9$, $R \cong \mathbb{Z}_9$.

If $char(R) = 3$, we can derive that $R \cong \mathbb{Z}_3[X]/(X^2)$. We will start off by showing that $R \ncong \mathbb{Z}_3 \bigoplus \mathbb{Z}_3$. Notice that $R \cong \mathbb{Z}_3 \bigoplus \mathbb{Z}_3$ is a direct sum of fields and is therefore von Neumann. Also, the primes $\mathfrak{P}$ in $R \cong \mathbb{Z}_3 \bigoplus \mathbb{Z}_3$ are of the form $\{(a, 0) \mid a \in \mathbb{Z}_3\}$ and $\{(0, b) \mid b \in \mathbb{Z}_3\}$. Since R is von Neumann, $R/\mathfrak{P} \cong R_{\mathfrak{P}}$. Therefore $R_{\mathfrak{P}} \cong \mathbb{Z}_3$, and hence $R_{\mathfrak{P}}$ is a field. Since we are assuming that there are no primes such that $R_{\mathfrak{P}}$ is a field, $R \ncong \mathbb{Z}_3 \bigoplus \mathbb{Z}_3$. We know that additively, R should behave like $\mathbb{Z}_3 \bigoplus \mathbb{Z}_3$. We know that R must consist of $\mathbb{Z}_3$ and another generator x. Therefore we can write R as $\{0, 1, 2, x, 2x, x+1, x+2, 2x+1, 2x+2\}$. Hence we know that $R \cong \mathbb{Z}_3[X]/(p(x))$ where $p(x)$ is a polynomial of degree 2. It is easy to verify that x^2 cannot equal $1, 2, x+1, x+2, -x+1$, and $-x+2$, since then x would have to be a unit. Notice that x cannot be a unit since Rx is a proper ideal. If we assume that $x^2 = x$ or $x^2 = -x$, we end up with a clique of size 3. We also end up with no nonzero nilpotent elements, which contradicts the fact that $cl(\mathfrak{N}) = 3$. This is because if $x^2 = x$, then $\mathbb{Z}_3[X]/(X^2 - X) \cong \mathbb{Z}_3[X]/(X(X-1)) \cong \mathbb{Z}_3 \bigoplus \mathbb{Z}_3$ by the Chinese Remainder Theorem, since (X) and $(X-1)$ are comaximal, and $\mathbb{Z}_3 \bigoplus \mathbb{Z}_3$ is reduced. The same holds in the case that $x^2 = -x$. Since $R \ncong \mathbb{Z}_3 \bigoplus \mathbb{Z}_3$, the only other possibility is for $x^2 = 0$. Hence, we conclude that $R \cong \mathbb{Z}_3[X]/(X^2)$.

Sub-case 2: Suppose now that $|Rx| = 2$. Then $Rx = (0, x)$. We claim that $\mathfrak{P} \neq Rx = Ann(\mathfrak{P})$. Take $y \in \mathfrak{P} \setminus Rx$. If $y^2 = 0$, then $\{0, x, y, x+y\}$ forms a clique of 4 elements in $\mathfrak{P}$ which is a contradiction to $cl(\mathfrak{N}) = 3$. Therefore $y^2 \neq 0$ and $\mathfrak{P} \neq Rx$.

In addition, since $t \in \mathfrak{P} \setminus Rx$, we can verify that $Ann(y) = \{0, x\} = Rx$. Since $y \in \mathfrak{P} \setminus Rx$, we know that $y \in \mathfrak{P} = Ann(x)$ and $y \notin Rx$. Therefore $yx = 0$ and $x \in Ann(y)$. For the reverse containment, let $t \in Rx \setminus Ann(y)$. Then $t = rx \notin Ann(y)$, which is a contradiction since $x \in Ann(\mathfrak{P})$ and $y \in \mathfrak{P}$. As a result, $Ann(y) = \{0, x\} = Rx$.

We also consider $y \in \mathfrak{P} \backslash Rx$ together with $(0, x) = Ann(\mathfrak{P})$ to show that there is an element $r \in R$ such that $ry = x$. Since $y \in \mathfrak{P} \setminus Rx$, we know that $y \in \mathfrak{N}$ and therefore $y^n = 0$ for some minimal $n \in \mathbb{N}$. Then $y^{n-1}y = y^n = 0$, giving us that $y^{n-1} \in Ann(y) = Rx$. Since n was minimal $y^{n-1} \neq 0$ and $y^{n-1} \in Rx$, we have that $y^{n-1} = x$. Let $r = y^{n-2}$. Therefore $ry = y^{n-2}y = x$ for some $r \in R$.

Now take $s \in \mathfrak{P} = Ann(x)$. Then $r(sy) = sx = 0$, so that $r \in Ann(sy)$. Therefore $Ann(sy) \not\subset Rx$, which requires that $sy \in Rx$.

Consider the exact sequence $0 \longrightarrow (0, x) \xrightarrow{f} \mathfrak{P} \xrightarrow{g} (0, x) \longrightarrow 0$ where $g(t) = ty$. Since $|(0, x)| = 2$, we have that $|\mathfrak{P}| = 4$. Since $R/\mathfrak{P} \cong Rx$, $|\mathfrak{P}| = 4$ and $|Rx| = 2$, by Lagrange's Theorem, $|R| = |\mathfrak{P}||Rx| = 8$. Then $char(R) = 2, 4$, or 8.

If $char(R) = 2$, the exact sequence tells us that $\mathfrak{P}$ consists of the elements $\{0, x, r, x + r\}$ where $x^2 = xr = 0$ and $ry = x$. If $r^2 = 0$ then $cl(\mathfrak{P}) = 4$. Therefore $r^2 \neq 0$. If $r^2 = r + x$, we have $r(1 - r) = r^2 - r = x$ which requires $r \in Rx$. Then $r^2 = x$. As a result, $\mathfrak{P} = \{0, r^2, r, r^2 + r\}$ where $r^3 = xr = 0$ and the units are $1, 1 + r, 1 + r^2$, and $1 + r + r^2$. Therefore $R \cong \mathbb{Z}_2[X]/(X^3)$.

Suppose $char(R) = 4$. We know that $\mathfrak{P} = \{0, r, r^2, r + r^2\}$ where $r^3 = 0$. Since $4 = 0$, we have that $2 \in \mathfrak{P}$. If $r = 2$ then $r^2 = 0$, which is a contradiction. If $r^2 + r = 2$ then $0 = (r^2 + r)^2 = r^4 + 2r^3 + r^2 = r^2$, which is also a contradiction. Therefore $r^2 = 2$ and $2r = r^3 = 0$.

We define a homomorphism $\phi : \mathbb{Z}_4[T] \to R$ by $\phi(T) = r$. Then ϕ is surjective. In addition, $T^2 - 2 \in ker(\phi)$ and $2T \in ker(\phi)$. Since $|\mathbb{Z}_4[T]/(2T, T^2 - 2)| = 8$, we have that $ker(\phi) = (2T, T^2 - 2)$ and therefore $R \cong \mathbb{Z}_4[X]/(2X, X^2 - 2)$.

If $char(R) = 8$, we have that $R \cong \mathbb{Z}_8$.

Therefore the following finite rings have zero-divisor graph with $\chi(\Gamma_0(R)) \leq 3$.

(1) $\chi(\Gamma_0(R)) = 1$.

   (a) $R = \{0\}$

(2) $\chi(\Gamma_0(R)) = 2$.

   (a) R is a finite field.

   (b) $R \cong \mathbb{Z}_2$

   (c) $R \cong \mathbb{Z}_2[X]/(X^2)$

(3) $\chi(\Gamma_0(R)) = 3$.

(a) R is a product of two finite fields.

(b) $R \cong k \times \mathbb{Z}_4$, k a finite field

(c) $R \cong k \times \mathbb{Z}_2[X]/(X^2)$, k a finite field

(d) $R \cong \mathbb{Z}_8$

(e) $R \cong \mathbb{Z}_9$

(f) $R \cong \mathbb{Z}_3[X]/(X^2)$

(g) $R \cong \mathbb{Z}_2[X]/(X^3)$

(h) $R \cong \mathbb{Z}_4[X]/(2X, X^2 - 2)$

7. THE COUNTEREXAMPLE

The results provided by Beck in [1], which we have discussed, give us the impression that the chromatic number of a zero-divisor graph may be equal to the clique number of a ring. This is precisely the conjecture made by Beck, which was proved to be false by Anderson and Naseer's counterexample in [6]. We will discuss this counterexample and provide a picture of the zero-divisor graph. The counterexample is the zero-divisor graph (including 0) of the ring

$$R = \mathbb{Z}_4[X, Y, Z]/(X^2 - 2, Y^2 - 2, Z^2, 2X, 2Y, 2Z, XY, XZ, YZ - 2).$$

In R, $X^2 = Y^2 = YZ = 2$ and $Z^2 = 2X = 2Y = 2Z = XY = XZ = 0$. Therefore every element belonging to R must have degree 1 and is of the form $a + bx + cy + dz$ where $a, b, c, d \in \{0, 1, 2, 3\}$. Since $2X = 2Y = 2Z = 0$, we can assume that $b, c, d \neq 2$. Since $3X = X$, $3Y = Y$ and $3Z = Z$ in R, we can also assume that $b, c, d \neq 3$. Therefore the possible values for a, b, c, and d are $a \in \{0, 1, 2, 3\}$ and $b, c, d \in \{0, 1\}$. Notice that the elements of $\mathfrak{M}$ are of the form $a + bx + cy + dz$ where $a \in \{0, 2\}$ and $b, c, d \in \{0, 1\}$ and the elements of $U(R)$ are of the form $a + bx + cy + dz$ where $a \in \{1, 3\}$ and $b, c, d \in \{0, 1\}$. As a result, R has a total of 32 elements and is a finite local ring with unique maximal ideal $\mathfrak{M} = \{0, 2, x, x+2, y, y+2, x+y, x+y+2, z, z+2, x+z, x+z+2, y+z, y+z+2, x+y+z, x+y+z+2\}$ consisting of 16 elements. It can easily be verified that $\mathfrak{M}$ is an ideal. We provide a multiplication table for $\mathfrak{M}$ on the next page and we exclude 0 and 2, since they annihilate all of $\mathfrak{M}$. The remaining 16 elements of R are units $\mathfrak{U}(R) = R \backslash \mathfrak{M} = 1 + \mathfrak{M} = \{1 + m \mid m \in \mathfrak{M}\} = \{1, 3, x+1, x+3, y+1, y+3, x+y+1, x+y+3, z+1, z+3, x+z+1, x+z+3, y+z+1, y+z+3, x+y+z+1, x+y+z+3\}$. Through some calculations, it can be verified that $1, 3, x+y+1, x+y+3, z+1, z+3, x+y+z+1, x+y+z+3$ are their own inverses and that each pair of elements $y+1$ and $y+3$, $x+1$ and $x+3$, $y+z+1$ and $y+z+3$, and $x+z+1$ and $x+z+3$ are inverses.

Table 7.1: Multiplication Table for $\mathfrak{M}$ as in [6].

	x	x+2	y	y+2	x+y	x+y+2	z	z+2	x+z	x+z+2	y+z	y+z+2	x+y+z	x+y+z+2
x	2	2	0	0	2	2	0	0	2	2	0	0	2	2
x+2	2	2	0	0	2	2	0	0	2	2	0	0	2	2
y	0	0	2	2	2	2	2	2	2	2	0	0	0	0
y+2	0	0	2	2	2	2	2	2	2	2	0	0	0	0
x+y	2	2	2	2	0	0	2	2	0	0	0	0	2	2
x+y+2	2	2	2	2	0	0	2	2	0	0	0	0	2	2
z	0	0	2	2	2	2	0	0	0	0	2	2	2	2
z+2	0	0	2	2	2	2	0	0	0	0	2	2	2	2
x+z	2	2	2	2	0	0	0	0	2	2	2	2	0	0
x+z+2	2	2	2	2	0	0	0	0	2	2	2	2	0	0
y+z	0	0	0	0	0	0	2	2	2	2	2	2	2	2
y+z+2	0	0	0	0	0	0	2	2	2	2	2	2	2	2
x+y+z	2	2	0	0	2	2	2	2	0	0	2	2	0	0
x+y+z+2	2	2	0	0	2	2	2	2	0	0	2	2	0	0

We also provide a picture of the zero-divisor graph of $\mathfrak{M}$ excluding 0 and 2.

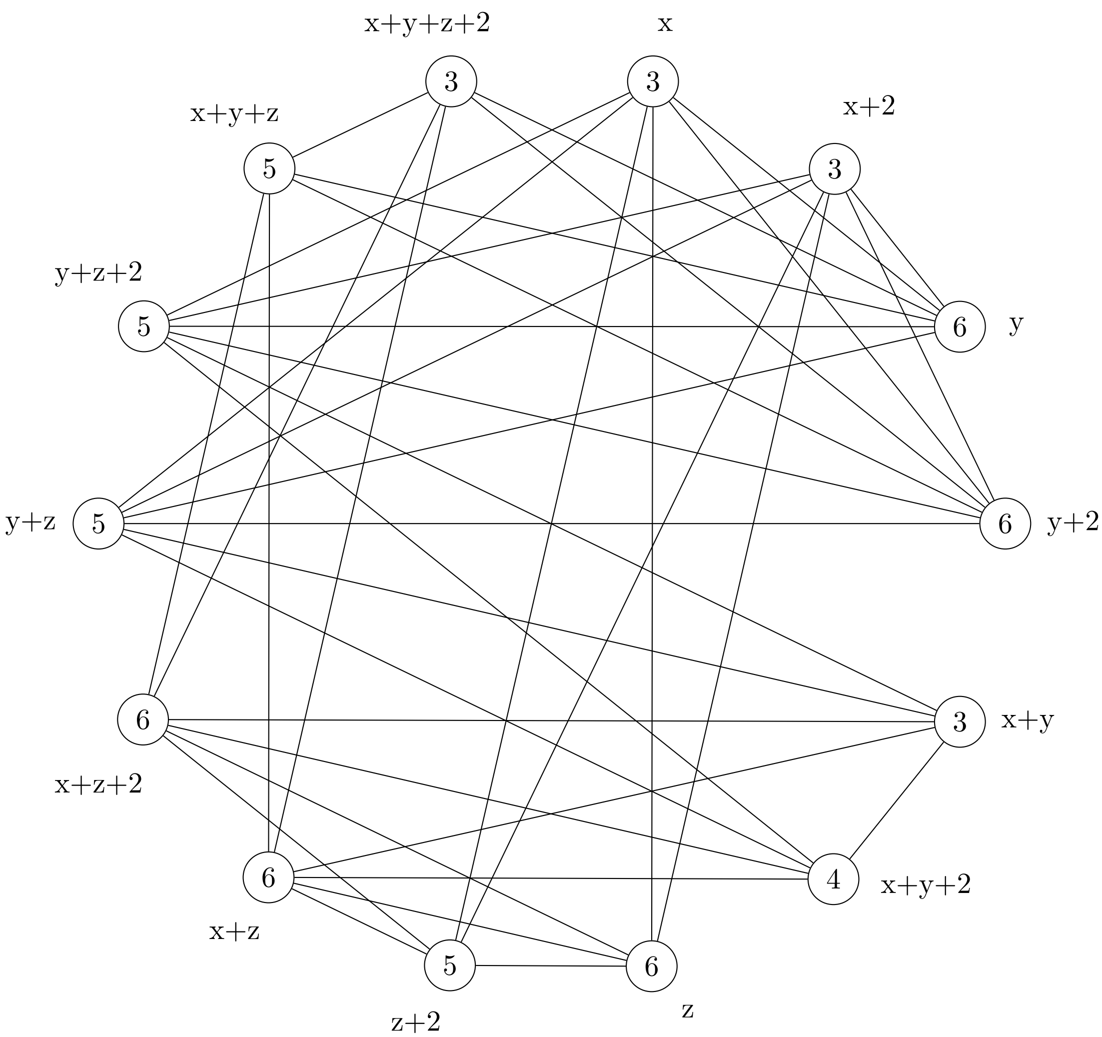

Figure 7.1: Zero-divisor graph of counterexample to $\chi(\Gamma_0(R)) = cl(R)$.

We will prove that $cl(R) = 5$ and $\chi(\Gamma_0(R)) = 6$ through a series of propositions. To prove that $cl(R) = 5$, it is enough to show that $cl(\mathfrak{M}) = 5$. We begin with eight facts that are easily verified using the multiplication table for $\mathfrak{M}$.

Remark 7.1. *Every maximal clique contains 0 and 2.*

The proof of this statement is straightforward because 0 and 2 both annihilate all of the elements in the maximal ideal $\mathfrak{M}$.

Remark 7.2. $\{0, 2, x, y, y + z\}$ *is a maximal clique, so* $cl(R) \geq 5$.

Using the multiplication table for $\mathfrak{M}$ we can observe that no other element can be added to enlarge this clique.

Proposition 7.3. *Any clique that contains x or $x+2$ has at most 5 elements.*

Proof. It is important to notice that a clique cannot contain both x and $x+2$. First, suppose x is in a clique. The possible candidate elements for the clique other than 0, 2 and x are either y or $y+2$, either $y+z$ or $y+z+2$, z, and $z+2$. If y is in the clique, then the elements $y+2$, z, and $z+2$ cannot be in the clique. The largest possible cliques in this case are $\{0,2,x,y,y+z\}$ and $\{0,2,x,y,y+z+2\}$. If $y+2$ is in the clique, then the elements y, z, and $z+2$ cannot be in the clique. Here, the largest possible cliques are $\{0,2,x,y+2,y+z\}$ and $\{0,2,x,y+2,y+z+2\}$. If $y+z$ is included in the clique, we must exclude z, $z+2$, and $y+z+2$. Hence the possible cliques are $\{0,2,x,y+z,y\}$, $\{0,2,x,y+z,y+2\}$. If $y+z+2$ is included in the clique, we must exclude z, $z+2$, and $y+z$ from the clique. Therefore the largest possible cliques are $\{0,2,x,y+z+2,y\}$ and $\{0,2,x,y+z+2,y+2\}$. If z or $z+2$ is included in the clique, we must exclude y, $y+2$, $y+z$ and $y+z+2$. Hence the largest possible clique is $\{0,2,x,z,z+2\}$. Thus any clique containing either x has at most 5 elements.

Now suppose that $x+2$ is in a clique. In this case, the candidate elements for the clique other than 0, 2, and $x+2$ are y or $y+2$, $y+z$ or $y+z+2$, z and $z+2$. If y is in the clique, then $y+2$, z, and $z+2$ cannot be in the clique. Therefore the largest possible cliques are $\{0,2,x+2,y,y+z\}$ and $\{0,2,x+2,y,y+z+2\}$. If $y+2$ is in the clique, the elements y, z, and $z+2$ cannot be in the clique. Hence the largest possible cliques are $\{0,2,x+2,y+2,y+z\}$ and $\{0,2,x+2,y+2,y+z+2\}$. If $y+z$ is in the clique, the elements $y+z+2$, z, and $z+2$ must be excluded from the clique. Therefore the largest possible cliques are $\{0,2,x+2,y+2,y+z\}$ and $\{0,2,x+2,y+2,y+z+2\}$. If $y+z+2$ is contained in the clique, the elements z, $z+2$, and $y+z$ cannot be in the clique. As a result, the largest possible cliques are $\{0,2,x+2,y+z+2,y\}$ and $\{0,2,x+2,y+z+2,y+2\}$. If either z or $z+2$ is included in the clique, y, $y+2$, $y+z$, and $y+z+2$ cannot be contained in the clique. Hence the possible largest clique is $\{0,2,x+2,z,z+2\}$. Thus any clique containing $x+2$ has at most 5 elements. $\square$

Proposition 7.4. *Any clique that contains y or $y+2$ has at most 5 elements.*

Proof. Suppose that y is in a clique. Then , $y+2$ cannot be in the clique. Since 0 and 2 both annihilate every element in the maximal ideal $\mathfrak{M}$, they are contained in every maximal clique. Since we know by the previous proposition that any clique containing x or $x+2$ must contain at most 5 elements, we will assume that the clique does not contain x or $x+2$. Therefore the possible elements contained in the clique aside from 0, 2, and y are either $y+z$ or $y+z+2$, $x+y+z$, and $x+y+z+2$. If either of $y+z$ or $y+z+2$ are contained in the clique, the elements $x+y+z$ and $x+y+z+2$ must be excluded. Hence the largest clique we can have is $\{0, 2, y, x+y+z, x+y+z+2\}$.

Now, suppose that $y+2$ is in the clique. Then the clique cannot contain y. Since 0 and 2 are contained in every maximal clique, they will also be in the clique with $y+2$. We will again assume that x and $x+2$ are not in the clique since any clique containing these two elements must contain at most 5 elements. The only possible candidates for elements in the clique in addition to 0, 2, and $y+2$ are one of either $y+z$ or $y+z+2$, $x+y+z$, and $x+y+z+2$. If $y+z$ or $y+z+2$ are contained in the clique, the elements $x+y+z$ and $x+y+z+2$ must be excluded. Hence the largest clique we can have is $\{0, 2, y+2, x+y+z, x+y+z+2\}$. In either case, any clique containing either of y or $y+2$ can contain at most 5 elements. □

Proposition 7.5. *Any clique that contains $x+y$ or $x+y+2$ has at most 5 elements.*

Proof. Notice that both $x+y$ and $x+y+2$ can both be contained in the same clique since $(x+y)(x+y+2) = 0$. We will start by assuming that 0, 2, $x+y$ and $x+y+2$ are contained in a clique. Possible candidates for other elements contained in the clique are $x+z$ or $x+z+2$ or $y+z$ or $y+z+2$. Since none of these four candidate elements annihilate the others, we can have at most one of them in the clique. Hence the possible maximal cliques containing 0, 2, $x+y$ and $x+y+2$ are $\{0, 2, x+y, x+y+2, x+z\}$, $\{0, 2, x+y, x+y+2, x+z+2\}$, $\{0, 2, x+y, x+y+2, y+z\}$, and $\{0, 2, x+y, x+y+2, y+z+2\}$. In any case, any clique containing $x+y$ and $x+y+2$ has at most 5 elements. □

Proposition 7.6. *Any clique that contains z or $z+2$ has at most 5 elements.*

Proof. Notice that both z and $z+2$ can both be contained in the same clique since $z(z+2) = 0$. Recall that 0 and 2 are contained in any maximal clique. Assume that 0, 2, z, and $z+2$ are all contained in the clique. Also, assume that x and $x+2$ are not contained in the clique since any clique containing these elements has at most 5 elements. The only other possible elements contained in the clique are either $x+z$ or $x+z+2$. Hence the possible maximal cliques are $\{0, 2, z, z+2, x+z\}$ and $\{0, 2, z, z+2, x+z+2\}$. Hence any clique containing z or $z+2$ has at most 5 elements. □

Proposition 7.7. *Any clique that contains $x + z$, $x + z + 2$, $y + z$, or $y + z + 2$ has at most 5 elements.*

Proof. Notice that 0 and 2 are contained in every maximal clique. We will assume that none of x, $x + 2$, y, $y + 2$, $x + y$, $x + y + 2$, z, and $z + 2$ are in the clique since any clique containing these elements must have at most 5 elements. Since only one of $x + z$, $x + z + 2$, $y + z$, or $y + z + 2$ can be contained in any clique, we have four cases. If either of $x + z$ or $x + z + 2$ are in the clique, then $x + y + z$ and $x + y + z + 2$ are the only other elements that can also be in the clique and therefore $\{0, 2, x + z, x + y + z, x + y + z + 2\}$ and $\{0, 2, x + z + 2, x + y + z, x + y + z + 2\}$ are maximal clique. If either of $y + z$ or $y + z + 2$ are in the clique, we have one pair of x and y, $x + 2$ and $y + 2$, or $x + y$ and $x + y + 2$. As a result, the possible maximal cliques are $\{0, 2, y + z, x, y\}$, $\{0, 2, y+z, x+2, y+2\}$, $\{0, 2, y+z, x+y, x+y+2\}$, $\{0, 2, y+z+2, x, y\}$, $\{0, 2, y+z+2, x+2, y+2\}$, $\{0, 2, y+z+2, x+y, x+y+2\}$. In any case, we have at most five elements in each maximal clique. □

Proposition 7.8. *Any clique that contains $x + y + z$ or $x + y + z + 2$ has at most 5 elements.*

Proof. Recall that every maximal clique contains 0 and 2. Notice that $x + y + z$ and $x + y + z + 2$ can both be in the same clique since they annihilate each other. The only other elements that can be in a clique with 0, 2, $x + y + z$ and $x + y + z + 2$ are one of y, $y + 2$, $x + z$, or $x + z + 2$. Therefore the possible maximal cliques are $\{0, 2, x+y+z, x+y+z+2, y\}$, $\{0, 2, x+y+z, x+y+z+2, y+2\}$, $\{0, 2, x + y + z, x + y + z + 2, x + z\}$, and $\{0, 2, x + y + z, x + y + z + 2, x + z + 2\}$. In any case, we have at most five elements in each maximal clique. □

Therefore we can conclude that by the previous six propositions, $cl(\mathfrak{M}) = 5$.

Next, we can verify that $\chi(\Gamma_0(R)) = 6$. Since $\{0, 2, x, y, y + z\}$ is a clique, we will require at least 5 colors for coloring the zero-divisor graph of R, call them 1, 2, 3, 4, and 5. Notice that 0 and 2 are adjacent to all elements in $\mathfrak{M}$, so they require their own colors. Therefore we assign color 1 to 0 and color 2 to 2. Consider the subgraph $\{0, 2, x, y, y + z, z, z + 2, x + y, x + y + 2, x + z\}$ of the zero-divisor graph of R. We will show that this subgraph cannot be colored with 5 colors. The coloring for the subgraph $\{x, y, y + z, z, z + 2, x + y, x + y + 2, x + z\}$ is demonstrated in Figure 7.2.

Using the multiplication table of elements in the maximal ideal of R, we can see that $xz = 0$, $x(z + 2) = 0$ and $z(z + 2) = 0$. Therefore x, z, and $z + 2$ are all adjacent to each other and each vertex requires its own colors. Hence we assign color 3 to x, color 4 to z and color 5 to $z + 2$.

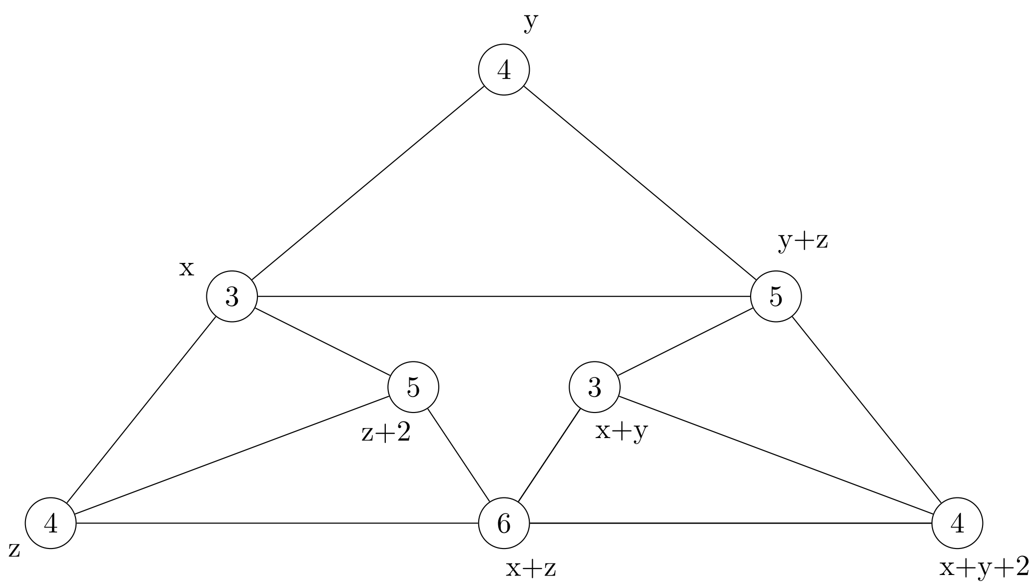

Figure 7.2: Coloring of subgraph $\{x, y, y+z, z, z+2, x+y, x+y+2, x+z\}$ as in [6].

Notice that $(x+y)(x+y+2) = 0$, which means that $x+y$ and $x+y+2$ must be assigned different colors. Also, we have that $(y+z)(x+y) = 0 = (y+z)(x+y+2)$. Therefore we can color $x+y$ with color 3 and $x+y+2$ with color 4.

Next, we will consider the vertex $x+z$ and notice that $(x+z)(x+y) = 0$, $(x+z)z = 0$ and $(x+z)(z+2) = 0$. Since $x+y$, z, and $z+2$ were assigned colors 3, 4, and 5 respectively, and $z+2$ is also adjacent to 0 and 2, which were colored with colors 1 and 2, we need a new color for $z+2$. Therefore $\chi(\Gamma_0(R)) \geq 6$. Since the partition $\{0\}, \{2\} \cup \mathfrak{U}(R), \{x, x+2, x+y, x+y+z\}, \{y, y+2, z, x+y+2\}, \{y+z, y+z+2, z+2, x+y+z+2\}, \{x+z, x+z+2\}$ of R is a coloring using 6 colors, we have that $\chi(\Gamma_0(R)) = 6$. Hence Beck's conjecture is false in general.

REFERENCES

[1] Istvan Beck, *Coloring of commutative rings*, Journal of Algebra **116** (1988), 208–226.

[2] Richard M. Foote David S. Dummit, *Abstract algebra*, third ed., John Wiley and Sons, Inc., 2004.

[3] Reinhard Diestel, *Graph theory*, Springer, 2010.

[4] Thomas W. Hungerford, *Algebra*, Springer, 1974.

[5] Irving Kaplansky, *Commutative rings*, The University of Chicago, 1970.

[6] D. D. Anderson, M. Naseer, *Beck's coloring of a commutative ring*, Journal of Algebra **159** (1993), 500–514.

[7] N. G. de Bruijn, P. Erd∅s, *A colour problem for infinite graphs and a problem in theory of relations*, Koninki. Nederl. Akademie Van Wetenschappen **54** (1953), 371–373.

[8] David F. Anderson, Andrea Frazier, Aaron Lauve, Philip S. Livingston, *The zero-divisor graph of a commutative ring, II*, Ideal Theoretic Methods in Commutative Algebra (Lecture Notes in Pure and Applied Mathematics) **220** (2001), 61–72.