NAK FOR EXT, ASCENT OF MODULE STRUCTURES, AND THE BLINDNESS OF EXTENDED MODULES

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BENJAMIN J. ANDERSON

The Supervisory Committee certifies that this ***disquisition*** complies with North Dakota State University's regulations and meets the accepted standards for the degree of

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SUPERVISORY COMMITTEE:

Dr. Sean Sather-Wagstaff
Chair

Dr. Jim Coykendall

Dr. Benton Duncan

Dr. Eugene Berry

Approved:

7/2/2012
Date

Dr. Dogan Comes
Department Chair

ABSTRACT

This dissertation investigates the interplay between properties of Ext modules and ascent of module structures along ring homomorphisms. First, we consider a flat local ring homomorphism $\varphi\colon (R, \mathfrak{m}, k) \to (S, \mathfrak{m}S, k)$. We show that if M is a finitely generated R-module such that $\operatorname{Ext}^i_R(S, M)$ satisfies NAK (e.g. if $\operatorname{Ext}^i_R(S, M)$ is finitely generated over S) for $i = 1, \ldots, \dim_R(M)$, then $\operatorname{Ext}^i_R(S, M) = 0$ for all $i \neq 0$ and M has an S-module structure via φ. We also provide explicit computations of $\operatorname{Ext}^1_R(S, M)$ to indicate how large it can be when M does not have a compatible S-module structure.

Next, we consider the properties of an R-module M that has a compatible S-module structure via the flat local ring homomorphism φ. Our results in this direction show that M cannot see the difference between the rings R and S. Specifically, many homological invariants of M are the same when computed over R and over S.

Finally, we investigate these ideas in the non-local setting. We consider a faithfully flat ring homomorphism $\varphi\colon R \to S$ such that for all $\mathfrak{m} \in \operatorname{m-Spec} R$, the map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism and the induced map $\varphi^*\colon \operatorname{Spec}(S) \to \operatorname{Spec}(R)$ is such that $\varphi^*(\operatorname{m-Spec}(S)) \subseteq \operatorname{m-Spec}(R)$, and show that if M is a finitely generated R-module such that $\operatorname{Ext}^i_R(S, M)$ satisfies NAK for $i = 1, \ldots, \dim_R(M)$, then M has an S-module structure via φ, and obtain the same Ext vanishing as in the local case.

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CHAPTER 1. INTRODUCTION

This dissertation is in the area of commutative algebra, that is, the study of commu- tative rings. Rings are sets that are endowed with addition and multiplication. Rings are used in a multitude of settings, from doing arithmetic on a clock, to solving systems of equations. Understanding rings allows us to answer a wide range of questions found in the areas of geometry and topology. For example, rings can help to decide whether two geometrical objects are the same or different.

In studying rings, we look at sets they act on. This is like studying groups by looking at sets that they act on, i.e., group actions. Rings act on sets called modules, which have the same axioms as vector spaces with the coefficients coming from a commutative ring. Information about a module's structure tells us information about the ring.

One idea in this vein is the following: Given a ring homomorphism $\varphi\colon R \to S$, describe how R-modules are related to S-modules. For instance, every S-module N has an R-module structure via restriction of scalars: $rn := \varphi(r)n$. On the other hand, an R-module M may or may not have an S-module structure, e.g., $\mathbb{Z}$ as a $\mathbb{Z}$-module is not a $\mathbb{Q}$-module. One can create a new R-module related to the original one that does have an S-module structure. Specifically, the R-modules $S \otimes_R M$ and $\mathrm{Hom}_R(S, M)$ are S-modules. However, this is rather unsatisfactory, as these operations yield modules different from M in general. The goal of this dissertation is, in a sense, to remedy this situation.

1.1. Conventions and Notations

Most of our definitions and notational conventions come from [5, 7, 15]. For the sake of clarity, we specify a few items here.

Throughout Chapters 2 and 3, $(R, \mathfrak{m})$ and $(S, \mathfrak{n})$ are commutative noetherian local rings and $\varphi\colon R \to S$ is a flat local homomorphism (i.e. such that S is flat over

R and $\varphi(\mathfrak{m}) \subseteq \mathfrak{n}$) with the property that $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism. Given an R-module M, the $\mathfrak{m}$-adic completion of M is denoted $\widehat{M}$.

Definition 1.1. Let R be a commutative ring. A sequence of R-module homomorphisms

$$M = \cdots \to M_{i+1} \xrightarrow{\partial^M_{i+1}} M_i \xrightarrow{\partial^M_i} M_{i-1} \xrightarrow{\partial^M_{i-1}} \cdots$$

is called an *R-complex* if $\partial^M_{i-1}\partial^M_i = 0$ for all i. We say M_i is the module in *degree i* in the R-complex M.

Definition 1.2. Let R be a commutative ring and let M be an R-complex. The ith *homology module* of an R-complex is defined as

$$\mathrm{H}_i(M) = \frac{\operatorname{Ker} \partial^M_i}{\operatorname{Im} \partial^M_{i+1}}.$$

We say that an R-complex is *exact* if $\mathrm{H}_i(M) = 0$ for all i.

Definition 1.3. Let R be a commutative ring and let X be an R-complex. For each $n \in \mathbb{Z}$ the *nth suspension* or *shift* of X is the R-complex $\Sigma^n X$ defined as $\Sigma^n X_i = X_{i-n}$ and $\partial^{\Sigma^n X}_i = (-1)^n \partial^X_{i-n}$. When $n = 1$ set $\Sigma X = \Sigma^1 X$.

Remark 1.4. Let R be a commutative ring and let X be an R-complex. There is an isomorphism of homologies $\mathrm{H}_i(\Sigma^n X) \cong \mathrm{H}_{i-n}(X)$ for all $i, n \in \mathbb{Z}$.

Definition 1.5. Let Y and Z be R-complexes. The *Hom-complex* $\operatorname{Hom}_R(Y, Z)$ is defined as follows: $\operatorname{Hom}_R(Y, Z)_i = \prod_{p\in\mathbb{Z}} \operatorname{Hom}_R(Y_p, Z_{p+i})$ for each $i \in \mathbb{Z}$. Each element $\alpha \in \operatorname{Hom}_R(Y, Z)_i$ is called a *homomorphism $\alpha\colon Y \to Z$ of degree i*. This homomorphism is a family $\alpha = (\alpha_p)_{p\in\mathbb{Z}} \in \prod_{p\in\mathbb{Z}} \operatorname{Hom}_R(Y_p, Z_{p+i})$ of linear maps $\alpha_p\colon Y_p \to Z_{p+i}$, with no requirement of commutativity.

Next, to define the differential $\partial^{\operatorname{Hom}_R(Y,Z)}_i\colon \operatorname{Hom}_R(Y, Z)_i \to \operatorname{Hom}_R(Y, Z)_{i-1}$ we take a family of linear maps $\alpha = (\alpha_p)_{p\in\mathbb{Z}} \in \operatorname{Hom}_R(Y, Z)_i$ and let the homomorphism

$\partial_i^{\mathrm{Hom}_R(Y,Z)}(\alpha) \in \mathrm{Hom}_R(Y,Z)_{i-1}$ of degree $i-1$ have p-th component

$$\partial_i^{\mathrm{Hom}_R(Y,Z)}(\alpha)_p = \partial_{p+i}^Y \alpha_p - (-1)^i \alpha_{p-1} \partial_p^X \colon Y_p \to Z_{p+i-1}.$$

Definition 1.6. Let X and Y be R-complexes. We define the *tensor-complex* $X \otimes_R Y$ as follows: For $l \in \mathbb{Z}$ the l-th module is given by $(X \otimes_R Y)_l = \coprod_{p \in \mathbb{Z}} X_p \otimes_R Y_{l-p}$. The l-th module $(X \otimes_R Y)_l$ is generated by elements $x_p \otimes y_{l-p}$ for $p \in \mathbb{Z}, x_p \in X_p$ and $y_{p-l} \in Y_{p-l}$. The differential $\partial_l^{X \otimes_R Y} \colon (X \otimes_R Y)_l \to (X \otimes_R Y)_{l-1}$ is given on a generator $x_p \otimes y_{l-p} \in X_p \otimes_R Y_{l-p} \subseteq (X \otimes_R Y)_l$ by

$$\partial_l^{X \otimes_R Y}(x_p \otimes y_{l-p}) = \partial_p^X(x_p) \otimes y_{l-p} + (-1)^p x_p \otimes \partial_{l-p}^Y(y_{l-p}),$$

which is an element in $(X_{p-1} \otimes_R Y_{l-p}) \oplus (X_p \otimes_R Y_{l-p-1}) \subseteq (X \otimes_R Y)_{l-1}$.

Definition 1.7. Let R be a commutative ring, and let X and Y be R-complexes. A *chain map* $F \colon X \to Y$ is a sequence $\{F_i \colon X_i \to Y_i\}_{i \in \mathbb{Z}}$ making the next diagram commute.

$$\begin{tikzcd}
X \arrow[d, "F"'] & \cdots \arrow[r, "\partial_{i+1}^X"] & X_i \arrow[r, "\partial_i^X"] \arrow[d, "F_i"'] & X_{i-1} \arrow[r, "\partial_{i-1}^X"] \arrow[d, "F_{i-1}"'] & \cdots \\
Y & \cdots \arrow[r, "\partial_{i+1}^Y"] & Y_i \arrow[r, "\partial_i^Y"] & Y_{i-1} \arrow[r, "\partial_{i-1}^Y"] & \cdots
\end{tikzcd}$$

We say that the chain map F is a *quasi-isomorphism* if for each index i the induced map $\mathrm{H}_i(F) \colon \mathrm{H}_i(X) \to \mathrm{H}_i(Y)$ given by $\mathrm{H}_i(F)(\overline{x}) = \overline{F_i(x)}$ is an isomorphism.

Definition 1.8. Let R be a commutative ring and let $f \colon X \to Y$ be a chain map.

The *mapping cone* of f is the R-complex $\operatorname{Cone}(f)$ defined as follows:

$$\operatorname{Cone}(f) = \cdots \to \begin{matrix} Y_i \\ \oplus \\ X_{i-1} \end{matrix} \xrightarrow{\begin{pmatrix} \partial_i^Y & f_{i-1} \\ 0 & -\partial_{i-1}^X \end{pmatrix}} \begin{matrix} Y_{i-1} \\ \oplus \\ X_{i-2} \end{matrix} \xrightarrow{\begin{pmatrix} \partial_{i-1}^Y & f_{i-2} \\ 0 & -\partial_{i-2}^X \end{pmatrix}} \begin{matrix} Y_{i-2} \\ \oplus \\ X_{i-3} \end{matrix} \to \cdots .$$

That is, the module in degree i is $\operatorname{Cone}(f)_i = Y_i \oplus X_{i-1}$ and the differential in degree i is $\partial_i^{\operatorname{Cone}(f)} = \begin{pmatrix} \partial_i^Y & f_{i-1} \\ 0 & -\partial_{i-1}^X \end{pmatrix}$.

Fact 1.9. For each chain map of R-complexes $f \colon Y \to Z$, the mapping cone $\operatorname{Cone}(f)$ gives rise to a short exact sequence $0 \to Z \to \operatorname{Cone}(f) \to \Sigma Y \to 0$. Hence we have the following associated long exact sequence on homology.

$$\cdots \to \mathrm{H}_i(Y) \xrightarrow{\mathrm{H}_i(f)} \mathrm{H}_i(Z) \to \mathrm{H}_i(\operatorname{Cone}(f)) \to \mathrm{H}_{i-1}(Y) \to \cdots .$$

Using the long exact sequence, it is straightforward to show that f is a quasi-isomorphism if and only if $\operatorname{Cone}(f)$ is exact.

We now define a very useful tool, the Koszul complex. This tool is used to detect the depth of a module, to detect quasi-isomorphisms, and to describe resolutions for quotients by regular sequences.

Definition 1.10. Let R be a commutative ring, and let $\mathbf{x} = x_1, \ldots, x_n \in R$. Set $L_0 = R$ with basis 1. Set $L_1 = R^n$ with basis $e_1, \ldots, e_n$. For $i = 2, \ldots, n$, let L_i denote the free R-module whose basis is the following set of formal symbols:

$$\{e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_i} \mid 1 \leqslant j_i < \cdots < j_i \leqslant n\}.$$

Let L be the sequence

$$L = 0 \to L_n \xrightarrow{\partial_n^L} L_{n-1} \xrightarrow{\partial_{n-1}^L} \cdots \xrightarrow{\partial_1^L} L_0 \to 0$$

with maps defined on the bases as follows. For $i = 1$, let $\partial_1^L \colon R^n \to R$ be given by $e_j \mapsto x_j$. For $i > 1$, let $\partial_i^L \colon L_i \to L_{i-1}$ be given by

$$e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_i} \mapsto \sum_{l=1}^{j} (-1)^{l+1} x_l e_{j_1} \wedge e_{j_2} \wedge \cdots \wedge e_{j_{l-1}} \wedge e_{j_{l+1}} \wedge \cdots \wedge e_{j_i}.$$

We call the complex L the *Koszul complex* on $\mathbf{x}$.

Another useful way to define and use the Kozul complex is in terms of a specific mapping cone as follows. This allows us to use either definition depending on which is more useful.

Definition 1.11. Let R be a commutative ring, and let M be an R-module. Let $\mathbf{x} = x_1, \ldots, x_n \in R$. The *Koszul complex* $K^R(\mathbf{x}; M)$ is constructed inductively on n as follows:

Base case: $n = 1$. Consider the module M as an R-complex concentrated in degree 0. For each $r \in R$, the map $\mu^r \colon M \to M$ given by $\mu^r(m) = rm$ is an R-module homomorphism and a chain map. Such a map is called a *homothety*. Taking the cone of the chain map μ^{x_1} is the Koszul complex on x_1. That is

$$K^R(x_1; M) := \mathrm{Cone}(\mu_1^x) = \qquad 0 \longrightarrow M \xrightarrow{\mu^{x_1}} M \longrightarrow 0$$

Inductive step: Assume that $n \geqslant 2$ and that $K^R(x_1, \ldots, x_{n-1}; M)$ has been constructed. Let $\mu^{x_n} \colon K^R(x_1, \ldots, x_{n-1}; M) \to K^R(x_1, \ldots, x_{n-1}; M)$ be the homothety given by $\mu^{x_n}(k) = x_n k$, and set

$$K^R(\mathbf{x}; M) = K^R(x_i \ldots, x_{n-1}, x_n; M) = \mathrm{Cone}(\mu^{x_n}).$$

When $M = R$, we write $K^R(\mathbf{x}) = K^R(\mathbf{x}; M)$.

Example 1.12. Let R be a commutative ring, and let M be an R-module. Let $x, y, z \in R$. It is straightforward to show the following:

$$K^R(x; M) \cong \qquad 0 \longrightarrow M \xrightarrow{x} M \longrightarrow 0$$

and

$$K^R(x, y; M) \cong \qquad 0 \longrightarrow M \xrightarrow{\begin{pmatrix} y \\ -x \end{pmatrix}} M^2 \xrightarrow{\begin{pmatrix} x & y \end{pmatrix}} M \longrightarrow 0$$

and $K^R(x, y, z; M) \cong$

$$0 \to M \xrightarrow{\begin{pmatrix} z \\ -y \\ x \end{pmatrix}} M^3 \xrightarrow{\begin{pmatrix} y & z & 0 \\ -x & 0 & z \\ 0 & -x & -y \end{pmatrix}} M^3 \xrightarrow{\begin{pmatrix} x & y & z \end{pmatrix}} M \to 0$$

by using the definition of the Koszul complex.

The remainder of this chapter is devoted to giving a general overview of the subsequent chapters.

1.2. Summary of Chapter 2

The origins of Chapter 2 begin with the following result of Buchweitz and Flenner [4, Theorem 2.3].

Theorem 1.13. *Assume that $(R, \mathfrak{m})$ is local, and let M be an R-module. If M is $\mathfrak{m}$-adically complete, then for each flat R-module F one has $\operatorname{Ext}^i_R(F, M) = 0$ for all $i \geqslant 1$.*

A natural question to ask is when does the converse hold? The converse fails with no extra assumptions on M, as seen in the following example.

Example 1.14. Let M be a non-zero injective R-module, and assume that $(R, \mathfrak{m})$ is a local ring of positive depth (e.g., R is a local domain and not a field). Then $\operatorname{Ext}^i_R(-, M) = 0$ for all $i \geqslant 1$. However, the fact that M is injective implies that it is divisible. Thus, we have $M = \mathfrak{m}M$, and it follows that $\widehat{M} = 0 \neq M$, so M is not complete.

What restrictions if any on M must we have to ensure the converse holds? The next result of Frankild and Sather-Wagstaff [9, Corollary 3.5] shows that the converse to Theorem 1.13 does hold when M is finitely generated.

Theorem 1.15. *Assume that $(R, \mathfrak{m})$ is local, and let M be a finitely generated R-module. Then the following conditions are equivalent.*

(i) *M is $\mathfrak{m}$-adically complete.*

(ii) *$\operatorname{Ext}^i_R(F, M) = 0$ for all $i \geqslant 1$ for each flat R-module F.*

(iii) *$\operatorname{Ext}^i_R(\widehat{R}, M) = 0$ for all $i \geqslant 1$.*

It is worth noting that the proof of this result is quite technical, relying heavily on the machinery of derived local homology and derived local cohomology.

Since the module M in Theorem 1.15 is finitely generated, a standard result shows that condition (i) is equivalent to the following: M has an $\widehat{R}$-module structure that is compatible with its R-module structure via the natural map $R \to \widehat{R}$. Thus, we consider the following ascent question, focusing on homological conditions.

Question 1.16. Given a ring homomorphism $\varphi\colon R \to S$ what conditions on an R-module M guarantees that it has an S-module structure compatible with its R-module structure via φ?

In the setting of Question 1.16, the module $S \otimes_R M$ has a natural S-module structure. Thus, if one had an R-module isomorphism $M \cong S \otimes_R M$, then one could transfer the S-module structure from $S \otimes_R M$ to M. One can similarly inflict an S-module structure on M if $M \cong \operatorname{Hom}_R(S, M)$. The following result of Frankild, Sather-Wagstaff, and R. Wiegand [10, Main Theorem 2.5] and Christensen and Sather-Wagstaff [6, Theorem 3.1 and Remark 3.2] shows that, when φ is a local homomorphism with properties like those of the natural map $R \to \widehat{R}$, these are in fact the only way for M to admit a compatible S-module structure.

Theorem 1.17. *Let $\varphi\colon (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local ring homomorphism such that the induced map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism. Let M be a finitely generated R-module, and consider the following conditions.*

(i) *M has an S-module structure compatible with its R-module structure via φ.*

(ii) $\operatorname{Ext}^i_R(S, M) = 0$ *for all* $i \geqslant 1$.

(iii) $\operatorname{Ext}^i_R(S, M)$ *is finitely generated over R for* $i = 1, \ldots, \dim(M)$.

(iv) *The natural map* $\operatorname{Hom}_R(S, M) \to M$ *given by* $f \mapsto f(1)$ *is an isomorphism.*

(v) *The natural map* $M \to S \otimes_R M$ *given by* $m \mapsto 1 \otimes m$ *is an isomorphism.*

(vi) $S \otimes_R M$ *is finitely generated over R.*

(vii) $\operatorname{Ext}^i_R(S, M)$ *is finitely generated over S for* $i = 1, \ldots, \dim_R(M)$.

Then conditions (i)–(vi) *are equivalent and imply condition* (vii). *If R is Gorenstein, then conditions* (i)–(vii) *are equivalent.*

The proof of this result is less technical than that of Theorem 1.15. But it does use the Amplitude Inequality of Foxby, Iyengar, and Iversen [8, 13]—a consequence of the New Intersection Theorem— and derived Gorenstein injective dimension.

The next result is the main theorem of Chapter 2. It contains several improvements on Theorem 1.17. First, it removes the Gorenstein hypothesis for the implication (vii) $\implies$ (i). Second, it further relaxes the conditions on the Ext-modules needed to obtain an S-module structure on M. Third, the proof is significantly less technical than the proofs of these earlier results, relying only on basic properties of Koszul complexes.

Definition 1.18. Assume that $(R, \mathfrak{m})$ is local. An R-module N *satisfies NAK* if $N = 0$ or $N/\mathfrak{m}N \neq 0$.

Remark 1.19. Let $\varphi\colon (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a local ring homomorphism. Given an S-module N, the condition "N satisfies NAK" is ambiguous since N is both and S-module and an R-module. For instance, if M is an R-module this is the case for $\operatorname{Ext}^i_R(S, M)$. When there is any danger of ambiguity, we write "satisfies NAK over S" or "satisfies NAK over R". It is worth noting that if N satisfies NAK over S, then it satisfies NAK over R because of the epimorphism $N/\mathfrak{m}N = N/\mathfrak{m}SN \twoheadrightarrow N/\mathfrak{n}N$. Furthermore, this reasoning shows that the converse holds if $\mathfrak{m}S = \mathfrak{n}$, e.g., if the induced map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism.

Theorem 1.20. *In Theorem* 1.17 *the conditions* (i)–(vii) *are equivalent, and they are equivalent to the following:*

(viii) $\operatorname{Ext}^i_R(S, M)$ *satisfies NAK for* $i = 1, \ldots, \dim_R(M)$.

We conclude this section by outlining the contents of Chapter 2. Section 2.1 summarizes foundational material needed for the proof of Theorem 1.20. Section 2.2 is devoted to the proof of this result. Finally, Section 2.3 consists of an example demonstrating how large $\mathrm{Ext}_R^i(S, M)$ is when it does not satisfy NAK, even over a relatively small ring. The results of Chapter 2 have been accepted for publication in the Proceedings of the American Mathematical Society

1.3. Summary of Chapter 3

The genesis of Chapter 3 is in the following observation. Given an R-module M with a compatible S-module structure as in Theorem 1.17, we have the following equivalences:

(i) $\mathrm{Hom}_R(S, M) \cong M \cong \mathrm{Hom}_R(R, M)$.

(ii) $S \otimes_R M \cong M \cong R \otimes_R M$.

(iii) $\mathrm{Ext}_R^i(S, M) = 0 = \mathrm{Ext}_R^i(R, M)$ for all $i \geqslant 1$.

(iv) $\mathrm{Tor}_i^R(S, M) = 0 = \mathrm{Tor}_i^R(R, M)$ for all $i \geqslant 1$.

Thus we begin to see that the R-module M cannot distinguish between the rings R and S. This gives rise to the following question.

Question 1.21. Given $\varphi\colon (R, \mathfrak{m}) \to (S, \mathfrak{n})$ flat local such that $R/\mathfrak{m} \cong S/\mathfrak{m}S$ and an R-module M that has a compatible S-module structure via φ, what properties of M over R are equivalent to properties of M over S?

Chapter 3 answers this question by establishing a collection of equalities between invariants computed over R and S. We also provide equivalences between certain Ext modules and equivalences between certain Auslander and Bass classes.

We conclude this section with a synopsis of Chapter 3. Section 3.1 is used to define the homological invariants used in Section 3.2 for the ease of the reader. Section 3.2 consists of the main results for this chapter.

1.4. Summary of Chapter 4

Chapter 4 is devoted to answering the following question. Do we need the local hypothesis on φ in Theorem 2.5 to obtain a unique compatible S-module structure via φ? An immediate problem arises when removing this hypothesis, in the fact that there are many ways to generalize the property $R/\mathfrak{m} \cong S/\mathfrak{m}S$. Section 4.1 discusses the various ways one can generalize this property to the non-local setting. We focus on the case where $\varphi\colon R \to S$ is faithfully flat such that $R/\mathfrak{m} \cong S/\mathfrak{m}S$ and for $\varphi^*\colon \operatorname{Spec} S \to \operatorname{Spec} R$ we have $\varphi^*(\text{m-Spec}\, S) \subseteq \text{m-Spec}\, R$. Using this property, we prove versions of Theorem 1.17 and Theorem 1.20 in this non-local setting in Theorem 4.22. Section 4.2 builds the various tools and background required for the proof of Theorem 4.22. Finally, Section 4.3 is devoted to the proof of this result.

CHAPTER 2. NAK FOR EXT AND ASCENT OF MODULE STRUCTURES

Let $\varphi\colon (R,\mathfrak{m}) \to (S,\mathfrak{n})$ be a flat local ring homomorphism such that the induced map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism. This chapter is devoted to the proof of Theorem 1.20; see Theorem 2.5. Note that the assumption $\mathfrak{n} = \mathfrak{m}S$ implies that an S-module N satisfies NAK over S if and only if N satisfies NAK over R; see Remark 1.19.

2.1. NAK and Ext Vanishing

This section contains the foundational material needed for the proof of the main results of this chapter.

Lemma 2.1. *Let i_0 be a fixed integer, let $\boldsymbol{x} = x_1,\ldots,x_n$ be a sequence in a commutative ring T, and let Y be an T-complex such that $\mathrm{H}_i(Y) = 0$ for all $i < i_0$. Then $\mathrm{H}_i(K^T(\boldsymbol{x}) \otimes_T Y) = 0$ for all $i < i_0$ and $\mathrm{H}_{i_0}(K^T(\boldsymbol{x}) \otimes_T Y) \cong \mathrm{H}_{i_0}(Y)/(\boldsymbol{x})\,\mathrm{H}_{i_0}(Y)$.*

Proof. We proceed by induction on n. For the base case let $\mu_Y^{x_1}: Y \to Y$ be the map given by $\mu_Y^{x_1}(y) = x_1 y$ and note that we have $K^T(x_1) \otimes_T Y \cong \mathrm{Cone}(\mu_Y^{x_1})$. Now since $\mu_Y^{x_1}$ is a chain map we have an exact sequence of complexes

$$0 \to Y \to \mathrm{Cone}(\mu_Y^{x_1}) \to \Sigma Y \to 0$$

Hence we can apply the long exact sequence to obtain the following exact sequence

$$\cdots \to H_i(Y) \to H_i(\mathrm{Cone}(\mu_Y^{x_1})) \to H_{i-1}(Y) \xrightarrow{x_1} H_{i-1}(Y) \to \cdots$$

If $i < i_0$ then by our assumption we have $H_i(Y) = 0 = H_{i-1}(Y)$. Therefore, from the exactness of the above sequence we have $H_i(\mathrm{Cone}(\mu_Y^{x_1})) = 0$ for all $i < i_0$. Now if

$i = i_0$ then we have the following exact sequence of T-modules

$$\cdots \to H_{i_0}(Y) \xrightarrow{x_1} H_{i_0}(Y) \to H_{i_0}(\mathrm{Cone}(\mu_Y^{x_1})) \to 0$$

Since the sequence is exact we have $H_{i_0}(\mathrm{Cone}(\mu_Y^{x_1})) \cong H_{i_0}(Y)/x_1 H_{i_0}(Y)$ as desired.

For the induction step we set $Y' = K^T(x_1, \ldots, x_{n-1}) \otimes_T Y$ and suppose that $H_i(Y') = 0$ for all $i < i_0$ and $H_{i_0}(Y') \cong H_{i_0}(Y)/(x_1, \ldots, x_{n-1})H_{i_0}(Y)$. Note that we have $K^T(\mathbf{x}) \otimes_T Y \cong \mathrm{Cone}(\mu_{Y'}^{x_n})$, so we can apply our base case to $\mathrm{Cone}(\mu_{Y'}^{x_n})$ to obtain $H_i(K^T(\mathbf{x}) \otimes_T Y) = 0$ for all $i < i_0$ and $H_{i_0}(K^T(\mathbf{x}) \otimes_T Y) \cong H_{i_0}(Y)/(\mathbf{x})H_{i_0}(Y)$. □

Lemma 2.2. *Let $\varphi\colon R \to S$ be a flat local ring homomorphism such that the induced map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism. Let $\mathbf{x} = x_1, \ldots, x_n \in R$ be a generating sequence for $\mathfrak{m}$.*

(a) *The natural map $\epsilon\colon K^R(\boldsymbol{x}) \to S \otimes_R K^R(\boldsymbol{x})$ is a quasi-isomorphism.*

(b) *The induced map $\bar{\epsilon}\colon \mathrm{Hom}_R(S \otimes_R K^R(\boldsymbol{x}), J) \to \mathrm{Hom}_R(K^R(\boldsymbol{x}), J)$ is a quasi-isomorphism, for each bounded above complex J of injective R-modules.*

Proof. Part (a) is from [10, 2.3]. For part (b) Suppose that ϵ is a quasi-isomorphism. That is, we have $\mathrm{H}_i(\mathrm{Cone}(\epsilon) = 0$ for all i. Note that we have $\mathrm{Cone}(\mathrm{Hom}_R(\epsilon, J)) \cong \boldsymbol{\Sigma}^1 \mathrm{Hom}_R(\mathrm{Cone}(\epsilon), J)$ by [7, 3.48]. This gives the first step in the following display.

$$\begin{aligned}
\mathrm{H}_i(\mathrm{Cone}(\mathrm{Hom}_R(\epsilon, J))) &\cong \mathrm{H}_i(\boldsymbol{\Sigma}^1 \mathrm{Hom}_R(\mathrm{Cone}(\epsilon), J)) \\
&= \mathrm{H}_{i-1}(\mathrm{Hom}_R(\mathrm{Cone}(\epsilon), J)) \\
&= 0.
\end{aligned}$$

The second step is from Remark 1.4. Since J is a bounded above complex of injective modules we have by [7, 6.5] $\mathrm{Hom}_R(-, J)$ preserves homological triviality giving the third step. Therefore, since the mapping cone $\operatorname{Cone}(\operatorname{Hom}_R(\epsilon, J))$ is exact, the chain map $\operatorname{Hom}_R(\epsilon, J)$ is a quasi-isomorphism. $\square$

The next result is from [10, proof of Theorem 2.5].

Lemma 2.3. *Let $\varphi\colon R \to S$ be a flat ring homomorphism, and let M be an R-module. Then $\operatorname{Ext}^i_R(S, M) = 0$ for all $i > \dim(R/\operatorname{Ann}_R(M))$. In particular, if M is finitely generated, then $\operatorname{Ext}^i_R(S, M) = 0$ for all $i > \dim_R(M)$.*

2.2. Main Results

This section contains the lemmata used for the proof of Theorem 1.20 along with the proof of this result.

Lemma 2.4. *Let $\varphi\colon R \to S$ be a flat local ring homomorphism such that the induced map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism. Let M be an R-module, and let $z \geqslant 1$. If $\operatorname{Ext}^i_R(S, M) = 0$ for all $i > z$ and $\operatorname{Ext}^z_R(S, M)$ satisfies NAK then $\operatorname{Ext}^z_R(S, M) = 0$.*

Proof. Let $\mathbf{x} = x_1, \ldots, x_n \in R$ be a generating sequence for $\mathfrak{m}$, and let J be an R-injective resolution of M. By assumption we have $\mathrm{H}_{-i}(\operatorname{Hom}_R(S, J)) \cong \operatorname{Ext}^i_R(S, M) = 0$ for all $i > z$, so we have $\mathrm{H}_{-i}(K^R(\mathbf{x}) \otimes_R \operatorname{Hom}_R(S, J)) = 0$ for all $i > z$ and

$$\begin{aligned} \mathrm{H}_{-z}(K^R(\mathbf{x}) \otimes_R \operatorname{Hom}_R(S, J)) &\cong \mathrm{H}_{-z}(\operatorname{Hom}_R(S, J))/\mathbf{x}\,\mathrm{H}_{-z}(\operatorname{Hom}_R(S, J)) \\ &\cong \operatorname{Ext}^z_R(S, M)/\mathbf{x}\operatorname{Ext}^z_R(S, M) \end{aligned}$$

by Lemma 2.1.

We claim that $\operatorname{Ext}^z_R(S, M)/\mathbf{x}\operatorname{Ext}^z_R(S, M) = 0$. To see this, consider the morphism $\alpha\colon \operatorname{Hom}_R(S, J) \to J$ given by $f \mapsto f(1)$. Apply $\operatorname{Hom}_R(K^R(\mathbf{x}), -)$ to obtain the top row of the following diagram

$$\begin{tikzcd}
\mathrm{Hom}_R(K^R(\mathbf{x}), \mathrm{Hom}_R(S, J)) \arrow[r, "\mathrm{Hom}_R(K^R(\mathbf{x}),\alpha)"] \arrow[d, "\cong"'] & \mathrm{Hom}_R(K^R(\mathbf{x}), J) \\
\mathrm{Hom}_R(S \otimes_R K^R(\mathbf{x}), J) \arrow[ur, "\simeq"'] &
\end{tikzcd}$$

where the vertical isomorphism is Hom-tensor adjointness and the diagonal quasi-isomorphism is from Lemma 2.2(b). It follows that $\mathrm{Hom}_R(K^R(\mathbf{x}), \alpha)$ is a quasi-isomorphism and thus $\mathrm{Cone}(\mathrm{Hom}_R(K^R(\mathbf{x}), \alpha))$ is exact. The fact that $K^R(\mathbf{x})$ is a self-dual and bounded complex of finitely generated free R-modules implies that the R-complexes $\mathrm{Cone}(\mathrm{Hom_R}(\mathrm{K^R}(\mathbf{x}), \alpha))$ and $\mathbf{\Sigma}^{-n} K^R(\mathbf{x}) \otimes_R \mathrm{Cone}(\alpha)$ are isomorphic. It follows that the complex $K^R(\mathbf{x}) \otimes_R \mathrm{Cone}(\alpha)$ is exact.

The long exact sequence in homology associated to $\mathrm{Cone}(\alpha)$ implies that

$$\mathrm{H}_i(\mathrm{Cone}(\alpha)) \cong \mathrm{Ext}_R^{1-i}(S, M) \tag{1}$$

for all $i \leqslant -1$, and that there is an exact sequence

$$\mathrm{H}_0(\mathrm{Cone}(\alpha)) \to \mathrm{Ext}_R^1(S, M) \to 0. \tag{2}$$

We have $\mathrm{Ext}_R^j(S, M) = 0$ for all $j > z$, by assumption, so the isomorphism (1) implies that $\mathrm{H}_i(\mathrm{Cone}(\alpha)) = 0$ for all $i < 1 - z$.

We consider two cases.

Case 1: $z \geqslant 2$. In this case, the isomorphism (1) shows that $\mathrm{H}_{1-z}(\mathrm{Cone}(\alpha)) \cong \mathrm{Ext}_R^z(S, M)$. From Lemma 2.1, we conclude that

$$0 = \mathrm{H}_{1-z}(K^R(\mathbf{x}) \otimes_R \mathrm{Cone}(\alpha)) \cong \mathrm{Ext}_R^z(S, M)/\mathbf{x}\,\mathrm{Ext}_R^z(S, M).$$

Case 2: $z = 1$. In this case, the exact sequence (2) conspires with the right-exactness of tensor product to explain the epimorphism in the next display:

$$0 = \mathrm{H}_0(K^R(\mathbf{x}) \otimes_R \mathrm{Cone}(\alpha)) \cong \frac{\mathrm{H}_0(\mathrm{Cone}(\alpha))}{\mathbf{x}\,\mathrm{H}_0(\mathrm{Cone}(\alpha))} \twoheadrightarrow \frac{\mathrm{Ext}^1_R(S,M)}{\mathbf{x}\,\mathrm{Ext}^1_R(S,M)}.$$

The other steps are justified as in Case 1. This implies that

$$\mathrm{Ext}^z_R(S,M)/\mathbf{x}\,\mathrm{Ext}^z_R(S,M) = \mathrm{Ext}^1_R(S,M)/\mathbf{x}\,\mathrm{Ext}^1_R(S,M) = 0.$$

This proves the claim. (It is worth noting that there is an alternate proof of this claim in [1].)

As $\mathrm{Ext}^z_R(S,M)$ satisfies NAK, the claim implies $\mathrm{Ext}^z_R(S,M) = 0$ as desired. □

Here is Theorem 1.20 from the introduction.

Theorem 2.5. *Let $\varphi\colon R \to S$ be a flat local ring homomorphism such that the induced map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism, and let M be a finitely generated R-module. If $\mathrm{Ext}^i_R(S,M)$ satisfies NAK for $i = 1,\dots,\dim_R(M)$ then $\mathrm{Ext}^i_R(S,M) = 0$ for all $i \neq 0$ and M has an S-module structure compatible with its R-module structure via φ.*

Proof. Lemma 2.3 implies that $z = \sup\{i \geqslant 0 \mid \mathrm{Ext}^i_R(S,M) \neq 0\} \leqslant \dim_R(M)$. Note that $z \leqslant 0$: if $z \geqslant 1$, then Lemma 2.4 implies that $\mathrm{Ext}^z_R(S,M) = 0$, a contradiction. It follows that $\mathrm{Ext}^i_R(S,M) = 0$ for all $i \neq 0$, and the remaining conclusions follow from Theorem 1.17. □

Remark 2.6. As we note in the introduction, our proof of this theorem removes the need to invoke the Amplitude Inequality in the proof of Theorem 1.17. Indeed, the Amplitude Inequality is used in the implication (iii) $\Longrightarrow$ (ii), which we prove directly in the proof of Theorem 2.5.

Remark 2.7. One can paraphrase Theorem 2.5 as follows: In Theorem 1.17(iii) one can replace the phrase "is finitely generated over R" with the phrase "satisfies NAK". It is natural to ask whether the same replacement can be done in Theorem 1.17(vi). In fact, this cannot be done because, given a finitely generated R-module M, the S-module $S \otimes_R M$ is finitely generated, so it automatically satisfies NAK, regardless of whether M has a compatible S-module structure.

2.3. Explicit Computations

Given a ring homomomorphism $\varphi\colon R \to S$ as in Theorem 2.5, if M is a finitely generated R-module that does *not* have a compatible S-module structure, then we know that $\operatorname{Ext}^i_R(S, M)$ does not satisfy NAK for some i. Hence, this Ext-module is quite large. This section is devoted to a computation showing how large this Ext-module is when $R \neq \widehat{R} = S$, even for the simplest ring R, e.g., for $R = k[X]_{(X)}$ where k is a field or for the localization $\mathbb{Z}_{p\mathbb{Z}}$. See Remark 2.10.

Lemma 2.8. *Let $\varphi\colon R \to S$ be a faithfully flat ring homomorphism, and let C be an R-module. Let $\mathfrak{m}$ be a maximal ideal of R, and assume that C is $\mathfrak{m}$-adically complete.*

(a) *Then $\operatorname{Ext}^i_R(S/R, C) = 0 = \operatorname{Ext}^i_R(S, C)$ for all $i \geqslant 1$.*

(b) *If R is local and the natural map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism, then $\operatorname{Hom}_R(S/R, C) = 0$, and C has an S-module structure compatible with its R-module structure via φ, and the natural maps $C \to \operatorname{Hom}_R(S, C) \to C$ are inverse isomorphisms.*

(c) *If R is local, then $\operatorname{Ext}^i_R(\widehat{R}/R, \widehat{R}) = 0 = \operatorname{Ext}^{i+1}_R(\widehat{R}, \widehat{R})$ for all $i \geqslant 0$, and the natural maps $\widehat{R} \to \operatorname{Hom}_R(\widehat{R}, \widehat{R}) \to \widehat{R}$ are inverse isomorphisms.*

Proof. (a) The fact that S is faithfully flat over R implies that φ is a pure monomorphism, and it follows that S/R is flat over R; see [15, Theorem 7.5]. Since S and S/R are flat over R, the desired vanishing follows from Theorem 1.13.

(b) Assume that R is local and the natural map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism. In particular, the ideal $\mathfrak{m}S \subset S$ is maximal.

Claim: the natural map $\widehat{\varphi}\colon \widehat{R} \to \widehat{S}$ between $\mathfrak{m}$-adic completions is an isomorphism. To see this, first note that the fact that $\mathfrak{m}S$ is maximal implies that $\widehat{S}$ is local with maximal ideal $\mathfrak{m}\widehat{S}$. Furthermore, the induced map $\widehat{R}/\mathfrak{m}\widehat{R} \to \widehat{S}/\mathfrak{m}\widehat{S}$ is equivalent to the isomorphism $R/\mathfrak{m} \to S/\mathfrak{m}S$, so it is an isomorphism. It follows from a version of Nakayama's Lemma [15, Theorem 8.4] that $\widehat{S}$ is a cyclic $\widehat{R}$-module. Since it is also faithfully flat, we deduce that $\widehat{\varphi}$ is an isomorphism, as claimed.

For each $n \in \mathbb{N}$, the induced map $R/\mathfrak{m}^n \to S/\mathfrak{m}^n S$ is an isomorphism: indeed, this map is equivalent to the induced map $\widehat{R}/\mathfrak{m}^n \to \widehat{S}/\mathfrak{m}^n\widehat{S}$ which is an isomorphism because $\widehat{R} \xrightarrow{\cong} \widehat{S}$. This justifies the last step in the following display:

$$(S/R) \otimes_R (R/\mathfrak{m}^n) \cong (S \otimes_R (R/\mathfrak{m}^n))/(R \otimes_R (R/\mathfrak{m}^n)) \cong (S/\mathfrak{m}^n S)/(R/\mathfrak{m}^n) = 0.$$

The above display explains the fifth isomorphism in the next sequence:

$$\begin{aligned}
\operatorname{Hom}_R(S/R, C) &\cong \operatorname{Hom}_R(S/R, \varprojlim_n C/\mathfrak{m}^n C) \\
&\cong \varprojlim_n \operatorname{Hom}_R(S/R, C/\mathfrak{m}^n C) \\
&\cong \varprojlim_n \operatorname{Hom}_R(S/R, \operatorname{Hom}_{R/\mathfrak{m}^n}(R/\mathfrak{m}^n, C/\mathfrak{m}^n C)) \\
&\cong \varprojlim_n \operatorname{Hom}_{R/\mathfrak{m}^n}((R/\mathfrak{m}^n) \otimes_R (S/R), C/\mathfrak{m}^n C) \\
&\cong \varprojlim_n \operatorname{Hom}_{R/\mathfrak{m}^n}(0, C/\mathfrak{m}^n C) \\
&= 0.
\end{aligned}$$

Now consider the exact sequence $0 \to R \to S \to S/R \to 0$ and part of the long exact sequence in $\operatorname{Ext}_R(-, C)$.

$$0 \to \underbrace{\operatorname{Hom}_R(S/R, C)}_{=0} \to \operatorname{Hom}_R(S, C) \to \underbrace{\operatorname{Hom}_R(R, C)}_{\cong C} \to \underbrace{\operatorname{Ext}^1_R(S/R, C)}_{=0}$$

It follows that the induced map $\alpha\colon \operatorname{Hom}_R(S, C) \to C$ is an isomorphism. It is straightforward to show that this is the evaluation map $f \mapsto f(1)$. Since C is complete, the isomorphism $\widehat{R} \cong \widehat{S}$ implies that C has an S-module structure that is compatible with its R-module structure via φ. From this, it follows that the map $\beta\colon C \to \operatorname{Hom}_R(S, C)$ given by $c \mapsto (s \mapsto sc)$ is a well-defined S-module homomorphism. Since the composition $\alpha\beta$ is the identity on C, it follows that α and β are inverse isomorphisms.

(c) This follows from parts (a) and (b) using $C = \widehat{R}$. $\square$

Proposition 2.9. *Let k be a field, let $R = k[X]_{(X)}$ denote the localized polynomial ring in one variable with completion $\widehat{R} = k[\![X]\!]$. Then R is a discrete valuation ring that is not complete, with $\mathfrak{m} = XR$. Set $E = E_R(k) = E_{\widehat{R}}(k)$, and consider the quotient fields $K = Q(R)$ and $\widetilde{K} = Q(\widehat{R})$. If $[\widetilde{K} : K] = \infty$, then there are an uncountable cardinal C and $\widehat{R}$-module isomorphisms*

$$\operatorname{Ext}^i_R(\widehat{R}, R) \cong \begin{cases} 0 & \text{if } i \neq 1 \\ E \oplus \widetilde{K}^{(C)} & \text{if } i = 1 \end{cases}$$

where $\widetilde{K}^{(C)}$ is the direct sum of copies of $\widetilde{K}$ indexed by C.

Proof. As a K-vector space and as an R-module, we have $\widetilde{K} \cong K^{(A)}$ for some infinite cardinal A. Note that since R and $\widehat{R}$ are discrete valuation rings with uniformizing parameter X, we have $K \cong R_X$ and $\widetilde{K} \cong \widehat{R}_X \cong K \otimes_R \widehat{R}$. Since K has no X-torsion, we have $\operatorname{Hom}_R(k, K) = 0$.

Claim 1: We have $\operatorname{Ext}^i_R(\widehat{R}, R) = 0$ for all $i \neq 1$. Since $\operatorname{id}_R(R) = 1$, it suffices to show that $\operatorname{Hom}_R(\widehat{R}, R) = 0$. From [10, Corollary 1.7] we know that $\operatorname{Hom}_R(\widehat{R}, R)$ is isomorphic to a complete submodule $I \subseteq R$. Since R is a discrete valuation ring, its non-zero submodules are all isomorphic to R, which is not complete. So we must have $\operatorname{Hom}_R(\widehat{R}, R) \cong I = 0$.

Claim 2: There is an R-module isomorphism $\operatorname{Hom}_R(\widehat{R}, \widehat{R}) \cong \widehat{R}$, and we have $\operatorname{Ext}^i_R(\widehat{R}, \widehat{R}) = 0$ for all $i \neq 0$. The isomorphism is from the following sequence of standard R-module isomorphisms:

$$\begin{aligned}
\operatorname{Hom}_R(\widehat{R}, \widehat{R}) &\cong \operatorname{Hom}_R(\widehat{R}, \varprojlim_n R/\mathfrak{m}^n) \\
&\cong \varprojlim_n \operatorname{Hom}_R(\widehat{R}, R/\mathfrak{m}^n) \\
&\cong \varprojlim_n \operatorname{Hom}_R(\widehat{R}/\mathfrak{m}^n\widehat{R}, R/\mathfrak{m}^n) \\
&\cong \varprojlim_n \operatorname{Hom}_R(R/\mathfrak{m}^n, R/\mathfrak{m}^n) \\
&\cong \varprojlim_n R/\mathfrak{m}^n \\
&\cong \widehat{R}.
\end{aligned}$$

The vanishing follows from [4, Theorem 2.3] which says that $\operatorname{Ext}^i_R(F, C) = 0$ for all flat R-modules F, for all complete R-modules C, for all $i \neq 0$.

Claim 3: There is an R-module isomorphism $\widehat{R}/R \cong \widetilde{K}/K$. In the following commutative diagram, the top row is a minimal R-injective resolution of R, and the bottom row is a minimal $\widehat{R}$-injective resolution of $\widehat{R}$:

$$\begin{tikzcd}
0 \arrow[r] & R \arrow[r] \arrow[d] & K \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \widehat{R} \arrow[r] & \widetilde{K} \arrow[r] & E \arrow[r] & 0.
\end{tikzcd} \tag{3}$$

The Snake Lemma yields an R-module isomorphism $\widehat{R}/R \cong \widetilde{K}/K$.

Claim 4: We have an R-module isomorphism $\operatorname{Hom}_R(\widehat{R}, \widehat{R}/R) \cong (\widetilde{K}/K)^{(B)}$ for some infinite cardinal $B > A$. This is from the next sequence of R-module isomorphisms where the first step is from Claim 3:

$$
\begin{aligned}
\operatorname{Hom}_R(\widehat{R}, \widehat{R}/R) &\cong \operatorname{Hom}_R(\widehat{R}, \widetilde{K}/K) \\
&\cong \operatorname{Hom}_R(\widehat{R}, \operatorname{Hom}_K(K, \widetilde{K}/K)) \\
&\cong \operatorname{Hom}_K(K \otimes_R \widehat{R}, \widetilde{K}/K) \\
&\cong \operatorname{Hom}_K(\widetilde{K}, \widetilde{K}/K) \\
&\cong \operatorname{Hom}_K(K^{(A)}, \widetilde{K}/K) \\
&\cong \operatorname{Hom}_K(K, \widetilde{K}/K)^A \\
&\cong (\widetilde{K}/K)^A \\
&\cong (\widetilde{K}/K)^{(B)}.
\end{aligned}
$$

The third step is by Hom-tensor adjointness, and the fourth step is from the isomorphism $\widetilde{K} \cong K \otimes_R \widehat{R}$ noted at the beginning of the proof, and the remaining steps are standard. Since A is infinite, we must have $B > A$, as claimed.

Claim 5: There is a cardinal C and an $\widehat{R}$-module isomorphism $\operatorname{Ext}^1_R(\widehat{R}, R) \cong E \oplus \widetilde{K}^{(C)}$. We compute $\operatorname{Ext}^1_R(\widehat{R}, R)$ using the injective resolution of R from the top row of (3). From Claim 1, this yields an exact sequence of $\widehat{R}$-module homomorphisms

$$0 \to \operatorname{Hom}_R(\widehat{R}, K) \to \operatorname{Hom}_R(\widehat{R}, E) \to \operatorname{Ext}^1_R(\widehat{R}, R) \to 0. \tag{4}$$

(One can also see this from the long exact sequence in $\operatorname{Ext}_R(\widehat{R}, -)$ associated to the the top row of (3), using Claim 1.) Since K and E are injective over R, the modules $\operatorname{Hom}_R(\widehat{R}, K)$ and $\operatorname{Hom}_R(\widehat{R}, E)$ are injective over $\widehat{R}$. (Given a ring homomorphism

$R \to S$ and an injective R-module J, the module $\mathrm{Hom}_R(S, J)$ is injective over S.) Because $\mathrm{Hom}_R(\widehat{R}, K)$ is injective over $\widehat{R}$, the sequence (4) splits. As $\mathrm{Hom}_R(\widehat{R}, E)$ is injective over $\widehat{R}$, it follows that $\mathrm{Ext}^1_R(\widehat{R}, R)$ is injective over $\widehat{R}$. So, there is an $\widehat{R}$-module isomorphism

$$\mathrm{Ext}^1_R(\widehat{R}, R) \cong E^{(D)} \oplus \widetilde{K}^{(C)} \tag{5}$$

for cardinals C and D where $D = \dim_k(\mathrm{Hom}_{\widehat{R}}(k, \mathrm{Ext}^1_R(\widehat{R}, R)))$. Since the sequence (4) splits, we have the third step in the next sequence:

$$\begin{aligned}
k &\cong \mathrm{Hom}_R(k, E) \\
&\cong \mathrm{Hom}_{\widehat{R}}(k, \mathrm{Hom}_R(\widehat{R}, E)) \\
&\cong \mathrm{Hom}_{\widehat{R}}(k, \mathrm{Hom}_R(\widehat{R}, K)) \oplus \mathrm{Hom}_{\widehat{R}}(k, \mathrm{Ext}^1_R(\widehat{R}, R)) \\
&\cong \mathrm{Hom}_R(k, K) \oplus \mathrm{Hom}_{\widehat{R}}(k, \mathrm{Ext}^1_R(\widehat{R}, R)) \\
&\cong \mathrm{Hom}_{\widehat{R}}(k, \mathrm{Ext}^1_R(\widehat{R}, R))
\end{aligned}$$

The second and fourth steps follow by Hom-tensor adjointness, and the fifth step follows from the vanishing $\mathrm{Hom}_R(k, K) = 0$ noted at the beginning of the proof. It follows that $D = 1$, so the claim follows from the isomorphism (5).

Now we complete the proof of the proposition. Because of Claim 5, we need only show that C is uncountable. Consider the exact sequence

$$0 \to R \to \widehat{R} \to \widehat{R}/R \to 0$$

and part of the associated long exact sequence induced by $\mathrm{Ext}_R(\widehat{R}, -)$:

$$\mathrm{Hom}_R(\widehat{R}, R) \to \mathrm{Hom}_R(\widehat{R}, \widehat{R}) \to \mathrm{Hom}_R(\widehat{R}, \widehat{R}/R) \to \mathrm{Ext}^1_R(\widehat{R}, R) \to \mathrm{Ext}^1_R(\widehat{R}, \widehat{R}).$$

Over R, this sequence has the following form by Claims 1,2, and 4 and Lemma 2.8(c):

$$0 \to \widehat{R} \to (\widetilde{K}/K)^{(B)} \to \mathrm{Ext}^1_R(\widehat{R}, R) \to 0.$$

Apply the functor $(-)_X$ to obtain the exact sequence of K-module homomorphisms

$$0 \to \widetilde{K} \to (\widetilde{K}/K)^{(B)} \to \mathrm{Ext}^1_R(\widehat{R}, R)_X \to 0$$

which therefore splits. Since $E_X = 0$, it follows from Claim 5 that over R we have

$$\widetilde{K}^{(C)} \cong \mathrm{Ext}^1_R(\widehat{R}, R)_X \cong (\widetilde{K}/K)^{(B)}/\widetilde{K} \cong (K^{(A)})^{(B)}/K^{(A)} \cong K^{(B)}/K^{(A)} \cong K^{(B)}.$$

The last two steps in this sequence follow from the fact that A and B are infinite cardinals such that $B > A$.

Suppose that C were countable. It would then follow that $C \leqslant A$, so we have

$$K^{(B)} \cong \widetilde{K}^{(C)} \cong (K^{(A)})^{(C)} \cong K^{(A)}.$$

It follows that $B = A$, contradicting the fact that $B > A$. It follows that C is uncountable, as desired. $\square$

Remark 2.10. Nagata [16, (E3.3)] shows that the assumption $[\widetilde{K} : K] = \infty$ in Proposition 2.9 is not automatically satisfied. On the other hand, the next result shows that the condition $[\widetilde{K} : K] = \infty$ is satisfied by the localizations $\mathbb{Z}_{p\mathbb{Z}}$ and $R = k[X]_{(X)}$.

Proposition 2.11. *Assume that R is a discrete valuation ring with $\mathfrak{m} = XR$ and such that $|R| = |k|$. For the quotient fields $K = Q(R)$ and $\widetilde{K} = Q(\widehat{R})$, we have $[\widetilde{K} : K] = \infty$.*

Proof. We claim that $|\widehat{R}| > |R|$. To show this, let $\{a_t\}_{t\in k} \subseteq R$ be a set of representatives of the elements of k. Then every element of $\widehat{R}$ has a unique representation $\sum_{i=0}^{\infty} a_{t_i} X^i$. It follows that $|\widehat{R}| = |k|^{\aleph_0} > |k| = |R|$, as claimed.

Suppose now that $[\widetilde{K} : K] = A < \infty$. The fact that K is infinite implies that

$$|K| = A|K| = |\widetilde{K}| = |\widehat{R}| > |R| = |K|$$

a contradiction. $\square$

The proof of Proposition 2.9 translates directly to give the following.

Proposition 2.12. *Let k be a field, let $R = k[X]_{(X)}$ denote the localized polynomial ring in one variable with completion $\widehat{R} = k[\![X]\!]$. Then R is a discrete valuation ring that is not complete, with $\mathfrak{m} = XR$. Set $E = E_R(k) = E_{\widehat{R}}(k)$, and consider the quotient fields $K = Q(R)$ and $\widetilde{K} = Q(\widehat{R})$. If $[\widetilde{K} : K] = A < \infty$, then there are $\widehat{R}$-module isomorphisms*

$$\operatorname{Ext}^i_R(\widehat{R}, R) \cong \begin{cases} 0 & \text{if } i \neq 1 \\ E \oplus \widetilde{K}^{A-2} & \text{if } i = 1. \end{cases}$$

Remark 2.13. It is worth noting that, in the notation of Proposition 2.12, we cannot have $A = 1$. Indeed, if $A = 1$, then we have $\widetilde{K} = K$, and the proof of Proposition 2.12 shows that $\widehat{R}/R \cong \widetilde{K}/K = 0$, contradicting the assumption that R is not complete.

On the other hand, Nagata [16, (E3.3)] shows how to build a ring R such that $A = 2$, which then has $\operatorname{Ext}^i_R(\widehat{R}, R) \cong E$ by Proposition 2.12.

CHAPTER 3. THE BLINDNESS OF EXTENDED MODULES

This chapter is predicated on the following idea: given a flat local ring homomorphism $\varphi\colon (R, \mathfrak{m}) \to (S, \mathfrak{m}S)$, such that the induced map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism, a finitely generated R-module M that has a compatible S-module structure cannot see the difference between R and S. This chapter establishes a collection of equalities between various invariants computed over R and S. We also provide equivalences between certain Ext modules and equivalences between certain Auslander and Bass classes.

3.1. Homological Invariants

This section documents definitions of homological invariants used in Section 3.2 for the readers convenience.

Definition 3.1. Let R be a commutative ring, and let M be an R-module. The *Krull dimension* of R is

$$\dim(R) = \sup\{n \geqslant 0 \mid \text{ there exists a chain } \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \text{ in } \operatorname{Spec}(R)\}$$

The *Krull dimension of* M is

$$\dim_R(M) = \sup\{n \geqslant 0 \mid \text{ there exists a chain } \mathfrak{p}_0 \subsetneq \mathfrak{p}_1 \subsetneq \cdots \subsetneq \mathfrak{p}_n \text{ in } \operatorname{Supp}_R(M)\}$$

where $\operatorname{Supp}_R(M) = \{\mathfrak{p} \in \operatorname{Spec}(R) \mid M_{\mathfrak{p}} \neq 0\}$.

Definition 3.2. Let R be a commutative ring, and let M be an R-module. An element $x \in R$ is *M-regular* if x is not a zero-divisor on M and $M \neq xM$. A sequence $\mathbf{x} = x_1, \ldots, x_n \in R$ is *M-regular* or is an *M-sequence* if x_1 is M-regular, and x_{i+1} is regular on $M/(x_1, \dots, x_i)M$ for $i = 1, \dots, n-1$. Let I be an ideal, and assume that $x_1, \dots, x_n \in I$. Then $x_1, \dots, x_n$ is a *maximal M-regular sequence* in I if $x_1 \dots, x_n$ is an M-regular sequence and, for all $b \in I$ the sequence $x_1, \dots, x_n, b$ is not M-regular. The *depth* of a module with respect to I, denoted $\operatorname{depth}_R(I; M)$, is the supremum of the lengths of maximal M-regular sequences in I. When $I = \mathfrak{m}$ we write $\operatorname{depth}_R(M)$ instead of $\operatorname{depth}_R(\mathfrak{m}; M)$.

Fact 3.3. All maximal M-regular sequences in I have the same length because the ring R is local by [3, Theorem 1.2.5]

Definition 3.4. Let M be an R-module. The *projective dimension* of M is

$$\operatorname{pd}_R(M) = \inf\{n \geqslant 0 \mid \text{ M has a projective resolution } P \text{ such that } P_i = 0 \text{ for } i > n\}.$$

The *injective dimension* of M is

$$\operatorname{id}_R(M) = \inf\{n \geqslant 0 \mid \text{ M has a injective resolution } I \text{ such that } I_{-i} = 0 \text{ for } i > n\}.$$

Definition 3.5. An R-module C is *totally reflexive* if it satisfies the following:

(1) C is finitely generated over R;

(2) the biduality map $\delta_C^R \colon C \to \operatorname{Hom}_R(\operatorname{Hom}_R(C, R), R)$ given by $\delta_C^R(c)(\phi) = \phi(c)$, is an isomorphism; and

(3) $\operatorname{Ext}_R^i(C, R) = 0 = \operatorname{Ext}_R^i(\operatorname{Hom}_R(C, R), R)$ for all $i \geqslant 1$.

Definition 3.6. Let M be a finitely generated R-module. A *G-resolution of M* is an exact sequence

$$G = \cdots \xrightarrow{\partial_{i+1}^G} G_i \xrightarrow{\partial_i^G} G_{i-1} \xrightarrow{\partial_{i-1}^G} \cdots \xrightarrow{\partial_1^G} G_0 \to M \to 0$$

such that each G_i is totally reflexive. The *G-dimension* of M is

$$\text{G-dim}_R(M) = \inf\{\sup\{n \geqslant 0 \mid G_n \neq 0\} \mid G \text{ is a G-resolution of } M\}.$$

Definition 3.7. An R-module J is said to be *Gorenstein injective* if there is an exact sequence

$$I = 0 \cdots \xrightarrow{\partial^I_{-i+1}} I_{-i} \xrightarrow{\partial^I_{-i}} I_{-i-1} \xrightarrow{\partial^I_{-i-1}} \cdots$$

such that $J \cong \operatorname{Ker}(\partial^I_0)$ and the complex $\operatorname{Hom}_R(E, I)$ is exact for every injective R-module E.

Definition 3.8. Let R be a commutative ring and M an R-module. A *G-injective resolution* of M is an exact sequence

$$G = 0 \to M \to G_0 \cdots \xrightarrow{\partial^G_{-i+1}} G_{-i} \xrightarrow{\partial^G_{-i}} G_{-i-1} \xrightarrow{\partial^G_{-i-1}} \cdots$$

such that each G_i is a Gorenstein injective module. The *Gorenstein injective dimension* of M is given by

$$\operatorname{G\,id}_R(M) = \inf\{\sup\{n \geqslant 0 \mid G_{-n} \neq 0\} \mid G \text{ is a G-injective resolution of } M\}.$$

Definition 3.9. An R-module C is *semidualizing* if it satisfies the following:

(1) C is finitely generated;

(2) the homothety map $\chi^R_C \colon R \to \operatorname{Hom}_R(C, C)$ given by $\chi^R_C(r)(c) = rc$, is an isomorphism; and

(3) $\operatorname{Ext}^i_R(C, C) = 0$ for all $i > 0$.

Definition 3.10. Let C be a finitely generated R-module. An R-module M is *totally C-reflexive* if it satisfies the following conditions:

(1) M is finitely generated over R;

(2) the biduality map $\delta_M^C \colon M \to \operatorname{Hom}_R(\operatorname{Hom}_R(M,C),C)$ given by $\delta_M^C(m)(\phi) = \phi(m)$, is an isomorphism; and

(3) $\operatorname{Ext}_R^i(M,C) = 0 = \operatorname{Ext}_R^i(\operatorname{Hom}_R(M,C),C)$ for all $i \geqslant 1$.

Definition 3.11. Let C be a semi-dualizing R-module and M a finitely generated R-module. A *G_C-resolution of M* is an exact sequence

$$G = \cdots \xrightarrow{\partial_{i+1}^G} G_i \xrightarrow{\partial_i^G} G_{i-1} \xrightarrow{\partial_{i-1}^G} \cdots \xrightarrow{\partial_1^G} G_0 \to M \to 0$$

such that each G_i is totally C-reflexive. The *G_C-dimension* of M is

$$\mathrm{G}_C\text{-}\mathrm{dim}_R(M) = \inf\{\sup\{n \geqslant 0 \mid G_n \neq 0\} \mid G \text{ is a } G_C\text{-resolution of } M\}.$$

Definition 3.12. Let C be a semi-dualizing R-module and M a finitely generated R-module. A *G_C-injective resolution of M* is an exact sequence

$$G = 0 \to M \to G_0 \cdots \xrightarrow{\partial_{-i+1}^G} G_{-i} \xrightarrow{\partial_{-i}^G} G_{-i-1} \xrightarrow{\partial_{-i-1}^G} \cdots$$

such that each G_i is a G_C-injective R-module. The *G_C-injective dimension* of M is

$$G_C - \mathrm{id}_R(M) = \inf\{\sup\{n \mid G_{-n} \neq 0\} \mid G \text{ is a } G_C\text{-injective resolution of } M\}.$$

Definition 3.13. Let $(R, \mathfrak{m})$ be a local ring. A *quasi-deformation* of R is a diagram of local ring homomorphisms

$$R \xrightarrow{\rho} R' \xleftarrow{\tau} Q$$

where ρ is flat, and τ is surjective with kernel generated by a Q-regular sequence. Let

M be a finitely generated R-module. The *complete intersection dimension* of M is

$$\text{CI-dim}_R(M) = \inf\{\mathrm{pd}_Q(R' \otimes_R M) - \mathrm{pd}_Q(R') \mid R \to R' \leftarrow Q \text{ is a quasi-deformation}\}.$$

Definition 3.14. Let $(R, \mathfrak{m})$ be a local ring and let M be an R-module. The *complete intersection injective dimension* of M is

$$\text{CI-id}_R(M) = \inf\{\mathrm{id}_Q(R' \otimes_R M) - \mathrm{pd}_Q(R') \mid R \to R' \leftarrow Q \text{ is a quasi-deformation}\}.$$

Definition 3.15. Let C be a finitely generated R-module. The *Auslander class* $\mathcal{A}_C(R)$ is the class of all R-modules M satisfying the following conditions:

(1) the natural map $\gamma^C_M \colon M \to \mathrm{Hom}_R(C, C \otimes_R M)$ given by $\gamma^C_M(m)(c) = m \otimes c$, is an isomorphism; and

(2) $\mathrm{Tor}^R_i(C, M) = 0 = \mathrm{Ext}^i_R(C, C \otimes_R M)$ for all $i \geqslant 1$.

Definition 3.16. Let C be a finitely generated R-module. The *Bass class* $\mathcal{B}_C(R)$ is the class of all R-modules M satisfying the following conditions:

(1) the natural map $\xi^C_M \colon C \otimes_R \mathrm{Hom}_R(C, M) \to M$ given by $\xi^C_M(c \otimes \phi) = \phi(c)$, is an isomorphism; and

(2) $\mathrm{Ext}^i_R(C, M) = 0 = \mathrm{Tor}^R_i(C, \mathrm{Hom}_R(C, M))$ for all $i \geqslant 1$.

Definition 3.17. Let $(R, \mathfrak{m})$ be a local ring and M an R-module. The ith *Bass number* of M is given by $\mu^i_R(M) = \dim_k \mathrm{Ext}^i_R(R/\mathfrak{m}, M)$.

Definition 3.18. Let $(R, \mathfrak{m})$ be a local ring and M an R-module. The ith *Betti number* of M is given by $\beta^R_i(M) = \dim_k \mathrm{Ext}^i_R(M, R/\mathfrak{m})$.

Definition 3.19. Let R be a commutative ring and M an R-module. The direct sum $R \oplus M$ can be equipped with the product: $(r, m) \cdot (r', m') = (rr', rm' + r'm)$.

This turns the module $R \oplus M$ into a ring called the *trivial extension* of R by M and is denoted by $R \ltimes C$.

3.2. Main Results

This section contains the main results demonstrating the blindness of M. We begin by noting the following useful isomorphisms between Ext modules:

Lemma 3.20. *Let $\varphi\colon R \to S$ be a flat ring homomorphism and let N be an S-module. If I is an injective resolution of N over S then I is an injective resolution of N over R.*

Proof. It suffices to show that if a module J is injective over S then J is injective over R. So let J be an injective module over S and consider the an exact sequence of R-modules

$$0 \to A \to B \to C \to 0$$

Since φ is flat then the following sequence is an exact sequence of S-modules.

$$0 \to S \otimes_R A \to S \otimes_R B \to S \otimes_R C \to 0.$$

Now J is injective over S, hence when we apply $\mathrm{Hom}_S(-, J)$ to the tensored sequence we obtain the top exact sequence in the next display:

$$\begin{tikzcd}
0 \arrow[r] & \mathrm{Hom}_S(S \otimes_R C, J) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_S(S \otimes_R B, J) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_S(S \otimes_R C, J) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_R(C, \mathrm{Hom}_S(S, J)) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_R(B, \mathrm{Hom}_S(S, J)) \arrow[r] \arrow[d, "\cong"] & \mathrm{Hom}_R(A, \mathrm{Hom}_S(S, J)) \arrow[r] \arrow[d, "\cong"] & 0 \\
0 \arrow[r] & \mathrm{Hom}_R(C, J) \arrow[r] & \mathrm{Hom}_R(B, J) \arrow[r] & \mathrm{Hom}_R(A, J) \arrow[r] & 0
\end{tikzcd}$$

where the vertical isomorphisms from the top row to the middle row are Hom-tensor adjointness and the vertical isomorphisms from the middle row to the bottom row are Hom cancelation. Thus J is injective over R. $\square$

Lemma 3.21. *Let $\varphi\colon (R,\mathfrak{m}) \to (S,\mathfrak{n})$ be a flat local ring homomorphism with $R/\mathfrak{m} \cong S/\mathfrak{m}S$. Let M be a finitely generated R-module that has a compatible S-module structure via φ. Then for $i \geqslant 1$ we have $\operatorname{Ext}^i_S(M, S/\mathfrak{m}S) \cong \operatorname{Ext}^i_R(M, R/\mathfrak{m})$ and $\operatorname{Ext}^i_S(S/\mathfrak{m}S, M) \cong \operatorname{Ext}^i_R(R/\mathfrak{m}, M)$.*

Proof. Let F be a free resolution of M over R. Notice that since φ is flat we have $S \otimes_R F$ is a free resolution of $S \otimes_R M$ over S. This gives us the following isomorphisms:

$$\begin{aligned}
\operatorname{Ext}^i_S(M, S/\mathfrak{m}S) &\cong \operatorname{Ext}^i_S(S \otimes_R M, S/\mathfrak{m}S) \\
&:= \mathrm{H}_{-i}(\operatorname{Hom}_S(S \otimes_R F, S/\mathfrak{m}S)) \\
&\cong \mathrm{H}_{i}(\operatorname{Hom}_R(F, \operatorname{Hom}_S(S, S/\mathfrak{m}S))) \\
&\cong \mathrm{H}_{-i}(\operatorname{Hom}_R(F, R/\mathfrak{m})) \\
&=: \operatorname{Ext}^i_R(M, R/\mathfrak{m}).
\end{aligned}$$

Where the first isomorphism is from $S \otimes_R M \cong M$, the second isomorphism is Hom-tensor adjointness, and the third isomorphism is Hom-cancellation. For the other variance let I be an injective resolution of M over S, which is an injective resolution of M over R by Lemma 3.20. Then we have:

$$\begin{aligned}
\operatorname{Ext}^i_S(S/\mathfrak{m}S, M) &:= \mathrm{H}_{-i}(\operatorname{Hom}_S(S/\mathfrak{m}S, I)) \\
&\cong \mathrm{H}_{-i}(\operatorname{Hom}_S(S \otimes_R R/\mathfrak{m}, I)) \\
&\cong \mathrm{H}_{-i}(\operatorname{Hom}_R(R/\mathfrak{m}, \operatorname{Hom}_S(S, I))) \\
&\cong \mathrm{H}_{-i}(\operatorname{Hom}_R(R/\mathfrak{m}, I)) \\
&=: \operatorname{Ext}^i_R(R/\mathfrak{m}, M).
\end{aligned}$$

Where the first isomorphism is from $S \otimes_R R/\mathfrak{m} \cong S/\mathfrak{m}S$, the second isomorphism is Hom-tensor adjointness, and the third isomorphism is Hom-cancellation. $\square$

Proposition 3.22. *Let $\varphi\colon (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local ring homomorphism such that the induced map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism. Let M be a finitely generated R-module that has a compatible S-module structure via φ. Then we have* $\dim_R(M) = \dim_S(M)$ *as well as* $\operatorname{depth}_S(M) = \operatorname{depth}_R(M)$.

Proof. From [3, Theorem A.11], we have for N an R-flat finite S-module

$$\dim_S(M \otimes_R N) = \dim_R(M) + \dim_S(N/\mathfrak{m}N).$$

Thus, with $S = N$ we have

$$\dim_S(M) = \dim_S(S \otimes_R M) = \dim_R(M) + \dim_S(S/\mathfrak{m}S) = \dim_R(M).$$

To show that M has the same depth we note that $\operatorname{depth}(S/\mathfrak{m}S) = 0$, since $\mathfrak{n} = \mathfrak{m}S$. Now from [19, Theorem IX.3.6] we have

$$\operatorname{depth}_S(M) = \operatorname{depth}_S(S \otimes_R M) = \operatorname{depth}_R(M) + \operatorname{depth}(S/\mathfrak{m}S) = \operatorname{depth}_R(M).$$

$\square$

Proposition 3.23. *Let $\varphi\colon (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local ring homomorphism such that the induced map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism. Let M be a finitely generated R-module that has a compatible S-module structure via φ. Then* $\operatorname{pd}_R(M) = \operatorname{pd}_S(M)$.

Proof. By [19, Theorem VII.3.14] there is an equality between the projective dimension and the largest index of a non-vanishing Ext module:

$$\operatorname{pd}_R(M) = \sup\{i \geqslant 0 \mid \operatorname{Ext}^i_R(M, R/\mathfrak{m}) \neq 0\}.$$

This characterization uses the fact that M is a finitely generated R-module. Now, by Lemma 3.21 we have $\operatorname{Ext}_S^i(M, S/\mathfrak{m}S) \cong \operatorname{Ext}_R^i(M, R/\mathfrak{m})$. Therefore, we have $\operatorname{Ext}_R^i(M, R/\mathfrak{m}) \neq 0$ if and only if $\operatorname{Ext}_S^i(M, S/\mathfrak{m}S) \neq 0$. Note that if no such value of i exists, then both projective dimensions are $-\infty$. □

Proposition 3.24. *Let $\varphi\colon (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local ring homomorphism such that the induced map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism. Let M be a finitely generated R-module that has a compatible S-module structure via φ. Then $\operatorname{id}_R(M) = \operatorname{id}_S(M)$.*

Proof. By [3, Proposition 3.1.14] there is an equality between the injective dimension and the largest index of a non-vanishing Ext module:

$$\operatorname{id}_R(M) = \sup\{i \geqslant 0 \mid \operatorname{Ext}_R^i(R/\mathfrak{m}, M) \neq 0\}.$$

This characterization uses the facts that R is local and M is a finitely generated R-module. Now, by Lemma 3.21 we have $\operatorname{Ext}_S^i(S/\mathfrak{m}S, M) \cong \operatorname{Ext}_R^i(R/\mathfrak{m}, M)$. Therefore, we have $\operatorname{Ext}_R^i(R/\mathfrak{m}, M) \neq 0$ if and only if $\operatorname{Ext}_S^i(S/\mathfrak{m}S, M) \neq 0$. Note that if no such value of i exists, then both injective dimensions are $-\infty$. □

Proposition 3.25. *Let $\varphi\colon (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local ring homomorphism such that the induced map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism. Let M be a finitely generated R-module that has a compatible S-module structure via φ. Then we have $G\operatorname{id}_R(M) = G\operatorname{id}_S(M)$.*

Proof. Since φ is a flat local ring homomorphism, we can apply [6, Theorem A] along with the fact that $\operatorname{Ext}_R^i(S, M) = 0$ for all $i \geqslant 1$ to obtain the first equality in the next sequence:

$$\operatorname{G\,id}_R(M) = \operatorname{G\,id}_S(\operatorname{Hom}_R(S, M)) = \operatorname{G\,id}_S(M).$$

Since M has a compatible S-module structure we have $\operatorname{Hom}_R(S, M) \cong M$. This gives the second equality. □

Proposition 3.26. *Let $\varphi\colon (R,\mathfrak{m}) \to (S,\mathfrak{n})$ be a flat local ring homomorphism such that the induced map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism. Let M be a finitely generated R-module that has a compatible S-module structure via φ, and let C be a semi-dualizing R-module. Then we have* $\mathrm{G}_C\text{-}\dim_R(M) = \mathrm{G}_{S\otimes_R C}\text{-}\dim_S(M)$.

Proof. Note that since φ is flat then by [20, Proposition 2.2.1] we have $S \otimes_R C$ is a semidualizing S-module. The isomorphism $S \otimes_R M \cong M$ justifies the first equality in the following display:

$$\mathrm{G}_{S\otimes_R C}\text{-}\dim_S(M) = \mathrm{G}_{S\otimes_R C}\text{-}\dim_S(S \otimes_R M) = \mathrm{G}_C\text{-}\dim_R(M).$$

Since S is flat over R we have $\mathrm{Tor}_i^R(S,M) = 0$ for all $i \geqslant 1$, so we obtain the second equality by [20, Theorem 6.3.1]. □

Corollary 3.27. *Let $\varphi\colon (R,\mathfrak{m}) \to (S,\mathfrak{n})$ be a flat local ring homomorphism such that the induced map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism. Let M be a finitely generated R-module that has a compatible S-module structure via φ. Then we have* $\mathrm{G}\text{-}\dim_R(M) = \mathrm{G}\text{-}\dim_S(M)$.

Proof. Let $C = R$ and apply Proposition 3.26. □

Lemma 3.28. *Let $\varphi\colon (R,\mathfrak{m}) \to (S,\mathfrak{n})$ be a flat local ring homomorphism such that the induced map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism. Let C be a semi-dualizing module. Then the map $\varphi'\colon R \ltimes C \to S \otimes_R (R \ltimes C)$ given by $x \mapsto 1 \otimes x$ is a flat local homomorphism.*

Proof. First we show that $R \ltimes C$ is local. Let $(0,c),(0,c') \in 0 \oplus C \subseteq R \ltimes C$ and notice that $(0,c)\cdot(0,c') = (0,0)$ for all $c,c' \in C$. Hence $0 \oplus C \subseteq \mathfrak{m}$ for all $\mathfrak{m} \in \text{m-Spec}(R \ltimes C)$. Therefore, each $\mathfrak{m} \in \text{m-Spec}(R \ltimes C)$ corresponds to a maximal ideal in $(R \ltimes C)/(0 \oplus C) \cong R$. Since R is local there can be only one maximal ideal in $R \ltimes C$, namely $\mathfrak{m} \oplus C$.

Next we show $S \otimes_R (R \ltimes C)$ is local. Let $\pi_1 \colon R \ltimes C \to R$ and $\pi_2 \colon R \ltimes C \to C$ be the natural surjections. Then the map $\beta \colon S \otimes_R (R \ltimes C) \to (S \otimes_R R) \ltimes (S \otimes_R C)$ given by $s \otimes (r, c) \mapsto (s \otimes r, s \otimes c)$ is an isomorphism by [12, Theorem 5.9]. Also, the map $\gamma \colon (S \otimes_R R) \ltimes (S \otimes_R C) \to S \ltimes (S \otimes_R C)$ given by $(s \otimes r, s' \otimes c) \mapsto (s \cdot \varphi(r), s' \otimes c)$ is an isomorphism by tensor cancelation. This explains the vertical isomorphism in the following commutative diagram.

$$\begin{tikzcd}
R \ltimes C \arrow[r, "\varphi'"] \arrow[dr, "\alpha"'] & S \otimes_R (R \ltimes C) \arrow[d, "\gamma\circ\beta", "\cong"'] \\
& S \ltimes (S \otimes_R C)
\end{tikzcd}$$

The diagonal map α is given by $(r, c) \mapsto (\varphi(r), 1 \otimes c)$, and φ' is the natural map given by $(r, c) \mapsto 1 \otimes (r, c)$. First we show that these maps are ring homomorphisms. Notice that β and γ are group homomorphisms so we need only check that they respect the multiplicative ring structures. Notice that it suffices to check that β and γ respect the multiplicative ring structure on simple tensors since we can extend to finite sums of simple tensors by the distributive property. Let $r, r' \in R$ and $s, s', t, t' \in S$ and $c, c' \in C$, and let $x = (s \otimes (r, c)) \in S \otimes_R (R \ltimes C)$ and $y = (s' \otimes (r', c')) \in S \otimes_R (R \ltimes C)$.

Then for β we have

$$\begin{aligned}
\beta(x)\beta(y) &= \beta(s \otimes (r, c)) \cdot \beta(s' \otimes (r', c')) \\
&= (s \otimes r, s \otimes c) \cdot (s' \otimes r', s' \otimes c') \\
&= ((s \otimes r)\,(s' \otimes r')\,,((s \otimes r)\,(s' \otimes c') + (s \otimes c)\,(s' \otimes r'))) \\
&= (ss' \otimes rr', (ss' \otimes rc') + (ss' \otimes r'c)) \\
&= (ss' \otimes rr', ss' \otimes (rc' + r'c)) \\
&= \beta\,(ss' \otimes (rr', rc' + r'c)) \\
&= \beta\,((s \otimes (r, c)) \cdot (s' \otimes (r', c'))) \\
&= \beta(xy).
\end{aligned}$$

Also, β respects identities since

$$\beta(1_{S\otimes_R(R\ltimes C)}) = \beta(1_S \otimes (1_R, 0)) = (1_S \otimes 1_R, 1_S \otimes 0) = (1_S \otimes 1_R, 0) = 1_{(S\otimes_R R)\ltimes(S\otimes_R C)}.$$

Let $u = (s\otimes r, t\otimes c) \in (S\otimes_R R)\ltimes(S\otimes_R C)$ and $v = (s'\otimes r', t'\otimes c') \in (S\otimes_R R)\ltimes(S\otimes_R C)$.

Then we have

$$\begin{aligned}
\gamma(u)\gamma(v) &= \gamma\left(s \otimes r, t \otimes c\right) \cdot \gamma\left(s' \otimes r', t' \otimes c'\right) \\
&= \left(s\varphi(r), t \otimes c\right) \cdot \left(s'\varphi(r'), t' \otimes c'\right) \\
&= \left(ss'\varphi(r)\varphi(r'), s\varphi(r)(t' \otimes c') + s'\varphi(r')(t \otimes c)\right) \\
&= \left(ss'\varphi(rr'), (s\varphi(r)t') \otimes c' + (s'\varphi(r')t) \otimes c\right) \\
&= \left(ss'\varphi(rr'), (st') \otimes (rc') + (s't) \otimes (r'c)\right) \\
&= \left(ss'\varphi(rr'), (s \otimes r)(t' \otimes c') + (s' \otimes r')(t \otimes c)\right) \\
&= \gamma((s \otimes r)(s' \otimes r'), (s \otimes r)(t' \otimes c') + (s' \otimes r')(t \otimes c)) \\
&= \gamma\left((s \otimes r, t \otimes c) \cdot (s' \otimes r', t' \otimes c')\right) \\
&= \gamma(uv).
\end{aligned}$$

Also, γ respects identities since

$$\gamma(1_{(S\otimes_R R)\ltimes(S\otimes_R C)}) = \gamma(1_S \otimes 1_R, 0) = (1_S\varphi(1_R), 1_S \otimes 0) = (1_S, 0) = 1_{S\ltimes(S\otimes_R C)}.$$

To show that α is a ring homomorphism notice that

$$\alpha(1_{R\ltimes C}) = \alpha(1_R, 0) = (\varphi(1_R), 1 \otimes 0) = (1_S, 0) = 1_{S\ltimes(S\otimes_R C)}.$$

So α respects identities. Now let $r, r', t \in R, s, s' \in S$ and $c, c' \in C$ and notice that

$$\begin{aligned}
\alpha(r,c) + \alpha(r',c') &= (\varphi(r), 1 \otimes c) + (\varphi(r'), 1 \otimes c') \\
&= (\varphi(r) + \varphi(r'), (1 \otimes c) + (1 \otimes c')) \\
&= (\varphi(r + r'), (1 \otimes (c + c')) \\
&= \alpha(r + r', c + c') \\
&= \alpha((r,c) + (r',c'))
\end{aligned}$$

and for the multiplicative structure:

$$\begin{aligned}
\alpha(r,c)\alpha(r',c') &= (\varphi(r), 1 \otimes c)(\varphi(r'), 1 \otimes c') \\
&= (\varphi(r)\varphi(r'), \varphi(r)(1 \otimes c') + \varphi(r')(1 \otimes c)) \\
&= (\varphi(rr'), 1 \otimes rc' + 1 \otimes r'c) \\
&= (\varphi(rr'), 1 \otimes (rc' + r'c)) \\
&= \alpha(rr', rc' + r'c) \\
&= \alpha((r,c)(r',c')).
\end{aligned}$$

Lastly we show that φ' is a ring homomorphism. Note that φ' respects identities since we have

$$\varphi'(1_{R \ltimes C}) = \varphi'(1_R, 0) = 1_S \otimes (1_R, 0) = 1_{S \otimes_R R \ltimes C}.$$

Let $r, r' \in R$ and $c, c' \in C$ then

$$\begin{aligned}\varphi'(r,c) + \varphi'(r,c') &= 1 \otimes (r,c) + 1 \otimes (r',c') \\ &= 1 \otimes ((r,c) + (r',c')) \\ &= \varphi'((r,c) + (r',c'))\end{aligned}$$

and

$$\begin{aligned}\varphi'(r,c)\varphi'(r,c) &= (1 \otimes (r,c))(1 \otimes (r',c')) \\ &= 1 \otimes ((r,c)(r',c')) \\ &= \varphi'((r,c)(r'c')).\end{aligned}$$

Therefore, all of the above maps are ring homomorphisms. The diagram commutes by the following equalities:

$$\gamma(\beta(\varphi'(r,c))) = \gamma(\beta(1 \otimes (r,c))) = \gamma(1 \otimes r, 1 \otimes c) = (\varphi(r), 1 \otimes c) = \alpha(r,c).$$

Since φ is flat then $S \otimes_R C$ is a semi-dualizing S-module by [20, Proposition 2.2.1]. Using the fact that S is local a similar argument as above shows that $S \ltimes (S \otimes_R C)$ is local with maximal ideal $\mathfrak{n} \oplus (S \otimes_R C)$. Hence $\gamma \circ \beta$ being an isomorphism implies that we have $S \otimes_R (R \ltimes C)$ is also local.

To see that φ' is a local homomorphism notice that $\alpha(\mathfrak{m} \oplus C) \subseteq \mathfrak{n} \oplus (S \otimes_R C)$ because $S \ltimes (S \otimes_R C)$ and φ are local. Now the fact that the diagram commutes implies that φ' is a local homomorphism.

To show that φ' is a flat homomorphism we need to show that $S \otimes_R (R \ltimes C)$ is flat as an R-module. Let $\mathcal{S}$ be an exact sequence of R-modules and notice that by tensor cancellation we have $\mathcal{S} \otimes_{R \ltimes C} (S \otimes_R (R \ltimes C)) \cong \mathcal{S} \otimes_R S$ which is exact since S

is flat over R. Therefore φ' is a flat local ring homomorophism. $\square$

Proposition 3.29. *Let $\varphi\colon (R,\mathfrak{m}) \to (S,\mathfrak{n})$ be a flat local ring homomorphism such that the induced map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism. Let M be a finitely generated R-module that has a compatible S-module structure via φ, and let C be a semi-dualizing R-module. Then $\mathrm{G}_C \operatorname{id}_R(M) = G_{S\otimes_R C} \operatorname{id}_S(M)$.*

Proof. Viewing M as an $R\ltimes C$-module we can apply Proposition 3.25 and Lemma 3.28 to obtain the second equality in the following display.

$$\begin{aligned} \mathrm{G}_C \operatorname{id}_R(M) &= \mathrm{G} \operatorname{id}_{R\ltimes C}(M) \\ &= \mathrm{G} \operatorname{id}_{S\ltimes(S\otimes_R C)}(M) \\ &= \mathrm{G}_{S\otimes_R C} \operatorname{id}_S(M) \end{aligned}$$

The first and the third equalities follow from [11, Theorem 2.16]. $\square$

Proposition 3.30. *Let $\varphi\colon (R,\mathfrak{m}) \to (S,\mathfrak{n})$ be a flat local ring homomorphism such that the induced map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism. Let M be a finitely generated R-module that has a compatible S-module structure via φ. Then we have* $\text{CI-dim}_R(M) = \text{CI-dim}_S(M)$.

Proof. By [17, Proposition 4.7(a)] we have the second equality below:

$$\text{CI-dim}_S(M) = \text{CI-dim}_S(S \otimes_R M) = \text{CI-dim}_R(M)$$

$\square$

Lemma 3.31. *Let $\varphi\colon (R,\mathfrak{m}) \to (S,\mathfrak{n})$ be a flat local ring homomorphism such that the induced map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism. Then $\widehat{R} \cong \widehat{S}$.*

Proof. First we claim that $R/\mathfrak{m}^t \cong S/\mathfrak{m}^t S$ for all $t \geqslant 1$. Let $\varphi' \colon R/\mathfrak{m} \to S/\mathfrak{m}S$ be the induced isomorphism and consider the following commutative diagram:

$$\begin{tikzcd} R/\mathfrak{m}^t \arrow[r, "\bar{\varphi}"] \arrow[d, "\tau_1"'] & S/\mathfrak{m}^t S \arrow[d, "\tau_2"] \\ R/\mathfrak{m} \arrow[r, "\varphi'", "\cong"'] & S/\mathfrak{m}S \end{tikzcd}$$

where the vertical maps are the natural surjections and $\bar{\varphi} \colon R/\mathfrak{m}^t \to S/\mathfrak{m}^t S$ is the induced map from φ. The diagram commutes by the following calculations:

$$\tau_2(\bar{\varphi}(r + \mathfrak{m}^t) = \tau_2(\varphi(r) + \mathfrak{m}^t S) = \varphi(r) + \mathfrak{m}S$$

and

$$\varphi'(\tau_1(r + \mathfrak{m}^t)) = \varphi'(r + \mathfrak{m}) = \varphi(r) + \mathfrak{m}S.$$

To see that $\bar{\varphi}$ is surjective, let $s + \mathfrak{m}^t S \in S/\mathfrak{m}^t S$. Now by [10, Lemma 1.1] there exists $r \in R$ such that $s - \varphi(r) \in \mathfrak{n}^t = \mathfrak{m}^t S$. Thus $\varphi(r)$ and s are in the same equivalence class in $S/\mathfrak{m}^t S$ and so we have $\bar{\varphi}(r + \mathfrak{m}^t) = s + \mathfrak{m}^t S$. To see that $\bar{\varphi}$ is injective suppose that $r + \mathfrak{m}^t \in \operatorname{Ker}(\bar{\varphi})$. Hence in S we have $\varphi(r) \in \mathfrak{m}^t S$. Now since φ is faithfully flat we have $r \in \mathfrak{m}^t$ [15, Theorem 7.5]. Thus $r + \mathfrak{m}^t = 0 + \mathfrak{m}^t$. Therefore, $\bar{\varphi}$ is an isomorphism. Since t was arbitrary this justifies the claim.

Now taking the inverse limits the claim provides the first isomorphism in the following display:

$$\widehat{R} = \varprojlim_t R/\mathfrak{m}^t \cong \varprojlim_t S/\mathfrak{m}^t S \cong \varprojlim_t S/\mathfrak{n}^t S = \widehat{S}$$

The second isomorphism is from the equality $\mathfrak{n} = \mathfrak{m}S$. $\square$

Proposition 3.32. *Let $\varphi \colon (R, \mathfrak{m}) \to (S, \mathfrak{n})$ be a flat local ring homomorphism such* *that the induced map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism. Let M be a finitely generated R-module that has a compatible S-module structure via φ and let C be a semi-dualizing R-module. Then $M \in \mathcal{A}_C(R)$ if and only if $M \in \mathcal{A}_{S\otimes_R C}(S)$, and $M \in \mathcal{B}_C(R)$ if and only if $M \in \mathcal{B}_{S\otimes_R C}(S)$.*

Proof. This results follows by [20, Proposition 3.4.6]. □

Proposition 3.33. *Let $\varphi\colon (R,\mathfrak{m}) \to (S,\mathfrak{n})$ be a flat local ring homomorphism such that the induced map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism. Let M be a finitely generated R-module that has a compatible S-module structure via φ. Then we have $\beta_i^R(M) = \beta_i^S(M)$ as well as $\mu_R^i(M) = \mu_S^i(M)$.*

Proof. Notice that by Lemma 3.21 we have $\operatorname{Ext}_R^i(M, R/\mathfrak{m}) \cong \operatorname{Ext}_S^i(M, S/\mathfrak{m}S)$ and $\operatorname{Ext}_S^i(S/\mathfrak{m}S, M) \cong \operatorname{Ext}_R^i(R/\mathfrak{m}, M)$. Hence we have both $\beta_i^R(M) = \beta_i^S(M)$ and $\mu_R^i(M) = \mu_S^i(M)$. □

Proposition 3.34. *Let $\varphi\colon (R,\mathfrak{m}) \to (S,\mathfrak{n})$ be a flat local ring homomorphism such that the induced map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism. Let M be a finitely generated R-module that has a compatible S-module structure via φ and let N be an S-module. Then we have $\operatorname{Ext}_S^i(N, M) \cong \operatorname{Ext}_R^i(N, M)$ and $\operatorname{Ext}_S^i(M, N) \cong \operatorname{Ext}_R^i(M, N)$.*

Proof. Let P be a projective resolution of M, and let I be an injective resolution of M over R. Note that since each module I_j in the injective resolution I is injective then $\operatorname{Hom}_R(S, I_j)$ is also injective. Hence we have $\operatorname{Hom}_R(S, I)$ is an injective resolution of $\operatorname{Hom}_R(S, M) \cong M$ over S since $\operatorname{Ext}_R^i(S, M) = 0$ by Theorem 2.5. Thus, we have the following isomorphisms:

$$\begin{aligned}
\operatorname{Ext}^i_S(N,M) &:= \mathrm{H}_{-i}(\operatorname{Hom}_S(N,\operatorname{Hom}_R(S,I))) \\
&\cong \mathrm{H}_{-i}(\operatorname{Hom}_R(S\otimes_S N, I)) \\
&\cong \mathrm{H}_{-i}(\operatorname{Hom}_R(N,I)) \\
&=: \operatorname{Ext}^i_R(N,M).
\end{aligned}$$

The first and second isomorphisms are the Hom-tensor adjointness and tensor cancellation isomorphisms respectively. Similarly, in the other variance, we have for each P_i in the projective resolution P the module $S \otimes_R P_i$ is projective over S. Hence $S \otimes_R P$ is a projective resolution of $S \otimes_R M \cong M$ over S since $\operatorname{Tor}^R_i(S,M) = 0$ because S is flat. Thus, we have the following isomorphisms:

$$\begin{aligned}
\operatorname{Ext}^i_S(M,N) &:= \mathrm{H}_{-i}(\operatorname{Hom}_S(S\otimes_R P, N)) \\
&\cong \mathrm{H}_{-i}(\operatorname{Hom}_R(P,\operatorname{Hom}_S(S,N))) \\
&\cong \mathrm{H}_{-i}(\operatorname{Hom}_R(P,N)) \\
&=: \operatorname{Ext}^i_R(M,N).
\end{aligned}$$

The first and second isomorphisms are Hom-tensor adjointness and Hom cancellation respectively. $\square$

CHAPTER 4. THE NON-LOCAL SETTING

This chapter is devoted to removing the local hypothesis from Theorem 1.17.

4.1. Some Daggers Are Better Than Others

There are many different ways one can remove the local hypothesis from Theorem 1.17. Specifically, with R and S noetherian, how does one interpret the condition $R/\mathfrak{m} \cong S/\mathfrak{m}S$ on φ when R or S is not local? Below is a list of a few of the different ways to consider generalizing this condition in the non-local setting. We use the notation m-Spec(R) to mean the set of maximal ideals of R. For completeness, the first dagger is the original local condition on φ.

($\dagger$) $\varphi\colon R \to S$ flat and local such that the induced map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism.

($\dagger'$) $\varphi\colon R \to S$ faithfully flat such that the induced map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism for all $\mathfrak{m} \in \text{m-Spec}\, R$.

($\hat{\dagger}$) $\varphi\colon R \to S$ faithfully flat such that for all $\mathfrak{n} \in \text{m-Spec}\, S$, the map $\varphi_{\mathfrak{n}}\colon R_{\mathfrak{p}} \to S_{\mathfrak{n}}$ satisfies ($\dagger$) where $\mathfrak{p} = \varphi^{-1}(\mathfrak{n})$.

($\tilde{\dagger}$) $\varphi\colon R \to S$ faithfully flat such that the induced map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism for all $\mathfrak{m} \in \text{m-Spec}\, R$, and the induced map $\varphi^*\colon \operatorname{Spec}(S) \to \operatorname{Spec}(R)$ satisfies $\varphi^*(\text{m-Spec}(S)) \subseteq \text{m-Spec}(R)$.

Remark 4.1. The containment $\varphi^*(\text{m-Spec}(S)) \subseteq \text{m-Spec}(R)$ in ($\tilde{\dagger}$) does not automatically follow from the assumption that $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism for all $\mathfrak{m} \in \text{m-Spec}\, R$; see Example 4.6.

We first show some of the implications between the above daggers.

Proposition 4.2. *With the above notation, the condition* $(\dagger)$ *implies* $(\tilde{\dagger})$*, and* $(\tilde{\dagger})$ *implies both* $(\dagger')$ *and* $(\hat{\dagger})$*.*

Proof. The implications $(\dagger) \Rightarrow (\tilde{\dagger}) \Rightarrow (\dagger')$ follow by definition. For $(\tilde{\dagger}) \Rightarrow (\hat{\dagger})$, let $\mathfrak{n} \in \text{m-Spec}(S)$. Notice that $\varphi_{\mathfrak{n}}$ is a flat local homomorphism by the local criterion for flatness [15, Theorem 7.1]. Also, $(\tilde{\dagger})$ implies that we have $\varphi^{-1}(\mathfrak{n}) = \mathfrak{m}$ for some $\mathfrak{m} \in \text{m-Spec}(R)$. So condition $(\tilde{\dagger})$ implies that $\mathfrak{n} = \mathfrak{m}S$. Let $\pi_{\mathfrak{m}}$ and $\pi_{\mathfrak{n}}$ be the natural surjections in the following commutative diagram

$$\begin{tikzcd}
R_{\mathfrak{m}} \arrow[r, "\varphi_{\mathfrak{n}}"] \arrow[d, "\pi_{\mathfrak{m}}"] & S_{\mathfrak{n}} \arrow[d, "\pi_{\mathfrak{n}}"] \\
R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}} \arrow[r, "\overline{\varphi_{\mathfrak{n}}}"] & S_{\mathfrak{n}}/\mathfrak{n}S_{\mathfrak{n}}
\end{tikzcd}$$

where $\overline{\varphi_{\mathfrak{n}}}$ is the induced map on the quotients $R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}}$ and $S_{\mathfrak{n}}/\mathfrak{n}S_{\mathfrak{n}}$. To see that the diagram commutes let $r/s \in R_{\mathfrak{m}}$ and notice that we have

$$\pi_{\mathfrak{n}}\left(\varphi_{\mathfrak{n}}\left(\frac{r}{s}\right)\right) = \pi_{\mathfrak{n}}\left(\frac{\varphi(r)}{\varphi(s)}\right) = \frac{\varphi(r)}{\varphi(s)} + \mathfrak{n}S_{\mathfrak{n}}$$

and

$$\overline{\varphi_{\mathfrak{n}}}\left(\pi_{\mathfrak{m}}\left(\frac{r}{s}\right)\right) = \overline{\varphi_{\mathfrak{n}}}\left(\frac{r}{s} + \mathfrak{m}R_{\mathfrak{m}}\right) = \frac{\varphi(r)}{\varphi(s)} + \mathfrak{n}S_{\mathfrak{n}}.$$

Thus $\pi_{\mathfrak{n}} \circ \varphi_{\mathfrak{n}} = \overline{\varphi_{\mathfrak{n}}} \circ \pi_{\mathfrak{m}}$ and the diagram commutes. Since localization commutes with quotients we have the first isomorphism in the following display:

$$S_{\mathfrak{n}}/\mathfrak{n}S_{\mathfrak{n}} \cong (S/\mathfrak{n})_{\mathfrak{n}} \cong S/\mathfrak{n} \cong S/\mathfrak{m}S$$

The second isomorphism is because $S/\mathfrak{n}$ is a field and the third follows from $\mathfrak{n} = \mathfrak{m}S$. Similarly, since $R/\mathfrak{m}$ is a field we have $R_{\mathfrak{m}}/\mathfrak{m}R_{\mathfrak{m}} \cong (R/\mathfrak{m})_{\mathfrak{m}} \cong R/\mathfrak{m}$. Therefore $\overline{\varphi_{\mathfrak{n}}}$ is an isomorphism and φ satisfies $(\hat{\dagger})$. Hence $(\tilde{\dagger})$ implies $(\hat{\dagger})$. $\square$

We know that $(\hat{\dagger})$ does not imply$(\dagger')$, and neither $(\dagger')$ nor $(\hat{\dagger})$ implies $(\tilde{\dagger})$, as seen in the following examples. First we need the following lemmas.

Lemma 4.3. *Let A and B be commutative rings. Let $\mathfrak{p} \in \operatorname{Spec}(A)$ and $\mathfrak{q} \in \operatorname{Spec}(B)$. Then the homomorphisms $f\colon (A \times B)_{\mathfrak{p}\times B} \to A_{\mathfrak{p}}$ given by $(a,b)/(x,y) \mapsto a/x$ and $g\colon (A \times B)_{A\times\mathfrak{q}} \to B_{\mathfrak{q}}$ given by $(a,b)/(b,y)$ are both isomorphisms.*

Proof. We prove that f is an isomorphism. The argument to show g is an isomorphism is similar. The homomorphism f is surjective since for any $a/x \in A_{\mathfrak{p}}$ we have $f((a,b)/(x,y)) = a/x$. To show that f is injective let $(a,b)/(x,y) \in \operatorname{Ker}(f)$ then $f((a,b)/(x,y)) = a/x = 0$. Therefore there exists $t \in R \setminus \mathfrak{p}$ such that $ta = 0$. Note that in $(A \times B)_{\mathfrak{p}\times B}$ the element $(t,0) \in (A \times B) \setminus (\mathfrak{p} \times B)$ so $(t,0)$ is an allowable denominator. Thus we have

$$\frac{(a,b)}{(x,y)} = \frac{(a,b)(t,0)}{(x,y)(t,0)} = \frac{(at,0)}{(x,y)(t,0)} = 0.$$

Hence $(a,b)/(x,y) = 0$ and f is an isomorphism. $\square$

Lemma 4.4. *Let A be a commutative ring and let $\mathfrak{p},\mathfrak{q} \in \operatorname{Spec}(A)$. Let the map $\varphi\colon A \to A_{\mathfrak{p}} \times A_{\mathfrak{q}}$ be given by $a \mapsto (a/1, a/1)$.*

(i) *$\varphi^{-1}(\mathfrak{p}A_{\mathfrak{p}} \times A_{\mathfrak{q}}) = \mathfrak{p}$ and $\varphi^{-1}(A_{\mathfrak{p}} \times \mathfrak{q}A_{\mathfrak{q}}) = \mathfrak{q}$.*

(ii) *The induced maps $\varphi_P\colon A_{\mathfrak{p}} \to (A_{\mathfrak{p}} \times A_{\mathfrak{q}})_{(\mathfrak{p}A_{\mathfrak{p}}\times A_{\mathfrak{q}})}$ and $\varphi_Q\colon A_{\mathfrak{p}} \to (A_{\mathfrak{p}} \times A_{\mathfrak{q}})_{(A_{\mathfrak{p}}\times\mathfrak{q}A_{\mathfrak{q}})}$ are isomorphisms where $P = \mathfrak{p}A_{\mathfrak{p}} \times A_q$ and $Q = A_{\mathfrak{p}} \times \mathfrak{q}A_{\mathfrak{q}}$.*

Proof. For part (i) we prove $\varphi^{-1}(\mathfrak{p}A_{\mathfrak{p}} \times A_{\mathfrak{q}}) = \mathfrak{p}$. The other equality is similar. Set $P' = \varphi^{-1}(\mathfrak{p}A_{\mathfrak{p}} \times A_{\mathfrak{q}}) = \{x \in A \mid \varphi(x) \in \mathfrak{p}A_{\mathfrak{p}} \times A_{\mathfrak{q}}\}$. Let $x \in \mathfrak{p}$ and note $\varphi(x) = (x/1, x/1) \in \mathfrak{p}A_{\mathfrak{p}} \times A_{\mathfrak{q}}$, since $x/1 \in \mathfrak{p}A_{\mathfrak{p}}$. Hence $\mathfrak{p} \subseteq P'$. For the other containment let $y \in P'$ and note that $\varphi(y) = (y/1, y/1) \in \mathfrak{p}A_{\mathfrak{p}} \times A_{\mathfrak{q}}$. Thus $y/1 \in \mathfrak{p}A_{\mathfrak{p}}$ and so $y \in A \cap \mathfrak{p}A_{\mathfrak{p}} = \mathfrak{p}$.

For part (ii) we prove that $\varphi_P\colon A_\mathfrak{p} \to (A_\mathfrak{p} \times A_\mathfrak{q})_{(\mathfrak{p}A_\mathfrak{p} \times A_\mathfrak{q})}$ is the inverse of the composition

$$(A_\mathfrak{p} \times A_\mathfrak{q})_{\mathfrak{p}A_\mathfrak{p} \times A_q} \xrightarrow[\cong]{f} (A_\mathfrak{p})_{\mathfrak{p}A_\mathfrak{p}} \xrightarrow[\cong]{\alpha} A_\mathfrak{p}$$

where f is the map from Lemma 4.3 with $A = A_\mathfrak{p}$ and $B = A_\mathfrak{q}$ and

$$\alpha\left(\frac{\left(\frac{a}{s}\right)}{\left(\frac{x}{u}\right)}\right) = \frac{au}{xs}.$$

The other inverse is similar. First note that for any

$$\frac{\left(\frac{a}{s}, \frac{b}{t}\right)}{\left(\frac{x}{u}, \frac{y}{v}\right)} \in (A_\mathfrak{p} \times A_\mathfrak{q})_{(\mathfrak{p}A_\mathfrak{p} \times A_\mathfrak{q})},$$

we have

$$\frac{\left(\frac{a}{s}, \frac{b}{t}\right)}{\left(\frac{x}{u}, \frac{y}{v}\right)} = \frac{\left(\frac{au}{1}, \frac{au}{1}\right)}{\left(\frac{xs}{1}, \frac{xs}{1}\right)}.$$

To see this notice that $(\frac{1}{1}, \frac{0}{1}) \in (A_\mathfrak{p} \times A_\mathfrak{q}) \setminus (\mathfrak{p}A_\mathfrak{p} \times A_\mathfrak{q})$. A routine computation shows

$$\left(\frac{1}{1}, \frac{0}{1}\right)\left[\left(\frac{a}{s}, \frac{b}{t}\right)\left(\frac{xs}{1}, \frac{xs}{1}\right) - \left(\frac{au}{1}, \frac{au}{1}\right)\left(\frac{x}{u}, \frac{y}{v}\right)\right] = 0.$$

Hence we have

$$(\varphi_P \circ \alpha \circ f)\left(\frac{\left(\frac{a}{s}, \frac{b}{t}\right)}{\left(\frac{x}{u}, \frac{y}{v}\right)}\right) = \varphi_P\left(\alpha\left(\frac{\left(\frac{a}{s}\right)}{\left(\frac{x}{u}\right)}\right)\right) = \varphi_P\left(\frac{au}{xs}\right) = \frac{\left(\frac{au}{1}, \frac{au}{1}\right)}{\left(\frac{xs}{1}, \frac{xs}{1}\right)} = \frac{\left(\frac{a}{s}, \frac{b}{t}\right)}{\left(\frac{x}{u}, \frac{y}{v}\right)}.$$

So $\varphi_P \circ \alpha \circ f = \mathrm{id}_{(A_\mathfrak{p} \times A_\mathfrak{q})_{(\mathfrak{p}A_\mathfrak{p} \times A_\mathfrak{q})}}$. Next, let $a/x \in A_\mathfrak{p}$ and notice that we have

$$(\alpha \circ f \circ \varphi_\mathfrak{p})\left(\frac{a}{x}\right) = \alpha\left(f\left(\frac{\left(\frac{a}{1}, \frac{a}{1}\right)}{\left(\frac{x}{1}, \frac{x}{1}\right)}\right)\right) = \alpha\left(\frac{\frac{a}{1}}{\frac{x}{1}}\right) = \frac{a \cdot 1}{x \cdot 1} = \frac{a}{x}.$$

Hence $\alpha \circ f \circ \varphi_P = \mathrm{id}_{A_\mathfrak{p}}$ and φ_P is the inverse of $\alpha \circ f$. $\square$

Example 4.5. Let $(R, \mathfrak{m})$ be a local ring and set $S = R \times R$. Consider $\varphi\colon R \to S$

given by $r \mapsto (r, r)$. First we claim that φ satisfies $(\hat{\dagger})$. To see this notice that if $\mathfrak{n} \in \text{m-Spec}(S)$ then $\mathfrak{n}$ is either $\mathfrak{m} \times R$ or $R \times \mathfrak{m}$. Since R is noetherian, S is flat over R, and $\mathfrak{m}S \neq S$ we know that S is faithfully flat over R, that is, φ is faithfully flat. Next set $\mathfrak{n}_1 = \mathfrak{m} \times R$ and $\mathfrak{n}_2 = R \times \mathfrak{m}$ and notice that Lemma 4.4 (i) implies that $\varphi^{-1}(\mathfrak{n}_i) = \mathfrak{m}$ for $i = 1, 2$. Thus the homomorphism $\varphi_{\mathfrak{n}_i} \colon R_{\mathfrak{m}} \to S_{\mathfrak{n}_i}$ given by $x/y \mapsto (x, x)/(y, y)$ is well-defined. Now by Lemma 4.4 (ii) the map $\varphi_{\mathfrak{n}_i} \colon R_{\mathfrak{m}} \to S_{\mathfrak{n}_i}$ is an isomorphism. Thus $\varphi_{\mathfrak{n}_i}$ satisfies $(\dagger)$ for $i = 1, 2$, and hence φ satisfies $(\hat{\dagger})$.

Next we claim that φ does not satisfy $(\dagger')$. Let $\mathfrak{m} \in \text{m-Spec}(R)$ and note that $\mathfrak{m}S = \mathfrak{m} \times \mathfrak{m}$. Thus $S/\mathfrak{m}S \cong R/\mathfrak{m} \times R/\mathfrak{m} \ncong R/\mathfrak{m}$.

Example 4.6. Let $(R, \mathfrak{m})$ be a local ring with a prime ideal $\mathfrak{p}$ such that $\mathfrak{p} \neq \mathfrak{m}$. Let $\varphi \colon R \to R \times R_{\mathfrak{p}}$ be given by $r \mapsto (r, r/1)$. Then φ is a faithfully flat ring homomorphism; indeed, both R and $R_{\mathfrak{p}}$ are flat, and R is faithfully flat, so $R \times R_{\mathfrak{p}}$ is faithfully flat over R. Consider the induced map $R/\mathfrak{m} \to R/\mathfrak{m} \times R_{\mathfrak{p}}/\mathfrak{m}R_{\mathfrak{p}}$. Since $R_{\mathfrak{p}}/\mathfrak{m}R_{\mathfrak{p}} = 0$, the induced map is an isomorphism. Hence φ satisfies $(\dagger')$. Also, φ satisfies $(\hat{\dagger})$. We already have shown that φ is a faithfully flat ring homomorphism. To see that $\varphi_{\mathfrak{n}}$ satisfies $(\dagger)$ set $\mathfrak{n}_1 = \mathfrak{m} \times R_{\mathfrak{p}}$ and $\mathfrak{n}_2 = R \times \mathfrak{p}R_{\mathfrak{p}}$. Lemma 4.4 (i) implies that $\mathfrak{p}_1 := \varphi^{-1}(\mathfrak{n}_1) = \mathfrak{m}$ and $\mathfrak{p}_2 := \varphi^{-1}(\mathfrak{n}_2) = \mathfrak{p}$. Thus the homomorphism $\varphi_{\mathfrak{n}_i} \colon R_{\mathfrak{p}_i} \to S_{\mathfrak{n}_i}$ given by $x/1 \mapsto (x, x/1)$ is well-defined. Now by Lemma 4.4 (ii) the map $\varphi_{\mathfrak{n}_i} \colon R_{\mathfrak{p}_i} \to S_{\mathfrak{n}_i}$ is an isomorphism. Thus $\varphi_{\mathfrak{n}_i}$ satisfies $(\dagger)$ for $i = 1, 2$, and hence φ satisfies $(\hat{\dagger})$.

To see that φ does not satisfy $(\tilde{\dagger})$ notice that $R \times \mathfrak{p}R_{\mathfrak{p}} \in \text{m-Spec}(R \times R_{\mathfrak{p}})$ and $\varphi^*(R \times \mathfrak{p}R_{\mathfrak{p}}) = \mathfrak{p}$ which is not in $\text{m-Spec}(R)$.

Remark 4.7. We do not know whether or not $(\dagger')$ implies $(\hat{\dagger})$.

To obtain the results of Theorem 1.17 in the non-local setting we need $(\tilde{\dagger})$; see Theorem 4.22. It is worth noting that not all of the lemmata used in the proof of Theorem 4.22 require the full strength of $(\tilde{\dagger})$.

4.2. Non-Local Lemmata

This section is dedicated to the lemmata required for the proof of Theorem 4.22. We begin with a non-local version of Nakayama's Lemma.

Lemma 4.8. *Let A be a commutative ring. If T is a finitely generated A-module such that $T = \mathfrak{m}T$ for all $\mathfrak{m} \in \text{m-Spec}(A)$, then $T = 0$.*

Proof. Let $\mathfrak{m} \in \text{m-Spec}(A)$ be arbitrary and suppose $T = \mathfrak{m}T$. Localizing we have $T_\mathfrak{m} = \mathfrak{m}T_m$. Since $\mathfrak{m}$ was arbitrary we have $T_\mathfrak{m} = \mathfrak{m}T_\mathfrak{m}$ for all $\mathfrak{m} \in \text{m-Spec}(A)$. Thus by Nakayama's Lemma [2, Proposition 2.6] we have $T_\mathfrak{m} = 0$ for all $\mathfrak{m} \in \text{m-Spec}(A)$. Therefore, by [2, Proposition 3.8] we have $A = 0$. □

The next result is a non-local version of [10, Lemma 1.1].

Lemma 4.9. *Let $\varphi\colon R \to S$ be a faithfully flat ring homomorphism such that the induced map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism for all $\mathfrak{m} \in \text{m-Spec}\, R$. Fix $\mathfrak{m}_0 \in \text{m-Spec}\, R$ and set $\mathfrak{n} = \mathfrak{m}_0 S$. For each $t \geqslant 1$ we have $\varphi(R) + \mathfrak{n}^t = S$.*

Proof. First notice that $S/\mathfrak{n}^{t+1}$ is finitely generated over R. To see this consider the descending chain

$$S/\mathfrak{n}^{t+1} \supset \mathfrak{n}/\mathfrak{n}^{t+1} \supset \mathfrak{n}^2/\mathfrak{n}^{t+1} \supset \cdots \supset \mathfrak{n}^t/\mathfrak{n}^{t+1} \supset 0$$

and notice that the ith quotient is isomorphic to $\mathfrak{n}^{i-1}/\mathfrak{n}^i \cong (S/\mathfrak{n})^{\alpha_i} \cong (R/\mathfrak{m}_0)^{\alpha_i}$ for some $\alpha_i \geqslant 0$. We claim that

$$\frac{\varphi(R) + \mathfrak{n}^t}{\mathfrak{n}^{t+1}} + \mathfrak{m}\frac{S}{\mathfrak{n}^{t+1}} = \frac{S}{\mathfrak{n}^{t+1}},$$

for all $\mathfrak{m} \in \text{m-Spec}\, R$. Notice that if $\mathfrak{m} \neq \mathfrak{m}_0$, then $\mathfrak{m}$ and $\mathfrak{m}_0$ are coprime, so

$\mathfrak{m}(S/\mathfrak{n}^{t+1}) = S/\mathfrak{n}^{t+1}$. In the case where $\mathfrak{m}_0 = \mathfrak{m}$ we have

$$\begin{aligned} \frac{\varphi(R)+\mathfrak{n}^t}{\mathfrak{n}^{t+1}} + \mathfrak{m}_0\frac{S}{\mathfrak{n}^{t+1}} &= \frac{\varphi(R)+\mathfrak{n}^t}{\mathfrak{n}^{t+1}} + \frac{\mathfrak{m}_0 S+\mathfrak{n}^{t+1}}{\mathfrak{n}^{t+1}} \\ &= \frac{\varphi(R)+\mathfrak{n}^t+\mathfrak{m}_0 S}{\mathfrak{n}^{t+1}} \\ &= \frac{\varphi(R)+\mathfrak{n}}{\mathfrak{n}^{t+1}} \\ &= \frac{S}{\mathfrak{n}^{t+1}} \end{aligned}$$

as claimed.

Since $S/\mathfrak{n}^{t+1}$ is finitely generated over R,we can apply Lemma 4.8 to M/N with $M = S/\mathfrak{n}^{t+1}$ and $N = (\varphi(R)+\mathfrak{n}^t)/\mathfrak{n}^{t+1}$ to obtain

$$\frac{\varphi(R)+\mathfrak{n}^t}{\mathfrak{n}^{t+1}} = \frac{S}{\mathfrak{n}^{t+1}}.$$

Thus we have $\varphi(R)+\mathfrak{n}^t = S$ for each $t \geqslant 1$. $\square$

The next result is a non-local version of Krull's Intersection Theorem.

Lemma 4.10. *Let A be a commutative noetherian ring and let N be a finitely generated A-module. Then $\bigcap_{\mathfrak{n}\in \text{m-Spec}(A)} \bigcap_{t\geqslant 1} \mathfrak{n}^t N = 0$.*

Proof. Let $n \in \bigcap_{\mathfrak{n}\in \text{m-Spec}(S)} \bigcap_{t\geqslant 1} \mathfrak{n}^t N$. Then $n \in \mathfrak{n}^t N$ for all $t \geqslant 1$ and for all $\mathfrak{n} \in \text{m-Spec}(S)$. Therefore, $n/1 \in N_\mathfrak{n}$ is contained in $\bigcap_t (\mathfrak{n}A_\mathfrak{n})^t N_\mathfrak{n}$. Now by Krull's Intersection Theorem [15, Theorem 8.9] we have $\bigcap_t (\mathfrak{n}A_\mathfrak{n})^t N_\mathfrak{n} = 0$. Hence $0 = n/1 \in N_\mathfrak{n}$ for all $\mathfrak{n} \in \text{m-Spec}(A)$. Thus, by [2, Proposition 3.8] we obtain $n = 0$. $\square$

The next result is a non-local version of [10, Proposition 1.2].

Lemma 4.11. *Let $\varphi\colon R \to S$ be a faithfully flat ring homomorphism such that the induced map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism for all $\mathfrak{m} \in \text{m-Spec}\, R$, and such that*

$\varphi^*(\text{m-Spec}(S)) \subseteq \text{m-Spec}(R)$. *If M and N are S-modules with N finitely generated, then* $\operatorname{Hom}_R(M,N) = \operatorname{Hom}_S(M,N)$.

Proof. The containment $\operatorname{Hom}_R(M,N) \supseteq \operatorname{Hom}_S(M,N)$ always holds so we only show the other containment. Let $f \in \operatorname{Hom}_R(M,N), m \in M$ and $s \in S$. We want to show that $sf(m) = f(sm)$. By Lemma 4.10, it suffices to show

$$sf(m) - f(sm) \in \bigcap_{\mathfrak{n} \in \text{m-Spec}(S)} \bigcap_t \mathfrak{n}^t N.$$

Fix an integer $t \geqslant 1$. Let $\mathfrak{n} \in \text{m-Spec}(S)$ and set $\mathfrak{m} = \varphi^*(\mathfrak{n})$. Notice that we have the following: $f(\mathfrak{n}^t M) = f(\mathfrak{m}^t M) = \mathfrak{m}^t f(M) \subseteq \mathfrak{n}^t N$. Use Lemma 4.9 to choose an element $r \in R$ such that $\varphi(r) - s \in \mathfrak{n}^t$. Then we have

$$\begin{aligned} f(sm) - sf(m) &= f(sm) - f(rm) + rf(m) - sf(m) \\ &= f((s - \varphi(r))m) + (\varphi(r) - s)f(m). \end{aligned}$$

Hence $f(sm) - sf(m) \in f((s-\varphi(r))M) + (\varphi(r)-s)N \subseteq f(\mathfrak{n}^t M) + \mathfrak{n}^t N = \mathfrak{n}^t N$. Since t and $\mathfrak{n}$ are arbitrary, we have $f(sm) - sf(m) \in \bigcap_{\mathfrak{n} \in \text{m-Spec}(S)} \bigcap_t \mathfrak{n}^t N = 0$. $\square$

The next result is a non-local version of [10, Lemma 1.4].

Lemma 4.12. *Let $\varphi\colon R \to S$ be a faithfully flat ring homomorphism such that the induced map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism for all $\mathfrak{m} \in \operatorname{m-Spec} R$, and such that $\varphi^*(\text{m-Spec}(S)) \subseteq \text{m-Spec}(R)$. Let N be a finitely generated S-module and let V be an R-submodule of N. Then ${}_RV$ has at most one compatible S-module structure. Specifically, if V has an S-module structure $(s,v) \mapsto s \circ v$ that is compatible with its R-module structure on V inherited from the S-module structure $(s,n) \mapsto s \cdot n$ on N, then $s \circ v = s \cdot v$ for all $s \in S$ and for all $v \in V$.*

Proof. Let $s \in S$ and $v \in V$ be given. Fix $t \geqslant 1$ and $\mathfrak{n} \in \text{m-Spec}(S)$. Use Lemma 4.9 to choose $r \in R$ such that $\varphi(r) - s \in \mathfrak{n}^t$ and set $\mathfrak{m} = \varphi^{-1}(\mathfrak{n})$. Notice that we have the following:

$$\begin{aligned}
\mathfrak{n}^t \circ V = (\mathfrak{m}^t S) \circ V &= \mathfrak{m}^t \circ (S \circ V) \\
&= \mathfrak{m}^t \circ V = \mathfrak{m}^t \cdot V \subseteq \mathfrak{m}^t \cdot N = \mathfrak{n}^t \cdot N.
\end{aligned} \tag{6}$$

Furthermore, we have

$$s \circ v - s \cdot v = s \circ v - r \circ v + r \cdot v - s \cdot v = (s - \varphi(r)) \circ v + (\varphi(r) - s) \cdot v.$$

Thus display (6) implies that $s \circ v - s \cdot v \in \mathfrak{n}^t \circ V + \mathfrak{n}^t \cdot V \subseteq \mathfrak{n}^t \cdot N$. Since t and $\mathfrak{n}$ are arbitrary then by Lemma 4.10 we have $s \circ v - s \cdot v = 0$. $\square$

The next result is a non-local version of [10, Proposition 1.5].

Lemma 4.13. *Let $\varphi \colon R \to S$ be a faithfully flat ring homomorphism such that the induced map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism for all $\mathfrak{m} \in \text{m-Spec}\, R$, and such that $\varphi^*(\text{m-Spec}(S)) \subseteq \text{m-Spec}(R)$. Let M be an R-module (not necessarily finitely generated) that is an R-submodule of some finitely generated S-module N. Let $\mathcal{V}(M)$ be the set of R-submodules of M that have S-module structures compatible with their R-module structures. Then $\mathcal{V}(M)$ is exactly the set of S-submodules of N that are contained in M. The set $\mathcal{V}(M)$ has a unique maximal element $V(M)$ and $V(M) = \{x \in M \mid Sx \subseteq M\} = \{x \in N \mid Sx \subseteq M\}$.*

Proof. Let $\mathcal{W}(M)$ be the set of S-submodules of N that are contained in M. For the first assertion let $A \in \mathcal{V}(M)$. That is, A is an R-submodule of M that has an S-module structure compatible with its R-module structure. In particular A is an S-module contained in M. Thus A is in $\mathcal{W}(M)$. For the other containment let

$B \in \mathcal{W}(M)$. Therefore B is contained in M and B is an S-module, and hence an R-module by restriction of scalars. Thus the S-module structure on B is compatible with the R-module structure obtained by restriction of scalars. Hence $B \in \mathcal{V}(M)$.

Now since N is a noetherian S-module, the set of S-submodules contained in M must have a maximal element $V(M)$. To see that $V(M)$ is unique suppose C is another maximal element, then $C + V(M)$ is another S-submodule in $\mathcal{V}(M)$ since $\mathcal{V}(M)$ is closed under sums. But then $V(M) \subset C + V(M)$, and by the maximality of $V(M)$ and C we must have $C = V(M)$.

Let $x \in V(M)$. Then the S-submodule $Sx \subseteq N$, is contained in M. Which shows

$$V(M) \subseteq \{x \in M \mid Sx \subseteq M\} \subseteq \{x \in N \mid Sx \subseteq M\} \subseteq V(M)$$

Where the last containment follows from $V(M)$ being the unique maximal element of $\mathcal{V}(M)$. Therefore all the containments are equalities. $\square$

The next result is a non-local version of [10, Proposition 1.6].

Lemma 4.14. *Let $\varphi\colon R \to S$ be a faithfully flat ring homomorphism such that the induced map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism for all $\mathfrak{m} \in \text{m-Spec}\, R$, and such that $\varphi^*(\text{m-Spec}(S)) \subseteq \text{m-Spec}(R)$. Let L be an S-module (not necessarily finitely generated). Let M be an R-submodule of some finitely generated S-module N, and let $V(M)$ be as in Lemma 4.13. Then the natural injection $\operatorname{Hom}_R(L, V(M)) \to \operatorname{Hom}_R(L, M)$ is an isomorphism.*

Proof. Let $g \in \operatorname{Hom}_R(L, M)$ and let $W = \operatorname{Im}(g)$. We need to show that $W \subseteq V(M)$. Let h be the composition $L \xrightarrow{g} M \xrightarrow{\subseteq} N$. Lemma 4.11 implies that h is S-linear, so $W = h(L)$ is an S-submodule of N, and therefore $W \subseteq V(M)$. $\square$

The next result is a non-local version of [10, Corollary 1.7].

Corollary 4.15. *Let $\varphi\colon R \to S$ be a faithfully flat ring homomorphism such that the induced map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism for all $\mathfrak{m} \in \operatorname{m-Spec} R$, and such that $\varphi^*(\text{m-Spec}(S)) \subseteq \text{m-Spec}(R)$. Let M be an R-submodule of a finitely generated S-module, and let $V(M)$ be as in Lemma 4.13. Then there is a commutative diagram of R-module homomorphisms.*

$$\begin{tikzcd}
\operatorname{Hom}_S(S, V(M)) \arrow[r, "\alpha", "\cong"'] \arrow[d, "\gamma"', "="] & V(M) \arrow[r, "\beta", "\subseteq"'] & M \\
\operatorname{Hom}_R(S, V(M)) \arrow[rr, "\cong", "\delta"'] & & \operatorname{Hom}_R(S, M) \arrow[u, "\epsilon"']
\end{tikzcd} \tag{7}$$

So $V(M)$ is the image of the natural map $\epsilon\colon \operatorname{Hom}_R(S, M) \to M$ given by $f \mapsto f(1)$ and ϵ is injective. Moreover, if M is finitely generated over R then so is $\operatorname{Hom}_R(S, M)$.

Proof. The isomorphism α is Hom cancellation, the equality γ is from Lemma 4.11, and the isomorphism δ is from Lemma 4.14. To see that the diagram commutes, let $\phi \in \operatorname{Hom}_S(S, V(M))$ then we have $\beta(\alpha(\phi)) = \beta(\phi(1)) = \phi(1)$ and $\epsilon(\delta(\gamma(\phi))) = \epsilon(\delta(\phi)) = \epsilon(\phi) = \phi(1)$.

Since each of the maps α, β, and γ are injective and the map δ is an isomorphism we have that ϵ is injective. Now if M is finitely generated over R, then so is the submodule $V(M) \cong \operatorname{Hom}_R(S, M)$. $\square$

4.3. Non-Local Main Results

This section contains the proof of our main non-local result, Theorem 4.22. We begin with the following non-local version of Definition 1.18.

Definition 4.16. An R-module N satisfies NAK if either $N = 0$ or there exists a maximal ideal $\mathfrak{m} \in \text{m-Spec}(R)$ such that $N/\mathfrak{m}N \neq 0$.

Note that Lemma 4.8 implies that every finitely generated R-module satisfies NAK. The next result is a non-local version of Lemma 2.4

Proposition 4.17. *Let $\varphi\colon R \to S$ be a faithfully flat ring homomorphism such that the induced map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism for all $\mathfrak{m} \in \operatorname{m-Spec} R$, and such that $\varphi^*(\operatorname{m-Spec}(S)) \subseteq \operatorname{m-Spec}(R)$. Let M be a finitely generated R-module, and let $z \geqslant 1$. If $\operatorname{Ext}^i_R(S, M) = 0$ for all $i > z$ and $\operatorname{Ext}^z_R(S, M)$ satisfies NAK, then we have $\operatorname{Ext}^z_R(S, M) = 0$.*

Proof. By way of contradiction suppose $\operatorname{Ext}^z_R(S, M) \neq 0$. Since $\operatorname{Ext}^z_R(S, M)$ satisfies NAK, there exists $\mathfrak{m} \in \operatorname{m-Spec}(R)$ such that $\operatorname{Ext}^z_R(S, M)/\mathfrak{m} \operatorname{Ext}^z_R(S, M) \neq 0$. Let $\mathbf{x} = x_1, \ldots, x_n \in R$ such that $(\mathbf{x})R = \mathfrak{m}$. The proof of Lemma 2.4 applies mutatis mutandis to show $\operatorname{Ext}^z_R(S, M)/\mathfrak{m} \operatorname{Ext}^z_R(S, M) = 0$, a contradiction. $\square$

Lemma 4.18. *Let $\varphi\colon R \to S$ be a faithfully flat ring homomorphism such that the induced map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism for all $\mathfrak{m} \in \operatorname{m-Spec} R$, and such that $\varphi^*(\operatorname{m-Spec}(S)) \subseteq \operatorname{m-Spec}(R)$. Then an R-module N has a compatible S-module structure if and only if the natural map $\iota\colon N \to S \otimes_R N$ is an isomorphism.*

Proof. One implication is clear. For the forward implication let $(s, x) \to s \cdot x$ be a compatible S-module structure on N. Notice that $S \otimes_R N$ has two compatible S-module structures, one from the multiplication in S, and the second from the S-module structure on N. With the first structure $S \otimes_R N$ is finitely generated over S. Hence $S \otimes_R N$ satisfies the hypotheses of Lemma 4.12, so these two S-module structures must be the same. Therefore, $1 \otimes (sx) = s(1 \otimes x) = s \otimes x$.

We claim that the multiplication map $\mu\colon S \otimes_R N \to N$ given by $s \otimes x \mapsto sx$ is the inverse of ι, which will prove the claim. In fact we have

$$(\mu \circ \iota)(x) = \mu(\iota(x)) = \mu(1 \otimes x) = x.$$

Hence we have $\mu \circ \iota = \mathrm{id}_N$. Also we have $\iota \circ \mu = \mathrm{id}_{S \otimes_R N}$ since,

$$(\iota \circ \mu)(s \otimes x) = \iota(sx) = 1 \otimes sx = s \otimes x.$$

Where the third equality is from the previous paragraph. $\square$

The next result is a non-local version of [10, Lemma 2.12]

Lemma 4.19. *Let $\varphi\colon R \to S$ be a faithfully flat ring homomorphism such that the induced map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism for all $\mathfrak{m} \in \operatorname{m-Spec} R$, and such that $\varphi^*(\text{m-Spec}(S)) \subseteq \text{m-Spec}(R)$. Let*

$$0 \to M' \to M \to M'' \to 0$$

be an exact sequence of finitely generated R-modules. Then M has a compatible S-module structure if and only if M' and M'' have compatible S-module structures.

Proof. Applying $S \otimes_R -$ to the given exact sequence, and using the flatness of S, we obtain the following commutative diagram with exact rows

$$\begin{tikzcd}
0 \arrow[r] & M' \arrow[r] \arrow[d, "\epsilon_1"] & M \arrow[r] \arrow[d, "\epsilon_2"] & M'' \arrow[r] \arrow[d, "\epsilon_3"] & 0 \\
0 \arrow[r] & S \otimes_R M' \arrow[r] & S \otimes_R M \arrow[r] & S \otimes_R M'' \arrow[r] & 0.
\end{tikzcd} \tag{8}$$

Note that each map ϵ_i is injective for $i = 1, 2, 3$. Thus by the Snake Lemma we see that ϵ_2 is an isomorphism if and only if ϵ_1 and ϵ_3 are isomorphisms. Thus M has a compatible S-module structure if and only if both M' and M'' have compatible S-module structures by Lemma 4.18. $\square$

Lemma 4.20. *Let $\varphi\colon R \to S$ be a flat ring homomorphism. Then $\operatorname{pd}_R(S) \leqslant \dim(R)$.*

Proof. This inequality follows from a result of Raynaud and Grunson [18, Seconde Partie, Theorem (3.2.6)] and Jensen [14, Proposition 6]. $\square$

Lemma 4.21. *Let $\varphi\colon R \to S$ be a faithfully flat ring homomorphism such that the induces map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism for all $\mathfrak{m} \in \operatorname{m-Spec} R$, and such that $\varphi^*(\text{m-Spec}(S)) \subseteq \text{m-Spec}(R)$. If N is an S-module then N satisfies NAK over R if and only if N satisfies NAK over S.*

Proof. If $N = 0$ then N satisfies NAK over R and over S. So we assume for the rest of the proof that $N \neq 0$.

($\Rightarrow$) Suppose N satisfies NAK over R. Then there exists a maximal ideal $\mathfrak{m} \subset R$ such that $N/\mathfrak{m}N \neq 0$. Note that the extension $\mathfrak{m}S$ is a maximal ideal in S because $R/\mathfrak{m} \to S/\mathfrak{m}S$ for all $\mathfrak{m} \in \text{m-Spec}(R)$. Set $\mathfrak{n} = \mathfrak{m}S$ and notice that $N/\mathfrak{n}M = N/\mathfrak{m}SN = N/\mathfrak{m}N \neq 0$. Therefore N satisfies NAK over S.

($\Leftarrow$) Suppose N satisfies NAK over S. Hence there exists a maximal ideal $\mathfrak{n} \subset S$ such that $N/\mathfrak{n}N \neq 0$. Set $\mathfrak{m} = \varphi^*(\mathfrak{n})$ which is a maximal ideal in R by the assumptions on φ. Then we have $0 \neq N/\mathfrak{n}M = N/\mathfrak{m}SN = N/\mathfrak{m}N$. Therefore N satisfies NAK over R. $\square$

The next result is a non-local version of [10, Main Theorem 2.5].

Theorem 4.22. *Let $\varphi\colon R \to S$ be a faithfully flat ring homomorphism such that the induces map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism for all $\mathfrak{m} \in \operatorname{m-Spec} R$, and such that $\varphi^*(\text{m-Spec}(S)) \subseteq \text{m-Spec}(R)$. Let M be a finitely generated R-module. Then the following conditions are equivalent:*

(i) *M has a compatible S-module structure.*

(ii) *the natural map $\iota\colon M \to S \otimes_R M$, $(x \mapsto 1 \otimes x)$ is bijective.*

(iii) *the natural map $\epsilon\colon \operatorname{Hom}_R(S, M) \to M$, $(f \mapsto f(1))$ is bijective.*

(iv) $S \otimes_R M$ *is finitely generated as an* R*-module.*

(v) $\mathrm{Ext}^i_R(S, M)$ *satisfies NAK over* R *for* $i = 1, \ldots, \dim_R(M)$.

(vi) $\mathrm{Ext}^i_R(S, M)$ *satisfies NAK over* S *for* $i = 1, \ldots, \dim_R(M)$.

(vii) $\mathrm{Ext}^i_R(S, M) = 0$ *for all* $i \neq 0$.

Proof. First notice that (iii) $\Rightarrow$ (i), (ii) $\Rightarrow$ (iv), (vii) $\Rightarrow$ (v), and (vii) $\Rightarrow$ (vi) follow easily. The equivalence of (i) $\Leftrightarrow$ (ii) is in Lemma 4.19. The equivalence of (v) $\Leftrightarrow$ (vi) is in Lemma 4.21.

(i) $\Rightarrow$ (iii). Note that since M is finitely generated as an S-module, Lemma 4.11 implies that $\mathrm{Hom}_R(S, M) = \mathrm{Hom}_S(S, M) \cong M$. Thus the map $\theta \colon M \to \mathrm{Hom}_R(S, M)$ given by $m \mapsto \{f_m \colon S \to M\}$ with $f_m(s) = sm$, is the inverse of ϵ.

(iv) $\Rightarrow$ (ii). Supposing (iv) we know that $S \otimes_R S \otimes_R M$ is finitely generated for the S-action in the first position. Now $S \otimes_R S \otimes_R M$ has two S-module structures, one for each of the first two positions. Let $\cdot$ denote the S-module structure in the first position and let $\circ$ denote the S-module structure in the second position. That is for $q, s, t \in S$, and $x \in M$

$$q \cdot (s \otimes t \otimes x) := (qs) \otimes t \otimes x,$$

and

$$q \circ (s \otimes t \otimes x) := s \otimes (qt) \otimes x.$$

Since (iv) implies that $S \otimes_R S \otimes_R M$ is finitely generated over S, then by Lemma 4.12 these two structures are the same:

$$(qs) \otimes t \otimes x = q \cdot (s \otimes t \otimes x) = q \circ (s \otimes t \otimes x) = s \otimes (qt) \otimes x.$$

We claim that the map $\beta\colon S \otimes_R S \otimes_R M \to S \otimes_R M$ given by $s \otimes t \otimes m \mapsto st \otimes m$ is the inverse of $1 \otimes \iota\colon S \otimes_R M \to S \otimes_R S \otimes_R M$. In fact we have

$$\begin{aligned}((1 \otimes \iota) \circ \beta)(s \otimes t \otimes x) &= (1 \otimes \iota)(st \otimes x)\\ &= (st) \otimes 1 \otimes x\\ &= s \otimes t \otimes x.\end{aligned}$$

Hence $(1 \otimes \iota) \circ \beta = \mathrm{id}_{S \otimes_R S \otimes_R M}$. We also have $\beta \circ (1 \otimes \iota) = \mathrm{id}_{S \otimes_R M}$ since,

$$(\beta \circ (1 \otimes \iota))(s \otimes x) = \beta(s \otimes 1 \otimes x) = s \otimes x.$$

Since φ is faithfully flat, we have ι is an isomorphism.

(v) $\Rightarrow$ (vii). First recall that $\mathrm{Ext}^i_R(S, M) = 0$ for every $i > \dim_R(M)$ by Lemma 4.20 and the proof of [10, Main Theorem 2.5]. We proceed by contradiction. Suppose that there exists a $j \geqslant 1$ such that $\mathrm{Ext}^j_R(S, M) \neq 0$, and set $m = \max\{i \mid \mathrm{Ext}^i_R(S, M) \neq 0\}$. Notice that $1 \leqslant m \leqslant \dim_R(M)$. Therefore, by Proposition 4.17 we have $\mathrm{Ext}^m_R(S, M) = 0$, a contradiction. Hence $\mathrm{Ext}^i_R(S, M) = 0$ for all $i \neq 0$.

(vii) $\Rightarrow$ (iii). Let J be an injective resolution of M and let α be the morphism $\alpha\colon \mathrm{Hom}_R(S, J) \to J$ given by $f \mapsto f(1)$. The long exact sequence in homology associated to $\mathrm{Cone}(\alpha)$ gives the following exact sequence

$$0 \to \mathrm{H}_1(\mathrm{Cone}(\alpha)) \to \mathrm{Hom}_R(S, M) \xrightarrow{\epsilon} M \to \mathrm{H}_0(\mathrm{Cone}(\alpha)) \to \mathrm{Ext}^1_R(S, M) \to 0.$$

Now $\mathrm{H}_1(\mathrm{Cone}(\alpha)) = 0$ because we know that ϵ is injective by Corollary 4.15. So we need to show that $\mathrm{H}_0(\mathrm{Cone}(\alpha)) = 0$. We proceed by contradiction. Assume that $0 \neq \mathrm{H}_0(\mathrm{Cone}(\alpha))$. Notice that by assumption (vii) we have $\mathrm{Ext}^1_R(S, M) = 0$. Hence from the sequence above that we obtain $\mathrm{H}_0(\mathrm{Cone}(\alpha)) \cong M/\operatorname{Im}(\epsilon)$. Set $N =$

$M/\operatorname{Im}(\epsilon)$. Also notice that since N is finitely generated over R, it satisfies NAK over R by Lemma 4.21. Thus there exists an ideal $\mathfrak{m} \in \text{m-Spec}(R)$ such that $N/\mathfrak{m}N \neq 0$. Let $\mathbf{x} = (x_1, \ldots, x_k)R = \mathfrak{m}$. Following the proof of Lemma 2.4 we have a quasi-isomorphism $\operatorname{Hom}_R(K^R(\mathbf{x}), \alpha) \colon \operatorname{Hom}_R(K^R(\mathbf{x}), \operatorname{Hom}_R(S, J)) \to \operatorname{Hom}_R(K^R(\mathbf{x}), J)$, so we have $\operatorname{Cone}(\operatorname{Hom}_R(K^R(\mathbf{x}), \alpha))$ is exact. The fact that $K^R(\mathbf{x})$ is a self-dual and bounded complex of finitely generated free R-modules implies that we have an isomorphism of complexes $\mathsf{\Sigma}^{-n} K^R(\mathbf{x}) \otimes_R \operatorname{Cone}(\alpha) \cong \operatorname{Cone}(\operatorname{Hom}_R(K^R(\mathbf{x}), \alpha))$. It follows that $K^R(\mathbf{x}) \otimes_R \operatorname{Cone}(\alpha)$ is exact. We claim that $\mathrm{H}_i(\operatorname{Cone}(\alpha)) = 0$ for all $i < 0$. Consider the following portion of the long exact sequence in homology associated to $\operatorname{Cone}(\alpha)$

$$\cdots \to \mathrm{H}_i(\operatorname{Hom}_R(S, M)) \to \mathrm{H}_i(J) \to \mathrm{H}_i(\operatorname{Cone}(\alpha)) \to \mathrm{H}_{i-1}(\operatorname{Hom}_R(S, J)) \to \cdots .$$

Notice that we have $\mathrm{H}_i(J) = 0$ for all $i \neq 0$ and by our Ext vanishing assumption $\mathrm{H}_i(\operatorname{Hom}_R(S, J)) \cong \operatorname{Ext}_R^{-i}(S, M) = 0$ for all $i \neq 0$. Thus we have $\mathrm{H}_i(\operatorname{Cone}(\alpha)) = 0$ for all $i < 0$. Applying Lemma 2.1 we have the following isomorphism:

$$0 = \mathrm{H}_0(K^R(\mathbf{x}) \otimes_R \operatorname{Cone}(\alpha)) \cong \mathrm{H}_0(\operatorname{Cone}(\alpha))/(\mathbf{x})\,\mathrm{H}_0(\operatorname{Cone}(\alpha)) \neq 0,$$

which is a contradiction. Thus, we have $\mathrm{H}_0(\operatorname{Cone}(\alpha)) = 0$ and ϵ is an isomorphism.

(i) $\Rightarrow$ (vii). (As we noted above, assumption (i) guarantees condition (ii).) Since M is finitely generated over R, it admits a filtration by R-submodules

$$0 = M_0 \subset M_1 \subset \cdots \subset M_t = M$$

such that $M_i/M_{i-1} \cong R/\mathfrak{p}_i$ for each $i = 1, \ldots, t$. We proceed by induction on t. For the base case $t = 1$, we have $M \cong R/\mathfrak{p}_1$, for some $\mathfrak{p}_1 \in \operatorname{Spec}(R)$. Now (ii) implies that

$M \cong S \otimes_R M$, so in this case we have $R/\mathfrak{p}_1 \cong S \otimes_R R/\mathfrak{p}_1$. Let Q be an R-projective resolution of S. Since S is flat, the complex $Q \otimes_R R/\mathfrak{p}_1$ is an $R/\mathfrak{p}_1$-projective resolution of $S \otimes_R R/\mathfrak{p}_1 \cong R/\mathfrak{p}_1$. Now

$$\begin{aligned} \operatorname{Hom}_R(Q, R/\mathfrak{p}_1) &\cong \operatorname{Hom}_R(Q, \operatorname{Hom}_{R/\mathfrak{p}_1}(R/\mathfrak{p}_1, R/\mathfrak{p}_1)) \\ &\cong \operatorname{Hom}_{R/\mathfrak{p}_1}(Q \otimes_R R/\mathfrak{p}_1, R/\mathfrak{p}_1). \end{aligned} \tag{9}$$

Therefore, $\operatorname{Ext}^i_R(S, M) \cong \operatorname{Ext}^i_R(S, R/\mathfrak{p}_1) \cong \operatorname{Ext}^i_{R/\mathfrak{p}_1}(R/\mathfrak{p}_1, R/\mathfrak{p}_1) = 0$ for $i \neq 0$.

Now for $t > 1$ assume the implication holds for each R-module M' admitting a prime filtration with fewer than t links. Consider the following exact sequence of R-modules:

$$0 \to M_{t-1} \to M \to M/M_{t-1} \to 0. \tag{10}$$

By Lemma 4.19 both M_{t-1} and M/M_{t-1} have S-module structures that are compatible with their R-module structures via φ. Also, both have filtrations that have less than t links, hence both M_{t-1} and M/M_{t-1} satisfy the induction hypothesis. Therefore, we have $\operatorname{Ext}^i_R(S, M_{t-1}) = 0 = \operatorname{Ext}^i_R(S, M/M_{t-1})$ for all $i > 0$. Now when we take the long exact sequence in $\operatorname{Ext}_R(S, -)$ associated to (10), we obtain $\operatorname{Ext}^i_R(S, M) = 0$ for all $i > 0$ as desired. $\square$

CHAPTER 5. FUTURE RESEARCH GOALS

One can continue the investigation into Question 1.16 by relaxing the hypothesis on φ in Theorem 1.17. For instance, what conditions do we need on an R-module M for the conclusions of Theorem 1.17 to hold when $R/\mathfrak{m} \to S/\mathfrak{m}S$ is assumed to be a finite field extension? One condition that is required is for the number of generators for M to be a multiple of the degree of the extension. An example of this is the field extension of $\mathbb{R}$ to $\mathbb{C}$, with $M = \mathbb{R}^{2n}$, for some integer n.

Another line of inquiry into Question 1.16 is to relax the finitely generated condition on the R-module M. That is, if we assume that M is not finitely generated what other conditions are required to attain the results of Theorem 1.17? For instance, if we assume that M is mini-max over R and that $R/\operatorname{Ann}_R(M) \cong S/\operatorname{Ann}_R(M)S$ then we have $M \cong S \otimes_R M$. Although it is not known if the other conditions of Theorem 1.17 hold.

Also one can continue the investigation into Question 1.21 in the non-local setting. That is, given a ring homomorphism $\varphi\colon R \to S$ faithfully flat such that the induced map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism, and for all $\mathfrak{m} \in \text{m-Spec}\, R$ the induced map $\varphi^*\colon \operatorname{Spec} S \to \operatorname{Spec} R$ satisfies $\varphi^*(\text{m-Spec}\, S) \subseteq \text{m-Spec}\, R$ and given an R-module M that has a compatible S-module structure via φ, what invariants computed over R are equal when computed over S?

One can also inquire about other non-local results from [10]. For instance given a ring homomorphism $\varphi\colon R \to S$ faithfully flat such that the induced map $R/\mathfrak{m} \to S/\mathfrak{m}S$ is an isomorphism, and for all $\mathfrak{m} \in \text{m-Spec}\, R$ the induced map $\varphi^*\colon \operatorname{Spec} S \to \operatorname{Spec} R$ satisfies $\varphi^*(\text{m-Spec}\, S) \subseteq \text{m-Spec}\, R$ and given a finitely generated R-module M, then does M have a compatible S-module structure if and only if $S = R + \mathfrak{p}S$ for every $\mathfrak{p} \in \operatorname{Min}_R(M)$ and/or for every $\mathfrak{p} \in \operatorname{Supp}_R(M)$?

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