ALMOST DEDEKIND DOMAINS AND ATOMICITY

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ABSTRACT

The objective of this dissertation was to determine the class of domains that are both almost Dedekind and atomic. To investigate this question we constructed a global object called the norm, and used it to determine properties that a domain must have to be both atomic and almost Dedekind. Additionally we use topological notions on the spectrum of a domain to determine atomicity. We state some theorems with regard to ACCP and class groups.

The lemmas and theorems in this dissertation answer in part the objective. We conclude with a chapter of future study that aims to approach a complete answer to the objective.

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CHAPTER 1. BACKGROUND

In this dissertation we will explore factorization in almost Dedekind domains. An almost Dedekind domain is a generalization of a Dedekind domain. There are several equivalent ways of defining a Dedekind domain, and in this paper we will focus our attention on two of these definitions. In the first two chapters we will provide the reader with background and definitions. We will motivate the study of factorization in almost Dedekind domains by discussing factorization in Dedekind domains. We will always assume R is commutative ring with identity.

1.1. Dedekind domains

Definition 1.1.1. *We say an integral domain D is Dedekind if every proper nonzero ideal of D factors (uniquely) as a product of prime ideals.*

Now while this definition gives a description of how Dedekind domains behave with respect to ideal factorization, we will be more interested in how domains behave with respect to factorization on an elemental level.

Definition 1.1.2. *Let D be an integral domain. We say $u \in D$ is a unit if there exists $w \in D$ such that $uw = 1$.*

Definition 1.1.3. *Let D be an integral domain. We say $\alpha \in D$ is an an atom (irreducible) if $\alpha = bc$ implies that b or c is a unit.*

Definition 1.1.4. *We say an integral domain D is atomic if every nonzero non-unital element $d \in D$ can be written as a product of atoms (irreducibles).*

The ideal factorization in a Dedekind domain leads to an elemental factorization. This leads to a well known fact (Theorem 1.1.6) about Dedekind domains.

Definition 1.1.5. *A domain is atomic if every nonzero non-unit can be written as a finite product of atoms.*

Theorem 1.1.6. *Every Dedekind domain is an atomic domain.*

This result about Dedekind domain motivates our study of atomicity of almost Dedekind domains. To motivate the generalization of Dedekind domains to almost Dedekind domains we present another definition of Dedekind, but we first introduce the notion of Noetherian.

Definition 1.1.7. *We say a domain D is Noetherian if every increasing chain of ideals stabilizes. Equivalently a domain D is Noetherian if every ideal is finitely generated.*

We let $\mathrm{Max}(D)$ denote the set of maximal ideals of the domain D. A Noetherian domain is a domain in which every non-empty set of ideals has a maximal element. We denote the localization of D at a maximal ideal M by D_M. We will discuss localizations in more detail in the in the next section.

Definition 1.1.8. *An integral domain D is Dedekind if it is Noetherian and D_M is a Noetherian valuation domain for all $M \in Max(D)$.*

We are now in a position to define the class of domains known as almost Dedekind domains.

Definition 1.1.9. *We say a domain D is almost Dedekind if D_M is a Noetherian valuation domain for all $M \in Max(D)$.*

We see that the only assumption that is dropped in this generalization is the assumption that the domain is Noetherian. Thus a Dedekind domain is almost Dedekind, and an almost Dedekind domain is Dedekind if and only if it is Noetherian. We have already stated that Dedekind domains are atomic, we will give a new proof of this fact in the third chapter. The question of atomicity in almost Dedekind domains is not so straightforward. We will see that some almost Dedekind domains are indeed atomic, while others are not atomic. We will present theorems that will determine atomicity in almost Dedekind domains. We will also discuss, paying attention to the factorization structure, common classes of almost Dedekind domains that appear in the literature. We will call a domain that is almost Dedekind, but not Dedekind, a purely almost Dedekind domain. For the remainder of this dissertation D will denote a purely almost Dedekind domain, unless otherwise stated.

Definition 1.1.10. *For any commutative ring R the Krull dimension (or dimension) of R is the maximum possible length of a chain $P_0 \subset P_1 \subset P_2 \subset \cdots \subset P_n$ of distinct prime ideals in R. The dimension is said to be infinite if R has arbitrarily long chains of distinct prime ideals.*

Now that we have the notion of dimension we give the classical characterization of almost Dedekind domains.

Definition 1.1.11. *We say an ideal $I \subseteq R$ is primary if $ab \in I$ and $a \notin I$ implies $b^n \in I$ for some positive integer n.*

Theorem 1.1.12. *Let D be a domain. The following are equivalent.*

1. *D is an almost Dedekind domain.*
2. *D is one-dimensional and primary ideals are prime powers.*
3. *If I, J, and H are non-zero ideals of D such that $IH = JH$, then $I = J$. (This is known as the cancellation law for ideals.)*

Proof. See Theorems 36.4 and 36.5 in [7]. □

Since an almost Dedekind domain is one-dimensional it follows that all nonzero prime ideals are maximal. Since maximal ideals are always prime, we will use the terms maximal and prime interchangeably when we are considering almost Dedekind domains. It will be understood that when we say maximal ideal or prime ideal, we are referring to an nonzero maximal ideal. In order to fully understand almost Dedekind domains it is necessary to understand localizations and valuations domains. It is here that we begin our journey.

1.2. Localizations

The idea of "localization" is a very powerful tool that plays a vital role in commutative algebra. The fact that "locally" an almost Dedekind domain is a Noetherian valuation domain will play a pivotal role in our analysis. A localization is essentially a ring of fractions where the set of denominators come from a multiplicatively closed set. (S is a closed multiplicative set if $0 \notin S$, $1 \in S$ and $a, b \in S$ implies $ab \in S$.) Since fractions have multiple representations it is necessary to define an equivalence relation. See 15.4 in [6]. Let S be a closed multiplicatively closed set of a ring R.

Definition 1.2.1. *We define R localized at S as*

$$S^{-1}R = \left\{ \frac{r}{s} \,|\, r \in R \text{ and } s \in S \right\}$$

along with the relation

$$\frac{r}{s} \sim \frac{r'}{s'} \quad \text{if and only if} \quad t(rs' - r's) = 0,$$

for some $t \in S$. We may also call $S^{-1}R$ the ring of fractions of R with respect to S.

We will deal mostly with localizing at a maximal ideal M, which means $S = R \setminus M$. Note maximal ideals are always prime, and the construction we present now is true for more generally for a prime ideal. Let R be a ring with prime ideal P. Now as P is prime $S = R \setminus P$ is a multiplicatively closed set. Thus we can consider $S^{-1}R$. The convention is to denote the localization at of R at P by R_P.

Definition 1.2.2. *A ring that has a unique maximal ideal is called a quasi-local ring.*

Proposition 1.2.3. *For any commutative ring R with 1, let R_P be the localization of R at the prime ideal P.*

(a) The ring R_P is a quasi-local ring.

(b) If R is an integral domain, then R_P is an integral domain.

(c) The prime ideals in R_P are in bijective correspondence with the prime ideals of R contained in P.

For a proof of the proposition see 15.4 in [6].

1.3. Valuation Domains and Valuations

For a given maximal ideal M of an almost Dedekind domain D, we have D_M is a Noetherian valuation domain. This will be used throughout our study. We let $K^\star$ denote the nonzero elements of K, where K is the field of fractions of D.

Definition 1.3.1. *Let K be a field and $K^\star$ denote the nonzero elements of K.*

1. A discrete valuation on a field K is a function $\nu : K^\star \to \mathbb{Z}$ satisfying:

(i) ν is surjective,

(ii) $\nu(xy) = \nu(x) + \nu(y)$ for all $x, y \in K^\star$,

(iii) $\nu(x + y) \geq \min\{\nu(x), \nu(y)\}$ for all $x, y \in K^\star$ with $x + y \neq 0$.

The subring $\{x \in K \mid \nu(x) \geq 0\} \cup \{0\}$ is called the valuation ring associated with ν.

2. An integral domain R is called a Noetherian valuation domain if D is the valuation ring of a discrete valuation ν on a field of fractions of D.

Proposition 1.3.2. *Suppose D is a Noetherian valuation domain with respect to the valuation ν, and let t be any element of D with $\nu(t) = 1$.*

1. A nonzero element $u \in D$ is a unit if and only if $\nu(u) = 0$.

2. *Every nonzero element $b \in D$ can be written in the form $b = ut^n$ for some unit $u \in R$ and some $n \geq 0$. Every nonzero element x in the field of fractions of D can be written in the form $x = ut^n$ for some unit $u \in R$ and some $n \in \mathbb{Z}$.*

3. *Every nonzero ideal of D is a principal ideal of the form (t^n) for some $n \geq 0$. In particular, D is a Noetherian ring.*

Proof. See [6]. $\square$

We now state a fact that will be pivotal in our study.

Fact 1.3.3. *Suppose $\nu_M(a) < \nu_M(b)$, then $\nu_M(a+b) = \nu_M(a)$. In other words the minimum in 1(iii) is realized, if the elements have different valuations.*

Proof. Let $a = ut^k$ and let $b = u't^l$ with $k < l$. Then $a + b = (u + u't^{l-k})t^k$ and note $(u + u't^{l-k})$ must be a unit, else $u \in M$ which is impossible. Thus $\nu_M(a+b) = k = \nu_M(a)$. $\square$

Theorem 1.3.4. *The following properties of a ring D are equivalent:*

1. *D is a discrete valuation domain;*

2. *R is a PID with a unique maximal ideal $P \neq 0$;*

3. *R is a UFD with a unique (up to associates) irreducible element t;*

4. *R is a Noetherian integral domain that is also a local ring whose unique maximal ideal is nonzero and principal;*

5. *R is a Noetherian, integrally closed, integral domain that is also a local ring of dimension 1.*

For proofs of the proposition and theorem see 16.2 in [6]. Let D be an almost Dedekind domain with $\mathrm{Max}(D)$ denoting the set of maximal ideals of D. Then for all $M \in \text{Max}(D)$ there exists a valuation $\nu_M : D_M \to \mathbb{Z}$. We will call these valuations local valuations. Furthermore u is a unit in D_M if $u \notin M$, that is $\nu_M(u) = 0$

Theorem 1.3.5. *For any commutative ring R with maximal ideals $Max(R)$. Then*

$$R = \bigcap_{M \in Max(R)} R_M.$$

Proof. See [7]. □

From the theorem and definition of localizing at a prime we get the following facts.

Theorem 1.3.6. *Let D be an almost Dedekind domain with field of fractions K, and let $a \in K$.*

1. *$u \in D$ is a unit only if u is not in any maximal ideal. That is $\nu_M(u) = 0$ for all $M \in Max(D)$.*
2. *$a \in D$ if and only if $\nu_M(a) \geq 0$ for all $M \in Max(D)$.*

Proof. The first part follows directly. For 2, we observe $a \in D$ if and only if for all M we have $a \in D_M$ for all $M \in \text{Max}(D)$. Which is if and only if $\nu_M(a) \geq 0$ for all $M \in \text{Max}(D)$. □

We will be using the local valuations throughout our study.

1.4. Class Groups

A Dedekind domain is a unique factorization domain (UFD) if and only if it is a principal ideal domain (PID) (see [6]). However, not all Dedekind domains are UFDs. In order to describe how far away a Dedekind domain is from being a UFD, one can look at its class group. Before defining the class group, let us recall some definitions. Let D denote an integral domain with field of fractions K.

Definition 1.4.1. *A fractional ideal I of D is a D-submodule of K such that there exists a nonzero $d \in D$ such that $dI \subseteq D$.*

We think of this special element d as clearing the denominators of I. We say that a fractional ideal I is invertible if there exists another fractional ideal J such that $IJ = \{a_1b_1 + a_2b_2 + \cdots + a_nb_n | a_i \in I, b_i \in J\} = D$. In a Dedekind domain all nonzero fractional ideals are invertible; in fact this is sometimes used as the definition of a Dedekind domain. We define the inverse of an ideal I to be $I^{-1} = \{k \in K | kI \subseteq D\}$. The set of invertible fractional ideals forms an abelian group, denoted by $\mathrm{Inv}(D)$. In an almost Dedekind domain every finitely generated (fractional) ideal is invertible.

Now clearly the nonzero principal ideals are invertible, for they are generated by merely one element. That is if $I \neq 0$ is principal, then $I = aR$ for some $a \in K \setminus \{0\}$, and we have $I^{-1} = a^{-1}R$. The set of principal ideals also form an abelian group, denoted by $\mathrm{Prin}(D)$.

Definition 1.4.2. *For an integral domain D we define the class group of D to be the quotient group:*

$$\mathfrak{C}(D) = Inv(D)/Prin(D).$$

It is clear that a Dedekind domain is a PID (hence a UFD) if and only if $\mathrm{Inv}(D) = \mathrm{Prin}(D)$ and this occurs if and only if $\mathfrak{C}(D)$ is trivial. Dedekind domains with non-trivial class groups are still atomic, but factorization is no longer unique.

In our study, we will use the class groups as an important tool. We will also use topological aspects of $\mathrm{Max}(D)$. The topological notions of dull and sharp maximal ideals will be introduced in Chapter 2. The idea will be that an atomic purely almost Dedekind domain cannot have "too many" principal ideals.

We present an example of a domain with a non-trivial class group.

Example 1.4.3. *Consider* $D = \mathbb{Z}[\sqrt{-5}]$. *It can be shown that* $2 \cdot 3 = 6 = (1 - \sqrt{-5})(1 + \sqrt{-5})$ *are distinct factorization of* 6, *hence we do not have unique factorization. The maximal ideal* $M = (2, 1 + \sqrt{-5})$ *is not principal, however* $M^2 = (2)$. *As it turns out the square of any non-principal maximal ideal is principal in* D. *Thus the class group is the group of order two. That is* $\mathfrak{C}(D) \cong \mathbb{Z}/2\mathbb{Z}$.

CHAPTER 2. EXAMPLES OF ALMOST DEDEKIND DOMAINS

In this chapter we will give specific examples of purely almost Dedekind domains, as well as general classes of almost Dedekind domains in the literature. The study of almost Dedekind domains trace back to N. Nakano in 1953, (see [14]). Since that time many authors have contributed to the understanding of almost Dedekind domains (see [7], [13], and [11]). Various algorithms for building purely almost Dedekind domains have been constructed. We will discuss these techniques and, in later chapters, give characterizations about domains constructed via the algorithms with respect to atomicity.

In 1974 A. Grams in [9] constructed a purely almost Dedekind domain that satisfied the ascending chain condition on principal ideals. This example is our first encounter with a purely almost Dedekind domain that is atomic. In this paper it is also shown that some purely almost Dedekind domains fail to be atomic. It is from here that we obtain our motivating question. Which almost Dedekind domains are atomic? This question has proved to be quite subtle. We will present several theorems that partially answer this question.

Before presenting the Grams' example, we need to develop some more machinery.

2.1. Max(D)

Set $\Delta_P = \text{Max}(D) \setminus P$. A domain D is said to have property $\sharp$ if for all P we have $D_P \not\subset \cap_{M \in \Delta_P} D_M$. In 1964, Gilmer showed in [8] that an almost Dedekind domain has property $\sharp$ if and only if it is Dedekind. In [11] this notion was adapted to an ideal characterization, given in the next definition.

Definition 2.1.1. *A maximal ideal $P \in Max(D)$ is said to be a sharp prime if $D_P \not\subset \cap_{M \in \Delta_P} D_M$. Equivalently, P is said to be sharp if it is the radical of finitely* *generated ideal.*

We will mainly use the equivalent notion that P is sharp if and only if there exists a finitely generated ideal I such that $I \subseteq P$ and I is not contained in any other maximal ideal. We will denote the set of sharp maximal ideals of D by $\mathrm{Max}_\sharp(D)$. The maximal ideals that are not sharp are called dull maximal ideals (see [11]). The set of dull primes of D will be denoted by $\mathrm{Max}_\dagger(D)$. The notions of dull and sharp have a topological aspect to them, which we will clarify with the next definition. Dull primes are "covered" by the other primes. This is also true for certain sharp primes. We introduction a new notion that makes this more precise.

Definition 2.1.2. *A sharp maximal ideal P is called hidden if $P \subseteq \cup_{M \in \Delta_P} M$.*

It should be noted that all dull primes satisfy the property in the definition, but we will only call sharp primes that satisfy the property as hidden. Hidden primes only arise if the class group is infinite.

For a given abelian group G we can find a Dedekind domain D with class group G. The result is in [1].

Example 2.1.3. *Let D be a Dedekind domain with class group $\mathbb{Z}$. And let P be a non-principal. Now all primes of a Dedekind domain are sharp, thus P is sharp. Suppose $P \nsubseteq \bigcup_{M \in \Delta_P} M$, then there is an $b \in P$ with b not any any other prime. Now since D is Dedekind we must be able to factor (b) into a finite product of prime ideals. But as b is only in P, the product can only contain P. Thus $(b) = P^n$ for some $n \geq 1$. But this is impossible since P is a non-principal prime in domain with class group $\mathbb{Z}$. Thus P must be hidden.*

Furthermore let us consider a sharp prime P that is not hidden. From the previous example it is clear that this sharp prime must be contained in a torsion class of the class group. That is $P^n = (b)$ for some $n \geq 1$ and some $b \in D$. Now it is clear that $\sqrt{(b)} = P$, since P^n is not contained in any other maximal ideal other than P. Thus the sharp primes that are not hidden are the radical of principal ideals. We restate in a lemma.

Lemma 2.1.4. *Let D be an almost Dedekind domain. If $P \subseteq D$ is a sharp prime and P is not hidden, then P is the radical of a principal ideal.*

Furthermore in an almost Dedekind domain a sharp prime is finitely generated. To see this we show that a sharp prime is always invertible, and use the fact that invertible ideals are always finitely generated. The following result is in [8].

Lemma 2.1.5. *Let D be an almost Dedekind domain. If P is a sharp prime then P is invertible.*

Proof. Let P be a sharp prime. Then $P = \sqrt{I}$ for some finitely generated ideal I. Now I is a P-primary ideal, hence it is a prime power. That is $I = P^n$. Now I is invertible, hence P^n is invertible. We conclude that P is invertible, hence finitely generated. □

Thus sharp primes are finitely generated and dull primes are not finitely generated. It is clear that if D is a purely almost Dedekind domain, then D must contain a non-finitely generated ideal, since it is not Noetherian. Thus a purely almost Dedekind domain must contain at least one dull prime.

An almost Dedekind domain that contains only sharp primes is called a sharp domain. It is shown in [8] that an almost Dedekind domain is sharp if and only if it is Dedekind. A domain that contains only dull primes will be called a dull domain. Most almost Dedekind domains encountered in the literature lie somewhere between these two extremes. That is, they contain a mixture of both dull and sharp primes.

2.2. Examples

In 1953 the first example of a non-Noetherian almost Dedekind domain appeared in a paper by Nakano (see [14]). This domain was constructed by adjoining p^{th} roots of unity to $\mathbb{Q}$. More precisely, let ζ_p be a primitive p^{th} root of unity for all nonzero primes p in $\mathbb{Z}$. Now consider $K = \mathbb{Q}(\zeta_2, \zeta_3, \cdots \zeta_p \cdots)$. The ring of integers R of K is almost Dedekind and not Noetherian.

While this is the first concrete example of a purely almost Dedekind domain, identifying $\text{Max}(R)$ (the set of maximal ideals) seems a difficult task. It is shown that when ζ_p is adjoined, the only prime ideal that ramifies is (p). Controlling the ramification will prove to be pivotal when constructing an almost Dedekind domain via integral extensions.

Another interesting example comes from C. Hashbarger (see [10]). In this case let

$$D = \mathbb{Z}[\sqrt{2}, \sqrt{3}, \cdots, \sqrt{p_n} \cdots],$$

where p_n is the n^{th} prime. It is shown that $\bar{D}$ (the integral closure of D) is an almost Dedekind domain. It should be noted that is the ring of integers of $\mathbb{Q}(\sqrt{2}, \sqrt{3}, \cdots, \sqrt{p_n} \cdots)$.

We can ask whether Nakano's and Hashbarger's examples are atomic. At this time the answers to these questions are not known, but the examples serve as concrete constructions of a purely almost Dedekind domain. We will present an example of an atomic purely almost Dedekind domain shortly, but first we need to introduce a few more theorems. We present three theorems used in the construction for context and later use.

Theorem 2.2.1. *Let D be a Dedekind domain with quotient field K, let M be a maximal ideal of D, let L be an n-dimensional normal extension field of K, and let $\bar{D}$ be the integral closure of D in L. If $\{M_i\}_{i=1}^{g}$ is the set of maximal ideals of $\bar{D}$ containing M, then the prime factorization of $M\bar{D}$ in $\bar{D}$ has the form $(M_i \cdots M_g)^e$,* *for some positive integer e. Moreover, $[\bar{D}/M_1 : D/M] = f < \infty$ and $[\bar{D}/M\bar{D} : D/M] = efg \leq n$. Equality holds if and only if $\bar{D}_N$ is a finite D_M-module, where $N = D - M$.*

Proof. See [3] and pp.493-501 in [7]. □

Definition 2.2.2. *Let P, Q, and U be maximal ideals of an almost Dedekind domain R. Let K denote the field of fractions of R. Let $K[t]$ be a simple quadratic extension. We say P decomposes (splits) in $K[t]$ if $P = P_1P_2$ with $P_1 \neq P_2$. We say Q ramifies in $K[t]$ if $Q = Q_1^2$. We say U is inert in $K[t]$ if U remains a maximal ideal.*

Theorem 2.2.3. *Let D be a Dedekind domain with quotient field K, and let $\{P_i\}_{i=1}^r$, $\{Q_i\}_{i=1}^s$, and $\{U_i\}_{i=1}^t$, where $r \geq 1$ be three collections of distinct maximal ideals of D, each with finite residue field. Then there exists a simple quadratic extension field $K(t)$ of D with t integral over D and separable over K such that if $\bar{D}$ is the integral closure of D in $K(t)$, each P_i is inertial with respect to $\bar{D}$, each Q_i ramifies with respect to $\bar{D}$, and each U_i decomposes with respect to $\bar{D}$.*

Proof. See [7] p. 585. □

Theorem 2.2.4. *Let G be a countable abelian group. Then there exists a Dedekind domain D whose class group $\mathfrak{C}(D)$ is isomorphic to G; moreover, D can be chosen so that it has a countably many maximal ideals $\{P_i\}_{i=1}^\infty$ and so that D/P_i is finite for each i.*

Proof. See [2]. □

Definition 2.2.5. *We say a domain D satisfies the ascending chain condition on principal ideal (ACCP) if every chain of principal ideals stabilizes. That is if $I_1 \subseteq I_2 \subset \cdots \subseteq I_k \subseteq \cdots$ is a chain of principal ideals, then there exists $n \in \mathbb{N}$ such that $I_j = I_n$ for all $j \geq n$.*

We are now in a position to consider the example constructed by A. Grams in [9]. The example, which appeared in 1974, shows that an almost Dedekind domain satisfying ACCP need not be Dedekind. Any domain that satisfies ACCP is atomic (see [3]).

For the construction, we let G be a countable abelian group that is not a torsion group, and let D_0 be a Dedekind domain and $\{P_i\}_{i=1}^{\infty}$ maximal ideals such that D_0/P_i is finite for each i and $\mathfrak{C}(D_0) \cong G$. Since the class group is not torsion there is a prime ideal, say P_1, with the property that P_1^n is not principal for each $n \in \mathbb{Z}^+$, and hence $P_1 \subset \cup_{i=2}^{\infty} P_i$. (In other words there is a hidden prime P_1)

Example 2.2.6 (Grams' Example)**.** *Let K be the quotient field of the domain D_0. There is a simple quadratic extension field $K(t_1)$, with t_1 integral over D_0 and separable over K, such that if D_1 is the integral closure of D_0 in $K(t_1) = F_1$, then P_1 decomposes with respect to D_1, say $P_1D_1 = M_1^{(1)}M_2^{(1)}$. Since D_1 is a Dedekind domain with the property that D_1/M is finite for each maximal ideal M, there exists a separable quadratic extension $F_1(t_2) = F_2$ such that t_2 is integral over D_1 and such that if D_2 is the integral closure of D_1 in F_2, then $M_2^{(1)}$ decomposes with respect to D_2, say $M_2^{(1)}D_2 = M_2^{(2)}M_3^{(2)}$, while $M_1^{(1)}$ and each prime ideal of D_1 lying over P_2 in D_0 is inertial with respect to D_2. Inductively, we can construct a sequence $\{F_i\}_{i=1}^{\infty}$ of extension fields of K such that if D_i is the integral closure of D_0 in F_i, then the following conditions hold for each i:*

1. $[F_{i+1} : F_i] = 2$.

2. *There are $i+1$ primes $\{M_j^{(i)}\}_{j=1}^{i+1}$ of D_i lying over P_1 in D_0. Further, $M_j^{(i)}$, for $j < i+1$, is inertial with respect to D_{i+1}, and $M_j^{(i+1)} = M_i^{(j)}D_{i+1}$. $M_{i+1}^{(i)}$ decomposes with respect to D_{i+1} as $M_{i+1}^{(i)}D_{i+1} = M_{i+1}^{(i+1)}M_{i+2}^{(i+2)}$.*

3. *Each prime ideal of D_i lying over any of the prime ideals $P_2, \cdots, P_{i+1}$ of D_0 is*

inertial with respect to D_{i+1}. Let $F = \cup_{i=1}^{\infty} F_i$ and $\bar{D} = \cup_{i=1}^{\infty} D_i$.

$\bar{D}$ is the desired example.

The proof that $\bar{D}$ satisfies ACCP (thus is atomic) is highly non-trivial and relies on a few facts. First a union of Dedekind domains is always one-dimensional and Prüfer. Secondly, a one-dimensional Prüfer domain is almost Dedekind if and only if it contains no idempotent maximal ideals. We restate the second fact in its complete form.

Theorem 2.2.7. *Let D be an integral domain with identity which is not a field. The following are equivalent:*

1. *D is an almost Dedekind domain.*
2. *The cancellation law for ideals holds in D.*
3. *D is a one-dimensional Prüfer domain and D contains no idempotent maximal ideal.*
4. *D is Prüfer domain and $\cap_{n=1}^{\infty} A^n = (0)$ for each proper ideal A of D.*

2.3. Classes of Almost Dedekind Domains

In this section we define several types of almost Dedekind domains that have appeared in the literature. First we start with class of domains introduced by A. Loper in [13] called glad domains.

Definition 2.3.1. *A domain D with quotient field K (different from D) is a glad domain provided:*

1) *$D = \cap_{\lambda \in \Lambda} V_\lambda$ where $\{V_\lambda | \ \lambda \in \Lambda\}$ is a family of Noetherian valuation overrings of the domain D. Let ν_λ be the normed additive valuation associated with V_λ, and let M_λ be the maximal ideal of V_λ.*

2) There is a monic polynomial $f \in D[x]$ with $deg(f) > 1$ such that for each $\lambda \in \Lambda$ and each $a \in V_\lambda$, $f(a)$ is a unit of V_λ.

3) For each nonzero $a \in D$ the set $\{\nu_\lambda(a) | \ \lambda \in \Lambda\}$ is bounded.

4) There exists $t \in D$ such that $tV_\lambda = M_\lambda$ for each $\lambda \in \Lambda$.

5) There exists a finite subset T of D which is a set representation for V_λ/M_λ for each $\lambda \in \Lambda$.

In [16], Rush gave the following alternate characterization.

Proposition 2.3.2. *Let D be an almost Dedekind domain with quotient field K (different from D). Then D is a glad domain if and only if the following statements hold.*

a) Each principal ideal of D contains a power of its radical.

b) The Jacobson radical $\mathcal{J}$ is a nonzero principal ideal.

c) There exists a finite subset T of D which is a set of representatives for D/P for each maximal ideal P of D.

Rush also established the following result concerning the intersection of finite number of glad overrings.

Proposition 2.3.3. *Let $D = D_1 \cap \cdots \cap D_n$ with each D_i a glad overring of D. Then the following hold.*

a) $D_i = D_S$ where $S = D - \cup\{Q \cap D | \ Q$ a maximal ideal of $D_i\}$

b) D is an almost Dedekind domain with finite residue fields.

c) D is a Bézout domain.

Loper introduced the idea of a sequence domain. We take his definition from [12].

Definition 2.3.4. *Let D be an almost Dedekind domain. We call D a sequence domain provided the following hold.*

1) The maximal ideals of D are $\{M_i |\ i \in \mathbb{N}\}$ and M_∞.

2) Each M_i is principal and M_∞ is not finitely generated.

3) The Jacobson radical $\mathcal{J}$ is nonzero.

Another class of domains we will wish to consider are SP-domains.

Definition 2.3.5. *A ring R in which every proper nonzero ideal is a product of radical ideals is called and SP-domain. (SP stands for semi-prime),*

The fact that SP-domains are almost Dedekind is not clear from the definition, but it was shown in 1978 in a paper by Vaughan and Yeagy, that they are indeed almost Dedekind. See [17].

In [15], Olberding characterized SP-domains within the class of almost Dedekind domains. We say a maximal ideal M of an almost Dedekind domain is critical if every finitely generated ideal contained in M is contained in the square of some maximal ideal of D.

Theorem 2.3.6. *Let D be an almost Dedekind domain. Then the following are equivalent.*

1) D is an SP-domain.

2) D has no critical maximal ideals.

3) If I is a proper finitely generated ideal of D, then $\sqrt{I}$ is finitely generated.

4) Every proper principal ideal is a product of radical ideals.

5) For every $a \in D$ the map $\gamma_a : Max(D) \to \mathbb{N}_0$ is upper semi-continuous and has finite image. Where $\gamma_a(M) = \nu_M(a)$.

6) For every proper ideal I of D, there exists radical ideals $J_1 \subseteq J_2 \subseteq \cdots \subseteq J_n$ such that $I = J_1 J_2 \cdots J_n$.

7) For every proper nonzero ideal I of D can be represented uniquely as a product $I = J_1 J_2 \cdots J_n$ of radical ideals J_i such that $J_1 \subseteq J_2 \subseteq \cdots \subseteq J_n$.

CHAPTER 3. NORMS AND NORMSETS

We have seen that an almost Dedekind domain is a generalization of a Dedekind domain. Our goal is to determine atomicity in an almost Dedekind domain. We will approach this question with many tools, but underlying most of our arguments will be the use of a map we will call the norm. Recall that locally an almost Dedekind domain is a Noetherian valuation domain. This is a very strong condition and we will take make use of it frequently.

3.1. The Norm

Let D be an almost Dedekind domain. Let $\mathrm{Max}(D)$ be the set of maximal ideals of D. Now for every $M \in \mathrm{Max}(D)$ we have a map

$$\nu_M : D_M \to \mathbb{N}_0.$$

Now we know that

$$D = \bigcap_{M \in \mathrm{Max}(D)} D_M.$$

Now for $b \in D$ we have $\nu_M(b) = 0$ if $b \notin M$ and $\nu_M(b) > 0$ if $b \in M$. We are now in a position to define our norm. For nonzero $b \in D$ we define the norm of b to be the net

$$N(b) = \big(\nu_M(b)\big)_{M \in \mathrm{Max}(D)} \subseteq \prod_{M \in \mathrm{Max}(D)} \mathbb{N}_0.$$

If u is a unit in D, then $N(u)$ is the zero net. We should note, that if $\mathrm{Max}(D)$ is countable then the norm of an element is merely a sequence. We will only draw distinctions in examples where it will be of benefit to the reader. We will see examples of domains with countably many and uncountably many maximal ideals.

We can define our local valuations on K the field of fractions of D. In this case, $\nu_M : K \to \mathbb{Z}$. Under this definition we see that $k \in K$ is in D if and only if for all M

$\nu_M(k) \geq 0$.

We define the addition of nets componentwise. That is

$$N(a) + N(b) := \big(\nu_M(a) + \nu_M(b)\big)_{M \in \mathrm{Max}(D)}.$$

Theorem 3.1.1. *Let D be an almost Dedekind domain. For all $a, b \in D$ we have $N(ab) = N(a) + N(b)$.*

Proof. $N(ab) = \big(\nu_M(ab)\big)_{M \in \mathrm{Max}(D)} = \big(\nu_M(a) + \nu_M(b)\big)_{M \in \mathrm{Max}(D)} = N(a) + N(b)$, by the properties of valuations. □

We wish to extend the notion of a norm to an ideal. For ideal $I \subset D$ and for a fixed maximal ideal M of $\mathrm{Max}(D)$ we define $\nu_M(I)$ to be k such that $I \subseteq M^k$ and $I \not\subset M^{k+1}$. Now we define the norm of an ideal as

$$N(I) = \big(\nu_M(I)\big)_{M \in \mathrm{Max}(D)} \subset \prod_{M \in \mathrm{Max}(D)} \mathbb{N}_0.$$

We should note that this is consistent with the norm of an element if we consider the principal ideal generated by that element.

Lemma 3.1.2. *Let $I \subseteq D$ be an ideal. We have*

$$N(I) = \big(\inf_{b \in I}\{\nu_M(b)\}\big)_{M \in Max(D)}.$$

Proof. $I \subset M^k$ if and only if for all $b \in I$ we have $b \in M^k$, which is if and only if $\nu_M(b) \geq k$. Now we defined $\nu_M(I)$ to be the minimal k such that $I \subset M^k$. Thus there must be a $b \in I$ that satisfies this infimum. Hence $N(I) = \big(\inf_{b \in I}\{\nu_M(b)\}\big)_{M \in \mathrm{Max}(D)}$. □

This will be a useful tool in calculating the norm of an ideal. It can also be used to establish a theorem about the norm of a product of ideals.

Theorem 3.1.3. *Let I, J be ideals of an almost Dedekind domain D, then $N(IJ) = N(I) + N(J)$.*

Proof. Recall $IJ = \sum a_i b_i$ where $a_i \in I$ and $b_i \in J$. Using this it is easy to see for $d \in IJ$ that $\nu_M(d) \geq \nu_M(IJ)$. Now we can find a and b for a fixed M such that $\nu_M(a) = \nu_M(I)$ and $\nu_M(b) = \nu_M(J)$ thus $\nu_M(ab) = \nu_M(I) + \nu_M(J)$. Therefore by the previous lemma we have $N(IJ) = N(I) + N(J)$. $\square$

Definition 3.1.4. *We say $N(a) \leq N(b)$ if for all $M \in Max(D)$ we have $\nu_M(a) \leq \nu_M(b)$. We say $N(a) < N(b)$ if $N(a) \leq N(b)$ and there exists an $M \in Max(D)$ with $\nu_M(a) < \nu_M(b)$.*

The use of this partial ordering will yield results with respect to atomicity. It should be noted that two elements are not necessarily comparable within this order, that is there maybe $a, b \in D$ such that $N(a) \nleq N(b)$ and $N(b) \nleq N(a)$. Now we present a very powerful lemma; the result's veracity is due to the local behavior of almost Dedekind domains.

Lemma 3.1.5. *Let D be an almost Dedekind domain and let $a, b \in D$. $N(a) \leq N(b)$ if and only if a divides b.*

Proof. Suppose $N(a) \leq N(b)$. We have $\frac{b}{a}$ is in the quotient field of D. Now $\nu_M(\frac{b}{a}) = \nu_M(b) - \nu_M(a) \geq 0$ for all $M \in \mathrm{Max}(D)$. Thus $\frac{b}{a} \in D_M$ for all M. Hence $\frac{b}{a} \in D$. We conclude that a divides b. Suppose a divides b. Then $\frac{b}{a} \in D$. Thus $\frac{b}{a} \in D_M$ for all $M \in \mathrm{Max}(D)$. Hence for all M we have $\nu_M(\frac{b}{a}) \geq 0$. Thus $\nu_M(b) \geq \nu_M(a)$ and we conclude that $N(a) \leq N(b)$. $\square$

If we wish to find a proper divisor of an element $b \in D$, all we will need to do is find an $a \in D$ such that $N(a) < N(b)$. We will use this idea to construct new divisors of elements. We will also use Lemma 3.1.5 to show when elements are atoms or are (finite) products of atoms. But first it should also be pointed out that this is not true for the traditional norm in a Dedekind domain. To see this let us consider an example.

Example 3.1.6. *Consider $D = \mathbb{Z}[\sqrt{-14}]$. The traditional norm is $N'(a+b\sqrt{-14}) = a^2+14b^2$. Now $N'(5+2\sqrt{-14}) = 81$ and $N'(3) = 9$ but 3 does not divide $5+2\sqrt{-14}$. But what about our norm? Let $M_1 = (3, 5+2\sqrt{-14})$ and $M_2 = (3, 5-2\sqrt{-14})$ It is easy to see that M_1 and M_2 are maximal in D and $3 \in M_1, M_2$. That is we have $\nu_{M_1}(3) = 2$ and $\nu_{M_2}(3) = 2$. Now $5+2\sqrt{-14} \in M_1$, but $5+2\sqrt{-14} \notin M_2$. If $5+2\sqrt{-14} \in M_2$, we would have $5+2\sqrt{-14}+5-2\sqrt{-14} = 10 \in M_2$ but 3 and 10 are coprime in D. Thus we would have $1 \in M_2$. Thus $5+2\sqrt{-14} \notin M_2$. Thus we have $\nu_{M_2}(5+2\sqrt{-14}) = 0$. We conclude that $N(3) \not< N(5+2\sqrt{-14})$. Thus our norm recognizes that 3 is not a divisor of $5+2\sqrt{-14}$.*

We now make some structural observations about the image of our norm map.

Definition 3.1.7. *Let D be an almost Dedekind domain. We define the normset of D to be*

$$\mathit{Norm}(D) = \{N(b) \mid b \in D\}.$$

It should be noted that $\mathrm{Norm}(D) \subseteq \prod_{M \in \mathrm{Max}(D)} \mathbb{N}_0$.

Theorem 3.1.8. *Let D be an almost Dedekind domain. $\mathit{Norm}(D)$ forms an additive monoid.*

Proof. First the zero net is in $\mathrm{Norm}(D)$ because D has an identity. Now if $N(a)$ and $N(b)$ are in $\mathrm{Norm}(D)$ (that is $a, b \in D$), then $N(a) + N(b) = N(ab) \in \mathrm{Norm}(D)$ because $ab \in D$. $\square$

It should be clear that our norm addition is associative. We should point out that we could define the normset of ideals by $\text{IdealNorm}(D) = \{N(I) | \, I \text{ is an ideal of } D\}$ This too will form a monoid under addition with the identity element realized by the entire domain D. The closure property comes from the lemma about the product of ideals.

Since the image under our norm map is well behaved in a structural sense, one might ask the question if we impose more conditions on D what is the effect on $\text{Norm}(D)$? Before asking and answering the question with regard to atomicity, let us first classify the elements of $\text{Norm}(D)$.

Theorem 3.1.9. *Let D be an almost Dedekind domain with $Max(D) = \{M_\lambda\}_{\lambda \in \Lambda}$. Then*

$$(e_\lambda)_{\lambda\in\Lambda} \in Norm(D) \iff \bigcap_{\lambda\in\Lambda} M_\lambda^{e_\lambda} \text{is a principal ideal.}$$

Where we take $M^0 = D$.

Proof. For the forward direction we take $a \in D$ with $N(a) = (e_\lambda)_{\lambda\in\Lambda}$. Recall $\nu_{M_\lambda}(a) = e_\lambda$ is equivalent to saying $a \in M_\lambda^{e_\lambda}$ and $a \notin M_\lambda^{e_\lambda+1}$. Thus for all λ we have $a \in M_\lambda^{e_\lambda}$, hence $a \in \cap_{\lambda\in\Lambda} M_\lambda^{e_\lambda}$. We will show this ideal is actually (a). Suppose $b \in \cap_{\lambda\in\Lambda} M_\lambda^{e_\lambda}$. Then for all λ we have $\nu_{M_\lambda}(b) \geq e_\lambda = \nu_{M_\lambda}(a)$. In other words we have $N(a) \leq N(b)$, hence a divides b. We conclude that

$$(a) = \bigcap_{\lambda\in\Lambda} M_\lambda^{e_\lambda} \ .$$

For the other direction we suppose $(a) = \cap_{\lambda\in\Lambda} M_\lambda^{e_\lambda}$. Then $N(a) = (e_\lambda)_{\lambda\in\Lambda}$. □

When we start to discuss atomicity in more general settings it will be necessary to be able to easily identify principal ideals.

Theorem 3.1.10. *An ideal I is principal if and only if there exists $b \in I$ with*

$N(b) = N(I)$.

Proof. Suppose I is principal, that is $I = (b)$ for some $b \in I$. Now for all $c \in I$ we have $b|c$. Thus $N(b) \leq N(c)$ hence b achieves the infimum for all maximal ideals, and we have $N(b) = N(I)$.

Now suppose there exists $b \in I$ with $N(b) = N(I)$. Then since b achieves the infimum at every maximal ideal, we have for all $c \in I$ $N(b) \leq N(c)$, hence $I = (b)$. $\square$

Now we will show that if D is an atomic almost Dedekind domain then $\text{Norm}(D)$ is an atomic monoid (additively). In [4] Coykendall describes the relation of the tradition normset to unique factorization. In Theorem 3.12 we prove a more general result with respect to atomicity and $\text{Norm}(D)$. To do this we establish an atomicity lemma. Recall $\alpha \in D$ is an atom if $\alpha = bc$ implies b or c is a unit in D. We say $N(a)$ is an atom in $\text{Norm}(D)$. if $N(a) = N(b) + N(c)$ implies $N(b)$ or $N(c)$ is the zero net.

Lemma 3.1.11. *$N(a)$ is an atom in Norm(D) if and only if a is an atom in D.*

Proof. Suppose $N(a)$ is an atom in $\text{Norm}(D)$. Suppose further $a = bc \in D$. Then $N(a) = N(b) + N(c)$ thus $N(b)$ or $N(c)$ must be the zero net. Thus b or c must be a unit, and a is an atom. Conversely, suppose a is atom in D. Suppose further $N(a) = N(b) + N(c)$. Now if both $N(b)$ and $N(c)$ are nonzero, then $N(b) < N(a)$, hence b would be a proper divisor of a. Hence a is not an atom. Thus $N(a)$ must be an atom in $\text{Norm}(D)$. $\square$

It should be noted that this is not true with regard to the traditional norm. For example in $\mathbb{Z}[\sqrt{-14}]$, we have $N'(5 + 2\sqrt{-14}) = 81$ and $N'(3) = 9$. Thus 81 is not an atom in the normset, but $5 + 2\sqrt{-14}$ is an atom in $\mathbb{Z}[\sqrt{-14}]$.

Theorem 3.1.12. *Let D be an almost Dedekind domain. D is atomic if and only if Norm(D) is an atomic monoid.*

Proof. Suppose D is atomic and consider $N(a)$. We have $a = \alpha_1\alpha_2\cdots\alpha_n$ a product of atoms. Now we have $N(a) = N(\alpha_1)+N(\alpha_2)+\cdots+N(\alpha_n)$ is an atomic "factorization" in $\mathrm{Norm}(D)$. Suppose $\mathrm{Norm}(D)$ is an atomic monoid. Consider $a \in D$. We write $N(a) = N(\alpha_1) + N(\alpha_2) + \cdots + N(\alpha_n)$ as an atomic factorization in $\mathrm{Norm}(D)$. We now see $a = u\alpha_1\alpha_2\cdots\alpha_n$ for some unit u is an atomic factorization in D. $\square$

Now while this is a complete (in some sense) characterization of the class of atomic almost Dedekind domains, it does leave one wanting, in the sense that actually computing the normset of an almost Dedekind domain is very difficult. The first step in such an endeavor would be to compute $\mathrm{Max}(D)$, which in many cases is a difficult task. We will pursue more practical ways of determining whether a given almost Dedekind domain is atomic, this will be the subject of the later chapters.

Another powerful observation we can make with regard to the normset is the following.

Theorem 3.1.13. *Let D be an almost Dedekind domain. D is a unique factorization domain (UFD) if and only if $\mathrm{Norm}(D)$ is a unique factorization monoid (UFM).*

Proof. Suppose D is a UFD, and let $b \in D$. Suppose $N(b) = N(\alpha_1) + N(\alpha_2) + \cdots + N(\alpha_n)$ and $N(b) = N(\beta_1) + N(\beta_2) + \cdots + N(\beta_m)$ are two atomic factorizations in $\mathrm{Norm}(D)$. By Lemma 3.1.11, we know that the α_i and β_i are atoms in D. Thus we have $b = u\alpha_1\alpha_2\cdots\alpha_n$ and $b = u'\beta_1\beta_2\cdots\beta_m$ are two atomic factorizations of b in D for some units u, u'. But D is a UFD, so it must be the case that $m = n$ and the β's are just a permutation of the α's up to associates. Hence the factorization in $\mathrm{Norm}(D)$ is unique.

The converse of the statement can be proved in the same manner by assuming $\mathrm{Norm}(D)$ is a UFM, and assuming $b \in D$ has two factorizations. $\square$

It should be noted that an almost Dedekind domain is a UFD if and only if it is a PID.

A half-factorial domain is an atomic domain such that if $b = \alpha_1\alpha_2\cdots\alpha_n$ and $b = \beta_1\beta_2\cdots\beta_m$ are two atomic factorizations then $n = m$.

Theorem 3.1.14. *Let D be an almost Dedekind domain. If D is a half-factorial domain (HFD) if and only if Norm(D) is a half-factorial monoid (HFM).*

Proof. Suppose D is an HFD and $b \in D$. Suppose further that $N(\alpha_1)+\cdots+N(\alpha_k) = N(b) = N(\beta_1)+\cdots+N(\beta_l)$ are two atomic factorization of $N(b)$ in Norm(D). Then $u\alpha_1\cdots\alpha_k = b = u'\beta_1\cdots\beta_l$ for some units $u, u' \in D$. But these are both atomic factorizations in D and D is an HFD, thus $k = l$. We conclude that Norm(D) is an HFM.

Conversely suppose Norm(D) is an HFM, and $\alpha_1\cdots\alpha_k = b = \beta_1\cdots\beta_l$ is an atomic factorization of $b \in D$. Then $N(\alpha_1)+\cdots+N(\alpha_k) = N(b) = N(\beta_1)+\cdots+N(\beta_l)$ is an atomic factorization of $N(b)$ in Norm(D). But since Norm(D) is an HFM we conclude that $k = l$ and D is an HFD.

□

There has been much discussion on atomic domains that do not have unique factorization or the half-factorial property. One of the notions that has arisen to quantify this is elasticity.

Definition 3.1.15. *Let D be an atomic domain. We define the elasticity of a nonzero non-unit $b \in D$ by*

$$\varrho(b) = \sup\{\frac{m}{n} |\, \alpha_1\alpha_2\cdots\alpha_n = b = \beta_1\beta_2\cdots\beta_m \textit{ for atoms } \alpha_i, \beta_j \in D\}.$$

We define the elasticity of D by

$$\varrho(D) = \sup\{\varrho(b) |\, b \textit{ is a nonzero non-unit in } D\}$$

Similarly we can define the elasticity of an atomic monoid.

Definition 3.1.16. *Let A be an atomic monoid. We define the elasticity of a nonzero $b \in A$ by*

$$\varrho(b) = \sup\{\frac{m}{n} \mid \alpha_1 + \alpha_2 + \cdots + \alpha_n = b = \beta_1 + \beta_2 + \cdots + \beta_m \text{ for atoms } \alpha_i, \beta_j \in A\}.$$

We define the elasticity of A by

$$\varrho(A) = \sup\{\varrho(b) \mid b \text{ is a nonzero non-unit in } A\}$$

Again, atomic factorizations in D are in one-to-one correspondence to the factorizations in $\mathrm{Norm}(D)$, thus we get the following theorem.

Theorem 3.1.17. *Let D be an atomic almost Dedekind domain, then $\varrho(D) = \varrho(Norm(D))$.*

Proof. We have $b = \alpha_1 \cdots \alpha_k$ if and only if $N(b) = N(\alpha_1) + \cdots + N(\alpha_k)$. That is the atomic factorizations in D are in one-to-one correspondence to the atomic factorizations in $\mathrm{Norm}(D)$. Let b be a nonzero non-unit in D. Then

$$\varrho(b) = \sup\{\frac{m}{n} \mid \alpha_1 \cdots \alpha_n = b = \beta_1 \cdots \beta_m \text{ for atoms } \alpha_i, \beta_j \in D\}$$

$$= \sup\{\frac{m}{n} \mid N(\alpha_1) + \cdots + N(\alpha_n) = N(b) = N(\beta_1) + \cdots + N(\beta_m)$$

$$\text{for atoms } N(\alpha_i), N(\beta_j) \in \mathrm{Norm}(D)\} = \varrho(N(b))$$

Thus the elasticity of a b in D is in agreement with the elasticity of $N(b)$ in $\mathrm{Norm}(D)$. Now

$$\varrho(D) = \sup\{\varrho(b) \mid b \text{ is a nonzero non-unit in } D\}$$

$$= \sup\{\varrho(N(b)) \mid N(b) \text{ is a nonzero non-unit in } \mathrm{Norm}(D)\} = \varrho(\mathrm{Norm}(D)).$$

□

It should be noted that this result is true for any almost Dedekind domain if we restrict our definition of elasticity to be the supremum taken over atomic elements. That is the elements that factor into a finite product of atoms.

3.2. More Examples

In this section we present two constructions of purely almost Dedekind domains. These constructions will be useful in the remainder this study. The focus at present will be their simplicity in the sense of calculating $\mathrm{Max}(D)$ and $\mathrm{Norm}(D)$.

Before constructing our examples, we recall that a union of Dedekind domain is a one-dimensional Prüfer domain. Further, a one-dimensional Prüfer domain is almost Dedekind if and only if $M \neq M^2$ for all maximal ideals M. We now present a theorem that we will use to show our examples are indeed almost Dedekind.

Theorem 3.2.1. *A one-dimensional Prüfer domain D is almost Dedekind if and only if it admits a non-trivial map*

$$N : D \to \prod_{M \in Max(D)} \mathbb{N}_0$$

with $N(ab) = N(a) + N(b)$ for all $a, b \in D$.

Proof. The forward direction has already been shown to be true by the construction of our norm map.

Assume D admits such a map. Then for any maximal ideal M, we consider the set $S = \{\nu_M(b) | b \in M\}$. S is a subset of $\mathbb{N}$ and contains a least element. Suppose b realizes this minimal value, then b cannot be in M^2. For if it were there would be an element of smaller value. Thus for all M we have $M \neq M^2$. We conclude that D is almost Dedekind. □

Example 3.2.2 (The D^q domain)**.** *Let $D = \mathbb{Z}_{(q)}$ for some prime q. Let K denote the quotient field of D. We can split (q) by adjoining $t_{1,1}$ a root of $x^2 - p$ for some prime p that is not a square modulo q. Let $K_1 = K[t_{1,1}]$. We have $(q) = (q_{1,1})(q_{1,2})$ where $(q_{1,1})$ and $(q_{1,2})$ are distinct. We set D_1 to be the integral closure of D in K_1. For the remainder of the construction D_i will be the integral closure of D_{i-1} in K_i and the $t_{i,j}$ are elements of the algebraic closure of K. Now by Theorem 2.7, we can find $t_{2,1}$ such that $(q_{1,1}) = (q_{2,1})(q_{2,2})$ while $(q_{1,2})$ remains inert. Similarly we can find $t_{2,2}$ such that $(q_{1,2}) = (q_{2,3})(q_{2,4})$ while $(q_{2,1})$ and $(q_{2,2})$ all remain inert. Set $K_2 = K_1[t_{2,1}, t_{2,2}]$ and D_2 to be the integral closure of D in K_2. We can keep finitely many primes inert as we build our extensions. We continue inductively finding $t_{i,1}$ such that $q_{i-1,1}$ splits and the other finitely many primes remain inert, then we find $t_{i,2}$ such that $q_{i-1,2}$ is the only prime that splits. We continue until we get $t_{i,2^i}$ with $q_{i,2^i}$ is the only prime that splits. Now we have $K_i = K_{i-1}[t_{i,1}, t_{i,2}, \cdots t_{i,2^i}]$ and D_i the integral closure of D in K_i. Now we set $D^q = \cup_{i=1}^{\infty} D_i$. Now D^q is an almost Dedekind domain. It should be noted that every D_i is a PID.*

We present a picture of how the ideal lattice is splitting.

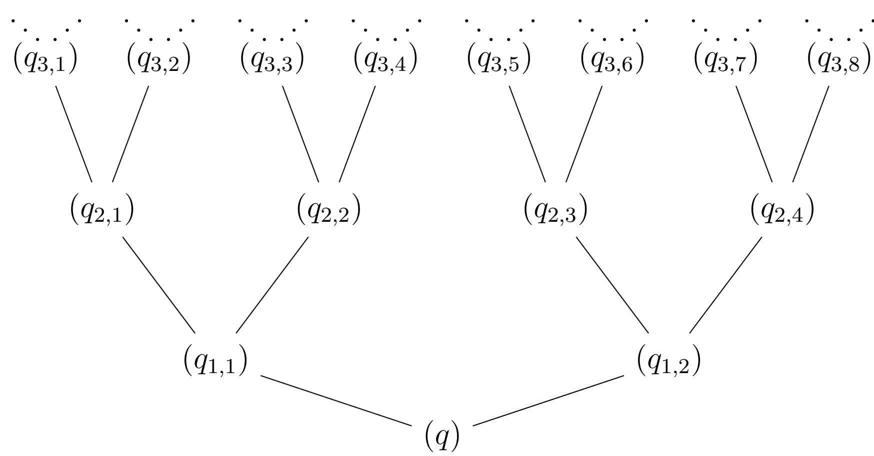

Figure 1: Ideal splitting lattice for D_q

We answer the following questions, before showing D^q is almost Dedekind. What are the maximal ideals of D? What is $\mathrm{Norm}(D)$?

Referring to the lattice, we see that $M = (q, q_{1,1}, q_{2,1}, \cdots q_{i,1}, \cdots)$ is a maximal ideal. M is not finitely generated. It should be noted that we do not need the first finite generators in the generating set, but they are included to better explain the structure of $\mathrm{Max}(D)$ and later $\mathrm{Norm}(D)$. $\mathrm{Max}(D)$ is an uncountable set, this is clear since our lattice is isomorphic to the binary tree. None of the ideals in $\mathrm{Max}(D)$ are finitely generated. If one starts at the root, q, and following any path to the top, we will get an infinite set that generates a maximal ideal. Thus D^q contains only dull primes, hence it is a dull domain.

Now that we have established the structure of $\mathrm{Max}(D)$, we turn our attention to $\mathrm{Norm}(D)$ We start by examining the value when we localize at the maximal ideal M constructed in the previous paragraph. In D^q_M we see that q and $q_{1,1}$ are associates, because $q_{1,2}$ is a unit in the localization. Similarly all of the generators of M are associates. Thus for all i we have $\nu_M(q_{i,1}) = 1$ and if $b \notin M$ we have $\nu_M(b) = 0$. More generally if t is a generator of a maximal ideal M', we have $\nu_M(t) = 1$.

Thus since q can be expressed as a generator of every maximal ideal we have $N(q) = (1)_{M \in \mathrm{Max}(D)}$. That is the norm of q is the net of all ones. More generally

$$N(q_{i,j}) = \begin{cases} 1 & q_{i,j} \in M \\ 0 & q_{i,j} \notin M \end{cases}$$

The elements $N(q_{i,j})$ generate the monoid $\mathrm{Norm}(D^q)$.

Additionally we observe for all M_λ we have $N(M_\lambda)$ is zero except for the λ^{th} "coordinate", which has value one.

Theorem 3.2.3. *D^q is an almost Dedekind domain.*

Proof. To see the result we note that q is an element of least value in every maximal ideal. Consider $b \in D^q$. Then $b \in (q_{i,j})$ for some i, j. But we know from above that q is an associate to $q_{i,j}$, hence b cannot have value less then q. Thus for every maximal ideal the value group associated to it is $\mathbb{N}_0$. Therefore applying Theorem 3.2.1 we see that D is almost Dedekind. □

Example 3.2.4 (The D^∞ domain)**.** *Let $D = \mathbb{Z}_{(q)}$ for some prime q. Let K denote the quotient field of D. We can split (q) by adjoining $t_{1,1}$ a root of $x^2 - p$ for some prime p that is not a square modulo q. Let $K_1 = K[t_{1,1}]$. We have $(q) = (q_1)(q_{1,2})$ where (q_1) and $(q_{1,2})$ are distinct. We set D_1 to be the integral closure of D in K_1. For the remainder of the construction D_i will be the integral closure of D_{i-1} in K_i and the $t_{i,j}$ are elements of the algebraic closure of K. Now by Theorem 2.7 we can find $t_{2,1}$ such that $(q_{1,2}) = (q_{2,1})(q_{2,2})$ in $D_1[t_{2,1}]$ while (q_1) remains inert. Now we find $t_{2,2}$ such that $(q_2)^2 = (q_{2,1})$ and the other two primes remain inert in $K_2 = K_1[t_{2,1}, t_{2,2}]$. Now we set D_2 to be the integral closure of D in K_2 Now we split $(q_{2,2}) = (q_{3,1})(q_{3,2})$ via another simple quadratic extension (add $t_{3,1}$) while keeping the three other primes inert. We then ramify $(q_{3,1})$ twice by using two simple quadratic extensions adding $t_{3,2}, t_{3,3}$. Thus yielding $(q_3)^4 = (q_{3,1})$. Now we set $K_3 = K_2[t_{3,1}, t_{3,2}, t_{3,3}]$. We set D_3 to be the integral closure of D in K_3. We continue by induction. In D_i we have $i+1$ namely $(q_1), (q_2), \cdots (q_i)$ and $q_{i,2}$ Now we add $t_{i,1}$ that splits $(q_{i,2}) = (q_{i+1,1})(q_{i+1,2})$ Now we adjoin elements $t_{i,2}, t_{i,3} \cdots t_{i,i}$ which all keep ramifying the prime $q_{i+1,1}$ such that $(q_{i+1})^{2^{i-1}} = (q_{i+1,1})$ while all other primes remain inert. Note we can do this because there are only finitely many primes at each step. Now we set $K_{i+1} = K_i[t_{i,1}, t_{i,2} \cdots t_{i,i}]$ and D_{i+1} to be the integral closure of D in K_{i+1}. Now we set $D^\infty = \cup_{i=1}^{\infty} D_i$. Now D^∞ is an almost Dedekind domain, moreover it is a sequence domain.*

Now that we have completely classified $\mathrm{Max}(D)$, let's start in on calculation $\mathrm{Norm}(D)$. First since $\mathrm{Max}(D)$ is countable, the normset is a set of sequences. $N(q_1) = (1, 0, 0 \cdots, 0, \cdots; 0)$ and $N(q_i)$ is zero in every entry except the i^{th} entry which is 1.

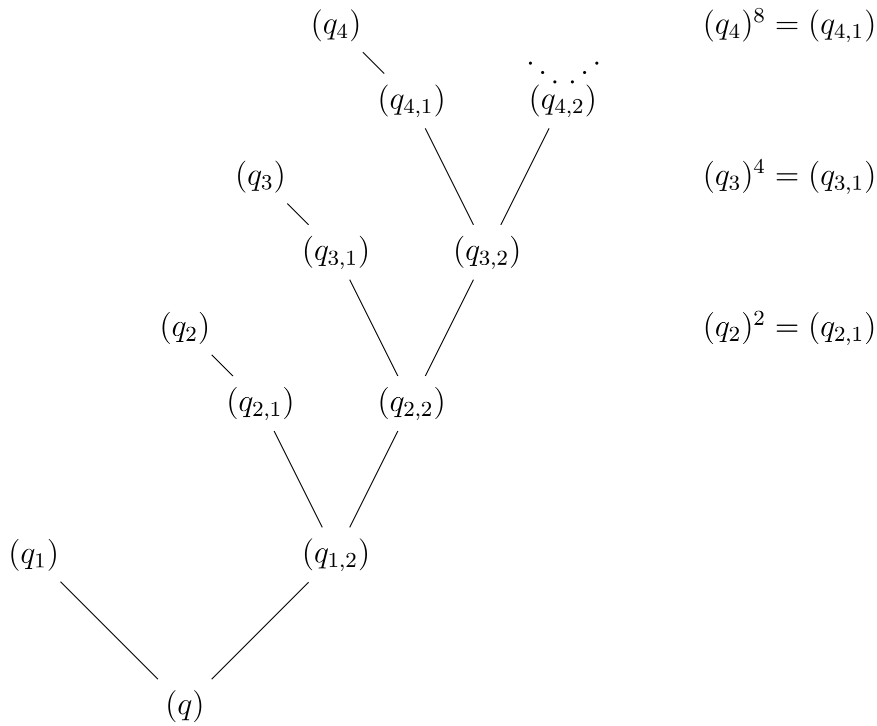

Figure 2: Ideal splitting lattice for D^∞

The entry after the semicolon denotes the value in M_∞.

Now $N(q) = (1, 2, 4, 8, \cdots 2^{i-1}, \cdots ; 1)$, and $N(q_{1,2}) = (0, 2, 4, 8, \cdots 2^{i-1} \cdots ; 1)$. The norm of $q_{i,2}$ is the sequence with the first i entries zero and the rest powers of 2. Now while this is a good start at describing the normset, we should point out that $(1, 1, 4, 8, \cdots, 2^{i-1}, \cdots ; 1)$ is also in the normset for it is $\frac{q}{q_2}$. We can easily grasp the elements of $\mathrm{Norm}(D^\infty)$; however writing the normset in a nice way is not practical.

Theorem 3.2.5. *D^∞ is an almost Dedekind domain.*

Proof. We will again use Theorem 3.2.1. The principal maximal ideals clearly have value group $\mathbb{N}_0$ as their generators are elements of least positive value. For M_∞, we showed above that q is an element of least positive value. Thus D_∞ is almost Dedekind. $\square$

We finish this section by noting that both D^q and D^∞ are Bézout domain and make a characterization about the Jacobson radical, $\mathcal{J} = \cap_{M \in \mathrm{Max}(D)} M$. First both domains are Bézout because they are direct limits of PIDs.

Theorem 3.2.6. *Let D be an almost Dedekind domain with $\mathcal{J} \neq 0$. If D contains an element b, such that $N(b) = (1)_{M \in Max(D)}$, then $\mathcal{J} = (b)$.*

Proof. Suppose such b exists. For all $c \in \mathcal{J}$, we have $N(b) \leq N(c)$. Hence $\mathcal{J} = (b)$. □

Now we can make a stronger statement about the Jacobson radical with respect to being principal if we restrict ourselves to sequence domains.

Theorem 3.2.7. *Let D be a sequence domain. Then $\mathcal{J}$ is principal if and only if there exists $b \in \mathcal{J}$ such that $N(b) = (1)_{M \in Max(D) \setminus M_\infty}$.*

Proof. Suppose there exists $b \in \mathcal{J}$ such that $N(b) = (1)_{M \in \mathrm{Max}(D) \setminus M_\infty}$. Now consider

$$S = \{\nu_{M_\infty}(b) | b \in \mathcal{J} \text{ and } N(b) = (1)_{M \in \mathrm{Max}(D) \setminus M_\infty}\} \subseteq \mathbb{N}.$$

Now there exist $b' \in \mathcal{J}$ such that b' achieves the minimal value of S. Now we have $\mathcal{J} = (b')$. Now suppose $\mathcal{J} = (b)$. With the exception of M_∞, all the maximal ideals are of the form $M_i = (m_i)$ and $\nu_{M_j}(m_i) = 1$ if $i = j$ and is zero if $i \neq j$. If $\nu_{M_i}(b) > 1$ for some i, then $\frac{b}{m_i} \in \mathcal{J}$ and $N(\frac{b}{m_i}) < N(b)$, which is impossible. Thus $(b) = (1)_{M \in \mathrm{Max}(D) \setminus M_\infty}$. □

Contained within this proof is the proof of a nice corollary.

Corollary 3.2.8. *In a sequence domain we always have $N(\mathcal{J}) = (1)_{M \in Max(D) \setminus M_\infty}$*

Corollary 3.2.9. *In a sequence domain, either $\mathcal{J}$ is principal or $\mathcal{J}$ is not finitely generated.*

Proof. If $\mathcal{J}$ is not principal, then for every $b \in \mathcal{J}$ we can find an m_i which generates one of the principal maximal ideals such that $\frac{b}{m_i} \in \mathcal{J}$. Thus we can never find an element in $\mathcal{J}$ that is minimal with respect to N. Thus $\mathcal{J}$ must not be finitely generated. □

3.3. Multiplicatively Closed Sets

In this section we will classify elements with regard to properties of their norms. We will let D be an almost Dedekind domain and $\mathrm{Max}(D)$ be the set of maximal ideals of D.

Definition 3.3.1. *We say that $b \in D$ is of finite norm if $N(b)$ has only finitely many entries that are nonzero. This is equivalent to saying that b is in only finitely many maximal ideals of Max(D). We will denote this by $N(b) < \infty$.*

Definition 3.3.2. *We say that $b \in D$ is of infinite norm, if $N(b)$ has infinitely many entries that are nonzero. That is b is in infinitely many maximal ideals. We will denote this by $N(b) = \infty$.*

Since a Dedekind domain is Noetherian, every element in a Dedekind domain is of finite norm. In a purely almost Dedekind domain (a domain that is almost Dedekind and not Dedekind) there must exist an element of infinite norm, else the domain would be Noetherian (a proof of this fact can be found in [7]).

Theorem 3.3.3. *If $b \in D$ and $N(b) < \infty$, then b can be written as a product of atoms.*

Proof. By Lemma 3.1.5 we know if a is a proper divisor of b, then $N(a) < N(b)$. Now there are only finitely many possible divisors. Thus we may express b as a product of atoms. □

Theorem 3.3.4. *If $b \in D$ with $N(b) = \infty$ and b can be written as a product of atoms, $b = \alpha_1 \alpha_2 \cdots \alpha_n$, then $N(\alpha_i) = \infty$ for some atom α_i.*

Proof. $N(b) = N(\alpha_1\alpha_2\cdots\alpha_n) = N(\alpha_1) + N(\alpha_2) + \cdots + N(\alpha_n)$ If $N(\alpha_i) < \infty$ for all i, then we would have $N(b) < \infty$. Thus for some i we have $N(\alpha_i) = \infty$. $\square$

Now we state a corollary that we will use repeatedly when working with an atomic purely almost Dedekind domain.

Corollary 3.3.5. *If D is a atomic purely almost Dedekind domain, then there is an atom, α, in D such that $N(\alpha) = \infty$.*

Definition 3.3.6. *We say $b \in D$ is bounded if $N(b)$ is a bounded net. That is if there exists a ρ such that $\nu_M(b) < \rho$ for all $M \in Max(D)$. We will call b unbounded if it is not bounded.*

Theorem 3.3.7. *If $b \in D$ is unbounded and $b = \alpha_1\alpha_2\cdots\alpha_n$ is a product of atoms, then α_i is unbounded for some i.*

Proof. $N(b) = N(\alpha_1\alpha_2\cdots\alpha_n) = N(\alpha_1) + N(\alpha_2) + \cdots + N(\alpha_n)$. Now if all the α_i were bounded, then b would be bounded. Thus there exists an i such that α_i is unbounded. $\square$

Now we switch our attention grouping these different types of elements into multiplicative sets.

Proposition 3.3.8. *The set $F = \{a|\, N(a) < \infty\}$ of elements of finite norm is a multiplicatively closed, saturated set.*

Proof. Take $a, b \in F$. Then ab is in only finitely many maximal ideals, hence ab is in F. Now suppose $b \in F$. If $a|\, b$, then $N(a) < N(b)$ hence we must have $a \in F$. $\square$

Now we know that any multiplicatively closed, saturated set is the set complement of a union of prime ideals. In an almost Dedekind domain (or any domain of dimension one) nonzero prime ideals are maximal. That is there exists $\{M_\lambda\}_{\lambda\in\Lambda} \subseteq \mathrm{Max}(D)$ such that

$$F = \left(\bigcup_{\lambda\in\Lambda} M_\lambda\right)^C.$$

Thus the maximal ideals in the union consist of only elements of infinite norm. It follows directly that these maximal ideals must be dull or hidden, for non-hidden sharp primes always have elements of finite norm.

Proposition 3.3.9. *The set $L = \{a|\, N(a) \text{is bounded}\}$ of bounded elements forms a multiplicatively closed, saturated set.*

Proof. Let a, b be in L. Now since $N(a)$ and $N(b)$ are bounded, we must have $N(ab) = N(a) + N(b)$ being bounded. Thus $ab \in L$. If b is in L and a divides b, then $N(a) < N(b)$. Thus $N(a)$ is bounded, and $a \in L$. $\square$

Now as before there are is a set $\{M_\lambda\}_{\lambda\in\Lambda} \subseteq \mathrm{Max}(D)$ such that

$$L = \left(\bigcup_{\lambda\in\Lambda} M_\lambda\right)^C.$$

Now an element is either bounded or it is unbounded. Thus the set of unbounded elements is

$$L^C = \bigcup_{\lambda\in\Lambda} M_\lambda\,.$$

Now as before we see a maximal ideal contained in the union must have every element being unbounded. Further an element a is unbounded only if $N(a) = \infty$. We have the following containments.

$$F \subseteq L \quad \text{and} \quad L^C \subseteq F^C\,.$$

What we are describing in a way, is how "badly" some maximal ideals behave with respect to the norm map. That is non-hidden sharp primes are “nice” for they are merely the radical of a principal ideal. Sharp primes that are hidden contain only elements of infinite norm, they are in this sense less well behaved. The same statement is true for dull primes. Then there is an even “worse” set of maximal ideals which we will call the unbounded primes. The unbounded primes consist of only unbounded elements. It is not clear whether a hidden prime can be an unbounded prime. It is however seen in the example of D^∞ that the dull prime M_∞ is unbounded.

CHAPTER 4. PROPERTIES OF ATOMIC ALMOST DEDEKIND DOMAINS

We start this section with the aim of determining whether a purely almost Dedekind domain is atomic. To begin with we make a definition.

4.1. Properties of Atomic Almost Dedekind Domains

Definition 4.1.1. *We say an almost domain D is bounded, or a bounded domain, if for all $b \in D$ we have b is of bounded norm.*

One should ask if this definition makes sense. In Chapter 2, we learned about glad and SP-domains. Both of these classes of domains are bounded. The construction D^q in Chapter 3 is another example of a bounded domain. However we know that the class of bounded domains is not the entire class of almost Dedekind domains (e.g., consider D^∞). We will call domains that are not bounded, unbounded domains.

We start by showing that if D is an atomic bounded domain, then $\mathcal{J} = 0$. One of the central techniques is to use the addition property of valuations. That is $\nu(a + b) \geq \min\{\nu(a), \nu(b)\}$ where we have equality if $\nu(a) \neq \nu(b)$. However using this idea is only fruitful if we can control the sum in all of the maximal ideals. We introduce a new notation for $b \in D$ we denote the set of maximal ideals that contain b by $\max(b)$. That is $\max(b) = \{M |\, b \in M\}$.

Theorem 4.1.2. *If D is an atomic purely almost Dedekind domain that is bounded, then the Jacobson radical $\mathcal{J} = \bigcap_{M \in Max(D)} M = 0$.*

Proof. Consider $S = \{n | b \in \mathcal{J}$ and $b = \alpha_1\alpha_2 \cdots \alpha_n$ is an atomic factorization$\}$. Now set k to be the minimal element of S.

Suppose $k = 1$. Then there exists an atom $\alpha \in \mathcal{J}$. Now we note that $\rho > \nu_M(\alpha) > 0$ for all M and some fixed integer ρ, since D is bounded. Now there exists

$b \in D$ with $b \notin \mathcal{J}$. We set $\xi = b^\rho + \alpha$ and note

$$\nu_M(\xi) = \begin{cases} \nu_M(\alpha) & M \in \max(b) \\ 0 & M \notin \max(b) \end{cases}$$

Thus we see $N(\xi) < N(\alpha)$, hence ξ divides α and ξ is not a unit. Hence α is not an atom.

Since there are no atoms in $\mathcal{J}$ and we are assuming that D is atomic, we can find $d \in \mathcal{J}$ such that $d = \beta_1\beta_2\cdots\beta_k$ where the β_i are atoms and k is chosen to be the minimal element of S. Since one the β_i's is in more than one maximal ideal, we will assume that β_1 is in more than one maximal ideal without loss of generality. Let $\beta_1 \in P, M'$. Set $b = \beta_2\beta_3\cdots\beta_k$, and note $b \notin \mathcal{J}$. Further we can choose P such that $b \notin P$ since $b \notin \mathcal{J}$.

We need to ensure that we create an element that overlaps with β_1 at some maximal ideal. Now we find $c \in M'$ with $c \notin P$. Now we note that $\rho > \nu_M(\beta_1) > 0$ for all $M \in \max(\beta_1)$ and some fixed integer ρ. Set $\xi = (bc)^\rho + \beta_1$. Thus we have

$$\nu_M(\xi) = \begin{cases} \nu_M(\beta_1) & M \in \max(bc) \cap \max(\beta_1) \\ 0 & M \in \max(bc) \setminus \max(\beta_1) \\ 0 & M \in \max(\beta_1) \setminus \max(bc) \end{cases}$$

Thus we see $N(\xi) < N(\beta_1)$, hence ξ divides β_1. Thus β_1 is not an atom. We conclude that the intersection must be trivial. $\square$

Now we state some important corollaries.

Corollary 4.1.3. *A glad domain is atomic if and only if it is a semi-local PID.*

Proof. A glad domain is bounded and has a nonzero Jacobson radical. Hence if it is atomic it can not be purely almost Dedekind, hence it must be Dedekind. A Dedekind domain with finitely many primes is a semi-local PID. $\square$

Corollary 4.1.4. *If D is an atomic SP-domain, then $\mathcal{J} = 0$.*

We could also use this theorem to state that a bounded sequence domain is never atomic, but in fact we are in position to show, in general, that a sequence domain is never atomic. Recall that sequence domains are almost Dedekind and are never Dedekind. All sharp primes of a sequence domain are principal. If $M_i = (m_i)$, we will call m_i a sharp atom, or the sharp atom associated with M_i.

Theorem 4.1.5. *Let D be a sequence domain, then D is not atomic. Further the only atoms in D are the generators of the principal maximal ideals.*

Proof. If D is atomic, then there exists an atom α that is contained in infinitely many maximal ideals. But that is $\alpha \in M_i = (m_i)$. But this is impossible for m_i would divide α. Thus D cannot be atomic. $\square$

We see from the above theorem that $\mathcal{J}$ plays an important role in determining atomicity. This motivates the study of almost Dedekind domains with nonzero Jacobson radical. It should be noted again that the Jacobson radical in the Grams example is zero.

4.2. Almost Dedekind Domains with nonzero Jacobson Radicals

Recall it was shown that if D is an atomic almost Dedekind domain then Norm(D) is an atomic monoid. We will use this to show that if D is an atomic almost Dedekind domain with a nonzero Jacobson radical then the sharp primes in Max(D) can be removed with atomicity preserved. But first we state a known result.

Theorem 4.2.1. *If D is an almost Dedekind domain with $\mathcal{J} \neq 0$ and M is a sharp prime in Max(D), then M is principal.*

Proof. See [11]. □

We can show that in an atomic almost Dedekind domain with $\mathcal{J} \neq 0$ that every sharp prime is principal, hence is generated by a sharp atom. The proof involves a technique that we call "blasting". The idea is to build an element b such that we can find a divisor a of b. We then create a new element $\xi = \frac{b}{a}$ that has the desired property. Recall that a divides b if and only if $N(a) < N(b)$. Using this we can often take a high enough power of b to ensure that a divides b. We now present a less general statement of the above theorem and prove it using this technique. The hope is that this serves as a sort of "warm up" for future proofs.

Lemma 4.2.2. *If D is an atomic almost Dedekind domain with $\mathcal{J} \neq 0$, then all sharp primes are principal.*

Proof. We know from [7], that if M is a sharp prime, there exists $m \in M$ such that $\sqrt{(m)} = M$. We take m to be the sharp atom associated with M. All we must show is that $\nu_M(m) = 1$, since $\nu_N(m) = 0$ for all $N \neq M$. Suppose $\nu_M(m) = r > 1$. Now we take $0 \neq d \in \mathcal{J}$, and we find $c \in D$ that is a generator of M, thus $\nu_M(c) = 1 < r$. We construct the element $\xi = cd^r \in \mathcal{J}$ and we note that $\nu_M(\xi) = r\nu_M(d) + 1$. Now we have $m^{\nu_M(d)}$ dividing ξ. Furthermore the quotient

$$t = \frac{\xi}{m^{\nu_M(d)}} \text{is in } \mathcal{J}$$

and $\nu_M(t) = 1 < r$. Thus $t + m$ is only in M and has value 1 on M. Thus we must have $\nu_M(m) = 1$, else m would not be an atom. Hence for any $b \in M$, we have m dividing b and M is principal as claimed. □

Corollary 4.2.3. *If D is an atomic almost Dedekind domain with $\mathcal{J} \neq 0$, then D contains no hidden primes.*

Proof. Recall hidden primes are sharp primes that are covered by the union of all other primes. As hidden primes are sharp they would need to be principal, thus they cannot be covered by the union. Hence there are no hidden primes. □

Now we see that if D is an atomic almost Dedekind domain with $\mathcal{J} \neq 0$, then D must not have too many sharp primes. For with every sharp prime in the case we have an associated sharp atom. If we have too many sharp atoms, it will be hard to keep an atomic factorization finite. This is stated more precisely in the next lemma.

Lemma 4.2.4. *If D is an atomic almost Dedekind domain with $\mathcal{J} \neq 0$, then D can only have finitely many sharp primes.*

Proof. Suppose D has infinitely many sharp primes. Let $0 \neq d \in \mathcal{J}$. Since D is atomic we factor d into atoms as $d = \alpha_1\alpha_2 \cdots \alpha_n$. Now α_i must be in infinitely many sharp primes for some i. But if $\alpha_i \in M_i$ for some sharp prime M_i then it must be divisible by m_i where m_i is the sharp atom associated with M_i. Thus this factorization is not an atomic factorization. We conclude that D has only finitely many sharp primes. □

Now that we know that an atomic almost Dedekind domain with $\mathcal{J} \neq 0$ has only finitely many sharp primes, one might ask what role do these sharp primes play? Is there a way of removing them and preserving atomicity? The answer is yes if there are only finitely many sharp primes, and is stated in the next lemma.

Lemma 4.2.5. *If D is an atomic almost Dedekind domain with a sharp prime $M = (\alpha)$, then $D[\alpha^{-1}]$ is atomic.*

Proof. Let $D' = D[\alpha^{-1}]$.

We note that α is not in any other maximal ideal of D, else M would not be maximal. Thus adjoining α^{-1} only annihilates M. That is $\mathrm{Max}(D) \setminus M = \mathrm{Max}(D')$.

Now for $b \in D$ we can write $b = \alpha^r \beta_1 \beta_2 \cdots \beta_n$ where $\nu_M(b) = r$ and none of the β_i are in M. Thus in $D' = D[\alpha^{-1}]$, $b = \beta_1 \beta_2 \cdots \beta_n$ as α becomes a unit. Now we only need to verify that atoms in D remain atoms in D' with the only exception being α. If $\beta \in D$ is an atom other than α then $\beta \notin M$. Now $N(\beta)$ is irreducible in the monoid $\mathrm{Norm}(D)$. Furthermore any factorization of $N(\beta)$ in $\mathrm{Norm}(D')$ would yield a factorization in $\mathrm{Norm}(D)$. Thus β remains an atom. $\square$

We are now in a position to state a very powerful observation, with some interesting corollaries. The above lemma showed that sharp primes are so well behaved, that they can be removed without affecting atomicity. In fact, if an element is a product of atoms with some of its atoms contained in a sharp prime we are able to write down its new factorization (in a suitable overring) when we annihilate the sharp prime. The new element has the exact same norm with the exception that the entry for M no longer exists for M is no longer a maximal ideal.

Recall a dull domain is a domain with only dull primes.

Theorem 4.2.6. *If D is an atomic purely almost Dedekind domain with $\mathcal{J} \neq 0$. Then there exists a dull domain D' derived from D that remains atomic.*

Proof. If D has no sharp primes there is nothing to prove. If D has sharp primes they are all principal. Moreover there can only be only finitely many principal primes. Let $(\alpha_1), (\alpha_2), \cdots (\alpha_n)$ be the list of sharp primes. Now we apply the previous lemma a finite number of times to arrive at $D' = D[\alpha_1^{-1}, \alpha_2^{-1}, \cdots \alpha_n^{-1}]$. Now we know D' is an atomic domain. Furthermore D' has no sharp primes, for we annihilated the sharp primes.

It suffices to show if $Q \subseteq D$ is a dull prime then $QD[\alpha^{-1}] \subseteq D[\alpha^{-1}]$ is dull. If $QD[\alpha^{-1}]$ is not dull, then it must be sharp. But since $\mathcal{J}(D[\alpha^{-1}]) \neq 0$, we must have $QD[\alpha^{-1}]$ being principal, say $QD[\alpha^{-1}] = (x)$. We may choose $x \in D$, but then x is contained only in $QD[\alpha^{-1}]$. So in D, x can only be contained in (α) and Q. But if $x \in (\alpha)$, then there is some positive integer n such that α^n completely divides x. Therefore $\frac{x}{\alpha^n}$ only in Q, which contradicts Q being dull. $\square$

We saw the power of boundedness at the start of this chapter. What can we say about a bounded almost Dedekind domain with a nonzero Jacobson radical? We present the following result. Recall an antimatter domain is a domain that contains no atoms. That is, if D is antimatter and $b \in D$ is a nonzero non-unit, then we can always find a divisor of b. For more on antimatter domains see [5]. We establish a lemma and some notation before presenting the result.

For b in D and $M_\alpha \in \text{Max}(D)$, we let $b_\alpha = \nu_{M_\alpha}(b)$. That is b_α is the integer equal to the value of b on M_α.

Lemma 4.2.7. *Let D be a bounded dull domain with $\mathcal{J} \neq 0$ and let $b \in D$ be a nonzero non-unit. There exists $c \in \mathcal{J}$ such that b divides c with the property that for some positive integer n and some maximal ideal $M \in max(b)$ we have $\nu_M(\frac{c}{b^n}) \neq 0$. We will say such a c is not a power of b on $max(b)$.*

Proof. Let $b \in D$. We find $c \in \mathcal{J}$ such that b divides c. To construct such an element we observe that if $\nu_M(b) < \rho$ for all $M \in \text{Max}(D)$ and $c \in \mathcal{J}$, then b divides c^ρ.

Now suppose $\nu_M(\frac{c}{b^n}) = 0$ for all $M \in \max(b)$ and some positive integer n. Then we find two maximal ideals M and M' of $\max(b)$ and an element $d \in M$ such that $d \notin M'$. Now $cd \in \mathcal{J}$ and cd has the desired property. That is $\nu'_M(\frac{cd}{b^n}) = 0$, but $\nu_M(\frac{cd}{b^n}) \neq 0$ It should be noted that $\max(b)$ is an infinite set since D is dull. $\square$

Theorem 4.2.8. *Let D be a bounded dull domain with $\mathcal{J} \neq 0$, then D is an antimatter domain.*

Proof. Take $b \in D$ with $\{M_\gamma\}_{\gamma \in \Gamma} = \max(b)$. We find $c \in \mathcal{J}$ such that b divides c and $\nu_M(\frac{c}{b}) \geq 1$ for some $M \in \max(b)$. Now as D is bounded there exists positive integers

ρ and π such that for all M we have $\rho > \nu_M(b)$ and $\pi > \nu_M(c)$. Now we consider the set

$$\left\{ \frac{\nu_M(b)}{\nu_M(c)} \middle| \, M \in \mathrm{Max}(D) \right\} \subset \mathbb{Q} \cap [0,1] \,.$$

This set is finite for there are only finitely many choices for the numerator and finitely many choices for the denominator, thus it must contain its supremum, say d. We find $\alpha \in \Gamma$ such that

$$\frac{\nu_{M_\alpha}(b)}{\nu_{M_\alpha}(c)} = d.$$

That is $\frac{b_\alpha}{c_\alpha} = d$. Now we consider b^{c_α} and c^{b_α}. We have for all maximal ideals M, $\nu_M(b^{c_\alpha}) = c_\alpha \nu_M(b)$ and $\nu_M(c^{b_\alpha}) = b_\alpha \nu_M(c)$. We claim b^{c_α} divides c^{b_α}. We verify by observing for all M we have

$$b_\alpha \nu_M(c) - c_\alpha \nu_M(b) = \nu_M(c) \left(b_\alpha - c_\alpha \left(\frac{\nu_M(b)}{\nu_M(c)} \right) \right) \geq \nu_M(c) \left(b_\alpha - c_\alpha \left(\frac{b_\alpha}{c_\alpha} \right) \right) = 0 \,.$$

Thus b^{c_α} divides c^{b_α}. Let Λ be the set of maximal ideals that do not contain the quotient. That is $\frac{c^{b_\alpha}}{b^{c_\alpha}} \in \mathrm{Max}(D) \setminus \Lambda \neq \emptyset$ where $\Lambda \subsetneq \Gamma$. Note the quotient is zero in the α^{th} slot, and is not zero on the entire set of Γ because of our insistence c is not a power of b on Γ. Now we have

$$N\left(b + \left(\frac{c^{b_\alpha}}{b^{c_\alpha}} \right)^\rho \right) = \begin{cases} 0 & M \in \Lambda \\ \nu_M(b) & M \in \Lambda^c \cap \Gamma \\ 0 & M \in \Gamma^c \end{cases} .$$

And we note

$$N\left(b + \left(\frac{c^{b_\alpha}}{b^{c_\alpha}} \right)^\rho \right) < N(b) \,.$$

Thus we have found a divisor of b. We conclude that D is an antimatter domain. $\square$

We use this result along with other work we have already to done to see the following corollary.

Corollary 4.2.9. *For any fixed $n \in \mathbb{N}$, there exists an almost Dedekind domain with exactly n atoms.*

Proof. We start with a Dedekind domain D with $n+1$ maximal ideals. It must be the case that all of these maximal ideals are principal and $\mathcal{J} \neq 0$. We now split one of the maximal ideals as we did in the construction of D^q leaving the remaining n maximal ideals inert. The result is a bounded almost Dedekind domain with a nonzero Jacobson radical with exactly n sharp primes and an uncountable number of dull primes. The previous result shows there are no atoms contained in any of the dull primes. $\square$

Now we have seen that if D is an atomic almost Dedekind domain with a nonzero Jacobson radical, then its "dull part" plays the major role. In particular we now know that an atomic purely almost Dedekind domain with only dull primes cannot be bounded. In fact much more is true.

4.3. Completely Unbounded Domains

Definition 4.3.1. *We call a domain D completely unbounded, if for all nonzero non-units $b \in D$ we have b unbounded. We will call these completely unbounded domains.*

It should be noted that this definition may be vacuous. At this time we know of no examples of completely unbounded domain. The definition is a necessary condition for a dull almost Dedekind domain with $\mathcal{J} \neq 0$ to be an atomic domain.

Theorem 4.3.2. *If D is an atomic dull domain with $\mathcal{J} \neq 0$, then D is completely unbounded.*

Proof. Suppose there exists a bounded element $b \in D$. Since we are assuming D is atomic, this implies there must exist a bounded atom. Thus we will take b to be a bounded atom. Now we find $c \in \mathcal{J}$ such that $b|c$ and c is not a multiple of b on $\Gamma = \max(b)$. Now if c were a bounded element we could find our divisor as we did in the previous theorem. Thus we need to circumnavigate the fact that c might be unbounded, but as we will see this just involves some observations.

Now we are assuming b is bounded so we find ρ with $\rho > \nu_M(b)$ for all $M \in \mathrm{Max}(D)$. Our previous proof relied upon finding the supremum of

$$\left\{ \frac{\nu_M(b)}{\nu_M(c)} \middle| M \in \mathrm{Max}(D) \right\} \subset \mathbb{Q}\,.$$

We knew such a supremum existed in the set, because the set was finite. But now that we do not have an upper bound on the value of c, the set might be infinite. However, we will see that the set contains its supremum. Let

$$\Sigma_1 = \left\{ \frac{1}{\nu_M(c)} \middle| M \in \mathrm{Max}(D) \text{ and } \nu_M(b) = 1 \right\}.$$

Now the set $\sigma_1 = \{\nu_M(c) | M \in \mathrm{Max}(D) \text{ and } \nu_M(b) = 1\}$ is a subset of $\mathbb{N}_0$. We should note that $\nu_M(c) \neq 0$ for any M, because $c \in \mathcal{J}$. Thus Σ_1 contains a least element, say τ_1. Now $\frac{1}{\tau_1}$ is the supremum of Σ_1. Now for $1 \leq i < \rho$ we define

$$\Sigma_i = \left\{ \frac{i}{\nu_M(c)} \middle| M \in \mathrm{Max}(D) \text{ and } \nu_M(b) = i \right\}.$$

Again we set $\sigma_i = \{\nu_M(c) | \nu_M(b) = i\}$. We find the least element τ_i of σ_i and note that $\tau_i \geq i$ since $b|c$. Now $\frac{i}{\tau_i}$ is the supremum of Σ_i. Now since b is bounded in value by ρ we see

$$\sup \left\{ \frac{\nu_M(b)}{\nu_M(c)} \middle| M \in \mathrm{Max}(D) \right\} = \sup(\Sigma_1, \Sigma_2, \cdots \Sigma_{\rho-1}) = \sup\left(\frac{1}{\tau_1}, \frac{2}{\tau_2}, \cdots \frac{\rho-1}{\tau_{\rho-1}} \right).$$

Thus the supremum exists and is in the set. Now we finish the proof just like in the previous theorem. We find an element that realizes the supremum and we use the "blasting" technique. Thus there must not exist any bounded elements in D. We conclude that D is a completely unbounded domain. □

It is not clear at the present whether one can construct a completely unbounded almost Dedekind domain. If no such completely unbounded domains exist, then we would be able to conclude that an atomic almost Dedekind domain with a nonzero Jacobson radical must, in fact, be Dedekind. Even if one can construct a completely unbounded domain, it is the belief of the author that it would most likely be impossible to construct and atomic completely unbounded domain. The rationale for this hypothesis is that unbounded elements have many possible divisors, and in order for a completely unbounded domain to be atomic it would need to contain unbounded atoms. This seems highly unlikely, but the question remains unanswered.

Without a solid construction proving the existence of completely unbounded domains, one should be cautious in stating results about said domains. In the attempt to disprove or discover their existence, a number of properties were discovered.

Theorem 4.3.3. *A completely unbounded domain D with $\mathcal{J} \neq 0$ must be a dull domain.*

Proof. Suppose D has a sharp prime M, then $M = (m)$ for some $m \in D$ Furthermore m is contained only in M hence $N(m)$ is bounded. □

It is possible that the sum of two unbounded elements might be bounded, this was part of the rationale in the next result. In hopes of disproving the existence of completely unbounded domains the following lemmas were established, although as of yet they have not led to any conclusions one way or the other about the existence of completely unbounded domains.

Let a, b be elements of a completely unbounded domain D We define the symmetrically related sets:

$$L = \{M \in \text{Max}(D) |\, \nu_M(a) < \nu_M(b)\},$$

$$E = \{M \in \text{Max}(D) |\, \nu_M(a) = \nu_M(b)\},$$

$$G = \{M \in \text{Max}(D) |\, \nu_M(a) > \nu_M(b)\}.$$

Lemma 4.3.4. *If D is a completely unbounded domain with $\mathcal{J} \neq 0$. Then for $a, b \in D$ with $a+b$ a non-unit we must have either $L' = \{\nu_M(a) |\, M \in L\}$ or $G' = \{\nu_M(b) |\, M \in G\}$ unbounded or E being infinite.*

Proof. Suppose L' and G' are both bounded and E is finite. Then

$$N(a+b) = \begin{cases} \nu_M(a) & M \in L \\ \nu_M(b) & M \in G \\ \nu_M(a+b) & M \in E \end{cases}$$

It is not clear what $\nu_M(a+b)$ is on E, since the values are equal on the maximal ideals of E. Now we must have $a + b$ being a bounded element which is impossible. □

With regard to the atoms we have the following.

Lemma 4.3.5. *Suppose α, β are atoms in a completely unbounded domain D such that $\alpha + \beta$ is a non-unit, then $E \neq \emptyset$.*

Proof. Suppose $E = \emptyset$ Then

$$N(\alpha+\beta) = \begin{cases} \nu_M(\alpha) & M \in L \\ \nu_M(\beta) & M \in G \end{cases}$$

But now $N(\alpha+\beta) < N(\alpha)$ and $(\alpha+\beta) | \alpha$. But α was an atom, thus we must have $E \neq \emptyset$. $\square$

We now present an approximation theorem for an almost Dedekind domain. We will then interpret the result in the case of completely unbounded domains.

Theorem 4.3.6. *Let D be an almost Dedekind domain. If $S \subset Max(D)$ is such that $\cap_{M\in S} M \neq \mathcal{J}$ then there exists a non-unit $b \in D$ such that $\nu_M(b) = 0$ for all $M \in S$.*

Proof. We find $b \in (\cap_{M\in S} M) \setminus \mathcal{J}$. Now since $b \notin \mathcal{J}$, there exist $r \in D$ such that $1 - rb$ is not a unit. Now $rb \in M$ for all $M \in S$. Hence $1 - rb \notin M$ for all $M \in S$, hence $\nu_M(1 - rb) = 0$ for all $M \in S$. $\square$

In terms of completely unbounded domains we have the following.

Corollary 4.3.7. *Let D be an almost Dedekind domain and let $0 \neq a \in \mathcal{J}$. For $n \in \mathbb{N}$ set $S_n = \{M | \nu_M(a) \geq n\}$. If D is completely unbounded, then $\cap_{M\in S_n} M = \mathcal{J}$.*

Proof. Suppose we have $\cap_{M\in S_n} M \neq \mathcal{J}$. Then there exists non-unit $b \in D$ such that $\nu_M(b) = 0$ for all $M \in S_n$. Now we have

$$N(b^n + a) = \begin{cases} 0 & M \in S_n \\ < n & M \notin S_n. \end{cases}$$

But then $b^n + a$ is a bounded element, hence D is not completely unbounded. $\square$

All of these conditions seem quite restrictive, it is the belief of the author that they will be useful in establishing the existence of a completely unbounded domain with a nonzero Jacobson radical or in the proof that no such almost Dedekind domain exists.

CHAPTER 5. CLASS GROUPS AND ALMOST DEDEKIND DOMAINS

We present the first theorem discovered in this study. The idea was that there cannot be "too many" principal ideals running around, else we will fail to have finite atomic factorizations. If the class group of a domain is finite, then sharp maximal ideals are always non-hidden. That is, they are radicals of principal ideals. This is clear since if M is sharp, then M is invertible. Further if the class group is finite we have a finite power of M being principal.

5.1. Class Groups and Atomicity

Theorem 5.1.1. *Let D be an atomic purely almost Dedekind domain with finite class group. Then D must have infinitely many dull primes.*

Proof. Suppose D has only finitely many dull primes. Suppose the order of the class group of D is r. Let $C_1, C_2, \cdots C_r$ be the classes of ideals. Now since D is atomic, it must contain an atom α of infinite norm, and since D has only finitely many dull primes, α must be contained in infinitely many sharp primes. Recall sharp primes are invertible and finitely generated. Thus there exists some class C_i (not the principal class) such that α is contained in $r+1$ distinct maximal ideals of class C_i; call these ideals $M_1, M_2, \cdots M_r, M_{r+1}$. Now

$$\alpha \in M_1 \cap M_2 \cap \cdots \cap M_r = M_1 M_2 \cdots M_r = (\beta)$$

since the product must be principal (any product of r ideals from C_i is principal). Thus we must have $(\alpha) = (\beta)$ since β cannot be a proper divisor of α. But then we also have

$$\alpha \in M_1 \cap M_2 \cap \cdots \cap M_{r-1} \cap M_{r+1} = M_1 M_2 \cdots M_{r-1} M_{r+1} = (\gamma)$$

and, as before, $(\alpha) = (\gamma)$. But this leads us to conclude that

$$M_1 M_2 \cdots M_{r-1} M_r = M_1 M_2 \cdots M_{r-1} M_{r+1}$$

But since D is almost Dedekind we have the cancellation law for ideals. Thus $M_r = M_{r+1}$. This is a contradiction. Thus α cannot be contained in $r+1$ ideals from the same class. Hence α can only be contained in only finitely many sharp primes. Thus α must be contained in infinitely many dull primes. We conclude that D must have infinitely many dull primes. $\square$

We can also use conditions on the class group to determine necessary properties for an almost Dedekind domain to be ACCP.

Theorem 5.1.2. *Suppose D is an purely almost Dedekind domain with only finitely many dull primes. If the class group, $\mathfrak{C}(D)$, has finite order then D does not satisfy ACCP.*

Proof. Set the order of $\mathfrak{C}(D) = r$. Now there exists an $\alpha \in D$ such that α is in infinitely many maximal ideals and contained in at least one of the dull primes. Since D has only finitely many dull primes α must be in infinitely many sharp maximal ideals, say $S = \{M_\lambda\}_{\lambda \in \Lambda}$ We set

$$N(\alpha) = \big(\nu_M(\alpha)\big)_{M \in \mathrm{Max}(D)}.$$

We fix a countable subset of S, say $M_1, M_2, M_3, \cdots$. Associated with these sharp primes are sharp atoms, m_i. Now M_i^r is a principal ideal, hence there is an element with value r in M_i and value zero in every other maximal ideal. Thus it must be the case that $\nu_{M_i}(m_i)$ divides r. (Recall Theorem 3.1.9.) Now we are ready to proceed. We consider α^r. We set $\xi_1 = \frac{\alpha^r}{m_1}$, $\xi_2 = \frac{\xi_1}{m_2}$, and continue inductively and set $\xi_n = \frac{\xi_{n-1}}{m_n}$.

Now we have the following chain of principal ideals.

$$(\alpha^r) \subsetneq (\xi_1) \subsetneq (\xi_2) \subsetneq \cdots \subsetneq (\xi_n) \subsetneq \cdots$$

Now this chain cannot end, else we would have an element of finite norm in a dull prime which is impossible. We conclude that D is not ACCP. $\square$

So we see that if D has a class group of finite order and satisfies ACCP, then D must have infinitely many dull primes. What can we say about atomicity with respect to a purely almost Dedekind domains with a zero Jacobson radical? This question is clearly more difficult, for we know from the Grams' example that a domain of this type can be atomic (or ACCP). Thus this class is strictly larger than the class of Dedekind domains. But we do have the following theorem that suggests that such a domain would need a large number of dull primes.

Theorem 5.1.3. *Suppose D is a purely almost Dedekind domain with a class group of order less than or equal to 2. If D has only finitely many dull primes, then D is not atomic.*

Proof. Suppose D is an atomic purely almost Dedekind domain. First note that if the class group is trivial, then every sharp prime is principal. Since D is purely almost Dedekind we can find an atom α contained in infinitely many maximal ideals. Now as D has only finitely many dull primes, α must be in infinitely many sharp maximal ideals. But now α is divisible by the sharp atoms associated to the sharp primes that contain it. Thus no such atom exists.

Suppose the class group has order 2. We fix a sharp prime M' with associated sharp atom m'. Again, we find an atom α that is contained in infinitely many sharp primes. We note that α is not contained in any principal primes, else it would not be an atom. For all non-principal sharp primes $\nu_M(\alpha) = 1$, else α would be divisible by the sharp atom associate with the sharp prime. The sharp atom has value at most 2 in its associated sharp prime. That is $\nu_M(\alpha) = 1$ for all sharp primes in $\max(\alpha)$.

Now we consider α^2 and note $\nu_M(\alpha^2) = 2$ for all sharp primes in $\max(\alpha)$. We know the sharp atom m' of M' has value 2 on M', and has value 0 everywhere else. Now we consider $\xi = \frac{\alpha^2}{m'}$ and we note that $\max(\alpha) \setminus M' = \max(\xi)$. Thus we have $\nu_M(\xi) = 2$ for all sharp primes in $\max(\xi)$ and $\nu_{M'}(\xi) = 0$. We know that ξ is not an atom for it is divisible by all of the sharp atoms of the sharp primes in $\max(\xi)$. So we factor ξ as a product of atoms, say $\xi = \beta_1\beta_2 \cdots \beta_k$. Now one of the β_i is contained in more than one sharp prime, without loss of generality, say β_1. But now $\nu_M(\beta_1) = 1$ on all the sharp primes contained in $\max(\beta_1)$. Note if this were not the case β_1 would be divisible by some sharp atom. However $\max(\beta_1) \subsetneq \max(\alpha)$, thus $N(\beta_1) < N(\alpha)$. Hence β_1 divides α and α is not an atom, and we conclude that D is not atomic. $\square$

Now one might be able to make a combinatorial argument for class groups of higher (but finite) order to get a similar result. It seems that an atomic purely almost Dedekind domain with any finitely ordered class group must contain infinitely many dull primes. At the present time this is an unresolved question.

CHAPTER 6. FUTURE STUDY

During this study, we have shown that an atomic purely almost Dedekind domain with $\mathcal{J} \neq 0$ must have infinitely many dull maximal ideals. We gave some rationale as to why this might be true if $\mathcal{J} = 0$ and D has a class group of finite order. This suggests a the following might be true.

If D is an atomic purely almost Dedekind domain, then D has infinitely many dull primes.

This is true in the Grams' example.

In the case that $\mathcal{J} \neq 0$, we have shown that if D is atomic, then there must exist an atomic completely unbounded domain with $\mathcal{J} \neq 0$. It would be nice to resolve whether such a completely unbounded domain exists. If such a domain does not exist we would have the following theorem.

If D is an atomic purely almost Dedekind, then $\mathcal{J} = 0$.

This would be a powerful theorem. The evidence seems to suggest it is true, but caution should be used.

These are two questions whose resolution would lead to a better understanding of atomic purely almost Dedekind domains. One more question, which this study lacks the muster to suggest in a strong way to be true, is the following.

If D is an atomic purely almost Dedekind domain, then D has a class group of infinite order.

If this is indeed true it might suggest that the class of atomic purely almost Dedekind domains are all very similar to the Grams' example. While this study did not answer the question of which purely almost Dedekind domains are atomic, it did narrow the choices dramatically. It would be nice to see a complete resolution of the question in the future.

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