CODUALIZING MODULES AND COMPLEXES

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Richard Kenneth Wicklein

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Richard Kenneth Wicklein

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SUPERVISORY COMMITTEE:

Sean Sather-Wagstaff
Chair

Jim Coykendall

Benton Duncan

Jane Schuh

Approved:

July 2, 2013
Date

Dogan Comez
Department Chair

ABSTRACT

A finitely generated R-module C is semidualizing if the R-module homomorphism $\chi_C^R : R \to \operatorname{Hom}_R(C, C)$ given by $\chi_C^R(r)(c) = rc$ is an isomorphism and $\operatorname{Ext}_R^i(C, C) = 0$ for all $i \geqslant 1$. When $(R, \mathfrak{m})$ is local, an artinian R-module T is quasidualizing if the map $\chi_T^{\widehat{R}^{\mathfrak{m}}} : \widehat{R}^{\mathfrak{m}} \to \operatorname{Hom}_R(T, T)$ is an isomorphism and $\operatorname{Ext}_R^i(T, T) = 0$ for all $i \geqslant 1$. In this dissertation we unify these two definitions under one "umbrella" definition. For an ideal $\mathfrak{a}$, an R-module M is $\mathfrak{a}$-codualizing if the R-module $\operatorname{Ext}_R^i(R/\mathfrak{a}, M)$ is finitely generated for all i, the small support of M is contained in $\operatorname{V}(\mathfrak{a})$, one has $\operatorname{Ext}_R^i(M, M) = 0$ for all $i \geqslant 1$, and the map $\chi_M^{\widehat{R}^{\mathfrak{a}}} : \widehat{R}^{\mathfrak{a}} \to \operatorname{Hom}_R(M, M)$ given by $\chi_T^{\widehat{R}^{\mathfrak{a}}}(r)(c) = rc$ is an isomorphism. We study the $\mathfrak{a}$-codualizing condition of modules and R-complexes. We show that $\mathbf{R}\Gamma_{\mathfrak{a}}(R)$ is always an example of an $\mathfrak{a}$-codualizing complex. We also study the Auslander and Bass classes in the context of $\mathfrak{a}$-codualizing complexes. In particular, we prove a version of Foxby equivalence in this context.

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1. INTRODUCTION

In this dissertation the term “ring” is short for “commutative, noetherian ring with identity.” The term “module” is short for “unital module.”

Throughout this dissertation let R be a ring, let $\mathfrak{a} \subsetneq R$ be a proper ideal of R, and let $\widehat{R}^{\mathfrak{a}}$ be the $\mathfrak{a}$-adic completion of R.

The study of dualities is fundamental to many branches of mathematics. A fundamental operation on a given field k and a vector space V is the duality $V \mapsto \operatorname{Hom}_k(V, k)$. The study of rings and modules in this manner can be traced back at least to work of Grothendieck and Hartshorne [22], Auslander and Bridger [2], and Foxby [17], using the following notion which is central to this dissertation.

Definition 1.1. A finitely generated R-module C is *semidualizing* if it satisfies the following conditions:

(i) the map $\chi_C^R : R \to \operatorname{Hom}_R(C, C)$ given by $\chi_C^R(r)(c) = rc$ is an isomorphism, and

(ii) $\operatorname{Ext}_R^i(C, C) = 0$ for all $i \geqslant 1$.

For any ring R, the ring itself as an R-module is an example of a semidualizing module. Also, D is a dualizing module if and only if D is semidualizing and has finite injective dimension. Loosely speaking, among finitely generated modules, semidualizing modules are good for studying dualities. For instance, the example of R gives the duality $M \mapsto \operatorname{Hom}_R(M, R)$ from [2], and when D is dualizing, this recovers Groethendieck’s local duality $M \mapsto \operatorname{Hom}_R(M, D)$ from [22].

However, in the study of dualities, semidualizing modules miss some important examples, such as Matlis duality: when $(R, \mathfrak{m}, k)$ is a local noetherian ring, this is the duality $M \mapsto \operatorname{Hom}_R(M, E)$ where E is the injective hull of the residue field (that is, E is the “smallest” injective module in which k can be embedded). Kubik [26] introduced the next definition to cover this deficiency.

Definition 1.2. Let $(R, \mathfrak{m})$ be local, and let T be an artinian R-module. The fact that T is artinian implies that it is $\mathfrak{m}$-torsion so it has the structure of an $\widehat{R}^{\mathfrak{m}}$-module. In particular, the map $\chi_T^{\widehat{R}^{\mathfrak{m}}} : \widehat{R}^{\mathfrak{m}} \to \operatorname{Hom}_R(T,T)$ given by $\chi_T^{\widehat{R}^{\mathfrak{m}}}(r)(c) = rc$ is a well-defined R-module homorphism. The R-module T is *quasidualizing* if it satisfies the following conditions:

(i) the map $\chi_T^{\widehat{R}^{\mathfrak{m}}} : \widehat{R}^{\mathfrak{m}} \to \operatorname{Hom}_R(T,T)$ is an isomorphism, and

(ii) $\operatorname{Ext}_R^i(T,T) = 0$ for all $i \geqslant 1$.

One always has an example of quasidualizing module since the injective hull of the residue field is a quasidualizing R-module [26, Example 1.17]. The point of this dissertation is to unify these two notions under the following "umbrella" notion.

Definition 1.3. An R-module M is $\mathfrak{a}$-*codualizing* if it satisfies the following conditions:

(i) the R-module $\operatorname{Ext}_R^i(R/\mathfrak{a}, M)$ is finitely generated for all i, and $\operatorname{V}(\mathfrak{a})$ contains the "small support" of M, that is, the set

$$\operatorname{supp}_R(M) := \{\mathfrak{p} \in \operatorname{Spec}(R) : \operatorname{Tor}_i^R(R/\mathfrak{p}, M)_{\mathfrak{p}} \neq 0 \text{ for some } i\},$$

(ii) $\operatorname{Ext}_R^i(M,M) = 0$ for all $i \geqslant 1$, and

(iii) the map $\chi_M^{\widehat{R}^{\mathfrak{a}}} : \widehat{R}^{\mathfrak{a}} \to \operatorname{Hom}_R(M,M)$ given by $\chi_T^{\widehat{R}^{\mathfrak{a}}}(r)(c) = rc$ is an isomorphism. This map is a well-defined R-module homomorphism as in Definition 1.2. See also Remark 3.3.

Note that one recovers the definitions of semidualizing and quasidualizing with $\mathfrak{a} = 0$ and $\mathfrak{a} = \mathfrak{m}$ respectively; see Propositions 3.5 and 3.6. There is some flexibility condition in (i) due to the following theorem of Melkersson [31], which does not assume that M is finitely generated; if M is finitely generated, then the equivalent conditions are automatically satisfied because R is assumed to be noetherian.

Fact 1.4. [31, Theorem 2.1] Let $\mathfrak{a} = (x_1, ..., x_n)$ and let M be an R-module. Then the following conditions are equivalent:

(i) the R-module $\mathrm{Ext}^i_R(R/\mathfrak{a}, M)$ is finitely generated over R for all i,

(ii) the R-module $\mathrm{Tor}^R_i(R/\mathfrak{a}, M)$ is finitely generated over R for all i, and

(iii) the Koszul homology modules $\mathrm{H}_i(\underline{x}; M)$ are finitely generated over R for $i = 0, ..., n$.

While modules are a well-defined setting to study the $\mathfrak{a}$-codualizing condition, it is more natural to consider this property in the derived category. This follows the tradition of studying dualizing properties in the derived category as in work of Grothedieck and Hartshorne [22], Avramov and Foxby [3], and Christensen [10]. See Chapter 2 for background information.

Definition 1.5. A homologically bounded R-complex X is $\mathfrak{a}$-*codualizing* if it satisfies the following conditions:

(i) $\mathrm{Ext}^i_R(R/\mathfrak{a}, X)$ is finitely generated for all i,

(ii) one has $\mathrm{supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$, and

(iii) the natural homothety map $\chi^{\widehat{R}^{\mathfrak{a}}}_C : \widehat{R}^{\mathfrak{a}} \to \mathbf{R}\mathrm{Hom}_R(X, X)$ is an isomorphism in the derived category $D(R)$.

We prove the following analogue of Melkersson's result for R-complexes in Theorem 3.15. It provides flexibility in studying the $\mathfrak{a}$-codualizing condition. It is the main result of Section 3.2.

Theorem 1.6. *Let M be a homologically bounded R-complex. Then the following conditions are equivalent:*

(i) the R-complex $K(\underline{x}) \otimes^{\mathbf{L}}_R M$ is homologically finite for some (equivalently, for every) generating sequence $\underline{x}$ of $\mathfrak{a}$,

(ii) the R-complex $M \otimes^{\mathbf{L}}_R R/\mathfrak{a}$ is homologically degree-wise finite, and

(iii) the R-complex $\mathbf{R}\mathrm{Hom}_R(R/\mathfrak{a}, M)$ is homologically degree-wise finite.

The next result shows that, as in the semidualizing and quasidualizing cases, one always has an example of an $\mathfrak{a}$-codualizing complex. It is Theorem 3.24, which is the main result of Section 3.3.

Theorem 1.7. *The R-complex $\mathbf{R}\Gamma_{\mathfrak{a}}(R)$ is $\mathfrak{a}$-codualizing.*

In the semidualizing case, two well-studied classes of complexes are the Auslander and Bass classes. We study these classes and show that the behavior in the $\mathfrak{a}$-codualizing case deviates from the semidualizing case in some surprising ways.

Definition 1.8. Let M be an $\mathfrak{a}$-codualizing R-complex. Let X and Y be homologically bounded R-complexes.

(a) The complex X is in the *Auslander class* $\mathcal{A}_M(R)$ if the R-complex $M \otimes^{\mathbf{L}}_R X$ is homologically bounded and the natural morphism $\gamma^M_X : X \to \mathbf{R}\mathrm{Hom}_R(M, M \otimes^{\mathbf{L}}_R X)$ is an isomorphism in $D(R)$.

(b) The complex Y is in the *Bass class* $\mathcal{B}_M(R)$ if the R-complex $\mathbf{R}\mathrm{Hom}_R(M, Y)$ is homologically bounded and the natural morphism $\xi^M_Y : M \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(M, Y) \to Y$ is an isomorphism in $D(R)$.

One of the main theorems about the Auslander and Bass Classes in the semidualizing case is so-called "Foxby equivalence." This notion connects the two classes via functors involving semidualizing complexes. To understand these classes in the $\mathfrak{a}$-codualizing case, we need to understand various support conditions. The following result is our version of Foxby equivalence in this setting. It is proved in Theorem 3.40. It is the main result of Section 3.4.

Theorem 1.9 (Foxby equivalence)**.** *Let M is $\mathfrak{a}$-codualizing complex. Then we have that the functors $\mathbf{R}\mathrm{Hom}_R(M, -) : \mathcal{B}_M(R) \to \mathcal{A}_M(R)$ and $M \otimes^{\mathbf{L}}_R - : \mathcal{A}_M(R) \to \mathcal{B}_M(R)$ are quasi-inverse equivalences. Further, we have the following.*

(a) An R-complex Y is in $\mathcal{B}_M(R)$ if and only if the R-complex $\mathbf{R}\mathrm{Hom}_R(M, Y) \in \mathcal{A}_M(R)$ and $\mathrm{supp}_R(Y) \subseteq \mathrm{V}(\mathfrak{a})$.

(b) If $X \in \mathcal{A}_M(R)$, then $M \otimes^{\mathbf{L}}_R X \in \mathcal{B}_M(R)$ and $\text{co-supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$.

(c) If $\dim(R) < \infty$ and $M \otimes^{\mathbf{L}}_R X \in \mathcal{B}_M(R)$ and $\text{co-supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$, then $X \in \mathcal{A}_M(R)$.

2. BACKGROUND

2.1. Homological Constructions

The origins of homological algebra can be traced back at least to the work of Cartan and Eilenberg [8]. We begin with some basic definitions and facts about homological algebra that are necessary for the subsequent chapters. For details on the constructions and proofs the interested reader may wish to consult Rotman [33].

Definition 2.1. A sequence of R-module homomorphisms

$$X = \cdots \longrightarrow X_{i+1} \xrightarrow{\partial^X_{i+1}} X_i \xrightarrow{\partial^X_i} X_{i-1} \longrightarrow \cdots$$

is a *chain complex* (or simply an *R-complex*) if $\partial^X_i \circ \partial^X_{i+1} = 0$ for each i. The module X_i is said to be in the i^{th} degree of X. The i^{th} *homology module* of X is

$$\mathrm{H}_i(X) = \mathrm{Ker}(\partial^X_i)/\,\mathrm{Im}(\partial^X_{i+1}).$$

We write $|x|$ to denote the degree of an element of X. By this we mean if $|x| = n$, then $x \in X_n$. Let $n \in \mathbb{Z}$. The *nth suspension* (or *shift*) of X is the complex $\Sigma^n X$ such that $(\Sigma^n X)_i := X_{i-n}$ and $\partial^{\Sigma^n X}_i = (-1)^n \partial^X_{i-n}$. We set $\Sigma X := \Sigma^1 X$.

Definition 2.2. Let X be an R-complex. If $X_i = 0$ for $i \gg 0$, $i \ll 0$, or $|i| \gg 0$, then X is *bounded above*, *bounded below*, or *bounded*, respectively. If $\mathrm{H}_i(X_i) = 0$ for $i \gg 0$, $i \ll 0$, or $|i| \gg 0$, then X is *homologically bounded above*, *homologically bounded below*, or *homologically bounded* respectively. If $\mathrm{H}_i(X_i) = 0$ for $|i| \gg 0$ and each module $\mathrm{H}_i(X_i)$ is finitely generated, then X is *homologically finite*.

Definition 2.3. Let X be an R-complex. The *infimum*, *supremum*, and *amplitude* of X are

$$\begin{aligned}\inf(X) &= \inf\{i \in \mathbb{Z} : \mathrm{H}_i(X) \neq 0\}\\ \sup(X) &= \sup\{i \in \mathbb{Z} : \mathrm{H}_i(X) \neq 0\}\\ \operatorname{amp}(X) &= \sup(X) - \inf(X).\end{aligned}$$

Definition 2.4. Let X and Y be R-complexes. A sequence $\{\alpha_i : X_i \to Y_i\}$ of R-module homomorphisms such that the following diagram commutes

$$\begin{tikzcd} X = \cdots \arrow[r] & X_i \arrow[r, "\partial_i^X"] \arrow[d, "\alpha_i"] & X_{i-1} \arrow[r] \arrow[d, "\alpha_{i-1}"] & \cdots \\ Y = \cdots \arrow[r] & Y_i \arrow[r, "\partial_i^Y"] & Y_{i-1} \arrow[r] & \cdots \end{tikzcd}$$

is a *chain map*. We denote this as $\alpha : X \to Y$. The chain map α is a *quasiisomorphism* if any induced map on homology is an isomorphism, that is, for all i, we have that the map $\mathrm{H}_i(\alpha) : \mathrm{H}_i(X) \to \mathrm{H}_i(Y)$ is an isomorphism. Quasiisomorphisms are identified by the symbol $\simeq$. The *category of R-complexes* is the category $\mathcal{C}(R)$ with objects equal to the R-complexes and morphisms equal to the chain maps.

We often use the mapping cone to study complexes and chain maps.

Definition 2.5. Let $f : X \to Y$ be a chain map. The *mapping cone* of f is the R-complex

$$\operatorname{Cone}(f) = \qquad \cdots \longrightarrow \begin{matrix} Y_i \\ \oplus \\ X_{i-1} \end{matrix} \xrightarrow{\begin{pmatrix} \partial_i^Y & f_{i-1} \\ 0 & -\partial_{i-1}^X \end{pmatrix}} \begin{matrix} Y_{i-1} \\ \oplus \\ X_{i-2} \end{matrix} \longrightarrow \cdots.$$

Fact 2.6. Let $f : X \to Y$ be a chain map. Then f is a quasiisomorphism if and only if $\operatorname{Cone}(f)$ is exact.

Definition 2.7. A chain map $\alpha : X \to Y$ is *null-homotopic* if there is a sequence of homomorphisms $\beta_i : X_i \to Y_{i+1}$ such that $\alpha_i = \beta_{i-1}\partial_i^X + \partial_{i+1}^Y \beta_i$ for all $i \in \mathbb{Z}$.

$$\begin{tikzcd}
X = \cdots \arrow[r] & X_{i+1} \arrow[r, "\partial_{i+1}^X"] \arrow[d, "\alpha_{i+1}"'] & X_i \arrow[r, "\partial_i^X"] \arrow[d, "\alpha_i"] \arrow[dl, dotted, "\beta_i"] & X_{i-1} \arrow[r] \arrow[d, "\alpha_{i-1}"] \arrow[dl, dotted, "\beta_{i-1}"] & \cdots \\
Y = \cdots \arrow[r] & Y_{i+1} \arrow[r, "\partial_{i+1}^Y"'] & Y_i \arrow[r, "\partial_i^Y"'] & Y_{i-1} \arrow[r] & \cdots
\end{tikzcd}$$

The chain map α is a *homotopy equivalence* if there exists a chain map $\gamma : Y \to X$ such that $\mathrm{id}_X - \gamma\alpha$ and $\mathrm{id}_Y - \alpha\gamma$ are null-homotopic.

Fact 2.8. Let $\alpha : X \to Y$ be a chain map.

(a) If α is an isomorphism, then α is a homotopy equivalence.

(b) If α is a homotopy equivalence, then α is a quasiisomorphism.

Definition 2.9. Let M be an R-module. An *augmented projective resolution* P^+ of M is an exact sequence of modules

$$P^+ = \cdots \longrightarrow P_{i+1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{\tau} M \longrightarrow 0$$

such that each P_i is projective over R. The complex

$$P = \cdots \longrightarrow P_{i+1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \longrightarrow 0$$

is a *projective resolution* of M.

Definition 2.10. Let M be an R-module. An *augmented flat resolution* F^+ of M is an exact sequence of modules

$$F^+ = \cdots \longrightarrow F_{i+1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \xrightarrow{\zeta} M \longrightarrow 0$$

such that each F_i is flat over R. The complex

$$F = \cdots \longrightarrow F_{i+1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \longrightarrow 0$$

is a *flat resolution* of M.

Definition 2.11. Let M be an R-module. An *augmented injective resolution* ${}^+J$ of M is an exact sequence of modules

$${}^+J = 0 \longrightarrow M \xrightarrow{\epsilon} J_0 \longrightarrow J_1 \longrightarrow \cdots \longrightarrow J_i \longrightarrow \cdots$$

such that each J_i is injective over R. The complex

$$J = 0 \longrightarrow J_0 \longrightarrow J_1 \longrightarrow \cdots \longrightarrow J_i \longrightarrow \cdots$$

is an *injective resolution* of M.

Remark 2.12. If M is an R-module with projective resolution P and injective resolution J, then P and J are not usually exact. In fact, we have $\mathrm{H}_i(P) = 0 = \mathrm{H}_i(J)$ for all $i \neq 0$ and $\mathrm{H}_0(P) \cong M \cong \mathrm{H}_0(J)$.

We next define two fundamental tools of homologically algebra. Note that these are defined using the projective and injective resolutions previously defined, and that these definitions are independent of choice of projective and injective resolution.

Definition 2.13. Let M and N be R-modules and P a projective resolution of M. Then $\mathrm{Ext}^i_R(M, N) := \mathrm{H}_{-i}(\mathrm{Hom}_R(P, N))$.

Fact 2.14. Let M and N be R-modules, P a projective resolution of M, and J an injective resolution of N. Then $\mathrm{Ext}^i_R(M, N) = \mathrm{H}_{-i}(\mathrm{Hom}_R(P, N)) \cong \mathrm{H}_{-i}(\mathrm{Hom}_R(M, J))$.

Definition 2.15. Let M and N be R-modules and P a projective resolution of M. Then $\operatorname{Tor}_i^R(M,N) := \mathrm{H}_i(P \otimes_R N)$.

Fact 2.16. Let M and N be R-modules, P a flat resolution of M, and Q a flat resolution of N. Then $\operatorname{Tor}_i^R(M,N) = \mathrm{H}_i(P \otimes_R N) \cong \mathrm{H}_i(M \otimes_R Q)$.

As much of our attention in the sequel is given to studying the $\mathfrak{a}$-codualizing condition in regards to complexes, we introduce some fundamental notions related to R-complexes, many of which are analogous to definitions related to R-modules.

Definition 2.17. Let X and Y be R-complexes.

1. The *Hom complex* $\operatorname{Hom}_R(X,Y)$ is defined as follows. In degree n, we have that $\operatorname{Hom}_R(X,Y)_n := \prod_{p\in\mathbb{Z}} \operatorname{Hom}_R(X_p, Y_{p+n})$ with differentials $\partial_n^{\operatorname{Hom}_R(X,Y)}(\{f_p\}) := \{\partial_{p+n}^Y f_p - (-1)^n f_{p-1}\partial_p^X\}$.

2. The *tensor product complex* $X \otimes_R Y$ is defined as follows. In degree n, we have $(X \otimes_R Y)_n := \oplus_{p\in\mathbb{Z}} X_p \otimes_R Y_{n-p}$. One defines the differentials on the generators as follows

$$\begin{aligned}&\partial_n^{X\otimes_R Y}(\ldots, 0, x_p \otimes y_{n-p}, 0, \ldots) = \\ &\qquad\qquad (\ldots, 0, \partial_p^X(x_p) \otimes y_{n-p}, (-1)^p x_p \otimes \partial_{n-p}^Y(y_{n-p}), 0, \ldots).\end{aligned}$$

Definition 2.18. Let $\alpha\colon X \to Y$ be a chain map, and let V be an R-complex. Then there are well-defined induced chain maps

$$\begin{aligned}
\operatorname{Hom}_R(V,\alpha)\colon \operatorname{Hom}_R(V,X) \to \operatorname{Hom}_R(V,Y) &\qquad f = \{f_p\} \mapsto \{\alpha_{p+|f|} f_p\}\\
\operatorname{Hom}_R(\alpha,V)\colon \operatorname{Hom}_R(Y,V) \to \operatorname{Hom}_R(X,V) &\qquad \{f_p\} \mapsto \{f_p \alpha_p\}\\
V \otimes_R \alpha\colon V \otimes_R X \to V \otimes_R Y &\qquad v_p \otimes x_q \mapsto v_p \otimes \alpha_q(x_q)\\
\alpha \otimes_R V\colon X \otimes_R V \to Y \otimes_R V &\qquad x_p \otimes v_q \mapsto \alpha_p(x_p) \otimes v_q.
\end{aligned}$$

The preceding definition describes the functorial properties of the constructions of Definition 2.17. These properties are analogous to the functorial properties in the context of R-modules.

Fact 2.19. It is straightforward to show that the constructions from Definition 2.18 are functorial, that is, that they respect compositions and identities.

Along with these constructions come a number of natural isomorphisms that are analogous to well-known results for modules. We record them here.

Fact 2.20. Let X, Y and Z be R-complexes.

Hom cancellation: $\mathrm{Hom}_R(R, X) \cong X$.

Tensor cancellation: $R \otimes_R X \cong X$.

Associativity: $X \otimes_R (Y \otimes_R Z) \cong (X \otimes_R Y) \otimes_R Z$.

Adjointness: $\mathrm{Hom}_R(X \otimes_R Y, Z) \cong \mathrm{Hom}_R(X, \mathrm{Hom}_R(Y, Z))$.

Definition 2.21. Let $\underline{x} = x_1, \ldots, x_n \in R$. The *Koszul complex* on x_i is defined as

$$K(x_i) = \qquad 0 \longrightarrow R \xrightarrow{x_i} R \longrightarrow 0$$

The Koszul complex on $\underline{x}$ is defined inductively as

$$K(\underline{x}) = K(x_1, \ldots, x_n) = K(x_1) \otimes_R \cdots \otimes_R K(x_n).$$

The Koszul complex may be defined as above or in terms of mapping cones or in terms of an exterior algebra. Each of these has its utility. For our purposes, the exterior algebra structure is particularly useful, so we describe it next.

Construction 2.22. Let $n \in \mathbb{N}$ and let $e_1, \ldots, e_n \in R^n$ be a basis. Let $\underline{x} = x_1, \ldots, x_n \in R$. The *ith exterior power of* R^n, denoted $\bigwedge^i R^n$, is the free R-module of rank $\binom{n}{i}$ with basis vectors given by the formal symbols of the form $e_{j_1} \wedge \cdots \wedge e_{j_i}$ with $1 \leqslant j_1 < \cdots < j_i \leqslant n$. In $\bigwedge^2 R^n$, define

$$e_{j_2} \wedge e_{j_1} := \begin{cases} -e_{j_1} \wedge e_{j_2} & \text{whenever } 1 \leqslant j_1 < j_2 \leqslant n \\ 0 & \text{whenever } 1 \leqslant j_1 = j_2 \leqslant n. \end{cases}$$

Extending this bilinearly, we define $\alpha \wedge \beta$ for all $\alpha, \beta \in \bigwedge^1 R^n = R^n$: write $\alpha = \sum_p \alpha_p e_p$ and $\beta = \sum_q \beta_q e_q$, and define

$$\alpha \wedge \beta = \left(\sum_p \alpha_p e_p\right) \wedge \left(\sum_q \beta_q e_q\right) = \sum_{p,q} \alpha_p \beta_q e_p \wedge e_q = \sum_{p<q} (\alpha_p \beta_q - \alpha_q \beta_p) e_p \wedge e_q.$$

For example, we have

$$(e_1 + e_2) \wedge (e_1 + e_3) = e_1 \wedge e_3 - e_1 \wedge e_2 + e_2 \wedge e_3.$$

This extends to a multiplication $\bigwedge^1 R^n \times \bigwedge^t R^n \to \bigwedge^{1+t} R^n$, which in turn extends (by induction on s) to a multiplication $\bigwedge^s R^n \times \bigwedge^t R^n \to \bigwedge^{s+t} R^n$ using the following formula when $1 \leqslant i_1 < \ldots < i_s \leqslant n$ and $1 \leqslant j_1 < \cdots < j_t \leqslant n$:

$$(e_{i_1} \wedge \cdots \wedge e_{i_s}) \wedge (e_{j_1} \wedge \cdots \wedge e_{j_t}) := e_{i_1} \wedge [(e_{i_2} \wedge \cdots \wedge e_{i_s}) \wedge (e_{j_1} \wedge \cdots \wedge e_{j_t})].$$

This multiplication is denoted as $(\alpha, \beta) \mapsto \alpha \wedge \beta$. When $s = 0$, since $\bigwedge^0 R^n = R$, the usual scalar multiplication $R \times \bigwedge^t R^n \to \bigwedge^t R^n$ describes the multiplication $\bigwedge^0 R^n \times \bigwedge^t R^n \to \bigwedge^t R^n$, and similarly when $t = 0$. This further extends to a well-defined multiplication on $\bigwedge R^n := \bigoplus_i \bigwedge^i R^n$:

$$e_{j_1} \wedge \cdots \wedge e_{j_t} := \begin{cases} 0 & \text{if } j_p = j_q \text{ for some } p \neq q \\ e_{j_1} \wedge (e_{j_2} \wedge \cdots \wedge e_{j_t}) & \text{if } j_p \neq j_q \text{ for all } p \neq q. \end{cases}$$

This endows the Koszul complex $K(\underline{x})$ with the structure of a graded commutative R-algebra. Using this notation, the differential on $K(\underline{x})$ is given by the following:

$$\partial_i^{KR(\mathbf{x})}(e_{j_1} \wedge \cdots \wedge e_{j_t}) = \sum_{s=1}^{t} (-1)^{s+1} x_{j_s} e_{j_1} \wedge \cdots \wedge \widehat{e_{j_s}} \wedge \cdots \wedge e_{j_t}.$$

Here, the hat signifies that a particular basis element has been removed. For instance, we have $e_1 \wedge \widehat{e_2} \wedge e_3 = e_1 \wedge e_3$.

Definition 2.23. Let $\underline{x} = x_1, \ldots, x_n \in R$. The *Čech complex on* $\underline{x}$ is the complex defined first for a single element

$$\check{\mathrm{C}}(x_1) = 0 \longrightarrow R \xrightarrow{f} R_{x_1} \longrightarrow 0$$

where $f : R \to R_{x_1}$ is defined by $f(r) = \frac{r}{1}$. Then one has

$$\check{\mathrm{C}}(\underline{x}) = \check{\mathrm{C}}(x_1) \otimes_R \ldots \otimes_R \check{\mathrm{C}}(x_n).$$

Definition 2.24. Let X be an R-complex.

(a) A *projective resolution* of X is a quasiisomorphism $P \xrightarrow{\simeq} X$, where P is a bounded below complex of projective modules.

(b) A *flat resolution* of X is a quasiisomorphism $F \xrightarrow{\simeq} X$, where F is a bounded below complex of flat modules.

(c) An *injective resolution* of X is a quasiisomorphism $X \xrightarrow{\simeq} J$, where J is a bounded above complex of injective modules.

Example 2.25. In the case that X is an R-module concentrated in degree 0, these definitions coincide with the definitions for modules. The augmented projective, flat, and injective resolutions

$$P^+ = \cdots \longrightarrow P_{i+1} \longrightarrow \cdots \longrightarrow P_1 \longrightarrow P_0 \xrightarrow{\tau} X \longrightarrow 0$$

$$F^+ = \cdots \longrightarrow F_{i+1} \longrightarrow \cdots \longrightarrow F_1 \longrightarrow F_0 \xrightarrow{\zeta} X \longrightarrow 0$$

$${}^+J = 0 \longrightarrow M \xrightarrow{\epsilon} J_0 \longrightarrow J_1 \longrightarrow \cdots \longrightarrow J_i \longrightarrow \cdots$$

give rise to the following quasiisomorphisms.

$$\begin{tikzcd}
P = \cdots \arrow[r] \arrow[d, "f"] & P_{i+1} \arrow[r] \arrow[d, "0"] & \cdots \arrow[r] & P_1 \arrow[r] \arrow[d, "0"] & P_0 \arrow[r] \arrow[d, "\tau"] & 0 \\
X = \cdots \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & X \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
F = \cdots \arrow[r] \arrow[d, "g"] & F_{i+1} \arrow[r] \arrow[d, "0"] & \cdots \arrow[r] & F_1 \arrow[r] \arrow[d, "0"] & F_0 \arrow[r] \arrow[d, "\zeta"] & 0 \\
X = \cdots \arrow[r] & 0 \arrow[r] & \cdots \arrow[r] & 0 \arrow[r] & X \arrow[r] & 0
\end{tikzcd}$$

$$\begin{tikzcd}
X = \quad 0 \arrow[r] \arrow[d, "h"] & X \arrow[r] \arrow[d, "\epsilon"] & 0 \arrow[r] \arrow[d, "0"] & \cdots \arrow[r] & 0 \arrow[r] \arrow[d, "0"] & \cdots \\
J = \quad 0 \arrow[r] & J_0 \arrow[r] & J_1 \arrow[r] & \cdots \arrow[r] & J_i \arrow[r] & \cdots
\end{tikzcd}$$

Note that these examples show quasiisomorphisms are not true isomorphisms, as there is no inverse chain map. This motivates our study of the derived category below.

Fact 2.26. Let X and Y be R-complexes.

(a) If X is bounded above, then X has an injective resolution.

(b) If Y is bounded below, then Y has a projective resolution. Therefore, there exists a flat resolution of Y.

Definition 2.27. Let X be a homologically bounded R-complex. The *injective dimension* of X is defined as

$$\mathrm{id}_R(X) = \inf\{\sup\{i \in \mathbb{Z} : J_{-i} \neq 0\} : \text{ J is an injective resolution of } X\}.$$

The *flat dimension* of X is defined as

$$\mathrm{fd}_R(X) = \inf\{\sup\{i \in \mathbb{Z} : F_i \neq 0\} : \text{ F is a flat resolution of } X\}.$$

2.2. Artinian And Torsion Modules

In this section we discuss artinian and torsion modules. This is essential for understanding the construction of "minimal injective resolutions" in Section 2.4 and understanding how the $\mathfrak{a}$-codualizing condition captures quasidualizing modules as an example.

Definition 2.28. Let M be an R-module. Set

$$\Gamma_{\mathfrak{a}}(M) = \{x \in M : \mathfrak{a}^n x = 0 \text{ for } n \gg 0\}.$$

The module M is $\mathfrak{a}$-*torsion* if $\Gamma_{\mathfrak{a}}(M) = M$.

Fact 2.29. Given an R-module homomorphism $f : M \to N$ we have $f(\Gamma_{\mathfrak{a}}(M)) \subseteq \Gamma_{\mathfrak{a}}(N)$.

Definition 2.30. Given an R-module homomorphism $f : M \to N$ the induced map $\Gamma_{\mathfrak{a}}(f) : \Gamma_{\mathfrak{a}}(M) \to \Gamma_{\mathfrak{a}}(N)$ is defined by restricting the domain and codomain of f.

Fact 2.31. It is straightforward to show that $\Gamma_{\mathfrak{a}}$ is a functor.

Fact 2.32. [30, Theorem 18.4] Let $\mathfrak{p} \in \operatorname{Spec}(R)$.

(a) If $x \in R \setminus \mathfrak{p}$, then multiplication by x is an automorphism on $E_R(R/\mathfrak{p})$.

(b) For any $x \in E_R(R/\mathfrak{p})$ there exists a positive integer n such that $\mathfrak{p}^n x = 0$. This is to say, $E_R(R/\mathfrak{p})$ is $\mathfrak{p}$-torsion. Moreover, if $\mathfrak{p} \in \mathrm{V}(\mathfrak{a})$, then $E_R(R/\mathfrak{p})$ is $\mathfrak{a}$-torsion.

The preceding fact can be stated succinctly as follows.

Fact 2.33. Let $\mathfrak{a}$ be an ideal of R and $\mathfrak{p} \in \operatorname{Spec}(R)$. Then we have

$$\Gamma_{\mathfrak{a}}(E_R(R/\mathfrak{p})) = \begin{cases} E_R(R/\mathfrak{p}) & \text{if } \mathfrak{p} \in \mathrm{V}(\mathfrak{a}) \\ 0 & \text{if } \mathfrak{p} \notin \mathrm{V}(\mathfrak{a}). \end{cases}$$

In a similar vein, we describe the behavior of injective hulls under localization.

Fact 2.34. [12, Theorems 3.3.3 and 3.3.8(vi)] Let U be a multiplicatively closed set in R and $\mathfrak{p} \in \operatorname{Spec}(R)$. Then we have

$$U^{-1}(E_R(R/\mathfrak{p})) = \begin{cases} E_R(R/\mathfrak{p}) \cong E_{U^{-1}R}(U^{-1}R/\mathfrak{p}U^{-1}R) & \text{if } \mathfrak{p} \cap U = \emptyset \\ 0 & \text{if } \mathfrak{p} \cap U \neq \emptyset. \end{cases}$$

Fact 2.35. [28, Fact 2.1(a)] If M is an $\mathfrak{a}$-torsion R-module, then the natural map $M \to \widehat{R}^{\mathfrak{a}} \otimes_R M$ is an isomorphism.

Fact 2.36. [12, Theorem 3.4.3] Let $(R, \mathfrak{m}, k)$ be local, and let E be the injective hull of the residue field k. Then an R-module M is artinian if and only if $M \subseteq E^n$ for some $n \geqslant 1$.

2.3. The Derived Category

One of the difficulties in working with complexes is that quasiisomorphisms are not invertible. To remedy this deficiency, Verdier [34] introduced the "derived category" of R-complexes. In this setting, quasiisomorphisms are formally inverted, turning them into isomorphisms.

Definition 2.37. The *derived category* of R is the category $\mathcal{D}(R)$ with objects equal to the R-complexes and morphisms $X \to Y$ equal to certain equivalence classes of diagrams of chain maps of the form $X \to J \xleftarrow{\simeq} Y$.

The definition of the equivalence relation used for the morphisms in $\mathcal{D}(R)$ is technical and is not used in the sequel, so we do not describe it here. On the other hand, the next fact documents some useful properties.

Fact 2.38. There is a natural functor $\mathcal{F}\colon \mathcal{C}(R) \to \mathcal{D}(R)$ given on objects by the formula $\mathcal{F}(X) = X$. For a chain map $\alpha\colon X \to Y$, the morphism $\mathcal{F}(\alpha)\colon X \to Y$ is the equivalence class of the diagram $X \xrightarrow{\alpha} Y \xleftarrow{=} Y$. The morphism $\mathcal{F}(\alpha)$ is an isomorphism in $\mathcal{D}(R)$ if and only if α is a quasiisomorphism.

Each morphism $\beta\colon X \to Y$ in $\mathcal{D}(R)$ induces a well defined R-module homomorphism $\mathrm{H}_i(\beta)\colon \mathrm{H}_i(X) \to \mathrm{H}_i(\beta)$ for each i. In the case that $\beta = \mathcal{F}(\alpha)$, then we have $\mathrm{H}_i(\mathcal{F}(\alpha)) = \mathrm{H}_i(\alpha)$.

To save space (and following tradition), we write α in place of $\mathcal{F}(\alpha)$. When we write α and there is a danger of confusion, we specify whether we are working in $\mathcal{C}(R)$ or $\mathcal{D}(R)$.

Remark 2.39. The category $\mathcal{D}(R)$ is "triangulated", which is a technical condition on a category similar to (but different from) being "abelian". In short, it means that $\mathcal{D}(R)$ comes equipped with a class of "distinguished triangles" which are diagrams

$$X \to Y \to Z \to \Sigma X$$

of morphisms in $\mathcal{D}(R)$ subject to certain technical axioms. To save space (and following tradition) we abbreviate the above distinguished triangle as

$$X \to Y \to Z \to$$

since the codomain of the third morphism is always the shift of the domain of the first morphism.

We give some important properties of the distinguished triangles in $\mathcal{D}(R)$ next.

Fact 2.40. We use the notation from Fact 2.38.

Every distinguished triangle $X \xrightarrow{\beta} Y \xrightarrow{\gamma} Z \xrightarrow{\sigma}$ in $\mathcal{D}(R)$ induces a long exact sequence in homology:

$$\cdots \xrightarrow{\mathrm{H}_{i+1}(\sigma)} \mathrm{H}_i(X) \xrightarrow{\mathrm{H}_i(\beta)} \mathrm{H}_i(Y) \xrightarrow{\mathrm{H}_i(\gamma)} \mathrm{H}_i(Z) \xrightarrow{\mathrm{H}_i(\sigma)} \mathrm{H}_{i-1}(X) \xrightarrow{\mathrm{H}_{i-1}(\beta)} \cdots .$$

Also, distinguished triangles can be "rotated" in the sense that the following diagrams are also distinguished triangles in $\mathcal{D}(R)$:

$$\Sigma^{-1}Z \xrightarrow{\Sigma^{-1}\sigma} X \xrightarrow{\beta} Y \xrightarrow{\gamma}$$
$$Y \xrightarrow{\gamma} Z \xrightarrow{\sigma} \Sigma X \xrightarrow{\Sigma\beta} .$$

Given any short exact sequence of chain maps

$$0 \to U \xrightarrow{\zeta} V \xrightarrow{\nu} W \to 0$$

there is distinguished triangle

$$U \xrightarrow{\zeta} V \xrightarrow{\nu} W \to$$

such that the induced long exact sequences from these two diagrams are the same. For instance, given a chain map $\alpha\colon X \to Y$, the standard mapping cone exact sequence

$$0 \to Y \xrightarrow{\epsilon} \mathrm{Cone}(\alpha) \xrightarrow{\tau} \Sigma X \to 0$$

induces a distinguished triangle

$$Y \xrightarrow{\epsilon} \mathrm{Cone}(\alpha) \xrightarrow{\tau} \Sigma X \xrightarrow{\Sigma\alpha}$$

which we can rotate into the form

$$X \xrightarrow{\alpha} Y \xrightarrow{\epsilon} \mathrm{Cone}(\alpha) \xrightarrow{\tau} .$$

Fact 2.41. Given a distinguished triangle $X \to Y \to Z \to$ and the associated long exact sequence in homology, if two of the complexes are homologically degree-wise finite, then so is the third one.

The derived category is the natural habitat for derived functors, which we describe next. Note that our definitions are not the most general, but they avoid certain technical constructions and suffice for our work.

Definition 2.42. Let X be a homologically bounded below R-complex and Y be any R-complex. Let $P \xrightarrow{\simeq} X$ be a projective resolution. Then the *right derived homomorphism complex* and *left derived tensor product* are defined respectively as

$$\mathbf{R}\mathrm{Hom}_R(X,Y) := \mathrm{Hom}_R(P,Y) \qquad\qquad X \otimes^{\mathbf{L}}_R Y := P \otimes_R Y.$$

For each i, we set

$$\mathrm{Ext}^i_R(X,Y) := \mathrm{H}_{-i}(\mathbf{R}\mathrm{Hom}_R(X,Y)) \qquad \mathrm{Tor}^R_i(X,Y) := \mathrm{H}_{-i}(X \otimes^{\mathbf{L}}_R Y).$$

We next discuss well-definedness and "balance" for derived functors.

Fact 2.43. Let X be a homologically bounded below R-complex with projective resolutions $P \xrightarrow{\simeq} X \xleftarrow{\simeq} Q$ and flat resolution $F \xrightarrow{\simeq} X$, and let Y be an R-complex. Then we have isomorphisms $\operatorname{Hom}_R(P,Y) \simeq \operatorname{Hom}_R(Q,Y)$ and $P \otimes_R Y \simeq F \otimes_R Y$ in $\mathcal{D}(R)$. It follows, in particular, that $\mathbf{R}\operatorname{Hom}_R(X,Y)$ and $X \otimes^{\mathbf{L}}_R Y$ are independent of choice of resolution (hence, well-defined) as are $\operatorname{Ext}^i_R(X,Y)$ and $\operatorname{Tor}^R_i(X,Y)$ for all i.

One can also define $Y \otimes^{\mathbf{L}}_R X$ with no boundedness condition on Y (still assuming that X is homologically bounded below) as $Y \otimes^{\mathbf{L}}_R X = Y \otimes_R P$. This is independent of P, as in the previous paragraph. Also, if Y is homologically bounded below, then it can be computed as $G \otimes_R X \simeq G \otimes_R F$ for any flat resolution $G \xrightarrow{\simeq} Y$.

Similarly, given a homologically bounded above R-complex Z, one can also define $\mathbf{R}\operatorname{Hom}_R(Y,Z)$ as $\mathbf{R}\operatorname{Hom}_R(Y,Z) = \operatorname{Hom}_R(Y,J)$ for any injective resolution $Z \xrightarrow{\simeq} J$. This is independent of J and agrees with the previous definition when Y is homologically bounded below.

Derived functors are actually functors, as follows.

Fact 2.44. Given a morphism $\alpha\colon X \to Y$ in $\mathcal{D}(R)$ and an R-complex V, there are well-defined induced morphisms in $\mathcal{D}(R)$.

$$\mathbf{R}\operatorname{Hom}_R(V,\alpha)\colon \mathbf{R}\operatorname{Hom}_R(V,X) \to \mathbf{R}\operatorname{Hom}_R(V,Y)$$

$$\mathbf{R}\operatorname{Hom}_R(\alpha,V)\colon \mathbf{R}\operatorname{Hom}_R(Y,V) \to \mathbf{R}\operatorname{Hom}_R(X,V)$$

$$V \otimes^{\mathbf{L}}_R \alpha\colon V \otimes^{\mathbf{L}}_R X \to V \otimes^{\mathbf{L}}_R Y$$

$$\alpha \otimes^{\mathbf{L}}_R V\colon X \otimes^{\mathbf{L}}_R V \to Y \otimes^{\mathbf{L}}_R V$$

These are essentially induced from Definition 2.18 and are appropriately functorial.[1] They

[1] According to our definitions, one needs to make reasonable boundedness assumptions to ensure that the domain and codomain of a given morphism are defined. To avoid dealing with a large number of cases, we leave the analysis of these assumptions to the interested reader.

also respect distinguished triangles, as follows. Given a distinguished triangle

$$X \xrightarrow{\alpha} Y \xrightarrow{\beta} Z \xrightarrow{\gamma}$$

in $\mathcal{D}(R)$, the induced diagrams

$$\mathbf{R}\mathrm{Hom}_R(V,X) \xrightarrow{\mathbf{R}\mathrm{Hom}(V,\alpha)} \mathbf{R}\mathrm{Hom}_R(V,Y) \xrightarrow{\mathbf{R}\mathrm{Hom}(V,\beta)} \mathbf{R}\mathrm{Hom}_R(V,Z) \xrightarrow{\mathbf{R}\mathrm{Hom}(V,\gamma)}$$
$$\mathbf{R}\mathrm{Hom}_R(Z,V) \xrightarrow{\mathbf{R}\mathrm{Hom}(\beta,V)} \mathbf{R}\mathrm{Hom}_R(Y,V) \xrightarrow{\mathbf{R}\mathrm{Hom}(\alpha,V)} \mathbf{R}\mathrm{Hom}_R(X,V) \xrightarrow{\Sigma\mathbf{R}\mathrm{Hom}(\gamma,V)}$$
$$V \otimes^{\mathbf{L}}_R X \xrightarrow{V\otimes^{\mathbf{L}}_R\alpha} V \otimes^{\mathbf{L}}_R Y \xrightarrow{V\otimes^{\mathbf{L}}_R\beta} V \otimes^{\mathbf{L}}_R Z \xrightarrow{V\otimes^{\mathbf{L}}_R\gamma}$$
$$X \otimes^{\mathbf{L}}_R V \xrightarrow{\alpha\otimes^{\mathbf{L}}_R V} Y \otimes^{\mathbf{L}}_R V \xrightarrow{\beta\otimes^{\mathbf{L}}_R V} Z \otimes^{\mathbf{L}}_R V \xrightarrow{\gamma\otimes^{\mathbf{L}}_R V}$$

are also distinguished triangles in $\mathcal{D}(R)$.

Definition 2.45. Let $P \xrightarrow{\simeq} X$ be a projective resolution and $Y \xrightarrow{\simeq} J$ be an injective resolution. The *left-derived local homology* and *right-derived local cohomology* complexes with respect to $\mathfrak{a}$ are defined respectively as follows:

$$\mathbf{L}\Lambda^{\mathfrak{a}}(X) := \Lambda^{\mathfrak{a}}(P) := \widehat{P}^{\mathfrak{a}} \qquad \mathbf{R}\Gamma_{\mathfrak{a}}(Y) = \Gamma_{\mathfrak{a}}(J).$$

Fact 2.46. The operations $\mathbf{R}\Gamma_{\mathfrak{a}}(-)$ and $\mathbf{L}\Lambda^{\mathfrak{a}}(-)$ are functorial (covariant) and respect distinguished triangles, as in Fact 2.44; see, e.g., Fact 2.29. Also, by [1, Section 1], we know that $\mathbf{L}\Lambda^{\mathfrak{a}}(X)$ can be computed as $\Lambda^{\mathfrak{a}}(F)$ for any flat resolution $F \xrightarrow{\simeq} X$. If X is homologically both degree-wise finite and bounded below, then there is a natural isomorphism $\mathbf{L}\Lambda^{\mathfrak{a}}(X) \simeq \widehat{R}^{\mathfrak{a}} \otimes^{\mathbf{L}}_R X$ by [21, Proposition 2.7].

Let $Y \xrightarrow{\simeq} J$ be an injective resolution, and let $P \xrightarrow{\simeq} X$ be a projective resolution. The natural chain maps $\Gamma_{\mathfrak{a}}(J) \to J$ and $P \to \widehat{P}^{\mathfrak{a}}$ induce morphisms $\mathbf{R}\Gamma_{\mathfrak{a}}(Y) \to Y$ and $X \to \mathbf{L}\Lambda^{\mathfrak{a}}(X)$, which are natural in Y and X; see [1, Theorem (0.3)*].

Let $\mathbf{x}$ be a generating sequence for $\mathfrak{a}$. Then one has $\mathbf{R}\Gamma_{\mathfrak{a}}(R) \simeq \check{C}(\mathbf{x})$. Moreover, if

X is homologically bounded, then there are natural isomorphisms

$$\mathbf{R}\Gamma_{\mathfrak{a}}(X) \simeq \mathbf{R}\Gamma_{\mathfrak{a}}(R) \otimes^{\mathbf{L}}_R X \simeq \check{C}(\mathbf{x}) \otimes^{\mathbf{L}}_R X$$
$$\mathbf{L}\Lambda^{\mathfrak{a}}(X) \simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(R), X) \simeq \mathbf{R}\mathrm{Hom}_R(\check{C}(\mathbf{x}), X).$$

See [1, Theorem (0.3)*] and [29, Proposition 3.1.2].

2.4. Support And Co-Support

Since Definition 1.3 uses the notion of the "small support" we record some definitions and facts here for the sequel.

Definition 2.47. Let M be an R-module.

1. The "small," or "homological," support of M is

$$\mathrm{supp}_R(M) = \{\mathfrak{p} \in \mathrm{Spec}(R) : \mathrm{Tor}_i^R(R/\mathfrak{p}, M)_{\mathfrak{p}} \neq 0 \text{ for some } i\}.$$

2. The "large" support of M is $\mathrm{Supp}_R(M) = \{\mathfrak{p} \in \mathrm{Spec}(R) : M_{\mathfrak{p}} \neq 0\}$.

There is always a containment of the small support in the large support. When M is a finitely generated R-module they are equal.

Definition 2.48. Let X be an R-complex.

(a) The "small," or "homological," support of X is

$$\mathrm{supp}_R(X) = \{\mathfrak{p} \in \mathrm{Spec}(R) : \kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_R X \not\simeq 0\}.$$

(b) The "large" support of X is $\mathrm{Supp}_R(X) = \{\mathfrak{p} \in \mathrm{Spec}(R) : X_{\mathfrak{p}} \not\simeq 0\}$.

Fact 2.49. Let X be an R-complex. Then $\mathrm{Supp}_R(\Gamma_a(X)) \subseteq \mathrm{V}(\mathfrak{a})$.

Fact 2.50. [16, Proposition 14.11] If X is a bounded below R-complex, then $X \simeq 0$ if and only if $\operatorname{supp}_R(X) = \emptyset$.

Fact 2.51. [20, Theorem 7.1(c)] Let X and Y be R-complexes. If X and Y are homologically bounded below, then $\operatorname{supp}_R(X \otimes_R^{\mathbf{L}} Y) = \operatorname{supp}_R(X) \bigcap \operatorname{supp}_R(Y)$.

Corollary 2.52. *Let X and Y be homologically bounded below R-complexes. Then we have* $\operatorname{supp}_R(X \otimes_R^{\mathbf{L}} Y) \subseteq \operatorname{supp}_R(X)$.

Definition 2.53. Let X be an R-complex. The *co-support* of X is defined as

$$\operatorname{co-supp}_R(X) = \{\mathfrak{p} \in \operatorname{Spec}(R) : \kappa(\mathfrak{p}) \otimes_{R_\mathfrak{p}}^{\mathbf{L}} \mathbf{R}\operatorname{Hom}_R(R_\mathfrak{p}, X) \not\simeq 0\}.$$

Proposition 2.54. *If X, Y are R-complexes where X is homologically bounded below or Y is homologically bounded above, then* $\operatorname{co-supp}_R(\mathbf{R}\operatorname{Hom}_R(X, Y)) \subseteq \operatorname{Supp}_R(X)$.

Proof. If $\mathfrak{p} \notin \operatorname{Supp}_R(X)$, then one has

$$\begin{aligned} \kappa(\mathfrak{p}) \otimes_{R_\mathfrak{p}}^{\mathbf{L}} \mathbf{R}\operatorname{Hom}_R(R_\mathfrak{p}, \mathbf{R}\operatorname{Hom}_R(X, Y)) &\simeq \kappa(\mathfrak{p}) \otimes_{R_\mathfrak{p}}^{\mathbf{L}} \mathbf{R}\operatorname{Hom}_R(R_\mathfrak{p} \otimes_R^{\mathbf{L}} X, Y) \\ &\simeq \kappa(\mathfrak{p}) \otimes_{R_\mathfrak{p}}^{\mathbf{L}} \mathbf{R}\operatorname{Hom}_R(X_\mathfrak{p}, Y) \\ &\simeq \kappa(\mathfrak{p}) \otimes_{R_\mathfrak{p}}^{\mathbf{L}} \mathbf{R}\operatorname{Hom}_R(0, Y) \\ &\simeq 0. \end{aligned}$$

Thus, $\mathfrak{p} \notin \operatorname{co-supp}_R(\mathbf{R}\operatorname{Hom}_R(M, Y))$, so $\operatorname{co-supp}_R(\mathbf{R}\operatorname{Hom}_R(M, Y)) \subseteq \operatorname{Supp}_R(M)$. $\square$

2.5. Minimal Resolutions

In this section, we discuss two types of minimal resolutions: minimal injective resolutions of homologically bounded above R-complexes and minimal flat resolutions of R-modules. We begin with two bookkeeping tools.

Definition 2.55. Let $(R, \mathfrak{m}, k)$ be a local ring. Let X be an R-complex.

(a) The i^{th} *Bass number* of X with respect to $\mathfrak{p}$ is $\mu^i_{\mathfrak{p}} = \operatorname{rank}_{\kappa(\mathfrak{p})}(\operatorname{Ext}^i_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}), X_{\mathfrak{p}})$. When the $\mu^i_{\mathfrak{m}}$ are finite, the *Bass series* of X is the formal series $I^X_R(t) = \sum_{i\in\mathbb{Z}} \mu^i_{\mathfrak{m}}(X)t^i$.

(b) The i^{th} *Betti number* of X is $\beta^R_i = \operatorname{rank}_k(\operatorname{Tor}^R_i(X, k))$. When the β^R_i are finite, the *Poincaré series* of X is the formal series $P^R_X(t) = \sum_{i\in\mathbb{Z}} \beta^R_i(X)t^i$.

Definition 2.56. Let $N \subseteq M$ be R-modules. Then N is an *essential submodule* of M if for each submodule L of M, the condition $N \cap L = 0$ implies $N = 0$.

Definition 2.57. [9] Let X be an R-complex. An injective resolution J of X is *minimal* if for all i the kernel of the differential $\partial^J_i : J_i \to J_{i+1}$ is an essential submodule of J_i.

Fact 2.58. Every homologically bounded above R-complex M has a minimal injective resolution $M \xrightarrow{\simeq} J$ such that $J_i = 0$ for all $i > \sup(M)$. Furthermore, if $M \xrightarrow{\simeq} I$ is another injective resolution, then there is an exact bounded above complex I' of injective R-modules and an isomorphism $I \cong J \oplus I'$; if I is minimal, then $I' = 0$. See [4, 2.11.3.5 Theorem and 2.12.2.1 Theorem].

Lemma 2.59. *Let M be a homologically bounded above R-complex, and let S be a multiplicatively closed subset of R. Given a (minimal) injective resolution $M \xrightarrow{\simeq} J$, the localization $S^{-1}M \xrightarrow{\simeq} S^{-1}J$ is a (minimal) injective resolution over $S^{-1}R$.*

Proof. Localization is exact, so it respects quasiisomorphisms, and we have a quasiisomorphism $S^{-1}M \xrightarrow{\simeq} S^{-1}J$ over $S^{-1}R$. Since R is noetherian, we know that each $S^{-1}J_i$ is injective over $S^{-1}R$, so the quasiisomorphism $S^{-1}M \xrightarrow{\simeq} S^{-1}J$ is an injective resolution. Furthermore, it is well-known that the "essential" property for submodules localizes (see, e.g., the proof of [7, Lemma 3.2.5]). Since localization is exact, if the original resolution is minimal over R, then the localized resolution is minimal over $S^{-1}R$. $\square$

Lemma 2.60. *Assume that $(R, \mathfrak{m}, k)$ is local, and let J be a minimal bounded above complex of injective R-modules, i.e., a minimal injective resolution of itself. Then the complex $\Gamma_{\mathfrak{m}}(J)$ is also a minimal bounded above complex of injective R-modules.*

Proof. $\Gamma_{\mathfrak{m}}(J)$ is a bounded above complex of injective R-modules by Fact 2.32(b). Thus, it remains to show that $\Gamma_{\mathfrak{m}}(J)$ is minimal. Since $\Gamma_{\mathfrak{m}}$ is left-exact, for each i we have

$$\operatorname{Ker}(\partial_i^{\Gamma_{\mathfrak{m}}(J)}) = \operatorname{Ker}(\Gamma_{\mathfrak{m}}(\partial_i^J)) = \Gamma_{\mathfrak{m}}(\operatorname{Ker}(\partial_i^J)) \subseteq \Gamma_{\mathfrak{m}}(J_{i-1}).$$

Since the inclusion $\operatorname{Ker}(\partial_i^J) \subseteq J_{i-1}$ is essential, it is straightforward to show that the inclusion $\Gamma_{\mathfrak{m}}(\operatorname{Ker}(\partial_i^J)) \subseteq \Gamma_{\mathfrak{m}}(J_{i-1})$ is also essential, so $\Gamma_{\mathfrak{m}}(J)$ is minimal. □

Fact 2.61. Let M be a homologically bounded R-complex. By [16, (14.20)], we have

$$\begin{aligned}\operatorname{supp}_R(M) &= \bigcup_{i\in\mathbb{Z}}\{\mathfrak{p} \in \operatorname{Spec}(R) \mid \mu_{\mathfrak{p}}^i(M) \neq 0\}\\ &= \{\mathfrak{p} \in \operatorname{Spec}(R) \mid \mathbf{R}\operatorname{Hom}_R(R/\mathfrak{p}, M)_{\mathfrak{p}} \not\simeq 0\}.\end{aligned}$$

Remark 2.62. Lemma 2.64 is used frequently in our proofs. Before we state and prove it, we give some background information about it.

In [5, Remark 9.2], the authors claim that Lemma 2.64 is proved in [19], but do not cite a specific result from [19]. As best we can ascertain, they are extrapolating from [19, 2.9 Remark], which only specifically deals with the case where R is a module. However, the proof given in [19, 2.9 Remark] is somewhat unconvincing, and we do not see how the claim of [5, Remark 9.2] follows. Furthermore, from [9, Remark 2.3], we learn that there is some confusion as to what is actually true in [5, Remark 9.2]. Thus, given the importance of this result for our work, we include a proof here, which may be along the lines of the intentions of [19, 2.9 Remark] and [5, Remark 9.2].

Lemma 2.63. *Assume that $(R, \mathfrak{m}, k)$ is local, and let M be a homologically bounded R-complex with minimal injective resolution $M \xrightarrow{\simeq} J$. Then $\mathfrak{m} \in \operatorname{supp}_R(M)$ if and only if $E_R(R/\mathfrak{m})$ is a summand of J_i for some i.*

Proof. Assume first that $\mathfrak{m} \in \operatorname{supp}_R(M)$. Fact 2.61 implies that

$$0 \not\simeq \mathbf{R}\mathrm{Hom}_R(R/\mathfrak{m}, M)_{\mathfrak{m}} \simeq \mathbf{R}\mathrm{Hom}_R(R/\mathfrak{m}, M) \simeq \mathrm{Hom}_R(R/\mathfrak{m}, J).$$

In particular, we have $0 \neq \mathrm{Hom}_R(R/\mathfrak{m}, J)$, so for some i we have

$$0 \neq \mathrm{Hom}_R(R/\mathfrak{m}, J)_i \cong (0 :_{J_i} \mathfrak{m}) \subseteq \Gamma_{\mathfrak{m}}(J_i).$$

Write $J_i \cong \oplus_{\mathfrak{p} \in \mathrm{Spec}(R)} E_R(R/\mathfrak{p})^{(\mu^i_{\mathfrak{p}})}$ for some sets $\mu^i_{\mathfrak{p}}$. It follows from Fact 2.32(b) that

$$0 \neq \Gamma_{\mathfrak{m}}(J_i) \cong \oplus_{\mathfrak{p} \in \mathrm{Spec}(R)} \Gamma_{\mathfrak{m}}(E_R(R/\mathfrak{p}))^{(\mu^i_{\mathfrak{p}})} \cong E_R(R/\mathfrak{m})^{(\mu^i_{\mathfrak{m}})}.$$

We conclude that $\mu^i_{\mathfrak{m}} \neq \emptyset$, so $E_R(R/\mathfrak{m})$ is a summand of J_i, as desired.

For the converse, assume that $E_R(R/\mathfrak{m})$ is a summand of J_i for some $i \in \mathbb{Z}$. It follows that $\Gamma_{\mathfrak{m}}(J) \neq 0$. Lemma 2.60 implies that $\Gamma_{\mathfrak{m}}(J)$ is a minimal injective resolution of itself, so it follows that $\Gamma_{\mathfrak{m}}(J) \not\simeq 0$. Since $\Gamma_{\mathfrak{m}}(J)_{\mathfrak{p}} = 0$ for all $\mathfrak{p} \in \mathrm{Spec}(R) \smallsetminus \{\mathfrak{m}\}$, we have $\emptyset \neq \operatorname{supp}_R(\Gamma_{\mathfrak{m}}(J)) \subseteq \operatorname{Supp}_R(\Gamma_{\mathfrak{m}}(J)) \subseteq \{\mathfrak{m}\}$. We conclude that $\mathfrak{m} \in \operatorname{supp}_R(\Gamma_{\mathfrak{m}}(J))$.

Consider the exact sequence

$$0 \to \Gamma_{\mathfrak{m}}(J) \xrightarrow{\subseteq} J \to J/\Gamma_{\mathfrak{m}}(J) \to 0.$$

Since $J/\Gamma_{\mathfrak{m}}(J)$ is a bounded above complex of injective R-modules, it is a direct summand of a minimal injective resolution $J' \simeq J/\Gamma_{\mathfrak{m}}(J)$. Fact 2.32(b) implies that for all i the module $E_R(R/\mathfrak{m})$ is not a summand of $J_i/\Gamma_{\mathfrak{m}}(J_i) = (J/\Gamma_{\mathfrak{m}}(J))_i$, so it is not a summand of J'. Thus, the first paragraph of this proof implies that $\mathfrak{m} \notin \operatorname{supp}_R(J/\Gamma_{\mathfrak{m}}(J))$. We need to show that $J/\Gamma_{\mathfrak{m}}(J)$ is homologically bounded. For this, it suffices to show that $\Gamma_{\mathfrak{m}}(J)$ is homologically bounded. This follows from the isomorphisms $\Gamma_{\mathfrak{m}}(J) \simeq \check{\mathrm{C}}(\mathfrak{m}) \otimes^{\mathbf{L}}_R J \simeq \check{\mathrm{C}}(\mathfrak{m}) \otimes^{\mathbf{L}}_R J'$, where J' is a truncation of J that is bounded and such that $J' \simeq J$. Since

$\check{\mathrm{C}}(\mathfrak{m})$ and J' are bounded, we have $\check{\mathrm{C}}(\mathfrak{m}) \otimes^{\mathbf{L}}_R J'$ (homologically) bounded. Applying the functor $\mathbf{R}\mathrm{Hom}_R(R/\mathfrak{m}, -) \simeq \mathbf{R}\mathrm{Hom}_R(R/\mathfrak{m}, -)_{\mathfrak{m}}$ to the above exact sequence, we obtain the distinguished triangle

$$\mathbf{R}\mathrm{Hom}_R(R/\mathfrak{m}, \Gamma_{\mathfrak{m}}(J)) \to \mathbf{R}\mathrm{Hom}_R(R/\mathfrak{m}, J) \to \mathbf{R}\mathrm{Hom}_R(R/\mathfrak{m}, J/\Gamma_{\mathfrak{m}}(J)) \to .$$

The condition $\mathfrak{m} \notin \mathrm{supp}_R(J/\Gamma_{\mathfrak{m}}(J))$ implies that $\mathbf{R}\mathrm{Hom}_R(R/\mathfrak{m}, J/\Gamma_{\mathfrak{m}}(J)) \simeq 0$, so the distinguished triangle implies that we have

$$\mathbf{R}\mathrm{Hom}_R(R/\mathfrak{m}, J) \simeq \mathbf{R}\mathrm{Hom}_R(R/\mathfrak{m}, \Gamma_{\mathfrak{m}}(J)) \not\simeq 0$$

as $\mathfrak{m} \in \mathrm{supp}_R(\Gamma_{\mathfrak{m}}(J))$. It follows that $\mathfrak{m} \in \mathrm{supp}_R(J) = \mathrm{supp}_R(M)$, as desired. □

Lemma 2.64. *Let M be a homologically bounded R-complex with minimal injective resolution $M \xrightarrow{\simeq} J$. Then* $\mathrm{supp}_R(M) = \cup_{i\in\mathbb{Z}}\{\mathfrak{p} \in \mathrm{Spec}(R) \mid E_R(R/\mathfrak{p})$ *is a summand of* $J_i\}$.

Proof. Let $\mathfrak{p} \in \mathrm{Spec}(R)$, and note that Lemma 2.59 implies that $M_{\mathfrak{p}} \xrightarrow{\simeq} J_{\mathfrak{p}}$ is a minimal injective resolution.

To prove the containment

$$\mathrm{supp}_R(M) \supseteq \cup_{i\in\mathbb{Z}}\{\mathfrak{p} \in \mathrm{Spec}(R) \mid E_R(R/\mathfrak{p}) \text{ is a summand of } J_i\}$$

assume that $E_R(R/\mathfrak{p})$ is a summand of J_i for some i. Then the module $E_R(R/\mathfrak{p})_{\mathfrak{p}} \cong E_{R_{\mathfrak{p}}}(R_{\mathfrak{p}}/\mathfrak{p}R_{\mathfrak{p}})$ is a summand of $(J_i)_{\mathfrak{p}}$ for some i. From Lemma 2.63, we conclude that $\mathfrak{p}R_{\mathfrak{p}} \in \mathrm{supp}_{R_{\mathfrak{p}}}(M_{\mathfrak{p}})$. It follows that

$$\emptyset \neq \mu^i_{\mathfrak{p}R_{\mathfrak{p}}}(M_{\mathfrak{p}}) = \mu^i_{\mathfrak{p}}(M).$$

So $\mathfrak{p} \in \mathrm{supp}_R(M)$ by Fact 2.61.

For the reverse containment, run the previous argument in reverse, using the fact that $(J_i)_{\mathfrak{p}}$ is a summand of J_i. $\square$

Proposition 2.65. *Let M be a homologically bounded R-complex. Then* $\operatorname{Supp}_R(M) \subseteq \operatorname{V}(\mathfrak{a})$ *if and only if* $\operatorname{supp}_R(M) \subseteq \operatorname{V}(\mathfrak{a})$.

Proof. $(\Rightarrow)$: This containment is always true since $\operatorname{supp}_R(M) \subseteq \operatorname{Supp}_R(M)$.
$(\Leftarrow)$: Assume $\operatorname{supp}_R(M) \subseteq \operatorname{V}(\mathfrak{a})$. Suppose $\mathfrak{p} \notin \operatorname{V}(\mathfrak{a})$. Then $\mathfrak{p} \notin \operatorname{supp}_R(M)$. It follows from Lemma 2.64 that $E_R(R/\mathfrak{p})$ does not occur as a summand in a minimal injective resolution J of M.

Now, for all $\mathfrak{q} \subseteq \mathfrak{p}$, the ideal $\mathfrak{q} \notin \operatorname{V}(\mathfrak{a})$. So $E_R(R/\mathfrak{q})$ does not occur in minimal injective resolution J. So in each degree of J we have

$$J_i = \oplus_{\mathfrak{q} \not\subseteq \mathfrak{p}} E_R(R/\mathfrak{q})^{(\mu^i_{\mathfrak{q}})}.$$

If $\mathfrak{q} \not\subseteq \mathfrak{p}$, then there exists $x \in \mathfrak{q} \setminus \mathfrak{p}$. So x is a unit in $R_{\mathfrak{p}}$ and $E_R(R/\mathfrak{q})$ is x-torsion. Then $E_R(R/\mathfrak{q})_{\mathfrak{p}} = 0$. Therefore, $(J_i)_{\mathfrak{p}} = 0$. It follows that $M_{\mathfrak{p}} \simeq J_{\mathfrak{p}} = 0$. That is, $\mathfrak{p} \notin \operatorname{Supp}_R(M)$. $\square$

Corollary 2.66. *If X is a homologically bounded R-complex, then* $\operatorname{Supp}_R(X)$ *is contained in the Zariski closure of* $\operatorname{supp}_R(X)$ *in* $\operatorname{Spec}(R)$.

We use of the notion of minimal flat resolutions in the sequel. The following provides the relevant background on the construction.

Definition 2.67. An R-module M is said to be *cotorsion* if for all flat modules F we have $\operatorname{Ext}^1_R(F, M) = 0$.

Remark 2.68. An R-module is M cotorsion if and only if $\operatorname{Ext}^i_R(F, M) = 0$ for all flat modules F and $i \geqslant 1$. This can be shown via a dimension-shifting argument in a projective resolution of F.

Definition 2.69. A submodule L of an R-module N is said to be a *pure submodule* if $0 \to A \otimes_R L \to A \otimes_R N$ is exact for all modules A. An R-module M is said to be *pure injective* if for every pure submodule $L \subseteq N$ of R-modules, the following sequence is exact.

$$\mathrm{Hom}_R(N, M) \to \mathrm{Hom}_R(L, M) \to 0$$

Definition 2.70. Let M be an R-module. A homomorphsm $\phi : F \to M$ where F is a flat R-module is said to be a *flat cover of* M if

1. any diagram with F' a flat R-module

$$\begin{tikzcd} F' \arrow[d, dotted] \arrow[dr] & \\ F \arrow[r, "\phi"] & M \end{tikzcd}$$

   can be completed to a commutative diagram, and

2. any diagram

$$\begin{tikzcd} F \arrow[d, dotted] \arrow[dr] & \\ F \arrow[r, "\phi"] & M \end{tikzcd}$$

   can only be completed to a commutative diagram using an automorphism of F.

Definition 2.71. A *minimal flat resolution* of an R-module M is an exact sequence

$$\cdots \longrightarrow F_i \xrightarrow{\partial_i} F_{i-1} \longrightarrow \cdots \longrightarrow F_0 \longrightarrow M \longrightarrow 0$$

such that each F_i is a flat cover of $\mathrm{Im}(\partial_i)$.

Fact 2.72. [6, Theorem 3] Every R-module has a flat cover. Therefore, every R-module has a minimal flat resolution.

2.6. Differential Graded Algebras

To prove our version of Melkersson's result, we need to work in the following more general setting. A useful reference for this subject is [4].

Definition 2.73. A *commutative differential graded R-algebra A* ("DG R-algebra" for short) is an R-complex A with a multiplication $A \times A \to A$ written $(a, b) \mapsto ab$ that satisfies the following conditions:

1. **associative**: for $a, b, c \in R$, one has $(ab)c = a(bc)$;
2. **distributive**: for $a, b, c \in R$ such that $|a| = |b|$, one has $(a + b) = ac + bc$;
3. **unital**: there is an element $1_A \in A_0$ such that for all $a \in A$, we have $1_A a = a$;
4. **graded**: for all $a, b \in A$, one has $ab \in A_{|a|+|b|}$;
5. **graded commutative**: for all $a, b \in A$, one has $ba = (-1)^{|a||b|}ab$, and $a^2 = 0$ whenever $|a|$ is odd;
6. **positively graded**: $A_i = 0$ for all $i < 0$; and
7. **Leibniz rule**: for $a, b \in A$, one has $\partial^A_{|a||b|}(ab) = \partial^A_{|a|}(a)b + (-1)^{|a|}a\partial^A_{|b|}(b)$.

Given a DG R-algebra A, the *underlying algebra* is the graded commutative R-algebra $A^\natural = \bigoplus_{i=0}^{\infty} A_i$.

Example 2.74.

1. The ring $R/\mathfrak{a}$ considered as an R-complex concentrated in degree 0 is a DG R-algebra.
2. The Koszul complex K on a given a sequence of elements $x_1, \dots, x_n \in R$ with the wedge product is a DG R-algebra. See Construction 2.22

Definition 2.75. A *morphism* $A \to B$ of DG R-algebras is a chain map $f : A \to B$ such that $f(aa') = f(a)f(a')$ for all $a, a' \in A$ and $f(1_A) = 1_B$.

Definition 2.76. Let A be a DG R-algebra. A *DG A-module* is an R-complex M with an operation called scalar multiplication $A \times M \to M$ written as $(a, m) \mapsto am$ that satisfies the following conditions:

1. **distributive**: for all $a, b \in A$ and $m, n \in M$, we have $(a + b)m = am + bm$ and $a(m + n) = am + an$;
2. **graded**: for all $a \in A$ and $m \in M$, we have $am \in M_{|a|+|m|}$;
3. **Leibniz rule**: for all $a \in A$ and for all $m \in M$, we have $\partial^M_{|a|+|m|}(am) = \partial^A_{|a|}(a)m + (-1)^{|a|}a\partial^M_{|m|}(m)$;
4. **unital**: we have $1_A m = m$ for all $m \in M$; and
5. **associative**: for all $a, b \in A$ and $m \in M$, we have $a(bm) = (ab)m$.

The *underlying $A^\natural$-module* associated to M is the $A^\natural$-module $M^\natural = \bigoplus_{j=-\infty}^{\infty} M_j$.

Definition 2.77. Let A be a DG R-algebra. A *morphism* of DG A-modules is a chain map $f\colon M \to N$ between DG A-modules that respects scalar multiplication: $f(am) = af(m)$. Isomorphisms in the category of DG A-modules are identified by the symbol $\cong$. A *quasiisomorphism* of DG A-modules is a morphism $M \to N$ such that each induced map $\mathrm{H}_i(M) \to \mathrm{H}_i(N)$ is an isomorphism, i.e., a morphism of DG A-modules that is a quasiisomorphism of R-complexes; these are identified by the symbol $\simeq$.

Definition 2.78. Let A be a DG R-algebra, and let M and N be DG A-modules. Given an integer n, a *DG A-module homomorphism of degree n* is an element $f \in \mathrm{Hom}_R(M, N)_n$ such that $f_{i+j}(am) = (-1)^{ni}af_j(m)$ for all $a \in A_i$ and $m \in M_j$. The graded submodule of $\mathrm{Hom}_R(M, N)$ consisting of all DG A-module homomorphisms $M \to N$ is denoted $\mathrm{Hom}_A(M, N)$.

Definition 2.79. Let A be a DG R-algebra, and let M be a DG A-module. A subset E of M is called a *semibasis* if it is a basis of the underlying $A^\natural$-module $M^\natural$. If M is bounded below and it has a semibasis, then M is called *semi-free*. A *degree-wise finite semi-free resolution* of a DG A-module M is a quasiisomorphism $F \xrightarrow{\simeq} M$ of DG A-modules such that F is semi-free with semibasis E such that the set $E \cap F_i$ is finite for all i.

Fact 2.80. [4, Theorem 2.11.3.3] Let A be a DG R-algebra such that A_i is finitely generated over R for all i. Let M be a DG A-module with $\inf(M) > -\infty$ such that $\mathrm{H}_i(M)$ is finitely generated over R for all i. Then M has a degree-wise finite semi-free resolution $F \xrightarrow{\simeq} M$ such that $F_i = 0$ for all $i < \inf(M)$.

Definition 2.81. Let A be a DG R-algebra, and let M, N be DG A-modules. The *tensor product* $M \otimes_A N$ is the quotient $(M \otimes_R N)/U$ where U is generated over R by the elements of the form $(am) \otimes n - (-1)^{|a||m|} m \otimes (an)$. Given an element $m \otimes n \in M \otimes_R N$, we denote the image in $M \otimes_A N$ as $m \otimes n$.

Keller [25] has shown that the category of DG modules over a DG R-algebra is rich enough to afford a derived category $\mathcal{D}(A)$. We shall not need the full strength of this construction, but we do use the following constructions which are independent of choice of resolution of M and respect isomorphisms $N \simeq N'$ in $\mathcal{D}(A)$.

Definition 2.82. Let A be a DG R-algebra such that A_i is finitely generated over R for all i. Let M and N be DG A-modules with $\inf(M) > -\infty$ such that $\mathrm{H}_i(M)$ is finitely generated over R for all i. Let $F \xrightarrow{\simeq} M$ be a degree-wise finite semi-free resolution. Then the *right derived homomorphism module* and *left derived tensor product* are defined respectively as

$$\mathbf{R}\mathrm{Hom}_A(M, N) := \mathrm{Hom}_R(F, N) \qquad\qquad M \otimes^{\mathbf{L}}_A N := F \otimes_R N.$$

3. CODUALIZING MODULES AND COMPLEXES

3.1. Codualizing Modules

As stated in the introduction, we seek to unify the notions of semidualizing and quasidualizing modules under one "umbrella" notion. We begin with a notion of "cofiniteness" that is due in spirit to Hartshorne.

Definition 3.1. An R-module M is $\mathfrak{a}$-*cofinite* if $\operatorname{supp}_R(M) \subseteq \operatorname{V}(\mathfrak{a})$ and the module $\operatorname{Ext}^i_R(R/\mathfrak{a}, M)$ is finitely generated for all i.

Proposition 3.2. *Let M be an R-module.*

(a) If $\mathfrak{a} = 0$, then M is 0-cofinite if and only if M is finitely generated.

(b) Let $(R, \mathfrak{m}, k)$ be local. If $\mathfrak{a} = \mathfrak{m}$, then M is $\mathfrak{m}$-cofinite if and only if it is artinian.

Proof. (a) ($\Leftarrow$): Assume M is a finitely generated R-module. Note that $\operatorname{Ext}^i_R(R/0, M) \cong \operatorname{Ext}^i_R(R, M)$. Also, we have $\operatorname{Ext}^i_R(R, M) = 0$ for all $i \geqslant 1$ and $\operatorname{Ext}^0_R(R, M) \cong M$. Since M is finitely generated, it follows that $\operatorname{Ext}^i_R(R/0, M)$ is finitely generated for all i. To see that M satisfies the support condition, consider the following

$$\operatorname{supp}_R(M) \subseteq \operatorname{Spec}(R) = \operatorname{V}(0)$$

which hold by definition.

($\Rightarrow$): Assume that M is 0-cofinite. As noted above we have $\operatorname{Ext}^0_R(R/0, M) \cong \operatorname{Ext}^0_R(R, M) \cong M$. By assumption, $\operatorname{Ext}^0_R(R, M)$ is finitely generated. Therefore, M is finitely generated.

(b) ($\Leftarrow$) Assume M is artinian. Since M is artinian over a local ring, a minimal injective resolution of M is of the form

$$^{+}J = 0 \to M \to E^{\mu_0} \to E^{\mu_1} \to \cdots$$

where E is the injective hull of the residue field and $\mu_i < \infty$ for all i. This follows from Fact 2.36 and the fact that artinian modules satsify the 2-of-3 condition. It follows that $\operatorname{Ext}^i_R(R/\mathfrak{m}, M)$ is a finite dimensional k-vector space for each i because the complex used compute $\operatorname{Ext}^i_R(R/\mathfrak{m}, M)$ in each degree is $\operatorname{Hom}_R(k, J)_i \cong k^{(\mu_i)}$. Then $\operatorname{Ext}^i_R(R/\mathfrak{m}, M)$ is finitely generated over R for all i. To see that M satisfies the support condition, consider the following

$$\operatorname{supp}_R(M) \subseteq \operatorname{Supp}_R(M) \subseteq \{\mathfrak{m}\} = \operatorname{V}(\mathfrak{m}).$$

The first containment is true for all modules, and the second follows from the fact that artinian modules are $\mathfrak{m}$-torsion [27, Fact 1.2(a)].

($\Rightarrow$) Assume that M is $\mathfrak{m}$-cofinite. So we have $\operatorname{supp}_R(M) \subseteq \{\mathfrak{m}\}$. By Lemma 2.64, a minimal injective resolution J of M satisfies the condition $J_i \cong E^{(\mu_i)}$ for all i where $\mu_i = \dim_k \operatorname{Ext}^i_R(R/\mathfrak{m}, M)$. Since M is $\mathfrak{m}$-cofinite, we have, $\mu_0 < \infty$. So, M is a submodule of the artinian module E^{μ_0}. Thus, M is artinian. $\square$

Remark 3.3. Assume that M is $\mathfrak{a}$-cofinite. Then the map $\chi_M^{\widehat{R}^{\mathfrak{a}}} : \widehat{R}^{\mathfrak{a}} \to \operatorname{Hom}_R(M, M)$ given by $\chi_T^{\widehat{R}^{\mathfrak{a}}}(r)(m) = rm$ is well-defined. Indeed, the fact that M satisfies the small support condition implies that M is $\mathfrak{a}$-torsion (this uses Fact 2.33 and Lemma 2.64). This endows M with an $\widehat{R}^{\mathfrak{a}}$-module structure that is compatible with the R-module structure on M by Fact 2.35. Hence, $\chi_M^{\widehat{R}^{\mathfrak{a}}}$ is well-defined.

We are now in a position to define the notion that recovers semidualizing and quasidualizing modules as examples.

Definition 3.4. An R-module M is $\mathfrak{a}$-*codualizing* if it is $\mathfrak{a}$-cofinite, the natural homothety map $\chi_M^{\widehat{R}^{\mathfrak{a}}} : \widehat{R}^{\mathfrak{a}} \to \operatorname{Hom}_R(M, M)$ is an isomorphism and $\operatorname{Ext}^i_R(M, M) = 0$ for all $i \geqslant 1$.

The following propositions show that the notion of an $\mathfrak{a}$-codualizing module is indeed the "umbrella" notion we set out to find because semidualizing modules and quasidualizing modules are recovered when $\mathfrak{a} = 0$ and $\mathfrak{a} = \mathfrak{m}$ respectively.

Proposition 3.5. *An R-module C is semidualizing if and only if C is 0-codualizing.*

Proof. The Ext-vanishing and isomorphism conditions in Definitions 1.1 and 3.1 are equivalent since $\widehat{R}^0 \cong R$. The rest of the proof follows from Proposition 3.2(a). $\square$

Proposition 3.6. *When $(R, \mathfrak{m})$ is local, an R-module T is quasidualizing if and only if T is $\mathfrak{m}$-codualizing.*

Proof. The Ext-vanishing and isomorphism conditions in Definitions 1.2 and 3.1 are equivalent. The remainder of the proof follows from Proposition 3.2(b). $\square$

Since there is existing research on the behavior of semidualizing and quasidualizing modules, we can look at how the preexisting research transfers to the setting of $\mathfrak{a}$-codualizing modules. We will begin with so-called Auslander and Bass classes.

Definition 3.7. Let M, A and B be R-modules.

1. Then A is in the *Auslander class* $\mathcal{A}_M^0(R)$ if for all $i \geqslant 1$ we have $\operatorname{Tor}_i^R(M, A) = 0 = \operatorname{Ext}_R^i(M, M \otimes_R A)$ and the natural map $\gamma_A^M \colon A \to \operatorname{Hom}_R(M, M \otimes_R A)$ defined by $\gamma_A^M(a)(m) = m \otimes a$ is an isomorphism.

2. The module B is in the *Bass class* $\mathcal{B}_M^0(R)$ if for all $i \geqslant 1$ we have $\operatorname{Ext}_R^i(M, B) = 0 = \operatorname{Tor}_i^R(M, \operatorname{Hom}_R(M, B))$ and the natural map $\xi_B^M \colon M \otimes_R \operatorname{Hom}_R(M, B) \to B$ defined by $\xi_B^M(m \otimes_R \psi) = \psi(m)$ is an isomorphism.

Remark 3.8. Our notation here deviates from the existing literature. We use $\mathcal{A}_M^0(R)$ and $\mathcal{B}_M^0(R)$ to distinguish them from $\mathcal{A}_M(R)$ and $\mathcal{B}_M(R)$, which we discuss below.

The following examples are well-known when R is noetherian, although, they are true in a more general setting, cf. [23].

Example 3.9. Let C be a semidualizing module (that is, 0-codualizing).

1. The free module R is in the Auslander class $\mathcal{A}_C(R)$.

2. If M has finite flat dimension, then M is in $\mathcal{A}_C(R)$.

3. The module C is in the Bass class $\mathcal{B}_C(R)$.

4. If M has finite injective dimension, then M is in $\mathcal{B}_C(R)$.

These examples raise natural questions about the behavior of the Foxby classes when the semidualizing module C is replaced with an arbitrary $\mathfrak{a}$-codualizing module. For example, one proves items 2 and 4 in the example by first showing the Auslander class contains all flat modules and the Bass class contains all injective module. Then one shows the Auslander and Bass classes satisfy the 2-of-3 condition, that is, given an exact sequence

$$0 \to M_1 \to M_2 \to M_3 \to 0$$

if two of the modules are in the Auslander or Bass class, then the third is in the Auslander or Bass class, as well.

The next proposition shows that one cannot expect $R \in \mathcal{A}_M^0(R)$ when M is an arbitrary $\mathfrak{a}$-codualizing module.

Proposition 3.10. *Let M be an $\mathfrak{a}$-codualizing module. Then R is in $\mathcal{A}_M^0(R)$ if and only if R is $\mathfrak{a}$-adically complete.*

Proof. ($\Rightarrow$): Assume that R is in $\mathcal{A}_M^0(R)$. Then we have isomorphisms

$$R \cong \operatorname{Hom}_R(M, M \otimes_R R) \cong \operatorname{Hom}_R(M, M) \cong \widehat{R}^{\mathfrak{a}}.$$

Therefore, R is $\mathfrak{a}$-adically complete.

($\Leftarrow$): Assume that R is $\mathfrak{a}$-adically complete. Since R is free over itself, we have $\operatorname{Tor}_i^R(M, R) = 0$ for all $i \geqslant 1$. Also, $\operatorname{Ext}_R^i(M, M \otimes_R R) \cong \operatorname{Ext}_R^i(M, M) = 0$ for all $i \geqslant 1$ because M is $\mathfrak{a}$-codualizing, where the isomorphism is tensor cancellation. Since

R is $\mathfrak{a}$-adically complete, we have $R = \widehat{R}^{\mathfrak{a}}$ and $\chi_M^{\widehat{R}^{\mathfrak{a}}} = \chi_M^R$. Let $F : M \otimes_R R \to M$ be the tensor cancelation map. To show $R \in \mathcal{A}_M(R)$ it suffices to show that the following diagram commutes:

$$\begin{tikzcd} R \arrow[r, "\gamma_R^M"] \arrow[dr, "\chi_M^R"', "\cong"] & \mathrm{Hom}_R(M, M \otimes_R R) \arrow[d, "\mathrm{Hom}_R(M,F)", "\cong"'] \\ & \mathrm{Hom}_R(M, M) \end{tikzcd}$$

This is done in the next computation:

$$(\mathrm{Hom}_R(M, F) \circ \gamma_R^M(r))(m) = F(\gamma_R^M(r)(m)) = F(m \otimes r) = rm = \chi_M^R(r)(m).$$

It follows that R is in $\mathcal{A}_M^0(R)$. □

The behavior of the Auslander and Bass classes for codualizing modules further deviates from the behavior in the semidualizing case by the next example. In this example, we see that modules of finite projective and flat dimension are not necessarily in the Auslander class and the Auslander class need not satisfy the 2-of-3 condition.

Example 3.11. Let k be a field, and let $R = k[[X]]$ be a power series ring in one variable. Note that this is a complete, local ring. Let E be the injective hull of the residue field. By the previous proposition, R is in $\mathcal{A}_E(R)$. Now consider $M = R/(X)R$. Since X is a regular element of R (that is a non-zero divisor), a free resolution of M is given by the Koszul complex on X.

Based on work in the semidualizing case, one may expect M to be in $\mathcal{A}_E(R)$, as it is a module with finite flat dimension. However, this module fails the to meet the definition of $\mathcal{A}_E^0(R)$ in two ways.

First, in the abstract, $\mathrm{Hom}_R(E, E \otimes_R M) \cong \mathrm{Hom}_R(E, 0) = 0$. So it is not possible for $\gamma_M^E \colon M \to \mathrm{Hom}_R(E, E \otimes_R M)$ to be an isomorphism.

Second, $\mathrm{Tor}_1^R(E, M) \cong k$ and $\mathrm{Tor}_i^R(E, M) = 0$ for all $i \neq 1$. To see this, observe that a free resolution of M is given by the Koszul complex on X, that is, the following R-complex

$$K(X) = \qquad 0 \longrightarrow R \xrightarrow{X} R \longrightarrow 0.$$

Then to compute $\operatorname{Tor}_i^R(E, M)$ we look at the induced complex

$$E \otimes_R K(X) = \qquad 0 \longrightarrow E \xrightarrow{X} E \longrightarrow 0.$$

From here one can compute $\operatorname{Tor}_1^R(E, M) \cong k$ and $\operatorname{Tor}_i^R(E, M) = 0$ for all $i \neq 1$. As in the semidualizing case, there is Tor-vanishing in every degree except for one, but it is in the wrong degree: to be in $\mathcal{A}_E^0(R)$, we need $\operatorname{Tor}_i^R(E, M) = 0$ for all $i \neq 0$.

This example is even more troubling because this shows that the Auslander class does not satisfy the 2-of-3 condition. The following is an exact sequence

$$0 \longrightarrow R \xrightarrow{X} R \longrightarrow M \longrightarrow 0$$

and the first two modules are in the Auslander class of E, but the third is not.

3.2. Changing Contexts

The deficiencies in Example 3.11 inspire a change of context. Although, as stated in the introduction, this change of contexts is natural and not solely due to the previous example. We change our focus from modules to complexes. We begin with a reformulation of the $\mathfrak{a}$-cofinite condition.

The main result of this section is Theorem 3.15 which is an analogue of Fact 1.4 for this context. DG algebra methods will play a prominent role; consult Section 2.6 for relevant background information.

Proposition 3.12. *Let M be a homologically bounded R-complex and $\mathfrak{a}$ an ideal of R. Then the following conditions are equivalent.*

(i) *The* R*-complex* $R/\mathfrak{a} \otimes^{\mathbf{L}}_R M$ *is homologically degree-wise finite.*

(ii) *The* R*-complex* $R/\mathfrak{b} \otimes^{\mathbf{L}}_R M$ *is homologically degree-wise finite for all ideals* $\mathfrak{b} \supseteq \mathfrak{a}$.

(iii) *The* R*-complex* $N \otimes^{\mathbf{L}}_R M$ *is homologically degree-wise finite for all finitely generated* R*-modules* N *such that* $\operatorname{Supp}_R(N) \subseteq \operatorname{V}(\mathfrak{a})$.

(iv) *The* R*-complex* $X \otimes^{\mathbf{L}}_R M$ *is homologically degree-wise finite for all homologically finite* R*-complexes* X *such that* $\operatorname{Supp}_R(X) \subseteq \operatorname{V}(\mathfrak{a})$.

(v) *The* R*-complex* $K(\underline{x}) \otimes^{\mathbf{L}}_R M$ *is homologically finite for some (equivalently, for every) generating sequence* $\underline{x}$ *of* $\mathfrak{a}$.

Proof. $(i) \Rightarrow (ii)$: One has the following commutative diagram

$$\begin{tikzcd} R \arrow[r] \arrow[dr] & R/\mathfrak{a} \arrow[d] \\ & R/\mathfrak{b}. \end{tikzcd}$$

By assumption, the complex $R/\mathfrak{a} \otimes^{\mathbf{L}}_R M$ is homologically degree-wise finite and bounded below over R, hence over $R/\mathfrak{a}$. From Fact 2.80, there exists a degree-wise finite free resolution F of $R/\mathfrak{a} \otimes^{\mathbf{L}}_R M$ over $R/\mathfrak{a}$. It follows from the associativity of tensor product that $R/\mathfrak{b} \otimes^{\mathbf{L}}_R M \simeq R/\mathfrak{b} \otimes^{\mathbf{L}}_{R/\mathfrak{a}} (R/\mathfrak{a} \otimes^{\mathbf{L}}_R M) \simeq R/\mathfrak{b} \otimes_{R/\mathfrak{a}} F$. This is degree-wise finite over $R/\mathfrak{b}$, hence over R. Therefore, we have that the complex $R/\mathfrak{b} \otimes^{\mathbf{L}}_R M$ is homologically degree-wise finite over R.

$(ii) \Rightarrow (iii)$: Assume that N is finitely generated such that $\operatorname{Supp}_R(N) \subseteq \operatorname{V}(\mathfrak{a})$. Then there exists a prime filtration $0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_t = N$ such that $N_i/N_{i-1} \cong R/\mathfrak{p}_i$ and $\mathfrak{p}_i \in \operatorname{Supp} N$. We proceed by induction on t.

Base case: Assume $t = 1$. Then $N \cong N_1/N_0 \cong R/\mathfrak{p}$, where $\mathfrak{a} \subseteq \mathfrak{p}$. Then by assumption $R/\mathfrak{p} \otimes^{\mathbf{L}}_R M$ is homologically degree-wise finite. Therefore, $N \otimes^{\mathbf{L}}_R M$ is homologically degree-wise finite.

Assume that $N \otimes_R^{\mathbf{L}} M$ is homologically degree-wise finite for all finitely generated R-modules N with $\operatorname{Supp}_R(N) \subseteq \mathrm{V}(\mathfrak{a})$ and having a prime filtration of length $t < m$. Let N have a prime filtration $0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_m = N$. Consider the short exact sequence $0 \to N_{m-1} \to N \to N/N_{m-1} \to 0$. Applying $- \otimes_R^{\mathbf{L}} M$, we obtain the distinguished triangle $N_{m-1} \otimes_R^{\mathbf{L}} M \to N \otimes_R^{\mathbf{L}} M \to N/N_{m-1} \otimes_R^{\mathbf{L}} M \to$. By the induction hypothesis, $N_{m-1} \otimes_R^{\mathbf{L}} M$ is homologically degree-wise finite. By the base case, $N/N_{m-1} \otimes_R^{\mathbf{L}} M$ is homologically degree-wise finite. Therefore, $N \otimes_R^{\mathbf{L}} M$ is homologically degree-wise finite by Fact 2.41.

$(iii) \Rightarrow (iv)$: Assume that X is homologically finite such that $\operatorname{Supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$. Then we have $\operatorname{Supp}_R(\mathrm{H}_i(X)) \subseteq \mathrm{V}(\mathfrak{a})$. By assumption $\mathrm{H}_i(X) \otimes_R^{\mathbf{L}} M$ is homologically degree-wise finite for all i. We proceed by induction on $\operatorname{amp}(X)$.

Base case: $\operatorname{amp}(X) = 0$. Then X has one non-zero homology module. Therefore, we have $X \simeq \Sigma^i \mathrm{H}_i(X)$. So $X \otimes_R^{\mathbf{L}} M$ is homologically degree-wise finite by the previous paragraph.

Assume that the result holds for all homologically finite complexes X' such that $\operatorname{amp}(X') < \operatorname{amp}(X)$ and $\operatorname{Supp}_R(X') \subseteq \mathrm{V}(\mathfrak{a})$. Let $s = \sup(X)$. Take a soft truncation of X at s, that is, set

$$X' = \quad 0 \to X_s/\operatorname{Im}(\partial_s^X) \to X_{s-1} \to \cdots \to X_j \to 0 \quad \simeq X.$$

The short exact sequence

$$0 \longrightarrow \Sigma^s \mathrm{H}_s(X) \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

gives rise to the following distinguished triangles. Then $\mathrm{H}_i(X'') \cong \mathrm{H}_i(X)$ for all $i < s$ and $\mathrm{H}_i(X'') = 0$ otherwise. Then the induction hypothesis applies to X'' and the base case applies for $\Sigma^s \mathrm{H}_s(X)$. So, Fact 2.41 yields the desired result for X.

$(iv) \Rightarrow (v)$: (v) is the special case $X = K(\underline{x})$ of (iv).

$(v) \Rightarrow (i)$: Set $K = K(\underline{x})$ and consider the following commutative diagram of DG R-algebra homomorphisms

$$\begin{tikzcd} R \arrow[r] \arrow[dr] & K(\underline{x}) \arrow[d] \\ & R/\mathfrak{a}. \end{tikzcd}$$

Since $K \otimes^{\mathbf{L}}_R M$ is homologically finite over R, it has a degree-wise finite semi-free resolution $X \xrightarrow{\simeq} K \otimes^{\mathbf{L}}_R M$ over K by Fact 2.80. It follows that the next complex is homologically degree-wise finite over $R/\mathfrak{a}$.

$$R/\mathfrak{a} \otimes_K X = R/\mathfrak{a} \otimes^{\mathbf{L}}_K (K \otimes^{\mathbf{L}}_R M) \simeq R/\mathfrak{a} \otimes^{\mathbf{L}}_R M$$

Therefore, $R/\mathfrak{a} \otimes^{\mathbf{L}}_R M$ is homologically degree-wise finite over R as well. $\square$

Lemma 3.13. *Let A be a DG R-algebra such that each A_i is finitely generated over R. Let B and N be DG A-modules that are homologically degree-wise finite over R such that B is homologically bounded below and N is homologically bounded above. Then $\mathbf{R}\mathrm{Hom}_A(B, N)$ is homologically degree-wise finite and bounded above.*

Proof. Let $F \xrightarrow{\simeq} B$ be a degree-wise finite semi-free resolution over A such that $F_i = 0$ for all $i < n = \inf(B)$; see Fact 2.80. We proceed by cases.

Case 1: N is homologically bounded. In this case, let $G \xrightarrow{\simeq} N$ be a degree-wise finite semi-free resolution over A. In particular, each G_i is finitely generated over R. Since N is homologically bounded above, say with $s = \sup(N) < \infty$, the truncation

$$N' = \quad 0 \to G_s/\operatorname{Im}(\partial^G_s) \xrightarrow{\overline{\partial^G_s}} G_{s-1} \xrightarrow{\partial^G_{s-1}} \cdots$$

is a DG A-module that is isomorphic to N in $\mathcal{D}(A)$ by [4, 2.11.4.1]. Furthermore, each module N'_i is finitely generated over R and $N'_i = 0$ for $|i| \gg 0$. The isomorphism $N \simeq N'$

in $\mathcal{D}(a)$ provides the isomorphisms

$$\mathbf{R}\mathrm{Hom}_A(B,N) \simeq \mathrm{Hom}_A(F,N) \simeq \mathrm{Hom}_A(F,N').$$

So it suffices to show that $\mathrm{Hom}_A(F,N')$ is homologically degree-wise finite. By definition, this is a sub-complex of $\mathrm{Hom}_R(F,N')$ which is degree-wise finitely generated since N' is bounded and all the F_i and N'_j are finitely generated over R. It follows that each module $\mathrm{Hom}_A(F,N')_n$ is finitely generated over R, hence so is each homology module, as desired.

Case 2: the general case. Let $i \in \mathbb{Z}$ be given, and consider the truncation

$$N'' = \quad \cdots \xrightarrow{\partial^N_{n+i+2}} N_{n+i+1} \xrightarrow{\partial^N_{n+i+1}} \mathrm{Ker}(\partial^N_{n+i}) \to 0.$$

The natural morphism $\alpha\colon N'' \to N$ induces isomorphisms $\mathrm{H}_j(\alpha)\colon \mathrm{H}_j(N'') \xrightarrow{\cong} \mathrm{H}_j(N)$ for all $j \geqslant n+i$, and we have $\mathrm{H}_j(N'') = 0$ for all $j < n+i$. In particular, the DG A-module N'' is homologically finite, so Case 1 implies that $\mathrm{H}_i(\mathbf{R}\mathrm{Hom}_A(B,N''))$ is finitely generated. Thus, it suffices to show that we have $\mathrm{H}_i(\mathbf{R}\mathrm{Hom}_A(B,N'')) \cong \mathrm{H}_i(\mathbf{R}\mathrm{Hom}_A(B,N))$. Consider the natural distinguished triangle

$$N'' \xrightarrow{\alpha} N \to N''' \to \tag{1}$$

where $N''' = \mathrm{Coker}(\alpha)$.

Claim: $\mathrm{H}_j(\mathbf{R}\mathrm{Hom}_A(B,N''')) = 0$ for all $j > i$. Since we have $\mathrm{H}_j(\alpha)\colon \mathrm{H}_j(N'') \xrightarrow{\cong} \mathrm{H}_j(N)$ for all $j \geqslant n+i$ and $\mathrm{H}_{n+i-1}(N'') = 0$, it follows from the long exact sequence in homology associated to the triangle (1) that $\mathrm{H}_j(N''') = 0$ for all $j \geqslant n+i$. In other words, we have $\sup(N''') < n+i$, and it follows that the truncation

$$N'''' = \quad 0 \to N'''_{n+i-1}/\operatorname{Im}(\partial^{N'''}_{n+i}) \xrightarrow{\overline{\partial^{N'''}_{n+i-1}}} {N'''}_{n+i-2} \xrightarrow{\partial^{N'''}_{n+i-2}} \cdots$$

is isomorphic to N''' in $\mathcal{D}(A)$. Since $F_m = 0$ for all $m < n$, we conclude that

$$0 = \operatorname{Hom}_R(F, N'''')_j \supseteq \operatorname{Hom}_A(F, N'''')_j$$

for all $j > i$. This implies that $\operatorname{Hom}_A(F, N'''')_j = 0$, and so $0 = \mathrm{H}_j(\operatorname{Hom}_A(F, N'''')) \cong \mathrm{H}_j(\mathbf{R}\operatorname{Hom}_A(B, N'''))$ for all $j > i$, as claimed.

Now consider the distinguished triangle

$$\mathbf{R}\operatorname{Hom}_A(B, N'') \xrightarrow{\mathbf{R}\operatorname{Hom}_A(B,\alpha)} \mathbf{R}\operatorname{Hom}_A(B, N) \to \mathbf{R}\operatorname{Hom}_A(B, N''') \to$$

induced from (1). Because of the claim, part of the long exact sequence in homology associated to this triangle has the following form

$$0 \to \mathrm{H}_i(\operatorname{Hom}_A(F, N'')) \to \mathrm{H}_i(\operatorname{Hom}_A(F, N)) \to 0.$$

We conclude that $\mathrm{H}_i(\mathbf{R}\operatorname{Hom}_A(B, N'')) \cong \mathrm{H}_i(\mathbf{R}\operatorname{Hom}_A(B, N))$, as desired. $\square$

Proposition 3.14. *Let M be a homologically bounded R-complex and $\mathfrak{a}$ an ideal of R. Then the following conditions are equivalent.*

(i) The R-complex $\mathbf{R}\operatorname{Hom}_R(R/\mathfrak{a}, M)$ is homologically degree-wise finite.

(ii) The complex $\mathbf{R}\operatorname{Hom}_R(R/\mathfrak{b}, M)$ is homologically degree-wise finite for all $\mathfrak{b} \supseteq \mathfrak{a}$.

(iii) The R-complex $\mathbf{R}\operatorname{Hom}_R(N, M)$ is homologically degree-wise finite for all finitely generated R-modules N such that $\operatorname{Supp}_R(N) \subseteq \mathrm{V}(\mathfrak{a})$.

(iv) The R-complex $\mathbf{R}\operatorname{Hom}_R(X, M)$ is homologically degree-wise finite for all homologically finite R-complexes X such that $\operatorname{Supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$.

(v) The R-complex $K(\underline{x}) \otimes^{\mathbf{L}}_R M$ is homologically finite for some (equivalently, for every) generating sequence $\underline{x}$ of $\mathfrak{a}$.

Proof. $(i) \Rightarrow (ii)$: One has the following commutative diagram

$$\begin{tikzcd} R \arrow[r] \arrow[dr] & R/\mathfrak{a} \arrow[d] \\ & R/\mathfrak{b}. \end{tikzcd}$$

By assumption, $\mathbf{R}\mathrm{Hom}_R(R/\mathfrak{a}, M)$ is homologically degree-wise finite over R, hence over $R/\mathfrak{a}$. By adjointness, we have

$$\mathbf{R}\mathrm{Hom}_R(R/\mathfrak{b}, M) \simeq \mathbf{R}\mathrm{Hom}_{R/\mathfrak{a}}(R/\mathfrak{b}, \mathbf{R}\mathrm{Hom}_R(R/\mathfrak{a}, M)).$$

We apply Lemma 3.13 with $A = R/\mathfrak{a}, B = R/\mathfrak{b}$, and $N = \mathbf{R}\mathrm{Hom}_R(R/\mathfrak{a}, M)$. We conclude that $\mathbf{R}\mathrm{Hom}_{R/\mathfrak{a}}(R/\mathfrak{b}, \mathbf{R}\mathrm{Hom}_R(R/\mathfrak{a}, M))$ is homologically degree-wise finite.

$(ii) \Rightarrow (iii)$: Assume that N is finitely generated such that $\mathrm{Supp}_R(N) \subseteq \mathrm{V}(\mathfrak{a})$. Then there exists a prime filtration $0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_t = N$ such that $N_i/N_{i-1} \cong R/\mathfrak{p}_i$ and $\mathfrak{p}_i \in \mathrm{Supp}\, N$. We proceed by induction on t.

Base case: Assume $t = 1$. Then $N \cong N_1/N_0 \cong R/\mathfrak{p}$, where $\mathfrak{a} \subseteq \mathfrak{p}$. Then by assumption $\mathbf{R}\mathrm{Hom}_R(R/\mathfrak{p}, M)$ is homologically degree-wise finite. Thus, $\mathbf{R}\mathrm{Hom}_R(N, M)$ is homologically degree-wise finite.

Assume that $\mathbf{R}\mathrm{Hom}_R(N, M)$ is homologically degree-wise finite for all finitely generated R-modules N with $\mathrm{Supp}_R(N) \subseteq \mathrm{V}(\mathfrak{a})$ and having a prime filtration of length $t < m$. Let N have a prime filtration $0 = N_0 \subseteq N_1 \subseteq \cdots \subseteq N_m = N$. Consider the short exact sequence $0 \to N_{m-1} \to N \to N/N_{m-1} \to 0$. Applying $\mathbf{R}\mathrm{Hom}_R(-, M)$, we obtain the distinguished triangle

$$\mathbf{R}\mathrm{Hom}_R(N/N_{m-1}, M) \to \mathbf{R}\mathrm{Hom}_R(N, M) \to \mathbf{R}\mathrm{Hom}_R(N_{m-1}, M) \to .$$

By the induction hypothesis, $\mathbf{R}\mathrm{Hom}_R(N_{m-1}, M)$ is homologically degree-wise finite. By the base case, $\mathbf{R}\mathrm{Hom}_R(N/N_{m-1}, M)$ is homologically degree-wise finite. Therefore, we have that $\mathbf{R}\mathrm{Hom}_R(N, M)$ is homologically degree-wise finite by Fact 2.41.

$(iii) \Rightarrow (iv)$: Assume that X is homologically finite such that $\mathrm{Supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$. Then we have $\mathrm{Supp}_R(\mathrm{H}_i(X)) \subseteq \mathrm{V}(\mathfrak{a})$. So by assumption $\mathbf{R}\mathrm{Hom}_R(\mathrm{H}_i(X), M)$ is homologically degree-wise finite for all i. We proceed by induction on $\mathrm{amp}(X)$.

Base case: $\mathrm{amp}(X) = 0$. Then X has one non-zero homology module. Thus, we have $X \simeq \boldsymbol{\Sigma}^i \mathrm{H}_i(X)$. So $\mathbf{R}\mathrm{Hom}_R(X, M)$ is homologically degree-wise finite by the previous paragraph.

Assume that the result holds for all homologically finite complexes X' such that $\mathrm{amp}(X') < \mathrm{amp}(X)$ and $\mathrm{Supp}_R(X') \subseteq \mathrm{V}(\mathfrak{a})$. Let $s = \sup(X)$. Take a soft truncation of X at s, that is, set

$$X' = \quad 0 \to X_s/\operatorname{Im}(\partial_s^X) \to X_{s-1} \to \cdots \to X_j \to 0 \quad \simeq X.$$

The short exact sequence

$$0 \longrightarrow \boldsymbol{\Sigma}^s \mathrm{H}_s(X) \longrightarrow X \longrightarrow X'' \longrightarrow 0$$

gives rise to the following distinguished triangles.

$$\boldsymbol{\Sigma}^s \mathrm{H}_s(X) \longrightarrow X \longrightarrow X'' \longrightarrow$$

$$\mathbf{R}\mathrm{Hom}_R(X'', M) \longrightarrow \mathbf{R}\mathrm{Hom}_R(X, M) \longrightarrow \mathbf{R}\mathrm{Hom}_R(\boldsymbol{\Sigma}^s \mathrm{H}_s(X), M) \longrightarrow$$

Then $\mathrm{H}_i(X'') \cong \mathrm{H}_i(X)$ for all $i < s$ and $\mathrm{H}_i(X'') = 0$ otherwise. Then the induction hypothesis applies to X'' and the base case applies for $\Sigma^s \mathrm{H}_s(X)$. So, Fact 2.41 yields the desired result for X.

$(iv) \Rightarrow (v)$: (v) is the special case $X = K(\underline{x})$ of (iv).

$(v) \Rightarrow (i)$: Set $K = K(\underline{x})$ and consider the following commutative diagram of DG R-algebra homomorphisms

$$\begin{tikzcd} R \arrow[r] \arrow[dr] & K(\underline{x}) \arrow[d] \\ & R/\mathfrak{a}. \end{tikzcd}$$

Assume $K \otimes_R^{\mathbf{L}} M$ is homologically finite. The shift isomorphism $K \sim \operatorname{Hom}_R(K, R)$ implies that $K \otimes_R^{\mathbf{L}} M$ is shift isomorphic over R to $\mathbf{R}\operatorname{Hom}_R(K, M)$. By adjointenss, we have $\mathbf{R}\operatorname{Hom}_R(R/\mathfrak{a}, M) \simeq \mathbf{R}\operatorname{Hom}_K(R/\mathfrak{a}, \mathbf{R}\operatorname{Hom}_R(K, M))$. We apply Lemma 3.13 with $A = K, B = R/\mathfrak{a}$, and $N = \mathbf{R}\operatorname{Hom}_R(K, M)$. Therefore, $\mathbf{R}\operatorname{Hom}_R(R/\mathfrak{a}, M)$ is homologically degree-wise finite over R. $\square$

The following result is Theorem 1.6 from the introduction.

Theorem 3.15. *Let M be a homologically bounded R-complex. Then the following conditions are equivalent.*

(i) The R-complex $K(\underline{x}) \otimes_R^{\mathbf{L}} M$ is homologically finite for some (equivalently for every) generating sequence $\underline{x}$ of $\mathfrak{a}$.

(ii) The R-complex $M \otimes_R^{\mathbf{L}} R/\mathfrak{a}$ is homologically degree-wise finite.

(iii) The R-complex $\mathbf{R}\operatorname{Hom}_R(R/\mathfrak{a}, M)$ is homologically degree-wise finite.

Proof. This is a immediate from Propositions 3.12 and 3.14. $\square$

Definition 3.16. A homologically bounded R-complex M is $\mathfrak{a}$-*cofinite* if M satisfies the equivalent conditions of Theorem 3.15 and $\operatorname{supp}_R(M) \subseteq \mathrm{V}(\mathfrak{a})$.

When M is an $\mathfrak{a}$-cofinite R-module, the map $\chi_M^{\widehat{R}^{\mathfrak{a}}}$ is well-defined because of the condition on the small support; see Remark 3.3. For complexes, the support condition is also used to show that $\chi_M^{\widehat{R}^{\mathfrak{a}}}$ is well-defined. However, one uses the injective resolution of M to endow M with an $\widehat{R}^{\mathfrak{a}}$-complex structure. We show this in Proposition 3.18(b).

Lemma 3.17. *Let M be a homologically bounded R-complex with $\operatorname{supp}_R(M) \subseteq V(\mathfrak{a})$.*

(a) The minimal injective resolution of M consists of $\mathfrak{a}$-torsion modules.

(b) The complex M has an injective resolution consisting of $\mathfrak{a}$-torsion modules.

(c) The natural morphism $\mathbf{R}\Gamma_{\mathfrak{a}}(M) \to M$ is an isomorphism in $\mathcal{D}(R)$.

(d) Every injective resolution of M consisting of $\mathfrak{a}$-torsion modules is an $\widehat{R}^{\mathfrak{a}}$-complex.

Proof. Let $M \xrightarrow{\simeq} J$ be a minimal injective resolution.

(a) By Lemma 2.64, for each i we have $J_i \cong \oplus_{\mathfrak{p}\in\operatorname{supp}_R(M)} E_R(R/\mathfrak{p})^{(\mu^i_{\mathfrak{p}})}$ for some sets $\mu^i_{\mathfrak{p}}$. Since each $\mathfrak{p} \in \operatorname{supp}_R(M)$ is in $V(\mathfrak{a})$, it follows that each summand $E_R(R/\mathfrak{p})^{(\mu^i_{\mathfrak{p}})}$ is $\mathfrak{a}$-torsion, so each J_i is $\mathfrak{a}$-torsion as well.

(b) Since M has a minimal injective resolution, this follows from part (a).

(c) Since each J_i is $\mathfrak{a}$-torsion, we have $\Gamma_{\mathfrak{a}}(J) = J$. As the natural morphism $\mathbf{R}\Gamma_{\mathfrak{a}}(M) \to M$ is represented by the inclusion $\Gamma_{\mathfrak{a}}(J) \xrightarrow{=} J$, it follows that the natural morphism is an isomorphism in $\mathcal{D}(R)$.

(d) Each module J_i is $\mathfrak{a}$-torsion, so it is an $\widehat{R}^{\mathfrak{a}}$-module by Fact 2.35. and each differential ∂^J_i is $\widehat{R}^{\mathfrak{a}}$-linear by [28, Lemma 2.2(a)]. $\square$

Proposition 3.18. *Let M be a homologically bounded R-complex with $\operatorname{supp}_R(M) \subseteq V(\mathfrak{a})$. Let $M \underset{\alpha}{\xrightarrow{\simeq}} J$ be an injective resolution of M consisting of $\widehat{R}^{\mathfrak{a}}$-modules.*

(a) The chain map $\chi^{\widehat{R}^{\mathfrak{a}}}_J \colon \widehat{R}^{\mathfrak{a}} \to \operatorname{Hom}_R(J, J)$ given by $\chi^{\widehat{R}^{\mathfrak{a}}}_J(r)(j) = rj$ is well-defined.

(b) The chain map $\chi^{\widehat{R}^{\mathfrak{a}}}_J \colon \widehat{R}^{\mathfrak{a}} \to \operatorname{Hom}_R(J, J)$ gives rise to a well-defined morphism $\chi^{\widehat{R}^{\mathfrak{a}}}_M \colon \widehat{R}^{\mathfrak{a}} \to \mathbf{R}\operatorname{Hom}_R(M, M)$ in $\mathcal{D}(R)$.

Proof. (a) For each $r \in \widehat{R}^{\mathfrak{a}}$, multiplication by r determines a well-defined chain map $J \xrightarrow{r} J$. It is straightforward to show that this implies that $\chi^{\widehat{R}^{\mathfrak{a}}}_J \colon \widehat{R}^{\mathfrak{a}} \to \operatorname{Hom}_R(J, J)$ is a well-defined chain map.

(b) Since $\mathbf{R}\operatorname{Hom}_R(M, M) \simeq \operatorname{Hom}_R(J, J)$, this follows from Fact 2.38. $\square$

Definition 3.19. Let M be a R-complex. M is $\mathfrak{a}$-*codualizing* if M is $\mathfrak{a}$-cofinite and the homothety map $\chi_M^{\widehat{R}^{\mathfrak{a}}} : \widehat{R}^{\mathfrak{a}} \to \mathbf{R}\mathrm{Hom}_R(M, M)$ is an isomorphism in $\mathcal{D}(R)$.

3.3. Building Examples

We now aim to provide an example of an $\mathfrak{a}$-codualizing complex. One should note that even in the case of semidualizing complexes (0-codualizing) it is challenging to find non-trivial examples.

Lemma 3.20. *Let N be a homologically bounded complex. Then* $\mathrm{supp}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(N)) \subseteq \mathrm{supp}_R(N) \cap \mathrm{V}(\mathfrak{a})$.

Proof. Note that $\mathrm{supp}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(R)) \subseteq \mathrm{V}(\mathfrak{a})$. Indeed, if J is an injective resolution of R, and $\mathfrak{p} \notin \mathrm{V}(\mathfrak{a})$, then $\Gamma_{\mathfrak{a}}(J)_{\mathfrak{p}} = 0$. Hence, $\mathfrak{p} \notin \mathrm{Supp}_R(\Gamma_{\mathfrak{a}}(J))$. Next, consider the isomorphisms $\mathbf{R}\Gamma_{\mathfrak{a}}(N) \cong \check{\mathrm{C}}(\mathfrak{a}) \otimes_R^{\mathbf{L}} N = \mathbf{R}\Gamma_{\mathfrak{a}}(R) \otimes_R^{\mathbf{L}} N$ in $\mathcal{D}(R)$. Then Fact 2.51 implies $\mathrm{supp}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(N)) = \mathrm{supp}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(R)) \cap \mathrm{supp}_R(N) \subseteq \mathrm{V}(\mathfrak{a}) \cap \mathrm{supp}_R(N)$. $\square$

The following is inspired by [11, Corollary 1].

Lemma 3.21. *Let M be a homologically bounded complex and let $\mathfrak{a} \subseteq \mathfrak{b}$ be ideals of R. Let $K(\mathfrak{a})$ denote the Koszul complex on a generating sequence for $\mathfrak{a}$, and let $K(\mathfrak{b})$ denote the Koszul complex on a generating sequence for $\mathfrak{b}$. If $K(\mathfrak{a}) \otimes_R^{\mathbf{L}} M$ is homologically finite, then $K(\mathfrak{b}) \otimes_R^{\mathbf{L}} M$ is homologically finite.*

Proof. This follows from Proposition 3.12. $\square$

Theorem 3.22. *Let M be an $\mathfrak{a}$-cofinite R-complex. If $\mathfrak{b}$ is an ideal of R such that $\mathfrak{a} \subseteq \mathfrak{b}$, then $\mathbf{R}\Gamma_{\mathfrak{b}}(M)$ is $\mathfrak{b}$-cofinite.*

Proof. Note that we have $\mathrm{supp}(\mathbf{R}\Gamma_{\mathfrak{b}}(M)) \subseteq \mathrm{V}(\mathfrak{b})$ by Lemma 3.20.

Let $K(\mathfrak{b})$ be a the Koszul complex on a generating sequence for the ideal $\mathfrak{b}$. We show the necessary finiteness condition by showing $K(\mathfrak{b}) \otimes_R^{\mathbf{L}} \mathbf{R}\Gamma_{\mathfrak{b}}(M)$ is homologically finite. Let $M \to J$ be a minimal injective resolution. By Lemma 2.64, we know $J_i = \oplus_{\mathfrak{p}\in \operatorname{supp}_R(M)} E_R(R/\mathfrak{p})^{(\mu^i_\mathfrak{p})}$.

Claim: if I is a complex of injective R-modules with $I_j \cong \oplus_{\mathfrak{p}\notin \mathrm{V}(\mathfrak{b})} E_R(R/\mathfrak{p})^{(\mu^i_\mathfrak{p})}$, then $\operatorname{supp}_R(I) \cap \mathrm{V}(\mathfrak{b}) = \emptyset$. Indeed, by Fact 2.34, if $\mathfrak{p} \in \mathrm{V}(\mathfrak{b})$, then $I_\mathfrak{p} = 0$. Therefore, $\operatorname{Supp}_R(I) \cap \mathrm{V}(\mathfrak{b}) = \emptyset$. It follows that $\operatorname{supp}_R(I) \cap \mathrm{V}(\mathfrak{b}) \subseteq \operatorname{Supp}_R(I) \cap \mathrm{V}(\mathfrak{b}) = \emptyset$. This proves the claim.

Now, consider the exact sequence

$$0 \longrightarrow \Gamma_\mathfrak{b}(J) \longrightarrow J \longrightarrow J/\Gamma_\mathfrak{b}(J) \longrightarrow 0. \tag{2}$$

This is degree-wise split because $\Gamma_\mathfrak{b}(J)$ is a complex of injective modules. Also, the complexes $\Gamma_\mathfrak{b}(J)$ and $J/\Gamma_\mathfrak{b}(J)$ are homologically bounded, as in the proof of Lemma 2.63. From Fact 2.33 it follows that we have

$$(J/\Gamma_\mathfrak{b}(J))_i \cong \bigoplus_{\mathfrak{p}\in \operatorname{supp}(M)\setminus \mathrm{V}(\mathfrak{b})} E_R(R/\mathfrak{p})^{(\mu^i_\mathfrak{p})}.$$

Note that this provides an injective resolution of $(J/\Gamma_\mathfrak{b}(J))$ that may not be minimal. Using Fact 2.51 and the claim with $I = J/\Gamma_\mathfrak{b}(J)$ we have

$$\begin{aligned}
\operatorname{supp}_R(K(\mathfrak{b}) \otimes_R J/\Gamma_\mathfrak{b}(J)) &= \operatorname{supp}_R(K(\mathfrak{b})) \cap \operatorname{supp}_R(J/\Gamma_\mathfrak{b}(J)) \\
&\subseteq \mathrm{V}(\mathfrak{b}) \cap \operatorname{supp}_R(J/\Gamma_\mathfrak{b}(J)) \\
&= \emptyset.
\end{aligned}$$

Hence, we have $K(\mathfrak{b}) \otimes_R J/\Gamma_\mathfrak{b}(J) \simeq 0$.

Apply $K(\mathfrak{b}) \otimes_R -$ to the short exact sequence (2).

$$0 \longrightarrow K(\mathfrak{b}) \otimes_R \Gamma_\mathfrak{b}(J) \longrightarrow K(\mathfrak{b}) \otimes_R J \longrightarrow K(\mathfrak{b}) \otimes_R J/\Gamma_\mathfrak{b}(J) \longrightarrow 0$$

Since $K(\mathfrak{b}) \otimes_R J/\Gamma_{\mathfrak{b}}(J) \simeq 0$, we have a quasiisomorphism

$$K(\mathfrak{b}) \otimes_R \Gamma_{\mathfrak{b}}(J) \xrightarrow{\simeq} K(\mathfrak{b}) \otimes_R J.$$

It follows that we have an isomorphism $K(\mathfrak{b}) \otimes^{\mathbf{L}}_R \mathbf{R}\Gamma_{\mathfrak{b}}(M) \simeq K(\mathfrak{b}) \otimes^{\mathbf{L}}_R M$ in $\mathcal{D}(R)$.

By assumption, $K(\mathfrak{a}) \otimes^{\mathbf{L}}_R M$ is homologically degree-wise finite. Since $\mathfrak{a} \subseteq \mathfrak{b}$, Lemma 3.21 implies $K(\mathfrak{b}) \otimes^{\mathbf{L}}_R M$ homologically degree-wise finite. Thus, the complex $K(\mathfrak{b}) \otimes^{\mathbf{L}}_R \mathbf{R}\Gamma_{\mathfrak{b}}(M)$ is homologically degree-wise finite. $\square$

Lemma 3.23. *If M is an $\mathfrak{a}$-codualizing R-complex and $\mathfrak{a} \subseteq \mathfrak{b}$, then there is an isomorphism $\widehat{R}^{\mathfrak{b}} \simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{b}}(M), \mathbf{R}\Gamma_{\mathfrak{b}}(M))$ in $\mathcal{D}(R)$.*

Proof. Let $\check{C}(\mathfrak{b})$ denote the Čech complex on a generating sequence for $\mathfrak{b}$, and let $M \xrightarrow{\simeq} J$ be a minimal injective resolution. Lemma 3.17(a) implies that J consists of $\mathfrak{a}$-torsion modules. By assumption, the morphism $\chi^{\widehat{R}^{\mathfrak{a}}}_M : \widehat{R}^{\mathfrak{a}} \to \mathbf{R}\mathrm{Hom}_R(M, M)$ is an isomorphism in $\mathcal{D}(R)$. Also, Fact 2.46 provides isomorphisms $\widehat{R}^{\mathfrak{b}} \cong \widehat{\widehat{R}^{\mathfrak{a}}}^{\mathfrak{b}} \simeq \mathbf{L}\Lambda^{\mathfrak{b}}(\widehat{R}^{\mathfrak{a}})$.

Our result follows from the next sequence of isomorphisms. We begin by applying the functor $\mathbf{L}\Lambda^{\mathfrak{b}}(-)$ to the isomorphism $\widehat{R}^{\mathfrak{a}} \simeq \mathbf{R}\mathrm{Hom}_R(M, M)$.

$$\begin{aligned}
\widehat{R}^{\mathfrak{b}} &\simeq \mathbf{L}\Lambda^{\mathfrak{b}}(\widehat{R}^{\mathfrak{a}}) \\
&\simeq \mathbf{L}\Lambda^{\mathfrak{b}}(\mathbf{R}\mathrm{Hom}_R(M, M)) \\
&\simeq \mathbf{R}\mathrm{Hom}_R(\check{C}(\mathfrak{b}), \mathbf{R}\mathrm{Hom}_R(M, M)) \\
&\simeq \mathbf{R}\mathrm{Hom}_R(M \otimes^{\mathbf{L}}_R \check{C}(\mathfrak{b}), M) \\
&\simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{b}}(M), M) \\
&\simeq \mathrm{Hom}_R(\Gamma_{\mathfrak{b}}(J), J) \\
&\simeq \mathrm{Hom}_R(\Gamma_{\mathfrak{b}}(J), \Gamma_{\mathfrak{b}}(J)) \\
&\simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{b}}(M), \mathbf{R}\Gamma_{\mathfrak{b}}(M))
\end{aligned}$$

The third and fifth isomorphisms are by Fact 2.46, and the fourth one is Hom-tensor adjointness. The seventh isomorphism is from [28, Lemma 2.2(b)], and the others are by definition. $\square$

The following result is Theorem 1.7 from the introduction.

Theorem 3.24. *The R-complex $\mathbf{R}\Gamma_{\mathfrak{a}}(R)$ is $\mathfrak{a}$-codualizing.*

Proof. Since R is semidualizing (0-codualizing), the $\mathfrak{a}$-cofinite condition on $\mathbf{R}\Gamma_{\mathfrak{a}}(R)$ is a consequence of Theorem 3.22. Lemma 3.23 shows that there is an abstract isomorphism $\widehat{R}^{\mathfrak{a}} \simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(R), \mathbf{R}\Gamma_{\mathfrak{a}}(R))$. It remains to show that the morphism

$$\chi_R^{\mathbf{R}\Gamma_{\mathfrak{a}}(R)} : \widehat{R}^{\mathfrak{a}} \to \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(R), \mathbf{R}\Gamma_{\mathfrak{a}}(R))$$

is an isomorphism.

Let $\alpha : R \to I$ be an injective resolution, and let $\beta : T \to \Gamma_{\mathfrak{a}}(I)$ be a flat resolution over $\widehat{R}^{\mathfrak{a}}$, and hence over R. By Fact 2.46 we have

$$\widehat{R}^{\mathfrak{a}} \simeq \mathbf{L}\Lambda^{\mathfrak{a}}(R) \simeq \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(R), R) \simeq \mathrm{Hom}_R(T, I).$$

Also $\mathrm{Hom}_R(T, I)$ an injective resolution over R and consists of $\widehat{R}^{\mathfrak{a}}$-modules. (This uses Hom-tensor adjointness.) Thus, there exists a quasiisomorphism $\widehat{R}^{\mathfrak{a}} \xrightarrow[\simeq]{\psi} \mathrm{Hom}_R(T, I)$.

We have the following diagram:

$$\begin{tikzcd}
\Gamma_{\mathfrak{a}}(I) \arrow[r, "\tilde{f}"] & I \arrow[drr, dotted, "\tilde{g}"'] & R \arrow[l, "\alpha"'] \arrow[r, "g_0"] \arrow[dr, "\psi\circ g_0"] & \widehat{R}^{\mathfrak{a}} \arrow[d, "\psi"] \\
 & & & \mathrm{Hom}_R(T, I)
\end{tikzcd}$$

The maps $\tilde{f}$ and g_0 are the natural ones, and the maps α and ψ are the quasiisomorphisms from the injective resolutions of R and $\widehat{R}^{\mathfrak{a}}$ respectively. Since I is a bounded above complex of injective R-modules, it is a semi-injective DG R-module. Then $\tilde{g}$ can be constructed such that the preceding diagram commutes up to homotopy by [4, Theorem 2.9.6.1]. The chain maps $\tilde{f}$ and $\tilde{g}$ represent the morphisms f and g in the following diagram

$$\mathbf{R}\Gamma_{\mathfrak{a}}(R) \xrightarrow{f} R \xrightarrow{g} \mathbf{L}\Lambda^{\mathfrak{a}}(R).$$

Consider the following diagram.

$$\begin{tikzcd}
\widehat{R}^{\mathfrak{a}} \arrow[r, "\chi_R^{\mathbf{L}\Lambda^{\mathfrak{a}}(R)}"] \arrow[d, "\chi_R^{\mathbf{R}\Gamma_{\mathfrak{a}}(R)}"'] & \mathbf{R}\mathrm{Hom}_R(\mathbf{L}\Lambda^{\mathfrak{a}}(R), \mathbf{L}\Lambda^{\mathfrak{a}}(R)) \arrow[d, "g^{\star}"] \\
\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(R), \mathbf{R}\Gamma_{\mathfrak{a}}(R)) \arrow[d, "f_{\star}"', "\simeq"] & \mathbf{R}\mathrm{Hom}_R(R, \mathbf{L}\Lambda^{\mathfrak{a}}(R)) \arrow[d, "\simeq"', "f^{\star}"] \\
\mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(R), R) \arrow[r, "\simeq", "g_{\star}"'] & \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(R), \mathbf{L}\Lambda^{\mathfrak{a}}(R))
\end{tikzcd}$$

To be clear, $(-)_{\star} = \mathbf{R}\mathrm{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(R), -)$ and $(-)^{\star} = \mathbf{R}\mathrm{Hom}_R(-, \mathbf{L}\Lambda^{\mathfrak{a}}(R))$ are functors on the derived category. The morphisms $f^{\star}$ and $g_{\star}$ are isomorphisms in $\mathcal{D}(R)$ by [1, Theorem 0.3]. The map $f_{\star}$ is an isomorphism by [1, Lemma 0.4.2]. We aim to show that $g^{\star}$ and $\chi_R^{\mathbf{L}\Lambda^{\mathfrak{a}}(R)}$ are isomorphisms and that the diagram commutes. We then conclude that $\chi_R^{\mathbf{R}\Gamma_{\mathfrak{a}}(R)}$ is an isomorphism.

The previous diagram is represented by the following diagram with $(-)_{*} = \mathrm{Hom}_R(\Gamma_a(I), -)$ and $(-)^{*} = \mathrm{Hom}_R(-, \mathrm{Hom}_R(T, I))$.

$$\begin{tikzcd}
\widehat{R}^{\mathfrak{a}} \arrow[r, "\chi_R^{\mathrm{Hom}_R(T,I)}"] \arrow[d, "\chi_R^{\Gamma_{\mathfrak{a}}(I)}"'] & \mathrm{Hom}_R(\mathrm{Hom}_R(T, I), \mathrm{Hom}_R(T, I)) \arrow[d, "\tilde{g}^{*}"] \\
\mathrm{Hom}_R(\Gamma_{\mathfrak{a}}(I), \Gamma_{\mathfrak{a}}(I)) \arrow[d, "\tilde{f}_{*}"'] & \mathrm{Hom}_R(I, \mathrm{Hom}_R(T, I)) \arrow[d, "\tilde{f}^{*}"] \\
\mathrm{Hom}_R(\Gamma_{\mathfrak{a}}(I), I) \arrow[r, "\tilde{g}_{*}"'] & \mathrm{Hom}_R(\Gamma_{\mathfrak{a}}(I), \mathrm{Hom}_R(T, I))
\end{tikzcd}$$

So we show that this diagram commutes. Let $z \in \widehat{R}^{\mathfrak{a}}$ and $w \in \Gamma_{\mathfrak{a}}(I)$. The third equality in the following sequence is from Lemma 2.2(a) of [28]

$$\begin{aligned}
[\tilde{g}_* \circ \tilde{f}_* \circ \chi_R^{\Gamma_{\mathfrak{a}}(I)}](z)(w) =& [\tilde{g} \circ \tilde{f} \circ [\chi_R^{\Gamma_{\mathfrak{a}}(I)}(z)]](w) \\
=& (\tilde{g} \circ \tilde{f})(zw) \\
=& z[(\tilde{g} \circ \tilde{f})(w)] \\
=& [(\chi_R^{\mathrm{Hom}_R(T,I)}(z)) \circ (\tilde{g} \circ \tilde{f})](w) \\
=& [\tilde{f}^* \circ \tilde{g}^* \circ \chi_R^{\mathrm{Hom}_R(T,I)}](z)(w).
\end{aligned}$$

The other equalities are by definition. Thus, the diagram commutes.

Next, we show that $\tilde{g}^*$ is an quasiisomorphism. To this end, note that the following diagram commutes, where the horizontal chain maps are Hom-tensor adjointness.

$$\begin{tikzcd}
\mathrm{Hom}_R(\widehat{R}^{\mathfrak{a}}, \mathrm{Hom}_R(T,I)) \arrow[r, "\theta_{\widehat{R}^{\mathfrak{a}}TI}", "\cong"'] \arrow[d, "g_0^*"] & \mathrm{Hom}_R(T \otimes_R \widehat{R}^{\mathfrak{a}}, I) \arrow[d, "(T\otimes_R g_0)^*", "\simeq"'] \\
\mathrm{Hom}_R(R, \mathrm{Hom}_R(T,I)) \arrow[r, "\theta_{RTI}", "\cong"'] & \mathrm{Hom}_R(T \otimes_R R, I)
\end{tikzcd}$$

The morphism $(T \otimes g_0)^*$ is a quasiisomorphism as follows. Let $T \xrightarrow{\beta} \Gamma_{\mathfrak{a}}(I)$ be the quasiisomorphism from the flat resolution of $\Gamma_{\mathfrak{a}}(I)$. Consider the following diagram.

$$\begin{tikzcd}
T \otimes_R R \arrow[r, "T\otimes g_0"] \arrow[d, "\beta\otimes R", "\simeq"'] & T \otimes_R \widehat{R}^{\mathfrak{a}} \arrow[d, "\beta\otimes\widehat{R}^{\mathfrak{a}}", "\simeq"'] \\
\Gamma_{\mathfrak{a}}(I) \otimes_R R \arrow[r, "\Gamma_{\mathfrak{a}}(I)\otimes g_0", "\cong"'] & \Gamma_{\mathfrak{a}}(I) \otimes_R \widehat{R}^{\mathfrak{a}}
\end{tikzcd}$$

The vertical chain maps are quasiisomorphism because R and $\widehat{R}^{\mathfrak{a}}$ are flat modules. The map $\Gamma_{\mathfrak{a}}(I) \otimes g_0$ is an isomorphism by Fact 2.35. This establishes the fact that g_0^* is an quasiisomorphism.

Since α and ψ are quasiisomorphisms and $\operatorname{Hom}_R(T, I)$ is an injective resolution, the chain maps α^* and ψ^* are quasiisomorphisms. Furthermore, the following diagram commutes up to homotopy, which establishes the fact that $\tilde{g}^*$ is an quasiisomorphism.

$$\begin{tikzcd}
\operatorname{Hom}_R(\operatorname{Hom}_R(T,I),\operatorname{Hom}_R(T,I)) \arrow[r, "\psi^*", "\simeq"'] \arrow[d, "\tilde{g}^*"] & \operatorname{Hom}_R(\widehat{R}^{\mathfrak{a}},\operatorname{Hom}_R(T,I)) \arrow[d, "g_0^*", "\simeq"'] \\
\operatorname{Hom}_R(I,\operatorname{Hom}_R(T,I)) \arrow[r, "\alpha^*", "\simeq"'] & \operatorname{Hom}_R(R,\operatorname{Hom}_R(T,I))
\end{tikzcd}$$

Next, consider the following string of isomorphisms. The first isomorphism is by Lemma 3.23 and the second is by [1, (0.3)].

$$\begin{aligned}
\widehat{R}^{\mathfrak{a}} &\simeq \mathbf{R}\operatorname{Hom}_R(\mathbf{R}\Gamma_{\mathfrak{a}}(R), \mathbf{R}\Gamma_{\mathfrak{a}}(R)) \\
&\simeq \mathbf{R}\operatorname{Hom}_R(\mathbf{L}\Lambda^{\mathfrak{a}}(R), \mathbf{L}\Lambda^{\mathfrak{a}}(R)) \\
&\simeq \mathbf{R}\operatorname{Hom}_R(\widehat{R}^{\mathfrak{a}}, \widehat{R}^{\mathfrak{a}})
\end{aligned}$$

This implies that $\operatorname{Ext}^i_R(\widehat{R}^{\mathfrak{a}}, \widehat{R}^{\mathfrak{a}}) = 0$ for all $i \geqslant 1$. It follows that showing the morphism $\chi_R^{\mathbf{L}\Lambda^{\mathfrak{a}}(R)} : \widehat{R}^{\mathfrak{a}} \to \mathbf{R}\operatorname{Hom}_R(\mathbf{L}\Lambda^{\mathfrak{a}}(R), \mathbf{L}\Lambda^{\mathfrak{a}}(R))$ is an isomorphism reduces to showing that the map $\chi_R^{\widehat{R}^{\mathfrak{a}}} : \widehat{R}^{\mathfrak{a}} \to \operatorname{Hom}_R(\widehat{R}^{\mathfrak{a}}, \widehat{R}^{\mathfrak{a}})$ is an isomorphism. It is straightforward to show that $\chi_R^{\widehat{R}^{\mathfrak{a}}}$ is a monomorphism. We wish to show it is onto. Let $\phi \in \operatorname{Hom}_R(\widehat{R}^{\mathfrak{a}}, \widehat{R}^{\mathfrak{a}})$ and set $\xi = \phi(1) \in \widehat{R}^{\mathfrak{a}}$. We would like to see that ϕ is given by multiplication by ξ.

It suffices to show $\phi(x) - x\xi = 0$ for all $x \in R$. This along with the containments $\mathfrak{a}\widehat{R}^{\mathfrak{a}} \subseteq J(\widehat{R}^{\mathfrak{a}}) \subseteq \widehat{R}^{\mathfrak{a}}$ tell us it suffices to show $\phi(x) - x\xi \in (\mathfrak{a}\widehat{R}^{\mathfrak{a}})^n = \mathfrak{a}^n\widehat{R}^{\mathfrak{a}}$ for all $n \geqslant 1$. (This uses the Krull Intersection Theorem, cf. [24, Theorem 4.4].) The natural map $\gamma : R/\mathfrak{a}^n \to \widehat{R}^{\mathfrak{a}}/\mathfrak{a}^n\widehat{R}^{\mathfrak{a}}$ is an isomorphism. So if $\overline{x} \in \widehat{R}^{\mathfrak{a}}/\mathfrak{a}^n\widehat{R}^{\mathfrak{a}}$, then there exists $\overline{y} \in R/\mathfrak{a}^n$ such that $\gamma(\overline{y}) = \overline{x}$. So there exists y in R such that $y - x$ is in $\mathfrak{a}^n\widehat{R}^{\mathfrak{a}}$. It follows that $(y-x)\xi$

and $\phi(x-y)$ are in $\mathfrak{a}^n\widehat{R}^{\mathfrak{a}}$. From the next computation

$$\begin{aligned}\phi(x)-x\xi &= \phi(x)-\phi(y)+\phi(y)-x\xi\\ &= \phi(x)-\phi(y)+y\phi(1)-x\xi\\ &= \phi(x-y)+(y-x)\xi\end{aligned}$$

we conclude that $\phi(x)-x\xi\in\mathfrak{a}^n\widehat{R}^{\mathfrak{a}}$ for all n. Therefore, $\chi_R^{\mathbf{L}\Lambda^{\mathfrak{a}}(R)}$ is an isomorphism, and the proof is complete. $\square$

We now turn our attention to a uniqueness result for local Gorenstein rings. Recall that ring is *Gorenstein* if it has finite injective dimension over itself.

Proposition 3.25. *Let $(R,\mathfrak{m},k)$ be Gorenstein and local. If M is $\mathfrak{a}$-codualizing, then $P_M^R(t)$ and $I_R^M(t)$ are monomials.*

Proof. Since R is Gorenstein and local, $\widehat{R}^{\mathfrak{a}}$ is local, Gorenstein, and $\dim(R)=\dim(\widehat{R}^{\mathfrak{a}})$. We consider the following isomorphisms in $\mathcal{D}(R)$.

$$\begin{aligned}\mathbf{R}\mathrm{Hom}_R(k,\widehat{R}^{\mathfrak{a}}) &\simeq \mathbf{R}\mathrm{Hom}_R(k,\mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(\widehat{R}^{\mathfrak{a}},\widehat{R}^{\mathfrak{a}}))\\ &\simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(k\otimes_R^{\mathbf{L}}\widehat{R}^{\mathfrak{a}},\widehat{R}^{\mathfrak{a}})\\ &\simeq \mathbf{R}\mathrm{Hom}_{\widehat{R}^{\mathfrak{a}}}(k,\widehat{R}^{\mathfrak{a}})\end{aligned}$$

The first isomorphism is Hom cancellation, the second is adjointness, and the third follows from the fact that the residue field of $\widehat{R}^{\mathfrak{a}}$ is isomorphic to k.

By assumption, the modules $\mathrm{Ext}_R^i(R/\mathfrak{a},M)$ and $\mathrm{Tor}_i^R(R/\mathfrak{a},M)$ are finitely generated. Lemma 3.21 implies that $\mathrm{Ext}_R^i(k,M)$ and $\mathrm{Tor}_i^R(k,M)$ are finite dimensional k-vector spaces. It follows that we have $I_{\widehat{R}^{\mathfrak{a}}}^{\widehat{R}^{\mathfrak{a}}}(t)=I_R^{\widehat{R}^{\mathfrak{a}}}(t)$. Since $\widehat{R}^{\mathfrak{a}}$ is Gorenstein, we conclude $I_R^{\widehat{R}^{\mathfrak{a}}}(t)=I_{\widehat{R}^{\mathfrak{a}}}^{\widehat{R}^{\mathfrak{a}}}(t)=t^d$.

The following is the same argument as Lemma 1.5.3(b) of [3]. The first and third isomorphisms are adjointness and the second and fourth are tensor cancellation. The fifth isomorphism is tensor evaluation, which applies since $k \otimes^{\mathbf{L}}_R M$ and $\mathbf{R}\mathrm{Hom}_R(k, M)$ are appropriately bounded homologically degree-wise finite complexes over k.

$$\begin{aligned}
\mathbf{R}\mathrm{Hom}_R(k, \mathbf{R}\mathrm{Hom}_R(M, M)) &\simeq \mathbf{R}\mathrm{Hom}_R(k \otimes^{\mathbf{L}}_R M, M) \\
&\simeq \mathbf{R}\mathrm{Hom}_R((k \otimes^{\mathbf{L}}_R M) \otimes_k k, M) \\
&\simeq \mathrm{Hom}_k(k \otimes^{\mathbf{L}}_R M, \mathbf{R}\mathrm{Hom}_R(k, M)) \\
&\simeq \mathrm{Hom}_k(k \otimes^{\mathbf{L}}_R M, k \otimes_k \mathbf{R}\mathrm{Hom}_R(k, M)) \\
&\simeq \mathrm{Hom}_k(k \otimes^{\mathbf{L}}_R M, k) \otimes_k \mathbf{R}\mathrm{Hom}_R(k, M)
\end{aligned}$$

It follows that

$$\begin{aligned}
\mathrm{Ext}^i_R(k, \mathbf{R}\mathrm{Hom}_R(M, M)) &= \mathrm{H}_{-i}(\mathbf{R}\mathrm{Hom}_R(k, \mathbf{R}\mathrm{Hom}_R(M, M))) \\
&\cong \mathrm{H}_{-i}(\mathrm{Hom}_k(k \otimes^{\mathbf{L}}_R M, k) \otimes_k \mathbf{R}\mathrm{Hom}_R(k, M)) \\
&\cong \oplus_{p+q=i} \mathrm{H}_{-p}(\mathrm{Hom}_k(k \otimes^{\mathbf{L}}_R M, k) \otimes_k \mathrm{H}_{-q}(\mathbf{R}\mathrm{Hom}_R(k, M)) \\
&\cong \oplus_{p+q=i} \mathrm{Hom}_k(\mathrm{Tor}^R_p(M, k), k) \otimes_k \mathrm{Ext}^q_R(k, M).
\end{aligned}$$

Subsequently, we have $P^R_M(t) I^M_R(t) = I_R^{\mathbf{R}\mathrm{Hom}_R(M,M)}(t) = I_R^{\widehat{R}^{\mathfrak{a}}}(t) = t^d$, and the desired result follows. $\square$

Corollary 3.26. *Let $(R, \mathfrak{m}, k)$ be a local, Gorenstein ring. If M is quasidualizing (that is, M is $\mathfrak{m}$-codualizing), then M is shift isomorphic to $E_R(R/\mathfrak{m})$.*

Proof. Proposition 3.25 implies that we have $I^M_R(t) = t^{-f}$ for some integer f. This implies that $\mathbf{R}\mathrm{Hom}_R(k, M) \simeq \Sigma^f k$. Since $\mathrm{supp}_R(M) \subseteq V(\mathfrak{m}) = \{\mathfrak{m}\}$, Fact 2.58 and Lemma 2.64

imply that a minimal injective resolution $M \xrightarrow{\simeq} J$ is of the form

$$J = \quad 0 \to E^{(\mu^{-s})} \xrightarrow{\partial^J_s} E^{(\mu^{-s+1})} \xrightarrow{\partial^J_{s-1}} \cdots$$

where $s = \sup(M)$ and $E = E_R(k)$. (Note that the isomorphism $\mathbf{R}\mathrm{Hom}_R(M, M) \simeq \widehat{R}^{\mathfrak{m}}$ implies $M \not\simeq 0$, so $\sup(M)$ is finite.) In particular, the homology module $\mathrm{H}_s(M) \neq 0$ is isomorphic to a submodule of the $\mathfrak{m}$-torsion module $E^{(\mu^s)}$, so $\mathrm{H}_s(M)$ is $\mathfrak{m}$-torsion. It follows that $\mathrm{Hom}_R(k, \mathrm{H}_s(M)) \neq 0$, so [18, Lemma 2.1(1)] implies that

$$f = \sup(\mathbf{R}\mathrm{Hom}_R(k, M)) = \sup(M) = s.$$

On the other hand, since $\mathrm{supp}_R(M) \subseteq \{\mathfrak{m}\}$, the fact that $\mu^i_{\mathfrak{m}}(M) = 0$ for all $i > -f$ implies that the injective dimension of M is $\mathrm{id}_R(M) \leqslant -f$, by [16, (13.5)]. That is, we have $J_i = 0$ for all $i < f \leqslant -\mathrm{id}_R(M) \leqslant s = f$, i.e., for all $i < s$. Thus, J has the form

$$J = \quad 0 \to E^{(\mu^{-s})} \to 0.$$

In other words, we have $M \simeq J = \Sigma^s E^{(\mu^{-s})}$. It remains to show that $|\mu^{-s}| = 1$. The condition $0 \not\simeq M \simeq \Sigma^s E^{(\mu^{-s})}$ implies that $|\mu^{-s}| \geqslant 1$. Suppose that $|\mu^{-s}| > 1$. It follows that E^2 is a summand of $E^{(\mu^{-s})}$. Given the isomorphisms

$$\widehat{R}^{\mathfrak{m}} \simeq \mathbf{R}\mathrm{Hom}_R(M, M) \simeq \mathrm{Hom}_R(E^{(\mu^{-s})}, E^{(\mu^{-s})})$$

It follows that $\mathrm{Hom}_R(E^2, E^2) \cong (\widehat{R}^{\mathfrak{m}})^4$ is a summand of $\widehat{R}^{\mathfrak{m}}$. From this, we conclude that $k^4 \cong (\widehat{R}^{\mathfrak{m}})^4/\mathfrak{m}(\widehat{R}^{\mathfrak{m}})^4$ is a summand of $\widehat{R}^{\mathfrak{m}}/\mathfrak{m}\widehat{R}^{\mathfrak{m}} \cong k$, which is impossible. We conclude that $|\mu^{-s}| = 1$, which completes the proof. $\square$

3.4. Foxby Classes

We have the prerequisites to revisit the notions of the Auslander and Bass classes in the context of $\mathfrak{a}$-codualizing complexes. In this setting we recover aspects of these classes in the semidualizing context that were lost in the case of $\mathfrak{a}$-codualizing modules; see Example 3.11. We also, recover key aspects of Foxby equivalence in the $\mathfrak{a}$-codualizing setting; see Theorem 1.9.

Definition 3.27. Let M an R-complex, and let X and Y be homologically bounded R-complexes.

1. The complex X is in the *Auslander class* $\mathcal{A}_M(R)$ if $M \otimes_R^{\mathbf{L}} X$ is homologically bounded and $\gamma_X^M : X \to \mathbf{R}\mathrm{Hom}_R(M, M \otimes_R^{\mathbf{L}} X)$ is an isomorphism in $\mathcal{D}(R)$.
2. The complex Y is in the *Bass class* $\mathcal{B}_M(R)$ if $\mathbf{R}\mathrm{Hom}_R(M, Y)$ is homologically bounded and $\delta_Y^M : M \otimes_R^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_R(M, Y) \to Y$ is an isomorphism in $\mathcal{D}(R)$.

Proposition 3.28. *Let M be an $\mathfrak{a}$-codualizing R-complex.*

1. *One has $\widehat{R}^{\mathfrak{a}} \in \mathcal{A}_M(R)$.*
2. *One has $M \in \mathcal{B}_M(R)$.*

Proof. Let $M \to J$ be a minimal injective resolution of M. Each module J_i is $\mathfrak{a}$-torsion by Lemma 3.17(b). So the natural map $\alpha : J \to J \otimes_R \widehat{R}^{\mathfrak{a}}$ is an isomorphism by Fact 2.35.

(1) From the previous paragraph we see that $J \otimes_R \widehat{R}^{\mathfrak{a}}$ is an injective resolution of $M \otimes_R^{\mathbf{L}} \widehat{R}^{\mathfrak{a}}$ over R. Since M is homologically bounded and $\widehat{R}^{\mathfrak{a}}$ is flat, $M \otimes_R^{\mathbf{L}} \widehat{R}^{\mathfrak{a}}$ is homologically bounded. It remains to check that the following diagram commutes

$$\begin{tikzcd} \widehat{R}^{\mathfrak{a}} \arrow[r, "\chi_M^{\widehat{R}^{\mathfrak{a}}}", "\simeq"'] \arrow[dr, "\gamma_{\widehat{R}^{\mathfrak{a}}}^M"'] & \mathbf{R}\mathrm{Hom}_R(M,M) \arrow[d, "\mathbf{R}\mathrm{Hom}_R(M,\alpha)", "\simeq"'] \\ & \mathbf{R}\mathrm{Hom}_R(M, M \otimes_R^{\mathbf{L}} \widehat{R}^{\mathfrak{a}}) \end{tikzcd}$$

We check that the next diagram commutes.

$$\begin{tikzcd}
\widehat{R}^{\mathfrak{a}} \arrow[r, "\chi_J^{\widehat{R}^{\mathfrak{a}}}", "\simeq"'] \arrow[dr, "\gamma^J_{\widehat{R}^{\mathfrak{a}}}"'] & \operatorname{Hom}_R(J,J) \arrow[d, "\operatorname{Hom}_R(J,\alpha)", "\simeq"'] \\
& \operatorname{Hom}_R(J, J \otimes_R \widehat{R}^{\mathfrak{a}})
\end{tikzcd}$$

Let $r \in \widehat{R}^{\mathfrak{a}}$ and $j \in J$. Degree-wise, the complex J has an $\widehat{R}^{\mathfrak{a}}$-module structure compatible with the R-module structure given as follows. The element j is annihilated by $\mathfrak{a}^l$ for some $l \geqslant 1$. Let $r_0 \in r$ such that $r - r_0 \in \mathfrak{a}^n \widehat{R}^{\mathfrak{a}}$. Then we have $rj := r_0 j$. The next computation shows that the previous diagram commutes.

$$\begin{aligned}
(\operatorname{Hom}_R(J,\alpha) \circ \chi_J^{\widehat{R}^{\mathfrak{a}}})(r)(j) =& \operatorname{Hom}_R(J,\alpha)(\chi_J^{\widehat{R}^{\mathfrak{a}}}(r))(j) \\
=& \alpha(\chi_J^{\widehat{R}^{\mathfrak{a}}}(r)(j)) \\
=& \alpha(rj) \\
=& rj \otimes 1 \\
=& r_0 j \otimes 1 \\
=& j \otimes r_0 \\
=& j \otimes r \\
=& \gamma^M_{\widehat{R}^{\mathfrak{a}}}(j)(r).
\end{aligned}$$

We justify the seventh equality as follows. By construction, we have $r - r_0 \in \mathfrak{a}^l \widehat{R}^{\mathfrak{a}}$. Then $r - r_0 = \sum_{i=1}^m x_i t_i$ where $x_i \in \mathfrak{a}^l$ and $t_i \in \widehat{R}^{\mathfrak{a}}$. So, we have

$$j \otimes r - j \otimes r_0 = j \otimes (r - r_0) = j \otimes \left(\sum_{i=1}^m x_i t_i \right) = \sum_{i=1}^m ((x_i j) \otimes t_i) = 0.$$

It follows that $j \otimes r = j \otimes r_0$. So, the diagram commutes and $\gamma^M_{\widehat{R}^{\mathfrak{a}}}$ is an isomorphism.

(2) By assumption, we have $\mathbf{R}\mathrm{Hom}_R(M,M) \simeq \widehat{R}^{\mathfrak{a}}$. This is homologically bounded. It remains to check that the following diagram commutes.

$$\begin{tikzcd}
M \otimes^{\mathbf{L}}_R \widehat{R}^{\mathfrak{a}} \arrow[r, "M\otimes^{\mathbf{L}}_R \chi^{\widehat{R}^{\mathfrak{a}}}_M", "\simeq"'] \arrow[dr, "\simeq"'] & M \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(M,M) \arrow[d, "\delta^M_M"] \\
& M
\end{tikzcd}$$

We check the commutativity of the following by showing $\delta^J_J \circ (J \otimes_R \chi^{\widehat{R}^{\mathfrak{a}}}_J) \circ \alpha = \mathrm{id}_J$.

$$\begin{tikzcd}
J \otimes_R \widehat{R}^{\mathfrak{a}} \arrow[r, "J\otimes_R \chi^{\widehat{R}^{\mathfrak{a}}}_J", "\simeq"'] \arrow[dr, "\simeq", "\alpha^{-1}"'] & J \otimes_R \mathrm{Hom}_R(J,J) \arrow[d, "\delta^J_J"] \\
& J
\end{tikzcd}$$

Let $j \in J$. We compute

$$\begin{aligned}
(\delta^J_J \circ (J \otimes_R \chi^{\widehat{R}^{\mathfrak{a}}}_J))(\alpha(j)) =& (\delta^J_J \circ (J \otimes_R \chi^{\widehat{R}^{\mathfrak{a}}}_J))(j \otimes 1_{\widehat{R}^{\mathfrak{a}}}) \\
=& \delta^J_J(j \otimes \chi^{\widehat{R}^{\mathfrak{a}}}_J(1_{\widehat{R}^{\mathfrak{a}}})) \\
=& \chi^{\widehat{R}^{\mathfrak{a}}}_J(1_{\widehat{R}^{\mathfrak{a}}})(j) \\
=& j.
\end{aligned}$$

Therefore, the diagram commutes and δ^M_M is an isomorphism. $\square$

Lemma 3.29. *Let M be an $\mathfrak{a}$-codualizing R-complex, and let $0 \not\simeq A \in \mathcal{D}(R)$.*

(a) *If $V(\mathfrak{a}) \cap \mathrm{supp}_R(A) \neq \emptyset$, then $M \otimes^{\mathbf{L}}_R A \not\simeq 0$.*

(b) *If A is homologically bounded and $V(\mathfrak{a}) \cap \mathrm{supp}_R(A) \neq \emptyset$, then $\mathbf{R}\mathrm{Hom}_R(M,A) \not\simeq 0$.*

(c) *If A is homologically bounded above and $V(\mathfrak{a}) \cap \mathrm{Ass}_R(\mathrm{H}_s(A)) \neq \emptyset$ where $s = \sup(A)$, then we have $\mathbf{R}\mathrm{Hom}_R(M,A) \not\simeq 0$.*

Proof. Let $K = K(\mathfrak{a})$ be the Koszul complex on a generating sequence for $\mathfrak{a}$. Then $K \otimes_R M$ is homologically finite over R.

Claim 1: To show that $\mathbf{R}\mathrm{Hom}_R(M, A) \not\simeq 0$, it suffices to show that there is a prime $\mathfrak{p} \in \mathrm{Spec}(R)$ such that

$$(K \otimes^{\mathbf{L}}_R M)_{\mathfrak{p}} \otimes^{\mathbf{L}}_{R_{\mathfrak{p}}} \kappa(\mathfrak{p}) \not\simeq 0 \not\simeq \mathbf{R}\mathrm{Hom}_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}), A_{\mathfrak{p}}). \tag{3}$$

Indeed, to show that $\mathbf{R}\mathrm{Hom}_R(M, A) \not\simeq 0$, it suffices to show that there is a prime $\mathfrak{p} \in \mathrm{Spec}(R)$ such that $(K \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(M, A))_{\mathfrak{p}} \not\simeq 0$. Given any $\mathfrak{p} \in \mathrm{Spec}(R)$, we have the following isomorphisms since $K \otimes^{\mathbf{L}}_R M$ is homologically finite over R:

$$\begin{aligned}(K \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(M, A))_{\mathfrak{p}} &\sim \mathbf{R}\mathrm{Hom}_R(K \otimes^{\mathbf{L}}_R M, A)_{\mathfrak{p}} \\ &\simeq \mathbf{R}\mathrm{Hom}_{R_{\mathfrak{p}}}((K \otimes^{\mathbf{L}}_R M)_{\mathfrak{p}}, A_{\mathfrak{p}}).\end{aligned}$$

Thus, it suffices to show there exists a prime $\mathfrak{p}$ such that $\mathbf{R}\mathrm{Hom}_{R_{\mathfrak{p}}}((K \otimes^{\mathbf{L}}_R M)_{\mathfrak{p}}, A_{\mathfrak{p}}) \not\simeq 0$. Given the isomorphisms

$$\begin{aligned}&\mathbf{R}\mathrm{Hom}_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p}), \mathbf{R}\mathrm{Hom}_{R_{\mathfrak{p}}}((K \otimes^{\mathbf{L}}_R M)_{\mathfrak{p}}, A_{\mathfrak{p}})) \\ &\qquad \simeq \mathbf{R}\mathrm{Hom}_{R_{\mathfrak{p}}}((K \otimes^{\mathbf{L}}_R M)_{\mathfrak{p}} \otimes^{\mathbf{L}}_{R_{\mathfrak{p}}} (\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p})), A_{\mathfrak{p}}) \\ &\qquad \simeq \mathbf{R}\mathrm{Hom}_{R_{\mathfrak{p}}}([(K \otimes^{\mathbf{L}}_R M)_{\mathfrak{p}} \otimes^{\mathbf{L}}_{R_{\mathfrak{p}}} \kappa(\mathfrak{p})] \otimes^{\mathbf{L}}_{\kappa(\mathfrak{p})} \kappa(\mathfrak{p}), A_{\mathfrak{p}}) \\ &\qquad \simeq \mathbf{R}\mathrm{Hom}_{\kappa(\mathfrak{p})}((K \otimes^{\mathbf{L}}_R M)_{\mathfrak{p}} \otimes^{\mathbf{L}}_{R_{\mathfrak{p}}} \kappa(\mathfrak{p}), \mathbf{R}\mathrm{Hom}_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}), A_{\mathfrak{p}}))\end{aligned}$$

it suffices to show that there is a prime $\mathfrak{p}$ such that

$$\mathbf{R}\mathrm{Hom}_{\kappa(\mathfrak{p})}((K \otimes^{\mathbf{L}}_R M)_{\mathfrak{p}} \otimes^{\mathbf{L}}_{R_{\mathfrak{p}}} \kappa(\mathfrak{p}), \mathbf{R}\mathrm{Hom}_{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}), A_{\mathfrak{p}})) \not\simeq 0.$$

In view of the Künneth formula, it suffices to find a prime $\mathfrak{p}$ satisfying (3).

Claim 2: To show that $M \otimes^{\mathbf{L}}_R A \not\simeq 0$, it suffices to show that there is a prime $\mathfrak{p} \in \operatorname{Spec}(R)$ such that

$$\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_\mathfrak{p}} (K \otimes^{\mathbf{L}}_R M)_\mathfrak{p} \not\simeq 0 \not\simeq \kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_\mathfrak{p}} A_\mathfrak{p}. \tag{4}$$

Indeed, to show that $M \otimes^{\mathbf{L}}_R A \not\simeq 0$, it suffices to find a prime $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $(K \otimes^{\mathbf{L}}_R (M \otimes^{\mathbf{L}}_R A))_\mathfrak{p} \not\simeq 0$. Given any $\mathfrak{p} \in \operatorname{Spec}(R)$, we have the next isomorphisms:

$$(K \otimes^{\mathbf{L}}_R (M \otimes^{\mathbf{L}}_R A))_\mathfrak{p} \simeq (K \otimes^{\mathbf{L}}_R M)_\mathfrak{p} \otimes^{\mathbf{L}}_{R_\mathfrak{p}} A_\mathfrak{p}$$

Thus, it suffices to show there exists a prime $\mathfrak{p}$ such that $(K \otimes^{\mathbf{L}}_R M)_\mathfrak{p} \otimes^{\mathbf{L}}_{R_\mathfrak{p}} A_\mathfrak{p} \not\simeq 0$. Consider

$$\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_\mathfrak{p}} [(K \otimes^{\mathbf{L}}_R M)_\mathfrak{p} \otimes^{\mathbf{L}}_{R_\mathfrak{p}} A_\mathfrak{p}] \simeq [\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_\mathfrak{p}} (K \otimes^{\mathbf{L}}_R M)_\mathfrak{p}] \otimes^{\mathbf{L}}_{\kappa(\mathfrak{p})} [\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_\mathfrak{p}} A_\mathfrak{p}].$$

Given these isomorphisms and the Künneth formula, it suffices to show that there is a prime $\mathfrak{p}$ satisfying (4). This completes the proof of Claim 2.

Claim 3: For all $\mathfrak{p} \in V(\mathfrak{a})$, we have $\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_\mathfrak{p}} (K \otimes^{\mathbf{L}}_R M)_\mathfrak{p} \not\simeq 0$. Indeed, since $K \otimes^{\mathbf{L}}_R M$ is homologically finite over R, it follows that $(K \otimes^{\mathbf{L}}_R M)_\mathfrak{p}$ is homologically finite over $R_\mathfrak{p}$. So, it suffices to show that $(K \otimes^{\mathbf{L}}_R M)_\mathfrak{p} \not\simeq 0$ by the statement following [10, (1.3.4)]. For this, it suffices to show $\mathbf{R}\mathrm{Hom}_{R_\mathfrak{p}}((K \otimes^{\mathbf{L}}_R M)_\mathfrak{p}, M_\mathfrak{p}) \not\simeq 0$: we use the next sequence

$$\begin{aligned}
\mathbf{R}\mathrm{Hom}_{R_\mathfrak{p}}((K \otimes^{\mathbf{L}}_R M)_\mathfrak{p}, M_\mathfrak{p}) &\simeq \mathbf{R}\mathrm{Hom}_R(K \otimes^{\mathbf{L}}_R M, M)_\mathfrak{p} \\
&\sim (K \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(M, M))_\mathfrak{p} \\
&\simeq (K \otimes^{\mathbf{L}}_R \widehat{R}^{\mathfrak{a}})_\mathfrak{p} \\
&\simeq K_\mathfrak{p} \\
&\not\simeq 0.
\end{aligned}$$

The last step is from the condition $\mathfrak{p} \in V(\mathfrak{a})$. This completes the proof of Claim 3.

(a) Let $\mathfrak{p} \in V(\mathfrak{a}) \cap \operatorname{supp}_R(A)$. Claim 3 implies that $\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_\mathfrak{p}} (K \otimes^{\mathbf{L}}_R M)_\mathfrak{p} \not\simeq 0$, and the condition $\mathfrak{p} \in \operatorname{supp}_R(A)$ implies that $\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_\mathfrak{p}} A_\mathfrak{p} \not\simeq 0$. Thus, Claim 2 implies that $M \otimes^{\mathbf{L}}_R A \not\simeq 0$.

(b) Assume that A is homologically bounded, and let $\mathfrak{p} \in V(\mathfrak{a}) \cap \operatorname{supp}_R(A)$. Claim 3 implies that $\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_\mathfrak{p}} (K \otimes^{\mathbf{L}}_R M)_\mathfrak{p} \not\simeq 0$. Since A is homologically bounded, the condition $\mathfrak{p} \in \operatorname{supp}_R(A)$ implies that $\mathbf{R}\mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), A_\mathfrak{p}) \not\simeq 0$. Thus, Claim 1 implies that $\mathbf{R}\mathrm{Hom}_R(M, A) \not\simeq 0$.

(c) Assume that A is homologically bounded above, and $\mathfrak{p} \in V(\mathfrak{a}) \cap \operatorname{Ass}_R(\mathrm{H}_s(A))$. In the next display, the first step follows from the fact that $A \not\simeq 0$ is homologically bounded above:

$$\pm\infty \neq \sup(A) = -\operatorname{depth}_{R_\mathfrak{p}}(A_\mathfrak{p}) = \sup(\mathbf{R}\mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), A_\mathfrak{p})).$$

The second step is from [10, (1.6.6)], and the third one is by definition. It follows that $\mathbf{R}\mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), A_\mathfrak{p}) \not\simeq 0$. As in the part (b), Claim 1 implies that $\mathbf{R}\mathrm{Hom}_R(M, A) \not\simeq 0$. $\square$

Lemma 3.30. *Let X be a homologically bounded below R-complex. Let $K(\mathfrak{a})$ denote the Koszul complex on generating sequence of $\mathfrak{a}$. If $\operatorname{supp}(X) \subseteq \mathrm{V}(\mathfrak{a})$ and $X \not\simeq 0$, then $K(\mathfrak{a}) \otimes^{\mathbf{L}}_R X \not\simeq 0$.*

Proof. Set $K = K(\mathfrak{a})$. Fact 2.51 tells us $\operatorname{supp}_R(X \otimes^{\mathbf{L}}_R K) = \operatorname{supp}_R(X) \cap \operatorname{supp}_R(K) = \operatorname{supp}_R(X) \cap \mathrm{V}(\mathfrak{a})$. By assumption, $\operatorname{supp}(X) \subseteq \mathrm{V}(\mathfrak{a})$. Therefore, $\operatorname{supp}_R(X \otimes^{\mathbf{L}}_R K) = \operatorname{supp}_R(X)$. It follows that $X \simeq 0$ if and only if $K \otimes^{\mathbf{L}}_R X \simeq 0$. $\square$

Lemma 3.31. *Let M be an $\mathfrak{a}$-codualizing R-complex. Let X and Y be homologically bounded R-complexes such that $\operatorname{supp}_R(X), \operatorname{supp}_R(Y) \subseteq \mathrm{V}(\mathfrak{a})$. Let $\alpha : X \to Y$ be a chain map.*

(a) One has $\operatorname{supp}_R(\operatorname{Cone}(\alpha)) \subseteq \mathrm{V}(\mathfrak{a})$.

(b) The following conditions are equivalent.

(i) α *is an isomorphism in* $\mathcal{D}(R)$*,*

(ii) $M \otimes^{\mathbf{L}}_R \alpha$ *is an isomorphism in* $\mathcal{D}(R)$*,*

(iii) $\mathbf{R}\mathrm{Hom}_R(M, \alpha)$ *is an isomorphism in* $\mathcal{D}(R)$*, and*

(iv) $K(\mathfrak{a}) \otimes^{\mathbf{L}}_R \alpha$ *is an isomorphism in* $\mathcal{D}(R)$*.*

Proof. (a) Consider the following exact sequence

$$0 \longrightarrow Y \longrightarrow \mathrm{Cone}(\alpha) \longrightarrow \Sigma X \longrightarrow 0$$

Therefore, the following diagram is a distinguished triangle.

$$Y \longrightarrow \mathrm{Cone}(\alpha) \longrightarrow \Sigma X \longrightarrow$$

Then for all $\mathfrak{p} \in \mathrm{Spec}(R)$, the following is a distinguished triangle

$$Y \otimes^{\mathbf{L}}_R \kappa(\mathfrak{p}) \longrightarrow \mathrm{Cone}(\alpha) \otimes^{\mathbf{L}}_R \kappa(\mathfrak{p}) \longrightarrow \Sigma X \otimes^{\mathbf{L}}_R \kappa(\mathfrak{p}) \longrightarrow$$

Let $\mathfrak{p} \notin \mathrm{V}(\mathfrak{a})$. Then $\mathfrak{p} \notin \mathrm{supp}_R(X)$ and $\mathfrak{p} \notin \mathrm{supp}_R(Y)$. Therefore, $Y \otimes^{\mathbf{L}}_R \kappa(\mathfrak{p}) \simeq 0 \simeq \Sigma X \otimes^{\mathbf{L}}_R \kappa(\mathfrak{p})$. The distinguished triangle above implies $\mathrm{Cone}(\alpha) \otimes^{\mathbf{L}}_R \kappa(\mathfrak{p}) \simeq 0$. It follows that $\mathfrak{p} \notin \mathrm{supp}_R(\mathrm{Cone}(\alpha))$.

(b) The implications $(i) \implies (ii)$ and $(i) \implies (iii)$ and $(i) \implies (iv)$ are standard. Consider the next distinguished triangle in each of the other implications.

$$X \longrightarrow Y \longrightarrow \mathrm{Cone}(\alpha) \longrightarrow \tag{5}$$

$(ii) \implies (i)$: Suppose $M \otimes^{\mathbf{L}}_R \alpha$ is an isomorphism. Then the distinguished triangle

(5) yields a second distinguished triangle

$$M \otimes^{\mathbf{L}}_R X \xrightarrow[\simeq]{M\otimes^{\mathbf{L}}_R \alpha} M \otimes^{\mathbf{L}}_R Y \longrightarrow M \otimes^{\mathbf{L}}_R \operatorname{Cone}(\alpha) \longrightarrow .$$

It follows that $M \otimes^{\mathbf{L}}_R \operatorname{Cone}(\alpha) \simeq 0$. By part (a), we have $\operatorname{supp}_R(\operatorname{Cone}(\alpha)) \subseteq \operatorname{V}(\mathfrak{a})$. By contradiction assume $\operatorname{Cone}(\alpha) \not\simeq 0$. Then there exists $\mathfrak{p} \in \operatorname{supp}_R(\operatorname{Cone}(\alpha)) \bigcap \operatorname{V}(\mathfrak{a})$. By Lemma 3.29(a), we conclude $M \otimes^{\mathbf{L}}_R \operatorname{Cone}(\alpha) \not\simeq 0$. This is a contradiction. Hence, we have $\operatorname{Cone}(\alpha) \simeq 0$, so α is an isomorphism in $\mathcal{D}(R)$.

$(iii) \implies (i)$: Suppose $\mathbf{R}\operatorname{Hom}_R(M, \alpha)$ is an isomorphism. Then the distinguished triangle (5) yields a second distinguished triangle

$$\mathbf{R}\operatorname{Hom}_R(M, X) \xrightarrow[\simeq]{\mathbf{R}\operatorname{Hom}_R(M,\alpha)} \mathbf{R}\operatorname{Hom}_R(M, Y) \longrightarrow \mathbf{R}\operatorname{Hom}_R(M, \operatorname{Cone}(\alpha)) \longrightarrow .$$

It follows that $\mathbf{R}\operatorname{Hom}_R(M, \operatorname{Cone}(\alpha)) \simeq 0$. By part (a), we have $\operatorname{supp}_R(\operatorname{Cone}(\alpha)) \subseteq \operatorname{V}(\mathfrak{a})$. By contradiction assume $\operatorname{Cone}(\alpha) \not\simeq 0$. Then there exists $\mathfrak{p} \in \operatorname{supp}_R(\operatorname{Cone}(\alpha)) \subseteq \operatorname{V}(\mathfrak{a})$. Therefore, there exists $\mathfrak{p} \in \operatorname{supp}_R(\operatorname{Cone}(\alpha)) \bigcap \operatorname{V}(\mathfrak{a})$. By Lemma 3.29(b), we conclude $\mathbf{R}\operatorname{Hom}_R(M, \operatorname{Cone}(\alpha)) \not\simeq 0$. This is a contradiction. Hence, we have $\operatorname{Cone}(\alpha) \simeq 0$, so α is an isomorphism in $\mathcal{D}(R)$.

$(iv) \implies (i)$: Suppose $K(\mathfrak{a}) \otimes^{\mathbf{L}}_R \alpha$ is an isomorphism. Then the distinguished triangle (5) yields a second distinguished triangle

$$K(\mathfrak{a}) \otimes^{\mathbf{L}}_R X \xrightarrow[\simeq]{K(\mathfrak{a})\otimes^{\mathbf{L}}_R \alpha} K(\mathfrak{a}) \otimes^{\mathbf{L}}_R Y \longrightarrow K(\mathfrak{a}) \otimes^{\mathbf{L}}_R \operatorname{Cone}(\alpha) \longrightarrow .$$

It follows that $K(\mathfrak{a}) \otimes^{\mathbf{L}}_R \operatorname{Cone}(\alpha) \simeq 0$. By part (a), we have $\operatorname{supp}_R(\operatorname{Cone}(\alpha)) \subseteq \operatorname{V}(\mathfrak{a})$. By contradiction assume $\operatorname{Cone}(\alpha) \not\simeq 0$. Then there exists $\mathfrak{p} \in \operatorname{supp}_R(\operatorname{Cone}(\alpha)) \subseteq \operatorname{V}(\mathfrak{a})$. Therefore, there exists $\mathfrak{p} \in \operatorname{supp}_R(\operatorname{Cone}(\alpha)) \bigcap \operatorname{V}(\mathfrak{a})$. By Lemma 3.30, we have $K(\mathfrak{a}) \otimes^{\mathbf{L}}_R \operatorname{Cone}(\alpha) \not\simeq 0$. This is a contradiction. Thus, we have $\operatorname{Cone}(\alpha) \simeq 0$. $\square$

For our version of Foxby equivalence, we need a variant of Lemma 3.29; see Proposition 3.37. For it, we require the ring to have finite Krull dimension.

Fact 3.32. [32, Corollary 3.2.7] If F is a flat R-module, then $\mathrm{pd}_R(F) \leqslant \dim(R)$.

Remark 3.33. If X is a homologically bounded R-complex and $\mathfrak{p} \in \mathrm{Spec}(R)$, then we have the following:

1. For all $Z \in \mathcal{D}(R)$ we have $\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_\mathfrak{p}} Z \simeq \kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{R} R_\mathfrak{p} \otimes^{\mathbf{L}}_{R_\mathfrak{p}} Z \simeq \kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{R} Z$.

2. If we further assume that $\dim(R) < \infty$, then $\mathrm{pd}_R(R_\mathfrak{p}) < \infty$ by Fact 3.32. Therefore, $\mathbf{R}\mathrm{Hom}_R(R_\mathfrak{p}, X)$ is a homologically bounded R-complex.

3. $\mathrm{Hom}_R(U, \coprod_\lambda V_\lambda)$ injects into $\mathrm{Hom}_R(U, \prod_\lambda V_\lambda) \cong \prod_\lambda \mathrm{Hom}_R(U, V_\lambda)$. So, if we have $\mathrm{Hom}_R(U, V_\lambda) = 0$ for all λ, then $\mathrm{Hom}_R(U, \coprod_\lambda V_\lambda) = 0$.

Lemma 3.34. *Let R be ring such that $\dim(R) < \infty$. If X is a homologically bounded R-complex, then $\mathfrak{p} \in \text{co-supp}_R(X)$ if and only if $\mathbf{R}\mathrm{Hom}_R(\kappa(\mathfrak{p}), X) \not\simeq 0$.*

Proof. By definition, $\mathfrak{p} \in \text{co-supp}_R(X)$ if and only if $\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_\mathfrak{p}} \mathbf{R}\mathrm{Hom}_R(R_\mathfrak{p}, X) \not\simeq 0$. By Remark 3.33.2 the R-complex $\mathbf{R}\mathrm{Hom}_R(R_\mathfrak{p}, X)$ is homologically bounded. Then by [16, Proposition 11.4], one has $\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_\mathfrak{p}} \mathbf{R}\mathrm{Hom}_R(R_\mathfrak{p}, X) \not\simeq 0$ if and only if

$$\mathbf{R}\mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), \mathbf{R}\mathrm{Hom}_R(R_\mathfrak{p}, X)) \not\simeq 0.$$

By adjointeness we have an isomorphism

$$\mathbf{R}\mathrm{Hom}_{R_\mathfrak{p}}(\kappa(\mathfrak{p}), \mathbf{R}\mathrm{Hom}_R(R_\mathfrak{p}, X)) \simeq \mathbf{R}\mathrm{Hom}_R(\kappa(\mathfrak{p}), X).$$

So, we have $\mathfrak{p} \in \text{co-supp}_R(X)$ if and only if $\mathbf{R}\mathrm{Hom}_R(\kappa(\mathfrak{p}), X) \not\simeq 0$. □

Lemma 3.35. *Let R be ring such that $\dim(R) < \infty$. If F is a non-zero flat module, then $\operatorname{co-supp}_R(F) \neq \emptyset$ and there exists a prime ideal $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $\sup(\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_\mathfrak{p}} \mathbf{R}\operatorname{Hom}_R(R_\mathfrak{p}, X)) = 0$.*

Proof. Let $0 \to F \to T^0 \to T^1 \to \cdots \to T^d \to 0$ be a minimal pure injective resolution of F. That is, each T^i is flat and cotorsion, the sequence is exact, each kernel is flat, and $d \leqslant \dim(R)$; cf. [15, Section 2]. For all i and $\mathfrak{p} \in \operatorname{Spec}(R)$ there exists $X^i_\mathfrak{p}$ such that $T^i \cong \prod_\mathfrak{p} \widehat{R^{(X^i_\mathfrak{p})}}^{\mathfrak{p}}$ by [14, Section 2]. The $X^i_\mathfrak{p}$ are uniquely determined by F, i, and $\mathfrak{p}$. The $X^i_\mathfrak{p}$ are the invariants $\pi_i(\mathfrak{p}, F)$ of [15].

By [14, Theorem 2.2], we have $\operatorname{co-supp}_R(T^i) = \{\mathfrak{p} \in \operatorname{Spec}(R) : X^i_\mathfrak{p} \neq 0\}$ for all i. In particular, $T^i \neq 0$ if and only if $\operatorname{co-supp}_R(T^i) \neq 0$. Since $F \neq 0$, we have $T^0 \neq 0$.

Now, let $\mathfrak{p}$ be maximal in $\operatorname{co-supp}_R(T^0)$ with respect to containment. In particular, $\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_\mathfrak{p}} \mathbf{R}\operatorname{Hom}_R(R_\mathfrak{p}, T^0) \neq 0$. From [15, Theorem 2.1] we have $\mathfrak{p} \notin \operatorname{co-supp}_R(T^i)$ for all $i \geqslant 1$. Moreover, for all $\mathfrak{q} \supseteq \mathfrak{p}$, we have $\mathfrak{q} \notin \operatorname{co-supp}_R(T^i)$ for all $i \geqslant 0$.

As each T^i is cotorsion and $R_\mathfrak{p}$ is flat, we have $\mathbf{R}\operatorname{Hom}_R(R_\mathfrak{p}, F) \simeq \operatorname{Hom}_R(R_\mathfrak{p}, T)$. Moreover, from the proof of [14, Theorem 2.7] each module $\operatorname{Hom}_R(R_\mathfrak{p}, T^i)$ is flat and cotorsion over $R_\mathfrak{p}$.

To be clear, the R-module $\operatorname{Hom}_R(R_\mathfrak{p}, T^i)$ is cotorsion because, for all flat $R_\mathfrak{p}$-modules and all $j \geqslant 1$, we have

$$\begin{aligned}
\operatorname{Ext}^j_{R_\mathfrak{p}}(L, \operatorname{Hom}_R(R_\mathfrak{p}, T^i)) &\cong \operatorname{Ext}^j_R(R_\mathfrak{p} \otimes_{R_\mathfrak{p}} L, T^i) \\
&\cong \operatorname{Ext}^j_R(L, T^i) \\
&= 0.
\end{aligned}$$

The first isomorphism is adjointness, the second is tensor cancellation, and the third is because T^i is cotorsion over R. Also, the $R_\mathfrak{p}$-module $\operatorname{Hom}_R(R_\mathfrak{p}, T^i)$ is flat because if M and N are flat R-modules, then $\operatorname{Hom}_R(M, N)$ is flat over R. This implies that $\operatorname{Hom}_R(R_\mathfrak{p}, T^i)$ is flat over $R_{\mathfrak{p}}$. This follows from the fact that $\operatorname{Hom}_R(R_{\mathfrak{p}}, T^i) \otimes_{R_{\mathfrak{p}}} - \cong \operatorname{Hom}_R(R_{\mathfrak{p}}, T^i) \otimes_R -$ as functors of $R_{\mathfrak{p}}$-modules.

Therefore, $\operatorname{Hom}_R(R_{\mathfrak{p}}, T)$ is a bounded flat resolution of $\mathbf{R}\operatorname{Hom}_R(R_{\mathfrak{p}}, F)$ over $R_{\mathfrak{p}}$. Thus, it is also a resolution over R. This allows us to conclude that

$$\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_{\mathfrak{p}}} \mathbf{R}\operatorname{Hom}_R(R_{\mathfrak{p}}, F) \simeq \kappa(\mathfrak{p}) \otimes_{R_{\mathfrak{p}}} \operatorname{Hom}_R(R_{\mathfrak{p}}, T).$$

Now, since $\mathfrak{p} \notin \operatorname{co-supp}_R(T^i)$ for all $i \geqslant 1$, we have

$$\kappa(\mathfrak{p}) \otimes_{R_{\mathfrak{p}}} \operatorname{Hom}_R(R_{\mathfrak{p}}, T) = \quad 0 \to \kappa(\mathfrak{p}) \otimes_{R_{\mathfrak{p}}} \operatorname{Hom}_R(R_{\mathfrak{p}}, T^0) \to 0 \to \cdots .$$

Since $\mathfrak{p} \in \operatorname{co-supp}_R(T^0)$, we have $\kappa(\mathfrak{p}) \otimes_{R_{\mathfrak{p}}} \operatorname{Hom}_R(R_{\mathfrak{p}}, T^0) \not\simeq 0$. Hence,

$$\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_{\mathfrak{p}}} \mathbf{R}\operatorname{Hom}_R(R_{\mathfrak{p}}, F) \simeq \kappa(\mathfrak{p}) \otimes_{R_{\mathfrak{p}}} \operatorname{Hom}_R(R_{\mathfrak{p}}, T) \simeq \kappa(\mathfrak{p}) \otimes_{R_{\mathfrak{p}}} \operatorname{Hom}_R(R_{\mathfrak{p}}, T^0) \not\simeq 0.$$

This implies that the supremum of $\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_{\mathfrak{p}}} \mathbf{R}\operatorname{Hom}_R(R_{\mathfrak{p}}, X)$ is 0. $\square$

Lemma 3.36. *Let R be ring such that $\dim(R) < \infty$. If $0 \not\simeq X$ is a homologically bounded R-complex, then $\operatorname{co-supp}_R(X) \neq \emptyset$.*

Proof. We proceed in cases.

Case 1: $\operatorname{fd}(X) = \infty$. Let $F \simeq X$ be a flat resolution of X, and shift X if necessary to assume $\sup(X) = 0$. We consider the following soft truncation of F. Define F' as follows

$$F' = \quad 0 \longrightarrow F_0/\operatorname{Im}(\partial_1^F) \xrightarrow{\epsilon} F_{-1} \xrightarrow{\partial_{-1}^F} \cdots \longrightarrow F_j \longrightarrow 0.$$

Note that $X \simeq F \simeq F'$. Let $\tau : G_0 \to F_0/\operatorname{Im}(\partial_1^F)$ be a flat cover. Then $\operatorname{Ker}(\tau) = C$ is cotorsion by [13, Lemma 2.2]. Let $C \xrightarrow{\iota} G_0$ be the inclusion map. Let

$$\tilde{G}^+ = \quad \cdots \longrightarrow G_2 \xrightarrow{\partial_1^{\tilde{G}}} G_1 \xrightarrow{\pi} C \longrightarrow 0$$

be a minimal flat resolution. By [13, Lemma 2.2] each G_i is cotorsion. We define the complex L as

$$L = \quad \cdots \longrightarrow G_2 \xrightarrow{\partial_1^{\tilde{G}}} G_1 \xrightarrow{\iota\circ\pi} G_0 \xrightarrow{\epsilon\circ\tau} F_{-1} \xrightarrow{\partial_{-1}^{F}} F_{-2} \longrightarrow \cdots .$$

Note that $L \simeq X$.

For each i, we have $G_i \cong \prod_{\mathfrak{p}} T_{\mathfrak{p},i}$ where $T_{\mathfrak{p},i}$ is a completion of a free $R_{\mathfrak{p}}$-module. Furthermore, all $\mathfrak{p}$ that occur have the property that $\kappa(\mathfrak{p}) \otimes_R^{\mathbf{L}} \operatorname{Hom}_R(R_{\mathfrak{p}}, \partial_i^{\tilde{G}}) = 0$ for all $i > 1$ by [14, Theorem 2.2]. Since C is cotorsion, the exact sequence

$$\cdots \longrightarrow \operatorname{Hom}_R(R_{\mathfrak{p}}, G_2) \longrightarrow \operatorname{Hom}_R(R_{\mathfrak{p}}, G_1) \longrightarrow \operatorname{Hom}_R(R_{\mathfrak{p}}, C) \longrightarrow 0$$

is an augmented flat resolution of $\operatorname{Hom}_R(R_{\mathfrak{p}}, C)$. Then for all $i \geqslant 1$ we have

$$\operatorname{Tor}_i^{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}), \operatorname{Hom}_R(R_{\mathfrak{p}}, C)) = \mathrm{H}_i(\kappa(\mathfrak{p}) \otimes_{R_{\mathfrak{p}}}^{\mathbf{L}} \operatorname{Hom}_R(R_{\mathfrak{p}}, \tilde{G})) = \kappa(\mathfrak{p}) \otimes_{R_{\mathfrak{p}}} \operatorname{Hom}_R(R_{\mathfrak{p}}, G_{i+1}).$$

In particular, if $T_{\mathfrak{p},i+1} \neq 0$, then $\operatorname{Tor}_i^{R_{\mathfrak{p}}}(\kappa(\mathfrak{p}), \operatorname{Hom}_R(R_{\mathfrak{p}}, C)) \neq 0$ by [14, Theorem 2.2].

Since $\operatorname{fd}(X) = \infty$, we also know $\operatorname{fd}(C) = \infty$, otherwise, $\tilde{G}$ would be bounded and $L \simeq X$ would be a bounded flat resolution. With the equalities above, this implies the set $\{i : \text{there exists } \mathfrak{p} \text{ such that } \operatorname{Tor}_i^R(\kappa(\mathfrak{p}), \mathbf{R}\operatorname{Hom}_R(R_{\mathfrak{p}}, C)) \neq 0\}$ is unbounded.

Consider the short exact sequence of complexes

$$0 \longrightarrow \Sigma\tilde{G} \longrightarrow L \longrightarrow L_{\leqslant 0} \longrightarrow 0$$

This a yields a distinguished triangle $\Sigma C \to X \to L_{\leqslant 0} \to$ in $\mathcal{D}(R)$. From Fact 3.32 $\operatorname{pd}_R(R_{\mathfrak{p}}) \leqslant \dim(R) < \infty$. So, there exists a projective resolution $P \simeq R_{\mathfrak{p}}$ such that $P_i = 0$ for all $i > \dim(R)$. Then $\mathbf{R}\operatorname{Hom}_R(R_{\mathfrak{p}}, L_{\leqslant 0}) \simeq \operatorname{Hom}_R(P, L_{\leqslant 0})$, where $\operatorname{Hom}_R(P, L_{\leqslant 0})$ is bounded complex of flat modules. Moreover, $\operatorname{Hom}_R(P, L_{\leqslant 0})_i = 0$ for all $i > 0$.

Therefore, $\kappa(\mathfrak{p}) \otimes_R \operatorname{Hom}_R(P, L_{\leqslant 0})_i = 0$ for all $i > 0$. This implies that $\mathrm{H}_i(\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(R_\mathfrak{p}, L_{\leqslant 0})) \cong \mathrm{H}_i(\kappa(\mathfrak{p}) \otimes_R \operatorname{Hom}_R(P, L_{\leqslant 0})) = 0$ for all $i > 0$. Hence, the supremum of $\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(R_\mathfrak{p}, L_{\leqslant 0})$ is less than or equal to 0 for all $\mathfrak{p}$.

There exists $\mathfrak{p}$ and $i \geqslant 1$ such that $\mathrm{H}_i(\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(R_\mathfrak{p}, \Sigma C)) \neq 0$. Then there exists a distinguished triangle

$$\begin{tikzcd}
\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(R_\mathfrak{p}, \Sigma C) \arrow[d] \\
\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(R_\mathfrak{p}, X) \arrow[d] \\
\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(R_\mathfrak{p}, L_{\leqslant 0}) \arrow[d] \\
{}
\end{tikzcd}$$

We know $\sup(\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(R_\mathfrak{p}, \Sigma C)) \geqslant 1$ and $\sup(\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(R_\mathfrak{p}, L_{\leqslant 0})) \leqslant 0$. The long exact sequence implies that $\sup(\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(R_\mathfrak{p}, X)) \geqslant 1$. In particular, we have that $\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(R_\mathfrak{p}, X) \not\simeq 0$ and hence, $\mathfrak{p} \in \text{co-supp}_R(X)$.

Case 2: $s = \mathrm{fd}_R(X) < \infty$. Let $F \simeq X$ such that F is a bounded complex of flat modules such that $F_i = 0$ for all $i > s$ and for all $i < j = \inf(X)$.

Claim: for all $\mathfrak{p} \in \operatorname{Spec}(R)$, we have $\sup(\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(R_\mathfrak{p}, X)) \leqslant s$. Indeed, $\mathrm{pd}_R(R_\mathfrak{p}) \leqslant \dim(R)$ from Fact 3.32. Therefore, there exists a projective resolution $P \simeq R_\mathfrak{p}$ of $R_\mathfrak{p}$ such that $P_i = 0$ for all $i > \dim(R)$ and $i < 0$. Then by definition, $\mathbf{R}\mathrm{Hom}_R(R_\mathfrak{p}, X) \simeq \operatorname{Hom}_R(P, F)$. By construction, $\operatorname{Hom}_R(P, F) = 0$ for all $i > s$ and $i < \dim(R) + j$, and $\operatorname{Hom}_R(P, F)_i$ is flat for all i. Hence, $\operatorname{Hom}_R(P, F)$ is a flat resolution of $\mathbf{R}\mathrm{Hom}_R(R_\mathfrak{p}, X)$. It follows that

$$\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(R_\mathfrak{p}, X) \simeq \kappa(\mathfrak{p}) \otimes_R \operatorname{Hom}_R(P, F).$$

The module $\kappa(\mathfrak{p})$ is concentrated in degree 0, and $\operatorname{Hom}_R(P, F)$ is concentrated in degrees $j - \dim(R) \leqslant i \leqslant s$. Therefore, the R-complex $\kappa(\mathfrak{p}) \otimes_R \operatorname{Hom}_R(P, F)$ is concentrated in degrees $j - \dim(R) \leqslant i \leqslant s$. Then

$$j - \dim(R) \leqslant \inf(\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(R_\mathfrak{p}, X)) \leqslant \sup(\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(R_\mathfrak{p}, X)) \leqslant s.$$

This proves the claim.

We proceed by induction on $s - j$. The base case follows from Lemma 3.35.

For the induction step, consider the following short exact sequence of complexes

$$0 \longrightarrow \Sigma^s F_s \longrightarrow F \longrightarrow F_{\leqslant s-1} \longrightarrow 0.$$

Since $s = \mathrm{fd}_R(X)$, we have $F_s \neq 0$. Lemma 3.35 implies there exists $\mathfrak{p}$ such that $\sup(\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(R_\mathfrak{p}, \Sigma^s F_s)) = s$. The claim above implies that we have

$$\sup(\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(R_\mathfrak{p}, F_{\leqslant s-1})) \leqslant s - 1.$$

Again there is a distinguished triangle

$$\begin{tikzcd}
\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(R_\mathfrak{p}, \Sigma^s F_s) \arrow[d] \\
\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(R_\mathfrak{p}, F) \arrow[d] \\
\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(R_\mathfrak{p}, F_{\leqslant s-1}) \arrow[d] \\
{}
\end{tikzcd}$$

The long exact sequence in homology implies $\sup(\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(R_\mathfrak{p}, F)) = s$. Therefore, we have $\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(R_\mathfrak{p}, F) \not\simeq 0$, so $\mathfrak{p} \in \text{co-supp}_R(X)$. $\square$

Next we record our analogue of Lemma 3.29(a).

Proposition 3.37. *Let M be an $\mathfrak{a}$-codualizing R-complex, and let $0 \not\simeq X$ be a homologically bounded R-complex. If* $\operatorname{co-supp}_R(X) \subseteq \operatorname{V}(\mathfrak{a})$*, then* $M \otimes^{\mathbf{L}}_R X \not\simeq 0$.

Proof. By Lemma 3.29 it suffices to find an ideal $\mathfrak{q} \in \operatorname{supp}_R(X) \cap \operatorname{V}(\mathfrak{a})$. Lemma 3.36 implies is a prime ideal $\mathfrak{p} \in \operatorname{co-supp}_R(X)$. Then Lemma 3.34 implies that we have $\mathbf{R}\operatorname{Hom}_R(\kappa(\mathfrak{p}), X) \not\simeq 0$.

Let J be a minimal injective resolution of X. Therefore,

$$0 \not\simeq \mathbf{R}\operatorname{Hom}_R(\kappa(\mathfrak{p}), X) \simeq \operatorname{Hom}_R(\kappa(\mathfrak{p}), J).$$

In particular, $\operatorname{Hom}_R(\kappa(\mathfrak{p}), J_i) = \operatorname{Hom}_R(\kappa(\mathfrak{p}), J)_i \neq 0$ for some i. By Lemma 2.64, we know that $\operatorname{supp}_R(X) = \cup_{i \in \mathbb{Z}}\{\mathfrak{q} \in \operatorname{Spec}(R) : E_R(R/\mathfrak{q})$ is a summand of $J_i\}$. Remark 3.33.3 provides a prime $\mathfrak{q} \in \operatorname{supp}_R(X)$ such that $\operatorname{Hom}_R(\kappa(\mathfrak{p}), E_R(R/\mathfrak{q})) \neq 0$. As $\kappa(\mathfrak{p})$ is $\mathfrak{p}$-torsion we have $\operatorname{Hom}_R(\kappa(\mathfrak{p}), \Gamma_{\mathfrak{p}}(E_R(R/\mathfrak{q}))) = \operatorname{Hom}_R(\kappa(\mathfrak{p}), E_R(R/\mathfrak{q})) \neq 0$. Therefore, $\Gamma_{\mathfrak{p}}(E_R(R/\mathfrak{q}) \neq 0$ and so $\mathfrak{p} \subseteq \mathfrak{q}$. The condition $\mathfrak{p} \in \operatorname{co-supp}_R(X) \subseteq \operatorname{V}(\mathfrak{a})$ implies that $\mathfrak{a} \subseteq \mathfrak{p} \subseteq \mathfrak{q}$. Hence, we have $\mathfrak{q} \in \operatorname{supp}_R(X) \cap \operatorname{V}(\mathfrak{a})$. $\square$

Lemma 3.38. *Let M be an $\mathfrak{a}$-codualizing complex, and X and Y are homologically bounded R-complexes such that* $\operatorname{co-supp}_R(X), \operatorname{co-supp}_R(Y) \subseteq \operatorname{V}(\mathfrak{a})$*. Let $\alpha : X \to Y$ be a chain map.*

(a) One has $\operatorname{co-supp}_R(\operatorname{Cone}(\alpha)) \subseteq \operatorname{V}(\mathfrak{a})$.

(b) The following conditions are equivalent:

(i) the morphism α is an isomorphism in $\mathcal{D}(R)$, and

(ii) the morphism $M \otimes^{\mathbf{L}}_R \alpha$ is an isomorphism in $\mathcal{D}(R)$.

Proof. (a) Consider the following distinguished triangle

$$X \xrightarrow{\alpha} Y \longrightarrow \mathrm{Cone}(\alpha) \longrightarrow \ .$$

Let $\mathfrak{p} \in \mathrm{Spec}(R)$. Then applying the functors $\mathbf{R}\mathrm{Hom}_R(R_\mathfrak{p}, -)$ and $\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_\mathfrak{p}} -$ in succession yields the following distinguished triangle.

$$\begin{tikzcd}
\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_\mathfrak{p}} \mathbf{R}\mathrm{Hom}_R(R_\mathfrak{p}, X) \arrow[d] \\
\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_\mathfrak{p}} \mathbf{R}\mathrm{Hom}_R(R_\mathfrak{p}, Y) \arrow[d] \\
\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_\mathfrak{p}} \mathbf{R}\mathrm{Hom}_R(R_\mathfrak{p}, \mathrm{Cone}(\alpha)) \arrow[d] \\

\end{tikzcd}$$

Let $\mathfrak{p} \notin \mathrm{V}(\mathfrak{a})$. Then $\mathfrak{p} \notin \text{co-supp}_R(X)$ and $\mathfrak{p} \notin \text{co-supp}_R(Y)$. So we have

$$\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_\mathfrak{p}} \mathbf{R}\mathrm{Hom}_R(R_\mathfrak{p}, Y) \simeq 0 \simeq \kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_\mathfrak{p}} \mathbf{R}\mathrm{Hom}_R(R_\mathfrak{p}, \Sigma X).$$

The distinguished triangle above implies $\kappa(\mathfrak{p}) \otimes^{\mathbf{L}}_{R_\mathfrak{p}} \mathbf{R}\mathrm{Hom}_R(R_\mathfrak{p}, \mathrm{Cone}(\alpha)) \simeq 0$. It follows that $\mathfrak{p} \notin \text{co-supp}_R(\mathrm{Cone}(\alpha))$.

(b) The implication $(i) \implies (ii)$ is standard.

$(ii) \implies (i)$: Suppose $M \otimes^{\mathbf{L}}_R \alpha$ is an isomorphism. As above, we have the distinguished triangle

$$X \xrightarrow{\alpha} Y \longrightarrow \mathrm{Cone}(\alpha) \longrightarrow .$$

This distinguished triangle yields a second distinguished triangle

$$M \otimes^{\mathbf{L}}_R X \xrightarrow[\simeq]{M \otimes^{\mathbf{L}}_R \alpha} M \otimes^{\mathbf{L}}_R Y \longrightarrow M \otimes^{\mathbf{L}}_R \mathrm{Cone}(\alpha) \longrightarrow .$$

By part (a), we have $\text{co-supp}_R(\text{Cone}(\alpha)) \subseteq \text{V}(\mathfrak{a})$. Assume that $\text{Cone}(\alpha) \not\simeq 0$. By Proposition 3.37, we conclude that $M \otimes^{\mathbf{L}}_R \text{Cone}(\alpha) \not\simeq 0$. This is a contradiction. Hence, we have $\text{Cone}(\alpha) \simeq 0$ □

The following is a corollary to Proposition 2.65.

Corollary 3.39. *Let M be an $\mathfrak{a}$-codualizing complex, and let X be an R-complex. If X is in $\mathcal{A}_M(R)$, then* $\text{co-supp}_R(X) \subseteq \text{V}(\mathfrak{a})$.

Proof. If $X \in \mathcal{A}_M(R)$, then $X \simeq \mathbf{R}\text{Hom}_R(M, M \otimes^{\mathbf{L}}_R X)$, so

$$\text{co-supp}_R(X) = \text{co-supp}_R(\mathbf{R}\text{Hom}_R(M, M \otimes^{\mathbf{L}}_R X)) \subseteq \text{V}(\mathfrak{a})$$

by Proposition 2.54. □

Now we are ready to state and prove our version Foxby equivalence, which is Theorem 1.9 in the introduction.

Theorem 3.40. *If M is $\mathfrak{a}$-codualizing complex, then $M \otimes^{\mathbf{L}}_R - : \mathcal{A}_M(R) \to \mathcal{B}_M(R)$ and $\mathbf{R}\text{Hom}_R(M, -) : \mathcal{B}_M(R) \to \mathcal{A}_M(R)$ are quasi-inverse equivalences. Further, we have the following.*

(a) $Y \in \mathcal{B}_M(R)$ if and only if $\mathbf{R}\text{Hom}_R(M, Y) \in \mathcal{A}_M(R)$ and $\text{supp}_R(Y) \subseteq \text{V}(\mathfrak{a})$.

(b) If $X \in \mathcal{A}_M(R)$, then $M \otimes^{\mathbf{L}}_R X \in \mathcal{B}_M(R)$ and $\text{co-supp}_R(X) \subseteq \text{V}(\mathfrak{a})$.

(c) If $\dim(R) < \infty$, then the converse of part (b) holds.

Proof. We begin by showing the forward implication of parts (a) and (b). The fact that the functors are quasi-inverse equivalences follows immediately from these two implications and the definitions of $\mathcal{A}_M(R)$ and $\mathcal{B}_M(R)$.

Let X and Y be homologically bounded R-complexes. Set $F = M \otimes^{\mathbf{L}}_R X$, and consider the morphisms $\delta^M_F : M \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(M, F) \to F$ and $\gamma^M_X : X \to \mathbf{R}\mathrm{Hom}_R(M, F)$. Then the morphism

$$F = M \otimes^{\mathbf{L}}_R X \xrightarrow{M \otimes^{\mathbf{L}}_R \gamma^M_X} M \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(M, F)$$

satisfies $\delta^M_F(M \otimes^{\mathbf{L}}_R \gamma^M_X) = \mathrm{id}_F$. It follows that $M \otimes^{\mathbf{L}}_R \gamma^M_X$ is an isomorphism if and only if δ^M_F is one.

If we further assume that $X \in \mathcal{A}_M(R)$, then X, F, and $\mathbf{R}\mathrm{Hom}_R(M, F)$ are homologically bounded R-complexes, and γ^M_X is an isomorphism. Corollary 3.39 yields the desired co-support condition. The morphism $M \otimes^{\mathbf{L}}_R \gamma^M_X$ is also an isomorphism since γ^M_X is one. From the above, we know δ^M_F is also an isomorphism, and therefore, we have $F \in \mathcal{B}_M(R)$. So $M \otimes^{\mathbf{L}}_R -$ is a functor from $\mathcal{A}_M(R) \to \mathcal{B}_M(R)$.

Set $G = \mathbf{R}\mathrm{Hom}_R(M, Y)$. Now, consider the morphisms $\delta^M_Y : M \otimes^{\mathbf{L}}_R G \to Y$ and $\gamma^M_G : G \to \mathbf{R}\mathrm{Hom}_R(M, M \otimes^{\mathbf{L}}_R G)$. Then the morphism

$$\mathbf{R}\mathrm{Hom}_R(M, M \otimes^{\mathbf{L}}_R G) \xrightarrow{\mathbf{R}\mathrm{Hom}_R(M, \delta^M_Y)} \mathbf{R}\mathrm{Hom}_R(M, Y) = G$$

satisfies $\mathbf{R}\mathrm{Hom}_R(M, \delta^M_Y)\gamma^M_G = \mathrm{id}_G$. If follows that $\mathbf{R}\mathrm{Hom}_R(M, \delta^M_Y)$ is an isomorphism if and only if γ^M_G is an isomorphism.

Assume that $Y \in \mathcal{B}_M(R)$. It follows from the Bass class isomorphism and Fact 2.51 that we have $\mathrm{supp}_R(Y) \subseteq \mathrm{V}(\mathfrak{a})$. Also, Y, G, and $M \otimes^{\mathbf{L}}_R G$ are homologically bounded R-complexes, and δ^R_Y is an isomorphism. Then $\mathbf{R}\mathrm{Hom}_R(M, \delta^M_Y)$ is also an isomorphism. Thus, γ^M_G is an isomorphism. It follows that $G \in \mathcal{A}_M(R)$, so $\mathbf{R}\mathrm{Hom}_R(M, -)$ is functor from $\mathcal{B}_M(R) \to \mathcal{A}_M(R)$.

We now prove the converse of part (a). Assume $G = \mathbf{R}\mathrm{Hom}_R(M, Y) \in \mathcal{A}_M(R)$ and $\mathrm{supp}_R(Y) \subseteq \mathrm{V}(\mathfrak{a})$. Then G and $M \otimes^{\mathbf{L}}_R G$ are homologically bounded R-complexes, and

γ_G^M is an isomorphism. From above, we know that $\mathbf{R}\mathrm{Hom}_R(M, \delta_Y^R)$ is an isomorphism. We would like to invoke Lemma 3.31 to conclude δ_Y^R is an isomorphism. It would follow that $Y \in \mathcal{B}_M(R)$.

We need to show that the complexes $M \otimes_R^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_R(M, Y)$ and Y satisfy the proper support condition. Corollary 2.52 implies $\mathrm{supp}_R(M \otimes_R^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_R(M, Y)) \subseteq \mathrm{supp}_R(M) \subseteq \mathrm{V}(\mathfrak{a})$. Therefore, Lemma 3.31 applies and δ_Y^R is an isomorphism. This establishes part (a).

We now show part (c). Assume $\dim(R) < \infty$ and $F = M \otimes_R^{\mathbf{L}} X \in \mathcal{B}_M(R)$ and $\text{co-supp}_R(X) \subseteq \mathrm{V}(\mathfrak{a})$. Then F and $\mathbf{R}\mathrm{Hom}_R(M, F)$ are homologically bounded R-complexes, and δ_F^M is an isomorphism. From above, the morphism

$$M \otimes_R^{\mathbf{L}} \gamma_X^R : M \otimes_R^{\mathbf{L}} X \to M \otimes_R^{\mathbf{L}} \mathbf{R}\mathrm{Hom}_R(M, F)$$

is an isomorphism. Proposition 2.54 implies $\mathrm{supp}_R(\mathbf{R}\mathrm{Hom}_R(M, F)) \subseteq \mathrm{V}(\mathfrak{a})$. Then Lemma 3.38 implies that the morphism γ_X^R is an isomorphism. Thus, $X \in \mathcal{A}_M(R)$. $\square$

If M is semidualizing over R, then part of Foxby equivalence [10, Theorem 4.6] states that, given a homologically bounded R-complex, one has $Y \in \mathcal{B}_M(R)$ if and only if $\mathbf{R}\mathrm{Hom}_R(M, Y) \in \mathcal{A}_M(R)$. Note that no mention is made of $\mathrm{supp}_R(Y)$. This is due to the fact that M is 0-codualizing in this case, so the condition $\mathrm{supp}_R(Y) \subseteq \mathrm{V}(0) = \mathrm{Spec}(R)$ is automatic. The following example shows that if M is $\mathfrak{a}$-codualizing and Y is an R-complex such that $\mathrm{supp}_R(Y) \not\subseteq \mathrm{V}(\mathfrak{a})$ and $\mathbf{R}\mathrm{Hom}_R(M, Y) \in \mathcal{A}_M(R)$, then Y need not be in $\mathcal{B}_M(R)$.

Example 3.41. Let k be a field and $R = k[X]_{(X)}$. Note that this is a local ring, and R is not complete with respect to its maximal ideal. Let E be the injective hull of the residue field. We first show $\mathbf{R}\mathrm{Hom}_R(E, R) \in \mathcal{A}_E(R)$. The minimal injective resolution J of R is

$$^{+}J = \quad 0 \longrightarrow R \longrightarrow k(X) \longrightarrow E \longrightarrow 0.$$

An application of $\mathrm{Hom}_R(E,-)$ to J yields the complex

$$\mathbf{R}\mathrm{Hom}_R(E,R) \simeq \mathrm{Hom}_R(E,J) = \quad 0 \longrightarrow \mathrm{Hom}_R(E,k(X)) \longrightarrow \mathrm{Hom}_R(E,E) \longrightarrow 0.$$

It is well-known that $\mathrm{Hom}_R(E,k(X)) = 0$ and $\mathrm{Hom}_R(E,E) \cong \widehat{R}^{\mathfrak{m}}$. It follows that $\mathbf{R}\mathrm{Hom}_R(E,R) \simeq \Sigma^{-1}\widehat{R}^{\mathfrak{m}}$. Proposition 3.28 implies $\Sigma^{-1}\widehat{R}^{\mathfrak{m}} \in \mathcal{A}_E(R)$.

Consider the isomorphisms

$$\Sigma^{-1}E \simeq E \otimes^{\mathbf{L}}_R \Sigma^{-1}\widehat{R}^{\mathfrak{m}} \simeq E \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(E,R).$$

It follows that the morphism $\delta^E_R : E \otimes^{\mathbf{L}}_R \mathbf{R}\mathrm{Hom}_R(E,R) \to R$ is not isomorphism. Hence, we have $R \notin \mathcal{B}_E(R)$. Notice we are not in the scope of Foxby equivalence because the support condition is not satisfied. Specifically, we have $\mathrm{supp}_R(R) \nsubseteq \mathrm{V}(\mathfrak{m}) = \{\mathfrak{m}\}$.

4. FUTURE WORK

There are still several questions related to the $\mathfrak{a}$-codualizing condition to answer.

Question 4.1. If M is $\mathfrak{a}$-codualizing complex, is $\mathbf{R}\Gamma_{\mathfrak{b}}(M)$ an $\mathfrak{b}$-codualizing complex?

We have proved part of this; see Theorem 3.22. It remains to show that the homothety morphism $\chi^{\widehat{R}^{\mathfrak{b}}}_{\mathbf{R}\Gamma_{\mathfrak{b}}(M)}$ is an isomorphism. It does not appear that the proof of Theorem 3.24 is easily adaptable to this question. If this were true, this would imply that there is injection from the set of shift-isomorphism classes of semidualizing complexes into the set of shift-isomorphism classes of $\mathfrak{a}$-codualizing complexes. This makes the next question natural.

Question 4.2. Do all $\mathfrak{a}$-codualizing complexes "come from" semidualizing ones?

Over a local Gorenstein ring, there are unique semidualizing and quasidualizing complexes up to shift-isomorphism. This makes the next question natural.

Question 4.3. If $(R, \mathfrak{m}, k)$ is local Gorenstein ring, is there a unique $\mathfrak{a}$-codualizing complex up to shift-isomorphism?

The next questions arise naturally from our study of Auslander and Bass classes.

Question 4.4. Can we remove the assumption $\dim(R) < \infty$ in Theorem 3.40?

Question 4.5. Let M be an $\mathfrak{a}$-codualizing complex. Is there an embedding (given an appropriate support condition) of complexes of finite flat dimension and finite injective dimension into $\mathcal{A}_M(R)$ and $\mathcal{B}_M(R)$, respectively?

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