VARIATIONAL METHODS FOR POLYCRYSTAL PLASTICITY AND RELATED TOPICS IN PARTIAL DIFFERENTIAL EQUATIONS

A Thesis
Submitted to the Graduate Faculty
of the
North Dakota State University
of Agriculture and Applied Science

By

Farhod B Abdullayev

In Partial Fulfillment of the Requirements
for the Degree of
DOCTOR OF PHILOSOPHY

Major Department:
Mathematics

April 2013

Fargo, North Dakota

North Dakota State University
Graduate School

Title

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The Supervisory Committee certifies that this ***disquisition*** complies with North Dakota State University's regulations and meets the accepted standards for the degree of

DOCTOR OF PHILOSOPHY

SUPERVISORY COMMITTEE:

Azer Akhmedov
Chair

Marian Bocea

Friedrich Littmann

Alan Denton

Approved:

06/25/2013
Date

Dogan Comez
Department Chair

ABSTRACT

In the first part of the thesis the effective yield set of ionic polycrystals is characterized by means of variational principles in L^∞ that are associated to supremal functionals acting on matrix-valued divergence-free fields. The second part of the thesis is concerned with the study of the asymptotic behavior, as $p \to \infty$, of the first and second eigenvalues and the corresponding eigenfunctions for the $p(x)$-Laplacian with Robin and Neumann boundary conditions, respectively, in an open, bounded domain $\Omega \subset \mathbb{R}^N$ with smooth boundary. We obtain uniform bounds for the sequences of eigenvalues (suitably rescaled), and we prove that the positive eigenfunctions converge uniformly in Ω to viscosity solutions of problems involving the ∞-Laplacian subject to appropriate boundary conditions.

ACKNOWLEDGMENTS

I would like to express my gratitude and appreciation to my Ph.D. Advisor, Professor Marian Bocea. He provided constant encouragement and support, excellent advice, and lots of good mathematical ideas throughout my graduate studies. With his enthusiasm, inspiration, and ability to explain things clearly, he helped me achieve my goals. I am also very grateful to Professors Azer Akhmedov, Friedrich Littmann, and Alan Denton for agreeing to be on my thesis committee.

I would like to acknowledge the support received from the U.S. National Science Foundation through my Ph.D. Advisor's Grant No. DMS-1156393. In particular, the research assistantship that I held during the academic year 2011-2012 allowed me to concentrate on finalizing work on various projects related to this thesis.

Finally, I wish to thank my family and friends for providing a loving and supportive environment and for helping me overcome difficulties during this period.

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1. INTRODUCTION

In this thesis we study several problems in Calculus of Variations and Partial Differential Equations that are motivated in part by the analysis of issues arising in Continuum Mechanics and Materials Science.

The definitions and auxiliary results that are needed throughout are collected in Chapter 2 of the thesis.

Chapter 3 of the thesis is based on the paper [1], motivated by recent work in connection with the mathematical derivation of various models related to polycrystal plasticity and the characterization of the effective yield of a polycrystal (see e.g., Kohn-Little [36], Garroni-Nesi-Ponsiglione [29], Goldsztein [31], [32], Garroni-Kohn [30], Bocea-Nesi [5], Bocea-Mihăilescu-Popovici [7], Bocea-Popovici [8]). Polycrystals are collections of grains, or single crystals, which are bonded together in different orientations. The yield of a single crystal is described by a closed convex subset K of $\mathbb{M}^{3\times 3}_{\mathrm{sym}}$, the space of symmetric 3×3 real matrices. Yield in a crystalline solid is associated with a finite number of slip systems, determined by pairs (n_k, m_k) of orthogonal vectors: n_k - the normal to the slip plane, and m_k - the direction of slip. Assuming that there are s slips systems present in the polycrystal, a typical yield set has the form

$$K = \left\{ A \in \mathbb{M}^{3\times 3}_{\mathrm{sym}} \ : \ \langle A, \mu_k \rangle \leq \tau_k, k = 1, \cdots, s \right\},$$

where $\mu_k := \frac{1}{2}\left(m_k \otimes n_k + n_k \otimes m_k\right)$ is the k-th slip tensor, and τ_k is the critical shear stress corresponding to the k-th slip system (n_k, m_k), $k = 1, \cdots, s$. The orientations of the grains in a polycrystal occupying a domain $\Omega \subset \mathbb{R}^3$ are described through a piecewise constant function $R : \Omega \to \mathrm{SO}(3)$, where for each point $x \in \Omega$ the rotation $R(x)$ indicates the orientation of the grain which contains that point. If K is the yield set of the basic crystal, the stress $\sigma : \Omega \to \mathbb{M}^{3\times3}_{\text{sym}}$ in the polycrystal occupying Ω is constrained to satisfy

$$\sigma(x) \in R(x)KR^T(x), \; x \in \Omega. \tag{1}$$

The set of all average stresses $\overline{\sigma} := \frac{1}{|\Omega|}\int_\Omega \sigma(x)dx$, where σ obeys the constraint (1), together with the equilibrium equation

$$\text{Div } \sigma = 0 \text{ in } \Omega, \tag{2}$$

is called the effective yield set of the polycrystal:

$$K_{\text{eff}} := \left\{ \overline{\sigma} := \frac{1}{|\Omega|}\int_\Omega \sigma(x)dx \; : \; (1) \text{ and } (2) \text{ hold} \right\}. \tag{3}$$

The definition of K_{eff} is the usual one in the polycrystal plasticity literature (see, e.g., Section 2 in [36], and references therein). The rigorous justification of the fact that this accurately describes the macroscopic behavior of the polycrystal follows from the homogenization theory (see, e.g., [35]). The solutions of the equilibrium problems at the (microscopic) length scale $\varepsilon > 0$ of the individual grains converge, as $\varepsilon \to 0$, to a solution of the equilibrium problem considered on the larger (macroscopic) scale. In the traditional model of polycrystal plasticity, the latter comes from a degenerate variational principle governed by an effective energy E_{hom}, obtained via homogenization, that is equal to zero at matrices corresponding to stresses which, in addition to solving the equilibrium equation, Div $\sigma = 0$, satisfy the constraint (1) at every point in the domain Ω occupied by the polycrystal, and it is equal to infinity otherwise. Hence, the definition of K_{eff} that is currently used in the literature on the subject (and which we have also adopted in the thesis) coincides with how one would formally need to define the yield set of the polycrystal, namely set it equal to the domain of the effective energy obtained via homogenization:

$$\mathrm{Dom}(E_{\mathrm{hom}}) := \{\sigma \in \mathbb{M}^{3\times 3} : E_{\mathrm{hom}}(\sigma) < \infty\} = \{\sigma \in \mathbb{M}^{3\times 3} : E_{\mathrm{hom}}(\sigma) = 0\}.$$

For details regarding the homogenization procedure from a deformation based (gradient fields) point of view we refer to the paper by Kohn and Little [36]. In the divergence-free case, a different derivation of the model, based on power-law regularization, is given in Bocea-Nesi [5].

The key issue in polycrystal plasticity is to understand the structure of the effective yield set K_{eff}, when the yield set K of the basic crystal is known, and when some information on the shapes and orientations of the grains present in the polycrystal is given. A similar problem arises in the analysis of models of dielectric breakdown and electrical resistivity, where an effective yield (strength) set is defined similarly, with a suitable modification of the pointwise constraint, and with (2) replaced by the requirement that the field $\sigma : \Omega \to \mathbb{R}^3$ be either curl-free or divergence-free, respectively (see [29], [30], and [5]). For example, in Garroni-Kohn [30], the pointwise constraint reads $\sigma(x) \in R(x)K$. The reason for the difference when compared to (1) is that [30] is concerned with two model problems, antiplane shear and plane stress, corresponding to gradient vector fields and divergence-free vector fields, respectively. For example, in the (two-dimensional, for simplicity) antiplane shear model, there are four basic slip systems with slip tensors $\pm\mu^{(1)}, \pm\mu^{(2)}$, where $\mu^{(1)} = \frac{1}{2}(\overrightarrow{\mathbf{i}} \otimes \overrightarrow{\mathbf{k}} + \overrightarrow{\mathbf{k}} \otimes \overrightarrow{\mathbf{i}})$ and $\mu^{(2)} = \frac{1}{2}(\mathbf{j} \otimes \overrightarrow{\mathbf{k}} + \overrightarrow{\mathbf{k}} \otimes \overrightarrow{\mathbf{j}})$, and with critical stresses equal to $\pm M$ and ± 1, respectively. The stress takes the particular form

$$\sigma(x) = \begin{pmatrix} 0 & 0 & \sigma_{13}(x) \\ 0 & 0 & \sigma_{23}(x) \\ \sigma_{13}(x) & \sigma_{23}(x) & 0 \end{pmatrix}, \ x \in \Omega.$$

Thus, σ can be identified with a vector field $\langle \sigma_{13}, \sigma_{23} \rangle$ in the plane. This is a simplification of the polycrystal plasticity setting which we consider in the thesis, where the stresses are divergence-free tensor fields (matrix valued, divergence free on every row). In the particular cases considered by Garroni and Kohn in [30], the pointwise constraint needs to be adapted to the fact that the stress $\sigma(x)$ is assumed to be a three-dimensional vector at any point $x \in \Omega$, so in their work the constraint becomes $\sigma(x) \in R(x)K \subset \mathbb{R}^3$.

When a direct description of the effective yield set is not available, the common approach has been to study the so-called Sachs and Bishop-Hill-Taylor bounds, which are the natural inner and outer bounds for this set (see [30], [31], [32], [36]).

During the last decade the issues described above have been undertaken in the framework of Γ-convergence. The first work in this direction is due to Garroni, Nesi, and Ponsiglione [29], who gave a mathematical derivation of first-failure dielectric breakdown models as limiting cases (via Γ-convergence) of various power-law models, leading to an efficient variational characterization of the effective yield set by means of variational principles associated to the limiting functionals. Bocea and Nesi [5] have considered the coresponding problems in the framework of $\mathcal{A}$-quasiconvexity, allowing for more general linear differential constraints on the underlying fields. In particular, the analysis in [5] leads to variational characterizations of the yield (strength) set in the framework of electrical resistivity, where the underlying fields are divergence-free. More recently these results have been extended in several directions (see, e.g., [6], [7]). First, it turns out that one can consider as a starting point more flexible power-law models where the exponent in the power-law regularization is allowed to depend on the point $x \in \Omega$. Second, the power-law functionals can be adapted to treat situations where the underlying fields take values in stress space $\mathbb{M}^{3\times 3}_{\mathrm{sym}}$, are divergence free, and where several (depending on the number of slip systems present in the basic crystal) distinct pointwise constraints are simultaneously verified. This is the case in some two-dimensional polycrystal plasticity models, such as antiplane shear and plane stress.

We propose an approach to the analysis of a three-dimensional model of polycrystal plasticity for which the work mentioned above does not apply. The focus will be on polycrystalline materials whose individual grains (crystallites) are assumed to be ionic crystals. This class of crystals was introduced in the celebrated work of Hutchinson [34], and it is representative in the modelling of crystalline materials exhibiting a deficient supply of slip systems.

Section 3.1 is devoted to the variational characterization of the effective yield set in Hutchinson's model. In Section 3.2 we prove a Γ-convergence result for the class of supremal functionals involved in the characterization of the effective yield set.

Chapters 4 and 5 of the thesis describe results obtained in [2] and [3]. They are devoted to the study of the asymptotic behavior, as $p \to \infty$, of the second and first eigenvalues and the corresponding eigenfunctions for the $p(x)$-Laplacian with Neumann and Robin boundary conditions, respectively. The analysis of partial differential equations with non-standard growth in the framework of variable exponent spaces has been the subject of an increasing interest during the last decade. We refer to the survey by Harjulehto, Hästö, Lê & Nuortio [33] for a comprehensive account of the developments up to 2010. In particular, a lot of attention has been paid to the study of eigenvalue problems for the $p(x)$-Laplace operator

$$-\Delta_{p(x)} := -\mathrm{div}\left(|\nabla u|^{p(x)-2}\nabla u\right) = \Lambda_{p(\cdot)}|u|^{p(x)-2}u$$

in open bounded domains $\Omega \subset \mathbb{R}^N$, subject to various boundary conditions. For example, in the case of Dirichlet boundary conditions, this equation has been analyzed in [24] (see also [25]), while the Neumann and Robin boundary conditions were studied later in [26] and [17], respectively.

More general eigenvalue problems for the $p(x)$-Laplacian have also been intensively studied in recent years. An excellent account of recent developments in this direction can be found in Mihăilescu's Ph.D. Thesis [41].

During the last several years, a number of papers have been devoted to the asymptotic analysis of solutions to partial differential equations involving the $p(x)$-Laplacian as $p(x) \to \infty$. We mention here the work of Manfredi, Rossi & Urbano [39], [40], Lindqvist & Lukkari [38], Pérez-Llanos & Rossi [44], [45], and Franzina & Lindqvist [28]. For the case of Dirichlet boundary conditions, the asymptotic behavior of the first eigenvalue/eigenfunction pairs associated to $-\Delta_{p(x)}$ has been studied in [44] (see also [28]), but the corresponding problems for other classes of boundary conditions have remained open. Chapters 4 and 5 of the thesis fit into this general area of investigation. In Chapter 4 our focus is on the Neumann eigenvalue problem for the $p(x)$-Laplacian:

$$\begin{cases} -\Delta_{p(x)} u = \Lambda_{p(\cdot)} |u|^{p(x)-2} u & \text{in } \Omega \\ \frac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega. \end{cases}$$

The analysis of the limiting behavior of this problem as $p \to \infty$ is undertaken in the following sense: we replace $p = p(x)$ above by $p_n = p_n(x)$, where $\{p_n\} \subset C^1(\overline{\Omega})$ is a sequence of functions that satisfies $p_n \to \infty, \nabla \ln p_n \to \xi \in C(\overline{\Omega}, \mathbb{R}^N)$, and $\frac{p_n}{n} \to q \in C(\overline{\Omega}, (0, +\infty))$ uniformly in Ω, and then we study what happens with the solutions of the problems at level n as $n \to \infty$. These conditions on the sequence p_n are typical in the literature (see, e.g. [40], [44], [45], or [38], [28] for the particular case $p_n(\cdot) = np(\cdot)$- corresponding to $\xi = \nabla \ln p$ and $q = p$). We refer to Chapter 4 for a list of possible choices of such sequences $\{p_n\}$. We prove that after eventually extracting a subsequence, the (positive) second eigenfunctions converge uniformly in $\Omega \subset \mathbb{R}^N$ to a viscosity solution of the problem

$$\begin{cases} \min \{-\Delta_\infty u_\infty - |\nabla u_\infty|^2 \ln |\nabla u_\infty| \langle \xi, \nabla u_\infty \rangle, |\nabla u_\infty|^q - \Lambda_\infty |u_\infty|^q\} = 0 & \text{in } \Omega \\ \frac{\partial u_\infty}{\partial \eta} = 0 & \text{on } \partial\Omega, \end{cases}$$

where Δ_∞ is the ∞-Laplace operator, $\Delta_\infty u := \sum_{i,j=1}^{N} u_{x_i} u_{x_j} u_{x_i x_j}$, and Λ_∞ is the limit of the sequence of (suitably rescaled) second eigenvalues. Chapter 5 is devoted to the Robin eigenvalue problem for the $p(x)$-Laplacian:

$$\begin{cases} -\Delta_{p(x)} u = \Lambda_{p(\cdot)} |u|^{p(x)-2} u & \text{in } \Omega \\ |\nabla u|^{p(x)-2} \frac{\partial u}{\partial \eta} + \beta |u|^{p(x)-2} u = 0 & \text{on } \partial\Omega. \end{cases}$$

We prove that the (positive) first eigenfunctions converge uniformly in $\Omega \subset \mathbb{R}^N$ to a viscosity solution of the problem

$$\begin{cases} \min \{-\Delta_\infty u - |\nabla u|^2 \ln |\nabla u| \langle \xi, \nabla u \rangle, |\nabla u|^q - \Lambda_\infty |u|^q\} = 0 & \text{in } \Omega \\ H(x, u, \nabla u) = 0 & \text{on } \partial\Omega, \end{cases}$$

where Λ_∞ is the limit of the sequence of (suitably rescaled) first eigenvalues, and $H : \Omega \times [0, \infty) \times \mathbb{R}^N \to \mathbb{R}$ is given by

$$H(x, r, \theta) = \begin{cases} \max \left\{ |r|^{q(x)} - |\theta|^{q(x)}, \langle \theta, \eta(x) \rangle \right\} & \text{if } r > 0 \\ \langle \theta, \eta(x) \rangle \chi_{(1,\infty)}(|\theta|) & \text{if } r = 0. \end{cases}$$

The plan of Chapters 4 and 5 is as follows: Sections 4.1 and 5.1 are devoted to the Neumann and Robin eigenvalue problems for $-\Delta_{p(x)}$, respectively, for the case where $p = p(x)$ is fixed. After recalling the definition of a weak solution for each of these problems, we revisit some details concerning the Ljusternik-Schnirelman existence theory in each case, and we show that continuous weak solutions are also solutions in the viscosity sense. We adopt the definition of viscosity solutions for second-order elliptic equations with fully nonlinear boundary conditions introduced by Barles in [4]. In Section 4.2 we state and prove the main result of the chapter, Theorem 3, regarding the convergence of the second eigenvalues and the corresponding positive eigenfunctions for the Neumann problem as $p(\cdot) \to \infty$. In Section 5.2 we prove (Theorem 4) the convergence of the first eigenvalues and the corresponding positive eigenfunctions associated to the Robin problem.

2. AUXILIARY RESULTS

2.1. Γ-convergence and $\mathcal{A}$-quasiconvexity

We first recall the definition of Γ-convergence [15], [16] in metric spaces. A thorough introduction to the subject may be found in [14] (see also [9], and [10]).

Definition 1. *Let X be a metric space. A sequence $\{I_n\}$ of functionals $I_n : X \to \overline{\mathbb{R}} := \mathbb{R} \cup \{+\infty\}$ is said to $\Gamma(X)$-converge to $I : X \to \overline{\mathbb{R}}$ (we write $\Gamma(X) - \lim_{n\to\infty} I_n = I$) if*

(i) *for every $u \in X$ and $\{u_n\} \subset X$ such that $u_n \to u$ in X, we have*

$$I(u) \leq \liminf_{n\to\infty} I_n(u_n);$$

(ii) *for every $u \in X$ there exists a* recovery sequence *$\{u_n\} \subset X$ such that $u_n \to u$ in X, and*

$$I(u) \geq \limsup_{n\to\infty} I_n(u_n).$$

Let $N, d, l \in \mathbb{N}$ be given, Ω be an open, bounded domain in $\mathbb{R}^N$, $1 < p < \infty$, and let p' be the Hölder conjugate exponent of p, that is, $1/p + 1/p' = 1$. Let $W^{-1,p}(\Omega; \mathbb{R}^l)$ be the dual of $W_0^{1,p'}(\Omega; \mathbb{R}^l)$. Given a family of linear operators $A^{(1)}, A^{(2)}, \cdots, A^{(N)} \in \mathrm{Lin}(\mathbb{R}^d; \mathbb{R}^l)$, consider the differential operator $\mathcal{A} : L^p(\Omega; \mathbb{R}^d) \to W^{-1,p}(\Omega; \mathbb{R}^l)$ defined by

$$\mathcal{A}v := \sum_{i=1}^{N} A^{(i)} \frac{\partial v}{\partial x_i}, \tag{4}$$

that is,

$$\langle \mathcal{A}v, u\rangle := \left\langle \sum_{i=1}^{N} A^{(i)} \frac{\partial v}{\partial x_i}, u \right\rangle = -\sum_{i=1}^{N} \int_{\Omega} A^{(i)} v \frac{\partial u}{\partial x_i} dx \text{ for all } u \in W_0^{1,p'}(\Omega;\mathbb{R}^l). \tag{5}$$

Definition 2. *The operator $\mathcal{A}$ satisfies the* constant rank property *if there exists $r \in \mathbb{N}$ such that*

$$\operatorname{rank}(\mathbb{A}(w)) = r \text{ for all } w = (w_1, \cdots, w_N) \in S^{N-1}, \tag{6}$$

where

$$\mathbb{A}(w) := \sum_{i=1}^{N} w_i A^{(i)} \in \operatorname{Lin}(\mathbb{R}^d;\mathbb{R}^l).$$

The constant rank property was introduced by Murat and Tartar in connection to the theory of compensated compactness (see, e.g., [42], [46], and [47]). We note that this restriction still allows the treatment of a broad class of differential constraints encountered in applications. Among these, we mention curl-free fields (gradients and partial gradients), divergence-free fields, higher order gradients, symmetrized gradients, and fields which satisfy Maxwell's equations.

For the applications that we discuss in the thesis $\mathcal{A}$ will be the divergence operator acting on fields which take values in the space of symmetric $N \times N$ matrices. Given a function $U \in L^p\left(\Omega; \mathbb{M}^{N\times N}\right)$, the differential operator $\mathcal{A}$ is given by

$$\mathcal{A}U := \operatorname{Div} U = \begin{pmatrix} \operatorname{div} U^{(1)} \\ \operatorname{div} U^{(2)} \\ \vdots \\ \operatorname{div} U^{(N)} \end{pmatrix},$$

where, for $i = 1, \cdots, N$, $U^{(i)}(x) := (U_{i1}(x), U_{i2}(x), \cdots, U_{iN}(x))$ stands for the i-th row of the matrix $U(x),\ x \in \Omega$. Thus, if we take $d = N^2, l = N$, and we define, for $i, k = 1, \cdots, N$ and $j = 1, \cdots, N^2$,

$$A_{ij}^{(k)} = \begin{cases} \delta_{i(j-(k-1)N)} & \text{if } (k-1)N + 1 \leq j \leq kN \\ 0 & \text{else,} \end{cases}$$

the differential constraint $\mathcal{A}U = 0$ can be written in the form (see (4))

$$\sum_{k=1}^{N} A^{(k)} \frac{\partial U}{\partial x_k} = 0.$$

Note that the constant rank condition (6) is satisfied since for every $w = (w_1, \cdots, w_N)$ in S^{N-1} we have

$$\ker(\mathbb{A}(w)) = \left\{ V \in \mathbb{M}^{N \times N} \ : \ wV = 0 \right\},$$

and thus $\dim(\ker \mathbb{A}(w)) = N^2 - N$.

We now recall the definition of $\mathcal{A}$-quasiconvexity, introduced in Fonseca & Müller [27] (see also [13]).

Definition 3. *A function $g : \mathbb{R}^d \to \mathbb{R}$ is said to be $\mathcal{A}$-quasiconvex if*

$$g(A) \leq \int_Q g(A + w(x))dx$$

for all $A \in \mathbb{R}^d$, all Q-periodic $w \in C^\infty(Q; \mathbb{R}^d)$ such that $\mathcal{A}w = 0$ and $\int_Q w(x)dx = 0$, where $Q = (0,1)^N$ is the unit cube in $\mathbb{R}^N$.

By Jensen's inequality, convex functions are $\mathcal{A}$-quasiconvex. It is shown in [27] that if $\mathcal{A}$ satisfies the constant rank property (6), $\Omega \subset \mathbb{R}^N$ is an open, bounded set, $(u,v):\Omega\to\mathbb{R}^m\times\mathbb{R}^d$ is measurable, and $g:\Omega\times\mathbb{R}^m\times\mathbb{R}^d\to\mathbb{R}$ is a normal integrand satisfying suitable growth assumptions, then $\mathcal{A}$-quasiconvexity of $g(x,u,\cdot)$ is a necessary and sufficient condition for the sequential lower semicontinuity of integral functionals of the form

$$(u,v)\mapsto\int_\Omega g(x,u(x),v(x))dx$$

along sequences such that $u_n\to u$ in measure, $v_n\rightharpoonup v$ weakly in L^p, and $\mathcal{A}v_n\to 0$ in $W^{-1,p}$. We will only need to use the following result from [27].

Proposition 1. (see [27, Theorem 3.7]) *Let $1\le p\le+\infty$ and suppose that $g:\Omega\times\mathbb{R}^m\times\mathbb{R}^d\to[0,+\infty)$ is a normal integrand such that $z\mapsto g(x,u,z)$ is $\mathcal{A}$-quasiconvex and continuous for a.e. $x\in\Omega$ and all $u\in\mathbb{R}^d$. If $1\le p<+\infty$, assume further that there exists a locally bounded function $a:\Omega\times\mathbb{R}^d\to[0,+\infty)$ such that*

$$0\le g(x,u,v)\le a(x,u)(1+|v|^p),$$

for a.e. $x\in\Omega$, and all $(u,v)\in\mathbb{R}^m\times\mathbb{R}^d$. If

$$u_n\to u \text{ in measure, } v_n\rightharpoonup v \text{ weakly (weakly}^* \text{ if } p=\infty) \text{ in } L^p(\Omega;\mathbb{R}^d),$$

and

$$\mathcal{A}v_n\to 0 \text{ in } W^{-1,p}(\Omega;\mathbb{R}^l)\ (\mathcal{A}v_n=0 \text{ if } p=\infty),$$

then

$$\int_\Omega g(x,u(x),v(x))dx\le\liminf_{n\to\infty}\int_\Omega g(x,u_n(x),v_n(x))dx.$$

2.2. Variable Exponent Lebesque and Sobolev Spaces

In this section, we provide a brief introduction to variable exponent Lebesque and Sobolev spaces. For more details we refer to the books by Diening, Harjulehto, Hästö & M. Ružička [19], Musielak [43], and the papers by Edmunds, Lang & Nekvinda [20], Edmunds & Rákosník [21], [22], and Kovacik & Rákosník [37].

Let $\Omega \subset R^N$ be an open set with smooth boundary, and let $|\Omega|$ stand for the N-dimensional Lebesgue measure of Ω. Given any continuous function $p : \overline{\Omega} \to (1, \infty)$, let $p^- := \inf_{x \in \Omega} p(x)$ and $p^+ := \sup_{x \in \Omega} p(x)$. The variable exponent Lebesgue space $L^{p(\cdot)}(\Omega)$ is defined by

$$L^{p(\cdot)}(\Omega) = \left\{ u : \Omega \to \mathbb{R} \text{ measurable } : \int_\Omega |u(x)|^{p(x)}\, dx < \infty \right\}.$$

It is a Banach space when endowed with the so-called Luxemburg norm

$$|u|_{p(\cdot)} := \inf \left\{ \mu > 0 \ : \ \int_\Omega \left| \frac{u(x)}{\mu} \right|^{p(x)} dx \leq 1 \right\}.$$

For constant functions p the space $L^{p(\cdot)}(\Omega)$ reduces to the classical Lebesgue space $L^p(\Omega)$, endowed with the standard norm

$$\|u\|_{L^p(\Omega)} := \left(\int_\Omega |u(x)|^p dx \right)^{1/p}.$$

$L^{p(\cdot)}(\Omega)$ is separable and reflexive if $1 < p^- \leq p^+ < +\infty$. If $0 < |\Omega| < \infty$ and if p_1, p_2 are variable exponents such that $p_1 \leq p_2$ in Ω then the embedding $L^{p_2(\cdot)}(\Omega) \hookrightarrow L^{p_1(\cdot)}(\Omega)$ is continuous, and its norm does not exceed $|\Omega| + 1$. We denote by $L^{p'(\cdot)}(\Omega)$ the conjugate space of $L^{p(\cdot)}(\Omega)$, where $1/p(x) + 1/p'(x) = 1$. The following version of

Hölder's inequality

$$\left|\int_{\Omega} uv\,dx\right| \leq \left(\frac{1}{p^-} + \frac{1}{p'^-}\right)|u|_{p(\cdot)}|v|_{p'(\cdot)},\ \forall\ u \in L^{p(\cdot)}(\Omega),\ v \in L^{p'(\cdot)}(\Omega) \tag{7}$$

holds. The modular of the space $L^{p(\cdot)}(\Omega)$ is the mapping $\rho_{p(\cdot)} : L^{p(\cdot)}(\Omega) \to \mathbb{R}$, defined by

$$\rho_{p(\cdot)}(u) := \int_{\Omega} |u(x)|^{p(x)} dx.$$

The variable exponent Sobolev space $W^{1,p(\cdot)}(\Omega)$ is defined by

$$W^{1,p(\cdot)}(\Omega) := \{u \in L^{p(\cdot)}(\Omega) :\ |\nabla u| \in L^{p(\cdot)}(\Omega)\}.$$

It becomes a Banach space when endowed with one of the equivalent norms

$$\|u\|_{p(\cdot)} := |u|_{p(\cdot)} + |\nabla u|_{p(\cdot)},$$

or

$$\|u\| := \inf\left\{\mu > 0;\ \int_{\Omega}\left(\left|\frac{\nabla u(x)}{\mu}\right|^{p(x)} + \left|\frac{u(x)}{\mu}\right|^{p(x)}\right)\ dx \leq 1\right\},$$

where in the definition of $\|u\|_{p(\cdot)}$, $|\nabla u|_{p(\cdot)}$ stands for the Luxemburg norm of $|\nabla u|$. Under very mild assumptions on the function p, the space $W^{1,p(\cdot)}(\Omega)$ is also separable and reflexive. Another important fact that we will use in the sequel is that the embedding $W^{1,p(\cdot)}(\Omega) \hookrightarrow C(\overline{\Omega})$ is compact and continuous if $p(x) \geq \alpha > N,\ \forall\ x \in \Omega$. The following extensions of the classical results for Lebesgue spaces are well-known (see, e.g., [19]).

Lemma 1. *Let $\{f_n\}$ be a sequence of measurable functions. If $f_n \to f$ and $|f_n(x)| \leq g(x)$ a.e. $x \in \Omega$ for some $f : \Omega \to \mathbb{R}$ measurable and $g \in L^{p(\cdot)}(\Omega)$, then $f_n \to f$ in $L^{p(\cdot)}(\Omega)$.*

Lemma 2. *Let* $\{u_n\} \subset L^{p(\cdot)}(\Omega)$ *and* $u \in L^{p(\cdot)}(\Omega)$. *The following statements are equivalent:*

(i) $\lim_{n\to\infty} |u_n - u|_{p(\cdot)} = 0$;

(ii) $\lim_{n\to\infty} \rho_{p(\cdot)}(u_n - u) = 0$;

(iii) $u_n \to u$ *in measure in* Ω *and* $\lim_{n\to\infty} \rho_{p(\cdot)}(u_n) = \rho_{p(\cdot)}(u)$.

3. A VARIATIONAL CHARACTERIZATION OF THE EFFECTIVE YIELD SET FOR IONIC POLYCRYSTALS

Let $\Omega \subset \mathbb{R}^N$ be a bounded, open domain, with sufficiently smooth boundary, and let $s \in \mathbb{N}$ be a positive integer. For $i = 1, 2, \cdots, s$, consider Carathéodory integrands $f_i : \Omega \times \mathbb{R}^d \to [0, +\infty)$ such that

$$f_i(x, \cdot) \ \text{ is } \mathcal{A}\text{-quasiconvex for a.e. } x \in \Omega. \tag{8}$$

Assume that there exists a constant $C > 0$ such that for every $i \in \{1, 2, \cdots, s\}$ we have

$$f_i(x, v) \le C\,(1 + |v|) \ \text{ for a.e. } x \in \Omega, \ \text{ and all } v \in \mathbb{R}^d. \tag{9}$$

Further, we assume that there exists a constant $c > 0$ such that

$$\sum_{i=1}^{s} f_i(x, v) \ge c|v| \text{ for a.e. } x \in \Omega, \ \text{ and all } v \in \mathbb{R}^d. \tag{10}$$

The effective yield set of a polycrystal can be characterized in several models of polycrystal plasticity by means of variational principles in L^∞ associated to Γ-limits of certain power-law functionals. Indeed, it is shown in [7, Theorem 5] that for suitable choices of the positive integers N, d, s, the differential operator $\mathcal{A}$, and of the functions $f_i (i = 1, \cdots, s)$ satisfying the conditions (8), (9), and (10), we have

$$K_{\text{eff}} = \left\{ \eta \in \mathbb{R}^d \ : \ f^{\text{eff}}_{s,\infty}(\eta) \le 1 \right\}, \tag{11}$$

where

$$f^{\text{eff}}_{s,\infty}(\eta) := \inf \left\{ \max_{i \in \{1, \cdots, s\}} \operatorname*{ess\,sup}_{x \in \Omega} f_i(x, w(x) + \eta) \ : \ w \in L^\infty(\Omega; \mathbb{R}^d), \int_\Omega w(x)\, dx = 0, \mathcal{A}w = 0 \right\}.$$

We will see in the following section that this result is not applicable to Hutchinson's model of ionic polycrystals, which is our focus here. It turns out (see Theorem 1 below) that in this case the effective yield set can be described in a similar way by means of variational principles adapted to this setting.

3.1. Ionic Polycrystals

The goal of this section is to characterize the effective yield set for ionic polycrystals, introduced by Hutchinson in [34]. In this model each individual grain has two different types of slip systems with critical stresses $\pm\tau_A$ and $\pm\tau_B$, which leads to a yield set K of the form

$$K = \left\{ \eta = (\eta_{ij}) \in \mathbb{M}^{3\times 3}_{\mathrm{sym}} : |\eta_{ii} - \eta_{jj}| \le \tau_A, |\eta_{ij}| \le \tau_B, \forall i,j \in \{1,2,3\}, i \neq j \right\}. \quad (12)$$

Let $R : \Omega \to \mathrm{SO}(3)$ be a piecewise constant rotation field, given by

$$R(x) = \begin{pmatrix} \cos\theta(x) & -\sin\theta(x) & 0 \\ \sin\theta(x) & \cos\theta(x) & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad (13)$$

where $\theta(x)$ is the angle of rotation describing the orientation of the grain which contains the point $x \in \Omega$ in the polycrystal occupying the region $\Omega \subset \mathbb{R}^3$. After computations, the pointwise constraint $\sigma(x) \in R(x)KR^T(x)$, $x \in \Omega$ (see (1)) on the stress field $\sigma : \Omega \to \mathbb{M}^{3\times 3}_{\mathrm{sym}}$ becomes

$$\begin{pmatrix} a_{11}(x) & a_{12}(x) & a_{13}(x) \\ a_{21}(x) & a_{22}(x) & a_{23}(x) \\ a_{31}(x) & a_{32}(x) & a_{33}(x) \end{pmatrix} \in K,$$

where, denoting by σ_{ij} $(i,j \in \{1,2,3\})$ the components of the stress field, $a_{ij} : \Omega \to \mathbb{R}$ are defined by

$$a_{11}(x) = \sigma_{11}(x)\cos^2\theta(x) + \sigma_{12}(x)\sin 2\theta(x) + \sigma_{22}(x)\sin^2\theta(x),$$

$$a_{12}(x) = \frac{\sigma_{22}(x) - \sigma_{11}(x)}{2}\sin 2\theta(x) + \sigma_{12}(x)\cos 2\theta(x),$$

$$a_{13}(x) = \sigma_{13}(x)\cos\theta(x) + \sigma_{23}(x)\sin\theta(x),$$

$$a_{22}(x) = \sigma_{11}(x)\sin^2\theta(x) - \sigma_{12}(x)\sin 2\theta(x) + \sigma_{22}(x)\cos^2\theta(x),$$

$$a_{23}(x) = -\sigma_{13}(x)\sin\theta(x) + \sigma_{23}(x)\cos\theta(x),$$

$$a_{33}(x) = \sigma_{33}(x),$$

and $a_{ij} = a_{ji}$ for all $i \neq j$. Taking into account the specific form (12) of the yield set, this can be written as

$$\sigma(x) \in \left\{\eta \in \mathbb{M}^{3\times 3}_{\text{sym}} \ : \ f_i(x,\eta) \leq 1 \text{ for } i \in \{1,\cdots,6\}\right\}, \tag{14}$$

where the functions $f_i : \Omega \times \mathbb{M}^{3\times 3}_{\text{sym}} \to [0,+\infty)$ $(i = 1,\cdots,6)$ are given by the following explicit formulas:

$$f_1(x,\eta) := \frac{1}{\tau_A}|(\eta_{11} - \eta_{22})\cos(2\theta(x)) + 2\eta_{12}\sin(2\theta(x))|, \tag{15}$$

$$f_2(x,\eta) := \frac{1}{\tau_A}|\eta_{11}\sin^2\theta(x) - \eta_{12}\sin(2\theta(x)) + \eta_{22}\cos^2\theta(x) - \eta_{33}|, \tag{16}$$

$$f_3(x,\eta) := \frac{1}{\tau_A}|\eta_{33} - \eta_{11}\cos^2\theta(x) - \eta_{12}\sin(2\theta(x)) - \eta_{22}\sin^2\theta(x)|, \tag{17}$$

$$f_4(x,\eta) := \frac{1}{\tau_B}\left|\eta_{12}\cos(2\theta(x)) + \frac{\eta_{22}-\eta_{11}}{2}\sin(2\theta(x))\right|, \tag{18}$$

$$f_5(x,\eta) := \frac{1}{\tau_B}|\eta_{23}\cos\theta(x) - \eta_{13}\sin\theta(x)|, \tag{19}$$

$$f_6(x,\eta) := \frac{1}{\tau_B}|\eta_{13}\cos\theta(x) + \eta_{23}\sin\theta(x)|. \tag{20}$$

It is easy to check that each f_i ($i \in \{1,\cdots,6\}$) satisfies (8) and (9). However, the coercivity condition (10) does not hold. This is precisely because in Hutchinson's model we are dealing with a deficient supply of slip systems. Since (10) is a key hypothesis in the proof of the characterization (11) of the yield set in [7], the variational characterization of the effective yield set for the model under consideration here does not follow from the analysis in that paper. To overcome this drawback, our strategy is to modify the yield set of the basic crystal by imposing an additional constraint, and then to show that the effective yield set can in fact be completely characterized by means of a family of variational principles parametrized by the corresponding critical shear stresses. Precisely, for each $m \in \mathbb{N}$, we introduce the modified yield sets

$$K^{(m)} = \left\{\eta \in \mathbb{M}^{3\times3}_{\text{sym}} : |\eta_{ii} - \eta_{jj}| \le \tau_A, |\eta_{ij}| \le \tau_B, \forall i,j \in \{1,2,3\},\ i \neq j,\ |tr(\eta)| \le m\right\} \tag{21}$$

where $tr(\eta)$ stands for the trace of the matrix $\eta \in \mathbb{M}^{3\times3}_{\text{sym}}$. If R is the rotation field defined by (13), the pointwise constraint on the stress field $\sigma : \Omega \to \mathbb{M}^{3\times3}_{\text{sym}}$ acting on the polycrystal occupying the domain Ω and whose individual grains have yield set $K^{(m)}$ reads:

$$\sigma(x) \in R(x)K^{(m)}R^T(x), \quad x \in \Omega. \tag{22}$$

It is easy to see that (22) can be written in a form similar to (14), that is,

$$\sigma(x) \in \left\{\eta \in \mathbb{M}^{3\times3}_{\mathrm{sym}} \ : \ f_i(x,\eta) \leq 1 \text{ for i} \in \{1,\cdots,7\}\right\}, \tag{23}$$

where $f_i : \Omega \times \mathbb{M}^{3\times3}_{\mathrm{sym}} \to [0,+\infty)$ $(i = 1,\cdots,6)$ are defined as before, and with $f_7 : \Omega \times \mathbb{M}^{3\times3}_{\mathrm{sym}} \to [0,+\infty)$ given by

$$f_7(x,\eta) := \frac{1}{m}|\eta_{11} + \eta_{22} + \eta_{33}|. \tag{24}$$

It is immediate that f_7 satisfies our hypotheses (8) and (9). We claim that (10) also holds (with $s = 7$). Indeed, we have the following

Lemma 3. *There exists $c > 0$ such that*

$$\sum_{i=1}^{7} f_i(x,\eta) \geq c|\eta| \text{ for a.e. } x \in \Omega \text{ and all } \eta \in \mathbb{M}^{3\times3}_{\mathrm{sym}}. \tag{25}$$

Proof. The computations are elementary. First, note that we have

$$\begin{aligned}|\eta_{12}| &= \left|\left(\eta_{12}\cos 2\theta + \frac{\eta_{22}-\eta_{11}}{2}\sin 2\theta\right)\cos 2\theta + \left(\eta_{12}\sin 2\theta + \frac{\eta_{11}-\eta_{22}}{2}\cos 2\theta\right)\sin 2\theta\right| \\ &\leq \left|\eta_{12}\cos 2\theta + \frac{\eta_{22}-\eta_{11}}{2}\sin 2\theta\right| + \left|\eta_{12}\sin 2\theta + \frac{\eta_{11}-\eta_{22}}{2}\cos 2\theta\right|.\end{aligned}$$

Thus,

$$|\eta_{12}| \leq \tau_B f_4(x,\eta) + \frac{\tau_A}{2} f_1(x,\eta) \text{ for a.e. } x \in \Omega \text{ and all } \eta \in \mathbb{M}^{3\times3}_{\mathrm{sym}}. \tag{26}$$

Similarly,

$$\begin{aligned}|\eta_{13}| &= |(\eta_{13}\cos\theta + \eta_{23}\sin\theta)\cos\theta - (\eta_{23}\cos\theta - \eta_{13}\sin\theta)\sin\theta| \\ &\leq |\eta_{13}\cos\theta + \eta_{23}\sin\theta| + |\eta_{23}\cos\theta - \eta_{13}\sin\theta|,\end{aligned}$$

and

$$
\begin{aligned}
|\eta_{23}| &= |(\eta_{23}\cos\theta - \eta_{13}\sin\theta)\cos\theta + (\eta_{13}\cos\theta + \eta_{23}\sin\theta)\sin\theta| \\
&\le |\eta_{23}\cos\theta - \eta_{13}\sin\theta| + |\eta_{13}\cos\theta + \eta_{23}\sin\theta|.
\end{aligned}
$$

Hence,

$$
|\eta_{13}| \le \tau_B \left(f_5(x,\eta) + f_6(x,\eta) \right), \tag{27}
$$

and

$$
|\eta_{23}| \le \tau_B \left(f_5(x,\eta) + f_6(x,\eta) \right), \tag{28}
$$

for a.e. $x \in \Omega$, and all $\eta \in \mathbb{M}^{3\times 3}_{\mathrm{sym}}$. Since

$$
|\eta_{11} - \eta_{22}| \le |(\eta_{11} - \eta_{22})\cos 2\theta + 2\eta_{12}\sin 2\theta| + 2\left|\eta_{12}\cos 2\theta + \frac{(\eta_{22} - \eta_{11})}{2}\sin 2\theta\right|,
$$

we have

$$
|\eta_{11} - \eta_{22}| \le \tau_A f_1(x,\eta) + 2\tau_B f_4(x,\eta) \text{ for a.e. } x \in \Omega \text{ and all } \eta \in \mathbb{M}^{3\times 3}_{\mathrm{sym}}. \tag{29}
$$

Next, observe that

$$
\begin{aligned}
|\eta_{11} + \eta_{22} - 2\eta_{33}| &= |(\eta_{11}\sin^2\theta - \eta_{12}\sin 2\theta + \eta_{22}\cos^2\theta - \eta_{33}) \\
&\qquad - (\eta_{33} - \eta_{11}\cos^2\theta - \eta_{12}\sin 2\theta - \eta_{22}\sin^2\theta)| \\
&\le |\eta_{11}\sin^2\theta - \eta_{12}\sin 2\theta + \eta_{22}\cos^2\theta - \eta_{33}| \\
&\qquad + |\eta_{33} - \eta_{11}\cos^2\theta - \eta_{12}\sin 2\theta - \eta_{22}\sin^2\theta|.
\end{aligned}
$$

Thus,

$$|\eta_{11}+\eta_{22}-2\eta_{33}| \le \tau_A f_2(x,\eta)+\tau_A f_3(x,\eta) \text{ for a.e. } x\in\Omega \text{ and all } \eta\in\mathbb{M}^{3\times3}_{\rm sym}. \quad (30)$$

Taking into account (24), we obtain that

$$|\eta_{11}+\eta_{22}| \le \frac{2m}{3} f_7(x,\eta)+\frac{\tau_A}{3} f_2(x,\eta)+\frac{\tau_A}{3} f_3(x,\eta), \quad (31)$$

for a.e. $x\in\Omega$, and all $\eta\in\mathbb{M}^{3\times3}_{\rm sym}$. In view of (29) and (31), we find

$$|\eta_{11}|+|\eta_{22}| \le \frac{2m}{3} f_7(x,\eta)+\frac{\tau_A}{3} f_2(x,\eta)+\frac{\tau_A}{3} f_3(x,\eta)+\tau_A f_1(x,\eta)+2\tau_B f_4(x,\eta). \quad (32)$$

for a.e. $x\in\Omega$, and all $\eta\in\mathbb{M}^{3\times3}_{\rm sym}$. Further, since

$$|\eta_{33}| \le \frac{1}{3}|\eta_{11}+\eta_{22}-2\eta_{33}|+\frac{1}{3}|\eta_{11}+\eta_{22}+\eta_{33}|,$$

(24) and (30) give

$$|\eta_{33}| \le \frac{\tau_A}{3} f_2(x,\eta)+\frac{\tau_A}{3} f_3(x,\eta)+\frac{m}{3} f_7(x,\eta) \text{ for a.e. } x\in\Omega \text{ and all } \eta\in\mathbb{M}^{3\times3}_{\rm sym}. \quad (33)$$

Overall, (26), (27), (28), (32), and (33) give

$$\begin{aligned}
|\eta| &\le |\eta_{11}|+|\eta_{22}|+|\eta_{33}|+\sqrt{2}|\eta_{12}|+\sqrt{2}|\eta_{13}|+\sqrt{2}|\eta_{23}| \\
&\le m f_7(x,\eta)+2\sqrt{2}\tau_B(f_6(x,\eta)+f_5(x,\eta))+(2+\sqrt{2})\tau_B f_4(x,\eta) \\
&\qquad +\frac{2\tau_A}{3}(f_3(x,\eta)+f_2(x,\eta))+\left(1+\frac{\sqrt{2}}{2}\right)\tau_A f_1(x,\eta) \\
&\le \max\left\{m,(2+\sqrt{2})\tau_B,\left(1+\frac{\sqrt{2}}{2}\right)\tau_A\right\}\sum_{i=1}^{7} f_i(x,\eta)
\end{aligned}$$

for a.e. $x \in \Omega$ and all $\eta \in \mathbb{M}^{3\times 3}_{\text{sym}}$. Thus, (25) holds, with

$$c = \left(\max \left\{ m, (2+\sqrt{2})\tau_B, \left(1+\frac{\sqrt{2}}{2}\right)\tau_A \right\} \right)^{-1}.$$

□

The remainder of this section is devoted to the variational characterization of the effective yield set in Hutchinson's model. To simplify the presentation, we will work in $\Omega = Q = (0,1)^3$ - the unit cube in $\mathbb{R}^3$. The definition (3) of the effective yield set becomes

$$K_{\text{eff}} := \left\{ \overline{\sigma} := \int_Q \sigma(x)dx \ : \ \text{Div } \sigma = 0 \text{ in } Q, \text{ and } \sigma(x) \in R(x)KR^T(x), \ x \in Q \right\}, \tag{34}$$

where K is defined by (12). We have already established (see (14)) that the pointwise constraint on the stress (1) may be written in the form

$$\sigma(x) \in \left\{ \eta \in \mathbb{M}^{3\times 3}_{\text{sym}} \ : \ f_i(x,\eta) \leq 1 \text{ for all } i = 1, \cdots, s \right\}, \ x \in Q, \tag{35}$$

where $s = 6$, and where $f_i : Q \times \mathbb{M}^{3\times 3}_{\text{sym}} \to \mathbb{R}$ $(i = 1, \cdots, 6)$ are defined by the formulas (15) through (20). These are Carathéodory integrands satisfying our hypotheses (8) and (9). However, the coercivity condition (10) does not hold (with $s = 6$), which makes the characterization (11) inapplicable. It is worth noting that if the yield set of the basic crystal is modified to be $K^{(m)}$, given by (21), rather than K (given by (12)), then (1) may be written in the form (35) with $s = 7$, where the additional function f_7 (which depends on m) is defined in (24). In view of our computations above, (10) does hold in this case, and thus (see [7]) the effective yield set of the modified polycrystal admits the variational characterization

$$K^{(m)}_{\text{eff}} = \left\{ \eta \in \mathbb{M}^{3\times 3}_{\text{sym}} \ : \ f^{m,\text{eff}}_{\infty}(\eta) \leq 1 \right\}, \tag{36}$$

where

$$f_\infty^{m,\text{eff}}(\eta) := \inf\left\{ \max_{i\in\{1,\cdots,7\}} \operatorname*{ess\,sup}_{x\in\Omega} f_i(x, w(x)+\eta) : w \in L^\infty(Q;\mathbb{M}^{3\times3}_{\text{sym}}), \int\limits_Q w(x)\ dx = 0,\ \text{Div}\ w = 0\right\}.$$

Note that the dependence of $f_\infty^{m,\text{eff}}(\eta)$ on m is realized through f_7 only.

The next result gives a characterization of the effective yield set of a ionic polycrystal in terms of the family of variational principles $\left\{f_\infty^{m,\text{eff}}\right\}$ defined above.

Theorem 1. *Let* K *and* K_{eff} *be given by (12) and (34), respectively. Then*

$$K_{\text{eff}} = \left\{\eta \in \mathbb{M}^{3\times3}_{\text{sym}}\ :\ \exists\ m\in\mathbb{N} \text{ s.t. } f_\infty^{m,\text{eff}}(\eta) \le 1\right\}. \tag{37}$$

Proof. First, note that in view of (35), with $f_i : Q\times\mathbb{M}^{3\times3}_{\text{sym}} \to \mathbb{R}\ \ (i=1,\cdots,6)$ defined by (15)-(20), we have

$$K_{\text{eff}} = \left\{\eta\in\mathbb{M}^{3\times3}_{\text{sym}}\ :\ \text{ there exists } \sigma\in L^\infty(Q;\mathbb{M}^{3\times3}_{\text{sym}}) \text{ such that } \eta = \int\limits_Q \sigma(x)dx,\right.$$
$$\left.\text{Div}\ \sigma = 0 \text{ in } Q,\ f_i(x,\sigma(x)) \le 1 \text{ for a.e. } x\in Q,\quad i=1,\cdots,6\right\}.$$

Equivalently,

$$K_{\text{eff}} = \left\{\eta\in\mathbb{M}^{3\times3}_{\text{sym}}\ :\ \text{ there exists } \sigma\in L^\infty(Q;\mathbb{M}^{3\times3}_{\text{sym}}) \text{ such that } \int\limits_Q \sigma(x)dx = 0,\right.$$
$$\left.\text{Div}\ \sigma = 0 \text{ in } Q,\ f_i(x,\sigma(x)+\eta) \le 1 \text{ for a.e. } x\in Q,\ i=1,\cdots,6\right\}. \tag{38}$$

Let $\eta\in K_{\text{eff}}$. There exists $\sigma\in L^\infty(Q;\mathbb{M}^{3\times3}_{\text{sym}})$ such that $\int\limits_Q \sigma(x)dx = 0,$ Div $\sigma = 0$ in Q, and with $f_i(x,\sigma(x)+\eta)\le 1$ for $\mathcal{L}^N-$a.e. $x\in Q$, and all $i=1,\cdots,6$. Thus,

$$\operatorname*{ess\,sup}_{x\in Q} f_i(x,\sigma(x)+\eta) \le 1 \text{ for every } i\in\{1,\cdots,6\}.$$

Let $m := [3(\|\sigma\|_{L^\infty(Q;\mathbb{M}^{3\times 3}_{\text{sym}})} + |\eta|)] + 1$, where $[x]$ stands for the integer part of the real number x. For a.e. $x \in Q$, we have

$$|\text{tr}(\sigma(x) + \eta)| \leq \sum_{i=1}^{3} |\sigma_{ii}(x) + \eta_{ii}| \leq 3(\|\sigma\|_{L^\infty(Q;\mathbb{M}^{3\times 3}_{\text{sym}})} + |\eta|) < m,$$

and thus, $\operatorname*{ess\,sup}_{x\in Q} f_7(x, \sigma(x) + \eta) \leq 1$. Overall, we have obtained that

$$\max_{i\in\{1,\cdots,7\}} \operatorname*{ess\,sup}_{x\in\Omega} f_i(x, \sigma(x) + \eta) \leq 1,$$

which gives that $f^{m,\text{eff}}_\infty(\eta) \leq 1$.

Conversely, let $\eta \in \mathbb{M}^{3\times 3}_{\text{sym}}$ be such that there exists $m \in \mathbb{N}$ with $f^{m,\text{eff}}_\infty(\eta) \leq 1$. Since

$$f^{m,\text{eff}}_\infty(\eta) = \inf\left\{ \max\left\{ \operatorname*{ess\,sup}_{x\in\Omega} f_i(x, \sigma(x) + \eta),\ i \in \overline{1,6}, \frac{1}{m} \operatorname*{ess\,sup}_{x\in\Omega} |\text{tr }(\sigma(x) + \eta)| \right\} :\right.$$
$$\left. \sigma \in L^\infty(Q; \mathbb{M}^{3\times 3}_{\text{sym}}), \int_Q \sigma(x)\ dx = 0,\ \text{Div } \sigma = 0 \text{ in } Q \right\} \quad (39)$$

there exists a sequence $\{\sigma_{m,n}\}_{n\in\mathbb{N}} \subseteq L^\infty(Q; \mathbb{M}^{3\times 3}_{\text{sym}})$ such that Div $\sigma_{m,n} = 0$ in Q, $\int_Q \sigma_{m,n}(x)dx = 0$ for all $n \in \mathbb{N}$, and

$$\max\left\{ \operatorname*{ess\,sup}_{x\in\Omega} f_i(x, \sigma_{m,n}(x) + \eta),\ i \in \overline{1,6}, \frac{1}{m} \operatorname*{ess\,sup}_{x\in\Omega} |\text{tr }(\sigma_{m,n}(x) + \eta)| \right\} \to f^{m,\text{eff}}_\infty(\eta) \quad (40)$$

as $n \to \infty$.

The coercivity condition (10) implies that the sequence $\{\sigma_{m,n}\}_{n\in\mathbb{N}}$ is bounded in $L^\infty(Q; \mathbb{M}^{3\times 3}_{\text{sym}})$. Thus, there exists a subsequence of $\{\sigma_{m,n}\}_{n\in\mathbb{N}}$ (not relabelled) and $\sigma_m \in L^\infty(Q; \mathbb{M}^{3\times 3}_{\text{sym}})$ such that $\sigma_{m,n} \rightharpoonup \sigma_m$ weakly* in $L^\infty(Q; \mathbb{M}^{3\times 3}_{\text{sym}})$ as $n \to \infty$, with Div $\sigma_m = 0$ and $\int_Q \sigma_m(x)dx = 0$. Let $x \in Q$ be a Lebesgue point for each of the $f_i(\cdot, \sigma_m(\cdot) + \eta),\ i = 1, \cdots, 6$. By Proposition 1 we deduce that for sufficiently small

$r > 0$ we have

$$\int_{B(x,r)} f_i(y, \sigma_m(y) + \eta) dy \leq \liminf_{n\to\infty} \int_{B(x,r)} f_i(y, \sigma_{m,n}(y) + \eta) dy, \ i = 1, \cdots, 6.$$

The integral on the right hand side is bounded above by

$$|B(x,r)| \max \left\{ \operatorname*{ess\,sup}_{x\in\Omega} f_i(x, \sigma_{m,n}(x) + \eta), \ i \in \overline{1,6}, \frac{1}{m} \operatorname*{ess\,sup}_{x\in\Omega} |\mathrm{tr}\ (\sigma_{m,n}(x) + \eta)| \right\},$$

and we deduce by (40) that

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} f_i(y, \sigma_m(y) + \eta) dy \leq f_\infty^{m,\mathrm{eff}}(\eta) \leq 1, \ i = 1, \cdots, 6. \tag{41}$$

Letting $r \to 0^+$, since almost every point $x \in Q$ is a Lebesgue point for all $f_i(\cdot, \sigma_m(\cdot) + \eta), i = 1, \cdots, 6$, we have that $f_i(x, \sigma_m(x) + \eta) \leq 1$ for a.e. $x \in Q, \ i = 1, \cdots, 6$. Taking (38) into account, we conclude that $\eta \in K_{\mathrm{eff}}$. □

3.2. A Γ-convergence Result

In this section we prove a Γ-convergence result for the class of supremal functionals governing the variational principles $f_\infty^{m,\mathrm{eff}}$ as the parameter m tends to ∞.

Theorem 2. *Let s be a positive integer, and for $i = 1, 2, \cdots, s$, let $f_i : \Omega \times \mathbb{R}^d \to [0, +\infty)$ be Carathéodory integrands satisfying (8), (9), and (10). Consider the sequence $\{F_m\}$ of functionals $F_m : L^\infty(\Omega; \mathbb{R}^d) \to [0, +\infty]$ defined by*

$$F_m(w) = \begin{cases} \max\left\{ \operatorname*{ess\,sup}_{x\in\Omega} f_i(x, w(x)), i \in \overline{1, s{-}1}, \frac{1}{m} \operatorname*{ess\,sup}_{x\in\Omega} f_s(x, w(x)) \right\} & \text{if } \ w \in L^\infty(\Omega; \mathbb{R}^d) \cap \ker\mathcal{A} \\ +\infty & \text{otherwise,} \end{cases}$$

and let $F_\infty : L^\infty(\Omega; \mathbb{R}^d) \to [0, +\infty]$ be defined by

$$F_\infty(w) = \begin{cases} \max_{i\in\{1,\cdots,s-1\}} \operatorname*{ess\,sup}_{x\in\Omega} f_i(x, w(x)) & \text{if } \ w \in L^\infty(\Omega; \mathbb{R}^d) \cap \ker\mathcal{A} \\ +\infty & \text{otherwise.} \end{cases}$$

Then

(i) *for every* $w \in L^\infty(\Omega;\mathbb{R}^d)$ *and* $\{w_m\} \subset L^\infty(\Omega,\mathbb{R}^d)$ *such that* $w_m \overset{*}{\rightharpoonup} w$ *weakly* in* $L^\infty(\Omega;\mathbb{R}^d)$ *we have*

$$F_\infty(w) \le \liminf_{m\to\infty} F_m(w_m); \tag{42}$$

(ii) *for every* $w \in L^\infty(\Omega;\mathbb{R}^d)$ *there exists a sequence* $\{w_m\} \subset L^\infty(\Omega;\mathbb{R}^d)$ *such that* $w_m \to w$ *in* $L^\infty(\Omega;\mathbb{R}^d)$, *and* $\lim_{m\to\infty} F_m(w_m) = F_\infty(w)$.

In particular, $\Gamma(L^\infty(\Omega;\mathbb{R}^d)) - \lim_{m\to\infty} F_m = F_\infty$.

Proof. Let $w \in L^\infty(\Omega;\mathbb{R}^d)$ and $\{w_m\} \subset L^\infty(\Omega;\mathbb{R}^d)$ be such that $w_m \overset{*}{\rightharpoonup} w$ in $L^\infty(\Omega;\mathbb{R}^d)$. Without loss of generality, and after extracting a subsequence if necessary, we may assume that

$$\liminf_{m\to\infty} F_m(w_m) = \lim_{m\to\infty} F_m(w_m) < +\infty.$$

Note that in view of our growth condition (9), we have that $f_i(\cdot, w(\cdot)) \in L^1(\Omega)$. Let $x \in \Omega$ be a Lebesgue point for $f_i(\cdot, w(\cdot)),\ i \in \{1, \cdots, s-1\}$. For any ball $B(x,r) \subset \Omega$ with sufficiently small radius we have, in view of Proposition 1,

$$\begin{aligned}\int_{B(x,r)} f_i(y, w(y))dy &\le \liminf_{m\to\infty} \int_{B(x,r)} f_i(y, w_m(y))dy \\ &\le \liminf_{m\to\infty} \int_{B(x,r)} \|f_i(\cdot, w_m(\cdot))\|_{L^\infty(\Omega)} dy,\end{aligned}$$

for every $i \in \{1, \cdots, s-1\}$. Thus,

$$\frac{1}{|B(x,r)|} \int_{B(x,r)} f_i(y, w(y))dy \le \liminf_{m\to\infty} \|f_i(\cdot, w_m(\cdot))\|_{L^\infty(\Omega)}.$$

Since almost every $x \in \Omega$ is a Lebesque point for $f_i(\cdot, w(\cdot))$, passing to the limit $r \to 0^+$ in the above inequality yields

$$f_i(x, w(x)) \leq \liminf_{k\to\infty} \|f_i(\cdot, w_m(\cdot))\|_{L^\infty(\Omega)} \quad \text{for a.e. } x \in \Omega.$$

We deduce that

$$\|f_i(\cdot, w(\cdot))\|_{L^\infty(\Omega)} \leq \liminf_{m\to\infty} \|f_i(\cdot, w_m(\cdot))\|_{L^\infty(\Omega)},$$

for every $i \in \{1, \cdots, s-1\}$. Thus,

$$\begin{aligned}\|f_i(\cdot, w(\cdot))\|_{L^\infty(\Omega)} &\leq \liminf_{m\to\infty} \max\left\{\operatorname*{ess\,sup}_{x\in\Omega} f_i(x, w_m(x)),\ i \in \overline{1, s-1}, \frac{1}{m} \operatorname*{ess\,sup}_{x\in\Omega} f_s(x, w_m(x))\right\} \\ &= \lim_{m\to\infty} F_m(w_m).\end{aligned}$$

Hence, (42) holds.

To prove (ii), let $w \in L^\infty(\Omega; \mathbb{R}^d)$, and note that since we only need to prove that

$$\limsup_{m\to\infty} F_m(w_m) \leq F_\infty(w),$$

we may assume, without loss of generality, that $F_\infty(w) < +\infty$, and thus $w \in L^\infty(\Omega; \mathbb{R}^d)$. It is now easy to show that the constant sequence $\{w_m\} = \{w\}$ is a recovery sequence for the Γ-limit. Indeed, since by the growth condition (9) we have $f_s(\cdot, w(\cdot)) \in L^\infty(\Omega)$, it follows that

$$\max\left\{\operatorname*{ess\,sup}_{x\in\Omega} f_i(x, w(x)),\ i \in \overline{1, s-1}, \frac{1}{m} \operatorname*{ess\,sup}_{x\in\Omega} f_s(x, w(x))\right\} = \max_{i\in\{1,\cdots,s-1\}} \operatorname*{ess\,sup}_{x\in\Omega} f_i(x, w(x))$$

for all $m \in \mathbb{N}$ sufficiently large. Thus,

$$\lim_{m\to\infty} F_m(w_m) = \lim_{m\to\infty} F_m(w) = F_\infty(w),$$

which concludes the proof. $\square$

It remains an open problem to determine whether the effective yield set for the Hutchinson's model can be characterized in terms of a variational principle involving supremal functionals which only depend on the mappings $f_1, f_2, \cdots, f_6$ as defined in the previous section. We conjecture that

$$\eta \in K_{\text{eff}} \text{ if and only if } f_\infty^{\text{eff}}(\eta) \leq 1,$$

where $f_\infty^{\text{eff}}(\eta)$ is given in terms of the Γ-limit, F_∞, defined in the statement of Theorem 2 (with $s = 6$ and $\mathcal{A} = \text{Div}$) by the formula

$$f_\infty^{\text{eff}}(\eta) := \inf \left\{ F_\infty(w(\cdot) + \eta) \ : w \in L^\infty(\Omega; \mathbb{M}^{3\times 3}_{\text{sym}}), \ \int_\Omega w(x)\ dx = 0, \ \text{Div } w = 0 \right\}.$$

4. THE NEUMANN EIGENVALUE PROBLEM

4.1. The Neumann Eigenvalue Problem for the $p(x)$-Laplacian

Let Ω be an open bounded domain with smooth boundary, and consider the Neumann eigenvalue problem for the $p(x)$-Laplacian

$$\begin{cases} -\Delta_{p(x)} u = \Lambda_{p(\cdot)} |u|^{p(x)-2} u & \text{in } \Omega \\ \frac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega, \end{cases} \tag{43}$$

where $\eta = \eta(x)$ stands for the outer unit normal to $\partial\Omega$ at $x \in \partial\Omega$.

Definition 4. *We say that $u \in W^{1,p(\cdot)}(\Omega)$ is a weak solution for the Neumann eigenvalue problem (43) if there exists $\Lambda_{p(\cdot)} \in \mathbb{R}$ such that*

$$\int_\Omega |\nabla u|^{p(x)-2} \nabla u \cdot \nabla v \; dx = \Lambda_{p(\cdot)} \int_\Omega |u|^{p(x)-2} uv \; dx, \quad \forall \; v \in W^{1,p(\cdot)}(\Omega). \tag{44}$$

If $u \neq 0$ we say that $\Lambda_{p(\cdot)}$ is an eigenvalue of (43), and that u is an eigenfunction corresponding to $\Lambda_{p(\cdot)}$.

Let $X := W^{1,p(\cdot)}(\Omega)$, and define the functionals $\mathcal{F}, \mathcal{G} : X \to \mathbb{R}$ by

$$\mathcal{F}(u) = \int_\Omega \frac{1}{p(x)} (|\nabla u|^{p(x)} + |u|^{p(x)}) dx \text{ and } \mathcal{G}(u) = \int_\Omega \frac{1}{p(x)} |u|^{p(x)} \; dx. \tag{45}$$

It is easy to see that $\mathcal{F}, \mathcal{G} \in C^1(X; \mathbb{R})$, and that for all $v \in X$ we have

$$\langle \mathcal{G}'(u), v \rangle_{X',X} = \int_\Omega |u|^{p(x)-2} uv \; dx$$

and

$$\langle \mathcal{F}'(u), v \rangle_{X',X} = \int_\Omega (|\nabla u|^{p(x)-2} \nabla u \cdot \nabla v + |u|^{p(x)-2} uv) \; dx,$$

where $\langle \cdot, \cdot \rangle_{X',X}$ stands for the usual duality pairing of X and X' (the topological dual of X). Consider the level set $S_{\mathcal{G}} := \{u \in X \ : \ \mathcal{G}(u) = 1\}$, and the eigenvalue problem

$$\mathcal{F}'(u) = \mu \mathcal{G}'(u), \quad u \in S_{\mathcal{G}}, \quad \mu \in \mathbb{R}. \tag{46}$$

The existence of a sequence of nonnegative eigenvalues $\mu_n \to 0^+$ as $n \to \infty$ for the problem (46) was established in [26]. It follows from the Ljusternik-Schnirelman theory (see, e.g., [11], [48]). We have $\mu_n = \sup\limits_{A \in \mathbb{A}_n} \inf\limits_{u \in A} \mathcal{F}(u)$, with

$$\mathbb{A}_n := \{A \subset S_{\mathcal{G}} \ : \ \mathcal{F}(u) > 0 \text{ on } A, \ A \text{ compact}, \ A = -A, \ \gamma(A) \geq n\},$$

where

$$\gamma(A) := \inf \left\{ k \in \mathbb{N} \ \mid \ \exists \ \ h : A \to \mathbb{R}^k \backslash \{0\}, \quad h \ \text{ odd and continuous} \right\}$$

is the genus of A. The eigenfunctions $u \in S_{\mathcal{G}}$ satisfy $\mathcal{F}'(u) = \mu \mathcal{G}'(u)$ or, equivalently, $\langle \mathcal{F}'(u), v \rangle_{X',X} = \mu \langle \mathcal{G}'(u), v \rangle_{X',X}$ for all $v \in X$. Hence,

$$\int\limits_{\Omega} |\nabla u|^{p(x)-2} \nabla u \cdot \nabla v \ dx = (\mu - 1) \int\limits_{\Omega} |u|^{p(x)-2} uv \ dx$$

for all $v \in W^{1,p(\cdot)}(\Omega)$, which means that u is a weak solution of problem (43) with $\Lambda_{p(\cdot)} = \mu - 1$.

The following definition of viscosity solutions for second-order elliptic equations with fully nonlinear boundary conditions can be found in [4] (see also [12]).

Definition 5. *Consider the boundary value problem*

$$\begin{cases} F(x, u, Du, D^2u) = 0 & \text{in } \Omega \\ H(x, u, Du) = 0 & \text{on } \partial\Omega. \end{cases} \tag{47}$$

(1) An upper semi-continuous function u is a viscosity subsolution of (47) if for every $\psi \in C^2(\overline{\Omega})$ such that $u - \psi$ has a maximum at the point $x_0 \in \overline{\Omega}$ with $u(x_0) = \psi(x_0)$ we have:

$$F(x_0, \psi(x_0), D\psi(x_0), D^2\psi(x_0)) \leq 0 \text{ if } x_0 \in \Omega,$$

and

$$\min\left\{H(x_0, \psi(x_0), D\psi(x_0)), F(x_0, \psi(x_0), D\psi(x_0), D^2\psi(x_0))\right\} \leq 0 \text{ if } x_0 \in \partial\Omega.$$

(2) A lower semi-continuous function u is a viscosity supersolution of (47) if for every $\varphi \in C^2(\overline{\Omega})$ such that $u - \varphi$ has a minimum at the point $x_0 \in \overline{\Omega}$ with $u(x_0) = \varphi(x_0)$ we have:

$$F(x_0, \varphi(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq 0 \text{ if } x_0 \in \Omega,$$

and

$$\max\left\{H(x_0, \varphi(x_0), D\varphi(x_0)), F(x_0, \varphi(x_0), D\varphi(x_0), D^2\varphi(x_0))\right\} \geq 0 \text{ if } x_0 \in \partial\Omega.$$

(3) We say that a continuous function u is a viscosity solution of (47) if it is both a viscosity subsolution and a viscosity supersolution of (47).

Remark 1. *As remarked in [4], if $H(x, r, \cdot)$ is strictly increasing in the normal direction to $\partial\Omega$ at x, that is, for all $R > 0$ there exists $\nu_R > 0$ such that*

$$H(x, r, \theta + \lambda\eta(x)) - H(x, r, \theta) \geq \nu_R\lambda \ \forall \ (x, r, \theta) \in \partial\Omega \times [-R, R] \times \mathbb{R}^N \text{ and } \lambda > 0, \quad (48)$$

the definitions of viscosity sub and supersolutions for problem (47) in Definition 5 *take a simpler form. Precisely,*

(1) *If u is a viscosity subsolution and $\psi \in C^2(\overline{\Omega})$ is such that $u - \psi$ has a maximum at the point $x_0 \in \overline{\Omega}$ with $u(x_0) = \psi(x_0)$ we have:*

$$F(x_0, \psi(x_0), D\psi(x_0), D^2\psi(x_0)) \leq 0 \text{ if } x_0 \in \Omega,$$

and

$$H(x_0, \psi(x_0), D\psi(x_0)) \leq 0 \text{ if } x_0 \in \partial\Omega.$$

(2) *If u is a viscosity supersolution and $\varphi \in C^2(\overline{\Omega})$ is such that $u - \varphi$ has a minimum at the point x_0 with $u(x_0) = \varphi(x_0)$, then*

$$F(x_0, \varphi(x_0), D\varphi(x_0), D^2\varphi(x_0)) \geq 0 \text{ if } x_0 \in \Omega,$$

and

$$H(x_0, \varphi(x_0), D\varphi(x_0)) \geq 0 \text{ if } x_0 \in \partial\Omega.$$

Our next goal in this section is to prove that continuous weak solutions of (43) are, in fact, viscosity solutions (see Proposition 2 below). Before we proceed, we note that the Neumann eigenvalue problem (43) takes the form (47), with the functions $F : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{M}_{\text{sym}}^{N \times N} \to \mathbb{R}$ and $H : \partial\Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ defined by

$$F(x, r, \theta, S) = -|\theta|^{p(x)-2} \left(\text{Tr}(S) + \ln|\theta| \langle \theta, \nabla p(x) \rangle\right) - (p(x) - 2)|\theta|^{p(x)-4} \langle S\theta, \theta \rangle - \Lambda_{p(\cdot)}|r|^{p(x)-2}$$

and

$$H(x, r, \theta) = \langle \theta, \eta \rangle,$$

where $\mathbb{M}_{\text{sym}}^{N\times N}$ is the space of $N \times N$ symmetric matrices, $\text{Tr}(S)$ stands for the trace of the matrix $S \in \mathbb{M}_{\text{sym}}^{N\times N}$, and $\langle \cdot, \cdot \rangle$ denotes the inner product in $\mathbb{R}^N$. Note that the function H defined above satisfies the strict monotonicity condition in Remark 1 with $\nu_R = 1$, since in this case we have

$$H(x, r, \theta + \lambda\eta(x)) - H(x, r, \theta) = \langle \theta + \lambda\eta(x), \eta(x)\rangle - \langle \theta, \eta(x)\rangle = \lambda|\eta(x)|^2 \geq \lambda$$

for all $(x, r, \theta) \in \partial\Omega \times [-R, R] \times \mathbb{R}^N$ and $\lambda > 0$.

Proposition 2. *Any continuous weak solution of (*43*) is a viscosity solution of (*43*).*

Proof. Let $u \in C(\overline{\Omega})$ be a weak solution of (43). To show that u is a viscosity supersolution of (43), let $x_0 \in \overline{\Omega}$, and consider a test function $\varphi \in C^2(\overline{\Omega})$ such that $u(x_0) = \varphi(x_0)$ and $u - \varphi$ has a minimum at x_0. If $x_0 \in \Omega$, we claim that we have

$$-\Delta_{p(x_0)}\varphi(x_0) - \Lambda_{p(\cdot)}|\varphi(x_0)|^{p(x_0)-2}\varphi(x_0) \geq 0.$$

Indeed, if we assume that this inequality does not hold, then there exists $r > 0$ such that $B(x_0, r) \subset \Omega$ and

$$-\Delta_{p(x)}\varphi(x) - \Lambda_{p(\cdot)}|\varphi(x)|^{p(x)-2}\varphi(x) < 0 \text{ for all } x \in B(x_0, r).$$

Taking r smaller, if necessary, we may assume that $u > \varphi$ in $B(x_0, r) \setminus \{x_0\}$. Let

$$m = \inf_{x\in\partial B(x_0,r)} (u - \varphi)(x) > 0,$$

and $\Phi(x) := \varphi(x) + \frac{m}{2}$. Note that $\Phi(x_0) > u(x_0)$, $\Phi(x) < u(x)$ for all $x \in \partial B(x_0, r)$, and

$$-\Delta_{p(x)}\Phi(x) - \Lambda_{p(\cdot)}|\varphi(x)|^{p(x)-2}\varphi(x) < 0 \text{ for all } x \in B(x_0, r). \tag{49}$$

Multiply (49) by $(\Phi - u)^+$ and integrate over $B(x_0, r)$ to get

$$\int\limits_{\{x\in B(x_0,r):\Phi(x)>u(x)\}} |\nabla\Phi|^{p(x)-2}\nabla\Phi\cdot\nabla(\Phi-u)dx$$

$$< \int\limits_{\{x\in B(x_0,r):\Phi(x)>u(x)\}} \Lambda_{p(\cdot)}|\varphi|^{p(x)-2}\varphi(\Phi-u)dx, \quad (50)$$

where we have used the fact that $(\Phi - u)^+ = 0$ on $\partial B(x_0, r)$. Extending $(\Phi - u)^+$ by zero outside $B(x_0, r)$, and using this extension as a test function in the weak formulation (44) gives

$$\int\limits_{\{x\in B(x_0,r):\Phi(x)>u(x)\}} |\nabla u|^{p(x)-2}\nabla u\cdot\nabla(\Phi-u)dx$$

$$= \int\limits_{\{x\in B(x_0,r):\Phi(x)>u(x)\}} \Lambda_{p(\cdot)}|u|^{p(x)-2}u(\Phi-u)dx. \quad (51)$$

After subtracting (51) from (50), using the fact that $u > \varphi$ on $B(x_0, r) \setminus \{x_0\}$, and using the elementary inequality (see, e.g., Chapter I in [18])

$$|a-b|^p \le 2^{p-1}\left(|a|^{p-2}a - |b|^{p-2}b\right)\cdot(a-b) \quad \text{for all } a, b \in \mathbb{R}^N \text{ and } p \ge 2, \qquad (52)$$

we obtain

$$\begin{aligned} 0 > &\int\limits_{\{x\in B(x_0,r):\Phi(x)>u(x)\}} \left(|\nabla\Phi|^{p(x)-2}\nabla\Phi - |\nabla u|^{p(x)-2}\nabla u\right)\cdot\nabla(\Phi-u)\,dx \\ &\qquad + \int\limits_{\{x\in B(x_0,r):\Phi(x)>u(x)\}} \Lambda_{p(\cdot)}\left(|u|^{p(x)-2}u - |\varphi|^{p(x)-2}\varphi\right)(\Phi-u)\,dx \\ \ge &\int\limits_{\{x\in B(x_0,r):\Phi(x)>u(x)\}} \left(|\nabla\Phi|^{p(x)-2}\nabla\Phi - |\nabla u|^{p(x)-2}\nabla u\right)\cdot\nabla(\Phi-u)\,dx \\ \ge &\frac{1}{2^{p^+-1}} \int\limits_{\{x\in B(x_0,r):\Phi(x)>u(x)\}} |\nabla\Phi-\nabla u|^{p(x)}\,dx \ge 0, \end{aligned}$$

which is clearly a contradiction. On the other hand, if $x_0 \in \partial\Omega$ we need to prove that

$$\max\left\{\frac{\partial\varphi}{\partial\eta}(x_0), -\Delta_{p(x_0)}\varphi(x_0) - \Lambda_{p(\cdot)}|\varphi(x_0)|^{p(x_0)-2}\varphi(x_0)\right\} \geq 0. \tag{53}$$

We proceed by contradiction. Assume that (53) does not hold. Then there exists $r > 0$ sufficiently small such that

$$\frac{\partial\varphi}{\partial\eta}(x) < 0 \tag{54}$$

and

$$-\Delta_{p(x)}\varphi(x) - \Lambda_{p(\cdot)}|\varphi(x)|^{p(x)-2}\varphi(x) < 0, \tag{55}$$

for all $x \in B(x_0, r) \cap \Omega$. For $r > 0$ sufficiently small we have $u(x) > \varphi(x)$ for all $x \in \left(\overline{B(x_0, r)} \setminus \{x_0\}\right) \cap \Omega$ and thus

$$m := \inf_{\partial B(x_0,r)\cap\overline{\Omega}} (u - \varphi)(x) > 0.$$

With $\Phi(x) := \varphi(x) + \frac{m}{2}$, note that $\Phi(x_0) > u(x_0)$, and that $\Phi(x) < u(x)$ for all $x \in \partial B(x_0, r) \cap \overline{\Omega}$. Multiplying (55) by $(\Phi - u)^+$ and integrating over $B(x_0, r) \cap \Omega$ gives

$$\int\limits_{B(x_0,r)\cap\Omega} |\nabla\varphi|^{p(x)-2}\nabla\varphi \cdot \nabla(\Phi - u)^+ \, dx - \int\limits_{\partial(B(x_0,r)\cap\Omega)} |\nabla\varphi|^{p(x)-2}\frac{\partial\varphi}{\partial\eta}(\Phi - u)^+ \, dx <$$

$$< \int\limits_{B(x_0,r)\cap\Omega} \Lambda_{p(\cdot)}|\varphi|^{p(x)-2}\varphi(\Phi - u)^+ \, dx. \tag{56}$$

Since $(\Phi - u)^+ = 0$ on $\partial B(x_0, r) \cap \overline{\Omega}$, we have

$$\int_{\partial(B(x_0,r)\cap\Omega)} |\nabla\varphi|^{p(x)-2}\frac{\partial\varphi}{\partial\eta}(\Phi - u)^+ \, dx = \int_{B(x_0,r)\cap\partial\Omega} |\nabla\varphi|^{p(x)-2}\frac{\partial\varphi}{\partial\eta}(\Phi - u)^+ \, dx.$$

Thus,

$$\begin{aligned}\int_{\{x\in B(x_0,r):\Phi(x)>u(x)\}\cap\Omega} & |\nabla\Phi|^{p(x)-2}\nabla\Phi\cdot\nabla(\Phi - u) \, dx \\ & < \int_{\{x\in B(x_0,r):\Phi(x)>u(x)\}\cap\partial\Omega} |\nabla\varphi|^{p(x)-2}\frac{\partial\varphi}{\partial\eta}(\Phi - u) \, dx \\ & \quad + \int_{\{x\in B(x_0,r):\Phi(x)>u(x)\}\cap\Omega} \Lambda_{p(\cdot)}|\varphi|^{p(x)-2}\varphi(\Phi - u) \, dx. \quad (57)\end{aligned}$$

Using the extension of $(\Phi - u)^+$ by zero outside $B(x_0, r) \cap \Omega$ as a test function in (44), we obtain

$$\begin{aligned}\int_{\{x\in B(x_0,r):\Phi(x)>u(x)\}\cap\Omega} & |\nabla u|^{p(x)-2}\nabla u\cdot\nabla(\Phi - u) \, dx \\ & = \int_{\{x\in B(x_0,r):\Phi(x)>u(x)\}\cap\Omega} \Lambda_{p(\cdot)}|u|^{p(x)-2}u(\Phi - u) \, dx. \quad (58)\end{aligned}$$

Thus, subtracting (58) from (57) leads to

$$\begin{aligned}\int_{\{x\in B(x_0,r):\Phi(x)>u(x)\}\cap\Omega} & \left(|\nabla\Phi|^{p(x)-2}\nabla\Phi - |\nabla u|^{p(x)-2}\nabla u\right)\cdot\nabla(\Phi - u) \, dx \\ & < \int_{\{x\in B(x_0,r):\Phi(x)>u(x)\}\cap\Omega} \Lambda_{p(\cdot)}\left(|\varphi|^{p(x)-2}\varphi - |u|^{p(x)-2}u\right)(\Phi - u) \, dx \\ & \quad + \int_{\{x\in B(x_0,r):\Phi(x)>u(x)\}\cap\partial\Omega} |\nabla\varphi|^{p(x)-2}\frac{\partial\varphi}{\partial\eta}(\Phi - u) \, dx.\end{aligned}$$

Since $r > 0$ was chosen sufficiently small so that (54) holds, we obtain that

$$\int_{\{x\in B(x_0,r):\Phi(x)>u(x)\}\cap\partial\Omega} |\nabla\varphi|^{p(x)-2}\frac{\partial\varphi}{\partial\eta}(\Phi-u)\ dx \le 0.$$

Thus,

$$\begin{aligned}\int_{\{x\in B(x_0,r):\Phi(x)>u(x)\}\cap\Omega} &\left(|\nabla\Phi|^{p(x)-2}\nabla\Phi-|\nabla u|^{p(x)-2}\nabla u\right)\cdot\nabla(\Phi-u)\ dx\\ &< \int_{\{x\in B(x_0,r):\Phi(x)>u(x)\}\cap\Omega} \Lambda_{p(\cdot)}\left(|\varphi|^{p(x)-2}\varphi-|u|^{p(x)-2}u\right)(\Phi-u)\ dx \le 0, \quad (59)\end{aligned}$$

where the last inequality follows from the fact that $u \ge \varphi$ on $B(x_0, r)\cap\Omega$. Applying (52) again, we deduce that

$$\begin{aligned}\frac{1}{2^{p^+-1}}\int_{\{x\in B(x_0,r):\Phi(x)>u(x)\}\cap\Omega} &|\nabla\Phi-\nabla u|^{p(x)}\ dx\\ &\le \int_{\{x\in B(x_0,r):\Phi(x)>u(x)\}\cap\Omega}\left(|\nabla\Phi|^{p(x)-2}\nabla\Phi-|\nabla u|^{p(x)-2}\nabla u\right)\cdot\nabla(\Phi-u)\ dx. \quad (60)\end{aligned}$$

Combining (59) and (60) gives

$$\int_{\{x\in B(x_0,r):\Phi(x)>u(x)\}\cap\Omega} |\nabla\Phi-\nabla u|^{p(x)}\ dx < 0,$$

which is a contradiction. We conclude that u is a viscosity supersolution of (43).

Next, we show that u is a viscosity subsolution of (43). Let $x_0 \in \overline{\Omega}$, and consider a test function $\psi \in C^2(\overline{\Omega})$ such that $u(x_0) = \psi(x_0)$ and $u - \psi$ has a maximum at x_0. If $x_0 \in \Omega$, we claim that we have

$$-\Delta_{p(x_0)}\psi(x_0) - \Lambda_{p(\cdot)}|\psi(x_0)|^{p(x_0)-2}\psi(x_0) \le 0.$$

Assuming that the above inequality does not hold, there exists $r > 0$ such that $B(x_0, r) \subset \Omega$ and

$$-\Delta_{p(x)}\psi(x) - \Lambda_{p(\cdot)}|\psi(x)|^{p(x)-2}\psi(x) > 0 \text{ for all } x \in B(x_0, r).$$

Taking r small, we may assume that $u < \psi$ in $B(x_0, r) \setminus \{x_0\}$. Let

$$m = \inf_{x \in \partial B(x_0,r)} (\psi - u)(x) > 0,$$

and set $\Psi(x) := \psi(x) - \frac{m}{2}$. Note that $\Psi(x_0) < u(x_0)$, $\Psi(x) > u(x)$ for all $x \in \partial B(x_0, r)$, and

$$-\Delta_{p(x)}\Psi(x) - \Lambda_{p(\cdot)}|\psi(x)|^{p(x)-2}\psi(x) > 0 \text{ for all } x \in B(x_0, r). \tag{61}$$

If we multiply (61) by $(\Psi - u)^-$ and integrate over $B(x_0, r)$, we obtain

$$\int_{\{x\in B(x_0,r):\Psi(x)<u(x)\}} |\nabla\Psi|^{p(x)-2}\nabla\Psi \cdot \nabla(u - \Psi)dx$$
$$> \int_{\{x\in B(x_0,r):\Psi(x)<u(x)\}} \Lambda_{p(\cdot)}|\psi|^{p(x)-2}\psi(u - \Psi)dx, \tag{62}$$

where we have used the fact that $(\Psi - u)^- = 0$ on $\partial B(x_0, r)$. Taking $(\Psi - u)^-$, extended by zero outside $B(x_0, r)$, as a test function in the weak formulation (44) gives

$$\int_{\{x\in B(x_0,r):\Psi(x)<u(x)\}} |\nabla u|^{p(x)-2}\nabla u \cdot \nabla(u - \Psi)dx$$
$$= \int_{\{x\in B(x_0,r):\Psi(x)<u(x)\}} \Lambda_{p(\cdot)}|u|^{p(x)-2}u(u - \Psi)dx. \tag{63}$$

After subtracting (62) from (63), using the fact that $u < \psi$ on $B(x_0, r) \setminus \{x_0\}$, and using (52) we obtain

$$
\begin{aligned}
0 > & \int_{\{x \in B(x_0,r) : \Psi(x) < u(x)\}} \left(|\nabla u|^{p(x)-2}\nabla u - |\nabla \Psi|^{p(x)-2}\nabla \Psi\right) \cdot \nabla(u - \Psi)\, dx \\
& \quad + \int_{\{x \in B(x_0,r) : \Psi(x) < u(x)\}} \Lambda_{p(\cdot)} \left(|\psi|^{p(x)-2}\psi - |u|^{p(x)-2}u\right)(u - \Psi)\, dx \\
\geq & \int_{\{x \in B(x_0,r) : \Psi(x) < u(x)\}} \left(|\nabla u|^{p(x)-2}\nabla u - |\nabla \Psi|^{p(x)-2}\nabla \Psi\right) \cdot \nabla(u - \Psi)\, dx \\
\geq & \frac{1}{2^{p^+ - 1}} \int_{\{x \in B(x_0,r) : \Psi(x) < u(x)\}} |\nabla u - \nabla \Psi|^{p(x)}\, dx \geq 0,
\end{aligned}
$$

which is a contradiction. Now, if $x_0 \in \partial\Omega$ we need to prove that

$$
\min\left\{ \frac{\partial \psi}{\partial \eta}(x_0), -\Delta_{p(x_0)}\psi(x_0) - \Lambda_{p(\cdot)}|\psi(x_0)|^{p(x_0)-2}\psi(x_0) \right\} \leq 0. \tag{64}
$$

We proceed by contradiction. Assume that (64) does not hold. Then there exists $r > 0$ sufficiently small such that

$$
\frac{\partial \psi}{\partial \eta}(x) > 0 \tag{65}
$$

and

$$
-\Delta_{p(x)}\psi(x) - \Lambda_{p(\cdot)}|\psi(x)|^{p(x)-2}\psi(x) > 0, \tag{66}
$$

for all $x \in B(x_0, r) \cap \Omega$. For $r > 0$ sufficiently small we have $u(x) < \psi(x)$ for all $x \in \left(\overline{B(x_0, r)} \setminus \{x_0\}\right) \cap \Omega$ and thus

$$
m := \inf_{\partial B(x_0,r) \cap \overline{\Omega}} (\psi - u)(x) > 0.
$$

Let $\Psi(x) := \psi(x) - \frac{m}{2}$, note that $\Psi(x_0) < u(x_0)$, and that $\Psi(x) > u(x)$ for all $x \in \partial B(x_0, r) \cap \overline{\Omega}$. Multiplying (66) by $(\Psi - u)^-$ and integrating over $B(x_0, r) \cap \Omega$ gives

$$\int_{B(x_0,r)\cap\Omega} |\nabla\psi|^{p(x)-2}\nabla\psi \cdot \nabla(\Psi - u)^- \, dx - \int_{\partial(B(x_0,r)\cap\Omega)} |\nabla\psi|^{p(x)-2}\frac{\partial\psi}{\partial\eta}(\Psi - u)^- \, dx$$
$$> \int_{B(x_0,r)\cap\Omega} \Lambda_{p(\cdot)}|\psi|^{p(x)-2}\psi(\Psi - u)^- \, dx. \quad (67)$$

Since $(\Psi - u)^- = 0$ on $\partial B(x_0, r) \cap \overline{\Omega}$, we have

$$\int_{\partial(B(x_0,r)\cap\Omega)} |\nabla\psi|^{p(x)-2}\frac{\partial\psi}{\partial\eta}(\Psi - u)^- \, dx = \int_{B(x_0,r)\cap\partial\Omega} |\nabla\psi|^{p(x)-2}\frac{\partial\psi}{\partial\eta}(\Psi - u)^- \, dx.$$

Thus,

$$\int_{\{x\in B(x_0,r):\Psi(x)<u(x)\}\cap\Omega} |\nabla\Psi|^{p(x)-2}\nabla\Psi \cdot \nabla(u - \Psi) \, dx$$
$$> \int_{\{x\in B(x_0,r):\Psi(x)<u(x)\}\cap\partial\Omega} |\nabla\psi|^{p(x)-2}\frac{\partial\psi}{\partial\eta}(u - \Psi) \, dx$$
$$+ \int_{\{x\in B(x_0,r):\Psi(x)<u(x)\}\cap\Omega} \Lambda_{p(\cdot)}|\psi|^{p(x)-2}\psi(u - \Psi) \, dx. \quad (68)$$

Using the extension of $(\Psi - u)^-$ by zero outside $B(x_0, r) \cap \Omega$ as a test function in (44), we obtain

$$\int_{\{x\in B(x_0,r):\Psi(x)<u(x)\}\cap\Omega} |\nabla u|^{p(x)-2}\nabla u \cdot \nabla(u - \Psi) \, dx$$
$$= \int_{\{x\in B(x_0,r):\Psi(x)<u(x)\}\cap\Omega} \Lambda_{p(\cdot)}|u|^{p(x)-2}u(u - \Psi) \, dx. \quad (69)$$

After subtracting (68) from (69) we have

$$\int\limits_{\{x\in B(x_0,r):\Psi(x)<u(x)\}\cap\Omega} \left(|\nabla u|^{p(x)-2}\nabla u - |\nabla\Psi|^{p(x)-2}\nabla\Psi\right)\cdot\nabla(u-\Psi)\ dx$$
$$< \int\limits_{\{x\in B(x_0,r):\Psi(x)<u(x)\}\cap\Omega} \Lambda_{p(\cdot)}\left(|u|^{p(x)-2}u - |\psi|^{p(x)-2}\psi\right)(u-\Psi)\ dx$$
$$- \int\limits_{\{x\in B(x_0,r):\Psi(x)<u(x)\}\cap\partial\Omega} |\nabla\psi|^{p(x)-2}\frac{\partial\psi}{\partial\eta}(u-\Psi)\ dx.$$

Since $r>0$ was chosen sufficiently small so that (65) holds, we obtain that

$$\int\limits_{\{x\in B(x_0,r):\Psi(x)<u(x)\}\cap\partial\Omega} |\nabla\psi|^{p(x)-2}\frac{\partial\psi}{\partial\eta}(u-\Psi)\ dx \geq 0.$$

Thus,

$$\int\limits_{\{x\in B(x_0,r):\Psi(x)<u(x)\}\cap\Omega} \left(|\nabla u|^{p(x)-2}\nabla u - |\nabla\Psi|^{p(x)-2}\nabla\Psi\right)\cdot\nabla(u-\Psi)\ dx$$
$$< \int\limits_{\{x\in B(x_0,r):\Psi(x)<u(x)\}\cap\Omega} \Lambda_{p(\cdot)}\left(|u|^{p(x)-2}u - |\psi|^{p(x)-2}\psi\right)(u-\Psi)\ dx \leq 0, \quad (70)$$

where the last inequality follows from the fact that $u\leq\psi$ on $B(x_0,r)\cap\Omega$. Applying (52) again, we deduce that

$$\frac{1}{2^{p^+-1}} \int\limits_{\{x\in B(x_0,r):\Psi(x)<u(x)\}\cap\Omega} |\nabla u-\nabla\Psi|^{p(x)}\ dx$$
$$\leq \int\limits_{\{x\in B(x_0,r):\Psi(x)<u(x)\}\cap\Omega} \left(|\nabla u|^{p(x)-2}\nabla u - |\nabla\Psi|^{p(x)-2}\nabla\Psi\right)\cdot\nabla(u-\Psi)\ dx. \quad (71)$$

Combining (70) and (71) gives

$$\int\limits_{\{x\in B(x_0,r):\Psi(x)<u(x)\}\cap\Omega} |\nabla u-\nabla\Psi|^{p(x)}\ dx < 0,$$

which is a contradiction. Thus u is a viscosity subsolution of (43).

4.2. The Asymptotic Behavior of the Second Eigenvalue/Eigenfunction Pairs

Consider a sequence of functions $\{p_n\} \subset C^1(\overline{\Omega})$ such that

$$1 < p_n^- := \min_{x \in \overline{\Omega}} p_n(x) \le p_n^+ := \max_{x \in \overline{\Omega}} p_n(x) < \infty, \ \forall \ n \in \mathbb{N}, \tag{72}$$

and satisfying the following assumptions

$$p_n \to \infty \ \text{ uniformly in } \ \overline{\Omega}, \tag{73}$$

$$\nabla \ln p_n \to \xi \ \text{ uniformly in } \ \overline{\Omega}, \tag{74}$$

and

$$\frac{p_n}{n} \to q \ \text{ uniformly in } \ \overline{\Omega}, \tag{75}$$

where $\xi \in C(\overline{\Omega}, \mathbb{R}^N)$, and $q \in C(\overline{\Omega}, (0, +\infty))$ is such that $q^- := \min_{x \in \overline{\Omega}} q(x) > 0$. Note that by (75) we have

$$\lim_{n \to \infty} \frac{p_n^-}{n} = q^-, \quad \lim_{n \to \infty} \frac{p_n^+}{n} = q^+ := \max_{x \in \overline{\Omega}} q(x). \tag{76}$$

Particular examples of sequences of functions $\{p_n\}$ which satisfy our assumptions are $p_n(x) = n$ (in which case $\xi(x) = 0, q(x) = 1$) corresponding to constant exponents and, for a suitably chosen $p \in C^1(\overline{\Omega})$, $p_n(x) = np(x)$ (in which case $\xi(x) = \nabla(\ln p(x))$ and $q(x) = p(x)$), $p_n(x) = np(x/n)$ $(\xi(x) = 0, q(x) = p(0))$, or $p_n(x) = n + p(x/n)$ $(\xi(x) = 0, q(x) = 1)$. We refer to [40], [44], or [45] for additional examples.

It was shown in [26, Theorem 3.2] that the first eigenvalue of the $p(x)$-Laplacian with Neumann boundary condition is zero, and that the second eigenvalue is strictly greater than the first eigenvalue. It is also known that the eigenfunctions do not change sign in Ω. In this section we analyze the asymptotic behavior of the positive second eigenfunctions of the $p_n(x)$-Laplacian with Neumann boundary conditions:

$$\begin{cases} -\Delta_{p_n(x)} u = \Lambda_{p_n(\cdot)} |u|^{p_n(x)-2} u & \text{in } \Omega \\ \frac{\partial u}{\partial \eta} = 0 & \text{on } \partial\Omega, \end{cases} \tag{77}$$

as $n \to \infty$. In what follows, we will denote the positive second eigenvalues by Λ_n^2. Following [26], they are given by

$$\Lambda_n^2 = \frac{\int_\Omega |\nabla u_n|^{p_n(x)} dx}{\int_\Omega |u_n|^{p_n(x)} dx}, \; n \in \mathbb{N}, \tag{78}$$

where $u_n \in W^{1,p_n(\cdot)}(\Omega)$ is the eigenfunction associated to Λ_n^2, a minimizer of the functional

$$W^{1,p_n(\cdot)}(\Omega) \ni u \mapsto \int_\Omega \frac{1}{p_n(x)} |\nabla u|^{p_n(x)} dx$$

among all $u \in W^{1,p_n(\cdot)}(\Omega)$ satisfying the constraint $\int_\Omega \frac{1}{p_n(x)} |u|^{p_n(x)} dx = 1$. For each $n \in \mathbb{N}$, we define

$$c_n^2 := \inf \left\{ \int_\Omega \frac{1}{p_n(x)} |\nabla u|^{p_n(x)} \;\; : u \in W^{1,p_n(\cdot)}(\Omega), \; \int_\Omega \frac{1}{p_n(x)} |u|^{p_n(x)} dx = 1 \right\}. \tag{79}$$

Proposition 3. *The sequence* $\left\{ (\Lambda_n^2)^{\frac{1}{n}} \right\}$ *is bounded.*

Proof. The proof follows closely the proof of Lemma 3.1 in [44]. In particular, since

$$c_n^2 \leq \inf \left\{ \int_\Omega \frac{1}{p_n(x)} |\nabla u|^{p_n(x)} \ : u \in W_0^{1,p_n(x)},\ \int_\Omega \frac{1}{p_n(x)} |u|^{p_n(x)} dx = 1 \right\},$$

it follows from [44] that $\left\{ (c_n^2)^{\frac{1}{n}} \right\}$ is bounded. Next, note that we have

$$\int_\Omega |u_n|^{p_n(x)} dx \geq \int_\Omega \frac{p_n^-}{p_n(x)} |u_n|^{p_n(x)} dx = p_n^- \int_\Omega \frac{|u_n|^{p_n(x)}}{p_n(x)} dx = p_n^-,$$

and thus, taking (78) into account, we obtain

$$\begin{aligned} 0 \leq \left(\Lambda_n^2\right)^{\frac{1}{n}} &\leq \left(\frac{1}{p_n^-}\right)^{\frac{1}{n}} \left(\int_\Omega |\nabla u_n|^{p_n(x)} dx \right)^{\frac{1}{n}} \leq \left(\frac{1}{p_n^-}\right)^{\frac{1}{n}} \left(\int_\Omega \frac{p_n^+}{p_n(x)} |\nabla u_n|^{p_n(x)} dx \right)^{\frac{1}{n}} \\ &= \left(\frac{p_n^+}{p_n^-}\right)^{\frac{1}{n}} \left(\int_\Omega \frac{|\nabla u_n|^{p_n(x)}}{p_n(x)} dx \right)^{\frac{1}{n}} = \left(\frac{p_n^+}{p_n^-}\right)^{\frac{1}{n}} \left(c_n^2\right)^{\frac{1}{n}} \end{aligned}$$

for all $n \in \mathbb{N}$. Since (74) implies the existence of a positive constant $C > 0$ such that the Harnack type inequality $p_n^+ \leq C p_n^-,\ \forall\, n \in \mathbb{N}$ holds (see [40] for details), we have

$$\lim_{n\to\infty} \left(\frac{p_n^+}{p_n^-}\right)^{\frac{1}{n}} = 1.$$

From the fact that the sequence $\left\{ (c_n^2)^{\frac{1}{n}} \right\}$ is bounded it now follows that $\left\{ (\Lambda_n^2)^{\frac{1}{n}} \right\}$ is also bounded, which concludes our proof. □

Theorem 3. *Let $\{p_n\}$ be a sequence of variable exponents satisfying (72)-(75) and, for $n \in \mathbb{N}$, let Λ_n^2 and $u_n \in W^{1,p_n(\cdot)}(\Omega)$ be the second eigenvalue and, respectively, the positive second eigenfunction corresponding to the Neumann problem (77). Then there exists $\Lambda_\infty \in \mathbb{R}$ and $u_\infty \in C(\overline{\Omega}) \setminus \{0\}$ such that, after eventually extracting a* *subsequence, we have*

$$\left(\Lambda_n^2\right)^{\frac{1}{n}} \to \Lambda_\infty \tag{80}$$

and

$$u_n \to u_\infty \ \text{ uniformly } \ \text{in } \ \overline{\Omega}, \tag{81}$$

as $n \to \infty$, where u_∞ is a nontrivial viscosity solution of the problem

$$\begin{cases} \min\left\{-\Delta_\infty u_\infty - |\nabla u_\infty|^2 \ln|\nabla u_\infty| \langle \xi, \nabla u_\infty\rangle, |\nabla u_\infty|^q - \Lambda_\infty |u_\infty|^q\right\} = 0 & \text{in } \Omega \\ \frac{\partial u_\infty}{\partial \eta} = 0 & \text{on } \partial\Omega. \end{cases} \tag{82}$$

Remark 2. *At points where the gradient is vanishing, the PDE in (82) is interpreted by assuming that the value of $v \mapsto |v|^2 \ln|v|$ at $v = 0$ is zero.*

Proof. Fix $m \in \mathbb{N}$ and choose $\varepsilon > 0$ such that $\varepsilon < q^-$. We have $\frac{p_n^-}{n} > q^- - \varepsilon > 0$ and $n > m$ for all $n \in \mathbb{N}$ sufficiently large. In view of Hölder's inequality, we have

$$\begin{aligned} \int_\Omega |u_n|^{\frac{mp_n(x)}{n}} dx &\leq \left(\int_\Omega |u_n|^{p_n(x)} dx\right)^{\frac{m}{n}} (|\Omega|)^{\frac{n-m}{n}} \leq \left(\int_\Omega |u_n|^{p_n(x)} dx\right)^{\frac{m}{n}} (|\Omega| + 1) \\ &\leq \left(p_n^+ \int_\Omega \frac{|u_n|^{p_n(x)}}{p_n(x)} dx\right)^{\frac{m}{n}} (|\Omega| + 1) = (p_n^+)^{\frac{m}{n}} (|\Omega| + 1). \end{aligned}$$

Since $\lim\limits_{n\to\infty} (p_n^+)^{\frac{m}{n}} = 1$, we obtain that

$$\int_\Omega |u_n|^{\frac{mp_n(x)}{n}} dx \quad \leq 2(|\Omega| + 1)$$

for $n \in \mathbb{N}$ sufficiently large. Using similar arguments we obtain, by Proposition 3, that there exists a constant $C = C(m) > 0$ such that

$$\int_\Omega |\nabla u_n|^{\frac{mp_n(x)}{n}} dx \le \Big(\int_\Omega |\nabla u_n|^{p_n(x)} dx \Big)^{\frac{m}{n}} (|\Omega|)^{\frac{n-m}{n}} \le \big(\Lambda_n^2\big)^{\frac{m}{n}} (p_n^+)^{\frac{m}{n}} (|\Omega|+1) \le C(m)$$

for all $n \in \mathbb{N}$ sufficiently large. Combining these inequalities, and taking into account the fact that $n \in \mathbb{N}$ was chosen sufficiently large so that $\frac{p_n(x)}{n} \ge \frac{p_n^-}{n} > q^- - \varepsilon$ in Ω, we deduce that the embedding $W^{1,\frac{mp_n(\cdot)}{n}}(\Omega) \subset W^{1,m(q^- -\varepsilon)}(\Omega)$ is continuous, and so the sequence $\{u_n\}$ is bounded in $W^{1,m(q^- -\varepsilon)}(\Omega)$. If we now choose $m \in \mathbb{N}$ sufficiently large such that $m(q^- - \varepsilon) > N$, it follows that the embedding of $W^{1,m(q^- -\varepsilon)}(\Omega)$ into $C(\overline{\Omega})$ is compact. Taking into account the reflexivity of the space $W^{1,m(q^- -\varepsilon)}(\Omega)$, we deduce that there exists a subsequence (not relabelled) of $\{u_n\}$ and a function $u_\infty \in C(\overline{\Omega})$ such that $u_n \rightharpoonup u_\infty$ weakly in $W^{1,m(q^- -\varepsilon)}(\Omega)$ and $u_n \to u_\infty$ uniformly in Ω.

Next, we prove that u_∞ is non-trivial. To this aim, recall that the second eigenfunctions satisfy the constraint $\displaystyle\int_\Omega \frac{|u_n|^{p_n(x)}}{p_n(x)} dx = 1$, which gives

$$\left(\int_\Omega |u_n|^{p_n(x)} dx \right)^{\frac{1}{n}} \ge (p_n^-)^{\frac{1}{n}}. \tag{83}$$

If $n \in \mathbb{N}$ is such that $\|u_n\|_\infty \le 1$, then $\|u_n\|_\infty^{p_n(\cdot)} \le \|u_n\|_\infty^{p_n^-}$ in Ω, and note that if $\|u_n\|_\infty > 1$ we have $\|u_n\|_\infty^{p_n(\cdot)} \le \|u_n\|_\infty^{p_n^+}$ in Ω. Thus,

$$\int_\Omega |u_n|^{p_n(x)} dx \le \int_\Omega \|u_n\|_\infty^{p_n(x)} dx \le |\Omega| \max\left\{ \|u_n\|_\infty^{p_n^-}, \|u_n\|_\infty^{p_n^+} \right\}.$$

Using (83), we obtain

$$\max\left\{ \|u_n\|_\infty^{p_n^-}, \|u_n\|_\infty^{p_n^+} \right\}^{\frac{1}{n}} \ge \left(\frac{p_n^-}{|\Omega|} \right)^{\frac{1}{n}}.$$

Letting $n \to \infty$ in the last inequality implies that $\max \left\{ \|u_\infty\|_\infty^{q^-}, \|u_\infty\|_\infty^{q^+} \right\} \geq 1$, which shows that $u_\infty \neq 0$ in Ω.

In view of what we just shown, and taking again into account Proposition 3, we may extract a subsequence (not relabelled) such that (80) and (81) hold. The rest of the proof is devoted to showing that u_∞ is a viscosity solution of (82).

Let $x_0 \in \Omega$, and $\varphi \in C^2(\Omega)$ be such that $u_\infty(x_0) = \varphi(x_0)$ and $u_\infty - \varphi$ has a minimum at x_0. The uniform convergence of u_n to u_∞ implies that there exists a sequence $\{x_n\} \subset \Omega$ such that $x_n \to x_0$, $u_n(x_n) = \varphi(x_n)$, and $u_n - \varphi$ has a minimum at x_n. Since for $n \in \mathbb{N}$ sufficiently large, Proposition 2 implies that u_n is a continuous viscosity solution of (77) with $\Lambda_{p_n(\cdot)} = \Lambda_n^2$, we have

$$\begin{aligned} &- |\nabla \varphi(x_n)|^{p_n(x_n)-2} \left(\Delta \varphi(x_n) + \ln |\nabla \varphi(x_n)| \langle \nabla p_n(x_n), \nabla \varphi(x_n) \rangle \right) \\ &- (p_n(x_n) - 2) |\nabla \varphi(x_n)|^{p_n(x_n)-4} \Delta_\infty \varphi(x_n) \geq \Lambda_n^2 |\varphi(x_n)|^{p_n(x_n)-2} \varphi(x_n). \end{aligned} \tag{84}$$

We will need to study two cases. First, if $u_\infty(x_0) > 0$, we have, for $n \in \mathbb{N}$ sufficiently large

$$\Lambda_n^2 |\varphi(x_n)|^{p_n(x_n)-2} \varphi(x_n) = \Lambda_n^2 |u_n(x_n)|^{p_n(x_n)-2} u_n(x_n) > 0,$$

and thus, by (84), we deduce that $|\nabla \varphi(x_n)| > 0$ for $n \in \mathbb{N}$ sufficiently large. Dividing both sides of (84) by $(p_n(x_n) - 2)|\nabla \varphi(x_n)|^{p_n(x_n)-4}$, we find

$$\begin{aligned} &\frac{-|\nabla \varphi(x_n)|^2 \left(\Delta \varphi(x_n) + \ln |\nabla \varphi(x_n)| \langle \nabla p_n(x_n), \nabla \varphi(x_n) \rangle \right)}{p_n(x_n) - 2} - \Delta_\infty \varphi(x_n) \\ &\qquad \geq \left(\frac{\left(\Lambda_n^2\right)^{1/n} |\varphi(x_n)|^{\frac{p_n}{n}(x_n) - \frac{2}{n}}}{|\nabla \varphi(x_n)|^{\frac{p_n}{n}(x_n) - \frac{4}{n}}} \right)^n \frac{\varphi(x_n)}{p_n(x_n) - 2}. \end{aligned}$$

Passing to the limit (supremum) as $n \to \infty$ and taking into account (74) leads to

$$- \Delta_\infty\varphi(x_0) - |\nabla\varphi(x_0)|^2 \ln|\nabla\varphi(x_0)|\langle\xi(x_0), \nabla\varphi(x_0)\rangle$$
$$\geq \limsup_{n\to\infty}\left[\left(\frac{(\Lambda_n^2)^{1/n}|\varphi(x_n)|^{\frac{p_n}{n}(x_n)-\frac{2}{n}}}{|\nabla\varphi(x_n)|^{\frac{p_n}{n}(x_n)-\frac{4}{n}}}\right)^n \frac{\varphi(x_n)}{p_n(x_n)-2}\right]. \quad (85)$$

In particular, we have

$$-\Delta_\infty\varphi(x_0) - |\nabla\varphi(x_0)|^2 \ln|\nabla\varphi(x_0)|\langle\xi(x_0), \nabla\varphi(x_0)\rangle \geq 0. \quad (86)$$

We claim that the inequality

$$|\nabla\varphi(x_0)|^{q(x_0)} - \Lambda_\infty|\varphi(x_0)|^{q(x_0)} \geq 0 \quad (87)$$

holds. Indeed, otherwise $|\nabla\varphi(x_0)|^{q(x_0)} < \Lambda_\infty|\varphi(x_0)|^{q(x_0)}$, and taking into account that (75) and (80) imply

$$\lim_{n\to\infty}\left(\frac{(\Lambda_n^2)^{1/n}|\varphi(x_n)|^{\frac{p_n}{n}(x_n)-\frac{2}{n}}}{|\nabla\varphi(x_n)|^{\frac{p_n}{n}(x_n)-\frac{4}{n}}}\right) = \frac{\Lambda_\infty|\varphi(x_0)|^{q(x_0)}}{|\nabla\varphi(x_0)|^{q(x_0)}} > 1, \quad (88)$$

we deduce that there exists $\varepsilon > 0$ such that

$$\frac{(\Lambda_n^2)^{1/n}|\varphi(x_n)|^{\frac{p_n}{n}(x_n)-\frac{2}{n}}}{|\nabla\varphi(x_n)|^{\frac{p_n}{n}(x_n)-\frac{4}{n}}} \geq 1+\varepsilon$$

for all $n \in \mathbb{N}$ sufficiently large. Hence,

$$\limsup_{n\to\infty}\left(\left(\frac{(\Lambda_n^2)^{1/n}|\varphi(x_n)|^{\frac{p_n}{n}(x_n)-\frac{2}{n}}}{|\nabla\varphi(x_n)|^{\frac{p_n}{n}(x_n)-\frac{4}{n}}}\right)^n \frac{\varphi(x_n)}{p_n(x_n)-2}\right) \geq \lim_{n\to\infty}\frac{(1+\varepsilon)^n}{n}\left(\frac{\varphi(x_n)}{\frac{p_n(x_n)-2}{n}}\right) = \infty,$$

which is a contradiction with (85). Thus, (87) holds, as claimed. Using (86) and (87) we deduce that in the case where $u_\infty(x_0) > 0$, we have

$$\min\Big\{ - \Delta_\infty\varphi(x_0) - |\nabla\varphi(x_0)|^2 \ln|\nabla\varphi(x_0)|\langle\xi(x_0), \nabla\varphi(x_0)\rangle,$$
$$|\nabla\varphi(x_0)|^{q(x_0)} - \Lambda_\infty|\varphi(x_0)|^{q(x_0)}\Big\} \geq 0. \quad (89)$$

If $u_\infty(x_0) = \varphi(x_0) = 0$, we either have $\nabla\varphi(x_0) \neq 0$ (in which case we can use very similar arguments to conclude that (86) and (87) hold), or else $\nabla\varphi(x_0) = 0$. For the latter, taking into account that $\Delta_\infty\varphi(x_0) = 0$ and Remark 2, we arrive at (86) again. On the other hand, (87) is clearly also true. We conclude that (89) holds.

Finally, let $x_0 \in \partial\Omega$, and assume that $u_\infty - \varphi$ has a minimum at a point $x_0 \in \partial\Omega$ and $u_\infty(x_0) = \varphi(x_0)$. Since u_n converges to u_∞ uniformly, we deduce that there exists $x_n \in \overline{\Omega}$ such that $x_n \to x_0$ and $u_n - \varphi$ has a minimum point at x_n. Since u_n is viscosity supersolution of (77) we obtain, in view of Remark 1 that $\frac{\partial\varphi}{\partial\eta}(x_n) \geq 0$, and hence

$$\frac{\partial\varphi}{\partial\eta}(x_0) = \lim_{n\to\infty} \frac{\partial\varphi}{\partial\eta}(x_n) \geq 0.$$

Therefore, if $x_0 \in \partial\Omega$, we have

$$\max\Big\{ \min\Big\{ -\Delta_\infty\varphi(x_0) - |\nabla\varphi(x_0)|^2 \ln|\nabla\varphi(x_0)|\langle\xi(x_0), \nabla\varphi(x_0)\rangle, |\nabla\varphi(x_0)|^{q(x_0)} - \Lambda_\infty|\varphi(x_0)|^{q(x_0)}\Big\}, \frac{\partial\varphi}{\partial\eta}(x_0)\Big\} \geq 0.$$

Overall, we have shown that u_∞ is a viscosity supersolution of (82). The proof of the fact that u_∞ is also a viscosity subsolution follows analogously. We conclude u_∞ is a viscosity solution of (82), which concludes the proof. □

5. THE ROBIN EIGENVALUE PROBLEM

5.1. The Robin Eigenvalue Problem for the $p(x)$-Laplacian

Consider the Robin eigenvalue problem for the $p(x)$-Laplacian in an open bounded domain Ω with smooth boundary

$$\begin{cases} -\Delta_{p(x)}u = \Lambda_{p(\cdot)}|u|^{p(x)-2}u & \text{in } \Omega \\ |\nabla u|^{p(x)-2}\frac{\partial u}{\partial \eta} + \beta|u|^{p(x)-2}u = 0 & \text{on } \partial\Omega, \end{cases} \tag{90}$$

where $\beta > 0$ is a given positive constant, and $\eta = \eta(x)$ stands for the outer unit normal to $\partial\Omega$ at $x \in \partial\Omega$.

Definition 6. *We say that $u \in W^{1,p(\cdot)}(\Omega)$ is a weak solution for the Robin eigenvalue problem (90) if there exists $\Lambda_{p(\cdot)} \in \mathbb{R}$ such that*

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u \cdot \nabla v \, dx + \int_{\partial\Omega} \beta|u|^{p(x)-2}uv \, dx = \Lambda_{p(\cdot)} \int_\Omega |u|^{p(x)-2}uv \, dx, \tag{91}$$

for all $v \in W^{1,p(\cdot)}(\Omega)$. If $u \neq 0$, we say that $\Lambda_{p(\cdot)}$ is an eigenvalue of (90), and that u is an eigenfunction corresponding to $\Lambda_{p(\cdot)}$.

Let $X := W^{1,p(\cdot)}(\Omega)$, and define the functionals $\mathcal{F}, \mathcal{G} : X \to \mathbb{R}$ by

$$\mathcal{F}(u) = \int_\Omega \frac{1}{p(x)}|u|^{p(x)} \, dx \tag{92}$$

and

$$\mathcal{G}(u) = \int_\Omega \frac{1}{p(x)}(|\nabla u|^{p(x)} + |u|^{p(x)})dx + \int_{\partial\Omega} \frac{\beta}{p(x)}|u|^{p(x)} \, dx. \tag{93}$$

It is easy to see that $\mathcal{F}, \mathcal{G} \in C^1(X;\mathbb{R})$, and that for all $v \in X$ we have

$$\langle \mathcal{F}'(u), v\rangle_{X',X} = \int_\Omega |u|^{p(x)-2}uv\ dx,$$

and

$$\langle \mathcal{G}'(u), v\rangle_{X',X} = \int_\Omega (|\nabla u|^{p(x)-2}\nabla u\cdot\nabla v + |u|^{p(x)-2}uv)\ dx + \int_{\partial\Omega} \beta|u|^{p(x)-2}uv\ dx.$$

Consider the level set $S_{\mathcal{G}} := \{u \in X\ :\ \mathcal{G}(u) = 1\}$, and the eigenvalue problem

$$\mathcal{F}'(u) = \mu\mathcal{G}'(u), \quad u \in S_{\mathcal{G}}, \quad \mu \in \mathbb{R}. \tag{94}$$

The existence of a sequence of nonnegative eigenvalues $\mu_n \to 0^+$ as $n \to \infty$ for the problem (94) follows from the Ljusternik-Schnirelman theory (see, e.g., [11], [48]). In fact, we have $\mu_n = \sup_{A\in\mathbb{A}_n}\inf_{u\in A}\mathcal{F}(u)$, with

$$\mathbb{A}_n := \{A \subset S_{\mathcal{G}}\ :\ \mathcal{F}(u) > 0 \text{ on } A,\ A \text{ compact},\ A = -A,\ \gamma(A) \geq n\},$$

where

$$\gamma(A) := \inf\left\{k \in \mathbb{N}\ \mid\ \exists\ \ h : A \to \mathbb{R}^k\backslash\{0\},\quad h\ \text{ odd and continuous}\right\}$$

is the genus of A. The eigenfunctions $u \in S_{\mathcal{G}}$ satisfy $\mathcal{F}'(u) = \mu\mathcal{G}'(u)$ or, equivalently, $\langle \mathcal{F}'(u), v\rangle_{X',X} = \mu\langle \mathcal{G}'(u), v\rangle_{X',X}$ for all $v \in X$. Hence, we have

$$\int_\Omega |\nabla u|^{p(x)-2}\nabla u\cdot\nabla v\ dx + \beta\int_{\partial\Omega} |u|^{p(x)-2}uv\ dx = \left(\frac{1}{\mu} - 1\right)\int_\Omega |u|^{p(x)-2}uv\ dx$$

for all $v \in W^{1,p(\cdot)}(\Omega)$, which means that u is a weak solution of problem (90) with

$\Lambda_{p(\cdot)} = \frac{1}{\mu} - 1$.

The key technical result that allows one to apply the Ljusternik-Schnirelman theory with the choice (92) and (93) of functionals $\mathcal{F}$ and $\mathcal{G}$, which is relevant for the Robin eigenvalue problem (90), is Proposition 4 below.

Proposition 4. *Let $\mathcal{F}, \mathcal{G} : X \to \mathbb{R}$ be the functionals defined in (92) and (93). The following conditions hold:*

(H1) $\mathcal{F}, \mathcal{G} \in C^1(X, \mathbb{R})$ are even functionals, and $\mathcal{F}(0) = \mathcal{G}(0) = 0$.

(H2) $\mathcal{F}'$ is strongly continuous, i.e $u_n \rightharpoonup u$ weakly in X implies $\mathcal{F}'(u_n) \to \mathcal{F}(u)$, and $\langle \mathcal{F}'(u), u \rangle_{X',X} = 0$, $u \in \overline{coS_{\mathcal{G}}}$ implies $\mathcal{F}(u) = 0$, where $\overline{coS_{\mathcal{G}}}$ stands for the closed convex hull of $S_{\mathcal{G}}$.

(H3) $\mathcal{G}'$ is continuous, bounded, and such that $u_n \to u$ in X as $n \to \infty$ whenever $u_n \rightharpoonup u$ weakly in X, $\mathcal{G}'(u_n) \rightharpoonup v$ in X', and $\langle \mathcal{G}'(u_n), u_n \rangle_{X',X} \to \langle v, u \rangle_{X',X}$.

(H4) The level set $S_{\mathcal{G}}$ is bounded, $\inf\limits_{u \in S_{\mathcal{G}}} \langle \mathcal{G}'(u), u \rangle_{X',X} > 0$, and

$$\langle \mathcal{G}'(u), u \rangle_{X',X} > 0, \ \lim_{t \to \infty} \mathcal{G}(tu) = +\infty, \ \forall u \in X \backslash \{0\}.$$

Proof. (H1) follows immediately from the definition of $\mathcal{F}$ and $\mathcal{G}$. To prove the first part of (H2), let $u_n \rightharpoonup u$ weakly in X. We seek to show that $\mathcal{F}'(u_n) \to \mathcal{F}'(u)$ in X'. To this aim, note that by Hölder's inequality (7), we have

$$\begin{aligned} |\langle \mathcal{F}'(u_n) - \mathcal{F}'(u), v \rangle_{X',X}| &= \left| \int\limits_{\Omega} (|u_n|^{p(x)-2} u_n - |u|^{p(x)-2} u) v dx \right| \\ &\leq 2 \left| |u_n|^{p(x)-2} u_n - |u|^{p(x)-2} u \right|_{p'(\cdot)} |v|_{p(\cdot)}. \end{aligned}$$

Thus,

$$\frac{|\langle \mathcal{F}'(u_n) - \mathcal{F}'(u), v\rangle_{X',X}|}{\|v\|_X} \le 2\left||u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u\right|_{p'(\cdot)}$$

for all $v \in X \setminus \{0\}$, which implies that

$$\sup_{\substack{v\in X\setminus\{0\}\\ \|v\|_{p(\cdot)}\le 1}} \frac{|\langle \mathcal{F}'(u_n) - \mathcal{F}'(u), v\rangle_{X',X}|}{\|v\|_X} \le 2\left||u_n|^{p(\cdot)-2}u_n - |u|^{p(\cdot)-2}u\right|_{p'(\cdot)}.$$

Hence,

$$\|\mathcal{F}'(u_n) - \mathcal{F}'(u)\|_{X'} \le 2\left||u_n|^{p(\cdot)-2}u_n - |u|^{p(\cdot)-2}u\right|_{p'(\cdot)}.$$

It remains to show that

$$\left||u_n|^{p(\cdot)-2}u_n - |u|^{p(\cdot)-2}u\right|_{p'(\cdot)} \to 0 \text{ as } n \to \infty. \tag{95}$$

Let $w_n := |u_n|^{p(\cdot)-2}u_n$ and $w =: |u|^{p(\cdot)-2}u$. Note that $w_n, w \in L^{p'(\cdot)}(\Omega)$. Since X is compactly embedded into $L^{p(\cdot)}(\Omega)$ we have $u_n \to u$ in $L^{p(\cdot)}(\Omega)$. Thus, there exists a subsequence of $\{u_n\}$ (not relabeled), and $g \in L^{p(\cdot)}(\Omega)$ such that $u_n \to u$ and $|u_n| \le g$ a.e. in Ω. It follows that $|w_n| = |u_n|^{p(\cdot)-1} \le g^{p(\cdot)-1} \in L^{p'(\cdot)}(\Omega)$. Therefore, by Lemma 1, we have that $w_n \to w$ in $L^{p'(\cdot)}(\Omega)$, which implies that (95) holds. The remaining assertion in (H2) follows easily from the definitions of $\mathcal{F}$ and $\mathcal{F}'$.

To show that (H3) holds, let $\{u_n\} \subset X$ be a sequence such that $u_n \rightharpoonup u$ weakly in X, $\mathcal{G}'(u_n) \rightharpoonup v$ in X', and $\langle \mathcal{G}'(u_n), u_n\rangle_{X',X} \to \langle v, u\rangle_{X',X}$. It follows that

$$\begin{aligned}&\lim_{n\to\infty} \langle \mathcal{G}'(u_n) - \mathcal{G}'(u), u_n - u\rangle_{X',X}\\ &= \lim_{n\to\infty} \left(\langle \mathcal{G}'(u_n), u_n\rangle_{X',X} - \langle \mathcal{G}'(u_n), u\rangle_{X',X} - \langle \mathcal{G}'(u), u_n - u\rangle_{X',X}\right) = 0.\end{aligned} \tag{96}$$

On the other hand, using the inequality (52) we obtain

$$\begin{aligned}\langle \mathcal{G}'(u_n) - \mathcal{G}'(u), u_n - u\rangle_{X',X} &= \int\limits_{\Omega} \left(|\nabla u_n|^{p(x)-2}\nabla u_n - |\nabla u|^{p(x)-2}\nabla u\right) \cdot (\nabla u_n - \nabla u)\, dx \\ &\quad + \int\limits_{\Omega} \left(|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u\right)(u_n - u)\, dx \\ &\quad + \beta \int\limits_{\partial\Omega} \left(|u_n|^{p(x)-2}u_n - |u|^{p(x)-2}u\right)(u_n - u)\, dx \\ &\geq \int\limits_{\Omega} \left(\frac{1}{2}\right)^{p(x)-1} \left(|\nabla u_n - \nabla u|^{p(x)} + |u_n - u|^{p(x)}\right)\, dx + \beta \int\limits_{\partial\Omega} \left(\frac{1}{2}\right)^{p(x)-1} |u_n - u|^{p(x)}\, dx \\ &\geq \left(\frac{1}{2}\right)^{p^+-1} \left(\int\limits_{\Omega} \left(|\nabla u_n - \nabla u|^{p(x)} + |u_n - u|^{p(x)}\right)\, dx + \beta \int\limits_{\partial\Omega} |u_n - u|^{p(x)}\, dx\right).\end{aligned}$$

Taking into account (96), we obtain

$$\lim_{n\to\infty} \int\limits_{\Omega} \left(|\nabla u_n - \nabla u|^{p(x)} + |u_n - u|^{p(x)}\right) dx = 0.$$

Thus, by Lemma 2, we have $|u_n - u|_{p(\cdot)} \to 0$ and $|\nabla u_n - \nabla u|_{p(\cdot)} \to 0$ as $n \to \infty$. We conclude that $u_n \to u$ in X. It remains to prove (H4). It is immediate that

$$\langle \mathcal{G}'(u), u\rangle_{X',X} = \int\limits_{\Omega} (|\nabla u|^{p(x)} + |u|^{p(x)})\, dx + \beta \int\limits_{\partial\Omega} |u|^{p(x)}\, dx > 0 \tag{97}$$

for all $u \in X\backslash\{0\}$. We claim that $\inf\limits_{u\in S_{\mathcal{G}}} \langle \mathcal{G}'(u), u\rangle_{X',X} > 0$. Indeed, if this was not the case, there would exist a sequence $\{u_n\} \in S_{\mathcal{G}}$ such that $\langle \mathcal{G}'(u_n), u_n\rangle_{X',X} \leq \frac{1}{n}$ for all $n \in \mathbb{N}$. Since $p(x) > 1$ for all $x \in \Omega$, we have

$$\begin{aligned}1 = \mathcal{G}(u_n) &= \int\limits_{\Omega} \frac{1}{p(x)} \left(|\nabla u_n|^{p(x)} + |u_n|^{p(x)}\right)\, dx + \int\limits_{\partial\Omega} \frac{\beta}{p(x)} |u_n|^{p(x)}\, dx \\ &\leq \langle \mathcal{G}'(u_n), u_n\rangle_{X',X} \leq \frac{1}{n}\end{aligned}$$

for all $n \in \mathbb{N}$, which is a contradiction. Finally, observe that for $u \in X \setminus \{0\}$ and $t > 1$ we have

$$\begin{aligned}
\mathcal{G}(tu) &= \left(\int\limits_{\Omega} \frac{1}{p(x)} (|\nabla(tu)|^{p(x)} + |tu|^{p(x)}) \; dx + \beta \int\limits_{\partial\Omega} \frac{1}{p(x)} |tu|^{p(x)} \; dx \right) \\
&= \left(\int\limits_{\Omega} \frac{1}{p(x)} (|\nabla u|^{p(x)} + |u|^{p(x)}) t^{p(x)} \; dx + \beta \int\limits_{\partial\Omega} \frac{1}{p(x)} |u|^{p(x)} t^{p(x)} \; dx \right) \\
&\geq \frac{t^{p^-}}{p^+} \langle \mathcal{G}'(u), u \rangle_{X',X}.
\end{aligned}$$

Hence, in view of (97), $\lim\limits_{t\to\infty} \mathcal{G}(tu) = \infty$. □

Our next goal in this section is to prove that continuous weak solutions of (90) are also viscosity solutions (see Proposition 5 below). Note that the Robin eigenvalue problem (90) takes the form (47), with $F : \overline{\Omega} \times \mathbb{R} \times \mathbb{R}^N \times \mathbb{M}^{N\times N}_{\text{sym}} \to \mathbb{R}$ and $H : \partial\Omega \times \mathbb{R} \times \mathbb{R}^N \to \mathbb{R}$ defined by

$$F(x, r, \theta, S) = -|\theta|^{p(x)-2} \left(\text{Tr}(S) + \ln|\theta| \langle \theta, \nabla p(x) \rangle \right) - (p(x) - 2)|\theta|^{p(x)-4} \langle S\theta, \theta \rangle - \Lambda_{p(\cdot)} |r|^{p(x)-2} r$$

and

$$H(x, r, \theta) = |\theta|^{p(x)-2} \langle \theta, \eta \rangle + \beta |r|^{p(x)-2} r.$$

Proposition 5. *Any continuous weak solution of (90) is a viscosity solution of (90).*

Proof. Let $u \in C(\overline{\Omega})$ be weak a solution of (90). To show that u is a viscosity supersolution of (90), let $x_0 \in \overline{\Omega}$, and consider a test function $\varphi \in C^2(\overline{\Omega})$ such that $u(x_0) = \varphi(x_0)$ and $u - \varphi$ has a minimum at x_0. If $x_0 \in \Omega$, we claim that we have

$$-\Delta_{p(x_0)} \varphi(x_0) - \Lambda_{p(\cdot)} |\varphi(x_0)|^{p(x_0)-2} \varphi(x_0) \geq 0.$$

Indeed, if we assume that this inequality does not hold, then there exists $r > 0$ such that $B(x_0, r) \subset \Omega$ and

$$-\Delta_{p(x)}\varphi(x) - \Lambda_{p(\cdot)}|\varphi(x)|^{p(x)-2}\varphi(x) < 0 \text{ for all } x \in B(x_0, r).$$

Taking r smaller, if necessary, we may assume that $u > \varphi$ in $B(x_0, r) \setminus \{x_0\}$. Set

$$m = \inf_{x \in \partial B(x_0,r)} (u - \varphi)(x) > 0,$$

and let $\Phi(x) := \varphi(x) + \frac{m}{2}$. Note that $\Phi(x_0) > u(x_0)$, $\Phi(x) < u(x)$ for all $x \in \partial B(x_0, r)$, and

$$-\Delta_{p(x)}\Phi(x) - \Lambda_{p(\cdot)}|\varphi(x)|^{p(x)-2}\varphi(x) < 0 \text{ for all } x \in B(x_0, r). \tag{98}$$

If we multiply (98) by $(\Phi - u)^+$ and integrate over $B(x_0, r)$, we obtain

$$\int_{\{x \in B(x_0,r):\Phi(x)>u(x)\}} |\nabla \Phi|^{p(x)-2}\nabla \Phi \cdot \nabla(\Phi - u)dx < \int_{\{x \in B(x_0,r):\Phi(x)>u(x)\}} \Lambda_{p(\cdot)}|\varphi|^{p(x)-2}\varphi(\Phi - u)dx, \tag{99}$$

where we have used the fact that $(\Phi - u)^+ = 0$ on $\partial B(x_0, r)$. Extending $(\Phi - u)^+$ by zero outside $B(x_0, r)$, and using this extension as a test function in weak formulation (91) gives

$$\int_{\{x \in B(x_0,r):\Phi(x)>u(x)\}} |\nabla u|^{p(x)-2}\nabla u \cdot \nabla(\Phi - u)dx = \int_{\{x \in B(x_0,r):\Phi(x)>u(x)\}} \Lambda_{p(\cdot)}|u|^{p(x)-2}u(\Phi - u)dx. \tag{100}$$

After subtracting (100) from (99), using the fact that $u > \varphi$ on $B(x_0, r) \setminus \{x_0\}$, and the inequality (52), we obtain

$$\begin{aligned}
0 >& \int\limits_{\{x\in B(x_0,r):\Phi(x)>u(x)\}} \left(|\nabla\Phi|^{p(x)-2}\nabla\Phi - |\nabla u|^{p(x)-2}\nabla u\right)\cdot\nabla(\Phi-u)\,dx \\
&\qquad + \int\limits_{\{x\in B(x_0,r):\Phi(x)>u(x)\}} \Lambda_{p(\cdot)}\left(|u|^{p(x)-2}u - |\varphi|^{p(x)-2}\varphi\right)(\Phi-u)\,dx \\
\geq& \int\limits_{\{x\in B(x_0,r):\Phi(x)>u(x)\}} \left(|\nabla\Phi|^{p(x)-2}\nabla\Phi - |\nabla u|^{p(x)-2}\nabla u\right)\cdot\nabla(\Phi-u)\,dx \\
\geq& \frac{1}{2^{p^+-1}}\int\limits_{\{x\in B(x_0,r):\Phi(x)>u(x)\}} |\nabla\Phi-\nabla u|^{p(x)}\,dx \geq 0,
\end{aligned}$$

which is clearly a contradiction. On the other hand, if $x_0\in\partial\Omega$ we need to prove that

$$\max\left\{|\nabla\varphi(x_0)|^{p(x_0)-2}\frac{\partial\varphi}{\partial\eta}(x_0)+\beta|\varphi(x_0)|^{p(x_0)-2}\varphi(x_0),\right. \\ \left. -\Delta_{p(x_0)}\varphi(x_0)-\Lambda_{p(\cdot)}|\varphi(x_0)|^{p(x_0)-2}\varphi(x_0)\right\}\geq 0$$

or, equivalently,

$$\max\left\{|\nabla\varphi(x_0)|^{p(x_0)-2}\frac{\partial\varphi}{\partial\eta}(x_0)+\beta|u(x_0)|^{p(x_0)-2}u(x_0),\right. \\ \left. -\Delta_{p(x_0)}\varphi(x_0)-\Lambda_{p(\cdot)}|\varphi(x_0)|^{p(x_0)-2}\varphi(x_0)\right\}\geq 0. \quad (101)$$

We again proceed by contradiction. Assume that (101) does not hold. Then there exists $r>0$ sufficiently small such that

$$|\nabla\varphi(x)|^{p(x)-2}\frac{\partial\varphi}{\partial\eta}(x)+\beta|u(x)|^{p(x)-2}u(x)<0 \quad (102)$$

and

$$-\Delta_{p(x)}\varphi(x)-\Lambda_{p(\cdot)}|\varphi(x)|^{p(x)-2}\varphi(x)<0, \quad (103)$$

for all $x \in B(x_0, r) \cap \Omega$. Assuming that r is small enough, we have $u(x) > \varphi(x)$ for all $x \in \left(\overline{B(x_0, r)} \setminus \{x_0\}\right) \cap \Omega$ and thus

$$m := \inf_{\partial B(x_0,r)\cap\overline{\Omega}} (u - \varphi)(x) > 0.$$

With $\Phi(x) := \varphi(x) + \frac{m}{2}$, note that $\Phi(x_0) > u(x_0)$, and that $\Phi(x) < u(x)$ for all $x \in \partial B(x_0, r) \cap \overline{\Omega}$. Multiplying (103) by $(\Phi - u)^+$ and integrating over $B(x_0, r) \cap \Omega$ gives

$$\int\limits_{B(x_0,r)\cap\Omega} |\nabla\varphi|^{p(x)-2}\nabla\varphi \cdot \nabla(\Phi - u)^+ \, dx - \int\limits_{\partial(B(x_0,r)\cap\Omega)} |\nabla\varphi|^{p(x)-2}\frac{\partial\varphi}{\partial\eta}(\Phi - u)^+ \, dx <$$
$$< \int\limits_{B(x_0,r)\cap\Omega} \Lambda_{p(\cdot)}|\varphi|^{p(x)-2}\varphi(\Phi - u)^+ \, dx. \tag{104}$$

Since $(\Phi - u)^+ = 0$ on $\partial B(x_0, r) \cap \overline{\Omega}$, we have

$$\int\limits_{\partial(B(x_0,r)\cap\Omega)} |\nabla\varphi|^{p(x)-2}\frac{\partial\varphi}{\partial\eta}(\Phi - u)^+ \, dx = \int\limits_{B(x_0,r)\cap\partial\Omega} |\nabla\varphi|^{p(x)-2}\frac{\partial\varphi}{\partial\eta}(\Phi - u)^+ \, dx.$$

Thus,

$$\int\limits_{\{x\in B(x_0,r):\Phi(x)>u(x)\}\cap\Omega} |\nabla\Phi|^{p(x)-2}\nabla\Phi \cdot \nabla(\Phi - u) \, dx$$
$$< \int\limits_{\{x\in B(x_0,r):\Phi(x)>u(x)\}\cap\partial\Omega} |\nabla\varphi|^{p(x)-2}\frac{\partial\varphi}{\partial\eta}(\Phi - u) \, dx$$
$$+ \int\limits_{\{x\in B(x_0,r):\Phi(x)>u(x)\}\cap\Omega} \Lambda_{p(\cdot)}|\varphi|^{p(x)-2}\varphi(\Phi - u) \, dx. \tag{105}$$

Using the extension of $(\Phi - u)^+$ by zero outside $B(x_0, r) \cap \Omega$ as a test function in (91), we obtain

$$\int_{\{x\in B(x_0,r):\Phi(x)>u(x)\}\cap\Omega} |\nabla u|^{p(x)-2}\nabla u\cdot\nabla(\Phi-u)\,dx$$
$$= -\beta \int_{\{x\in B(x_0,r):\Phi(x)>u(x)\}\cap\partial\Omega} |u|^{p(x)-2}u(\Phi-u)\,dx$$
$$+ \int_{\{x\in B(x_0,r):\Phi(x)>u(x)\}\cap\Omega} \Lambda_{p(\cdot)}|u|^{p(x)-2}u(\Phi-u)\,dx. \quad (106)$$

Subtracting (106) from (105) gives

$$\int_{\{x\in B(x_0,r):\Phi(x)>u(x)\}\cap\Omega} \left(|\nabla\Phi|^{p(x)-2}\nabla\Phi-|\nabla u|^{p(x)-2}\nabla u\right)\cdot\nabla(\Phi-u)\,dx$$
$$< \int_{\{x\in B(x_0,r):\Phi(x)>u(x)\}\cap\Omega} \Lambda_{p(\cdot)}\left(|\varphi|^{p(x)-2}\varphi-|u|^{p(x)-2}u\right)(\Phi-u)\,dx$$
$$+ \int_{\{x\in B(x_0,r):\Phi(x)>u(x)\}\cap\partial\Omega} \left(|\nabla\varphi|^{p(x)-2}\frac{\partial\varphi}{\partial\eta}+\beta|u|^{p(x)-2}u\right)(\Phi-u)\,dx.$$

Recalling that $r>0$ was chosen sufficiently small so that, in particular, (102) holds, we obtain that

$$\int_{\{x\in B(x_0,r):\Phi(x)>u(x)\}\cap\partial\Omega} \left(|\nabla\varphi|^{p(x)-2}\frac{\partial\varphi}{\partial\eta}+\beta|u|^{p(x)-2}u\right)(\Phi-u)\,dx\le 0.$$

Thus,

$$\int_{\{x\in B(x_0,r):\Phi(x)>u(x)\}\cap\Omega} \left(|\nabla\Phi|^{p(x)-2}\nabla\Phi-|\nabla u|^{p(x)-2}\nabla u\right)\cdot\nabla(\Phi-u)\,dx$$
$$< \int_{\{x\in B(x_0,r):\Phi(x)>u(x)\}\cap\Omega} \Lambda_{p(\cdot)}\left(|\varphi|^{p(x)-2}\varphi-|u|^{p(x)-2}u\right)(\Phi-u)\,dx\le 0, \quad (107)$$

where the last inequality follows from the fact that $u\ge\varphi$ on $B(x_0,r)\cap\Omega$. In view of (52) we have

$$\frac{1}{2^{p^+-1}} \int\limits_{\{x\in B(x_0,r):\Phi(x)>u(x)\}\cap\Omega} |\nabla\Phi - \nabla u|^{p(x)}\, dx$$
$$\leq \int\limits_{\{x\in B(x_0,r):\Phi(x)>u(x)\}\cap\Omega} \left(|\nabla\Phi|^{p(x)-2}\nabla\Phi - |\nabla u|^{p(x)-2}\nabla u\right)\cdot\nabla(\Phi-u)\, dx. \quad (108)$$

Combining (107) and (108) leads to a contradiction. We conclude that u is a viscosity supersolution of (90).

Next, we prove that u is a viscosity subsolution of (90). Let $x_0 \in \overline{\Omega}$, and consider a test function $\psi \in C^2(\overline{\Omega})$ such that $u(x_0) = \psi(x_0)$ and $u - \psi$ has a maximum at x_0. If $x_0 \in \Omega$, we claim that we have

$$-\Delta_{p(x_0)}\psi(x_0) - \Lambda_{p(\cdot)}|\psi(x_0)|^{p(x_0)-2}\psi(x_0) \leq 0.$$

Assuming that the above inequality does not hold, there exists $r > 0$ such that $B(x_0, r) \subset \Omega$ and

$$-\Delta_{p(x)}\psi(x) - \Lambda_{p(\cdot)}|\psi(x)|^{p(x)-2}\psi(x) > 0 \text{ for all } x \in B(x_0, r).$$

Taking r small, we may assume that $u < \psi$ in $B(x_0, r) \setminus \{x_0\}$. Let

$$m = \inf_{x\in\partial B(x_0,r)} (\psi - u)(x) > 0,$$

and set $\Psi(x) := \psi(x) - \frac{m}{2}$. Note that $\Psi(x_0) < u(x_0)$, $\Psi(x) > u(x)$ for all $x \in \partial B(x_0, r)$, and

$$-\Delta_{p(x)}\Psi(x) - \Lambda_{p(\cdot)}|\psi(x)|^{p(x)-2}\psi(x) > 0 \text{ for all } x \in B(x_0, r). \quad (109)$$

If we multiply (109) by $(\Psi - u)^-$ and integrate over $B(x_0, r)$, we obtain

$$\int\limits_{\{x\in B(x_0,r):\Psi(x)<u(x)\}} |\nabla\Psi|^{p(x)-2}\nabla\Psi\cdot\nabla(u-\Psi)dx$$

$$> \int\limits_{\{x\in B(x_0,r):\Psi(x)<u(x)\}} \Lambda_{p(\cdot)}|\psi|^{p(x)-2}\psi(u-\Psi)dx, \quad (110)$$

where we have used the fact that $(\Psi-u)^-=0$ on $\partial B(x_0,r)$. Taking $(\Psi-u)^-$, extended by zero outside $B(x_0,r)$, as a test function in the weak formulation (91) gives

$$\int\limits_{\{x\in B(x_0,r):\Psi(x)<u(x)\}} |\nabla u|^{p(x)-2}\nabla u\cdot\nabla(u-\Psi)dx$$

$$= \int\limits_{\{x\in B(x_0,r):\Psi(x)<u(x)\}} \Lambda_{p(\cdot)}|u|^{p(x)-2}u(u-\Psi)dx. \quad (111)$$

After subtracting (110) from (111), using the fact that $u<\psi$ on $B(x_0,r)\setminus\{x_0\}$, and using (52) we obtain

$$\begin{aligned}
0 > &\int\limits_{\{x\in B(x_0,r):\Psi(x)<u(x)\}} \left(|\nabla u|^{p(x)-2}\nabla u-|\nabla\Psi|^{p(x)-2}\nabla\Psi\right)\cdot\nabla(u-\Psi)\ dx\\
&+\int\limits_{\{x\in B(x_0,r):\Psi(x)<u(x)\}} \Lambda_{p(\cdot)}\left(|\psi|^{p(x)-2}\psi-|u|^{p(x)-2}u\right)(u-\Psi)\ dx\\
\geq &\int\limits_{\{x\in B(x_0,r):\Psi(x)<u(x)\}} \left(|\nabla u|^{p(x)-2}\nabla u-|\nabla\Psi|^{p(x)-2}\nabla\Psi\right)\cdot\nabla(u-\Psi)\ dx\\
\geq &\frac{1}{2^{p^+-1}}\int\limits_{\{x\in B(x_0,r):\Psi(x)<u(x)\}} |\nabla u-\nabla\Psi|^{p(x)}\ dx\geq 0,
\end{aligned}$$

which is a contradiction. On the other hand, if $x_0\in\partial\Omega$ we need to prove that

$$\min\left\{|\nabla\psi(x_0)|^{p(x_0)-2}\frac{\partial\psi}{\partial\eta}(x_0)+\beta|\psi(x_0)|^{p(x_0)-2}\psi(x_0),\right.$$
$$\left.-\Delta_{p(x_0)}\psi(x_0)-\Lambda_{p(\cdot)}|\psi(x_0)|^{p(x_0)-2}\psi(x_0)\right\}\leq 0$$

or, equivalently,

$$\min\left\{|\nabla\psi(x_0)|^{p(x_0)-2}\frac{\partial\psi}{\partial\eta}(x_0)+\beta|u(x_0)|^{p(x_0)-2}u(x_0),\right.$$
$$\left.-\Delta_{p(x_0)}\psi(x_0)-\Lambda_{p(\cdot)}|\psi(x_0)|^{p(x_0)-2}\psi(x_0)\right\}\leq 0. \quad (112)$$

We again proceed by contradiction. Assume that (112) does not hold. Then there exists $r>0$ sufficiently small such that

$$|\nabla\psi(x)|^{p(x)-2}\frac{\partial\psi}{\partial\eta}(x)+\beta|u(x)|^{p(x)-2}u(x)>0 \quad (113)$$

and

$$-\Delta_{p(x)}\psi(x)-\Lambda_{p(\cdot)}|\psi(x)|^{p(x)-2}\psi(x)>0, \quad (114)$$

for all $x\in B(x_0,r)\cap\Omega$. Assuming that r is small enough, we have $u(x)<\psi(x)$ for all $x\in\left(\overline{B(x_0,r)}\setminus\{x_0\}\right)\cap\Omega$ and thus

$$m:=\inf_{\partial B(x_0,r)\cap\overline{\Omega}}(\psi-u)(x)>0.$$

With $\Psi(x):=\psi(x)-\frac{m}{2}$, note that $\Psi(x_0)<u(x_0)$, and that $\Psi(x)>u(x)$ for all $x\in\partial B(x_0,r)\cap\overline{\Omega}$. Multiplying (114) by $(\Psi-u)^-$ and integrating over $B(x_0,r)\cap\Omega$ gives

$$\int_{B(x_0,r)\cap\Omega}|\nabla\psi|^{p(x)-2}\nabla\psi\cdot\nabla(\Psi-u)^-\,dx-\int_{\partial(B(x_0,r)\cap\Omega)}|\nabla\psi|^{p(x)-2}\frac{\partial\psi}{\partial\eta}(\Psi-u)^-\,dx>$$
$$>\int_{B(x_0,r)\cap\Omega}\Lambda_{p(\cdot)}|\psi|^{p(x)-2}\psi(\Psi-u)^-\,dx. \quad (115)$$

Since $(\Psi-u)^-=0$ on $\partial B(x_0,r)\cap\overline{\Omega}$, we have

$$\int_{\partial(B(x_0,r)\cap\Omega)} |\nabla\psi|^{p(x)-2}\frac{\partial\psi}{\partial\eta}(\Psi-u)^- \, dx = \int_{B(x_0,r)\cap\partial\Omega} |\nabla\psi|^{p(x)-2}\frac{\partial\psi}{\partial\eta}(\Psi-u)^- \, dx.$$

Thus,

$$\begin{aligned}\int_{\{x\in B(x_0,r):\Psi(x)<u(x)\}\cap\Omega} &|\nabla\Psi|^{p(x)-2}\nabla\Psi\cdot\nabla(u-\Psi)\, dx \\ &> \int_{\{x\in B(x_0,r):\Psi(x)<u(x)\}\cap\partial\Omega} |\nabla\psi|^{p(x)-2}\frac{\partial\psi}{\partial\eta}(u-\Psi)\, dx \\ &\quad + \int_{\{x\in B(x_0,r):\Psi(x)<u(x)\}\cap\Omega} \Lambda_{p(\cdot)}|\psi|^{p(x)-2}\psi(u-\Psi)\, dx. \end{aligned} \tag{116}$$

Using the extension of $(\Psi - u)^-$ by zero outside $B(x_0, r) \cap \Omega$ as a test function in (91), we obtain

$$\begin{aligned}\int_{\{x\in B(x_0,r):\Psi(x)<u(x)\}\cap\Omega} &|\nabla u|^{p(x)-2}\nabla u\cdot\nabla(u-\Psi)\, dx \\ &= -\beta \int_{\{x\in B(x_0,r):\Psi(x)<u(x)\}\cap\partial\Omega} |u|^{p(x)-2}u(u-\Psi)\, dx \\ &\quad + \int_{\{x\in B(x_0,r):\Psi(x)<u(x)\}\cap\Omega} \Lambda_{p(\cdot)}|u|^{p(x)-2}u(u-\Psi)\, dx. \end{aligned} \tag{117}$$

Subtracting (116) from (117) gives

$$\begin{aligned}\int_{\{x\in B(x_0,r):\Psi(x)<u(x)\}\cap\Omega} &\left(|\nabla u|^{p(x)-2}\nabla u - |\nabla\Psi|^{p(x)-2}\nabla\Psi\right)\cdot\nabla(u-\Psi)\, dx \\ &< \int_{\{x\in B(x_0,r):\Psi(x)<u(x)\}\cap\Omega} \Lambda_{p(\cdot)}\left(|u|^{p(x)-2}u - |\psi|^{p(x)-2}\psi\right)(u-\Psi)\, dx \\ &\quad - \int_{\{x\in B(x_0,r):\Psi(x)<u(x)\}\cap\partial\Omega} \left(|\nabla\psi|^{p(x)-2}\frac{\partial\psi}{\partial\eta} + \beta|u|^{p(x)-2}u\right)(u-\Psi)\, dx.\end{aligned}$$

Recalling that $r > 0$ was chosen sufficiently small so that (113) holds, we obtain that

$$\int_{\{x \in B(x_0,r):\Psi(x)<u(x)\}\cap\partial\Omega} \left(|\nabla\psi|^{p(x)-2}\frac{\partial\psi}{\partial\eta} + \beta|u|^{p(x)-2}u\right)(u-\Psi)\ dx \geq 0.$$

Thus,

$$\begin{aligned}\int_{\{x \in B(x_0,r):\Psi(x)<u(x)\}\cap\Omega} &\left(|\nabla u|^{p(x)-2}\nabla u - |\nabla\Psi|^{p(x)-2}\nabla\Psi\right)\cdot\nabla(u-\Psi)\ dx \\ &< \int_{\{x \in B(x_0,r):\Psi(x)<u(x)\}\cap\Omega} \Lambda_{p(\cdot)}\left(|u|^{p(x)-2}u - |\psi|^{p(x)-2}\psi\right)(u-\Psi)\ dx \leq 0, \end{aligned} \tag{118}$$

where the last inequality follows from the fact that $u \leq \psi$ on $B(x_0,r)\cap\Omega$. In view of (52) we have

$$\begin{aligned}\frac{1}{2^{p^+-1}}\int_{\{x \in B(x_0,r):\Psi(x)<u(x)\}\cap\Omega} &|\nabla u - \nabla\Psi|^{p(x)}\ dx \\ &\leq \int_{\{x \in B(x_0,r):\Psi(x)<u(x)\}\cap\Omega} \left(|\nabla u|^{p(x)-2}\nabla u - |\nabla\Psi|^{p(x)-2}\nabla\Psi\right)\cdot\nabla(u-\Psi)\ dx. \end{aligned} \tag{119}$$

Combining (118) and (119) leads to a contradiction. We conclude that u is a viscosity subsolution of (90). □

5.2. The Asymptotic Behavior of the First Eigenvalue/Eigenfunction Pairs

Consider a sequence of functions $\{p_n\} \subset C^1(\overline{\Omega})$ satisfying the assumptions (72), (73), (74), and (75).

The goal of this section is to analyze the asymptotic behavior of the positive first eigenfunctions and the corresponding eigenvalues of the $p_n(\cdot)$-Laplacian with Robin boundary conditions as $n \to \infty$. The relevant problem is

$$\begin{cases} -\Delta_{p_n(x)}u = \Lambda_{p_n(\cdot)}|u|^{p_n(x)-2}u & \text{in } \Omega \\ |\nabla u|^{p_n(x)-2}\frac{\partial u}{\partial\eta} + \beta|u|^{p_n(x)-2}u = 0 & \text{on } \partial\Omega. \end{cases} \tag{120}$$

Our assumption (73) and the embeddings available for variational exponent Sobolev spaces (see Section 2.2) imply that the weak solutions $u = u_n$ of (120) are continuous for $n \in \mathbb{N}$ sufficiently large. Also, it is known (see, e.g. [23], [24], or the survey [33]) that for each $n \in \mathbb{N}$ fixed, we have either $u_n > 0$ in Ω or $u_n < 0$ in Ω (first eigenfunctions do not change sign in Ω). Throughout the rest of the paper we will restrict our attention to the positive first eigenfunctions $u_n > 0,\ n \in \mathbb{N}$. The complementary case of negative eigenfunctions should follow using very similar arguments, although the limiting problem will likely require different boundary conditions.

In view of our analysis of problem (90) in the previous section corresponding to fixed (but possibly nonconstant) exponents, classical variational and duality arguments imply that the first eigenvalues, which we denote in what follows by Λ_n^1, are given by

$$\Lambda_n^1 = \frac{\int\limits_\Omega |\nabla u_n|^{p_n(x)} dx + \beta \int\limits_{\partial\Omega} |u_n|^{p_n(x)} dx}{\int\limits_\Omega |u_n|^{p_n(x)} dx}, \tag{121}$$

where $u_n \in W^{1,p_n(\cdot)}(\Omega)$ is the eigenfunction associated to Λ_n^1, which minimizes the functional

$$W^{1,p_n(\cdot)}(\Omega) \ni u \mapsto \int\limits_\Omega \frac{1}{p_n(x)} |\nabla u|^{p_n(x)} dx + \beta \int\limits_{\partial\Omega} \frac{1}{p_n(x)} |u|^{p_n(x)} dx$$

among all $u \in W^{1,p_n(\cdot)}(\Omega)$ satisfying the constraint $\int\limits_\Omega \frac{1}{p_n(x)} |u|^{p_n(x)} dx = 1$. For each $n \in \mathbb{N}$, we define

$$c_n^1 = \inf \left\{ \int\limits_\Omega \frac{|\nabla u|^{p_n(x)}}{p_n(x)} dx + \beta \int\limits_{\partial\Omega} \frac{|u|^{p_n(x)}}{p_n(x)} dx \ : \ u \in W^{1,p_n(\cdot)}(\Omega)\ ,\ \int\limits_\Omega \frac{|u|^{p_n(x)}}{p_n(x)} dx = 1 \right\} \tag{122}$$

Proposition 6. *The sequence* $\left\{ (\Lambda_n^1)^{\frac{1}{n}} \right\}$ *is bounded.*

Proof. We first show that the sequence $\left\{(c_n^1)^{\frac{1}{n}}\right\}$ is bounded. Let $a > 0$ be a positive constant such that

$$\int_\Omega \frac{(a \cdot \operatorname{dist}(x, \partial\Omega))^{p_n(x)}}{p_n(x)}\, dx = 1, \text{ for all } n \in \mathbb{N}.$$

The existence of a constant a as above has been proven in [44]. Using the function $u(x) = a \cdot \operatorname{dist}(x, \partial\Omega)$ as a test function in (122), we obtain for $n \in \mathbb{N}$ sufficiently large,

$$\begin{aligned}
(c_n^1)^{\frac{1}{n}} &\leq \left(\int_\Omega \frac{|\nabla(a \cdot \operatorname{dist}(x, \partial\Omega))|^{p_n(x)}}{p_n(x)} dx + \int_{\partial\Omega} \frac{\beta |a \cdot \operatorname{dist}(x, \partial\Omega)|^{p_n(x)}}{p_n(x)} dx \right)^{\frac{1}{n}} \\
&= \left(\int_\Omega \frac{a^{p_n(x)}}{p_n(x)} dx \right)^{\frac{1}{n}} \leq \left(\max\{a^{p_n^+}, a^{p_n^-}\} \frac{|\Omega|}{p_n^-} \right)^{\frac{1}{n}} \leq \max\{a^{q^+ + q^-/2}, a^{q^-/2}\} \left(\frac{|\Omega|}{p_n^-} \right)^{\frac{1}{n}}.
\end{aligned}$$

Since, in view of (76), $\lim_{n\to\infty} \left(\frac{|\Omega|}{p_n^-} \right)^{\frac{1}{n}} = 1$, we deduce that $\left\{(c_n^1)^{\frac{1}{n}}\right\}$ is bounded, as claimed. Next, note that we have

$$\int_\Omega |u_n|^{p_n(x)} dx \geq \int_\Omega \frac{p_n^-}{p_n(x)} |u_n|^{p_n(x)} dx = p_n^- \int_\Omega \frac{|u_n|^{p_n(x)}}{p_n(x)} dx = p_n^-,$$

and thus, taking (121) into account, we obtain

$$\begin{aligned}
0 \leq (\Lambda_n^1)^{\frac{1}{n}} &\leq \left(\frac{1}{p_n^-} \right)^{\frac{1}{n}} \left(\int_\Omega |\nabla u_n|^{p_n(x)} dx + \beta \int_{\partial\Omega} |u_n|^{p_n(x)} dx \right)^{\frac{1}{n}} \\
&\leq \left(\frac{1}{p_n^-} \right)^{\frac{1}{n}} \left(\int_\Omega \frac{p_n^+}{p_n(x)} |\nabla u_n|^{p_n(x)} dx + \beta \int_{\partial\Omega} \frac{p_n^+}{p_n(x)} |u_n|^{p_n(x)} dx \right)^{\frac{1}{n}} \\
&= \left(\frac{p_n^+}{p_n^-} \right)^{\frac{1}{n}} \left(\int_\Omega \frac{|\nabla u_n|^{p_n(x)}}{p_n(x)} dx + \beta \int_{\partial\Omega} \frac{|u_n|^{p_n(x)}}{p_n(x)} dx \right)^{\frac{1}{n}} = \left(\frac{p_n^+}{p_n^-} \right)^{\frac{1}{n}} (c_n^1)^{\frac{1}{n}}
\end{aligned}$$

for all $n \in \mathbb{N}$. Since, as already observed in the previous chapter (see page 45), there exists a positive constant $C > 0$ such that the Harnack type inequality $p_n^+ \leq C p_n^-$, $\forall\, n \in \mathbb{N}$ holds, we have

$$\lim_{n\to\infty} \left(\frac{p_n^+}{p_n^-}\right)^{\frac{1}{n}} = 1.$$

From the fact that the sequence $\left\{(c_n^1)^{\frac{1}{n}}\right\}$ is bounded, we deduce that $\left\{(\Lambda_n^1)^{\frac{1}{n}}\right\}$ is bounded as well. This concludes the proof. $\square$

Proposition 7. *For $n \in \mathbb{N}$, let $u_n \in W^{1,p_n(\cdot)}(\Omega)$ be the positive eigenfunction corresponding to the first eigenvalue, Λ_n^1. Then there exists a subsequence of $\{u_n\}$ which converges uniformly in Ω, as $n \to \infty$, to some function $u_\infty \in C(\overline{\Omega}) \setminus \{0\}$.*

Proof. Fix $m \in \mathbb{N}$ and choose $\varepsilon > 0$ such that $\varepsilon < q^-$. We have $\frac{p_n^-}{n} > q^- - \varepsilon > 0$ and $n > m$ for all $n \in \mathbb{N}$ sufficiently large. By Hölder's inequality,

$$\begin{aligned}\int_\Omega |u_n|^{\frac{mp_n(x)}{n}} dx &\leq \left(\int_\Omega |u_n|^{p_n(x)} dx\right)^{\frac{m}{n}} (|\Omega|)^{\frac{n-m}{n}} \leq \left(\int_\Omega |u_n|^{p_n(x)} dx\right)^{\frac{m}{n}} (|\Omega|+1)\\ &\leq \left(p_n^+ \int_\Omega \frac{|u_n|^{p_n(x)}}{p_n(x)} dx\right)^{\frac{m}{n}} (|\Omega|+1) = (p_n^+)^{\frac{m}{n}} (|\Omega|+1).\end{aligned}$$

Since $\lim_{n\to\infty} (p_n^+)^{\frac{m}{n}} = 1$, we obtain that

$$\int_\Omega |u_n|^{\frac{mp_n(x)}{n}} dx \quad \leq 2(|\Omega|+1)$$

for $n \in \mathbb{N}$ sufficiently large. Using similar arguments we obtain, in view of Proposition 6, that there exists a constant $C = C(m) > 0$ such that

$$\int_\Omega |\nabla u_n|^{\frac{mp_n(x)}{n}} dx \leq \Big(\int_\Omega |\nabla u_n|^{p_n(x)} dx\Big)^{\frac{m}{n}} (|\Omega|)^{\frac{n-m}{n}} \leq \left(\Lambda_n^1\right)^{\frac{m}{n}} (p_n^+)^{\frac{m}{n}} (|\Omega|+1) \leq C(m)$$

for all $n \in \mathbb{N}$ sufficiently large. Combining these inequalities with the continuity of the embedding $W^{1,\frac{mp_n(\cdot)}{n}}(\Omega) \subset W^{1,m(q^- -\varepsilon)}(\Omega)$ (here we use the fact that $n \in \mathbb{N}$ was chosen large enough so that $\frac{p_n(x)}{n} \geq \frac{p_n^-}{n} > q^- - \varepsilon$ in Ω), we deduce that the sequence $\{u_n\}$ is bounded in $W^{1,m(q^- -\varepsilon)}(\Omega)$. To guarantee that the embedding of $W^{1,m(q^- -\varepsilon)}(\Omega)$ into $C(\overline{\Omega})$ is compact, choose $m \in \mathbb{N}$ sufficiently large such that $m(q^- - \varepsilon) > N$. Taking into account the reflexivity of the space $W^{1,m(q^- -\varepsilon)}(\Omega)$, it follows that there exists a subsequence (not relabelled) of $\{u_n\}$ and a function $u_\infty \in C(\overline{\Omega})$ such that $u_n \rightharpoonup u_\infty$ weakly in $W^{1,m(q^- -\varepsilon)}(\Omega)$ and $u_n \to u_\infty$ uniformly in Ω.

It remains to prove that $u_\infty \neq 0$. To this aim, recall that the first eigenfunctions satisfy the normalization $\int\limits_\Omega \frac{|u_n|^{p_n(x)}}{p_n(x)} dx = 1$, which implies that

$$\left(\int\limits_\Omega |u_n|^{p_n(x)} dx \right)^{\frac{1}{n}} \geq (p_n^-)^{\frac{1}{n}}. \tag{123}$$

If $n \in \mathbb{N}$ is such that $\|u_n\|_\infty \leq 1$, then $\|u_n\|_\infty^{p_n(\cdot)} \leq \|u_n\|_\infty^{p_n^-}$ in Ω, while if $\|u_n\|_\infty > 1$, we have $\|u_n\|_\infty^{p_n(\cdot)} \leq \|u_n\|_\infty^{p_n^+}$ in Ω. Therefore

$$\int\limits_\Omega |u_n|^{p_n(x)} dx \leq \int\limits_\Omega \|u_n\|_\infty^{p_n(x)} dx \leq |\Omega| \max \left\{ \|u_n\|_\infty^{p_n^-}, \|u_n\|_\infty^{p_n^+} \right\}.$$

Thus, using (123), we obtain

$$\max \left\{ \|u_n\|_\infty^{p_n^-}, \|u_n\|_\infty^{p_n^+} \right\}^{\frac{1}{n}} \geq \left(\frac{p_n^-}{|\Omega|} \right)^{\frac{1}{n}}.$$

Passing to the limit, $n \to \infty$, in this inequality implies that $\max \left\{ \|u_\infty\|_\infty^{q^-}, \|u_\infty\|_\infty^{q^+} \right\} \geq 1$, which concludes the proof. □

Theorem 4. *Let $\{p_n\}$ be a sequence of variable exponents satisfying (72)-(75) and, for each $n \in \mathbb{N}$, let Λ_n^1 and u_n be the first eigenvalue and the positive first eigenfunction* *corresponding to the Robin problem (*120*). Then there exists* $\Lambda_\infty \in \mathbb{R}$ *and* $u_\infty \in C(\overline{\Omega}) \setminus \{0\}$ *such that, after eventually extracting a subsequence, we have*

$$\left(\Lambda_n^1\right)^{\frac{1}{n}} \to \Lambda_\infty \tag{124}$$

and

$$u_n \to u_\infty \ \text{ uniformly } \ \text{in } \ \overline{\Omega}, \tag{125}$$

where u_∞ *is a nontrivial viscosity solution of the problem*

$$\begin{cases} \min\left\{-\Delta_\infty u_\infty - |\nabla u_\infty|^2 \ln|\nabla u_\infty| \langle \xi, \nabla u_\infty\rangle, |\nabla u_\infty|^q - \Lambda_\infty |u_\infty|^q\right\} = 0 & \text{in } \Omega \\ H(x, u_\infty, \nabla u_\infty) = 0 & \text{on } \partial\Omega. \end{cases} \tag{126}$$

Here, $H : \Omega \times [0, \infty) \times \mathbb{R}^N \to \mathbb{R}$ *is given by*

$$H(x, r, \theta) = \begin{cases} \max\left\{|r|^{q(x)} - |\theta|^{q(x)}, \langle \theta, \eta(x)\rangle\right\} & \text{if } r > 0 \\ \langle \theta, \eta(x)\rangle \chi_{(1,\infty)}(|\theta|) & \text{if } r = 0, \end{cases}$$

where $\chi_{(1,\infty)}$ *stands for the characteristic function of the interval* $(1, \infty)$.

Remark 3. *Just as in the previous chapter, at points where the gradient is vanishing, the PDE in (*126*) is interpreted by assuming that the value of* $v \mapsto |v|^2 \ln|v|$ *at* $v = 0$ *is zero.*

Proof. In view of Propositions 6 and 7, we may extract a subsequence (not relabelled) such that (124) and (125) hold. We will show that u_∞ is a viscosity solution of (126). Let $x_0 \in \Omega$, and $\varphi \in C^2(\Omega)$ be such that $u_\infty(x_0) = \varphi(x_0)$ and $u_\infty - \varphi$ has a minimum at x_0. The uniform convergence of u_n to u_∞ implies that there exists a sequence $\{x_n\} \subset \Omega$ such that $x_n \to x_0$, $u_n(x_n) = \varphi(x_n)$, and $u_n - \varphi$ has a minimum at x_n.

Since, by Proposition 5, u_n is (for $n \in \mathbb{N}$ large) a continuous viscosity solution of (120) with $\Lambda_{p_n(\cdot)} = \Lambda_n^1$, we have

$$\begin{aligned} &- |\nabla\varphi(x_n)|^{p_n(x_n)-2} \left(\Delta\varphi(x_n) + \ln|\nabla\varphi(x_n)|\langle\nabla p_n(x_n), \nabla\varphi(x_n)\rangle\right) \\ &- (p_n(x_n)-2)\,|\nabla\varphi(x_n)|^{p_n(x_n)-4}\Delta_\infty\varphi(x_n) \geq \Lambda_n^1|\varphi(x_n)|^{p_n(x_n)-2}\varphi(x_n). \end{aligned} \tag{127}$$

We will need to study two cases. First, if $u_\infty(x_0) > 0$, we have

$$\Lambda_n^1|\varphi(x_n)|^{p_n(x_n)-2}\varphi(x_n) = \Lambda_n^1|u_n(x_n)|^{p_n(x_n)-2}u_n(x_n) > 0,$$

and thus, by (127), we deduce that $|\nabla\varphi(x_n)| > 0$ for $n \in \mathbb{N}$ sufficiently large. Dividing both sides of (127) by $(p_n(x_n)-2)|\nabla\varphi(x_n)|^{p_n(x_n)-4}$, we find

$$\begin{aligned} &\frac{-|\nabla\varphi(x_n)|^2\left(\Delta\varphi(x_n) + \ln|\nabla\varphi(x_n)|\langle\nabla p_n(x_n), \nabla\varphi(x_n)\rangle\right)}{p_n(x_n)-2} - \Delta_\infty\varphi(x_n) \\ &\qquad\geq \left(\frac{(\Lambda_n^1)^{1/n}\,|\varphi(x_n)|^{\frac{p_n}{n}(x_n)-\frac{2}{n}}}{|\nabla\varphi(x_n)|^{\frac{p_n}{n}(x_n)-\frac{4}{n}}}\right)^n \frac{\varphi(x_n)}{p_n(x_n)-2}. \end{aligned}$$

Letting $n \to \infty$, and taking into account (74) gives

$$\begin{aligned} &- \Delta_\infty\varphi(x_0) - |\nabla\varphi(x_0)|^2\ln|\nabla\varphi(x_0)|\langle\xi(x_0), \nabla\varphi(x_0)\rangle \\ &\qquad\geq \limsup_{n\to\infty}\left[\left(\frac{(\Lambda_n^1)^{1/n}|\varphi(x_n)|^{\frac{p_n}{n}(x_n)-\frac{2}{n}}}{|\nabla\varphi(x_n)|^{\frac{p_n}{n}(x_n)-\frac{4}{n}}}\right)^n \frac{\varphi(x_n)}{p_n(x_n)-2}\right]. \end{aligned} \tag{128}$$

In particular, we have that

$$-\Delta_\infty\varphi(x_0) - |\nabla\varphi(x_0)|^2\ln|\nabla\varphi(x_0)|\langle\xi(x_0), \nabla\varphi(x_0)\rangle \geq 0. \tag{129}$$

We now claim that

$$|\nabla\varphi(x_0)|^{q(x_0)} - \Lambda_\infty|\varphi(x_0)|^{q(x_0)} \geq 0. \tag{130}$$

Indeed, otherwise $|\nabla\varphi(x_0)|^{q(x_0)} < \Lambda_\infty|\varphi(x_0)|^{q(x_0)}$, and taking into account that (75) and (124) imply

$$\lim_{n\to\infty}\left(\frac{(\Lambda_n^1)^{1/n}|\varphi(x_n)|^{\frac{p_n}{n}(x_n)-\frac{2}{n}}}{|\nabla\varphi(x_n)|^{\frac{p_n}{n}(x_n)-\frac{4}{n}}}\right) = \frac{\Lambda_\infty|\varphi(x_0)|^{q(x_0)}}{|\nabla\varphi(x_0)|^{q(x_0)}} > 1, \tag{131}$$

we deduce that there exists $\varepsilon > 0$ such that

$$\frac{(\Lambda_n^1)^{1/n}|\varphi(x_n)|^{\frac{p_n}{n}(x_n)-\frac{2}{n}}}{|\nabla\varphi(x_n)|^{\frac{p_n}{n}(x_n)-\frac{4}{n}}} \geq 1+\varepsilon$$

for all $n \in \mathbb{N}$ sufficiently large. We are led to

$$\limsup_{n\to\infty}\left(\left(\frac{(\Lambda_n^1)^{1/n}|\varphi(x_n)|^{\frac{p_n}{n}(x_n)-\frac{2}{n}}}{|\nabla\varphi(x_n)|^{\frac{p_n}{n}(x_n)-\frac{4}{n}}}\right)^n \frac{\varphi(x_n)}{p_n(x_n)-2}\right) \geq \lim_{n\to\infty}\frac{(1+\varepsilon)^n}{n}\left(\frac{\varphi(x_n)}{\frac{p_n(x_n)-2}{n}}\right) = \infty,$$

which is a contradiction with (128). Hence, (130) holds, as claimed. From (129) and (130) we deduce that

$$\min\Big\{ -\Delta_\infty\varphi(x_0) - |\nabla\varphi(x_0)|^2\ln|\nabla\varphi(x_0)|\langle\xi(x_0),\nabla\varphi(x_0)\rangle, \\ |\nabla\varphi(x_0)|^{q(x_0)} - \Lambda_\infty|\varphi(x_0)|^{q(x_0)}\Big\} \geq 0. \tag{132}$$

On the other hand, if $u_\infty(x_0) = \varphi(x_0) = 0$, we either have $\nabla\varphi(x_0) \neq 0$, (if so, we can use very similar arguments to conclude that (129) and (130) hold), or else $\nabla\varphi(x_0) = 0$, in which case, taking into account that $\Delta_\infty\varphi(x_0) = 0$ and Remark 3, we again obtain (129), while (130) is clearly also true in this case. Overall, we conclude that (132) holds. The proof of the fact that

$$\min\Big\{ -\Delta_\infty\psi(x_0) - |\nabla\psi(x_0)|^2\ln|\nabla\psi(x_0)|\langle\xi(x_0),\nabla\psi(x_0)\rangle, \\ |\nabla\psi(x_0)|^{q(x_0)} - \Lambda_\infty|\psi(x_0)|^{q(x_0)}\Big\} \leq 0 \tag{133}$$

whenever $\psi \in C^2(\Omega)$ is such that $u_\infty(x_0) = \psi(x_0)$ and $u_\infty - \psi$ has a strict maximum at $x_0 \in \Omega$, proceeds along the same lines. We omit the details.

It remains to show that the boundary condition in (126) is satisfied in the viscosity sense. Let $x_0 \in \partial\Omega$. We need to consider four cases. First, assume that $u_\infty - \varphi$ has a minimum at a point $x_0 \in \partial\Omega$ with $u_\infty(x_0) = \varphi(x_0) > 0$. Using the uniform convergence of u_n to u_∞ we deduce that there exists $x_n \in \overline{\Omega}$ such that $x_n \to x_0$ and $u_n - \varphi$ has a minimum point at x_n. If $x_n \in \Omega$ for infinitely many $n \in \mathbb{N}$, we can argue as above to obtain

$$\min\Big\{ -\Delta_\infty\varphi(x_0) - |\nabla\varphi(x_0)|^2 \ln|\nabla\varphi(x_0)|\langle\xi(x_0), \nabla\varphi(x_0)\rangle, \\ |\nabla\varphi(x_0)|^{q(x_0)} - \Lambda_\infty|\varphi(x_0)|^{q(x_0)}\Big\} \geq 0. \quad (134)$$

On the other hand, if there exists $N \in \mathbb{N}$ such that $x_n \in \partial\Omega$ for all $n \geq N$, we have

$$\max\Big\{ -\Delta_{p_n(x_n)}\varphi(x_n) - \Lambda_n^1|\varphi(x_n)|^{p_n(x_n)-2}\varphi(x_n), \\ |\nabla\varphi(x_n)|^{p_n(x_n)-2}\frac{\partial\varphi}{\partial\eta}(x_n) + \beta|\varphi(x_n)|^{p_n(x_n)-2}\varphi(x_n)\Big\} \geq 0 \quad (135)$$

for all $n \geq N$. If there exists a subsequence $\{k_n\}_{n\geq N}$ such that

$$-\Delta_{p_{k_n}(x_{k_n})}\varphi(x_{k_n}) - \Lambda_{k_n}^1|\varphi(x_{k_n})|^{p_{k_n}(x_{k_n})-2}\varphi(x_{k_n}) \geq 0$$

for all $n \geq N$, then we can again argue as before to deduce that (134) holds. Otherwise, by taking N larger if necessary, we may assume without loss of generality that we have

$$|\nabla\varphi(x_n)|^{p_n(x_n)-2}\frac{\partial\varphi}{\partial\eta}(x_n) + \beta|\varphi(x_n)|^{p_n(x_n)-2}\varphi(x_n) \geq 0 \quad \forall n \geq N.$$

Since $\varphi(x_0) > 0$, it follows that $\varphi(x_n) > 0$ for sufficiently large $n \in \mathbb{N}$, and thus, after dividing both sides above by $|\varphi(x_n)|^{p_n(x_n)-2}$, we obtain

$$\left(\left|\frac{\nabla\varphi(x_n)}{\varphi(x_n)}\right|^{\frac{p_n}{n}(x_n)-\frac{2}{n}}\right)^n \frac{\partial\varphi}{\partial\eta}(x_n) \geq -\beta\varphi(x_n). \tag{136}$$

We claim that either $\left|\frac{\nabla\varphi(x_0)}{\varphi(x_0)}\right|^{q(x_0)} \leq 1$ or $\frac{\partial\varphi}{\partial\eta}(x_0) \geq 0$. Indeed, if $\left|\frac{\nabla\varphi(x_0)}{\varphi(x_0)}\right|^{q(x_0)} > 1$ and $\frac{\partial\varphi}{\partial\eta}(x_0) < 0$ then letting $n \to \infty$ in (136) leads to

$$-\infty = \lim_{n\to\infty}\left(\left|\frac{\nabla\varphi(x_n)}{\varphi(x_n)}\right|^{\frac{p_n}{n}(x_n)-\frac{2}{n}}\right)^n \frac{\partial\varphi}{\partial\eta}(x_n) \geq \lim_{n\to\infty}(-\beta\varphi(x_n)) = -\beta\varphi(x_0),$$

which is a contradiction. Therefore, $\max\left\{|\varphi(x_0)|^{q(x_0)} - |\nabla\varphi(x_0)|^{q(x_0)}, \frac{\partial\varphi}{\partial\eta}(x_0)\right\} \geq 0$. Overall, taking (134) into account, we have the following inequality

$$\begin{aligned}&\max\Bigg\{\max\left\{|\varphi(x_0)|^{q(x_0)} - |\nabla\varphi(x_0)|^{q(x_0)}, \frac{\partial\varphi}{\partial\eta}(x_0)\right\},\\ &\min\left\{-\Delta_\infty\varphi(x_0) - |\nabla\varphi(x_0)|^2\ln|\nabla\varphi(x_0)|\langle\xi(x_0),\nabla\varphi(x_0)\rangle, |\nabla\varphi(x_0)|^{q(x_0)} - \Lambda_\infty|\varphi(x_0)|^{q(x_0)}\right\}\Bigg\} \geq 0.\end{aligned}$$

If $u_\infty - \varphi$ has a minimum at a point $x_0 \in \partial\Omega$ with $u_\infty(x_0) = \varphi(x_0) = 0$, let $x_n \in \overline{\Omega}$ be such that $x_n \to x_0$ and x_n is a minimum point for $u_n - \varphi$. If $x_n \in \Omega$ for infinitely many n, we can argue as before to deduce that (134) holds, while if there exists $N \in \mathbb{N}$ such that $x_n \in \partial\Omega$ for all $n \in \mathbb{N}$, then (135) holds for all such n. Arguing as in the previous case, it suffices to analyze what happens when

$$|\nabla\varphi(x_n)|^{p_n(x_n)-2}\frac{\partial\varphi}{\partial\eta}(x_n) + \beta|\varphi(x_n)|^{p_n(x_n)-2}\varphi(x_n) \geq 0 \quad \forall n \geq N.$$

Since $\varphi(x_0) = 0$, we have $\lim\limits_{n\to\infty}\left(\beta|\varphi(x_n)|^{p_n(x_n)-2}\varphi(x_n)\right) = 0$. Thus,

$$\liminf_{n\to\infty}\left(\left(|\nabla\varphi(x_n)|^{\frac{p_n}{n}(x_n)-\frac{2}{n}}\right)^n \frac{\partial\varphi}{\partial\eta}(x_n)\right) \geq 0. \tag{137}$$

We claim that

$$\chi_{(1,\infty)}\left(|\nabla\varphi(x_0)|^{q(x_0)}\right)\frac{\partial\varphi}{\partial\eta}(x_0)\geq 0. \tag{138}$$

To see why this is true, note that if $|\nabla\varphi(x_0)|>1$ then (137) implies that $\frac{\partial\varphi}{\partial\eta}(x_0)\geq 0$, and since we clearly have $\chi_{(1,\infty)}\left(|\nabla\varphi(x_0)|^{q(x_0)}\right)=1$ in this case, we obtain (138). On the other hand, (138) clearly holds if $|\nabla\varphi(x_0)|\leq 1$. We deduce that

$$\max\Bigg\{\chi_{(1,\infty)}\left(|\nabla\varphi(x_0)|^{q(x_0)}\right)\frac{\partial\varphi}{\partial\eta}(x_0),$$
$$\min\Big\{-\Delta_\infty\varphi(x_0)-|\nabla\varphi(x_0)|^2\ln|\nabla\varphi(x_0)|\langle\xi(x_0),\nabla\varphi(x_0)\rangle,|\nabla\varphi(x_0)|^{q(x_0)}-\Lambda_\infty|\varphi(x_0)|^{q(x_0)}\Big\}\Bigg\}\geq 0.$$

For the remaining two cases, assume that $u_\infty-\psi$ has a maximum at $x_0\in\partial\Omega$, and consider the case where $u_\infty(x_0)=\psi(x_0)>0$. As before, the uniform convergence of u_n to u_∞ implies the existence of a maximum point $x_n\in\overline{\Omega}$ of $u_n-\psi$ such that $x_n\to x_0$. If $x_n\in\Omega$ for infinitely many $n\in\mathbb{N}$, we can argue as in the case where x_0 was an interior point to obtain

$$\min\Big\{-\Delta_\infty\psi(x_0)-|\nabla\psi(x_0)|^2\ln(|\nabla\psi(x_0)|)\langle\xi(x_0),\nabla\psi(x_0)\rangle,$$
$$|\nabla\psi(x_0)|^{q(x_0)}-\Lambda_\infty|\psi(x_0)|^{q(x_0)}\Big\}\leq 0. \tag{139}$$

If, on the other hand, there exists $N\in\mathbb{N}$ such that $x_n\in\partial\Omega$ for all $n\geq N$, we have, since u_n is a viscosity subsolution of (120) (with $\Lambda_{p_n(\cdot)}$ replaced by Λ_n^1),

$$\min\Big\{-\Delta_{p_n(x_n)}\psi(x_n)-\Lambda_n^1|\varphi(x_n)|^{p_n(x_n)-2}\psi(x_n),$$
$$|\nabla\psi(x_n)|^{p_n(x_n)-2}\frac{\partial\psi}{\partial\eta}(x_n)+\beta|\psi(x_n)|^{p_n(x_n)-2}\psi(x_n)\Big\}\leq 0$$

for all $n\geq N$. If, for a subsequence $\{k_n\}_{n\geq N}$, the left hand side above is equal to $-\Delta_{p_{k_n}(x_{k_n})}\varphi(x_{k_n}) - \Lambda^1_{k_n}|\varphi(x_{k_n})|^{p_{k_n}(x_{k_n})-2}\varphi(x_{k_n})$, we again argue as in the interior point case to deduce that (139) holds. Otherwise, we may assume without loss of generality (take N larger, if necessary) that

$$|\nabla\psi(x_n)|^{p_n(x_n)-2}\frac{\partial\psi}{\partial\eta}(x_n) + \beta|\psi(x_n)|^{p_n(x_n)-2}\psi(x_n) \leq 0 \text{ for all } n \geq N.$$

Since we are in the case where $\psi(x_0) > 0$, we may assume that $\psi(x_n) > 0$ for all $n \in \mathbb{N}$ sufficiently large, which allows us to rewrite the previous inequality in the form

$$\left(\left|\frac{\nabla\psi(x_n)}{\psi(x_n)}\right|^{\frac{p_n}{n}(x_n)-\frac{2}{n}}\right)^n \frac{\partial\psi}{\partial\eta}(x_n) \leq -\beta\psi(x_n). \tag{140}$$

We claim that

$$|\psi(x_0)|^{q(x_0)} - |\nabla\psi(x_0)|^{q(x_0)} \leq 0 \quad \text{and} \quad \frac{\partial\psi}{\partial\eta}(x_0) \leq 0.$$

Indeed, if $|\nabla\psi(x_0)|^{q(x_0)} < |\psi(x_0)|^{q(x_0)}$, passing to the limit in (140) gives

$$0 = \lim_{n\to\infty}\left(\left|\frac{\nabla\psi(x_n)}{\psi(x_n)}\right|^{\frac{p_n}{n}(x_n)-\frac{2}{n}}\right)^n \frac{\partial\psi}{\partial\eta}(x_n) \leq \lim_{n\to\infty}(-\beta\psi(x_n)) = -\beta\psi(x_0) < 0,$$

a contradiction. Also, if $\frac{\partial\psi}{\partial\eta}(x_0) > 0$, we obtain

$$0 \leq \liminf_{n\to\infty}\left(\left|\frac{\nabla\psi(x_n)}{\psi(x_n)}\right|^{p_n(x_n)-2} \frac{\partial\psi}{\partial\eta}(x_n)\right) \leq \lim_{n\to\infty}(-\beta\psi(x_n)) = -\beta\psi(x_0) < 0,$$

which is again a contradiction. Therefore

$$\max\left\{|\psi(x_0)|^{q(x_0)} - |\nabla\psi(x_0)|^{q(x_0)}, \frac{\partial\psi}{\partial\eta}(x_0)\right\} \leq 0.$$

Overall, we have shown that

$$\min\Bigg\{\max\Big\{|\psi(x_0)|^{q(x_0)}-|\nabla\psi(x_0)|^{q(x_0)},\frac{\partial\psi}{\partial\eta}(x_0)\Big\},$$
$$\min\Big\{-\Delta_\infty\psi(x_0)-|\nabla\psi(x_0)|^2\ln|\nabla\psi(x_0)|\langle\xi(x_0),\nabla\psi(x_0)\rangle,|\nabla\psi(x_0)|^{q(x_0)}-\Lambda_\infty|\psi(x_0)|^{q(x_0)}\Big\}\Bigg\}\leq 0.$$

Finally, assume that $u_\infty-\psi$ has a maximum at $x_0\in\partial\Omega$ with $u_\infty(x_0)=\psi(x_0)=0$. Let $x_n\in\overline{\Omega}$ be a maximum point of $u_n-\psi$ such that $x_n\to x_0$. If $x_n\in\Omega$ for infinitely many $n\in\mathbb{N}$, then the inequality (139) can be deduced via the usual arguments, while if $x_n\in\partial\Omega$ for all $n\geq N$, we can again restrict our attention to the case where

$$|\nabla\psi(x_n)|^{p_n(x_n)-2}\frac{\partial\psi}{\partial\eta}(x_n)+\beta|\psi(x_n)|^{p_n(x_n)-2}\psi(x_n)\leq 0,\ \forall\ n\geq N.$$

The fact that $\psi(x_0)=0$ implies $\lim\limits_{n\to\infty}\big(\beta|\psi(x_n)|^{p_n(x_n)-2}\psi(x_n)\big)=0$, and hence we must have

$$\limsup_{n\to\infty}\left(\left(|\nabla\psi(x_n)|^{\frac{p_n}{n}(x_n)-\frac{2}{n}}\right)^n\frac{\partial\psi}{\partial\eta}(x_n)\right)\leq 0. \tag{141}$$

We still need to show that

$$\chi_{(1,\infty)}\left(|\nabla\psi(x_0)|^{q(x_0)}\right)\frac{\partial\psi}{\partial\eta}(x_0)\leq 0.$$

The inequality clearly holds if $|\nabla\psi(x_0)|\leq 1$, while if $|\nabla\psi(x_0)|>1$, it follows from the fact that $\frac{\partial\psi}{\partial\eta}(x_0)\leq 0$, which is a consequence of (141). We are now able to conclude that

$$\min\Bigg\{\chi_{(1,\infty)}\left(|\nabla\psi(x_0)|^{q(x_0)}\right)\frac{\partial\psi}{\partial\eta}(x_0),$$
$$\min\Big\{-\Delta_\infty\psi(x_0)-|\nabla\psi(x_0)|^2\ln|\nabla\psi(x_0)|\langle\xi(x_0),\nabla\psi(x_0)\rangle,|\nabla\psi(x_0)|^{q(x_0)}-\Lambda_\infty|\psi(x_0)|^{q(x_0)}\Big\}\Bigg\}\leq 0,$$

which completes the proof. □

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