GORENSTEIN DIMENSION OVER SOME RINGS OF THE FORM $R \oplus C$

A Dissertation
Submitted to the Graduate Faculty
of the
North Dakota State University
of Agriculture and Applied Science

By

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In Partial Fulfillment of the Requirements
for the Degree of
DOCTOR OF PHILOSOPHY

Major Department:
Mathematics

August 2015

Fargo, North Dakota

NORTH DAKOTA STATE UNIVERSITY

Graduate School

Title

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The supervisory committee certifies that this dissertation complies with North Dakota State University's regulations and meets the accepted standards for the degree of

DOCTOR OF PHILOSOPHY

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ABSTRACT

Commutative algebra is the study of commutative rings and other abstract structures based on commutative rings. Modules can be viewed as a common generalization of several of those structures, and some invariants, e.g. homological dimensions, of modules are used to characterize certain properties of the base ring. Some generalizations of such invariants include C-Gorenstein dimensions, where C is a semidualizing module over a commutative noetherian ring.

Holm and Jørgensen [16] investigate some connections between C-Gorenstein dimensions of an R-complex and Gorenstein dimensions of the same complex viewed as a complex over the "trivial extension" $R \ltimes C$. I generalize some of their results to a certain type of retract diagram. I also investigate some examples of such retract diagrams, namely D'Anna and Fontana's amalgamated duplication [8] and Enescu's pseudocanonical cover [9].

ACKNOWLEDGEMENTS

This dissertation would not be completed without the support of many people.

First, I am deeply grateful to my dissertation advisor, Dr. Sean Sather-Wagstaff, whose extraordinary guidance has been crucial to my graduate career. He provided me with a supportive environment where I could grow as a mathematician. His understanding and support have been instrumental in helping me get through some difficult times. He has been my mentor in every sense of the word.

I am also grateful to several past and present faculty members at the department. Specifically, Dr. Jim Coykendall and Dr. Benton Duncan have gone above and beyond what is required to help me with my career opportunities and professional development. I am also thankful to Dr. Davis Cope for writing recommendation letters to my potential employers. I am fortunate to have been surrounded by such professors.

I also appreciate the help of all of my dissertation committee members, including Dr. Kristi Groberg, for allowing me to arrange a dissertation defense on a short notice.

My fellow graduate students and several staff members, including Melanie Milam and Jacalyn Benson, of the department have also made my life easier during my time here at NDSU. Their companionship and help are greatly appreciated. I express my thanks here to fellow graduate students, Aaron Feickert and Jonathan Totushek, as well. They spent a considerable effort to write the LaTeX class file, on which this dissertation is built.

On a more personal note, this dissertation owes its existence to my parents. Without the arduous effort and sacrifice of my parents and my sister, I would not be here pursuing a higher education. Their confidence in me has kept me going.

Finally, I am grateful for my wife. My marriage to her is one of the best things that have happened in my life. Without her patience, understanding and loving support, my life in graduate school would have been difficult.

DEDICATION

This dissertation is dedicated to my father, who used to begin his answers to my elementary mathematical questions with something akin to *Dr. Sheldon Cooper*'s "It's a warm summer evening in ancient Greece."

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1. INTRODUCTION

This chapter serves as a brief introduction to *commutative algebra*, *module theory* and *homological dimensions*, leading to my research. The goal is to motivate and illustrate my research, and this chapter, with the exception of its last section, strives to achieve this goal by using a minimal amount of technical details.

1.1. Commutative Algebra

Mathematics has evolved over the course of human history from a primitive skill set of counting things in everyday life to a beautiful discipline that is increasingly abstract, and yet, firmly grounded in reality. This very process of abstraction has made mathematics versatile and progressive.

We present *commutative algebra* as a case in point. To introduce commutative algebra, we first describe its building blocks, *commutative rings*. At the heart of a commutative ring are the concepts based on elementary arithmetic operations of addition, subtraction and multiplication. Among these operation, we use addition and multiplication below to demonstrate the nature of a commutative ring.

Numbers can be of course added and multiplied together. For example, we consider the set of integers, $\{\ldots, -2, -1, 0, 1, 2, \ldots\}$. The sum of two integers is an integer, and so is the product of two integers. This property is called *closure under addition and multiplication* of integers. In other words, adding and multiplying two integers does not yield a result that is outside the realm of integers. (It is worthwhile to note here that division does not behave in the same way with integers: an integer divided by another nonzero integer does not always yield an integer.) We also know that these arithmetic operations of addition and multiplication have some other nice properties. For example, multiplication of two integers has a property called *commutativity*, i.e., for any two integers a and b, we have $ab = ba$. Addition and multiplication of integers also satisfy a property called *distributivity*, i.e., for any integers a, b and c, we have $a(b + c) = ab + ac$ and $(a + b)c = ac + bc$.

However, we also discover that integers are *not* the only mathematical entities that can be added and multiplied together. There are other mathematical objects, such as polynomials, that can be added and multiplied together as well. To illustrate this fact more precisely, consider polynomials in one variable with integer coefficients. For example, $f = 2 - x + 3x^2$ and $g = -1 + x$ are polynomials with integer coefficients. The sum of these two polynomials f and g is

$$\begin{aligned} f + g &= (2 - x + 3x^2) + (-1 + x) \\ &= (2 - 1) + (-x + x) + 3x^2 \\ &= 1 + 3x^2 \end{aligned}$$

which is also a polynomial with integer coefficients. Likewise, the product of these two polynomials f and g is

$$\begin{aligned} fg &= (2 - x + 3x^2)(-1 + x) \\ &= (2 - x + 3x^2)(-1) + (2 - x + 3x^2)x \\ &= -2 + x - 3x^2 + 2x - x^2 + 3x^3 \\ &= -2 + 3x - 4x^2 + 3x^3 \end{aligned}$$

which is also a polynomial with integer coefficients. We therefore notice that polynomials in one variable with integer coefficients also have this *closure under addition and multiplication* property. Also note that, for any polynomials f, g and h, we have $fg = gf$, $f(g + h) = fg + fh$ and $(f + g)h = fh + gh$, which are commutativity and distributivity, just like for integers.

Besides numbers and polynomials, there are numerous other types of mathematical objects that can be added and multiplied together with the same underlying abstract properties. In other words, these properties, such as closure under addition and multiplication, commutativity and distributivity, are not exclusive to a single type of mathematical entity. It is therefore more efficient and insightful for us to extract these underlying concepts, removing any dependencies on specific mathematical objects, and conceptualize an abstract construct called *a commutative ring.* A commutative ring is defined as *any* set of elements that can be added and multiplied together, satisfying the aforementioned properties, such as closure, commutativity and distributivity, along with a few other properties that we did not mention.

The set of integers, denoted as $\mathbb{Z}$, therefore is an example of a commutative ring, where elements are integers, addition is the usual addition of numbers and multiplication is also the usual multiplication of numbers. The set of polynomials in one variable with integer coefficients, denoted as $\mathbb{Z}[X]$, is also an example of a commutative ring, where elements are such polynomials, addition is polynomial addition and multiplication is polynomial multiplication.

One of the many benefits of such an abstraction is efficiency. Once a mathematical result, such as a theorem or a lemma, concerning commutative rings has been proved, that result can be readily applied for all commutative rings, including integers and polynomials, without having to discover similar results for each and every commutative ring separately. For example, commutativity and distributivity properties can be used to prove the formula $(a+b)^2 = a^2 + 2ab + b^2$ for any a and b in a commutative ring. Therefore, this formula holds regardless of whether a and b are integers, or a and b are polynomial functions. As long as the things we are adding and multiplying satisfy the said properties, in this case commutativity and distributivity, the formula is guaranteed to hold.

Another benefit of abstraction is the greater insight it can provide. Recounting the formula above, consider the following: instead of extracting the underlying concepts, if we proved the formula separately, say once for integers and once for polynomials, then it would not be obvious that the real reason the formula holds is because of commutativity and distributivity, not just because we are working with integers or polynomials.

Commutative algebra is the study of commutative rings and other abstract structures that are built upon commutative rings. (We discuss three such structures, namely *fields*, *vector spaces* and *modules*, in some details in the following sections.) It has far-reaching applications ranging from error-correcting codes and data encryption to algebraic geometry and theoretical physics. Interactions between commutative algebra and other branches of mathematics, such as geometry, topology and combinatorics, have also advanced mathematics as a discipline.

1.2. Fields, Vector Spaces and Modules

Among the various structures in commutative algebra, we focus below on structures called *modules*. To introduce the notion of modules, we first need to consider two other structures called *fields* and *vector spaces*.

Wherein a commutative ring is defined as a set where its elements can be added and multiplied together obeying some rules, a field is defined as a commutative ring, where we can divide any element by any nonzero element, obeying some rules, such as closure under division. For example, the set of rational numbers, i.e., integers and fractions, satisfies the property of closure under division because dividing any fraction by any nonzero fraction always yields a fraction. However, as we have noted in the previous section, the set of integers does not satisfy the closure under division property: for example, dividing the integer 3 by the nonzero integer 2 yields 3/2 or 1.5, which is not an integer. Therefore, while the set of rational numbers, denoted as $\mathbb{Q}$, is an example of a field, the set $\mathbb{Z}$ is not. The set of real numbers, i.e., any number that can be represented by a point on the number line, denoted as $\mathbb{R}$, is also another example of a field. (See Figure 1.1.)

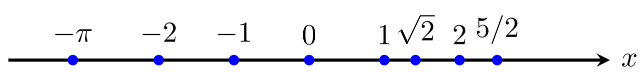


Figure 1.1. A few examples of real numbers drawn in blue.

A *vector space* is an abstract construct that is based on a field. Loosely, a vector space is a set with two operations called *vector addition* and *scalar multiplication*. To visualize a vector space, consider, for example, two-dimensional vectors, i.e., arrows that can be drawn on a plane. Such a vector is represented by an ordered pair of numbers, e.g., $\mathbf{u} = \langle 1, 2 \rangle$. If $\mathbf{v} = \langle -1, 0 \rangle$, we can add the two vectors together to get another two-dimensional vector $\mathbf{u} + \mathbf{v} = \langle 1, 2 \rangle + \langle -1, 0 \rangle = \langle 1 - 1, 2 + 0 \rangle = \langle 0, 2 \rangle$. This is an example of vector addition. To visualize scalar multiplication, imagine stretching or scaling the vector $\mathbf{u}$ by a factor of 2 to get a new vector, i.e., $\mathbf{w} = 2\mathbf{u} = 2\langle 1, 2 \rangle = \langle 2 \times 1, 2 \times 2 \rangle = \langle 2, 4 \rangle$, and shrinking the vector $\mathbf{w}$ by a factor of 2 to get the original vector $\mathbf{u}$ back, i.e., $\mathbf{u} = (1/2)\mathbf{w} = \langle 1, 2 \rangle$. Both of these operations, multiplying $\mathbf{u}$ by 2, and multiplying $\mathbf{w}$ by (1/2), are examples of scalar multiplication, where *scalars* used are 2 and 1/2, respectively. See Figure 1.2.

This set of two-dimensional vectors on a plane, denoted as $\mathbb{R}^2$, with any real number as a scalar, is an example of a vector space. Note that, in this example, any vector stretched by a nonzero factor k can be shrunk again to the original size by multiplying the vector by an appropriate

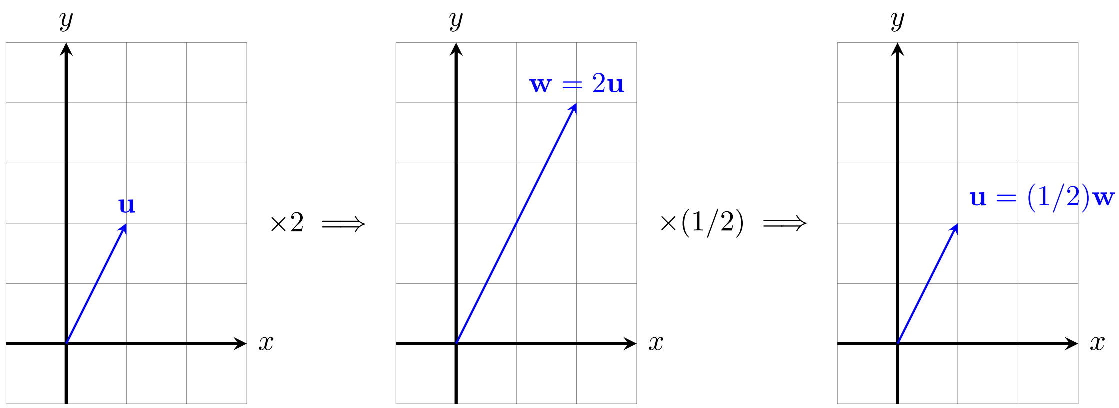


Figure 1.2. Stretching and shrinking vectors by scalar multiplication

scalar, i.e., $1/k$. In order for such a property to hold, division among scalars must be allowed. In other words, scalars must come from a field. Therefore, an abstract vector space can be loosely defined as a set with two operations called vector addition and scalar multiplication that follows some rules, and with scalars from a field. For example, we say that $\mathbb{R}^2$ is a vector space over the field of scalars $\mathbb{R}$. In short, we say $\mathbb{R}^2$ is a $\mathbb{R}$-vector space. There are also other more esoteric examples of vector spaces.

A *module* is a vector space, where scalars do not necessarily come from a field, i.e., division among scalars does not necessarily work. For example, consider the same example, $\mathbb{R}^2$, we used above. However, instead of using $\mathbb{R}$, which is a field, for scalars, suppose that the scalars come from $\mathbb{Z}$, which is only a commutative ring, *not* a field. Then, $\mathbb{R}^2$ is not a vector space over $\mathbb{Z}$, but $\mathbb{R}^2$ is a module over commutative ring $\mathbb{Z}$. In short, we say $\mathbb{R}^2$ is not a $\mathbb{Z}$-vector space, but a $\mathbb{Z}$-module. To visualize this, we can consider a figure similar to Figure 1.2. (See Figure 1.3.) In this case however, suppose that scalars come from $\mathbb{Z}$. The first scalar multiplication is allowed because 2 is an integer. However, the second scalar multiplication is not valid in this case because $1/2$ is not an integer. Therefore, we can very loosely describe a module as a vector space where *shrinking a stretched vector back to its original size need not work*. In other words, in a module, a scalar multiplication operation does not have to be reversible.

There are numerous other important examples of modules, such as *homomorphism modules*. However, they are almost impossible to describe without providing a substantial background. One example of a module that is relatively easy to describe is $\mathbb{Z}^2$, the set of lattice points, i.e., points with integer coordinates. (See Figure 1.4.)

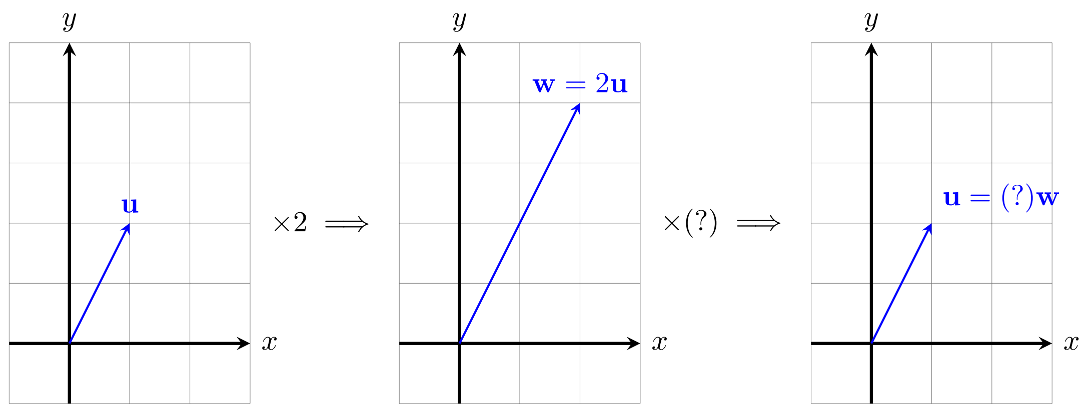


Figure 1.3. No integer scalar can be used to multiply **w** back to **u**.

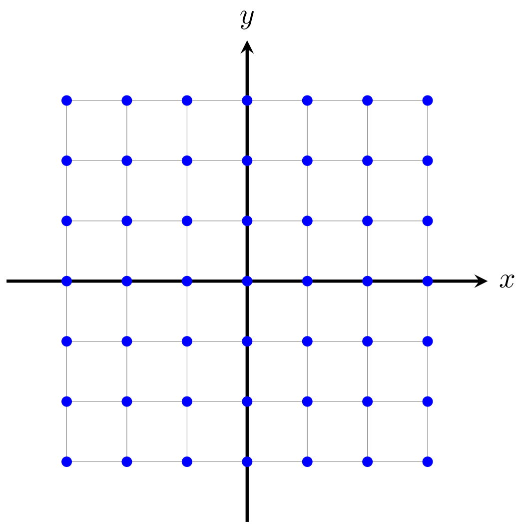


Figure 1.4. The set of lattice points, $\mathbb{Z}^2$, is a $\mathbb{Z}$-module. (Lattice points are drawn in blue.)

The set $\mathbb{Z}^2$ is indeed a $\mathbb{Z}$-module, with vectors being arrows starting at the origin and ending at a lattice point, and scalars being integers.

In summary, a module is a generalization of a vector space, where the scalars need not come from a field; a commutative ring is enough. On the other hand, a vector space can be viewed as a special kind of a module. For a commutative ring R, an R-module is an R-vector space if and only if R is a field.

1.3. Module Theory

One may ask at this point why we wish to focus on modules among different abstract constructs in commutative algebra. There are many good reasons to do so, and they may vary from one group of mathematicians to another. We present two of them as our reasons to study module theory.

One first reason is their ubiquity in commutative algebra. Although we introduced modules as generalized vector spaces, the notion of module in fact can be viewed as a *common generalization* [2, p. 17] of several other important structures in commutative algebra. Since any reasonable description of those structures requires a substantial amount of background, we list some of them below without explaining what they are. The names of those structures are in italics.

Suppose that R is a commutative ring and M is an R-module:

(i) The ring R itself is an R-module, and any *ideal* in R is an R-module.

(ii) If R is a field, then M is an *R-vector space*, as seen in the previous section.

(iii) If R is $\mathbb{Z}$, then M is an *abelian group*.

(iv) If R is a *group-algebra* of a *finite abelian group* G over the field k, i.e., $R = k[G]$, then M is a *k-representation of G*.

As with any good generalization, since the notion of a module is a generalization for several other structures, a result about modules can be interpreted for a wide range of objects with relative ease.

Our second reason is the strong interplay between a module and its ring. For example, we can rephrase (ii) from the above list as follows: a commutative ring R is a field if and only if every R-module is a vector space. In other words, we can characterize a ring based on its modules. Our maxim, therefore, is, "to study a ring, study its modules." More examples of this deep connection between a ring and its modules are presented in the sections below.

1.4. Dimensions as Invariants

Suppose that we are studying a class of modules to see if it possesses a certain property, say, property P. If the setting of our problem is specific enough, i.e., if we know enough about the class of modules and the property P, the answer is usually either "yes" or "no." Suppose that the answer is "no." It is natural to ask how far off our modules are from having this property P. In other words, we need to quantify the "defect" in our modules that prevents them from having the property P. This is sometimes done by defining a certain *dimension* as an invariant to measure the "defect" in a module. In many cases, the smaller this number is, the closer the module is to having property P. This invariant is sometimes extremely useful to characterize the base ring of the module as well.

To illustrate this phenomenon of measuring a "defect," we consider a similar situation involving numbers. There are, of course, different types of numbers, such as integers, rational numbers, real numbers, and so on. Suppose that we are interested in the numbers 3, 3/2, 2/3 and π, and we ask ourselves for each of these four numbers, is it an integer? The answer is yes for 3, but no for all the rest. One way to see that is to write each of them down as a decimal number:

$$3 = 3. \qquad 3/2 = 1.5 \qquad 2/3 = 0.\overline{6} \qquad \pi = 3.14159265\ldots$$

where "$\overline{6}$" means that the digit 6 repeats endlessly, and "..." means non-terminating non-recurring string of digits. The number 3 is an integer of course because it does not need any decimal expansion; it is a whole number. In other words, no digit or symbol is necessary after the decimal point to write 3. The rest of the numbers are not integers obviously. However, in some sense, 3/2 and 2/3 are closer to being an integer than π is. After all, both 3/2 and 2/3 are rational numbers, and we note that 3/2 is just half of an integer 3, and 2/3 is just a third of integer 2. However, we can say no such things for π. This difference in severity of the "defect of not being an integer" shows up in the length of the decimal expansion of the number; terminating or recurring decimal expressions can be rewritten as rational numbers (fractions) and are closer to being integers; but non-terminating non-recurring decimal expressions represent *irrational* (real but not rational) numbers, and they are further away from being integers.

Similarly, we measure the "defect" of a module not possessing a certain property. We do this by using an appropriate dimension. Suppose that we are studying a property called *projectivity* of an R-module M. Just as we ask if a certain number is an integer, we can ask if a certain module M is *projective*. If the answer is "no," then we would like to measure how far it is from being projective. Just as the length of a decimal expansion of a number can be used to gauge how far it is from being an integer, we can use a special kind of sequence called *projective resolution* of the module to quantify how close the module is to being projective:

$$\cdots P_n \to \cdots P_1 \to P_0 \to M \to 0.$$

The length of this projective resolution is called the *projective dimension* of M, denoted by $\operatorname{pd}_R M$. As in the case of numbers, where a number is an integer if and only if it requires no digit after the decimal point, the module M is projective if and only if $\operatorname{pd}_R M = 0$. Just as a terminating (or recurring) decimal expansion of the number indicates that it is a rational number, a terminating projective resolution of M, i.e., when $\operatorname{pd}_R M < \infty$, indicates that the base ring of the module M has a special property called *regularity*. The finiteness of $\operatorname{pd}_R M$ indicates that the module is particularly nice, and when all R-modules have this property, the ring R is *regular*. This result is called the Regularity Theorem and was discovered by Auslander-Buchsbaum [5] and Serre [22].

1.5. Homological Dimensions

As in the famous theorem of Auslander-Buchsbaum [5] and Serre [22] where projective dimension of R-modules is used to characterize regularity of R, Auslander and Bridger introduced Gorenstein dimension in [4] to characterize Gorenstein rings: a local ring R is Gorenstein if and only if every finitely generated R-module M has finite Gorenstein dimension, i.e., $\text{G-dim}_R M < \infty$. To extend similar results to non-finitely generated R-modules, Enochs and Jenda introduced Gorenstein projective dimension [10]. In particular, a local ring is Gorenstein if and only if every (finitely generated) R-module M has finite Gorenstein projective dimension, i.e., $\operatorname{Gpd}_R M < \infty$; see [6, 12]. Enochs and Jenda also studied the Gorenstein injective dimension Gid and, with Torrecillas [11], the Gorenstein flat dimension Gfd.

Semidualizing R-modules, first introduced by Foxby in [13] and later studied by Vasconcelos [24] and Golod [14], arise naturally in the study of the connection between R and its modules:

a finitely generated R-module C is *semidualizing* if $R \cong \mathrm{Hom}_R(C,C)$ and $\mathrm{Ext}^i_R(C,C) = 0$ for all $i \geqslant 1$. For example, the ring R itself is a semidualizing R-module. As the duality properties with respect to the ring R of totally reflexive modules are useful in studying Gorenstein dimension, the duality properties with respect to semidualizing R-modules are useful in generalizing totally reflexive modules and Gorenstein dimension. For example, Golod introduced the G_C-dimension in [14] and proved a formula of the same type as the Auslander-Buchsbaum formula and the Auslander-Bridger formula.

Holm and Jørgensen extended the G_C-dimension in [16] by introducing three new homological dimensions called the C-Gorenstein projective, C-Gorenstein injective and C-Gorenstein flat dimensions, denoted as $C\text{-}\mathrm{Gpd}_R(M)$, $C\text{-}\mathrm{Gid}_R(M)$, and $C\text{-}\mathrm{Gfd}_R(M)$, respectively, for an R-complex M. They also proved how these new dimensions coincide with Enochs, Jenda and Torrecillas' Gorenstein dimensions over the trivial extension $R \ltimes C$ [16, Theorem 2.16]. (The trivial extension is used to put a module inside a commutative ring to study the link between them. See Section 3.3 below for some details.) This means that for an R-module M, one has

$$C\text{-}\mathrm{Gpd}_R(M) = \mathrm{Gpd}_{R\ltimes C}(M)$$
$$C\text{-}\mathrm{Gid}_R(M) = \mathrm{Gid}_{R\ltimes C}(M)$$
$$C\text{-}\mathrm{Gfd}_R(M) = \mathrm{Gfd}_{R\ltimes C}(M).$$

In this paper, we generalize the above result to a triple (R,S,C) in the setting of the following retract diagram

$$\begin{array}{ccc} R & \xrightarrow{f} & S \\ & {\scriptstyle \mathrm{id}_R}\searrow & \downarrow{\scriptstyle g} \\ & & R \end{array}$$

where R and S are commutative rings, C is an R-module, f and g are ring homomorphisms, and id_R is the identity map on R, satisfying the following properties:

(i) $C \cong \mathrm{Ker}\, g$,

(ii) $\mathrm{Hom}_R(S,C) \cong S$ as S-modules, and

(iii) $\mathrm{Ext}^i_R(S,C) = 0$ for all $i \geqslant 1$.

It should be noted here that the triple $(R, R \ltimes C, C)$ satisfies the above conditions if C is semidualizing over R; see Section 3.3 below. Therefore, the following result generalizes Holm and Jørgensen's Theorem 2.16 [16]; see Theorem 3.2.21 below.

Theorem A. *In the setting of the above retract diagram, if C is a semidualizing R-module, then for every homologically left-bounded R-complex M and every homologically right-bounded R-complex N one has*

$$C\text{-}\mathrm{Gid}_R(M) = \mathrm{Gid}_S(M)$$

$$C\text{-}\mathrm{Gpd}_R(N) = \mathrm{Gpd}_S(N)$$

$$C\text{-}\mathrm{Gfd}_R(N) = \mathrm{Gfd}_S(N).$$

Along the way we also prove the following characterization of semidualizing modules; see Theorem 3.2.6.

Theorem B. *In the setting of the above retract diagram, if C is a finitely generated R-module, then the following conditions are equivalent:*

(a) C is semidualizing over R;

(b) R is Gorenstein projective over S and $\mathrm{Ann}_R(C) = 0$; and

(c) C is Gorenstein projective over S and $\mathrm{Ann}_R(C) = 0$.

We show that $S = R \ltimes C$ is not the only example of a ring satisfying our generalized settings set forth in the retract diagram above. See Theorems 3.3.9 and 3.3.14, along with their corollaries.

Theorem C. *The following examples satisfy the hypotheses of Theorem A, i.e. the triple (R, S, C) satisfies the conditions (i) through (iii) in the above retract diagram if we replace S with each of the following rings:*

(a) *D'Anna and Fontana's [8] amalgamated duplication $S = R \bowtie C$, and*

(b) *Enescu's [9] pseudocanonical cover $S = S(h)$, when h is a square in R.*

In particular, part (a) recovers Theorem 3.2 of Salimi, Tavasoli and Yassemi [20], and part (b) regarding pseudocanonical covers is a new result.

We end this introduction by summarizing the contents of the remainder of this dissertation. Chapter 2 contains background material, Chapter 3 contains our new results, which are to appear in [3], and Chapter 4 contains further research goals that are currently in progress.

2. PRELIMINARIES

This chapter is a collection of some preliminary definitions and properties to be used in presenting my current and future research in Chapters 3 and 4. From this point on, we assume throughout this dissertation that R is commutative noetherian ring with identity. We also note here that any module in this dissertation is assumed to be a unital bimodule based on a commutative ring with identity.

2.1. Notations for Some Classes of Modules

In this section, we set the notations for some classes of modules that are ubiquitous in our research.

Notation 2.1.1. Let C be an R-module.

(a) Let $\mathcal{I}$ be the class of injective R-modules.

(b) Let $\mathcal{P}$ be the class of projective R-modules.

(c) Let $\mathcal{F}$ be the class of flat R-modules.

(d) Let $\mathcal{I}_C$ be the class of R-modules isomorphic to $\operatorname{Hom}_R(C, I)$ for some injective R-module I.

(e) Let $\mathcal{P}_C$ be the class of R-modules isomorphic to $C \otimes_R P$ for some projective R-module P.

(f) Let $\mathcal{F}_C$ be the class of R-modules isomorphic to $C \otimes_R F$ for some flat R-module F.

2.2. Foxby Classes

The following two classes, known collectively as *Foxby classes*, are associated with a finitely generated R-module C. These definitions first appeared in [13], and they are studied in conjunction with various homological dimensions, such as the G-dimension in [27], the C-projective dimension in [23] and the Gorenstein projective dimension in [26].

Definition 2.2.1. Let C be a finitely generated R-module. The *Auslander class* $\mathcal{A}_C(R)$ is the class of all R-modules M such that

(a) the natural map $\gamma_M^C : M \to \operatorname{Hom}_R(C, C \otimes_R M)$, defined as $\gamma_M^C(m)(c) := c \otimes_R m$ for all $m \in M$ and $c \in C$, is an isomorphism; and

(b) $\operatorname{Tor}_i^R(C,M)=0=\operatorname{Ext}_R^i(C,C\otimes_R M)$ for all $i\geqslant 1$.

Definition 2.2.2. Let C be a finitely generated R-module. The *Bass class* $\mathcal{B}_C(R)$ is the class of all R-modules M such that

(a) the evaluation map $\xi_M^C : C\otimes_R \operatorname{Hom}_R(C,M)\to M$, defined as $\xi_M^C(c\otimes_R \psi):=\psi(c)$ for all $c\in C$ and $\psi\in\operatorname{Hom}_R(C,M)$, is an isomorphism; and

(b) $\operatorname{Ext}_R^i(C,M)=0=\operatorname{Tor}_i^R(C,\operatorname{Hom}_R(C,M))$ for all $i\geqslant 1$.

2.3. Resolutions, Coresolutions and Complete Resolutions

Resolutions and coresolutions are used to define, characterize and study modules. They are defined as follows.

Definition 2.3.1. Let M be an R-module, and let $\mathcal{A}$ be a class of R-modules. Then an *augmented $\mathcal{A}$-resolution $\underline{X}^+$ of M* is an exact sequence of R-modules of the form

$$\underline{X}^+ = \cdots \xrightarrow{\partial_2^X} X_1 \xrightarrow{\partial_1^X} X_0 \to M \to 0$$

where $X_i\in\mathcal{A}$ for each integer $i\geqslant 0$. The R-complex

$$\underline{X} = \cdots \xrightarrow{\partial_2^X} X_1 \xrightarrow{\partial_1^X} X_0 \to 0$$

is the associated *$\mathcal{A}$-resolution* of M.

Definition 2.3.2. Let N be an R-module, and let $\mathcal{B}$ be a class of R-modules. Then an *augmented $\mathcal{B}$-coresolution ${}^+\underline{Y}$ of N* is an exact sequence of R-modules of the form

$${}^+\underline{Y} = \quad 0 \to N \to Y_0 \xrightarrow{\partial_0^Y} Y_{-1} \xrightarrow{\partial_{-1}^Y} \cdots$$

where $Y_j\in\mathcal{B}$ for each integer $j\leqslant 0$. The R-complex

$$\underline{Y} = \quad 0 \to Y_0 \xrightarrow{\partial_0^Y} Y_{-1} \xrightarrow{\partial_{-1}^Y} \cdots$$

is the associated *$\mathcal{B}$-coresolution* of N.

As we can see in the above definitions, resolutions are bounded on the right whereas coresolutions are bounded on the left. We define below several types of *complete resolutions*, which are not necessarily bounded on the either side. It should be noted here that several items from Notation 2.1.1 are used in the following definitions.

Definition 2.3.3. Let C be an R-module.

(a) A *complete $\mathcal{I}_C\mathcal{I}$-resolution* $\underline{X}$ of R-modules is an exact sequence of R-modules of the form

$$\underline{X} = \cdots \xrightarrow{\partial_2^X} X_1 \xrightarrow{\partial_1^X} X_0 \xrightarrow{\partial_0^X} X_{-1} \xrightarrow{\partial_{-1}^X} \cdots$$

such that the module $X_i \in \mathcal{I}_C$ for each integer $i \geqslant 1$, the module $X_j \in \mathcal{I}$ for each integer $j \leqslant 0$, and the sequence $\operatorname{Hom}_R(A, \underline{X})$ is exact for each $A \in \mathcal{I}_C$.

(b) A *complete $\mathcal{P}\mathcal{P}_C$-resolution* $\underline{X}$ of R-modules is an exact sequence of R-modules of the form

$$\underline{X} = \cdots \xrightarrow{\partial_2^X} X_1 \xrightarrow{\partial_1^X} X_0 \xrightarrow{\partial_0^X} X_{-1} \xrightarrow{\partial_{-1}^X} \cdots$$

such that the module $X_i \in \mathcal{P}$ for each integer $i \geqslant 0$, the module $X_j \in \mathcal{P}_C$ for each integer $j \leqslant -1$, and the sequence $\operatorname{Hom}_R(\underline{X}, A)$ is exact for each $A \in \mathcal{P}_C$.

(c) A *complete $\mathcal{F}\mathcal{F}_C$-resolution* $\underline{X}$ of R-modules is an exact sequence of R-modules of the form

$$\underline{X} = \cdots \xrightarrow{\partial_2^X} X_1 \xrightarrow{\partial_1^X} X_0 \xrightarrow{\partial_0^X} X_{-1} \xrightarrow{\partial_{-1}^X} \cdots$$

such that the module $X_i \in \mathcal{F}$ for each integer $i \geqslant 0$, the module $X_j \in \mathcal{F}_C$ for each integer $j \leqslant -1$, and the sequence $A \otimes_R \underline{X}$ is exact for each $A \in \mathcal{I}_C$.

2.4. C-Gorenstein-Injectivity, -Projectivity and -Flatness

Using complete resolutions, we next define C-Gorenstein injectivity, C-Gorenstein projectivity and C-Gorenstein flatness.

Definition 2.4.1. Let C be an R-module.

(a) An R-module M is *C-Gorenstein injective* if there is a complete $\mathcal{I}_C\mathcal{I}$-resolution $\underline{X}$, as in Definition 2.3.3(a), such that $\operatorname{Ker} \partial_0^X \cong M$.

(b) An R-module M is *C-Gorenstein projective* if there is a complete $\mathcal{PP}_C$-resolution $\underline{X}$, as in Definition 2.3.3(b), such that $\operatorname{Coker} \partial_1^X \cong M$.

(c) An R-module M is *C-Gorenstein flat* if there is a complete $\mathcal{FF}_C$-resolution $\underline{X}$, as in Definition 2.3.3(c), such that $\operatorname{Coker} \partial_1^X \cong M$.

When $C = R$, Definition 2.4.1 reduces to the definitions of *Gorenstein injectivity*, *Gorenstein projectivity* and *Gorenstein flatness* of Enochs, Jenda, and Torrecillas [10, 11], with complete $\mathcal{I}_C\mathcal{I}$-resolution, complete $\mathcal{PP}_C$-resolution and complete $\mathcal{FF}_C$-resolution becoming *complete injective resolution*, *complete projective resolution* and *complete flat resolution*, respectively. We also note that Definition 2.4.1 is equivalent to [16, Definition 2.7]. This fact is stated specifically in the following lemmas.

Lemma 2.4.2. *Let C and M be R-modules. Then M is C-Gorenstein injective if and only if the following conditions are satisfied.*

(a) *For each $A \in \mathcal{I}_C$, we have $\operatorname{Ext}_R^i(A, M) = 0$ for all $i \geqslant 1$.*

(b) *M admits an augmented $\mathcal{I}_C$-resolution $\underline{Y}^+$ such that $\operatorname{Hom}_R(A, \underline{Y}^+)$ is exact for each $A \in \mathcal{I}_C$.*

Proof. To prove the forward implication, we assume that M is C-Gorenstein injective. Then, by Definition 2.4.1(a), there exists a complete $\mathcal{I}_C\mathcal{I}$-resolution $\underline{X}$ such that $\operatorname{Ker} \partial_0^X \cong M$. By "slicing" $\underline{X}$, we have the following commutative diagram

$$\begin{array}{c} \underline{X} = \cdots \xrightarrow{\partial_3^X} X_2 \xrightarrow{\partial_2^X} X_1 \xrightarrow{\partial_1^X} X_0 \xrightarrow{\partial_0^X} X_{-1} \xrightarrow{\partial_{-1}^X} \cdots \\ X_1 \xrightarrow{\pi} M \xrightarrow{\epsilon} X_0 \\ 0 \to M \to 0 \end{array} \tag{2.4.2.1}$$

where all three sequences, namely $\underline{X}$,

$$^{+}\underline{I} := 0 \to M \xrightarrow{\epsilon} \underbrace{X_0}_{\text{degree } 0} \xrightarrow{\partial_0^X} X_{-1} \xrightarrow{\partial_{-1}^X} \cdots, \text{ and}$$

$$\underline{Y}^+ := \cdots \xrightarrow{\partial_3^X} X_2 \xrightarrow{\partial_2^X} \underbrace{X_1}_{\text{degree } 0} \xrightarrow{\pi} M \to 0$$

are all exact. By construction, $^+\underline{I}$ is an augmented $\mathcal{I}$-coresolution of M, and $\underline{Y}^+$ is an augmented $\mathcal{I}_C$-resolution of M.

For any $A \in \mathcal{I}_C$, let $(-)_*$ denote the left-exact covariant functor $\operatorname{Hom}_R(A,-)$. Applying $(-)_*$ to (2.4.2.1) yields the following diagram.

$$\begin{array}{ccccccccccc}
\cdots \xrightarrow{(\partial_3^X)_*} & (X_2)_* & \xrightarrow{(\partial_2^X)_*} & (X_1)_* & & \xrightarrow{\qquad(\partial_1^X)_*\qquad} & & (X_0)_* & \xrightarrow{(\partial_0^X)_*} & (X_{-1})_* & \xrightarrow{(\partial_{-1}^X)_*} \cdots \\
 & & & & \searrow^{(\pi)_*} & & \nearrow^{(\epsilon)_*} & & & & \\
 & & & & & (M)_* & & & & & \\
 & & & & \nearrow & & \searrow & & & & \\
 & & & 0 & & & & 0 & & &
\end{array}$$

The above diagram is commutative by functoriality of the covariant functor $(-)_*$, as follows: $\left(\partial_1^X\right)_* = (\epsilon \circ \pi)_* = (\epsilon)_* \circ (\pi)_*$. Since $\underline{X}$ is a complete $\mathcal{I}_C\mathcal{I}$-resolution, the sequence $(\underline{X})_*$ is exact. We note that $(^+\underline{I})_*$ is also exact – at degrees 1 and 0 by the left-exactness of $(-)_*$ and at degree i for each integer $i \leqslant -1$ by the exactness of $(\underline{X})_*$. Since $(\underline{X})_*$ and $(^+\underline{I})_*$ are both exact, so is $\left(\underline{Y}^+\right)_*$.

Since $^+\underline{I}$ is an augmented $\mathcal{I}$-coresolution of M, we have $\underline{I}$ as an $\mathcal{I}$-coresolution of M, which we use to compute $\operatorname{Ext}_R^i(A,M)$ for each integer $i \geqslant 1$ as follows.

$$\begin{aligned}
\operatorname{Ext}_R^i(A,M) &\cong H_{-i}(\operatorname{Hom}_R(A,\underline{I})) & \\
&\cong H_{-i}\left(\left(^+\underline{I}\right)_*\right) & \text{(since } i \geqslant 1\text{)} \\
&= 0 & \text{(since } \left(^+\underline{I}\right)_* \text{ is exact)}
\end{aligned}$$

satisfying condition (a). Since $\underline{Y}^+$ is an augmented $\mathcal{I}_C$-resolution of M such that $\left(\underline{Y}^+\right)_*$ is exact, condition (b) is satisfied as well. This concludes the proof of the forward implication of the lemma.

To prove the reverse implication, we assume that the conditions (a) and (b) are satisfied. For any $A \in \mathcal{I}_C$, let $(-)_*$ denote the left-exact covariant functor $\operatorname{Hom}_R(A,-)$. Since M is an R-module, it admits an augmented $\mathcal{I}$-coresolution

$$^+\underline{I} := 0 \to M \xrightarrow{\epsilon} \underbrace{I_0}_{\text{degree } 0} \xrightarrow{\partial_0^I} I_{-1} \xrightarrow{\partial_{-1}^I} \cdots .$$

By condition (b), M admits an augmented $\mathcal{I}_C$-resolution

$$\underline{Y}^+ := \cdots \xrightarrow{\partial_2^Y} Y_1 \xrightarrow{\partial_1^Y} \underbrace{Y_0}_{\text{degree } 0} \xrightarrow{\pi} M \to 0$$

such that $\left(\underline{Y}^+\right)_*$ is exact. Using ${}^+\underline{I}$ and $\underline{Y}^+$, we "splice together" a sequence $\underline{X}$ of R-modules as follows:

$$\begin{array}{ccccccccccc}
\underline{X} := \cdots & \xrightarrow{\partial_2^Y} & \underbrace{Y_1}_{\text{degree } 2} & \xrightarrow{\partial_1^Y} & Y_0 & & \xrightarrow{\partial_1^X := \epsilon\circ\pi} & & I_0 & \xrightarrow{\partial_0^I} & \underbrace{I_{-1}}_{\text{degree } -1} \xrightarrow{\partial_{-1}^I} \cdots \\
 & & & & & \searrow^{\pi} & & \nearrow^{\epsilon} & & & \\
 & & & & & & M & & & & \\
 & & & & & \nearrow & & \searrow & & & \\
 & & & & 0 & & & & 0 & &
\end{array} \tag{2.4.2.2}$$

that is, set $X_i := I_i$ and $\partial_i^X := \partial_i^I$ for each integer $i \leqslant 0$; set $X_j := Y_{j-1}$ for each integer $j \geqslant 1$, and $\partial_j^X := \partial_{j-1}^Y$ for each integer $j \geqslant 2$; and set $\partial_1^X := \epsilon \circ \pi$. We note that this diagram is commutative by construction. Since both $\underline{I}^+$ and $\underline{Y}^+$ are exact, so is $\underline{X}$ by a diagram chase.

Applying $(-)_*$ to (2.4.2.2) yields the following diagram.

$$\begin{array}{ccccccccccc}
\cdots & \xrightarrow{(\partial_2^Y)_*} & (Y_1)_* & \xrightarrow{(\partial_1^Y)_*} & (Y_0)_* & & \xrightarrow{(\partial_1^X)_*} & & (I_0)_* & \xrightarrow{(\partial_0^I)_*} & (I_{-1})_* \xrightarrow{(\partial_{-1}^I)_*} \cdots \\
 & & & & & \searrow^{(\pi)_*} & & \nearrow^{(\epsilon)_*} & & & \\
 & & & & & & (M)_* & & & & \\
 & & & & & \nearrow & & \searrow & & & \\
 & & & & 0 & & & & 0 & &
\end{array}$$

The above diagram is commutative by functoriality of the covariant functor $(-)_*$, as follows: $\left(\partial_1^X\right)_* = (\epsilon \circ \pi)_* = (\epsilon)_* \circ (\pi)_*$. We claim that $({}^+\underline{I})_*$ is also exact. Exactness of $({}^+\underline{I})_*$ at $(M)_*$ and $(I_0)_*$ is by the left-exactness of the covariant functor $(-)_*$. For each integer $i \geqslant 1$, we have

$$\begin{aligned}
H_{-i}\left(({}^+\underline{I})_*\right) &= H_{-i}\left((\underline{I})_*\right) && \text{(since } i \geqslant 1\text{)} \\
&\cong \operatorname{Ext}_R^i(A, M) && \text{(since } \underline{I} \text{ is an } \mathcal{I}\text{-coresolution of } M\text{)} \\
&= 0 && \text{(by condition (a).)}
\end{aligned}$$

Therefore $({}^{+}\underline{I})_*$ is exact. We note that $\left(\underline{Y}^{+}\right)_*$ is also exact by condition (b). Since both $({}^{+}\underline{I})_*$ and $\left(\underline{Y}^{+}\right)_*$ are exact, so is $(\underline{X})_*$ by a diagram chase. Therefore $\underline{X}$ is a complete $\mathcal{I}_C\mathcal{I}$-resolution, such that $\operatorname{Ker}\partial_0^X = \operatorname{Ker}\partial_0^I \cong M$ by the exactness of ${}^{+}\underline{I}$. Therefore M is C-Gorenstein injective. □

Lemma 2.4.3. *Let C and M be R-modules. Then M is C-Gorenstein projective if and only if the following conditions are satisfied.*

(a) *For each $A \in \mathcal{P}_C$, $\operatorname{Ext}^i_R(M, A) = 0$ for all $i \geqslant 1$.*

(b) *M admits an augmented $\mathcal{P}_C$-coresolution ${}^{+}\underline{Y}$ such that $\operatorname{Hom}_R({}^{+}\underline{Y}, A)$ is exact for each $A \in \mathcal{P}_C$.*

Proof. The proof is similar to that of Lemma 2.4.2. Instead of "slicing" and "splicing together" a complete $\mathcal{I}_C\mathcal{I}$-resolution, we use a complete $\mathcal{P}\mathcal{P}_C$-resolution. □

Lemma 2.4.4. *Let C and M be R-modules. Then M is C-Gorenstein flat if and only if the following conditions are satisfied.*

(a) *For each $A \in \mathcal{I}_C$, $\operatorname{Tor}_i^R(A, M) = 0$ for all $i \geqslant 1$.*

(b) *M admits an augmented $\mathcal{F}_C$-coresolution ${}^{+}\underline{Y}$ such that $A \otimes_R ({}^{+}\underline{Y})$ is exact for each $A \in \mathcal{I}_C$.*

Proof. The proof is similar to that of Lemma 2.4.2. Instead of "slicing" and "splicing together" a complete $\mathcal{I}_C\mathcal{I}$-resolution, we use a complete $\mathcal{F}\mathcal{F}_C$-resolution. □

2.5. Complexes and Homological Dimensions

Regardless of its origin in algebraic topology, an R-complex as a standalone algebraic structure can be viewed as a generalization of an R-module.

Definition 2.5.1. An *R-complex* X is a sequence of R-module homomorphisms

$$X = \cdots \xrightarrow{\partial^X_{i+2}} X_{i+i} \xrightarrow{\partial^X_{i+1}} X_i \xrightarrow{\partial^X_i} X_{i-1} \xrightarrow{\partial^X_{i-1}} \cdots$$

where $\partial^X_j \partial^X_{j+1} = 0$ for all $j \in \mathbb{Z}$. The R-module X_i is called the *module in degree* i. For any integer j, the R-complex X is said to be *concentrated in degree* j if $X_i = 0$ for all $i \neq j$.

Slightly abusing the notation, an R-module M can be viewed as an R-complex M that is concentrated in degree 0 with $M_0 = M$:

$$M = 0 \to M \to 0.$$

As an R-complex is a generalization of an R-module, several structures and invariants associated with modules also have their complex counterparts. Instead of presenting both the module versions and the complex versions of the definitions of those structures and invariants, we collect in this section the more general complex versions only. To introduce the complex versions, however, we require some background provided below.

Definition 2.5.2. Given an R-complex X, the *homology module in degree* i, denoted as $H_i(X)$, is defined as follows:

$$H_i(X) := \operatorname{Ker} \partial_i^X / \operatorname{Im} \partial_{i-1}^X.$$

The *homology complex* $H(X)$ is defined as follows:

$$H(X) := \cdots \xrightarrow{0} H_{i+i}(X) \xrightarrow{0} H_i(X) \xrightarrow{0} H_{i-1}(X) \xrightarrow{0} \cdots$$

i.e., $H(X)$ is an R-complex with $H(X)_j := H_j(X)$ and $\partial_j^{H(X)} := 0$ for all $j \in \mathbb{Z}$.

It should be noted here that an R-complex X coincides with the notion of an exact sequence of R-modules if and only if the homology complex $H(X)$ is the zero complex, i.e., $H(X) = 0$.

Definition 2.5.3. A *left-bounded R-complex* X is an R-complex X such that, for some integer u, we have $X_j = 0$ for all $j > u$. Similarly, a *right-bounded R-complex* X is an R-complex X such that, for some integer l, we have $X_j = 0$ for all $j < l$. A *bounded R-complex* is an R-complex that is both left-bounded and right-bounded.

There is also a notion of being *homologically bounded* as follows.

Definition 2.5.4. An R-complex X is *homologically left-bounded* [resp. *homologically right-bounded*, *homologically bounded*] if the homology complex $H(X)$ is *left-bounded* [resp. *right-bounded*, *bounded*].

Just as the category of R-modules contains R-modules as objects and R-module homomorphisms as morphisms or arrows, the category of R-complexes contains R-complexes as objects and *chain maps* between them as morphisms or arrows.

Definition 2.5.5. If X and Y are R-complexes, then a *chain map* $f : X \to Y$ is a family of R-module homomorphisms $f_i : X_i \to Y_i$ making the following diagram

$$\begin{array}{ccccccccc}
X & = \cdots \longrightarrow & X_{i+1} & \xrightarrow{\partial^X_{i+1}} & X_i & \xrightarrow{\partial^X_i} & X_{i-1} & \longrightarrow \cdots \\
{\scriptstyle f}\big\downarrow & & \big\downarrow{\scriptstyle f_{i+1}} & & \big\downarrow{\scriptstyle f_i} & & \big\downarrow{\scriptstyle f_{i-1}} & \\
Y & = \cdots \longrightarrow & Y_{i+1} & \xrightarrow{\partial^Y_{i+1}} & Y_i & \xrightarrow{\partial^Y_i} & Y_{i-1} & \longrightarrow \cdots
\end{array}$$

commutative, i.e., $f_j\partial^X_{j+1} = \partial^Y_{j+1}f_{j+1}$ for all $j \in \mathbb{Z}$. The *identity chain map* $\mathrm{id}_X : X \to X$ is the family of identity R-module homomorphisms $\mathrm{id}_{X_j} : X_j \to X_j$ for all $j \in \mathbb{Z}$.

Definition 2.5.6. If $f : X \to Y$ is a chain map from an R-complex X to an R-complex Y, then the *induced chain map* $H(f) : H(X) \to H(Y)$ is a chain map from the R-complex $H(X)$ to the R-complex $H(Y)$ with $H(f)_i(\overline{x}) = \overline{f_i(x)}$ for all $i \in \mathbb{Z}$ and for all $\overline{x}$ in $H_i(X)$.

A central notion in complexes is the notion of a *quasi-isomorphism*.

Definition 2.5.7. A chain map $f : X \to Y$ is an *isomorphism of R-complexes* if there exists a chain map $g : Y \to X$ such that $fg = \mathrm{id}_Y$ and $gf = \mathrm{id}_X$. A chain map $f : X \to Y$ is a *quasi-isomorphism* if the induced chain map $H(f) : H(X) \to H(Y)$ is an isomorphism of R-complexes.

Definition 2.5.8. Given R-complexes X and Y, we say X and Y are *equivalent*, denoted as $X \simeq Y$, if there exist an R-complex Z and two quasi-isomorphisms $f : X \to Z$ and $g : Y \to Z$, i.e. $X \underset{f}{\xrightarrow{\simeq}} Z \underset{g}{\xleftarrow{\simeq}} Y$.

We note here that by formally inverting quasi-isomorphisms, one can build the derived category of R-modules, denoted as $\mathcal{D}(R)$, from the category of R-modules. The construction of $\mathcal{D}(R)$ is quite technical; see [25] and [15] for details. The point is, if $X \simeq Y$, then X and Y are isomorphic to each other as objects in $\mathcal{D}(R)$.

The notions defined above are used to define C-Gorenstein injective, projective and flat resolutions [16, Defintion 2.9].

Definition 2.5.9. Let M be a homologically left-bounded R-complex and N be a homologically right-bounded R-complex.

(a) A *C-Gorenstein injective resolution* of M is a left-bounded R-complex I of C-Gorenstein injective R-modules, such that $I \simeq M$ in $\mathcal{D}(R)$.

(b) A *C-Gorenstein projective resolution* of N is a right-bounded R-complex P of C-Gorenstein projective R-modules, such that $P \simeq N$ in $\mathcal{D}(R)$.

(c) A *C-Gorenstein flat resolution* of N is a right-bounded R-complex F of C-Gorenstein flat R-modules, such that $F \simeq N$ in $\mathcal{D}(R)$.

Using the resolutions defined above, we can define C-Gorenstein dimensions as follows.

Definition 2.5.10. Let M be a homologically left-bounded R-complex and N be a homologically right-bounded R-complex.

(a) The *C-Gorenstein injective dimension* of M, denoted as $C\text{-}\mathrm{Gid}_R\, M$, is defined as follows:

$$C\text{-}\mathrm{Gid}_R\, M := \inf \left\{ \sup\{i \in \mathbb{Z} \mid X_{-i} \neq 0\} \;\middle|\; \begin{array}{c} X \text{ is a } C\text{-Gorenstein injective} \\ \text{resolution of } M \end{array} \right\}.$$

(b) The *C-Gorenstein projective dimension* of N, denoted as $C\text{-}\mathrm{Gpd}_R\, N$, is defined as follows:

$$C\text{-}\mathrm{Gpd}_R\, N := \inf \left\{ \sup\{i \in \mathbb{Z} \mid X_{i} \neq 0\} \;\middle|\; \begin{array}{c} X \text{ is a } C\text{-Gorenstein projective} \\ \text{resolution of } N \end{array} \right\}.$$

(c) The *C-Gorenstein flat dimension* of N, denoted as $C\text{-}\mathrm{Gfd}_R\, N$, is defined as follows:

$$C\text{-}\mathrm{Gfd}_R\, N := \inf \left\{ \sup\{i \in \mathbb{Z} \mid X_{i} \neq 0\} \;\middle|\; \begin{array}{c} X \text{ is a } C\text{-Gorenstein flat} \\ \text{resolution of } N \end{array} \right\}.$$

None of the sets over which the infimum is taken is empty because Examples 2.8 in [16] can be used to construct the required C-Gorenstein resolutions.

It should also be noted here that the above C-Gorenstein injective [resp. projective, flat] dimension coincides with Gorenstein injective [resp. projective, flat] dimension when $C = R$, just as C-Gorenstein injectivity [resp. projectivity, flatness] coincides with Gorenstein injectivity [resp. projectivity, flatness] when $C = R$.

Although it is not necessary for the above definitions, we end this chapter with a brief description of the right derived functor $\mathbf{R}\mathrm{Hom}_R(-,-)$. We use it only a few times below as a convenient language. It is derived from the homomorphism functor for R-complexes.

Definition 2.5.11. Let X and Y be R-complexes. The *Hom complex* $\mathrm{Hom}_R(X,Y)$ is defined as follows. For each integer n, set $\mathrm{Hom}_R(X,Y)_n := \prod_{p\in\mathbb{Z}} \mathrm{Hom}_R(X_p, Y_{p+n})$ and $\partial_n^{\mathrm{Hom}_R(X,Y)}(\{f_p\}) := \{\partial_{p+n}^Y f_p - (-1)^n f_{p-1}\partial_p^X\}$.

Definition 2.5.12. Let X and Y be R-complexes such that either X is homologically right-bounded or Y is homologically left-bounded. The functor $\mathbf{R}\mathrm{Hom}_R(X,Y)$ is defined as the equivalence class of R-complexes of either $\mathrm{Hom}_R(P,Y)$ or $\mathrm{Hom}_R(X,I)$, where P is a right-bounded complex of projective R-modules such that $P \simeq X$ or where I is a left-bounded complex of injective R-modules such that $I \simeq Y$.

3. SEMIDUALIZING MODULES AND GORENSTEIN DIMENSIONS

I present in this chapter my results extending Holm and Jørgensen's Theorem 2.16 in [16] to a more general setting, several examples illustrating our results, and a new characterization of semidualizing modules based on the generalized setting. These results are taken from my article [3].

3.1. Preliminary Preparations

We prepare in this section a few results to fit our needs in our generalized setting. We first note that a couple of results of Ishikawa [18] extend to a slightly more general setting.

Lemma 3.1.1. *Let $f : R \to S$ be a ring homomorphism. Let S be finitely generated as an R-module, let M be an R-module, and let N be an injective R-module. Then the natural map*

$$\Theta_{S,M,N} : S \otimes_R \mathrm{Hom}_R(M, N) \to \mathrm{Hom}_R(\mathrm{Hom}_R(S, M), N)$$

defined as $(\Theta_{S,M,N}(s \otimes_R \psi))(\phi) = \psi(\phi(s))$ for each $s \otimes_R \psi \in S \otimes_R \mathrm{Hom}_R(M, N)$ and each $\phi \in \mathrm{Hom}_R(S, M)$, is an S-module isomorphism.

Proof. By Ishikawa's Hom evaluation result [18, Lemma 1.6], the map $\Theta_{S,M,N}$ is an R-module isomorphism. One readily checks that $\Theta_{S,M,N}$ is also an S-module homomorphism, hence it is an S-module isomorphism. □

Lemma 3.1.2. *Let $f : R \to S$ be a ring homomorphism. Let S be finitely generated as an R-module, let M be an R-module, and let N be a flat R-module. Then the natural map*

$$\Omega_{S,M,N} : \mathrm{Hom}_R(S, M) \otimes_R N \to \mathrm{Hom}_R(S, M \otimes_R N)$$

defined as $\Omega_{S,M,N}(\psi \otimes_R n)(s) = \psi(s) \otimes_R n$ for each $\psi \otimes_R n \in \mathrm{Hom}_R(S, M) \otimes_R N$ and each $s \in S$, is an S-module isomorphism.

Proof. The proof is similar to that of Lemma 3.1.1, using Ishikawa's Tensor evaluation result, Lemma 1.1 in [18], instead. □

We also collect here some properties of injectivity, projectivity and flatness associated with restriction of scalars. A version of this result for $S = R \ltimes C$ is found in [17, Lemma 3.1].

Lemma 3.1.3. *Let $f : R \to S$ be a ring homomorphism.*

(a) *Each injective S-module J is a direct summand of* $\operatorname{Hom}_R(S, I)$ *for some injective R-module I.*

(b) *Each projective S-module Q is a direct summand of $S \otimes_R P$ for some projective R-module P.*

Proof. (a) Since J is also an R-module via f, we have an exact sequence $0 \to J \to I$ of R-modules for some injective R-module I. Applying the left-exact $\operatorname{Hom}_R(S, -)$ to this exact sequence, noting that $\operatorname{Hom}_S(S, J)$ is an S-submodule of $\operatorname{Hom}_R(S, J)$, and using Hom cancellation, we obtain the following S-module iso/mono-morphisms.

$$J \xrightarrow{\cong} \operatorname{Hom}_S(S, J) \hookrightarrow \operatorname{Hom}_R(S, J) \hookrightarrow \operatorname{Hom}_R(S, I).$$

Since J is injective over S, this composite monomorphism splits as desired.

(b) This part is proved dually. $\square$

3.2. Theorems A and B

The main point of this section is to prove Theorems A and B from the introduction.

Property 3.2.1. Let R and S be rings, and let C be an R-module. Then the triple (R, S, C) *satisfies Property 3.2.1* if there is a commutative retract diagram

$$\begin{array}{ccc} R & \xrightarrow{f} & S \\ & \searrow^{\mathrm{id}_R} & \downarrow g \\ & & R \end{array}$$

of ring homomorphisms, with id_R the identity map on R, such that $\operatorname{Hom}_R(S, C) \cong S$ as S-modules and $\operatorname{Ext}^i_R(S, C) = 0$ for all $i \geqslant 1$.

Remark 3.2.2. We first note from the above retract diagram that R is a finitely generated S-module. We also note that if a triple (R, S, C) satisfies Property 3.2.1, then $\mathbf{R}\operatorname{Hom}_R(S, C) \simeq S$ in the derived category $\mathcal{D}(S)$. In other words, if $\underline{I}$ is an injective resolution of C over R, then Property 3.2.1 implies that $\operatorname{Hom}_R(S, \underline{I})$ is an injective resolution of the S-module S.

Property 3.2.3. Let R and S be rings, and let C be an R-module. Then the triple (R, S, C) *satisfies Property 3.2.3* if it satisfies Property 3.2.1 and $C \cong \operatorname{Ker} g$ as R-modules.

Remark 3.2.4. We here note that if (R, S, C) satisfies Property 3.2.3, it follows that $S \cong R \oplus C$ as R-modules because the short exact sequence $0 \to \operatorname{Ker} g \to S \xrightarrow{g} R \to 0$ splits. We also note that the sequence splits because Property 3.2.1 implies that f yields a splitting on the right.

We next state and prove versions of several lemmas of Holm and Jørgensen [16, 17] in the general setting of Properties 3.2.1 and 3.2.3.

Lemma 3.2.5. *Let R and S be rings, and let C be an R-module. If (R, S, C) satisfies Property 3.2.1, then the following statements hold:*

(a) *For any R-module M, we have $\operatorname{Ext}^i_S(M, S) \cong \operatorname{Ext}^i_R(M, C)$ as S-modules for all $i \geqslant 0$.*

(b) *For all $i \geqslant 1$, we have $\operatorname{Ext}^i_S(R, S) = 0$ and $\operatorname{Hom}_S(R, S) \cong C$ as S-modules.*

Proof. (a) Argue as in [17, Lemma 3.2 (ii)] with the ring S taking the place of the trivial extension $R \ltimes C$. The essential point is to use Hom-tensor adjointness with the injective resolution $\operatorname{Hom}_R(S, \underline{I})$ of S, as described in Remark 3.2.2.

(b) This is the special case of part (a) where $M = R$. $\square$

The following is Theorem B from the introduction.

Theorem 3.2.6. *Let R and S be rings, and let C be a finitely generated R-module such that (R, S, C) satisfies Property 3.2.1. Then the following are equivalent:*

(a) C is semidualizing over R;

(b) R is Gorenstein projective over S and $\operatorname{Ann}_R(C) = 0$; and

(c) C is Gorenstein projective over S and $\operatorname{Ann}_R(C) = 0$.

Proof. To prove that (a) implies (b), we assume that C is semidualizing over R. Using Lemma 3.2.5, we note that

$$\operatorname{Ext}^i_S(\operatorname{Hom}_S(R, S), S) \cong \operatorname{Ext}^i_S(C, S) \cong \operatorname{Ext}^i_R(C, C).$$

This is equal to 0 for all $i \geqslant 1$ and isomorphic to R when $i = 0$ because C is semidualizing over R. Again, using the Ext-vanishing from Lemma 3.2.5(b), this means that R is Gorenstein projective over S by [6, Proposition 2.2.2]. We also note that $\mathrm{Ann}_R(C)$ is the kernel of the homothety map $\chi_C^R : R \to \mathrm{Hom}_R(C, C)$, which is 0 because C is semidualizing over R.

To prove that (b) implies (c), we recall that $\mathrm{Hom}_S(-, S)$ preserves the class of finitely generated Gorenstein projective S-modules by [6, Observation 1.1.7]. This proves the desired implication because $C \cong \mathrm{Hom}_S(R, S)$ as S-modules by Lemma 3.2.5(b).

To prove that (c) implies (a), we assume that C is Gorenstein projective over S and $\mathrm{Ann}_R(C) = 0$. Since C is finitely generated over R, it is also finitely generated over S. Therefore, by [6, Theorem 4.2.6], we have

$$\mathrm{Ext}^i_S(C, S) = 0 = \mathrm{Ext}^i_S(\mathrm{Hom}_S(C, S), S)$$

for all $i \geqslant 1$ and the biduality map

$$\delta_C^S : C \to \mathrm{Hom}_S(\mathrm{Hom}_S(C, S), S)$$

is an S-module isomorphism. Using Lemma 3.2.5, we have

$$\mathrm{Ext}^i_R(C, C) = 0 = \mathrm{Ext}^i_R(\mathrm{Hom}_R(C, C), C)$$

for all $i \geqslant 1$ and the biduality map

$$\delta_C^S : C \to \mathrm{Hom}_S(\mathrm{Hom}_S(C, S), S) \cong \mathrm{Hom}_R(\mathrm{Hom}_R(C, C), C)$$

is an R-module isomorphism. Since $\mathrm{Ann}_R(C) = 0$, it follows that C is semidualizing over R by Fact 1.1 in [21]. □

Remark 3.2.7. The assumption $\mathrm{Ann}_R(C) = 0$ is essential in Theorem 3.2.6; see [21, Example 1.2].

Lemma 3.2.8. *Let R and S be rings, let N be a finitely generated R-module, and let C be a semidualizing R-module. If (R, S, C) satisfies Property 3.2.1, and if N is Gorenstein projective* *as an S-module, then the module* $\mathrm{Hom}_R(N, I)$ *is Gorenstein injective over* S *for any injective* R*-module* I.

Proof. Since N is Gorenstein projective over S, the module N has a complete projective resolution $\underline{P}$ over S. Moreover, since N is finitely generated over R (hence over S as well) $\underline{P}$ can be chosen to consist of finitely generated S-modules by [6, Theorems 4.1.4 and 4.2.6]. As in the proof of [17, Lemma 3.3 (ii)], it is straightforward to show that $\mathrm{Hom}_S(\underline{P}, \mathrm{Hom}_R(S, I))$ is a complete injective resolution of $\mathrm{Hom}_R(N, I)$ over S. □

We next establish a version of [17, Lemma 3.3 (ii)] for our general setting.

Proposition 3.2.9. *Let* R *and* S *be rings, and let* C *be a semidualizing* R*-module. If* (R, S, C) *satisfies Property 3.2.1, then for any injective* R*-module* I*, the modules* $\mathrm{Hom}_R(C, I)$ *and* $\mathrm{Hom}_R(R, I) \cong I$ *are Gorenstein injective over* S.

Proof. The modules C and R are Gorenstein projective over S by Theorem 3.2.6. Thus, the duals $\mathrm{Hom}_R(C, I)$ and $\mathrm{Hom}_R(R, I) \cong I$ are Gorenstein injective over S by Lemma 3.2.8. □

Next we prove a version of [17, Lemma 3.4] in the general setting.

Lemma 3.2.10. *Let* R *and* S *be rings, and let* C *be a semidualizing* R*-module. If* (R, S, C) *satisfies Property 3.2.3, then for any injective* R*-module* J*, we have*

$$\mathrm{Ext}^i_S(\mathrm{Hom}_R(S, J), -) \cong \mathrm{Ext}^i_R(\mathrm{Hom}_R(C, J), -)$$

for $i \geqslant 0$ *as functors on the category of* S*-modules.*

Proof. Argue as in the proof of [17, Lemma 3.4] that

$$\mathrm{Hom}_R(S, J) \cong \mathrm{Hom}_R(\mathrm{Hom}_R(S, C), J) \cong S \otimes_R \mathrm{Hom}_R(C, J)$$

as S-modules using Lemma 3.1.1 and the fact that (R, S, C) satisfies Property 3.2.3 (hence Property 3.2.1). If $\underline{P}$ is a projective resolution over R of $\mathrm{Hom}_R(C, J)$, one can argue that $S \otimes_R \underline{P}$ is a projective resolution over S of $S \otimes_R \mathrm{Hom}_R(C, J) \cong \mathrm{Hom}_R(S, J)$. This uses the facts that $S \cong R \oplus C$

as R-modules and $\mathrm{Hom}_R(C, J)$ belongs to $\mathcal{A}_C(R)$ since J is an injective R-module. Using this projective resolution over S of $\mathrm{Hom}_R(S, J)$ and Hom-tensor adjointness, one can obtain the desired isomorphism. □

As a consequence of the above lemma, we have the following proposition.

Proposition 3.2.11. *Let R and S be rings, and let C be a semidualizing R-module. If (R, S, C) satisfies Property 3.2.3 and M is an R-module, then for each $i \geqslant 0$, we have $\mathrm{Ext}^i_R(\mathrm{Hom}_R(C, J), M) = 0$ for every injective R-module J if and only if $\mathrm{Ext}^i_S(U, M) = 0$ for every injective S-module U.*

Proof. As in [16, Corollary 2.3 (1)], this follows from Lemmas 3.1.3(a) and 3.2.10. □

Lemma 3.2.12. *Let R and S be rings, and let C be a semidualizing R-module. If the triple (R, S, C) satisfies Property 3.2.3 and M is an R-module that is Gorenstein injective over S, then there exists a short exact sequence of R-modules*

$$0 \to M' \to \mathrm{Hom}_R(C, I) \to M \to 0$$

for some injective R-module I such that

1. *M' is Gorenstein injective over S*
2. *the above sequence is $\mathrm{Hom}_R(\mathrm{Hom}_R(C, J), -)$-exact for any injective R-module J.*

Proof. The proof begins similarly to that of [17, Lemma 4.1]. Since M is Gorenstein injective over S, it has a complete injective resolution. From this, we can construct the following short exact sequence of S-modules

$$0 \to N \to K \to M \to 0$$

where K is injective over S, N is Gorenstein injective over S and the sequence is $\mathrm{Hom}_S(L, -)$-exact for each injective S-module L, particularly for $L = \mathrm{Hom}_R(S, J)$ where J is any injective R-module.

As in the proof of [17, Lemma 4.1], we can use Lemma 3.1.3(a) to assume without loss of generality that the above sequence is of the form

$$0 \to N \xrightarrow{\epsilon} \mathrm{Hom}_R(S, I) \xrightarrow{\eta} M \to 0 \qquad (3.2.12.1)$$

for some injective R-module I.

Note that we cannot make use of a specific ring structure of S as in the proof of [17, Lemma 4.1], so we use Lemma 3.1.1 instead. Since $S \cong \mathrm{Hom}_R(S, C)$ as S-modules by Property 3.2.1, we have

$$\mathrm{Hom}_R(S, I) \cong \mathrm{Hom}_R(\mathrm{Hom}_R(S, C), I) \cong S \otimes_R \mathrm{Hom}_R(C, I) \qquad (3.2.12.2)$$

as S-modules, where the second isomorphism is by Lemma 3.1.1. (We note that Lemma 3.1.1 is applicable here because I is injective over R, and S is finitely generated over R.) The isomorphisms in (3.2.12.2) enable us to replace $\mathrm{Hom}_R(S, I)$ in (3.2.12.1) with $S \otimes_R \mathrm{Hom}_R(C, I)$ to obtain the top row of the following diagram.

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \xrightarrow{\epsilon'} & S \otimes_R \mathrm{Hom}_R(C, I) & \xrightarrow{\eta'} & M & \longrightarrow & 0 \\
 & & \downarrow{\scriptstyle \psi\circ\epsilon'} & & \downarrow{\scriptstyle \psi} & & \| & & \\
0 & \longrightarrow & M' := \mathrm{Ker}\,\phi & \hookrightarrow & \mathrm{Hom}_R(C, I) & \xrightarrow{\phi} & M & \longrightarrow & 0
\end{array} \qquad (3.2.12.3)$$

The maps ψ and ϕ are defined as follows. For any $s \otimes_R \beta \in S \otimes_R \mathrm{Hom}_R(C, I)$, set $\psi(s \otimes_R \beta) := s\beta$, where the scalar multiplication is afforded by the S-module structure on the R-module $\mathrm{Hom}_R(C, I)$. For any β in $\mathrm{Hom}_R(C, I)$, set $\phi(\beta) := \eta'(1_S \otimes_R \beta)$. It is routine to check that both ψ and ϕ are well-defined S-module homomorphisms and that the diagram (3.2.12.3) is commutative.

As in [17, Lemma 4.1], we can show that the bottom row of the diagram (3.2.12.3) satisfies the desired properties. □

Lemma 3.2.13. *Let R and S be rings, and let C be an R-module such that (R, S, C) satisfies Property 3.2.3. Let M be an R-module that is C-Gorenstein injective over R. Then there exists a short exact sequence of S-modules*

$$0 \to M' \to U \to M \to 0$$

where U is injective over S, M' is C-Gorenstein injective over R and the above sequence is $\mathrm{Hom}_S(V, -)$-exact for any V injective over S.

Proof. The proof is similar to [16, Lemma 2.11], using Lemma 3.1.1 as in the previous result. □

Using the lemmas proved above in the general setting of the retract diagram, we can generalize some propositions and theorems as in [16] and [17].

Proposition 3.2.14. *Let R and S be rings, and let C be a semidualizing R-module, such that the triple (R, S, C) satisfies Property 3.2.3. Then, for any R-module M, M is C-Gorenstein injective over R if and only if M is Gorenstein injective over S.*

Proof. This is proved similarly as in [16, Proposition 2.13 (1)]. □

We need the dual versions of Proposition 3.2.11, and Lemmas 3.2.10, 3.2.12 and 3.2.13, to prove the projective and flat versions of Proposition 3.2.14. They are stated next for the sake of completeness.

Lemma 3.2.15. *Let R and S be rings, and let C be a semidualizing R-module. If (R, S, C) satisfies Property 3.2.3, then for any projective R-module Q, we have*

$$\operatorname{Ext}_S^i(-, S \otimes_R Q) \cong \operatorname{Ext}_R^i(-, C \otimes_R Q)$$

for all $i \geqslant 0$ as functors on S-modules.

Proof. This is the dual of Lemma 3.2.10 using Lemma 3.1.2 and $\operatorname{Hom}_R(S, \underline{I})$ as the injective resolution over S of $\operatorname{Hom}_R(S, C \otimes_R Q)$ where $\underline{I}$ is an injective resolution of $C \otimes_R Q$. □

Proposition 3.2.16. *Let R and S be rings, and let C be a semidualizing R-module. If (R, S, C) satisfies Property 3.2.3 and M is an R-module, then for each $i \geqslant 0$, we have $\operatorname{Ext}_R^i(M, C \otimes_R P) = 0$ for every projective R-module P if and only if $\operatorname{Ext}_S^i(M, V) = 0$ for every projective S-module V.*

Proof. This is the dual of Proposition 3.2.11. □

Lemma 3.2.17. *Let R and S be rings, and C be a semidualizing R-module. If the triple (R, S, C) satisfies Property 3.2.3 and M is an R-module that is Gorenstein projective over S, then there exists a short exact sequence of R-modules*

$$0 \to M \to C \otimes_R P \to M' \to 0$$

for some projective R-module P such that

1. *M' is Gorenstein projective over S*

2. *the above sequence is $\mathrm{Hom}_R(-, C \otimes_R Q)$-exact for any projective R-module Q.*

Proof. This is the dual of Lemma 3.2.12, using Lemma 3.1.2. □

Lemma 3.2.18. *Let R and S be rings, and let C be an R-module such that (R, S, C) satisfies Property 3.2.3. Let M be an R-module that is C-Gorenstein projective over R. Then there exists a short exact sequence of S-modules*

$$0 \to M \to W \to M' \to 0$$

where W is projective over S, M' is C-Gorenstein projective over R and the above sequence is $\mathrm{Hom}_S(-, Y)$-exact for any projective S-module Y.

Proof. This is the dual of Lemma 3.2.13. □

Using the above results, one can prove the injective version of Proposition 3.2.14.

Proposition 3.2.19. *Let R and S be rings, and let C be a semidualizing R-module, such that the triple (R, S, C) satisfies Property 3.2.3. Then, an R-module M is C-Gorenstein projective if and only if M is Gorenstein projective over S.*

Proof. Argue similarly as in the proof of Proposition 3.2.14 using Lemmas 3.2.15, 3.2.17, 3.2.18 and Proposition 3.2.16 instead. □

The flat version of Proposition 3.2.14 can be proved by essentially the same techniques as in the proof of [16, Proposition 2.15].

Proposition 3.2.20. *Let R and S be rings, and let C be a semidualizing R-module, such that the triple (R, S, C) satisfies Property 3.2.3. Then, for any R-module M, M is C-Gorenstein flat over R if and only if M is Gorenstein flat over S.*

Proof. Argue as in the beginning of the proof of [16, Proposition 2.15], using Hom-tensor adjointness, that for any faithfully injective R-module E, the module M is C-Gorenstein flat if and only if the module $\mathrm{Hom}_R(M, E)$ is C-Gorenstein injective.

Since $\operatorname{Hom}_R(S,E)$ is faithfully injective over S for any faithfully injective R-module E, one has $\operatorname{Gfd}_S M = \operatorname{Gid}_S(\operatorname{Hom}_S(M, \operatorname{Hom}_R(S,E)))$ by [6, Theorem 6.4.2]. Moreover, since

$$\operatorname{Hom}_S(M, \operatorname{Hom}_R(S,E)) \cong \operatorname{Hom}_R(M,E)$$

by Hom-tensor adjointness and tensor cancellation, we have

$$\operatorname{Gfd}_S M = \operatorname{Gid}_S(\operatorname{Hom}_R(M,E)).$$

The above two facts, combined with Proposition 3.2.14, give the desired result. □

The last result of this section is Theorem A.

Theorem 3.2.21. *Let R and S be rings, and let C be a semidualizing R-module. If (R,S,C) satisfies Property 3.2.3, then for any homologically left-bounded R-complex M and any homologically right-bounded R-complex N, one has*

$$C\text{-}\operatorname{Gid}_R M = \operatorname{Gid}_S M$$

$$C\text{-}\operatorname{Gpd}_R N = \operatorname{Gpd}_S N$$

$$C\text{-}\operatorname{Gfd}_R N = \operatorname{Gfd}_S N$$

Proof. This follows from Propositions 3.2.14, 3.2.19 and 3.2.20 as in [16, Theorem 2.16]. We prove the equality $C\text{-}\operatorname{Gid}_R M = \operatorname{Gid}_S M$ as follows.

Since every C-Gorenstein injective R-module is Gorenstein injective over S by Proposition 3.2.14, we have $C\text{-}\operatorname{Gid}_R M \geqslant \operatorname{Gid}_S M$.

For the opposite inequality, let $n := \operatorname{Gid}_S M < \infty$. Let I be a left-bounded complex of injective R-modules such that $I \simeq M$ in the derived category $\mathcal{D}(R)$. Each I_i is injective over R, hence Gorenstein injective over S by Proposition 3.2.9. Let $M' := \operatorname{Ker}(I_{-n} \to I_{-n-1})$. By Theorem 3.3 in [7], M' is Gorenstein injective over S. By Proposition 3.2.14, M' is C-Gorenstein injective over R, and so are $I_0, \ldots, I_{-n+1}$. Let $I' := \cdots \to I_{-n+1} \to M' \to 0$. Since $I' \simeq I \simeq M$, $C\text{-}\operatorname{Gid}_R M \leqslant n$.

The projective and flat versions are proved similarly. □

3.3. Examples

This section lists three examples of rings of the form $R \oplus C$, where C is an R-module, that are suitable for our general retract diagrams. The first example recovers Theorem 2.16 in [16] as a special case of Theorem 3.2.21. The rest of this section is devoted to two similar constructions, which can be recovered as special cases of the retract diagram in Property 3.2.1. In particular, I prove in this section Theorem C from the introduction.

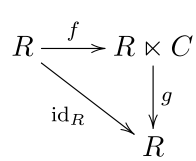
3.3.1. Trivial Extension of a Ring by a Module

In this section, we apply Theorem 3.2.21 to Nagata's *trivial extension* of a ring R by an R-module C [19], recovering Holm and Jørgensen's Theorem 2.16 [16].

Definition/Notation 3.3.1. Let R be a ring, and let C be an R-module. Then define a multiplication structure on $R \oplus C$ as follows: for each (r, c) and (r', c') in $R \oplus C$, we define $(r, c)(r', c') = (rr', rc' + r'c)$. The group $R \oplus C$ with this multiplication structure is a ring with $(1_R, 0)$ as the multiplication identity. We denote this ring as $R \ltimes C$, and call it the trivial extension of R by C.

This process of defining a multiplication structure on $R \oplus C$ is called the "idealization" of C, and it is sometimes used to transfer some properties from ideals to modules [1]. It is routine to check that $R \ltimes C$ fits into a retract diagram satisfying Property 3.2.1. We collect this information in the following lemmas.

Lemma 3.3.2. *Let R be a ring, and let C be an R-module. Then the diagram*



where $f(r) := (r, 0)$ and $g(r, c) := r$ for each $r \in R$ and $c \in C$, is a commutative diagram of ring homomorphisms such that $\operatorname{Ker} g \cong C$ *as R-modules.*

Proof. It is routine to check that the maps f and g are well-defined and they respect addition. We only check that f and g respect multiplication. Let $r, r' \in R$ and $c, c' \in C$. Then,

$$f(rr') = (rr', 0) = (rr', r(0) + r'(0)) = (r, 0)(r', 0) = f(r)f(r'),$$

and

$$g((r,c)(r',c')) = g(rr', rc' + r'c) = rr' = g(r,c)g(r',c).$$

By construction of g, we have $\operatorname{Ker} g \cong C$ as R-modules. □

Lemma 3.3.3. *Let R be a ring. If C is a semidualizing R-module, then* $\operatorname{Hom}_R(R \ltimes C, C) \cong R \ltimes C$ *as $R \ltimes C$-modules, and* $\operatorname{Ext}^i_R(R \ltimes C, C) = 0$ *for all $i \geqslant 1$.*

Proof. We first note that the $R \ltimes C$-module structure of $\operatorname{Hom}_R(R \ltimes C, C)$ comes from $R \ltimes C$ in the first slot. Specifically, for any (r,c) and (s,d) in $R \ltimes C$, and for any R-module homomorphism φ from $R \ltimes C$ to C, we have $((r,c)\varphi)(s,d) = \varphi((r,c)(s,d)) = \varphi(rs, rd + sc)$. Since $R \ltimes C \cong R \oplus C$ as R-modules, we know that $\operatorname{Hom}_R(R \ltimes C, C) \cong \operatorname{Hom}_R(C,C) \oplus C$ as R-modules.

Since C is assumed to be semidualizing over R, the natural homothety $R \to \operatorname{Hom}_R(C,C)$ is an R-module isomorphism, hence there is a natural R-module isomorphism $\Theta : R \ltimes C \to \operatorname{Hom}_R(R \ltimes C, C)$. Tracing all the natural isomorphisms involved, we see that the natural R-module isomorphism $\Theta : R \ltimes C \to \operatorname{Hom}_R(R \ltimes C, C)$ is defined by $(r,c) \mapsto \phi^{(r,c)}$, where $\phi^{(r,c)}$ is defined for any $(r'', c'') \in R \ltimes C$ as $\phi^{(r,c)}(r'', c'') = rc'' + r''c$. It is routine to check that this natural R-module isomorphism Θ is *also* an $R \ltimes C$-module isomorphism.

Finally, we note that we already have $\operatorname{Ext}^i_R(R \ltimes C, C) \cong \operatorname{Ext}^i_R(C,C)$ as R-modules. Since C is semidualizing over R, we have $\operatorname{Ext}^i_R(C,C) \cong 0$ for all $i \geqslant 1$, and $\operatorname{Ext}^i_R(R \ltimes C, C) \cong 0$ for all $i \geqslant 1$ as well. □

We next prove that the triple $(R, R \ltimes C, C)$ satisfies Property 3.2.1.

Theorem 3.3.4. *Let R be a ring, and set $S := R \bowtie C$. If C is semidualizing as an R-module, then (R,S,C) satisfies Property 3.2.3.*

Proof. Lemma 3.3.2, combined with Lemma 3.3.3, yields the desired result. □

Since $(R, R \ltimes C, C)$ satisfies Property 3.2.3, Theorem 3.2.21 can be applied to recover Holm and Jørgensen's Theorem 2.16 [16].

Corollary 3.3.5. *Let R be a ring, and let C be an ideal in R such that C is semidualizing over R. Then, for any homologically left-bounded R-complex M and any homologically right-bounded R-complex N, one has*

$$C\text{-}\mathrm{Gid}_R\, M = \mathrm{Gid}_{R\ltimes C}\, M$$

$$C\text{-}\mathrm{Gpd}_R\, N = \mathrm{Gpd}_{R\ltimes C}\, N$$

$$C\text{-}\mathrm{Gfd}_R\, N = \mathrm{Gfd}_{R\ltimes C}\, N.$$

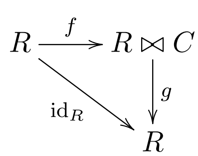
3.3.2. Amalgamated Duplication of a Ring along an Ideal

The following construction is due to D'Anna and Fontana [8].

Definition/Notation 3.3.6. Let R be a ring, and let C be an ideal in R. Then define a multiplication structure on $R \oplus C$ as follows: for each (r, c) and (r', c') in $R \oplus C$, we define $(r, c)(r', c') = (rr', rc' + r'c + cc')$. The group $R \oplus C$ with this multiplication structure is a ring with $(1_R, 0)$ as the multiplicative identity [8]. We denote this ring as $R \bowtie C$, and call it the amalgamated duplication of R along C.

It is routine to check that $R \bowtie C$ fits into a retract diagram satisfying Property 3.2.1. We collect this information in the following lemma.

Lemma 3.3.7. *Let R be a ring, and let C be an ideal in R. Then the diagram*



where $f(r) := (r, 0)$ and $g(r, c) := r$ for each $r \in R$ and $c \in C$, is a commutative diagram of ring homomorphisms such that $\operatorname{Ker} g \cong C$ *as R-modules.*

Proof. By construction, f and g are well-defined functions, and the diagram commutes. Since f and g are natural injection and surjection, respectively, they respect addition in $R \bowtie C$. Therefore it is enough to check that f and g respect multiplication in $R \bowtie C$. Let $r, r' \in R$ and $c, c' \in C$.

Then

$$\begin{aligned}
f(rr') &= (rr', 0) \\
&= (rr', r.0 + r'.0 + 0.0) \\
&= (r, 0)(r', 0) \\
&= f(r)f(r') \\
f(1_R) &= (1_R, 0) \\
g\left((r, c)(r', c')\right) &= g(rr', rc' + r'c + cc') \\
&= rr' \\
&= g(r, c)g(r', c') \\
g(1_R, 0) &= 1_R.
\end{aligned}$$

By construction of g, we have $\operatorname{Ker} g \cong C$. □

We prove next that the triple $(R, R \bowtie C, C)$ satisfies Property 3.2.1.

Lemma 3.3.8. *Let R be a ring, and let C be an ideal in R. If C is a semidualizing R-module, then $\operatorname{Hom}_R(R \bowtie C, C) \cong R \bowtie C$ as $R \bowtie C$-modules, and $\operatorname{Ext}^i_R(R \bowtie C, C) = 0$ for all $i \geqslant 1$.*

Proof. We first note that the $R \bowtie C$-module structure of $\operatorname{Hom}_R(R \bowtie C, C)$ comes from $R \bowtie C$ in the first slot. Specifically, for any (r, c) and (s, d) in $R \bowtie C$, and for any R-module homomorphism φ from $R \bowtie C$ to C, we have $((r, c)\varphi)(s, d) = \varphi((r, c)(s, d)) = \varphi(rs, rd+sc+cd)$. Since $R \bowtie C \cong R \oplus C$ as R-modules, we know that $\operatorname{Hom}_R(R \bowtie C, C) \cong \operatorname{Hom}_R(C, C) \oplus C$ as R-modules.

Since C is assumed to be semidualizing over R, the natural homothety $R \to \operatorname{Hom}_R(C, C)$ is an R-module isomorphism, hence there is a natural R-module isomorphism $\Theta : R \bowtie C \to \operatorname{Hom}_R(R \bowtie C, C)$. Tracing all the natural isomorphisms involved, we see that the natural R-module isomorphism $\Theta : R \bowtie C \to \operatorname{Hom}_R(R \bowtie C, C)$ is defined by $(r, c) \mapsto \phi^{(r,c)}$, where $\phi^{(r,c)}$ is defined for any $(r'', c'') \in R \bowtie C$ as $\phi^{(r,c)}(r'', c'') = rc'' + r''c$.

However, unlike in the case of $R \ltimes C$, the natural R-module isomorphism Θ is *not* an $R \bowtie C$-module isomorphism. We therefore use Θ to construct a new map Φ from $R \bowtie C$ to $\operatorname{Hom}_R(R \bowtie C, C)$, and we prove that Φ is indeed an $R \bowtie C$-module isomorphism.

Define $\Phi : R \bowtie C \to \mathrm{Hom}_R(R \bowtie C, C)$ as $\Phi(r, c) := \Theta(r + c, c)$ for any (r, c) in $R \bowtie C$. It is then immediate that Φ is bijective, and it is also routine to check that Φ is indeed an $R \bowtie C$-module homomorphism with respect to the module structures noted above.

Finally, we note that we already have $\mathrm{Ext}^i_R(R \bowtie C, C) \cong \mathrm{Ext}^i_R(C, C)$ as R-modules. Since C is semidualizing over R, we have $\mathrm{Ext}^i_R(C, C) \cong 0$ for all $i \geqslant 1$, and $\mathrm{Ext}^i_R(R \bowtie C, C) \cong 0$ for all $i \geqslant 1$ as well. □

The next result is Theorem C(a) from the introduction.

Theorem 3.3.9. *Let R be a ring, let C be an ideal in R, and set $S := R \bowtie C$. If C is semidualizing as an R-module, then (R, S, C) satisfies Property 3.2.3.*

Proof. Lemma 3.3.7, combined with Lemma 3.3.8, yields the desired result. □

Since $(R, R \bowtie C, C)$ satisfies Property 3.2.3, Theorem 3.2.21 can be applied to recover the following result of Salimi et. al. [20].

Corollary 3.3.10. *Let R be a ring, and let C be an ideal in R such that C is semidualizing over R. Then, for any homologically left-bounded R-complex M and any homologically right-bounded R-complex N, one has*

$$C\text{-}\mathrm{Gid}_R\, M = \mathrm{Gid}_{R \bowtie C}\, M$$

$$C\text{-}\mathrm{Gpd}_R\, N = \mathrm{Gpd}_{R \bowtie C}\, N$$

$$C\text{-}\mathrm{Gfd}_R\, N = \mathrm{Gfd}_{R \bowtie C}\, N.$$

3.3.3. Pseudocanonical Cover

In this section, we apply Theorem 3.2.21 to pseudocanonical covers introduced by Enescu in [9].

Definition/Notation 3.3.11. Let R be a ring, let $h \in R$, and let C be an ideal in R. We define a ring structure on $R \oplus C$ by defining $(r, c)(r', c') = (rr' + cc'h, rc' + r'c)$ for each $(r, c), (r', c') \in R \oplus C$. The group $R \oplus C$ with this multiplication structure, denoted as $S(h)$, is indeed a ring with $(1_R, 0)$ as its multiplicative identity [9], and is called the *pseudocanonical cover of R via h*.

We construct a retract diagram similar to the one in Property 3.2.1 using $S(h)$.

Lemma 3.3.12. *Let R be a ring, let C be an ideal and let $h \in R$ such that $h = r_0^2$ for some $r_0 \in R$. Then the diagram*

$$\begin{array}{ccc} R & \xrightarrow{f} & S(h) \\ & \searrow^{\mathrm{id}_R} & \downarrow g \\ & & R \end{array}$$

where $f(r) := (r, 0)$ and $g(r, c) := r + cr_0$ for each $r \in R$ and $c \in C$, is a commutative diagram of ring homomorphisms such that $\operatorname{Ker} g \cong C$ as R-modules.

Proof. By construction, f and g are well-defined functions making the diagram commute. It is routine to check that f is a ring homomorphism and that g respects addition. To check that g respects multiplication as well, let $r, r' \in R$ and $c, c' \in C$. Then

$$\begin{aligned} g\left((r,c)(r',c')\right) &= g(rr' + cc'h, rc' + r'c) \\ &= rr' + cc'h + rc'r_0 + r'cr_0 \\ &= r(r' + c'r_0) + cc'r_0^2 + r'cr_0 \\ &= r(r' + c'r_0) + cr_0(c'r_0 + r') \\ &= (r + cr_0)(r' + c'r_0) \\ &= g(r,c)g(r',c') \end{aligned}$$

where we used the fact that $h = r_0^2$.

We note that $\operatorname{Ker} g$ is the R-submodule of $S(h)$ consisting of all elements of the form $(-cr_0, c)$ with $c \in C$. Therefore one can readily prove that the map from C to $\operatorname{Ker} g$ that sends c to $(-cr_0, c)$ is indeed an R-module isomorphism. $\square$

Lemma 3.3.13. *Let R be a ring, let C be an ideal in R, and let $h \in R$ such that $h = r_0^2$ for some $r_0 \in R$. If C is semidualizing over R, then $\operatorname{Hom}_R(S(h), C) \cong S(h)$ as $S(h)$-modules, and $\operatorname{Ext}^i_R(S(h), C) = 0$ for all $i \geqslant 1$.*

Proof. We first note that the $S(h)$-module structure of $\operatorname{Hom}_R(S(h), C)$ comes from $S(h)$ in the first slot. Since $S(h) \cong R \oplus C$ as R-modules, we know that

$$\mathrm{Hom}_R(S(h), C) \cong \mathrm{Hom}_R(C, C) \oplus C$$

as R-modules.

Since C is assumed to be semidualizing over R, we have $\mathrm{Hom}_R(C, C) \cong R$ as R-modules, hence $S(h) \cong \mathrm{Hom}_R(S(h), C)$ as R-modules. Tracing the composition of all the natural R-module isomorphisms above, we have an R-module isomorphism $\Theta : S(h) \to \mathrm{Hom}_R(S(h), C)$ sending $(r, c) \mapsto \varphi^{(r,c)}$, where $\varphi^{(r,c)}$ is defined for any $(r'', c'') \in S(h)$ as $\varphi^{(r,c)}(r'', c'') = rc'' + r''c$. It is routine to check that Θ is also an $S(h)$-module homomorphism.

Finally, we have that $\mathrm{Ext}^i_R(S(h), C) \cong \mathrm{Ext}^i_R(C, C)$ as R-modules for all $i \geqslant 1$. Since C is semidualizing over R, we have $\mathrm{Ext}^i_R(C, C) = 0$ for all $i \geqslant 1$, hence $\mathrm{Ext}^i_R(S(h), C) = 0$ as well. □

The next result is Theorem C(b) from the introduction.

Theorem 3.3.14. *Let R be a ring, let C be an ideal in R, let $h \in R$ such that $h = r_0^2$ for some $r_0 \in R$, and let $S(h)$ be the pseudocanonical cover of R via h. If C is semidualizing as an R-module, then $(R, S(h), C)$ satisfies Property 3.2.3.*

Proof. Lemma 3.3.12 combined with Lemma 3.3.13 yields the desired result. □

We end this section by applying Theorem 3.2.21 to $S(h)$.

Corollary 3.3.15. *Let R be a ring, let C be an ideal in R such that C is semidualizing over R, and let $h \in R$ such that $h = r_0^2$ for some $r_0 \in R$. Then, for any homologically left-bounded R-complex M and any homologically right-bounded R-complex N, one has*

$$C\text{-}\mathrm{Gid}_R M = \mathrm{Gid}_{S(h)} M$$

$$C\text{-}\mathrm{Gpd}_R N = \mathrm{Gpd}_{S(h)} N$$

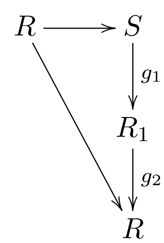
$$C\text{-}\mathrm{Gfd}_R N = \mathrm{Gfd}_{S(h)} N$$

Proof. Since $(R, S(h), C)$ satisfies Property 3.2.3, this is a direct application of Theorem 3.2.21. □

3.4. Remarks

It is natural to ask if the general setting we discuss above characterizes the situation where an R-module M is C-Gorenstein projective/injective/flat over R if and only if M is Gorenstein projective/injective/flat over S. For example, one can ask if Property 3.2.3 is a necessary condition for Theorem 3.2.19. This fails in general, and the following is a counterexample.

Example 3.4.1. Let C be a semidualizing module, and set $R_1 := R \ltimes C$ and $S := R_1 \ltimes R_1$. Let g_1 and g_2 be appropriate canonical projections, making the following diagram commute, and let $g := g_2 \circ g_1$.



We note that M is C-Gorenstein projective over R if and only if it is Gorenstein projective over R_1, if and only if Gorenstein projective over S by [16, Proposition 2.13]. However, the triple (R, S, C) does not satisfy Property 3.2.3 because $\operatorname{Ker} g \ncong C$; noting that $S \cong R \oplus (C \oplus R \oplus C)$ as R-modules and $g = g_2 \circ g_1$, we have $\operatorname{Ker} g \cong R \oplus C^2$, which is different from C.

We finally note here that the R-module structure on S in the previous example is not by accident. If we assume that a retract diagram in our general setting exists, i.e., there exists a ring homomorphism $f : R \to S$ such that $g \circ f = \mathrm{id}_R$, then g is a split surjection. This implies that $S \cong R \oplus \operatorname{Ker} g$ as R-modules as in the above example.

4. FURTHER RESEARCH GOALS

I am currently working to simplify the retract diagram in Property 3.2.3 into a module-finite extension $S \to R$ with some assumptions on the existence of a dualizing S- and R-complexes. As part of this, I am working to understand "local-global behavior" for C-Gorenstein homological dimensions. For instance, if R is a commutative noetherian ring and M is an R-module, then the local-global behavior of the injective dimension of M can be stated as follows:

$$\operatorname{id}_R M = \sup\left\{\operatorname{id}_{R_\mathfrak{p}} M_\mathfrak{p} \,|\, \mathfrak{p} \in \operatorname{Spec} R\right\} = \sup\left\{\operatorname{id}_{R_\mathfrak{m}} M_\mathfrak{m} \,|\, \mathfrak{m} \in \operatorname{Max} R\right\}.$$

I am working to understand the analogous formula for C-Gorenstein injective dimension, (e.g. for Gorenstein injective dimension) and similar formulas for (C-)Gorenstein projective dimension and (C-)Gorenstein flat dimension.

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