L^1 APPROXIMATION IN DE BRANGES SPACES

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The supervisory committee certifies that this dissertation complies with North Dakota State University's regulations and meets the accepted standards for the degree of

DOCTOR OF PHILOSOPHY

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ABSTRACT

In this thesis we study bandlimited approximations to various functions. Bandlimited functions have compactly supported Fourier transforms, which is a desirable feature in many applications. In particular, we address the problem of determining best approximations that minimize a weighted integral error. By utilizing the theory of Hilbert spaces of entire functions developed by L. de Branges, we are able to obtain optimal solutions for several weighted approximation problems. As an application, we determine extremal majorants and minorants that vanish at a prescribed point for a class of functions, which may be used to remove contributions from undesirable points.

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DEDICATION

To my wife, Emily, and my parents, Ted and Deb.

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1. INTRODUCTION

We say that an entire function $F : \mathbb{C} \to \mathbb{C}$ is of *exponential type* $\delta \geq 0$ if for every $\varepsilon > 0$ there exists a constant C_ε such that

$$|F(z)| \leq C_\varepsilon e^{(\delta+\varepsilon)|z|} \tag{1.1}$$

holds for all $z \in \mathbb{C}$. By the famed Paley-Wiener theorem (see Theorem 2.2.2), functions of exponential type $2\pi\delta$ have (distributional) Fourier transforms supported in the interval $[-\delta, \delta]$, i.e., they are δ-bandlimited. Given this remarkable connection along with the vast applications of bandlimited functions (some of which we describe below), one of the classical problems at the intersection of approximation theory and harmonic analysis asks: how well can a given real-valued function be approximated on the real line by a δ-bandlimited function? In this thesis we address the following $L^1(\mathbb{R}, \mu)$-approximation problem and variations of it:

Problem 1.0.1. *Given $f : \mathbb{R} \to \mathbb{R}$, $\delta > 0$, and μ a Borel measure on $\mathbb{R}$, can we find an entire function $K : \mathbb{C} \to \mathbb{C}$ of exponential type $2\pi\delta$ such that the integral*

$$\|K - f\|_{L^1(\mathbb{R},\mu)} := \int_{-\infty}^{\infty} |K(x) - f(x)| \, d\mu(x) \tag{1.2}$$

is minimized?

Such a function is called a *best approximation* to f in $L^1(\mathbb{R}, \mu)$-norm. In the case of the Lebesgue measure this problem is classical with a large collection of works dating back to the late 1930s by Akhiezer, Bernstein, Krein, Nagy, and many others. Influential works include those of M.G. Krein [41] and B. Sz.-Nagy [62] in which they construct best approximations for large classes of functions. Accounts of these results (in English) as well as additional information on L^1-approximations can be found in the books of H. Shapiro [61] and A.F. Timan [63]. Recently, M. Ganzburg [29] has extended the Krein-Nagy approach to obtain best $L^1(\mathbb{R}, dx)$-approximations to locally integrable functions on $\mathbb{R}$.

It is a well-known result in the theory of $L^1(\mathbb{R},\mu)$-approximation that a best approximation K from $\mathcal{A}(\delta)$, the space of all entire functions of exponential type δ, can be characterized by the sign change pattern of $K-f$ (see e.g., [24, Theorem 10.1] or [61, Theorem 4.2.2]). In particular (see Theorem 5.1.2), a function $K \in \mathcal{A}(\delta)$ is a best approximation to f in $L^1(\mathbb{R},\mu)$-norm if (1.2) is finite, i.e., $K - f \in L^1(\mathbb{R},\mu)$, and $\psi = \operatorname{sgn}(K-f)$ is orthogonal to $\mathcal{A}^1(\delta,\mu)$, the space of all entire functions of exponential type δ whose restrictions to the real line are integrable with respect to μ.[1] It is worth mentioning that this 'sign change' result for $L^1(\mathbb{R},\mu)$-approximations (as well as $L^p(\mathbb{R},\mu)$-approximations) is a natural extension of the Chebyshev alternation theorem[2] from the late 19th century which studies the problem of best uniform approximations of continuous functions on $[a,b]$ with polynomials (see e.g., [24, Theorem A]).

In Chapter 5 we identify sign change patterns ψ that are high-pass for $\mathcal{A}^1(\delta,\mu)$, i.e., orthogonal to $\mathcal{A}^1(\delta,\mu)$. Here μ is taken to be a so-called Hermite-Biehler weight, that is $d\mu_E(x) = |E(x)|^{-2}\,dx$ for some entire function $E:\mathbb{C}\to\mathbb{C}$ that satisfies $|E(\overline{z})| < |E(z)|$ for all z in the open upper half-plane $\mathbb{C}^+$. These Hermite-Biehler weights are particularly well-suited for the problem of determining extremal signatures as they allow us to utilize the remarkable theory of the Hilbert spaces of entire functions developed by L. de Branges in the 1960s (see Chapter 3) as well as Hardy spaces in the upper half-plane (see Section 2.3). It is a well-known result (and we reprove it in Theorem 5.1.2) that best approximations can frequently be obtained as solutions of an interpolation problem.

Problem 1.0.2. *Given $f:\mathbb{R}\to\mathbb{R}$, $\delta>0$, μ a Borel measure on $\mathbb{R}$ with polynomial growth, and ψ a high-pass function for $\mathcal{A}^1(\delta,\mu)$ with $\psi^2=1$ a.e. on $\mathbb{R}$, can we construct an entire function K of exponential type δ such that $K-f\in L^1(\mathbb{R},\mu)$ and*

$$\psi(x) = \operatorname{sgn}(K(x)-f(x)) \tag{1.3}$$

for almost every real x?

Under reasonable assumptions on the Hermite-Biehler function E, we are able to identify Laguerre-Pólya functions F, i.e., uniform limits of polynomials with only real zeros, so that $\psi =$

[1] Here $\operatorname{sgn}(x)$ is the signum function defined by $\operatorname{sgn}(x) = 1$ for $x > 0$, $\operatorname{sgn}(x) = -1$ for $x < 0$, and $\operatorname{sgn}(0) = 0$.

[2] Even though the alternation theorem commonly bears Chebyshev's name he did not actually prove it. It was proved independently by Blichfeldt and Kirchberger in the early 1900s.

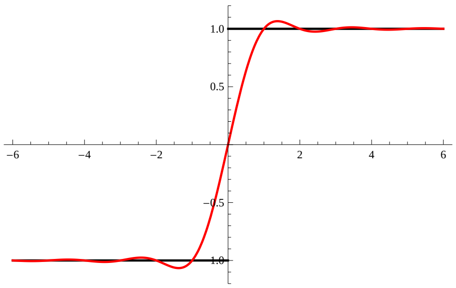

Figure 1.1: Best approximation to the signum function (of type π) in $L^1(\mathbb{R}, dx)$-norm

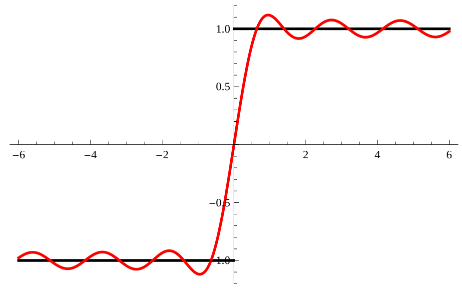

Figure 1.2: Best approximation to the signum function (of type π) in $L^1(\mathbb{R}, \frac{dx}{x^2+1})$-norm

$\operatorname{sgn}(F)$ is high-pass for $\mathcal{A}^1(\delta, \mu_E)$. In Chapter 4 we give a general method to construct entire functions K that interpolate a given function $f : \mathbb{R} \to \mathbb{R}$ at the zeros of a Laguerre-Pólya function F, so that

$$\operatorname{sgn}(F(x)) = \operatorname{sgn}(K(x) - f(x)) \tag{1.4}$$

for almost every real x. Using these interpolations, we are able to construct (unique) best approximations in $L^1(\mathbb{R}, \mu_E)$-norm for a class of truncated Laplace transforms and their odd extensions, which includes the signum function (see Figures 1.1 and 1.2). Moreover, by utilizing the recent interpolation results for classes of even functions in [11, 13] by Carneiro and Littmann, we are able to construct best approximations in $L^1(\mathbb{R}, \mu_E)$-norm for large classes of even, odd, and truncated functions.

In applications, it is convenient to consider a variant of Problem 1.0.1 where the entire function K further satisfies the majorizing constraint, $K(x) \geq f(x)$ for all real x. In this case, such a function is called an *extremal majorant* of f. Analogously, we can define an *extremal minorant* of f. Below we mention some of the history as well as applications of these so-called one-sided approximations.

In the 1930s, A. Beurling considered the one-sided $L^1(\mathbb{R}, dx)$-approximation problem for the signum function, $\operatorname{sgn}(x)$. He observed that the entire function

$$B_{2\pi\delta}(z) = \left(\frac{\sin(\pi\delta z)}{\pi}\right)^2 \left\{ \sum_{n=0}^{\infty} \frac{1}{(\delta z - n)^2} - \sum_{n=-\infty}^{-1} \frac{1}{(\delta z - n)^2} + \frac{2}{\delta z} \right\} \tag{1.5}$$

satisfies the following properties:

1. $B_{2\pi\delta}$ is of exponential type $2\pi\delta$,

2. $B_{2\pi\delta}$ majorizes the signum function on the real line (see Figure 1.3), i.e., $B_{2\pi\delta}(x) \geq \operatorname{sgn}(x)$ for all real x,

3. $\displaystyle\int_{-\infty}^{\infty} \{B_{2\pi\delta}(x) - \operatorname{sgn}(x)\}\, dx = \delta^{-1}$.

Moreover, Beurling showed that if F is any entire function of exponential type $2\pi\delta$ with $F(x) \geq \operatorname{sgn}(x)$ for all real x and $F \neq B_{2\pi\delta}$, then $\int_{-\infty}^{\infty}\{F(x) - \operatorname{sgn}(x)\}\, dx > \delta^{-1}$, hence the function $B_{2\pi\delta}$ is the extremal majorant of the signum function. Using this function, he was able to obtain an inequality for almost periodic functions on the real line. Beurling never published these results (though he did present his findings at a Harmonic Analysis seminar in 1942), and the first appearance of his results in the literature can be found in a classical survey paper on L^1-approximations and their applications [64] by J.D. Vaaler. For additional information on some of Beurling's early results see [4].

In the 1970s, A. Selberg [59, 60] independently discovered the function $B_{2\pi\delta}$ in his work to obtain a sharp form of the large sieve inequality of analytic number theory. Selberg observed that the function $B_{2\pi\delta}$ could be used to majorize and minorize the characteristic function of an interval

$$\chi_{[a,b]}(x) = \begin{cases} 1 & \text{if } a \leq x \leq b, \\ 0 & \text{if } x < a \text{ or } b < x, \end{cases} \tag{1.6}$$

where $a < b$. With the observation that for $x \neq a$ and $x \neq b$ we have $\chi_{[a,b]}(x) = 2^{-1}\{\operatorname{sgn}(b - x) + \operatorname{sgn}(x - a)\}$, Selberg showed that the entire function

$$C_{2\pi\delta}(z) = 2^{-1}\left(B_{2\pi\delta}(b - z) + B_{2\pi\delta}(z - a)\right) \tag{1.7}$$

majorizes $\chi_{[a,b]}$ on the real line (see Figure 1.4), is of exponential type $2\pi\delta$, and satisfies

$$\int_{-\infty}^{\infty} \left\{C_{2\pi\delta}(x) - \chi_{[a,b]}\right\}\, dx = \frac{1}{\delta}. \tag{1.8}$$

Moreover, Selberg showed that $C_{2\pi\delta}$ is an extremal majorant if and only if $\delta(b - a)$ is an integer. The extremal problem for the characteristic function of an interval when $\delta(b - a)$ is not an integer

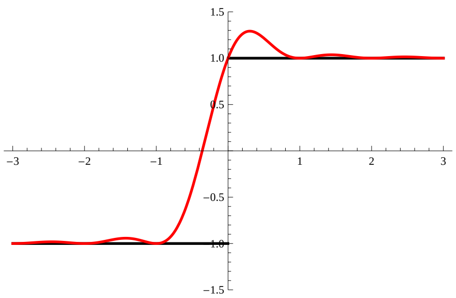

Figure 1.3: Beurling's extremal majorant of the signum function (of type 2π)

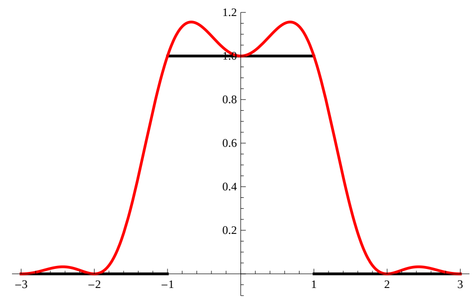

Figure 1.4: Selberg's extremal majorant of the characteristic function of an interval (of type 2π)

has since been resolved by Donoho and Logan [25] when $\delta(b-a) < 1$ with the remaining cases settled by F. Littmann [49].

Since Beurling and Selberg's investigations, the theory of one-sided $L^1(\mathbb{R}, dx)$-approximation problems (now commonly referred to as Beurling-Selberg extremal problems) has been extensively developed over the past 30 years with solutions for these extremal problems now available for large classes of even, odd, and truncated functions [9, 10, 12, 13, 14, 15, 16, 25, 33, 47, 48, 49, 50, 64]. As these extremal majorants and minorants have compactly supported Fourier transforms (by the Paley-Wiener Theorem) and minimal L^1-errors (which often translates into optimal constants in certain inequalities), they have proven extremely useful in a variety of inequalities and number theoretical applications such as Hilbert-type inequalities [16, 33, 48, 64], bounds for the Riemann zeta function and functions related to it [6, 7, 8, 9, 17, 31], the pair correlation of zeros of the Riemann zeta-function [8, 28, 54], and Erdös-Turan discrepancy inequalities [46, 64] along with problems in signal processing [25, 49].

To highlight the utility of these extremal problems in connection with the Riemann zeta-function we sketch out Chandee and Soundararajan's argument to improve the bound of $\zeta(z)$ along the critical line, $\mathrm{Re}(z) = 1/2$, under the Riemann Hypothesis (see [17]). For additional information on the history and applications of Beurling-Selberg extremal problems with the Riemann zeta-function see the survey paper [6] by E. Carneiro.

By Hadamard's factorization formula for the Riemann ξ function[3] and Stirling's formula, Chandee and Soundararajan obtain the following representation for $\log|\zeta(z)|$ along the critical line.

[3]Here the Riemann ξ function is the entire function $\xi(s) = 2^{-1}s(s-1)\pi^{-s/2}\Gamma(2^{-1}s)\zeta(s)$.

For large t,

$$\log|\zeta(\frac{1}{2}+it)| = \log t - \frac{1}{2}\sum_{\gamma} f(t-\gamma) + \mathcal{O}(1) \tag{1.9}$$

where $f(x) = \log\left(\frac{4+x^2}{x^2}\right)$ and the sum runs over the nontrivial zeros $\rho = \frac{1}{2}+i\gamma$ of $\zeta(z)$.

By replacing f with a function of exponential type that minorizes it (which exists under the framework of [15]), they invoke a so-called Guinand-Weil explicit formula [37, Theorem 5.2] to connect the sum over the nontrivial zeros of $\zeta(z)$ to a sum of the Fourier transform of the minorant evaluated at prime powers. Since the minorant is chosen to be of exponential type (i.e., bandlimited) its Fourier transform has compact support and the resulting sum only has finitely many terms. Moreover, the choice of optimal minorant of f in $L^1(\mathbb{R}, dx)$-norm minimizes the additional contribution from the Fourier transform of the minorant evaluated at $t = 0$ that also appears in the explicit formula. By carefully analyzing the remaining terms, Chandee and Soundararajan are able to improve the current bound for the size of $\zeta(z)$ on the critical line (currently the best bound under the Riemann Hypothesis).

In general, an explicit formula is an identity that relates the values $h(\rho)$, where $\zeta(\rho) = 0$ (or $L(\chi, \rho) = 0$ for a general L-function with Dirichlet character χ) and h is a smooth function, to the values of the Fourier transform of h. Under the assumption of the Riemann Hypothesis (RH) and writing $\rho = \frac{1}{2}+i\gamma$ the series will only involve real values γ, and h is frequently chosen to be a solution to the Beurling-Selberg extremal problem. Without the assumption of RH (or GRH for general L-functions) the series will involve non-real values of γ. A tool that may aid in accommodating for these additional points are extremal functions that vanish at given points in the upper half-plane and respectively in the lower half-plane[4].

Problem 1.0.3. *Given $f : \mathbb{R} \to \mathbb{R}$, $\delta > 0$, and $\alpha \in \mathbb{C}^+$, can we find an entire function F of exponential type δ such that $F(\alpha) = 0$, $F(x) \geq f(x)$ for all real x, and the integral*

$$\int_{-\infty}^{\infty} \{F(x) - f(x)\}\, dx \tag{1.10}$$

is minimal?

[4]This type of extremal problem can also be used in other situations where contribution from particular points needs to be eliminated (e.g., zeros and poles).

This variant of the Beurling-Selberg extremal problem was first considered in [39] for the characteristic function of an interval $\chi_{[a,b]}$. By multiplying Selberg's (non-vanishing) majorant and minorant of $\chi_{[a,b]}$ with an expression that vanishes at α, M. Kelly found non-optimal majorants and minorants of $\chi_{[a,b]}$ that vanish at a given point in the upper half-plane and obtained bounds for the L^1-error as a function of δ.

In Chapter 6 we study how imposing vanishing conditions affects the construction of extremal functions, and we find optimal majorants and minorants of the signum function (i.e., Beurling's Problem), monomials, and the Poisson kernel that vanish at a given point $\alpha = ib$ along the imaginary axis. Moreover, using the result for monomials, i.e., polynomials which only have one term, we are able to construct non-optimal majorants and minorants of polynomials in $\mathbb{R}[x]$, i.e., polynomials with real coefficients, that satisfy the vanishing condition at ib. Here the approach is to modify the approximated function $f : \mathbb{R} \to \mathbb{R}$ and encode the vanishing condition into a new measure (so that the vanishing condition may be dropped). This modification is done by noticing that any majorant $F \in \mathcal{A}(\delta)$ of f that vanishes at $ib \in i\mathbb{R}$ also vanishes at $-ib$ (since F is real entire and therefore $F(\overline{z}) = \overline{F(z)}$ for all z), hence F is necessarily of the form

$$F(z) = F_b(z)(z^2 + b^2) \tag{1.11}$$

where $F_b \in \mathcal{A}(\delta)$ is a majorant of $f_b(x) = f(x)(x^2 + b^2)^{-1}$. Thus, we seek to answer the following (non-vanishing) weighted one-sided $L^1(\mathbb{R}, (x^2 + b^2)\,dx)$ extremal problem:

Problem 1.0.4. *Given $f : \mathbb{R} \to \mathbb{R}$, $\delta > 0$, and $b > 0$, can we find an entire function $F_b : \mathbb{C} \to \mathbb{C}$ of exponential type δ such that $F_b(x) \geq f(x)(x^2 + b^2)^{-1}$ for all real x and the integral*

$$\int_{-\infty}^{\infty} \left\{ F_b(x) - \frac{f(x)}{x^2 + b^2} \right\} (x^2 + b^2)\,dx \tag{1.12}$$

is minimal?

In Section 6.6 we consider this problem when f is taken to be the signum function, a monomial, or the Poisson kernel. In fact, in Sections 6.2, 6.3, and 6.4 we are able to solve this problem for a more general class of Hermite-Biehler weights μ_E. By constructing a suitable Hermite-Biehler function E (see Section 6.5) we are able to apply the general weighted one-sided $L^1(\mathbb{R}, \mu_E)$-

approximation results to solve the vanishing problem for the signum function, monomials, and the Poisson kernel (Theorems 6.6.1, 6.6.3, and 6.6.6, respectively). For these functions, we show that prescribing vanishing at $\alpha = ib$ substantially affects the integral value for small values of δ, but the vanishing condition only leads to a small change if δ becomes large.

The problem of solving weighted one-sided $L^1(\mathbb{R}, \mu)$-approximation problems was first studied in [35]. In this remarkable work, J.J. Holt and J.D. Vaaler extend Beurling and Selberg's solutions (majorants/minorants of the signum function) to weighted $L^1(\mathbb{R}, \mu)$-norms. Moreover, they use these results to construct majorants and minorants to characteristic functions of Euclidean balls in $\mathbb{R}^N$ while minimizing the $L^1(\mathbb{R}^N, d\mathbf{x})$-norm[5] and obtain a multi-dimensional version of the large-sieve inequality of analytic number theory. The key step in their results is to show that the weighted one-sided $L^1(\mathbb{R}, \mu)$ extremal problem has a close connection to the theory of Hilbert spaces of entire functions developed by L. de Branges during the 1960s. These so-called de Branges spaces generalize the classical Paley-Wiener spaces and Fourier Analysis. In fact, many of the de Branges spaces used in applications can be viewed as weighted Paley-Wiener spaces (see Section 3.6) making them an ideal starting point for solving weighted $L^1(\mathbb{R}, \mu)$ approximation problems (where the tools of Fourier analysis and classical interpolation results in the Paley-Wiener spaces are no longer available). This approach of using de Branges spaces to study weighted one-sided $L^1(\mathbb{R}, \mu)$ extremal problems has also proven useful in several recent works [10, 13, 50] with results now known for classes of even, odd, and truncated functions. Very recently, the one-sided approximation problem for the characteristic function of an interval in weighted L^1-norm was solved in [8] and applied to improve existing bounds for the pair correlation of zeros of the Riemann zeta-function, under the Riemann Hypothesis. Other recent interesting approximation problems that have utilized the theory of de Branges spaces include [32, 39, 49, 50]. In Chapter 5 we show that the best (two-sided) $L^1(\mathbb{R}, \mu)$-approximation problem (Problem 1.0.1) also has a close connection to de Branges spaces, and in Chapter 6 we extend the known results for Beurling-Selberg extremal problems in de Branges spaces.

This thesis is structured as follows. In Chapter 2 we describe general facts about some classical spaces of analytic functions including the L^p Paley-Wiener Spaces, Hardy Spaces in the upper half-plane, and functions of bounded type.

[5] This work is the first to treat the case of one-sided L^1-approximation in higher dimensions.

In Chapter 3 we give an introduction to the theory of Hilbert Spaces of Entire Functions developed by L. de Branges and record basic facts about these spaces. This includes that these spaces are reproducing kernel Hilbert spaces with some remarkable generalizations of properties of the Paley-Wiener spaces (e.g., interpolation formulas, orthogonal sets, and Parseval's formula). Moreover, we describe some equivalent characterizations of these de Branges spaces which further strengthen the relationship between de Branges spaces and weighted Paley-Wiener spaces. Lastly, we prove an interpolation formula for these spaces which will be used to show that some best $L^1(\mathbb{R},\mu)$ approximations are unique (see Section 5.4).

In Chapter 4 we give a general method to construct entire functions K that interpolate a given function $f:\mathbb{R}\to\mathbb{R}$ at the zeros of a Laguerre-Pólya function F, so that

$$x \mapsto F(x)\,(K(x)-f(x)) \tag{1.13}$$

is of one sign for all real x. These functions will serve as candidates for the best approximation and one-sided approximation problems of Chapters 5 and 6. The construction in Chapter 4 is based on the general method of obtaining interpolations at the zeros of Laguerre-Pólya functions used in [13] and [35] (which generalize the methods for creating extremal majorants and minorants with respect to the Lebesgue measure in [33] by Graham and Vaaler).

In Chapter 5 we study the problem of best (two-sided) approximations in $L^1(\mathbb{R},\mu)$-norm. In Sections 5.2 and 5.3 we identify extremal signatures for Hermite-Biehler weights $d\mu_E(x) = |E(x)|^{-2}\,dx$ and describe general sign change properties of high-pass functions for μ_E.

In Section 5.4 we use the sign change results of Section 5.2 and 5.3 along with the interpolations constructed in Chapter 4 to determine best approximations in $L^1(\mathbb{R},\mu_E)$-norm for a class of odd and truncated functions including the signum function, Heaviside function, truncated Poisson kernel, and truncated exponential. Lastly, in Section 5.5 we consider the special cases of best approximations to the Poisson and conjugate Poisson kernels. In these cases the best approximations in $L^1(\mathbb{R},\mu_E)$-norm are explicit and the $L^1(\mathbb{R},\mu_E)$-error is remarkably simple.

In Chapter 6 we solve the extremal problem with vanishing condition described above. In Sections 6.2, 6.3, and 6.4, we solve the one-sided $L^1(\mathbb{R},\mu_E)$-approximation problem for the class of functions $f_b(x) = f(x)(x^2+b^2)^{-1}$ when f is taken to be the signum function, a monomial, or the

Poisson kernel with respect to a large family of Hermite-Biehler weights. The key step in connecting the extremal problems in $L^1(\mathbb{R}, \mu_E)$-norm with the vanishing problem is that the weighted Paley-Wiener space $\mathcal{A}^2(\delta, (x^2 + b^2)\,dx)$ is isometrically equal to a de Branges space $\mathcal{A}^2(\delta, |E_b(x)|^{-2}\,dx)$ where E_b is of bounded type. In Section 6.5 we show this isometry and describe many important properties about the space $\mathcal{A}^2(\delta, |E_b(x)|^{-2}\,dx)$.

2. SPACES OF ANALYTIC FUNCTIONS

2.1. Introduction

In this chapter we give an overview of some spaces of analytic functions that will be needed throughout this work. We begin by stating the Paley-Wiener Theorem which allows us to describe functions of exponential type in terms of the support of their Fourier transforms. Using this we define the classical Paley-Wiener spaces and related Bernstein spaces (i.e., L^p Paley-Wiener spaces). These spaces are Banach spaces of entire functions and we state some well-known results about these spaces including interpolation formulas and Parseval's formula (Theorem 2.2.6). Many of these results have remarkable generalizations in weighted Paley-Wiener spaces or so-called de Branges spaces (see Chapters 3 and 6) which will be used to show extremal properties as well as uniqueness (in some cases) of best approximations in weighted L^1-norm.

The formulation of these de Branges spaces requires some notions and results from function theory in the upper half-plane. In Section 2.3 we review some basic facts about the Hardy Spaces in the upper half-plane, denoted $H^p(\mathbb{C}^+)$. In Lemma 2.8.2 we show that functions belonging to $H^p(\mathbb{C}^+)$ are of so-called bounded type, i.e., they can be represented as the quotient of two bounded functions in the upper half-plane. Functions of bounded type have a canonical 'inner-outer' factorization (Lemma 2.6.4) from which we define the notion of 'mean type.' By Krein's theorem (Lemma 2.7.10) the mean type is a natural generalization of exponential type for functions that may not be entire. Lastly, in Lemmas 2.8.1 and 2.8.2 we prove necessary and sufficient conditions for a function to belong to a Hardy Space in the upper half-plane using the mean type and integrability conditions of the boundary function. This chapter is based on [23], [53], and [57] and more details (as well as omitted proofs) about these spaces of analytic functions in the upper half-plane can be found there.

We denote the field of real numbers by $\mathbb{R}$, the field of complex numbers by $\mathbb{C}$, and the open upper half-plane by $\mathbb{C}^+ := \{z \in \mathbb{C} \mid \mathrm{Im}(z) > 0\}$ where $\mathrm{Im}(z)$ and $\mathrm{Re}(z)$ denote the real and imaginary parts of $z \in \mathbb{C}$, respectively. The space of all analytic functions on the upper half-plane is denoted by $H(\mathbb{C}^+)$, and $H^\infty(\mathbb{C}^+)$ is its subset containing all bounded analytic functions on $\mathbb{C}^+$.

2.2. Bernstein and Paley-Wiener spaces

For $f \in L^1(\mathbb{R})$, we define the Fourier transform of f, denoted $\widehat{f}$, by

$$\widehat{f}(t) = \int_{-\infty}^{\infty} f(x) e^{-2\pi i x t}\, dx. \tag{2.1}$$

The function $\widehat{f}$ is continuous on $\mathbb{R}$ and by the Riemann-Lebesgue Lemma $\widehat{f}(t) \to 0$ as $|t| \to \infty$. We extend the definition of the Fourier transform to $L^p(\mathbb{R})$, $1 \leq p \leq \infty$, in the usual way (cf. [36] and [53]).

Definition 2.2.1. A function $f \in L^p(\mathbb{R})$, $1 \leq p \leq \infty$, is called δ-*bandlimited* if its Fourier transform vanishes outside $[-\delta, \delta]$. We define the classical Paley-Wiener space, denoted $\mathcal{PW}_\delta$, as the space of all continuous δ-bandlimited functions.

For a function f belonging to $\mathcal{PW}_\delta$ with $f \in L^2(\mathbb{R})$, we have that $\widehat{f}(t) = 0$ for all $|t| > \delta$ hence $\widehat{f} \in L^1(\mathbb{R})$ and therefore

$$f(x) = \int_{-\delta}^{\delta} \widehat{f}(t) e^{2\pi i x t}\, dt \tag{2.2}$$

for all real x. By Morera's Theorem, the integral extends to an entire function on $\mathbb{C}$ given by

$$F(z) = \int_{-\delta}^{\delta} \widehat{f}(t) e^{2\pi i z t}\, dt. \tag{2.3}$$

For $z = x + iy$, we have that

$$|F(z)| \leq \int_{-\delta}^{\delta} |\widehat{F}(t)| e^{-2\pi y t}\, dt \leq C e^{2\pi\delta|z|}. \tag{2.4}$$

Since $F(x) = f(x)$ on $\mathbb{R}$, it follows that f extends to an entire function of exponential type $2\pi\delta$.

By the Paley-Wiener Theorem (Theorem 2.2.2), the converse is also true (this is the deep part of the theory). We say that an entire function $F : \mathbb{C} \to \mathbb{C}$ belongs to $L^p(\mathbb{R}, dx)$ if the restriction of F to the real line belongs to $L^p(\mathbb{R}, dx)$.

Theorem 2.2.2 (Paley-Wiener)**.** *Let $F : \mathbb{C} \to \mathbb{C}$ be an entire function such that $F \in L^2(\mathbb{R}, dx)$. The following are equivalent:*

1. *F is of exponential type $2\pi\delta$.*

2. $\widehat{F}(t) = 0$ *for all* $|t| > \delta$.

Proof. See e.g., [65, Theorem 7.2]. □

For $\delta > 0$ and $1 \leq p \leq \infty$, we define the L^p Paley-Wiener space or Bernstein space, denoted $\mathcal{A}^p(\delta, dx)$, as the space of all entire functions F of exponential type δ that belong to $L^p(\mathbb{R}, dx)$. These spaces are Banach spaces, and for $p = 2$, the space $\mathcal{A}^2(2\pi\delta, dx)$ is a Hilbert space with standard L^2-inner product. Moreover, by the Paley-Wiener Theorem we have $\mathcal{PW}_\delta = \mathcal{A}^2(2\pi\delta, dx)$ (if we identify elements in $\mathcal{PW}_\delta$ with their extensions to entire functions).

The following lemma (see [5, Theorem 6.7.1]) gives that, unlike the spaces $L^p(\mathbb{R}, dx)$, the spaces $\mathcal{A}^p(\delta, dx)$ are nested.

Lemma 2.2.3 ([5, Theorem 6.7.1])**.** *Let* $\delta \geq 0$. *If* $F : \mathbb{C} \to \mathbb{C}$ *is an entire function of exponential type* δ *that belongs to* $L^p(\mathbb{R}, dx)$*, for some* $1 \leq p < \infty$*, then for all real* y

$$\int_{-\infty}^{\infty} |F(x+iy)|^p \, dx \leq e^{p\delta|y|} \int_{-\infty}^{\infty} |F(x)|^p \, dx. \tag{2.5}$$

Moreover, $F \in L^q(\mathbb{R}, dx)$ *for every* $q > p$.

Proof. The proof is based on an application of the Phragmen-Lindelöf theorem. It is somewhat technical and we refer to [5, Theorem 6.7.7]. In fact, [5, Theorem 6.7.7] shows that (2.5) holds for functions that are of exponential type on the closed upper half-plane. □

Remark 2.2.4. For simplicity, we have stated the Paley-Wiener theorem for functions belonging to $L^2(\mathbb{R}, dx)$; however, using Lemma 2.2.3 we see that the Paley-Wiener Theorem holds for entire functions belonging to $L^p(\mathbb{R}, dx)$, $1 \leq p \leq 2$. In fact, the theory can also be carried out for distributions with compact support (see e.g., [36, Section 7.3]).

Remark 2.2.5. Containment in the other direction still does not hold. For example, for all $w \in \mathbb{C}$ the entire function $z \mapsto K_\delta(w, z) := \frac{\sin(\delta(\overline{w}-z))}{\pi(\overline{w}-z)}$ is of exponential type δ and belongs to $L^2(\mathbb{R}, dx)$, but it does not belong to $L^1(\mathbb{R}, dx)$.

One of the remarkable properties about the Paley-Wiener space $\mathcal{A}^2(2\pi\delta, dx)$ is that for every $w \in \mathbb{C}$

$$F(w) = \left\langle F, \frac{\sin(2\pi\delta(\overline{w} - \cdot))}{\pi(\overline{w} - \cdot)} \right\rangle = \langle F, K_{2\pi\delta}(w, \cdot) \rangle \tag{2.6}$$

holds for all $F \in \mathcal{A}^2(2\pi\delta,\, dx)$. Hence, the Paley-Wiener space is a so-called Reproducing Kernel Hilbert Space (RKHS). More information about RKHS will be mentioned in Section 3.2.

Using the reproducing kernel property along with the fact that the collection of entire functions $\{z \mapsto \frac{\sin(\pi z)}{\pi(z-n)}\}_{n\in\mathbb{Z}}$ forms an orthonormal basis for $\mathcal{A}^2(\pi, dx)$, one can deduce the famed Shannon-Whittaker-Kotel'nikov (SWK) interpolation formula (see e.g., [65, Theorem 7.19]), which provides a way to reconstruct any bandlimited function from its samples taken at equally spaced nodes on the real line.

Theorem 2.2.6 ([65, Theorem 7.19])**.** *Let $F \in \mathcal{A}^2(\pi, dx)$. Then*

$$F(z) = \frac{\sin(\pi z)}{\pi} \sum_{n\in\mathbb{Z}} (-1)^n \frac{F(n)}{z-n} \tag{2.7}$$

where the expression on the right-hand side of (2.7) *converges uniformly on compact subsets of* $\mathbb{C}$*. Moreover,*

$$\int_{-\infty}^{\infty} |F(t)|^2 \, dt = \sum_{n\in\mathbb{Z}} |F(n)|^2. \tag{2.8}$$

Remark 2.2.7. By Theorem 2.2.3, the results of the previous theorem hold for any function $F \in \mathcal{A}^p(\pi, dx)$, $1 \le p \le 2$.

From this we can obtain interpolation formulas for functions of exponential type that are bounded on the real axis (see e.g., [65, Equation 7.22]).

Corollary 2.2.8. *Let $F \in \mathcal{A}^\infty(\pi, dx)$. Then*

$$F(z) = \frac{\sin(\pi z)}{\pi} \left\{ F'(0) + \frac{F(0)}{z} + \sum_{n\in\mathbb{Z}\setminus\{0\}} (-1)^n F(n) \left(\frac{1}{z-n} + \frac{1}{n} \right) \right\} \tag{2.9}$$

where the expression on the right-hand side of (2.9) *converges uniformly on compact subsets of* $\mathbb{C}$.

Remark 2.2.9. By applying the previous results to the function $G(z) = F((2\delta)^{-1}z)$, we obtain similar representations for functions of exponential type $2\pi\delta$.

In Section 3.8, we prove a generalization of Corollary 2.2.8 for the de Branges space $\mathcal{A}^\infty(\delta, \mu_E)$.

2.3. Hardy spaces in the upper half-plane

In this section we state some classical results about Hardy Spaces in the upper half-plane $H^p(\mathbb{C}^+)$. These Hardy spaces originated in the context of complex analytic and Fourier analysis in the early twentieth century. The deep connection between the theories of Fourier Analysis and Hardy Spaces have made these spaces extremely useful in a variety of applications. We follow the notation and presentation of [53]. For further information about Hardy spaces on the unit disc as well as upper half-plane see [26], [30], or [40].

Definition 2.3.1. For $1 \leq p < \infty$, define the Hardy Space on the upper half-plane, $H^p(\mathbb{C}^+)$, as the space of functions F that are analytic in the upper half-plane and satisfy

$$\|F\|_{H^p(\mathbb{C}^+)} := \sup_{0<y<\infty} \left(\int_{-\infty}^{\infty} |F(x+iy)|^p dx \right)^{1/p} < \infty. \tag{2.10}$$

For $p = \infty$, $H^\infty(\mathbb{C}^+)$ is the space of all bounded analytic functions in the upper half-plane. The norm for each function $F \in H^\infty(\mathbb{C}^+)$ is given by

$$\|F\|_{H^\infty(\mathbb{C})} := \sup_{z\in\mathbb{C}^+} |F(z)|. \tag{2.11}$$

It is a classical result in Hardy Space Theory (see e.g., [53, Chapter 13]) that the spaces $H^p(\mathbb{C}^+)$, $1 \leq p \leq \infty$, are Banach spaces which are isomorphic to the real Hardy Spaces

$$H^p(\mathbb{R}) = \left\{ f \in L^p(\mathbb{R}) : \int_{-\infty}^{\infty} \frac{f(t)}{t - \bar{z}} dt = 0, \text{ for all } z \in \mathbb{C}^+ \right\} \tag{2.12}$$

and

$$H^\infty(\mathbb{R}) = \left\{ f \in L^\infty(\mathbb{R}) : \int_{-\infty}^{\infty} \frac{f(t)}{(t - \bar{z})(t+i)} dt = 0, \text{ for all } z \in \mathbb{C}^+ \right\} \tag{2.13}$$

respectively. In particular, we will make use of the following characterization result (see [53, Theorems 13.2 - 13.5]). Here $\|\cdot\|_p$ denotes the standard $L^p(\mathbb{R}, dx)$-norm and $F_y(x) = F(x+iy)$.

Theorem 2.3.2 ([53, Theorems 13.2 - 13.5])**.** *Let $1 \leq p \leq \infty$. Let F be an analytic function in the upper half-plane $\mathbb{C}^+$. Then $F \in H^p(\mathbb{C}^+)$ if and only if there exists a unique $f \in H^p(\mathbb{R})$ such that*

$$F(z) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{(x-t)^2 + y^2}\, dt \tag{2.14}$$

in $\mathbb{C}^+$. If so, for almost all $x \in \mathbb{R}$,

$$\lim_{y \to 0} F(x+iy) = f(x) \tag{2.15}$$

and

$$\|F\|_{H^p(\mathbb{C}^+)} = \|f\|_p. \tag{2.16}$$

Moreover, for $1 \leq p < \infty$ we have

$$\lim_{y \to 0} \|F_y - f\|_p = 0, \tag{2.17}$$

while if $p = \infty$, then

$$\lim_{y \to 0} \|(F_y - f)\varphi\|_\infty = 0 \tag{2.18}$$

for every $\varphi \in L^1(\mathbb{R})$.

It turns out that the real Hardy spaces can be characterized by the support of their Fourier transforms.

Lemma 2.3.3 ([53, Theorem 13.6])**.** *Let $f \in L^1(\mathbb{R})$. Then $f \in H^1(\mathbb{R})$ if and only if $\widehat{f}(t) = 0$ for all $t \leq 0$.*

Before giving the proof of Lemma 2.3.3, we make the following observation. For $f \in H^1(\mathbb{R}) \subseteq L^1(\mathbb{R})$ we have that $\widehat{f}(0) = 0$, i.e.,

Corollary 2.3.4. *If $f \in H^1(\mathbb{R})$, then*

$$\widehat{f}(0) = \int_{-\infty}^{\infty} f(t)\, dt = 0. \tag{2.19}$$

Proof of Lemma 2.3.3. We follow the proof of [53, Theorem 13.6]. Let $f \in L^1(\mathbb{R})$. If $z \in \mathbb{C}^+$, then the function $g : \mathbb{R} \to \mathbb{C}$ defined by

$$g(t) = \begin{cases} 0 & t > 0 \\ -e^{2\pi i \overline{z} t} & t \leq 0 \end{cases} \tag{2.20}$$

belongs to $L^1(\mathbb{R})$ and has Fourier transform given by

$$\widehat{g}(t) = \frac{1}{2\pi i} \frac{1}{t - \overline{z}}. \tag{2.21}$$

Hence,

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t - \overline{z}} \, dt = \int_{-\infty}^{\infty} f(t)\widehat{g}(t) \, dt = \int_{-\infty}^{\infty} \widehat{f}(t) g(t) \, dt = -\int_{-\infty}^{0} \widehat{f}(t) e^{2\pi i \overline{z} t} \, dt \tag{2.22}$$

for all $z \in \mathbb{C}^+$.

If $\widehat{f}(t) = 0$ for all $t \leq 0$, equation (2.22) implies that

$$\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{f(t)}{t - \overline{z}} \, dt = 0 \tag{2.23}$$

for all $z \in \mathbb{C}^+$, hence $f \in H^1(\mathbb{R})$.

On the other hand, for $f \in H^1(\mathbb{R})$, it follows from (2.12) and equation (2.22) that

$$\int_{-\infty}^{0} \widehat{f}(t) e^{2\pi i \overline{z} t} \, dt = 0 \tag{2.24}$$

for all $z \in \mathbb{C}^+$. As $\widehat{f}$ is continuous on $\mathbb{R}$, it follows by the uniqueness theorem for the Fourier transform (see e.g., [53, Corollary 11.11]) that $\widehat{f}(t) = 0$ for all $t \leq 0$. □

In fact, for functions $f \in H^p(\mathbb{R}) \subseteq L^p(\mathbb{R})$, $1 \leq p \leq 2$, one obtains that $\widehat{f}(t) = 0$ for almost every $t \leq 0$ (see e.g., [53, Theorem 13.6]), which gives an alternative definition for the real Hardy Spaces, for $1 \leq p \leq 2$,

$$H^p(\mathbb{R}) = \{f \in L^p(\mathbb{R}) \mid \widehat{f}(t) = 0 \text{ for almost every } t \leq 0\}. \tag{2.25}$$

2.4. Nevanlinna representation

In this section we state Nevanlinna's Representation for functions that are analytic in the upper half-plane with non-negative real part there, i.e., belong to the Nevanlinna Class $\mathcal{N}_0(\mathbb{C}^+)$.

Definition 2.4.1. If μ is Borel measure on $\mathbb{R}$ that satisfies

$$\int_{\mathbb{R}} \frac{d|\mu|(t)}{1+t^2} < \infty, \tag{2.26}$$

then its *Herglotz integral*, $H_\mu : \mathbb{C}^+ \to \mathbb{C}$, is defined as

$$H_\mu(z) = \frac{i}{\pi} \int_{\mathbb{R}} \left(\frac{1}{z-t} + \frac{t}{1+t^2} \right) d\mu(t). \tag{2.27}$$

Notice that the modified Cauchy kernel in (2.27) satisfies

$$\frac{1}{z-t} + \frac{t}{1+t^2} = -\frac{1+tz}{t-z}\frac{1}{1+t^2}. \tag{2.28}$$

Using (2.26), it follows that H_μ is a well-defined function which is analytic in the upper half-plane. Moreover, if μ is a real Borel measure on $\mathbb{R}$, then since

$$\operatorname{Re}\left(\frac{i}{\pi}\left(\frac{1}{z-t} + \frac{t}{1+t^2}\right)\right) = \operatorname{Re}\frac{i}{\pi}\frac{1}{z-t} = \frac{y}{\pi}\frac{1}{(t-x)^2+y^2} \tag{2.29}$$

we have that

$$\operatorname{Re} H_\mu(z) = \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{d\mu(t)}{(t-x)^2+y^2} = P_y * d\mu(x) \geq 0 \tag{2.30}$$

for all $z \in \mathbb{C}^+$, where $P_y(t) = \frac{1}{\pi}\frac{y}{t^2+y^2}$ and $*$ denotes convolution, i.e., $f * d\nu(t) = \int_{-\infty}^{\infty} f(t-x)\, d\nu(x)$.

Definition 2.4.2. We say that an analytic function in the upper half-plane belongs to the Nevanlinna Class for $\mathbb{C}^+$, denoted $\mathcal{N}_0(\mathbb{C}^+)$, if its real part is non-negative on $\mathbb{C}^+$.

Example 2.4.3.

1. For all $b, \alpha \in \mathbb{R}$, $\alpha \leq 0$, the functions $z \mapsto ib + i\alpha z$ belong to the Nevanlinna Class.

2. The functions $z \mapsto z^{1/2}$ and $z \mapsto -i\log(z)$, with appropriately chosen branch cuts, are analytic and have non-negative real part on $\mathbb{C}^+$, hence they belong to the Nevanlinna Class.

3. If μ is a non-negative Borel measure on $\mathbb{R}$ which satisfies

$$\int_{\mathbb{R}} \frac{d\mu(t)}{1+t^2} < \infty, \tag{2.31}$$

then by (2.30) it follows that the Herglotz Integral of μ, H_μ, belongs to $\mathcal{N}_0(\mathbb{C}^+)$.

It turns out that all functions belonging to the Nevanlinna class are combinations of (1) and (3) (see e.g., [23, Theorems 3 - 4] and [57, Theorem 5.3]).

Theorem 2.4.4 (Nevanlinna Representation)**.** *Every analytic function F on the upper half-plane $\mathbb{C}^+$ such that $\operatorname{Re} F(z) \geq 0$ for all $z \in \mathbb{C}^+$ has a representation*

$$F(z) = ib + i\alpha z + \frac{i}{\pi}\int_{\mathbb{R}} \left(\frac{1}{z-t} + \frac{t}{1+t^2}\right) d\mu(t) \tag{2.32}$$

where $b, \alpha \in \mathbb{R}$, $\alpha \leq 0$, and μ is a nonnegative Borel measure on $\mathbb{R}$ which satisfies

$$\int_{\mathbb{R}} \frac{d\mu(t)}{1+t^2} < \infty. \tag{2.33}$$

Sketch of proof. Since F is analytic on $\mathbb{C}^+$, then by the Cauchy-Riemann equations the functions $\operatorname{Re} F$ and $\operatorname{Im} F$ are harmonic on $\mathbb{C}^+$. Hence, $\operatorname{Re} F$ is both harmonic and non-negative on $\mathbb{C}^+$ which implies it has a Poisson representation in the upper half-plane (see e.g., [57, Theorem 5.2]), i.e.,

$$\operatorname{Re} F(x+iy) = -\alpha y + \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{d\mu(t)}{(t-x)^2+y^2} \tag{2.34}$$

where $\alpha \leq 0$ and μ is a nonnegative Borel measure on $\mathbb{R}$ such that $\int_{\mathbb{R}}(1+t^2)^{-1}\, d\mu(t) < \infty$. It follows that the function

$$G(z) = F(z) - i\alpha z - \frac{i}{\pi}\int_{\mathbb{R}} \left(\frac{1}{z-t} + \frac{t}{1+t^2}\right) d\mu(t) \tag{2.35}$$

is analytic and satisfies $\operatorname{Re} G \equiv 0$ on the upper half-plane, thus G is some imaginary constant which proves (2.32). $\square$

Theorem 2.4.5 (Fatou)**.** *Let $F \in \mathcal{N}_0(\mathbb{C}^+)$ with Nevanlinna representation given by* (2.32)*. If*

$$d\mu = G\,dx + d\sigma \tag{2.36}$$

is the Lebesgue-Radon-Nikodym decomposition of μ where σ is singular with respect to the Lebesgue measure, then

$$\lim_{y\to 0} \operatorname{Re}(F(x+iy)) = G(x) \tag{2.37}$$

almost everywhere on $\mathbb{R}$.

Proof. This is Fatou's Theorem, see for example [57, Theorem 5.5]. □

2.5. Functions of bounded type

Definition 2.5.1. A function F that is analytic in the upper half-plane $\mathbb{C}^+$ is said to be of *bounded type* in the upper half-plane if F is the quotient of two bounded and analytic functions on $\mathbb{C}^+$, i.e., there exist P and Q belonging to $H^\infty(\mathbb{C}^+)$ such that $F = P/Q$. We denote the space of all functions of bounded type in $\mathbb{C}^+$ by $\mathcal{N}(\mathbb{C}^+)$.

Functions of bounded type in the lower half-plane $\mathbb{C}^- = \{z \in \mathbb{C} \mid \operatorname{Im}(z) < 0\}$ are defined analogously. Naturally, any bounded function is of bounded type. The definition of bounded type implies that finite sums and products of functions of bounded type are also of bounded type.

Remark 2.5.2. For $x > 0$, let $\log^+(x)$ denote the positive part of $\log(x)$, i.e.,

$$\log^+(x) = \begin{cases} \log(x) & \text{if } x \geq 1 \\ 0 & \text{else.} \end{cases} \tag{2.38}$$

If F is of bounded type and $F = P/Q$, then we can assume without loss of generality that P is bounded by 1 and $0 < |Q| \leq 1$. Hence,

$$\log^+|F| \leq \log^+|P| + \log^+|Q^{-1}| = -\log|Q|. \tag{2.39}$$

As $-\log|Q|$ is harmonic in $\mathbb{C}^+$, we see that any function of bounded type in $\mathbb{C}^+$ has a harmonic majorant on $\mathbb{C}^+$. In fact, the converse of this is true as well (cf. [57, Theorem 3.20]). Hence, an equivalent definition for an analytic function in the upper half-plane to be of bounded type is that $\log^+|F|$ has a harmonic majorant on $\mathbb{C}^+$.

Next we mention two direct observations about functions of bounded type.

Lemma 2.5.3. *Let F belong to the Nevanlinna Class $\mathcal{N}_0(\mathbb{C}^+)$. Then F is of bounded type in the upper half-plane.*

Proof. Since F has non-negative real part in the upper half-plane, we have that $F/(1+F)$ and $1/(1+F)$ are bounded analytic functions in the upper half-plane. Therefore,

$$F = \frac{F/(1+F)}{1/(1+F)} \tag{2.40}$$

is of bounded type in the upper half-plane. $\square$

Lemma 2.5.4. *Let P be a polynomial. Then P is of bounded type in the upper half-plane.*

Proof. Since the product of functions of bounded type is of bounded type, it is sufficient to show the result for $p(z) = z - z_0$ where $z_0 \in \mathbb{C}$. Notice that $z \mapsto (z - z_0)/(z + i)$ and $z \mapsto 1/(z + i)$ are bounded analytic functions in the upper half-plane. Therefore,

$$z - z_0 = \frac{(z - z_0)/(z + i)}{1/(z + i)} \tag{2.41}$$

is of bounded type in the upper half-plane. $\square$

2.6. Nevanlinna factorization

One of the classical approaches to studying analytic functions is through their zeros and singularities. For functions of bounded type there is a canonical 'inner-outer' factorization (Theorem 2.6.4). Here the 'inner' part accounts for the zeros of the function as well as the possible singular behavior of the function near the real line.

Similar to the Hardy spaces, the boundary function of a function of bounded type exists almost everywhere on the real line [57, Theorem 5.6].

Definition 2.6.1. A function $F \in H^\infty(\mathbb{C}^+)$ is called *inner* for the upper half-plane if

$$|f(x)| = \left| \lim_{y \to 0^+} F(x + iy) \right| = 1 \tag{2.42}$$

for almost all $x \in \mathbb{R}$.

Example 2.6.2.

1. If $\alpha \leq 0$, the entire function $z \mapsto e^{-i\alpha z}$ is an inner function for the upper half-plane.

2. If $z_0 \in \mathbb{C}^+$, the Blaschke factor
$$b_{z_0}(z) = \frac{z - z_0}{z - \overline{z_0}} \tag{2.43}$$
is an inner function for the upper half-plane, since
$$|b_{z_0}(x)| = \left|\frac{x - z_0}{x - \overline{z_0}}\right| = \left|\frac{x - x_0 - iy_0}{x - x_0 + iy_0}\right| = 1 \tag{2.44}$$
for all real x.

3. Let $\{z_n\}$ be a sequence of complex numbers in the upper half-plane with no accumulation point in $\mathbb{C}^+$ and let A denote the set of all accumulation points on the real line. If
$$\sum_n \frac{\mathrm{Im}(z_n)}{|i + z_n|^2} = \sum_n \frac{y_n}{x_n^2 + (y_n + 1)^2} < \infty \tag{2.45}$$
let
$$B(z) = \lim_{N \to \infty} \prod_{n=1}^{N} \frac{\left|\frac{i - z_n}{i - \overline{z_n}}\right|}{\frac{i - z_n}{i - \overline{z_n}}} \frac{z - z_n}{z - \overline{z_n}} = \lim_{N \to \infty} \prod_{n=1}^{N} \frac{|z_n^2 + 1|}{z_n^2 + 1} \frac{z - z_n}{z - \overline{z_n}}. \tag{2.46}$$
By (2.45) the partial product on the right side of (2.46) converges uniformly on compact subsets of $\mathbb{C} \setminus \{A \cup \{\overline{z}_1, \overline{z}_2, ...\}\}$ (see e.g., [53, Theorem 13.13] or [23, Problem 23]) and so (2.46) defines an analytic function in the upper half-plane and a meromorphic function on $\mathbb{C}$. As each factor is an inner function, it follows that B is an inner function. We call such an inner function a *Blaschke product*.

4. If σ is a positive singular Borel measure on $\mathbb{R}$ such that
$$\int_{\mathbb{R}} \frac{d\sigma(t)}{1 + t^2} < \infty \tag{2.47}$$
let
$$S_\sigma(z) = \exp\left(-\frac{i}{\pi} \int_{\mathbb{R}} \left(\frac{1}{z - t} + \frac{t}{1 + t^2}\right) d\sigma(t)\right) = \exp\left(-H_\sigma(z)\right). \tag{2.48}$$

By Theorem 2.4.4, H_σ defines an analytic function in the upper half-plane with non-negative real part there. Hence, S_σ is analytic and bounded by 1 in the upper half-plane. As σ is singular, Fatou's Theorem (Theorem 2.4.5) implies that the boundary function of $\mathrm{Re}(H_\sigma(z))$ is zero almost everywhere on $\mathbb{R}$. It follows that S_σ is an inner function for the upper half-plane which we call a *singular inner function.*

Notice that the product of two inner functions is again inner; hence, $e^{-i\alpha z}B(z)S_\sigma(z)$ is inner. In fact, every inner function has such a decomposition up to a multiplicative constant of modulus one (this follows as a consequence of the Nevanlinna Factorization Theorem (Theorem 2.6.4)).

Definition 2.6.3. An analytic function F defined on $\mathbb{C}^+$ is called *outer* if

$$F(z) = \alpha \exp\left(\frac{i}{\pi}\int_{-\infty}^{\infty}\left(\frac{1}{z-t}+\frac{t}{1+t^2}\right)\log K(t)\,dt\right) \tag{2.49}$$

where $|\alpha| = 1$ and $K(t) > 0$ a.e. on $\mathbb{R}$ such that

$$\int_{-\infty}^{\infty}\frac{|\log K(t)|}{1+t^2}\,dt < \infty. \tag{2.50}$$

If F is an outer function, then the function K satisfies $K(x) = |F(x)|$ a.e., where $F(x)$ is the boundary function of $F(z)$.

For functions of bounded type, we state the following canonical factorization theorem due to Nevanlinna (see e.g., [57, Theorem 6.13] or [23, Theorem 9]).

Theorem 2.6.4 (Nevanlinna's Factorization)**.** *Let $F \not\equiv 0$ be an analytic function of bounded type in the upper half-plane $\mathbb{C}^+$. Then*

$$F(z) = e^{-ivz}B(z)\frac{S_{\sigma_+}(z)}{S_{\sigma_-}(z)}G(z) \tag{2.51}$$

for $z \in \mathbb{C}^+$, where v is a real number, B is a convergent Blaschke product formed with the zeros of F in the upper half-plane, G is an outer function, and $\sigma_\pm$ are non-negative singular Borel measures on the real line satisfying

$$\int_{-\infty}^{\infty}\frac{d\sigma_\pm(t)}{1+t^2} < \infty. \tag{2.52}$$

Moreover, if F has analytic continuation across some open interval I on the real line, then the restrictions of the measure $\sigma_{\pm}$ to I is zero.

Corollary 2.6.5. *Let $F \in H^{\infty}(\mathbb{C}^{+})$ be an inner function that has an analytic continuation across the real line. Then*

$$F(z) = e^{-i\alpha z + iq} B(z) \tag{2.53}$$

where $\alpha \leq 0$ and q are real numbers and B is a convergent Blaschke product formed with the zeros of F in the upper half-plane. Moreover, if F is non-constant, then there exists a continuous, increasing, and real-valued function φ such that

$$F(x) = e^{2i\varphi(x)} \tag{2.54}$$

for all real x.

Proof. The factorization (2.53) is immediate from Theorem 2.6.4. The argument for (2.54) is essentially that of the proof and following comments of [35, Lemma 13]. As the function φ will be extremely important for the development of de Branges spaces (Chapter 3) as well as extremal signatures (Chapter 5), we include the proof for completeness.

Since F is inner and analytic on an open set containing the closed half-plane, we have that $|F(x)| = 1$ for all real x. It follows that there is an open, simply connected set S that contains the real line such that F is analytic and does not vanish on S. Hence, there exists an analytic function $\varphi : S \to \mathbb{C}$ such that

$$F(z) = e^{2i\varphi(z)} \tag{2.55}$$

in S. As $|F(x)| = 1$ on $\mathbb{R}$, it follows that φ is real valued on the real line. From the identities (2.53) and (2.55) we obtain

$$\varphi'(z) = -\frac{i}{2}\frac{F'(z)}{F(z)} = -\frac{\alpha}{2} + \sum_{n} \frac{y_n}{(z - z_n)(z - \bar{z}_n)} \tag{2.56}$$

for every $z \in S$ and

$$\varphi'(x) = -\frac{\alpha}{2} + \sum_{n} \frac{y_n}{(x - x_n)^2 + y_n^2} \tag{2.57}$$

for all real x. Since F is non-constant, it follows from (2.55) and (2.57) that either $\alpha < 0$ or F vanishes at some point in the upper half-plane; therefore, by (2.57) we conclude that $\varphi'(x) > 0$ for all real x. $\square$

2.7. Mean type

Definition 2.7.1. Following de Branges, we refer to the real number v in (2.6.4) as the *mean type* of F, denoted $v(F)$.

The mean type of a function which is identically zero is taken to be $-\infty$. In practice, we can compute the mean type of a given function by either taking an average radial limit or studying the growth of the function along the imaginary axis.

Theorem 2.7.2 ([23, Theorem 10])**.** *Let $F \not\equiv 0$ belong to $\mathcal{N}(\mathbb{C}^+)$. The mean type of F satisfies*

$$v(F) = \lim_{R\to\infty} \frac{2}{\pi R} \int_0^\pi \log|F(Re^{i\theta})| \sin\theta \, d\theta \tag{2.58}$$

and

$$v(F) = \limsup_{y\to\infty} \frac{\log|F(iy)|}{y}. \tag{2.59}$$

Proof. These formulas have direct proofs if F does not have any zeros. In the case that F has zeros see [57, Theorem 6.15] or [23, Theorem 10]. $\square$

Remark 2.7.3. Theorem 2.7.2 shows that $v(F)$ is unique, hence the definition of mean type is well-defined.

Remark 2.7.4. If follows from the definition of mean type and (2.51) that the mean type of a Blaschke product is zero.

From this we make the following useful observation.

Corollary 2.7.5. *Let $F \not\equiv 0$ and $G \not\equiv 0$ be functions of bounded type in $\mathbb{C}^+$. Then*

$$v(FG) = v(F) + v(G) \tag{2.60}$$

and

$$v(F + G) = \max\{v(F), v(G)\}. \tag{2.61}$$

Example 2.7.6.

1. The mean type of any non-zero polynomial is zero. Using Corollary 2.7.5 it is sufficient to show this for $P(z) = z - z_0$ where $z_0 \in \mathbb{C}$. In this case we have that

$$|v(P)| = \left|\limsup_{y\to\infty} \frac{\log|P(iy)|}{y}\right| = \left|\limsup_{y\to\infty} \frac{\log|iy - z_0|}{y}\right| \leq \left|\limsup_{y\to\infty} \frac{\log(y + |z_0|)}{y}\right| = 0. \quad (2.62)$$

2. The mean type of any bounded analytic function in the upper half-plane is non-positive, since

$$v(F) = \limsup_{y\to\infty} \frac{\log|F(iy)|}{y} \leq \limsup_{y\to\infty} \frac{C}{y} = 0. \quad (2.63)$$

The following lemma shows that a function in the Nevanlinna Class has non-positive mean type. Recall from Lemma 2.5.3 that functions in the Nevanlinna Class are of bounded type.

Lemma 2.7.7 ([23, Problem 30])**.** *Let F be an analytic function with non-negative real part in the upper half-plane. Then F has non-positive mean type.*

Proof. If $F \equiv 0$, then $\nu(F) = -\infty$. Assume now that $F \not\equiv 0$. Since F is analytic on $\mathbb{C}^+$ and $\operatorname{Re} F(z) \geq 0$ for all $z \in \mathbb{C}^+$, the function $G = F^{1/2}$ is an analytic function with non-negative real part in $\mathbb{C}^+$ that satisfies

$$|G(z)| \leq \sqrt{2}\operatorname{Re} G(z) \quad (2.64)$$

for all $z \in \mathbb{C}^+$. It follows from Theorem 2.4.4 that there exists a non-negative Borel measure on $\mathbb{R}$ which satisfies

$$\int_{\mathbb{R}} \frac{d\mu(t)}{1+t^2}\, dt < \infty \quad (2.65)$$

and $\alpha \geq 0$ such that

$$\operatorname{Re} G(x+iy) = \alpha y + \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{d\mu(t)}{(t-x)^2 + y^2}\, dt. \quad (2.66)$$

Hence, for $y \geq 1$, we have

$$\operatorname{Re} G(iy) = \alpha y + \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{d\mu(t)}{t^2 + y^2}\, dt \leq \alpha y + \frac{y}{\pi}\int_{-\infty}^{\infty} \frac{d\mu(t)}{1+t^2}\, dt \leq Cy \quad (2.67)$$

for some $C \geq 0$. The case $C = 0$ can be excluded because then $\operatorname{Re}(G(iy)) = 0$, then $G(iy) = 0$ by (2.64) and $F \equiv 0$.

Applying (2.65) and (2.67) we find that

$$\log|F(iy)| = \log|G(iy)|^2 = 2\log|G(iy)| \leq 2\log(\sqrt{2}\mathrm{Re}\, G(iy)) \leq 2(\log\sqrt{2} + \log(Cy)) \tag{2.68}$$

for $y \geq 1$. Dividing through by y and taking limits gives

$$v(F) = \limsup_{y\to\infty} \frac{\log|F(iy)|}{y} \leq 0 \tag{2.69}$$

hence F has non-positive mean type. $\square$

In fact, one can obtain that the mean type of any non-zero analytic function with non-negative real part in the upper half-plane is zero (see e.g., [23, Problem 30]). However, we only need that such functions have non-positive mean type.

Lemma 2.7.8. *Let F be a function of bounded type in the upper half-plane. If there exists a function G of bounded type and non-negative mean type in the upper half-plane such that such that GF is bounded in the upper half-plane, then F has non-positive mean type.*

Proof. Since GF is bounded in the upper half-plane, it has non-positive mean type. As $v(G) \geq 0$, it follows from Corollary 2.7.5 that

$$v(F) \leq v(G) + v(F) = v(GF) \leq 0. \tag{2.70}$$

$\square$

Remark 2.7.9. Recall that any polynomial has zero mean type. Hence, if there exists a polynomial P such that PF is bounded, then F has non-positive mean type.

An entire function $F : \mathbb{C} \to \mathbb{C}$, not identically zero, has exponential type $\tau = \tau(F)$ if

$$\tau(F) = \limsup_{|z|\to\infty} \frac{\log|F(z)|}{|z|} < \infty. \tag{2.71}$$

If $F : \mathbb{C} \to \mathbb{C}$ is an entire function, we define the entire function $F^* : \mathbb{C} \to \mathbb{C}$ by

$$F^*(z) := \overline{F(\overline{z})}. \tag{2.72}$$

By a classical result of M.G. Krein (see Lemma 2.7.10), an entire function F is of exponential type if it is of bounded type in the upper and lower half-planes (i.e., F and F^* belong to $\mathcal{N}(\mathbb{C}^+)$). With this the mean type generalizes the notation of exponential type for functions that may not be entire. Recall that $\log^+(x)$ denotes the positive part of $\log(x)$ (see Remark 2.5.2).

Lemma 2.7.10 (Krein)**.** *Let $F : \mathbb{C} \to \mathbb{C}$ be an entire function. The following are equivalent:*

1. *F and F^* have bounded type in the upper half-plane.*
2. *F has exponential type and*

$$\int_{-\infty}^{\infty} \frac{\log^+ |F(x)|}{1+x^2}\, dx < \infty. \tag{2.73}$$

Moreover, if either and therefore both of these conditions hold the exponential type of F is the maximum of the mean types of F and F^* in the upper half-plane.

Proof. This is [57, Theorem 6.17] see also [42]. □

We denote by $\mathcal{B}$ the set of entire functions F which satisfy either and therefore both of the conditions of the previous Lemma[1].

We conclude the section on mean type by stating Cauchy's formula for the upper half-plane (see e.g., [23, Theorem 12] or [57, Theorem 5.19]).

Theorem 2.7.11 (Cauchy's Formula)**.** *Let F be a function that is analytic and of bounded type in the upper half-plane with non-positive mean type and continuous extension to the closed half-plane. If $F \in L^p(\mathbb{R}, dx)$, for some $1 \leq p < \infty$, then*

$$F(z) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{F(t)}{t-z}\, dt \tag{2.74}$$

and

$$0 = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{F(t)}{t-\overline{z}}\, dt \tag{2.75}$$

for all $z \in \mathbb{C}^+$.

[1] This set is also commonly referred to as the *Cartwright Class*.

2.8. Connection between Hardy spaces and functions of bounded type

It is the goal of this section to describe the connections between between Hardy Spaces in the upper half-plane and functions of bounded type.

Lemma 2.8.1. *Let $1 \leq p \leq \infty$. Let F be an analytic function on the upper half-plane $\mathbb{C}^+$ with continuous extension to the closed half-plane $\overline{\mathbb{C}^+}$. The following are equivalent:*

1. *$F \in H^p(\mathbb{C}^+)$.*

2. *F has bounded type and non-positive mean type in the upper half-plane and $\|F\|_p < \infty$.*

There are many (equivalent) formulations of Lemma 2.8.1 in literature (see e.g, [2, Theorem 2.2], [57, Theorem 5.23], [30, Theorem 5.4]). This particular formulation gives us an easier characterization of de Branges spaces (see Theorem 3.3.4). We prove the equivalent statements of Lemma 2.8.1 in the following two lemmas. In particular, Lemma 2.8.2 shows that the class of functions of bounded type and non-positive mean type in the upper half-plane[2] is a natural upper limit of the Hardy spaces $H^p(\mathbb{C}^+)$.

Lemma 2.8.2. *Let $1 \leq p \leq \infty$. If $F \in H^p(\mathbb{C}^+)$, then F has bounded type and non-positive mean type in the upper half-plane and the boundary function $f(x) = \lim_{y \to 0} F(x + iy)$ exists for almost every real x and satisfies $\|f\|_p < \infty$.*

Proof. If $p = \infty$, then F is bounded on the upper half-plane and hence of bounded type with non-positive mean type in the upper half-plane and bounded boundary function.

For $1 \leq p < \infty$, Theorem 2.3.2 gives there exists a unique $f \in H^p(\mathbb{R}) \subseteq L^p(\mathbb{R})$ such that

$$F(z) = \frac{y}{\pi} \int_{-\infty}^{\infty} \frac{f(t)}{(x-t)^2 + y^2}\, dt \tag{2.76}$$

for all $\mathbb{C}^+$, and $f(x) = \lim_{y \to 0} F(x + iy)$ for almost every real x.

Let $\operatorname{Re} f(t) = u_+(t) - u_-(t)$, where u_+ and u_- are non-negative functions on $\mathbb{R}$. Since $f \in H^p(\mathbb{R}) \subseteq L^p(\mathbb{R})$, we have that

$$\int_{-\infty}^{\infty} \frac{u_\pm(t)}{1+t^2}\, dt \leq \int_{-\infty}^{\infty} \frac{|f(t)|}{1+t^2}\, dt < \infty. \tag{2.77}$$

[2] A sufficient condition for a function to belong to the so-called Smirnov class, $\mathcal{N}^+(\mathbb{C}^+)$, is that it is of bounded type and non-positive mean type in the upper half-plane (cf. [2, Theorem 2.3]).

Thus, the Herglotz integrals

$$U_{\pm}(z) = \frac{i}{\pi}\int_{\mathbb{R}}\left(\frac{1}{z-t}+\frac{t}{1+t^2}\right)u_{\pm}(t)d\,t \tag{2.78}$$

are analytic functions with non-negative real parts in the upper half-plane. By construction, $\mathrm{Re}(F-(U_+-U_-)) \equiv 0$ on $\mathbb{C}^+$, hence $F-(U_+-U_-)$ is an imaginary constant, say ib. By replacing U_+ with $U_+ + ib$, we have that $F(z) = U_+(z) - U_-(z)$ on $\mathbb{C}^+$. Since U_+ and U_- have non-negative real part in the upper half-plane, Lemmas 2.5.3 and 2.65 imply they are of bounded type with non-positive mean type. Hence F is of bounded type, and $v(F) = v(U_+ - U_-) \leq v(U_+) + v(U_-) \leq 0$. $\square$

For the other direction we have the following. Here we again use the notation $F_y(x) = F(x+iy)$.

Lemma 2.8.3. *Let $1 \leq p < \infty$. Let F be a function of bounded type and nonpositive mean type in $\mathbb{C}^+$. If there exists $f \in L^p(\mathbb{R})$ so that $\|F_y - f\|_p \to 0$ as $y \to 0^+$, then $F \in H^p(\mathbb{C}^+)$ (and hence $f \in H^p(\mathbb{R})$).*

Proof. Let $\tau \leq 0$ be the mean type of F in $\mathbb{C}^+$. Let G be the analytic function in $\mathbb{C}^+$ defined by

$$G(z) = e^{i\tau z}F(z). \tag{2.79}$$

By construction G has exponential type 0 in $\mathbb{C}^+$. For $\beta > 0$ the function $G_\beta(z) = G(z+i\beta)$ is analytic on $\{z \in \mathbb{C} \,|\, \mathrm{Im}(z) > -\beta\}$ and has exponential type 0 there. Lemma 2.2.3 gives

$$\int_{-\infty}^{\infty} |G_\beta(x+iy)|^p dx \leq \int_{-\infty}^{\infty} |G_\beta(x)|^p dx < \infty \tag{2.80}$$

for all $y \geq 0$, hence for any $\beta > 0$ we have

$$\sup_{\beta<y<\infty}\int_{-\infty}^{\infty} |G(x+iy)|^p dx < \infty. \tag{2.81}$$

Let $\varepsilon > 0$ and choose $\beta_0 > 0$ so that $\|G_\beta - f\|_p < \varepsilon$ for all $0 < \beta \leq \beta_0$. It follows that $\|G_\beta\|_p \leq \|f\|_p + \varepsilon$ for $0 < \beta \leq \beta_0$, hence

$$\sup_{0<y<\beta_0} \int_{-\infty}^{\infty} |G(x+iy)|^p dx \leq \|f\|_p + \varepsilon < \infty. \tag{2.82}$$

This combined with (2.81) implies that $G \in H^p(\mathbb{C}^+)$. Since $|e^{-i\tau z}| = e^{\tau y} \leq 1$ for $y > 0$, it follows that $F \in H^p(\mathbb{C}^+)$. $\square$

3. DE BRANGES SPACES

3.1. Introduction

In the late 1950s Louis de Branges founded a beautiful theory of Hilbert spaces of entire functions which generalizes the classical Paley-Wiener spaces and Fourier analysis (contained in the text [23] see also [18, 19, 20, 21, 22]). In recent years, this theory has proven extremely useful in a variety of (somewhat unrelated) contexts. Noteworthy examples are the following: spectral theory of canonical systems and Schrödinger operators [27] and [56], reformulations of the Riemann Hypothesis for Dirichlet-Riemann L-functions [43], as well as sampling and interpolation problems [52].

It is the goal of this chapter to introduce the theory of these so-called de Branges spaces or weighted Paley-Wiener spaces and record basic facts about these spaces. In addition, we describe some generalizations, alternative definitions, and interpolation formulas that will be useful for L^1 approximation problems.

3.2. Reproducing kernel Hilbert spaces

Let $\mathcal{H}$ be a non-zero Hilbert space of entire functions with the property that for each complex number w the evaluation functional $F \mapsto F(w)$ is bounded on $\mathcal{H}$, i.e., for any $w \in \mathbb{C}$ there exists a constant C_w such that

$$|F(w)| \leq C_w \|F\|_{\mathcal{H}}. \tag{3.1}$$

By Riesz's theorem, for each complex number w there exists an entire function $z \mapsto K(w, z)$ in $\mathcal{H}$ such that

$$F(w) = \langle F, K(w, \cdot) \rangle_{\mathcal{H}} \tag{3.2}$$

for all $F \in \mathcal{H}$ where $\langle \cdot, \cdot \rangle_{\mathcal{H}}$ is the inner product in $\mathcal{H}$. The function K is called the *reproducing kernel* for the Hilbert space $\mathcal{H}$. We say a Hilbert Space is a *reproducing kernel Hilbert space* (RKHS) if the evaluation functional is bounded. Using (3.2) we have

$$K(w, w) = \langle K(w, \cdot), K(w, \cdot) \rangle_{\mathcal{H}} = \|K(w, \cdot)\|_{\mathcal{H}}^2 \tag{3.3}$$

for all complex w, and by the Cauchy-Schwarz inequality it follows that

$$|F(w)|^2 \leq \|F\|_{\mathcal{H}}^2 \|K(w,\cdot)\|_{\mathcal{H}}^2 = \|F\|_{\mathcal{H}}^2 K(w,w) \tag{3.4}$$

for all complex w and all F in $\mathcal{H}$. Notice that if $K(w,w) > 0$, then there is equality in (3.4) if and only if $F(z) = \gamma K(w,z)$ for some complex γ. Moreover, if a sequence of functions $\{F_n\}$ converges to F in $\mathcal{H}$, i.e., $\|F_n - F\|_{\mathcal{H}} \to 0$ as $n \to \infty$, then (3.4) implies that $F_n \to F$ pointwise on $\mathbb{C}$ and uniformly on any subset of $\mathbb{C}$ where the function $z \mapsto K(z,z)$ is bounded.

Example 3.2.1.

1. The Hardy Space $H^2(\mathbb{C}^+)$ is a reproducing kernel Hilbert space with reproducing kernel

$$K(w,z) = \frac{1}{2\pi i}\frac{1}{\overline{w} - z} \tag{3.5}$$

   for $w,\ z \in \mathbb{C}^+$.

2. The Paley-Wiener space $\mathcal{A}^2(2\pi\delta, dx)$, entire functions of exponential type $2\pi\delta$ that are square integrable on the real line, is a reproducing kernel Hilbert space with reproducing kernel

$$K(w,z) = \frac{\sin(2\pi\delta(\overline{w} - z))}{\pi(\overline{w} - z)} \tag{3.6}$$

   for $w,\ z \in \mathbb{C}$.

3.3. De Branges spaces

The classical proof that the Paley-Wiener space is a reproducing kernel space relies on Fourier Analysis. In the late 1950s Louis de Branges discovered a new approach to this fact using Cauchy's Formula for the upper half-plane (Lemma 2.7.11) and basic properties of orthogonal sets. It is this approach which allowed de Branges to generalize the Paley-Wiener spaces. These weighted Paley-Wiener spaces or de Branges spaces are also reproducing kernel Hilbert spaces (Theorem 3.3.8) with remarkable generalizations of the SWK Interpolation Theorem and Parseval's formula (Theorem 3.5.5). Further information and complete proofs about the results stated in following sections can be found in [23] as well as [1], [32], and [35].

Recall that if $F : \mathbb{C} \to \mathbb{C}$ is an entire function, we define the entire function $F^* : \mathbb{C} \to \mathbb{C}$ by

$$F^*(z) := \overline{F(\overline{z})}. \tag{3.7}$$

We say that an entire function F is real entire if $F^*(z) = F(z)$ for all z (equivalently, the restriction of F to the real line is real valued).

Definition 3.3.1. The Hermite-Biehler class, denoted $\mathcal{HB}$, is defined as the set of all entire functions E, satisfying

$$|E^*(z)| < |E(z)| \tag{3.8}$$

for all $z \in \mathbb{C}^+$.

This property implies that a Hermite-Biehler function does not have any zeros in the upper half-plane. It should be mentioned that the terminology "Hermite-Biehler" is not uniform throughout literature. In fact, when working with de Branges spaces many authors refer to these functions as *de Branges functions*. For a detailed discussion on the class of Hermite-Biehler functions (including representation theorems) see Chapter VII of [45].

Example 3.3.2. For all $\delta > 0$, the entire function $E(z) = e^{-i\delta z}$ is Hermite-Biehler, since

$$|E^*(z)| = e^{-\delta \mathrm{Im}(z)} < e^{\delta \mathrm{Im}(z)} = |E(z)| \tag{3.9}$$

for all $z \in \mathbb{C}^+$.

For a given E, define

$$A(z) := \frac{1}{2}\left(E(z) + E^*(z)\right) \text{ and } B(z) := \frac{i}{2}\left(E(z) - E^*(z)\right). \tag{3.10}$$

The functions A and B are real entire functions with only real zeros (from (3.8)) such that

$$E(z) = A(z) - iB(z). \tag{3.11}$$

If one views Hermite-Biehler functions as a generalization of the exponential (see Example 3.3.2), then the functions A and B are generalizations of sine and cosine. In Lemmas 3.3.11 and 3.5.4 we will explore additional properties of these functions.

Definition 3.3.3. Given a Hermite-Biehler function E and $p \in [1, \infty]$, define the L^p de Branges space associated to E, denoted $\mathcal{H}^p(E)$, as the space of all entire functions $F : \mathbb{C} \to \mathbb{C}$ such that both ratios F/E and F^*/E are of bounded type and of non-positive mean type in the upper half-plane and

$$\|F\|_{E,p} := \|F/E\|_p = \left(\int_{-\infty}^{\infty} \left| \frac{F(t)}{E(t)} \right|^p dt \right)^{1/p} < \infty \tag{3.12}$$

if p is finite, and

$$\|F\|_{E,\infty} := \|F/E\|_\infty = \sup_{x \in \mathbb{R}} |F(x)/E(x)| < \infty \tag{3.13}$$

if $p = \infty$.

These spaces are Banach spaces (see [1] as well as [32, Section 3]), and by Lemma 2.8.1 and Lemma 3.3.7 (below), we have a more elegant description of $\mathcal{H}^p(E)$.

Lemma 3.3.4. *Let $1 \leq p \leq \infty$. Let E be a Hermite-Biehler function and F be an entire function. Then $F \in \mathcal{H}^p(E)$ if and only if F/E and F^*/E belong to $H^p(\mathbb{C}^+)$.*

In the case $p = 2$, $\mathcal{H}^2(E)$ is a Hilbert space [23, Theorem 21] with inner product given by

$$\langle F, G \rangle_E = \int_{-\infty}^{\infty} \frac{F(t)\overline{G(t)}}{|E(t)|^2} \, dt \tag{3.14}$$

for all $F, G \in \mathcal{H}^2(E)$. We refer to this space as the de Branges space associated to E and write $\mathcal{H}(E) = \mathcal{H}^2(E)$ and $\|\cdot\|_E = \|\cdot\|_{E,2}$.

The prototypical example of a de Branges space is the Paley-Wiener space $\mathcal{A}^2(2\pi\delta, dx)$. In this case $E_\delta(z) = e^{-2\pi i \delta z}$. The conditions that F/E_δ and F^*/E_δ have non-positive mean type imply that F and F^* do not grow faster E_δ, i.e., this condition is essentially equivalent to the statement that F has exponential type $2\pi\delta$. In Theorem 3.6.1, the connection between F/E and F^*/E having non-positive mean type and F being of exponential type is made formal for Hermite-Biehler functions with bounded type.

A fundamental result of de Branges is the recognition that $\mathcal{H}(E)$ is a reproducing kernel Hilbert space (RKHS), see Theorem 3.3.8 below. The reproducing kernel is given by

$$K_E(w,z) = \frac{E(z)E^*(\overline{w}) - E^*(z)E(\overline{w})}{2\pi i(\overline{w}-z)} = \frac{B(z)A(\overline{w}) - A(z)B(\overline{w})}{\pi(z-\overline{w})} \tag{3.15}$$

for $z \neq \overline{w}$ and when $z = \overline{w}$ we find that

$$K_E(\overline{z},z) = \frac{E(z)\overline{E'(z)} - E^*(z)E'(z)}{2\pi i} = \frac{B'(z)A(z) - A'(z)B(z)}{\pi}. \tag{3.16}$$

Notice that for every complex w, the function $z \mapsto K_E(w,z)$ is an entire function.

Remark 3.3.5. For $\alpha \in \mathbb{R}$ and $k > 0$, the Hermite-Biehler functions

$$E_\alpha(z) = e^{i\alpha}E(z) := A_\alpha(z) - iB_\alpha(z) \tag{3.17}$$

and

$$E_k(z) = kA(z) - \frac{i}{k}B(z) \tag{3.18}$$

give the same reproducing kernel as E, hence they generate the same space $\mathcal{H}(E)$. In fact, if M is a real 2×2 matrix with $\det(M) = 1$ and $E_M = A_M - iB_M$ where

$$\begin{pmatrix} A_M \\ B_M \end{pmatrix} = M \begin{pmatrix} A \\ B \end{pmatrix}, \tag{3.19}$$

then

$$K_{E_M}(w,z) = \frac{B_M(z)A_M(\overline{w}) - A_M(z)B_M(\overline{w})}{\pi(z-\overline{w})} = \frac{B(z)A(\overline{w}) - A(z)B(\overline{w})}{\pi(z-\overline{w})} = K_E(w,z). \tag{3.20}$$

hence $\mathcal{H}(E_M) = \mathcal{H}(E)$ isomorphically. Thus, there is an $SL(2,\mathbb{R})$-structure on the Hermite-Biehler functions that generate a given de Branges space. Furthermore, if E and $\widetilde{E}$ are Hermite-Biehler functions such that $\mathcal{H}(E) = \mathcal{H}(\widetilde{E})$ isomorpically, then $\widetilde{E} = E_M$ for some real 2×2 matrix with determinant 1 (see [20, Theorem 1]).

The following lemmas will be used to show that that $\mathcal{H}(E)$ is a RKHS. The first shows that the kernel belongs to $\mathcal{H}^p(E)$ for all $1 < p \leq \infty$. The second gives that for any function $F \in \mathcal{H}^p(E)$, $1 < p \leq \infty$, the quotients F/E and F/E^* do not have singularities on the real axis, hence they have continuous extensions to the closed half-plane $\overline{\mathbb{C}^+}$ (which allows us to apply Cauchy's Formula (Theorem 2.7.11)). We follow the notation and proofs of de Branges [23] making the appropriate changes for p when needed.

Lemma 3.3.6. *Let $w \in \mathbb{C}$. Then $z \mapsto K(w,z)/E(z)$ has bounded type and non-positive mean type in $\mathbb{C}^+$ and*

$$\|K(w,\cdot)\|_{E,p}^p = \int_{-\infty}^{\infty} \left| \frac{K(w,t)}{E(t)} \right|^p dt < \infty \tag{3.21}$$

for all $1 < p < \infty$ and

$$\|K(w,\cdot)\|_{E,\infty} = \sup_{t\in\mathbb{R}} |K(w,t)/E(t)| < \infty. \tag{3.22}$$

Proof. Since E is Hermite-Biehler, it does not have any zeros in the upper half-plane. By (3.15) we have that

$$\frac{2\pi i(\overline{w} - z)K(w,z)}{E(z)} = \overline{E(w)} - \frac{E^*(z)}{E(z)} E(\overline{w}). \tag{3.23}$$

As $|E^*(z)E(z)^{-1}| < 1$ on $\mathbb{C}^+$ it follows that the right hand side of (3.23) is bounded in the half-plane, hence of bounded type. As $2\pi i(\overline{w} - z)$ is of bounded type in the upper half-plane with zero mean type, it follows by Lemma 2.7.8 that the quotient $z \mapsto K(w,z)/E(z)$ is also of bounded type with non-positive mean type in $\mathbb{C}^+$.

Notice that the right hand side of (3.23) has a zero at $\overline{w}$. Hence, for every complex number w the quotient

$$\frac{K(w,t)}{E(t)} = \frac{E(t)\overline{E(w)} - \overline{E(t)}E(\overline{w})}{2\pi i(\overline{w} - t)E(t)} \tag{3.24}$$

defines a continuous and bounded function of t on the real line (where the left side is defined by continuity at the point $t = \overline{w}$ if w is real). This proves the case $p = \infty$.

Using (3.24) it follows that

$$\begin{aligned}\int_{-\infty}^{\infty}\left|\frac{K(w,t)}{E(t)}\right|^p dt &= \int_{|\overline{w}-t|\le 1}\left|\frac{K(w,t)}{E(t)}\right|^p dt + \int_{|\overline{w}-t|>1}\left|\frac{K(w,t)}{E(t)}\right|^p dt \\ &\le C + \int_{|\overline{w}-t|>1}\left|\frac{E(t)\overline{E(w)}-\overline{E(t)}E(\overline{w})}{2\pi i(\overline{w}-t)E(t)}\right|^p dt \\ &\le C + \frac{|E(w)|^p+|E(\overline{w})|^p}{(2\pi)^p}\int_{|\overline{w}-t|>1}\left|\frac{1}{\overline{w}-t}\right|^p dt \\ &< \infty\end{aligned} \tag{3.25}$$

for all $1 < p < \infty$, which proves the theorem. □

Lemma 3.3.7. *Let $1 \le p \le \infty$. Let E be a Hermite-Biehler function. If E has a zero of order $r > 0$ at a real point x_0, then any $F \in \mathcal{H}^p(E)$ must have a zero of order at least r at x_0.*

Proof. Let $p \in [1, \infty)$. Assume that F does not have a zero of order at least r at x_0, then

$$F(z) = F_0(z)(z - x_0)^m \tag{3.26}$$

for some $0 \le m < r$ and entire F_0 such that $F_0(x_0) \ne 0$. Similarly, since E has a zero of order r at x_0, it follows that $E(z) = E_0(z)(z - x_0)^r$ where E_0 is entire and $E_0(x_0) \ne 0$. Since F_0 and E_0 are entire and non-zero at x_0, it follows by continuity that there exists $\varepsilon > 0$ and $\delta > 0$ such that $|E_0/F_0| \ge \varepsilon$ on $(x_0 - \delta, x_0 + \delta)$. Since $r > m$ it follows that

$$\int_{-\infty}^{\infty}\left|\frac{F(t)}{E(t)}\right|^p dt = \int_{-\infty}^{\infty}\left|\frac{F_0(t)}{E_0(t)}\right|^p \frac{1}{|t-x_0|^{p(r-m)}}\,dt \ge \varepsilon^p\int_{x_0-\delta}^{x_0+\delta}\frac{1}{|t-x_0|^{p(r-m)}}\,dt = \infty. \tag{3.27}$$

This contradicts the assumption that $F \in \mathcal{H}^p(E)$, thus F must have a zero of order at least r at x_0. The case when $p = \infty$ is immediate. □

Theorem 3.3.8. *Let E be a Hermite-Biehler function. For every complex number w and any $p \in [1, \infty)$, $z \mapsto K(w, z)$ belongs to $\mathcal{H}^{p'}(E)$, where $1/p + 1/p' = 1$, and*

$$F(w) = \langle F(t), K(w,t)\rangle_E = \int_{-\infty}^{\infty} F(t)\overline{K(w,t)}|E(t)|^{-2}\,dt \tag{3.28}$$

for every F in $\mathcal{H}^p(E)$.

Proof. Let F belong to $\mathcal{H}^p(E)$. Since F/E and F^*/E have non-positive mean type and belong to $L^p(\mathbb{R}, dx)$, Lemma 3.3.7 implies they have continuous extensions to the closed half-plane. By Cauchy's formula (Theorem 2.7.11), applied to F/E, it follows that

$$F(w)/E(w) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{F(t)/E(t)}{t-w}\, dt \tag{3.29}$$

for $z \in \mathbb{C}^+$, and

$$0 = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{F(t)/E(t)}{t-w}\, dt \tag{3.30}$$

for all $w \in \mathbb{C}^-$. Similarly, applying Cauchy's formula to F^*/E gives

$$F^*(w)/E(w) = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\overline{F(t)}/E(t)}{t-w}\, dt \tag{3.31}$$

for $w \in \mathbb{C}^+$, and

$$0 = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{\overline{F(t)}/E(t)}{t-w}\, dt \tag{3.32}$$

for all $w \in \mathbb{C}^-$. Taking conjugates in (3.31) and (3.32) leads to

$$F(w)/E^*(w) = -\frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{F(t)/\overline{E}(t)}{t-w}\, dt \tag{3.33}$$

for $z \in \mathbb{C}^-$, and

$$0 = \frac{1}{2\pi i} \int_{-\infty}^{\infty} \frac{F(t)/\overline{E}(t)}{t-w}\, dt \tag{3.34}$$

for all $w \in \mathbb{C}^+$.

Formulas (3.29), (3.30), (3.33), and (3.34) imply that for all non-real w,

$$\begin{aligned}
F(w) &= \frac{1}{2\pi i}\int_{-\infty}^{\infty} F(t)\frac{E^*(t)E(w) - E(t)E^*(w)}{t-w}\frac{dt}{|E(t)|^2}\,dt \\
&= \int_{-\infty}^{\infty} F(t)K(t,w)\,dt \\
&= \int_{-\infty}^{\infty} F(t)\overline{K(w,t)}\,dt \\
&= \langle F, K(w,\cdot)\rangle_E.
\end{aligned} \tag{3.35}$$

If w is real, choose a sequence $\{w_n\}$ of nonreal numbers such that $w = \lim_n w_n$ and apply a standard limiting argument (see [23, Theorem 19]) to obtain (3.28). □

Remark 3.3.9. By the Cauchy-Schwarz inequality, for $F \in \mathcal{H}^p(E)$, $1 \le p < \infty$, we have that

$$|F(w)| \le \|F\|_{E,p}\|K(w,\cdot)\|_{E,p'} \tag{3.36}$$

for all complex w.

The following statements record some additional properties of the kernel K as well as the functions A and B.

Lemma 3.3.10. *Let $w \in \mathbb{C}$. If $\mathrm{Im}(w) \neq 0$ then $K(w,w) > 0$ and if w is real then $0 < K(w,w)$ if and only if $E(w) \neq 0$.*

Proof. This proof is essentially that of [35, Lemma 11], and we include it here for completeness.

If $w = x + iy$ with $y \neq 0$ then (3.15) implies

$$4\pi y K(w,w) = |E(x+iy)|^2 - |E^*(x+iy)|^2. \tag{3.37}$$

Since E is Hermite-Biehler, it follows that $K(w,w) > 0$. Assume now that w is real. If $E(w) = 0$, then by (3.16) we have

$$K(w,w) = \frac{E(w)\overline{E'(w)} - \overline{E(w)}E'(w)}{2\pi i} = 0. \tag{3.38}$$

On the other hand, if $K(w,w) = 0$, then

$$0 = K(w,w) = \langle K(w,\cdot), K(w,\cdot)\rangle_{\mathcal{H}(E)} = \|K(w,\cdot)\|^2_{\mathcal{H}(E)}. \tag{3.39}$$

Thus, $z \mapsto K(w,z)$ is identically zero, and by (3.15), we have

$$0 = E(z)\left(\overline{E(w)} - \frac{E^*(z)}{E(z)}E(w)\right) \tag{3.40}$$

for all $z \in \mathbb{C}^+$.

If $E(w) \neq 0$, then since E is non-zero on $\mathbb{C}^+$ we have

$$\frac{\overline{E(w)}}{E(w)} = \frac{E^*(z)}{E(z)} \tag{3.41}$$

for all $z \in \mathbb{C}^+$. It follows that $|E^*(z)| = |E(z)|$ for all $z \in \mathbb{C}^+$ which contradicts the fact that E is Hermite-Biehler. Thus, $E(w) = 0$. □

Lemma 3.3.11. *Let E be a Hermite-Biehler function with no real zeros. The following conditions are equivalent:*

1. *$E(-z) = E^*(z)$ for all z.*

2. *The function $z \mapsto E(iz)$ is real entire.*

3. *A is even and B is odd with a simple zero at the origin.*

Proof. To prove that (1) and (2) are equivalent note that (1) implies that

$$\overline{E(i\overline{z})} = \overline{E(\overline{-iz})} = E^*(-iz) = E(iz) \tag{3.42}$$

for all complex z, hence $z \mapsto E(iz)$ is real entire. Notice (3.42) holds if we assume $z \mapsto E(iz)$ is real entire, and letting $w = -iz$ in (3.42) gives (1).

To show that (1) implies (3), notice that $E(-z) = E^*(z)$ gives $A = 2^{-1}(E + E^*)$ is even and $B = i2^{-1}(E - E^*)$ is odd with $B(0) = 0$. If the zero at 0 is not simple, then (3.16) gives $K(0,0) = 0$ which implies that $E(0) = 0$ (by Lemma 3.3.10), a contradiction.

For the other direction, we show that (3) implies (2). Recall that A and B are real entire functions, i.e., $A^* = A$ and $B^* = B$, hence

$$\overline{E(i\overline{z})} = A^*(-iz) + iB^*(-iz) = A(-iz) + iB(-iz) = A(iz) - iB(iz) = E(iz) \tag{3.43}$$

for all complex z and (2) follows. $\square$

Remark 3.3.12. Under the assumptions of the previous lemma we see that if x is real then $A(ix) \in \mathbb{R}$ and $B(ix) \in i\mathbb{R}$, hence $(A(ix))^2 > 0$ and $(B(ix))^2 < 0$ for all $x \in \mathbb{R}$. Moreover, since E is Hermite-Biehler, it follows that $iB(ix) = -2^{-1}(E(ix) - E(-ix)) < 0$ for $x > 0$.

3.4. Axiomatic de Branges spaces

One can easily deduce that a given de Branges space $\mathcal{H}(E)$ satisfies the following properties:

- H1. If $F \in \mathcal{H}(E)$ and w is a non-real zero of F, then the function $z \mapsto F(z)\frac{z-\overline{w}}{z-w}$ belongs to $\mathcal{H}(E)$ and has the same norm as F.
- H2. For every nonreal number $w \in \mathbb{C}$, the evaluation functional $F \mapsto F(w)$ is continuous.
- H3. If $F \in \mathcal{H}(E)$, then $F^* \in \mathcal{H}(E)$ and has the same norm as F.

One of the classical theorems of de Branges [23, Theorem 23] states that if $\mathcal{H}$ is a nontrivial Hilbert space of entire functions that satisfies H1 − H3 (so-called *axiomatic de Branges space*), then there exists a Hermite-Biehler function E such that $\mathcal{H}$ is equal isometrically to $\mathcal{H}(E)$. In fact, by (H2) an axiomatic de Branges spaces is a RKHS, and a Hermite-Biehler function for which $\mathcal{H} = \mathcal{H}(E)$ holds is

$$E(z) = \frac{i\sqrt{\pi}(\overline{w} - z)K(w,z)}{(\mathrm{Im}(w)K(w,w))^{1/2}} \tag{3.44}$$

where w is any point in the upper half-plane (see e.g., [8, Proposition 20]). Notice that the Hermite-Biehler function is not uniquely determined (which we have already seen in Remark 3.3.5).

3.5. Orthonormal sets in $\mathcal{H}(E)$

Lemma 3.5.1. *Let E be a Hermite-Biehler function. Then there exists a continuous, increasing, and real-valued function φ such that*

$$\frac{E^*(x)}{E(x)} = e^{2i\varphi(x)} \tag{3.45}$$

for $x \in \mathbb{R}$.

Proof. Since $|E^*(z)| < |E(z)|$ for all $z \in \mathbb{C}^+$, it follows that $D(z) = \frac{E^*(z)}{E(z)}$ defines an analytic function on $\mathbb{C}^+$ with analytic continuation to the closed half-plane that satisfies $|D(z)| < 1$ on $\mathbb{C}^+$ and $|D(x)| = 1$ on $\mathbb{R}$. Hence, D is a non-constant inner function and (3.45) follows from Lemma 2.6.5. $\square$

Definition 3.5.2. If E is a Hermite-Biehler function, then any function φ that satisfies Lemma 3.5.1 is referred to as a *phase function* associated to E.

The phase function associated to E is uniquely defined up to an additive constant πk where k is any integer. In particular, by Lemma 2.6.5, if φ is any such function, then

$$\varphi'(x) = -\frac{\alpha}{2} + \sum_k \frac{y_k}{(x - x_k)^2 + y_k^2} \tag{3.46}$$

for all $x \in \mathbb{R}$, where $\{z_n\}$ are the zeros of E and $\alpha = \nu(E^*/E) \leq 0$. Moreover, equations (3.45) and (3.16) give

$$\varphi'(x) = \pi K(x, x)|E(x)|^{-2} \tag{3.47}$$

for all real x.

Example 3.5.3.

1. The Hermite-Biehler function $z \mapsto e^{-i\tau z}$ has phase $\varphi(x) = \tau x$.

2. The Hermite-Biehler function $z \mapsto z e^{-i\tau z}$ also has phase $\varphi(x) = \tau x$.

3. The Hermite-Biehler function $z \mapsto e^{-i\tau z}(z + i)$ has phase $\tau(x) = \tau x + \arctan(x)$.

Using the phase function, we can record useful properties of the functions A and B.

Lemma 3.5.4. *Let $E = A - iB$ be a Hermite-Biehler function with no real zeros. Then the zeros of A and B are simple and interlace.*

Proof. Recall that $A = 2^{-1}(E + E^*)$ and $B = i2^{-1}(E - E^*)$. Using (3.45) it follows that

$$\begin{aligned} A(x) &= e^{i\varphi(x)} E(x) \cos(\varphi(x)) \\ B(x) &= e^{i\varphi(x)} E(x) \sin(\varphi(x)). \end{aligned} \tag{3.48}$$

Since E does not have any real zeros and φ is continuous, real-valued, and increasing on the real-line, the result follows. $\square$

Let α be a real number, and define

$$\mathcal{T}_\alpha := \{t \in \mathbb{R} \mid \varphi(t) \equiv \alpha \bmod \pi\}. \tag{3.49}$$

We denote by $\mathcal{T}_F$ the set of zeros of F. For a Hermite-Biehler E with no real zeros, it follows from (3.48) that $\mathcal{T}_0 = \mathcal{T}_B$ and $\mathcal{T}_{\pi/2} = \mathcal{T}_A$. For a general Hermite-Biehler E and $\alpha \in \mathbb{R}$, we have $\mathcal{T}_\alpha = \mathcal{T}_{E^{-1}B_\alpha}$ for all α, where $B_\alpha = e^{i\alpha}E - e^{-i\alpha}E^*$.

Orthogonal sets in a de Branges space can also be constructed via phase functions, which yield the remarkable generalization of Parseval's formula (Theorem 3.5.5) for norms in the space (see [23, Theorem 22]).

Notice that if $a, b \in \mathcal{T}_\alpha$ with $a \neq b$, then by Theorem 3.3.8 and (3.45) we obtain

$$\begin{aligned}\langle K(a,\cdot), K(b,\cdot)\rangle_E &= K(a,b)\\ &= \frac{E(b)E^*(a) - E^*(b)E(a)}{2\pi i(a-b)}\\ &= \frac{E(b)E^*(a)(1 - e^{2i(\varphi(b)-\varphi(a))})}{2\pi i(a-b)}\\ &= 0.\end{aligned} \tag{3.50}$$

Next, define $X_E = \{x \in \mathbb{R} \mid E(x) \neq 0\}$. If $\xi \in X_E$, then by Lemma 3.3.10 we have $\|K(\xi,\cdot)\|_E \neq 0$. Hence, the function

$$z \mapsto \frac{K(\xi,z)}{\|K(\xi,\cdot)\|_E} \tag{3.51}$$

belongs to $\mathcal{H}(E)$ and has norm 1.

Theorem 3.5.5 ([23, Theorem 22]). *Let E be a Hermite-Biehler function with φ an associated phase function. Let $\alpha \in \mathbb{R}$ be a real number such that $\mathcal{T}_\alpha \cap X_E$ is not empty. Then the collection of entire functions*

$$\mathcal{K}_\alpha := \left\{ z \mapsto \frac{K(\xi,z)}{\|K(\xi,\cdot)\|_E} \mid \xi \in \mathcal{T}_\alpha \cap X_E \right\} \tag{3.52}$$

forms an orthonormal set in $\mathcal{H}(E)$.

Moreover, if $B_\alpha = e^{i\alpha}E - e^{-i\alpha}E^* \notin \mathcal{H}(E)$, then $\mathcal{K}_\alpha$ is an orthonomal basis for $\mathcal{H}(E)$, and for all $F \in \mathcal{H}(E)$ the identity

$$\|F\|_E^2 = \int_{-\infty}^{\infty} |F(t)|^2 \frac{dt}{|E(t)|^2} = \sum_{\xi \in \mathcal{T}_\alpha} \left| \frac{F(\xi)}{E(\xi)} \right|^2 \frac{\pi}{\varphi'(\xi)} = \sum_{\xi \in \mathcal{T}_\alpha} \frac{|F(\xi)|^2}{K(\xi,\xi)} \tag{3.53}$$

holds.

Remark 3.5.6. There is at most one $\alpha \in [0, \pi)$ such that $B_\alpha = e^{i\alpha}E - e^{-i\alpha}E^* \in \mathcal{H}(E)$ (see [23, Problem 46]) for if B_α and B_β belong to $\mathcal{H}(E)$ with $\alpha \not\equiv \beta \bmod \pi$, then

$$E = \frac{1}{e^{2i\alpha} - e^{2i\beta}}(e^{i\alpha}B_\alpha - e^{i\beta}B_\beta) \in \mathcal{H}(E), \tag{3.54}$$

a contradiction.

The previous theorem implies an interpolation formula in de Branges spaces. Namely, any function $F \in \mathcal{H}(E)$ can be recovered from its samples $\{F(\xi)\}_{\xi \in \mathcal{T}_\alpha}$. To see this, expand $F \in \mathcal{H}(E)$ with respect to the orthogonal basis $\mathcal{K}_\alpha$ to obtain

$$F(z) = \sum_{\xi \in \mathcal{T}_\alpha} \left\langle F, \frac{K(\xi,\cdot)}{\|K(\xi,\cdot)\|_E} \right\rangle_E \frac{K(\xi,z)}{\|K(\xi,\cdot)\|_E} = \sum_{\xi \in \mathcal{T}_\alpha} F(\xi) \frac{K(\xi,z)}{K(\xi,\xi)} \tag{3.55}$$

in $\mathcal{H}(E)$. By (3.53) along with the fact that $\mathcal{H}(E)$ is a reproducing kernel Hilbert space (see Section 3.2) it follows that the convergence is also uniform and absolute on compact subsets of $\mathbb{C}$ (see [23, Problem 47 and Theorem 22]).

3.6. Hermite-Biehler functions with bounded type

In the case that the Hermite-Biehler function is of bounded type in the upper half-plane (which is the case for most of the de Branges spaces used in applications) there is an easier characterization of functions in the L^p de Branges spaces (Theorem 3.6.1). The statement for $p = 2$ is due to Holt and Vaaler [35, Lemma 12], and a similar result is given by H. Dym in [27, Lemma 3.5]. Recall that an entire function F is said to belong to $\mathcal{B}$ if F is of exponential type and

$$\int_{-\infty}^{\infty} \frac{\log^+ |F(x)|}{1 + x^2}\, dx < \infty. \tag{3.56}$$

By Krein's theorem (Theorem 2.7.10) this is equivalent to F and F^* being of bounded type in the upper half-plane. Moreover, the exponential type of F is then the max of the mean types of F and F^*.

Theorem 3.6.1. *Let E be a Hermite-Biehler function of bounded type in the upper half-plane and F be an entire function. For $1 \le p < \infty$, the following conditions are equivalent:*

1. *F belongs to $\mathcal{H}^p(E)$.*
2. *F belongs to $\mathcal{B}$, $\max\{v(F), v(F^*)\} \le v(E)$, and $F/E \in L^p(\mathbb{R}, dx)$,*
3. *F has exponential type, $\tau(F) \le \tau(E)$, and $F/E \in L^p(\mathbb{R}, dx)$.*

Definition 3.6.2. For $1 \le p \le \infty$, $\delta > 0$, and μ a Borel measure on $\mathbb{R}$, we define the weighted L^p Paley-Wiener space, denoted $\mathcal{A}^p(\delta, \mu)$, as the space of all entire functions F of exponential type δ that belong to $L^p(\mathbb{R}, \mu)$.

Remark 3.6.3. Under the assumptions of the previous lemma, we see that

$$\mathcal{H}^p(E) = \mathcal{A}^p(\tau(E), |E(x)|^{-p}\, dx). \tag{3.57}$$

In particular, $\mathcal{H}(E) = \mathcal{H}^2(E) = \mathcal{A}^2(\tau(E), |E(x)|^{-2}\, dx)$ and $\mathcal{H}^1(E^2) = \mathcal{A}^1(2\tau(E), |E(x)|^{-2}\, dx)$.

For the proof of Theorem 3.6.1 we will make use of the following Lemmas. The first records many useful properties of Hermite-Biehler functions that are also of bounded type. The second states well known facts about $z \mapsto \log^+ |z|$ and the last is Jensen's Inequality (see e.g., [23, Problem 32]).

Lemma 3.6.4. *Let E be a Hermite-Biehler function of bounded type in the upper half-plane. Then*

1. *E^{-1} is of bounded type with $v(E) = -v(E^{-1})$.*
2. *E^* is of bounded type and $v(E^*) \le v(E)$.*
3. *$E \in \mathcal{B}$ and $\tau(E) = v(E)$.*
4. *The mean type of E is non-negative.*

Proof. Since E does not have any zeros in the upper half-plane, $z \mapsto E^{-1}(z)$ defines an analytic function on the upper half-plane. As E is of bounded type, it follows that E^{-1} is also of bounded type, and by (2.59) we have that

$$v(E^{-1}) = \lim_{y\to\infty} \frac{\log|E^{-1}(iy)|}{y} = -\lim_{y\to\infty} \frac{\log|E(iy)|}{y} = -v(E). \tag{3.58}$$

Since E is Hermite-Biehler, it follows that E^*/E is bounded by 1 in the upper half-plane, hence of bounded type in the upper half-plane, and $v(E^*/E) \le 0$. Thus, E^* has bounded type in the upper half-plane and

$$v(E^*) = v(E^*E^{-1}E) = v(E^*/E) + v(E) \le v(E). \tag{3.59}$$

Krein's Theorem (Theorem 2.7.10) then implies that $E \in \mathcal{B}$ and E is of exponential type with $v(E) = \max\{v(E), v(E^*)\} = \tau(E) \ge 0$.

□

Lemma 3.6.5. *Let $1 \le p < \infty$. Then the inequalities*

$$\log^+|a| \le p^{-1}\log(1+|a|^p) \tag{3.60}$$

and

$$\log^+|ab| \le \log^+|a| + \log^+|b| \tag{3.61}$$

hold for all complex numbers a and b.

Proof. For $|a| \le 1$, (3.60) is immediate. For $|a| > 1$, we have that

$$p^{-1}\log(1+|a|^p) - \log^+|a| = p^{-1}\log(1+|a|^p) - \log|a| = p^{-1}\log(1+|a|^{-p}) \ge 0 \tag{3.62}$$

and (3.60) follows. Inequality (3.61) follows directly from properties of the logarithm. □

Lemma 3.6.6 (Jensen's Inequality)**.** *If f is a Borel measurable function, then*

$$\frac{y}{\pi}\int_{-\infty}^{\infty} \frac{\log|f(t)|}{(t-x)^2+y^2} \le \log\left(\frac{y}{\pi}\int_{-\infty}^{\infty} \frac{|f(t)|}{(t-x)^2+y^2}\,dt\right) \tag{3.63}$$

holds for all real x and $y > 0$.

We turn to the proof of Theorem 3.6.1.

Proof of Theorem 3.6.1. The proof follows that of Holt and Vaaler [35, Lemma 12] making the appropriate changes for p when needed.

Suppose that F belongs to $\mathcal{H}^p(E)$. Then $F/E \in L^p(\mathbb{R}, dx)$, F/E and F^*/E have bounded type in $\mathbb{C}^+$, and $\max\{v(F/E), v(F^*/E)\} \leq 0$. Since E has bounded type in $\mathbb{C}^+$, it follows that F and F^* have bounded type in $\mathbb{C}^+$,

$$v(F) = v(FE^{-1}E) = v(F/E) + v(E) \leq v(E), \tag{3.64}$$

and

$$v(F^*) = v(F^*E^{-1}E) = v(F^*/E) + v(E) \leq v(E). \tag{3.65}$$

Hence,

$$\max\{v(F), v(F^*)\} \leq \max\{v(F/E), v(F^*/E)\} + v(E) \leq v(E) \tag{3.66}$$

which shows that condition (2) holds.

By Krein's Theorem (Theorem 2.7.10) we have that (2) implies (3).

Suppose that F satisfies condition (3). Applying inequality (3.60), Jensen's inequality, and that $F/E \in L^p(\mathbb{R}, dx)$ gives

$$\begin{aligned}
&\int_{-\infty}^{\infty} (1+x^2)^{-1} \log^+ |F(x)E(x)^{-1}|\, dx \\
&\quad\leq p^{-1} \int_{-\infty}^{\infty} (1+x^2)^{-1} \log\left(1 + |F(x)E(x)^{-1}|^p\right)\, dx \\
&\quad\leq \pi p^{-1} \log\left(\pi^{-1} \int_{-\infty}^{\infty} (1+x^2)^{-1} \left(1 + |F(x)E(x)^{-1}|^p\right)\, dx\right) \\
&\quad= \pi p^{-1} \log\left(1 + \pi^{-1} \int_{-\infty}^{\infty} (1+x^2)^{-1} |F(x)E(x)^{-1}|^p\, dx\right) \\
&\quad\leq \pi p^{-1} \log\left(1 + \pi^{-1} \int_{-\infty}^{\infty} |F(x)E(x)^{-1}|^p\, dx\right) \\
&\quad< \infty.
\end{aligned} \tag{3.67}$$

Using inequality (3.61), the above inequality, and that $E \in \mathcal{B}$ we obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty} (1+x^2)^{-1} \log^+ |F(x)|\, dx \\
&\qquad \leq \int_{-\infty}^{\infty} (1+x^2)^{-1} \log^+ |F(x)E(x)^{-1}|\, dx + \int_{-\infty}^{\infty} (1+x^2)^{-1} \log^+ |E(x)|\, dx \\
&\qquad < \infty.
\end{aligned} \tag{3.68}$$

Hence F belongs to $\mathcal{B}$ which gives F and F^* are of bounded type. Moreover, since E^{-1} is of bounded type (Lemma 3.6.4), it follows that F/E and F^*/E are of bounded type and by Krein's Theorem we find that

$$\max\{v(F/E), v(F^*/E)\} \leq \max\{v(F), v(F^*)\} - v(E) = \tau(F) - \tau(E) \leq 0, \tag{3.69}$$

hence F belongs to $\mathcal{H}^p(E)$. □

3.7. UU^* decomposition

We conclude the discussion of L^p de Branges spaces by mentioning a key lemma that allows us to connect the L^1 and L^2 theories. The following lemma gives that any function in $\mathcal{H}^1(E^2)$ that is non-negative on $\mathbb{R}$ can be written as a square of a function belonging to $\mathcal{H}^2(E)$. The proof of Lemma 3.7.1 is that of [13, Lemma 14] (which is essentially contained in the proof of [35, Theorem 15]).

Lemma 3.7.1 ([13, Lemma 14]). *Let E be a Hermite-Biehler function of bounded type in $\mathbb{C}^+$ with exponential type $\tau(E)$. Let $F : \mathbb{C} \to \mathbb{C}$ be a real entire function of exponential type $2\tau(E)$ that satisfies*

$$F(x) \geq 0 \tag{3.70}$$

for all $x \in \mathbb{R}$ and

$$\int_{-\infty}^{\infty} \frac{F(x)}{|E(x)|^2}\, dx < \infty. \tag{3.71}$$

Then there exists $U \in \mathcal{H}(E)$ such that

$$F(z) = U(z)U^*(z). \tag{3.72}$$

Proof. By Theorem 3.6.1 we have that $F \in \mathcal{H}^1(E^2)$ and F is of bounded type with $v(F) \le v(E^2) = 2v(E)$. Since F has bounded type in $\mathbb{C}^+$, if $z_1, z_2, ..., z_n, ...$ are the zeros of F in $\mathbb{C}^+$ listed with appropriate multiplicity then

$$\sum_n \frac{\mathrm{Im}(z_n)}{|i+z_n|^2} = \sum_n \frac{y_n}{x_n^2 + (y_n+1)^2} < \infty \tag{3.73}$$

where $z_n = x_n + iy_n$ and $y_n > 0$. Let

$$D(z) = \lim_{N\to\infty} \prod_{n=1}^{N} \frac{\left|\frac{i-z_n}{i-\overline{z_n}}\right|}{\frac{i-z_n}{i-\overline{z_n}}} \frac{z-z_n}{z-\overline{z_n}} = \lim_{N\to\infty} \prod_{n=1}^{N} \frac{|z_n^2+1|}{z_n^2+1} \frac{z-z_n}{z-\overline{z_n}} \tag{3.74}$$

be the corresponding Blaschke product. Since F is real entire and non-negative on the real line, the function $D^{-1}F$ is an entire function with all of its zeros having even multiplicity and no zeros in $\mathbb{C}^+$ (note that since F is real entire (i.e., $F = F^*$), the zeros of F in the lower half-plane are precisely $\{\overline{z_n}\}$, and since F is non-negative on $\mathbb{R}$, any zero of F on $\mathbb{R}$ must have even multiplicity). Hence there exists an entire function U such that

$$D^{-1}F = U^2. \tag{3.75}$$

Since $DD^* = 1$ and F is real entire ($F = F^*$), we have $FD = (U^*)^2$. This with (3.75) shows that $F^2 = (UU^*)^2$. Since F is non-negative on the real line, we conclude that

$$F(z) = U(z)U^*(z). \tag{3.76}$$

It remains to show that $U \in \mathcal{H}(E)$. Using (3.75) and Lemma 2.6.4, we find that U has bounded type in $\mathbb{C}^+$. By (3.75) and (3.76) we have $U^* = DU$. Since D is of bounded type and has zero mean type (follows from the fact that D is a Blaschke product, see Remark 2.7.4), the function U^* is then of bounded type and $v(U^*) = v(U)$. By Krein's Theorem (Theorem 2.7.10), it follows that $U \in \mathcal{B}$. Using (3.76) we find that

$$2\max\{v(U), v(U^*)\} = 2v(U) = v(F) \le v(E^2) = 2v(E). \tag{3.77}$$

Since $F \in \mathcal{H}^1(E^2)$, we have

$$\int_{-\infty}^{\infty} \frac{|U(x)|^2}{|E(x)|^2}\, dx = \int_{-\infty}^{\infty} \frac{F(x)}{|E(x)|^2}\, dx < \infty, \tag{3.78}$$

and it follows using Theorem 3.6.1 that U belongs to $\mathcal{H}^2(E) = \mathcal{H}(E)$. $\square$

3.8. Interpolation formulas for $\mathcal{H}^\infty(E)$

We end the Chapter on de Branges spaces by proving an analogue of the interpolation formulas for functions of exponential type that are bounded on the real line (Theorem 2.2.8).

Theorem 3.8.1. *Let $E = A - iB$ be a Hermite-Biehler function with no real zeros such that $B \notin \mathcal{H}(E)$. If $F \in \mathcal{H}^\infty(E)$, then for all $t_m \in \mathcal{T}_B$*

$$F(z) = B(z) \left\{ \sum_{\substack{t_n \in \mathcal{T}_B \\ t_n \neq t_m}} \frac{F(t_n)}{B'(t_n)} \left(\frac{1}{z - t_n} + \frac{1}{t_n - t_m} \right) + \frac{F(t_m)}{B'(t_m)(z - t_m)} + C_{F,E,m} \right\}, \tag{3.79}$$

where

$$C_{F,E,m} = \frac{F'(t_m)A(t_m) - F(t_m)A'(t_m) + 2A'(t_m)}{A(t_m)B'(t_m)} + \frac{B''(t_m)}{2B'(t_m)^2} \tag{3.80}$$

and the the expression on the right-hand side of (3.79) *converges uniformly on compact subsets of* $\mathbb{C}$.

Remark 3.8.2. In the case that E has no real zeros and $E^*(z) = E(-z)$ we have that A is even and B is odd with a simple zero at the origin (see Lemma 3.3.11), hence $A'(0) = 0$ and $B''(0) = 0$. In this case, applying the above result with the zero $t_m = 0$ simplifies the above constant $C_{F,E,0}$ to

$$C_{F,E,0} = \frac{F'(0)}{B'(0)}. \tag{3.81}$$

Remark 3.8.3. Applying Theorem 3.8.1 with $E(z) = e^{-i\pi z} = \cos(\pi z) - i \sin(\pi z)$ and $t_m = 0$ gives Corollary 2.2.8.

In the proof of Theorem 3.8.1 we will make use of the following lemma found in [32] which follows from the Stieltjes inversion formula [23, Theorem 3] and Theorem 3.5.5 [23, Theorem 22].

Lemma 3.8.4 ([32, Lemma 3]). *Let E be a Hermite-Biehler function with no real zeros. If $B \notin \mathcal{H}(E)$, then for all $t_m \in \mathcal{T}_B$*

$$\sum_{\substack{t_n \in \mathcal{T}_B \\ t_n \neq t_m}} \frac{|A(t_n)|}{|B'(t_n)|(1+t_n^2)} < \infty \tag{3.82}$$

and

$$\sum_{\substack{t_n \in \mathcal{T}_B \\ t_n \neq t_m}} \frac{A(t_n)}{B'(t_n)} \left(\frac{1}{z-t_m} + \frac{1}{t_n - t_m} \right) = \frac{A(z)}{B(z)} - \frac{A'(t_m)}{B'(t_m)} + \frac{A(t_m)B''(t_m)}{2B'(t_m)^2} + \frac{A(t_m)}{B'(t_m)(z-t_m)} \tag{3.83}$$

where the series converges uniformly on compact sets of $\mathbb{C}$ that do not contain points from $\mathcal{T}_B$.

We turn to the proof of Theorem 3.8.1.

Proof of Theorem 3.8.1. Let $F \in \mathcal{H}^\infty(E)$ and $t_m \in \mathcal{T}_B$. By Lemma 3.5.4, we have that the zeros of A and B are simple and interlace, hence $A(t_m) \neq 0$.

Define the entire function

$$R_m(z) = \begin{cases} \frac{F(z)A(t_m) - F(t_m)A(z)}{z - t_m} & \text{if } z \neq 0 \\ F'(t_m)A(t_m) - F(t_m)A'(t_m) & \text{if } z = 0. \end{cases} \tag{3.84}$$

Since F/E and F^*/E are of bounded type with non-positive mean type and A/E and A^*/E are bounded in the upper half-plane, we have that R_m/E and R_m^*/E are of bounded type with non-positive mean type in the upper half-plane. Moreover, since R_m is continuous on $\mathbb{R}$ and F/E and A/E are bounded on $\mathbb{R}$ it follows that $\|R_m\|_{E,2} < \infty$ (similar to the proof of Lemma 3.3.6), hence $R_m \in \mathcal{H}^2(E)$.

Applying (3.55) to R_m we obtain

$$\begin{aligned} &F(z)A(t_m) - F(t_m)A(z) \\ &= B(z) \left\{ \frac{F'(t_m)A(t_m) - F(t_m)A'(t_m)}{B'(t_m)} + \sum_{\substack{t_n \in \mathcal{T}_B \\ t_n \neq t_m}} \frac{F(t_n)A(t_m) - F(t_m)A(t_n)}{B'(t_n)} \left(\frac{1}{z-t_n} + \frac{1}{t_n - t_m} \right) \right\} \end{aligned} \tag{3.85}$$

for every $z \in \mathbb{C}$. This leads to the following representation

$$F(z) = \frac{F(t_m)}{A(t_m)} A(z) + B(z) \left\{ \frac{F'(t_m) - \frac{F(t_m)}{A(t_m)} A'(t_m)}{B'(t_m)} \right. \tag{3.86}$$

$$\left. + \sum_{\substack{t_n \in \mathcal{T}_B \\ t_n \neq t_m}} \frac{F(t_n) - \frac{F(t_m)}{A(t_m)} A(t_n)}{B'(t_n)} \left(\frac{1}{z - t_n} + \frac{1}{t_n - t_m} \right) \right\}.$$

Applying Lemma 3.8.4 gives

$$F(z) = B(z) \left\{ \sum_{\substack{t_n \in \mathcal{T}_B \\ t_n \neq t_m}} \frac{F(t_n)}{B'(t_n)} \left(\frac{1}{z - t_n} + \frac{1}{t_n - t_m} \right) + \frac{1}{z - t_m} \frac{F(t_m)}{B'(t_m)} \right. \tag{3.87}$$

$$\left. + \frac{F'(t_m) - \frac{F(t_m)}{A(t_m)} A'(t_m)}{B'(t_m)} - \frac{F(t_m)}{A(t_m)} \left(\frac{1}{2B'(t_m)^2} \left(A(t_m) B''(t_m) - 2A'(t_m) B'(t_m) \right) \right) \right\}$$

$$= B(z) \left\{ \sum_{\substack{t_n \in \mathcal{T}_B \\ t_n \neq t_m}} \frac{F(t_n)}{B'(t_n)} \left(\frac{1}{z - t_n} + \frac{1}{t_n - t_m} \right) + \frac{F(t_m)}{B'(t_m)(z - t_m)} + C_{F,E,m} \right\}$$

for every $z \in \mathbb{C}$.

As $E(t_n) = 2^{-1} A(t_n)$ for all t_n such that $B(t_n) = 0$, we have

$$\sum_{\substack{t_n \in \mathcal{T}_B \\ t_n \neq t_m}} \frac{F(t_n)}{B'(t_n)} \left(\frac{1}{z - t_n} + \frac{1}{t_n - t_m} \right) = \frac{1}{2} \sum_{\substack{t_n \in \mathcal{T}_B \\ t_n \neq t_m}} \frac{F(t_n)}{E(t_n)} \frac{A(t_n)}{B'(t_n)} \left(\frac{1}{z - t_n} + \frac{1}{t_n - t_m} \right). \tag{3.88}$$

Since F/E is bounded on the real line, it follows from Lemma 3.8.4 that the series in (3.88) converses uniformly on compact sets of $\mathbb{C} \setminus \mathcal{T}_B$, and the result follows. □

In later sections we will make use of the interpolation formula (Theorem 3.8.1) to prove uniqueness of best approximations. The following lemma shows that if the phase does not grow too quickly then $\mathcal{H}^1(E^2)$ is contained in $\mathcal{H}^\infty(E^2)$. The proof of Lemma 3.8.5 is that of [32, Lemma 9].

Lemma 3.8.5 ([32, Lemma 9])**.** *Let E be a Hermite-Biehler function such that $\varphi'(x)$ is bounded. Then $\mathcal{H}^1(E) \subset \mathcal{H}^\infty(E)$.*

Proof. Recall that $\varphi'(x) = \pi K(x,x)|E(x)|^{-2}$ for all real x (see (3.47)). Hence, there exists a $C > 0$ such that $K(x,x)|E(x)|^{-2} \leq C$ for all real x. Since

$$\|K(w,\cdot)\|_{E,\infty}^2 = \sup_{x\in\mathbb{R}} \left|\frac{K(w,x)}{E(x)}\right|^2 \tag{3.89}$$

and $K(w,x)^2 \leq K(w,w)K(x,x)$, it follows that

$$\|K(w,\cdot)\|_{E,\infty}^2 \leq CK(w,w) \leq C^2|E(w)|^2 \tag{3.90}$$

for all real w.

Let $F \in \mathcal{H}^1(E)$. By Cauchy-Schwartz (see (3.36)) we have that

$$|F(w)/E(w)| \leq \|F\|_{E,1}\|K(w,\cdot)\|_{E,\infty}/|E(w)| \leq C\|F\|_{E,1} < \infty \tag{3.91}$$

for all real w, hence $F \in \mathcal{H}^\infty(E)$. □

Remark 3.8.6. It should be mentioned that $\varphi'(x)$ is not always bounded.

The entire function

$$E(z) = \prod_{n\in\mathbb{Z}\setminus\{0\}} \left(1 + \frac{z}{n + i/|n|}\right) \tag{3.92}$$

is Hermite-Biehler. Using (3.46) we have that

$$\varphi'(x) = \sum_{n\in\mathbb{Z}\setminus\{0\}} \frac{1/|n|}{(x-n)^2 + n^{-2}} \tag{3.93}$$

which is not bounded on $\mathbb{R}$.

4. INTERPOLATIONS AT ZEROS OF LAGUERRE-PÓLYA-FUNCTIONS

4.1. Introduction

Let $\mathcal{M}_b(\mathbb{R}^+)$ be the collection of all signed Borel measures on $[0,\infty)$ with distribution function $V(x) := \nu([0,x])$ satisfying the following inequality

$$0 \le V(x) \le C \tag{4.1}$$

for some constant C and all $x \ge 0$, and $\mathcal{M}_b^+(\mathbb{R}^+)$ is its sub-collection of non-negative measures.

For $\nu \in \mathcal{M}_b(\mathbb{R}^+)$ we define $\mathcal{L}\{\nu\}$, the Laplace transform of ν, by

$$\mathcal{L}\{\nu\}(z) = \int_{[0,\infty)} e^{-\lambda z}\, d\nu(\lambda). \tag{4.2}$$

Notice that an integration by parts in (4.2) along with (4.1) gives that $\mathcal{L}\{\nu\}$ is a well-defined function that is analytic in the half-plane $\mathrm{Re}(z) > 0$.

We further define $f_\nu^+ : \mathbb{C} \to \mathbb{C}$, the truncated Laplace transform of ν, by

$$f_\nu^+(z) = \begin{cases} \int_{[0,\infty)} e^{-\lambda z}\, d\nu(\lambda) & \text{if } \mathrm{Re}(z) > 0; \\ 0 & \text{if } \mathrm{Re}(z) \le 0, \end{cases} \tag{4.3}$$

and its odd extension

$$\widetilde{f}_\nu(z) := f_\nu^+(z) - f_\nu^+(-z). \tag{4.4}$$

Since $\mathcal{L}\{\nu\}$ is analytic in the right half-plane, it follows that f_ν^+ and $\widetilde{f}_\nu$ are analytic on both $\mathrm{Re}(z) < 0$ and $\mathrm{Re}(z) > 0$.

In this chapter, we show how to construct entire functions K that interpolate f_ν^+ at the elements of a given discrete subset $\mathcal{T} \subset \mathbb{R}$ so that $K - f_\nu^+$ has no sign changes between consecutive elements of $\mathcal{T}$. These interpolations will be the basis for the construction of extremal functions in Sections 5.4, 6.2, and 6.6. Under relatively mild assumptions, the set $\mathcal{T}$ forms the zero set of a so-called Laguerre-Pólya function (i.e., uniform limits of polynomials with only real zeros). These Laguerre-Pólya functions belong to the Pólya class which has a close connection with functions of bounded type and the theory of de Branges Spaces (see Lemmas 4.2.3 and 4.2.4). A key feature of Laguerre-Pólya functions is that their reciprocals can be represented as inverse Laplace transforms of a totally positive frequency function. This along with the Laplace transform representation of f_ν^+ makes Laguerre-Pólya functions ideal for constructing interpolations of f_ν^+.

In Sections 4.2 and 4.3 we record various properties of Laguerre-Pólya functions and their associated frequency functions. In Sections 4.4, 4.5, 4.6, and 4.7 we construct interpolations of f_ν^+ (and $\widetilde{f_\nu}$) using even and odd Laguerre-Pólya functions. The construction of these interpolations is based on the general method of obtaining interpolations at the zeros of Laguerre-Pólya functions used in [13] and [35]. This approach has also proven effective in [10] and [11].

Below we mention some interesting functions that fall under the umbrella of the measures described above.

Example 4.1.1.

1. For $c \geq 0$, the Dirac measure, δ_c, belongs to $\mathcal{M}_b^+(\mathbb{R}^+)$, and

$$f_{\delta_c}^+(z) = \begin{cases} e^{-cz} & \text{if } \mathrm{Re}(z) > 0; \\ 0 & \text{if } \mathrm{Re}(z) \leq 0. \end{cases} \tag{4.5}$$

   In particular, for $c = 0$, we have that $f_{\delta_0}^+$ is the Heaviside step function and $\widetilde{f_{\delta_0}}$ is the signum function.

2. For $q \geq 0$ and $\alpha > 0$, the measure $d\nu_{q,\alpha}(\lambda) = \lambda^q e^{-\alpha\lambda}\, d\lambda$ belongs to $\mathcal{M}_b^+(\mathbb{R}^+)$, and

$$f_{\nu_{q,\alpha}}^+(z) = \begin{cases} \frac{\Gamma(q+1)}{(z+\alpha)^{q+1}} & \text{if } \mathrm{Re}(z) > 0; \\ 0 & \text{if } \mathrm{Re}(z) \leq 0. \end{cases} \tag{4.6}$$

3. For $a > 0$, the (signed) measure $d\nu_a(\lambda) = \sin(a\lambda)\,d\lambda$ belongs to $\mathcal{M}_b(\mathbb{R}^+)$, and $f^+_{\nu_a}$ is the truncated Poisson kernel

$$f^+_{\nu_a}(z) = \begin{cases} \frac{a}{z^2+a^2} & \text{if } \mathrm{Re}(z) > 0; \\ 0 & \text{if } \mathrm{Re}(z) \leq 0. \end{cases} \tag{4.7}$$

4.2. Pólya and Laguerre-Pólya classes

Definition 4.2.1. The *Pólya class*, denoted $\mathcal{P}$, is defined as the set of all entire functions E that satisfy the following properties:

1. E has no zeros in the upper half-plane.

2. $|E^*(z)| \leq |E(z)|$ for all $z \in \mathbb{C}^+$.

3. For each real x, $y \mapsto |E(x+iy)|$ is a non-decreasing function of $y > 0$.

If $a \geq 0$ and $\mathrm{Im}(b) \leq 0$, then by direct verification we see that the Gaussian, e^{-az^2}, and the exponential, e^{bz}, are of Pólya class. Also, if c is a constant, then the polynomial $z - c$ is of Pólya class if and only if c has non-positive imaginary part. The definition implies that the product of any two functions of Pólya class is again of Pólya class, hence any polynomial having only zeros in the closed lower half-plane belongs to $\mathcal{P}$. Moreover, if $\{P_n(z)\}$ is a sequence of polynomials of Pólya class such that

$$\lim_{n\to\infty} P_n(z) = E(z) \tag{4.8}$$

uniformly on bounded sets, then the entire function E is also of Pólya class.

Functions belonging to Pólya class can be characterized by their Hadamard factorization.

Theorem 4.2.2 ([23, Theorem 7])**.** *Let $E \in \mathcal{P}$. If $\{z_n\}$ is the sequence of non-zero zeros of E (counted with multiplicity), then*

$$\sum_n \frac{1 - y_n}{x_n^2 + y_n^2} < \infty. \tag{4.9}$$

Moreover, there exist $a \geq 0$ and b with $\mathrm{Im}(b) \leq 0$ such that

$$E(z) = E^{(r)}(0)\frac{z^r}{r!}e^{-az^2}e^{bz}\prod_n\left(1 - \frac{z}{z_n}\right)e^{z\mathrm{Re}\frac{1}{z_n}}. \tag{4.10}$$

The following lemmas describe the connection between the Pólya class and functions of bounded type in the upper half-plane as well as the Nevanlinna Class (analytic functions with non-negative real parts in the upper half-plane). The proofs (which we omit) of Lemmas 4.2.3 and 4.2.4 are somewhat technical and rely on a variant of Phragmén-Lindelöf Principle [23, Theorem 1]. A generalization of Lemma 4.2.3 can be found in [38, Theorem 1.3], and Lemma 4.2.4 is Theorem 15 of [23].

Lemma 4.2.3 ([38, Theorem 1.3] and [23, Problem 34]). *Let E be an entire function which has no zeros in the upper half-plane and which satisfies the inequality $|E^*(z)| \leq |E(z)|$ for all $z \in \mathbb{C}^+$. Then E is of Pólya class if there exists an entire function F of Pólya class such that $E/F \in \mathcal{N}(\mathbb{C}^+)$.*

Lemma 4.2.4 ([23, Theorem 15]). *Let $E \in \mathcal{HB}$ with $E(0) = 1$. Let $\log E(z)$ be defined in such a way that it is analytic in the upper half-plane with limit zero at the origin. Then $E \in \mathcal{P}$ if and only if*

$$\operatorname{Re}\left\{ i\frac{\log E(z)}{z} \right\} \geq 0 \tag{4.11}$$

for all $z \in \mathbb{C}^+$.

In this work we are primarily interested in Hermite-Biehler functions that are also of bounded type. Applying Lemma 4.2.3 with $F \equiv 1$ gives the following extremely useful corollary.

Corollary 4.2.5. *If E is a Hermite-Biehler function of bounded type in the upper half-plane, then $E \in \mathcal{P}$.*

We define the Laguerre-Pólya class as the subclass of functions $F \in \mathcal{P}$ that are real-valued on the real line and only have real zeros. By (4.10) we have the following:

Definition 4.2.6. The *Laguerre-Pólya* class, denoted $\mathcal{LP}$, consists of all entire functions of the form

$$F(z) = Cz^r e^{-az^2} e^{bz} \prod_n \left(1 - \frac{z}{x_n}\right) e^{z/x_n} \tag{4.12}$$

where $a \geq 0$, $r \in \mathbb{N}_0$, C, b, x_n $(n \in \mathbb{N}_0)$ are real, and

$$\sum_n \frac{1}{x_n^2} < \infty. \tag{4.13}$$

It is a classical result of Laguerre (see [44] or [34, Chapter 3]) that all functions in $\mathcal{LP}$ are the uniform limit on compact sets of $\mathbb{C}$ of polynomials with only real zeros.

Example 4.2.7. From the Hadamard factorization (4.12), we can easily read off many interesting examples of functions belonging to the Laguerre-Pólya class. For example, we have that

$$\sin \pi z = \pi z \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{n^2}\right) = \pi z \prod_{\substack{n \in \mathbb{Z} \\ n \neq 0}} \left(1 - \frac{z}{n}\right) e^{z/n} \tag{4.14}$$

and

$$\cos \pi z = \prod_{n=1}^{\infty} \left(1 - \frac{z^2}{(n-1/2)^2}\right) = \prod_{n \in \mathbb{Z}} \left(1 - \frac{z}{n-1/2}\right) e^{z/(n-1/2)} \tag{4.15}$$

are $\mathcal{LP}$-functions. Also, the Gamma function, $\Gamma(z)$, satisfies

$$\frac{1}{\Gamma(z)} = z e^{\gamma z} \prod_{n=1}^{\infty} \left(1 + \frac{z}{n}\right) e^{-z/n} \tag{4.16}$$

where γ is Euler's constant and its reciprocal, $1/\Gamma$, belongs to $\mathcal{LP}$.

Lemma 4.2.8. *Assume that $E = A - iB$ is a Hermite-Biehler function of bounded type in the upper half-plane. Then the functions A and B are in the Laguerre-Pólya class.*

Proof. Since E^*/E is bounded on $\mathbb{C}^+$, it follows that A/E is bounded on $\mathbb{C}^+$ and hence of bounded type. Since $A = A^*$, Lemma 4.2.3 (applied to A) implies that that A is in the Pólya class. The same argument shows that B is in the Pólya class. Since A and B are real entire and only have real zeros, it follows that they are in the Laguerre-Pólya Class. $\square$

For a Laguerre-Pólya function F, we say that F has finite degree $\mathcal{N} = \mathcal{N}(F)$ if $a = 0$ in (4.12) and F has exactly $\mathcal{N}$ zeros counted with multiplicity. Otherwise we set $\mathcal{N}(F) = \infty$. We denote by $\mathcal{T}_F$ the set of zeros of F. From the Hadamard Factorization (4.12) we easily deduce the following lemma.

Lemma 4.2.9 ([34, Theorem 3.5.3])**.** *If F is a Laguerre-Pólya function with $\mathcal{N}(F) \geq k$, then for any $R > 0$*

$$\frac{1}{|F(x+iy)|} = \mathcal{O}(|y|^{-k}) \tag{4.17}$$

as $|y| \to \infty$, *uniformly in* $|x| \le R$.

If F is a Laguerre-Pólya function with $\mathcal{N}(F) \ge 2$ (which will include the case that $\mathcal{N}(F) = \infty$) and $c \in \mathbb{R} \setminus \mathcal{T}_F$, we define the frequency function[1] g_c by

$$g_c(t) = \frac{1}{2\pi i} \int_{c-i\infty}^{c+i\infty} \frac{e^{tz}}{F(z)}\, dz. \tag{4.18}$$

Notice that by Lemma 4.2.9 this integral converges absolutely. Let τ_1 and τ_2 be two consecutive elements from $\mathcal{T}_F$ and $c \in (\tau_1, \tau_2)$. A Fourier inversion (see also [34, Corollary 5.4]) shows that

$$\frac{1}{F(z)} = \int_{-\infty}^{\infty} e^{-zt} g_c(t)\, dt \tag{4.19}$$

in the strip $\tau_1 < \mathrm{Re}(z) < \tau_2$. Moreover, an application of the residue theorem shows that $g_c = g_d$ for $c, d \in (\tau_1, \tau_2)$.

If $\mathcal{N}(F) = 1$ we can still define g_c as a Cauchy principal value to give (4.19), and if $\mathcal{N}(F) = 0$ we have that (4.19) holds with $g_c(t)\, dt$ being a suitable Dirac measure, though for this investigation we are not interested in these cases.

Example 4.2.10. By (4.14), the entire functions $\sin \pi z$ and $\frac{\sin \pi z}{z}$ are Laguerre-Pólya functions. With $g(t) := \frac{\pi^{-1}}{e^{-t}+1}$, we have

$$\frac{1}{\sin \pi z} = \int_{-\infty}^{\infty} e^{-zt} \frac{\pi^{-1}}{e^{-t}+1}\, dt = \int_{-\infty}^{\infty} e^{-zt} g(t)\, dt \tag{4.20}$$

for $0 < \mathrm{Re}(z) < 1$ and

$$\frac{z}{\sin \pi z} = \int_{-\infty}^{\infty} e^{-zt} \frac{\pi^{-1} e^{-t}}{(e^{-t}+1)^2}\, dt = \int_{-\infty}^{\infty} e^{-zt} g'(t) dt \tag{4.21}$$

for $-1 < \mathrm{Re}(z) < 1$. Moreover, $\sin^2 \pi z$ and $(\frac{\sin \pi z}{z})^2$ are also Laguerre-Pólya functions. Using the multiplication formula for the Laplace transform it follows that

$$\frac{1}{\sin^2 \pi z} = \int_{-\infty}^{\infty} e^{-zt} \frac{\pi^{-2} t e^t}{e^t - 1}\, dt = \int_{-\infty}^{\infty} e^{-zt} g * g(t)\, dt \tag{4.22}$$

[1] The function g_0 is a frequency function in the language of probability provided $F(0) = 1$.

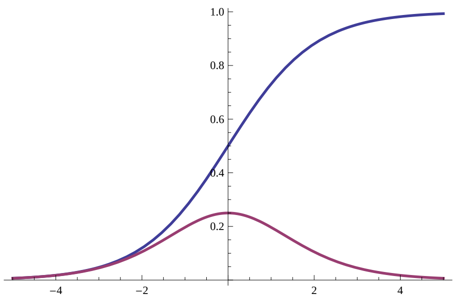

Figure 4.1: Graph of g and its derivative

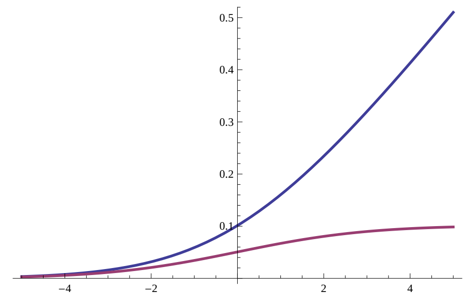

Figure 4.2: Graph of $g * g$ and its derivative

for $0 < \mathrm{Re}(z) < 1$ and

$$\frac{z^2}{\sin^2 \pi z} = \int_{-\infty}^{\infty} e^{-zt} \frac{\pi^{-2} e^t (e^t(t-2)+t+2)}{(e^t-1)^3}\, dt = \int_{-\infty}^{\infty} e^{-zt} g' * g'(t) dt \tag{4.23}$$

for $-1 < \mathrm{Re}(z) < 1$, where $*$ denotes convolution, i.e., $f * g(t) = \int_{-\infty}^{\infty} f(t-x)g(x)\, dx$.

The functions $g(t) = \frac{\pi^{-1}}{e^{-t}+1}$ (which is associated with an odd $\mathcal{LP}$-function with a simple zero at the origin) and $g * g(t) = \frac{\pi^{-2} t e^t}{e^t - 1}$ (which is associated with an even $\mathcal{LP}$-function with a double zero at the origin) will be useful to keep in mind while reading the following sections. The graphs of g and $g * g$ (as well as their derivatives) are included in Figures 4.1 and 4.2, respectively.

Remark 4.2.11. Let $F \in \mathcal{LP}$ have a zero of order k at the origin and at least one (smallest) positive zero τ_+ and one (largest) negative zero τ_- and define $g_{\tau_+/2}$ by (4.18). Integrating by parts k times in (4.19) gives

$$\frac{1}{z^{-k}F(z)} = \int_{-\infty}^{\infty} e^{-zt} g^{(k)}_{\tau_+/2}(t)\, dt \tag{4.24}$$

for all z with $0 < \mathrm{Re}(z) < \tau_+$. Since $z^{-k}F(z)$ is non-zero at the origin, convergence extends to $\tau_- < \mathrm{Re}(z) < \tau_+$. Moreover, the function $H(z) = z^{-k}F(z)$ is in $\mathcal{LP}$ and if we define h_0 by (4.18) (applied to H), it follows that $h_0 = g^{(k)}_{\tau_+/2}$.

4.3. Properties of the frequency function

As part of a series of papers on total positivity, I.J. Schoenberg [58] gave an intrinsic characterization of the functions g that may occur as Laplace inverse transformations of $\mathcal{LP}$ functions. In Lemmas 4.3.1 and 4.3.2, we record some of the key properties of these functions. For more information and complete proofs of these facts, see the excellent account of this theory in [34, Chapters II-V].

Lemma 4.3.1 ([34, Chapter IV, Theorem 5.1 and Theorem 5.3])**.** *Let $F \in \mathcal{LP}$ have degree $\mathcal{N} \geq 2$ and let g_c be defined by* (4.18) *where $c \in \mathbb{R} \setminus \mathcal{T}_F$. The following propositions hold:*

1. *The function $g_c \in C^{\mathcal{N}-2}(\mathbb{R})$ and is real valued.*

2. *The function g_c is of one sign, and its sign equals the sign of $F(c)$.*

3. *If $c = 0$, the function $g_0^{(k)}$ has exactly k sign changes on the real line for $k = 0, 1, ..., \mathcal{N} - 2$.*

Proof. Let $k \leq \mathcal{N} - 2$. By Lemma 4.2.9 we have that (4.18) may be differentiated k times under the integral sign. Since the resulting integral converges uniformly, $g_c^{(k)}$ exists and is continuous. Since F is real entire, it follows that g_c is real valued which shows (1).

Items (2) and (3) are [34, Chapter IV, Theorem 5.1 and Theorem 5.3]. □

Lastly, we require knowledge of the rate of decay of these functions (and their derivatives) on the real line (see e.g., [34, Chapter IV, Theorem 5.1]).

Lemma 4.3.2 ([34, Chapter IV, Theorem 5.1])**.** *Let $F \in \mathcal{LP}$ have degree $\mathcal{N}$. If τ_1 and τ_2 are two consecutive elements in $\mathcal{T}_F$ and $c \in (\tau_1, \tau_2)$, then for $0 \leq n \leq \mathcal{N} - 2$ there exists polynomials P_n and Q_n such that*

$$\begin{aligned} |g_c^{(n)}(t)| &\leq |P_n(t)| e^{\tau_1 t} \quad \text{as} \quad t \to \infty \\ |g_c^{(n)}(t)| &\leq |Q_n(t)| e^{\tau_2 t} \quad \text{as} \quad t \to -\infty. \end{aligned} \tag{4.25}$$

For even and odd Laguerre-Pólya functions, we describe the parity of the derivatives of these frequency functions and evaluate their Fourier sine and cosine transforms.

Lemma 4.3.3. *Let F be an even $\mathcal{LP}$-function with a double zero at the origin and at least one (smallest) positive zero τ_+. Assume that F is positive in $(0, \tau_+)$, and define $g = g_{\tau_+/2}$ by* (4.18)*.*

1. *The derivatives g' and g'' exist and are nonnegative on the real line.*

2. *The function g'' is even.*

3. *If $-\tau_+ < \mathrm{Im}(\xi) < \tau_+$, then*

$$\int_{-\infty}^{\infty} g''(\lambda) \cos(\xi \lambda) \, d\lambda = -\frac{\xi^2}{F(i\xi)}. \tag{4.26}$$

In particular, F is real-valued on the imaginary axis.

4. If $-\tau_+ < \mathrm{Im}(\xi) < \tau_+$ and $w \in \mathbb{R}$, then

$$\int_{-\infty}^{\infty} g''(w-\lambda)\sin(\xi\lambda)\,d\lambda = -\frac{\xi^2\sin(\xi w)}{F(i\xi)}. \tag{4.27}$$

5. If $\xi > 0$, then

$$F(i\xi) < 0. \tag{4.28}$$

Proof. Since F has at least four zeros counted with multiplicity, Lemma 4.3.1 with $c = \tau_+/2$ implies that g' and g'' exist and are real valued. Two integration by parts in (4.18) show that

$$\frac{z^j}{F(z)} = \int_{-\infty}^{\infty} e^{-zt} g^{(j)}(t)dt \qquad (j \in \{0,1,2\}) \tag{4.29}$$

for all z with $0 < \mathrm{Re}(z) < \tau_+$. Since $z^{-j}F(z)$ is in $\mathcal{LP}$ for $j \in \{0,1,2\}$, Lemma 4.3.1 with $k = 0$ implies that $g^{(j)}$ has no sign changes on the real line. Evaluation of (4.29) at $z = \tau_+/2$ shows that these derivatives are nonnegative on the real line (by Lemma 4.3.1 part (2)). Since $z^{-2}F(z)$ is non-zero at the origin, (4.29) extends to $-\tau_+ < \mathrm{Re}(z) < \tau_+$ for $j = 2$. Since $z^{-2}F(z)$ is an even function of z, (4.18) with $c = 0$ gives

$$g''(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \frac{(iu)^2 e^{iut}}{F(iu)} du \tag{4.30}$$

which implies that g'' is even.

Let ξ such that $-\tau_+ < \mathrm{Im}(\xi) < \tau_+$. Since (4.29) extends to $-\tau_+ < \mathrm{Re}(z) < \tau_+$, we have

$$\int_{-\infty}^{\infty} g''(\lambda)\cos(\xi\lambda)\,d\lambda = \frac{1}{2}\left(\frac{(i\xi)^2}{F(i\xi)} + \frac{(-i\xi)^2}{F(-i\xi)}\right), \tag{4.31}$$

and (4.26) follows since F is even. Since $\lambda \mapsto g''(\lambda)\sin\xi\lambda$ is odd, the trigonometric identity $\sin\xi(\lambda - w) = \cos\xi w \sin\xi\lambda - \cos\xi\lambda\sin\xi w$ implies

$$\int_{-\infty}^{\infty} g''(\lambda)\sin\xi(\lambda - w)d\lambda = -\sin\xi w\int_{-\infty}^{\infty} g''(\lambda)\cos\xi\lambda d\lambda. \tag{4.32}$$

An application of (4.26) gives (4.27).

To show (4.28), let $0 < x_1 < x_2 < \ldots$ be the positive zeros of F. Since F is even and has a double zero at the origin, it follows that

$$F(z) = Cz^2 e^{-az^2} \prod_n \left(1 - \frac{z^2}{x_n^2}\right) \tag{4.33}$$

for some C and a with $a > 0$. Since F is positive in $(0, x_1)$, we have that

$$F\left(\frac{x_1}{2}\right) = C\left(\frac{x_1}{2}\right)^2 e^{-a\left(\frac{x_1}{2}\right)^2} \prod_n \left(1 - \frac{\left(\frac{x_1}{2}\right)^2}{x_n^2}\right) > 0 \tag{4.34}$$

which gives that $C > 0$. It follows that

$$F(i\xi) = C(i\xi)^2 e^{-a(i\xi)^2} \prod_n \left(1 - \frac{(i\xi)^2}{x_n^2}\right) = -C\xi^2 e^{a\xi^2} \prod_n \left(1 + \frac{\xi^2}{x_n^2}\right) < 0. \tag{4.35}$$

□

Analogously, for odd $\mathcal{LP}$-functions we have the following.

Lemma 4.3.4. *Let F be an odd $\mathcal{LP}$-function with a simple zero at the origin and at least one (smallest) positive zero τ_+. Assume that F is positive in $(0, \tau_+)$, and define $g = g_{\tau_+/2}$ by (4.18).*

1. *The derivative g' exists, is non-negative on the real line and even.*

2. *If $-\tau_+ < \mathrm{Im}(\xi) < \tau_+$, then*

$$\int_{-\infty}^{\infty} g'(\lambda) \cos(\xi\lambda)\, d\lambda = \frac{i\xi}{F(i\xi)}. \tag{4.36}$$

   In particular, F is imaginary on the imaginary axis.

3. *If $-\tau_+ < \mathrm{Im}(\xi) < \tau_+$ and $w \in \mathbb{R}$, then*

$$\int_{-\infty}^{\infty} g'(w - \lambda) \sin(\xi\lambda)\, d\lambda = \sin(\xi w) \frac{i\xi}{F(i\xi)}. \tag{4.37}$$

4. *If $\xi > 0$, then*

$$iF(i\xi) < 0. \tag{4.38}$$

Proof. The proof is nearly identical to Lemma 4.3.3. Since the specific integral values will be important in the construction of interpolations, we include the proof for completeness.

Since F has at least three zeros counted with multiplicity, Lemma 4.3.1 with $c = \tau_+/2$ implies that g' exist and is real valued. One integration by parts in (4.18) shows that

$$\frac{z^j}{F(z)} = \int_{-\infty}^{\infty} e^{-zt} g^{(j)}(t)dt \qquad (j \in \{0, 1\}) \tag{4.39}$$

for all z with $0 < \mathrm{Re}(z) < \tau_+$. Since $z^{-j}F(z)$ is in $\mathcal{LP}$ for $j \in \{0,1\}$, Lemma 4.3.1 with $k = 0$ implies that $g^{(j)}$ has no sign changes on the real line. Evaluation of (4.39) at $z = \tau_+/2$ shows that g' is non-negative on the real line. Since $z^{-1}F(z)$ is non-zero at the origin, (4.39) extends to $-\tau_+ < \mathrm{Re}(z) < \tau_+$ for $j = 1$. Since $z^{-1}F(z)$ is an even function of z, (4.18) with $c = 0$ gives

$$g'(t) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{iue^{iut}}{F(iu)} du \tag{4.40}$$

which implies that g' is even.

Let ξ such that $-\tau_+ < \mathrm{Im}(\xi) < \tau_+$. Since (4.39) extends to $-\tau_+ < \mathrm{Re}(z) < \tau_+$, we have

$$\int_{-\infty}^{\infty} g'(\lambda)\cos(\xi\lambda)\, d\lambda = \frac{1}{2}\left(\frac{i\xi}{F(i\xi)} + \frac{-i\xi}{F(-i\xi)}\right), \tag{4.41}$$

and (4.36) follows since F is odd. Since $\lambda \mapsto g'(\lambda)\sin\xi\lambda$ is odd, the trigonometric identity $\sin\xi(\lambda - w) = \cos\xi w \sin\xi\lambda - \cos\xi\lambda\sin\xi w$ implies

$$\int_{-\infty}^{\infty} g'(\lambda)\sin\xi(\lambda - w)d\lambda = -\sin\xi w \int_{-\infty}^{\infty} g'(\lambda)\cos\xi\lambda d\lambda. \tag{4.42}$$

An application of (4.36) gives (4.37).

To show (4.38), let $0 < x_1 < x_2 < ...$ be the positive zeros of F. Since F is odd and has a simple zero at the origin, it follows that

$$F(z) = Cze^{-az^2}\prod_n \left(1 - \frac{z^2}{x_n^2}\right) \tag{4.43}$$

for some C and a with $a > 0$. Since F is positive in $(0, x_1)$, we have that

$$F\left(\frac{x_1}{2}\right) = Cx_1/2e^{-a\left(\frac{x_1}{2}\right)^2} \prod_n \left(1 - \frac{\left(\frac{x_1}{2}\right)^2}{x_n^2}\right) > 0 \tag{4.44}$$

which gives that $C > 0$. It follows that

$$iF(i\xi) = -C\xi e^{-a(i\xi)^2} \prod_n \left(1 - \frac{(i\xi)^2}{x_n^2}\right) = -C\xi e^{a\xi^2} \prod_n \left(1 + \frac{\xi^2}{x_n^2}\right) < 0. \tag{4.45}$$

□

4.4. Interpolations in $\mathcal{LP}$

Recall that if $\nu \in \mathcal{M}_b(\mathbb{R}^+)$, we defined the truncated Laplace transform $f_\nu^+ : \mathbb{C} \to \mathbb{C}$ by

$$f_\nu^+(z) := \begin{cases} \int_{[0,\infty)} e^{-\lambda z}\, d\nu(\lambda) & \text{if } \mathrm{Re}(z) > 0; \\ 0 & \text{if } \mathrm{Re}(z) \leq 0. \end{cases} \tag{4.46}$$

In this section, for a given Laguerre-Pólya function F we utilize the fact that $1/F$ and f_ν^+ are represented as inverse Laplace transforms to construct an entire function that interpolates f_ν^+ at the zeros of F.

Let F be a Laguerre-Pólya function, and denote by τ_+ the smallest positive zero of F (if no such zero exists, we set $\tau_+ = \infty$). Let $g = g_{\tau_+/2}$ by (4.18). Define the function $g * d\nu : \mathbb{R} \to \mathbb{R}$ by

$$g * d\nu(w) = \int_0^\infty g(w - \lambda)\, d\nu(\lambda). \tag{4.47}$$

If F has at least two zeros and $\tau_+ < \infty$, Lemma 4.3.2 with $\tau_2 = \tau_+$ implies that $g(t)$ and $g'(t)$ decay exponentially as $t \to -\infty$. Hence $g * d\nu(w)$ is finite for every $w \in \mathbb{R}$, and an integration by parts gives for all real w

$$g * d\nu(w) = \int_0^\infty g(w - \lambda)\, d\nu(\lambda) = \int_0^\infty g'(w - \lambda) V(\lambda)\, d\lambda = g' * V(w). \tag{4.48}$$

We will require bounds for $g * d\nu$ and its derivatives.

Lemma 4.4.1. *Let F be an even $\mathcal{LP}$-functions with a double zero at the origin and at least one (smallest) positive zero τ_+, and at least five zeros (counted with multiplicity). Assume that F is positive in $(0, \tau_+)$, and define $g = g_{\tau_+/2}$ by (4.18). The inequalities*

$$0 \leq g^{(n)} * d\nu(w) \leq Cg^{(n)}(w) \tag{4.49}$$

hold for $n \in \{0, 1\}$ and all real w, and for $n = 2$ and $w \leq 0$.

Proof. From (4.48) we see that

$$g * d\nu(w) = \int_0^\infty g(w - \lambda)\, d\nu(\lambda) = \int_0^\infty g'(w - \lambda)V(\lambda)\, d\lambda. \tag{4.50}$$

The functions g, g', and g'' are non-negative on $\mathbb{R}$ by Lemma 4.3.3, and g''' has exactly one sign change on $\mathbb{R}$ by Lemma 4.3.1 applied to g''. Since g'' is even, the sign change is located at the origin. As $0 \leq V(x) \leq C$ on $[0, \infty)$, it follows that for all real w and $n \in \{0, 1\}$, as well as for $n = 2$ and $w \leq 0$,

$$0 \leq g^{(n)} * d\nu(w) = \int_0^\infty g^{(n+1)}(w - \lambda)V(\lambda)\, d\lambda \leq C \int_0^\infty g^{(n+1)}(w - \lambda)\, d\lambda = Cg^{(n)}(w) \tag{4.51}$$

which gives the desired inequality. □

Analogously, for odd Laguerre-Pólya functions, we have the following.

Lemma 4.4.2. *Let F be an odd $\mathcal{LP}$-function with a simple zero at the origin and at least one (smallest) positive zero τ_+, and at least five zeros (counted with multiplicity). Assume that F is positive in $(0, \tau_+)$, and define $g = g_{\tau_+/2}$ by (4.18). The inequality*

$$0 \leq g * d\nu(w) \leq Cg(w) \tag{4.52}$$

holds all real w, and

$$0 \leq g' * d\nu(w) \leq Cg'(w) \tag{4.53}$$

holds for all $w \leq 0$.

The next lemma investigates two interpolations of f_ν^+ in $\mathrm{Re}(z) \geq 0$ and $\mathrm{Re}(z) \leq \tau_+$, respectively, and shows that they are representations of a single entire function $z \mapsto A(F, \nu, z)$ which interpolates f_ν^+ at the zeros of F. See [13] for a similar construction for the cutoff of an exponential function.

Lemma 4.4.3. *Let $F \in \mathcal{LP}$ satisfy the assumptions of Lemma 4.4.1 (or Lemma 4.4.2). Assume that F is positive in $(0, \tau_+)$, and define $g = g_{\tau_+/2}$ by (4.18). Define*

$$\begin{aligned} A_1(F,\nu,z) &= F(z)\int_{-\infty}^{0} g * d\nu(w) e^{-zw}\,dw \quad \text{for} \quad \mathrm{Re}(z) < \tau_+ \\ A_2(F,\nu,z) &= f_\nu^+(z) - F(z)\int_{0}^{\infty} g * d\nu(w) e^{-zw}\,dw \quad \text{for} \quad \mathrm{Re}(z) > 0. \end{aligned} \tag{4.54}$$

Then $z \mapsto A_1(F,\nu,z)$ is analytic in $\mathrm{Re}(z) < \tau_+$, $z \mapsto A_2(F,\nu,z)$ is analytic in $\mathrm{Re}(z) > 0$, and these functions are restrictions of an entire function $z \mapsto A(F,\nu,z)$. Moreover, there exists a constant $c > 0$ so that

$$|A(F,a,z)| \leq c(1 + |F(z)|) \tag{4.55}$$

for all $z \in \mathbb{C}$, and

$$A(F,\nu,\xi) = f_\nu^+(\xi) \tag{4.56}$$

for all $\xi \in \mathbb{R}$ with $F(\xi) = 0$.

Proof. Inequality (4.25) with $\tau_2 = \tau_+$ implies that $g'(t)$ decays exponentially as $t \to -\infty$. It follows from Lemma 4.4.1 (or Lemma 4.4.2) that

$$0 \leq g * d\nu(w) \leq C g(w) \tag{4.57}$$

for all real w, and (4.25) with $\tau_2 = \tau_+$ applied to g for $t \to -\infty$ implies that the integral defining $A_1(F,\nu,z)$ converges absolutely in $\mathrm{Re}(z) < \tau_+$. Inequality (4.25) implies with $\tau_1 = 0$ that g has polynomial growth on the positive real axis, hence the integral in the definition of $A_2(F,a,z)$ converges absolutely for $\mathrm{Re}(z) > 0$. It follows that A_1 and A_2 are analytic functions in $\mathrm{Re}(z) < \tau_+$ and $0 < \mathrm{Re}(z)$, respectively.

To prove that A_1 and A_2 are analytic continuations of each other it suffices to prove that they are equal in the strip $0 < \mathrm{Re}(z) < \tau_+$. Starting with the identity

$$f_\nu^+(z) = \int_0^\infty e^{-z\lambda}\, d\nu(\lambda), \tag{4.58}$$

which is valid for $\mathrm{Re}(z) > 0$, and combining this with (4.19) gives for $0 < \mathrm{Re}(z) < \tau_+$

$$\begin{aligned} f_\nu^+(z) &= F(z) \int_{-\infty}^\infty \int_0^\infty e^{-z(w+\lambda)} g(w)\, d\nu(\lambda)\, dw \\ &= F(z) \int_{-\infty}^\infty \int_0^\infty e^{-zw} g(w-\lambda) d\nu(\lambda)\, dw \\ &= F(z) \int_{-\infty}^\infty g * d\nu(w) e^{-zw}\, dw. \end{aligned} \tag{4.59}$$

Inserting this in (4.54) shows that $A_2(F,\nu,z) = A_2(F,\nu,z)$ for $0 < \mathrm{Re}(z) < \tau_+$. To prove (4.55) we note that inequality (4.57) implies in $\mathrm{Re}(z) \le \tau_+/2$

$$|A(F,\nu,z)| \le |F(z)| \int_{-\infty}^0 |g * d\nu(w) e^{-zw}|\, dw \le C|F(z)| \int_{-\infty}^0 g(w) e^{-\tau_+ w/2}\, dw, \tag{4.60}$$

and an analogous calculation holds in $\mathrm{Re}(z) \ge \tau_+/2$. Identity (4.56) follows from the definition of A. □

Starting with the function $A(F,\nu,z)$, we construct interpolations K of f_ν^+ that interpolate f_ν^+ at the zeros of F so that $K - f_\nu^+$ has no sign changes between two consecutive zeros of F. This is accomplished by selecting the value at the origin appropriately. In the following sections we construct these interpolations of f_ν^+ (as well as $\widetilde{f}_\nu$) by using odd and even Laguerre-Pólya functions.

4.5. Interpolations for f_ν^+ – odd $\mathcal{LP}$ functions

In this section, assume that $\nu \in \mathcal{M}_b^+(\mathbb{R}^+)$. Let F be an odd $\mathcal{LP}$ function that satisfies the assumptions of Lemma 4.4.2. We define $z \mapsto K(F,\nu,z)$ by the construction of Lemma 4.4.3:

$$K(F,\nu,z) = A(F,\nu,z) + g * d\nu(0)\frac{F(z)}{z}. \tag{4.61}$$

By construction K is an entire function such that $K(F,\nu,\xi) = f_\nu^+(\xi)$ for all real $\xi \in \mathcal{T}_F \setminus \{0\}$.

Theorem 4.5.1. *Let F be an odd $\mathcal{LP}$-function with a simple zero at the origin and at least one (smallest) positive zero τ_+. Assume that F is positive in $(0, \tau_+)$, and define $g = g_{\tau_+/2}$ by (4.18). Then*

$$F(x)\left\{f_\nu^+(x) - K(F,\nu,x)\right\} \geq 0 \tag{4.62}$$

for all $x \in \mathbb{R}$, and

$$|f_\nu^+(x) - K(F,\nu,x)| = \mathcal{O}\left(\frac{F(x)}{1+x^2}\right) \tag{4.63}$$

for all real x.

Proof. An expansion of the second term in (4.61) in a Laplace transform together with (4.54) gives

$$f_\nu^+(x) - K(F,\nu,x) = -F(x)\int_{-\infty}^0 (g * d\nu(w) - g * d\nu(0))e^{-xw}\,dw \tag{4.64}$$

for $x < 0$ and

$$f_\nu^+(x) - K(F,\nu,x) = F(x)\int_0^\infty (g * d\nu(w) - g * d\nu(0))e^{-xw}\,dw. \tag{4.65}$$

for $x > 0$. By Lemma 4.4.2 we have $0 \leq g * d\nu(w) \leq Cg(w)$ for all real w. Inequality (4.25) with $\tau_2 = \tau_+$ and $\tau_1 = 0$ applied to g implies that g has exponential decay on the negative real axis and polynomial growth on the positive real axis, hence the integrals in (4.64) and (4.65) converge.

By assumption F is an odd $\mathcal{LP}$-function that is positive in $(0, \tau_+)$, and Lemma 4.3.4 gives that g' is non-negative on the real line. Since ν is non-negative, it follows that

$$g * d\nu(w) - g * d\nu(0) = \int_0^\infty (g(w-\lambda) - g(-\lambda))\,d\nu(\lambda) \leq 0 \tag{4.66}$$

for all $w \leq 0$ and $g * d\nu(w) - g * d\nu(0) \geq 0$ for all $w \geq 0$. Multiplying through by F in (4.64) and (4.65) gives (4.62).

Since $z^{-1}F(z)$ is an even Laguerre-Pólya function that is non-zero at the origin, Lemma 4.3.2 with $-\tau_1 = \tau_2 = \tau_+$ implies that $g''(t)$ and $g^{(3)}(t)$ decay exponentially as $|t| \to \infty$. Hence $\int_{-\infty}^\infty |g^{(3)}(t)|\,dt < \infty$, and (4.1) gives

$$\left|g'' * d\nu(w)\right| = \left|\int_0^\infty g^{(3)}(w-\lambda)V(\lambda)\,d\lambda\right| \leq C\int_0^\infty \left|g^{(3)}(w-\lambda)\right|\,d\lambda < \infty \tag{4.67}$$

for all real w. Two integration by parts in equations (4.64) and (4.65) then give (4.63).

$\square$

From these results we can easily construct interpolations for the odd extension of f_ν^+,

$$\widetilde{f}_\nu(z) := f_\nu^+(z) - f_\nu^+(-z). \tag{4.68}$$

If F is a Laguerre-Pólya function that satisfies the assumptions of Theorem 4.5.1 define

$$\widetilde{K}(F,\nu,z) = K(F,\nu,z) - K(F,\nu,-z). \tag{4.69}$$

Using Theorem 4.5.1 and the fact that F is odd we have

$$\begin{aligned} F(x)\left\{\widetilde{f}_\nu(x) - \widetilde{K}(F,\nu,x)\right\} &= F(x)\left\{f_\nu^+(x) - K(F,\nu,x) - (f_\nu^+(-x) - K(F,\nu,-x))\right\} \qquad (4.70)\\ &= F(x)\left\{f_\nu^+(x) - K(F,\nu,x)\right\} + F(-x)\left\{f_\nu^+(-x) - K(F,\nu,-x)\right\}\\ &\geq 0 \end{aligned}$$

holds for all real x.

Theorem 4.5.2. *Let F be an odd $\mathcal{LP}$-function with a simple zero at the origin and at least one (smallest) positive zero τ_+. Assume that F is positive in $(0,\tau_+)$, and define $g = g_{\tau_+/2}$ by (4.18). Then*

$$F(x)\left\{\widetilde{f}_\nu(x) - \widetilde{K}(F,\nu,x)\right\} \geq 0 \tag{4.71}$$

for all $x \in \mathbb{R}$, and

$$\left|\widetilde{K}(F,\nu,x) - \widetilde{f}_\nu(x)\right| = \mathcal{O}\left(\frac{F(x)}{1+x^2}\right) \tag{4.72}$$

for all real x.

4.6. Interpolations for the truncated Poisson kernel – odd $\mathcal{LP}$ functions

The goal of this section is to study the interpolation problem for f_ν^+ for the signed measure $d\nu_a(\lambda) = \sin(a\lambda)\,d\lambda$. Notice that for $a > 0$ we have

$$V_a(\lambda) = \nu_a([0,\lambda)) = \frac{1-\cos(a\lambda)}{a} \leq \frac{2}{a}, \tag{4.73}$$

hence ν_a belongs to $\mathcal{M}_b(\mathbb{R}^+)$. Moreover, we have that that

$$\int_0^\infty e^{-\lambda z}\, d\nu_a(\lambda) = \int_0^\infty e^{-\lambda z}\, \sin(a\lambda)\, d\lambda = \frac{a}{a^2+z^2} \tag{4.74}$$

for $\mathrm{Re}(z) > 0$. Hence $f^+_{\nu_a}$ is the truncated Poisson kernel

$$f^+_{\nu_a}(z) = \begin{cases} \frac{a}{a^2+z^2} & \text{if } \mathrm{Re}(z) > 0; \\ 0 & \text{if } \mathrm{Re}(z) \le 0. \end{cases} \tag{4.75}$$

Since ν_a belongs $\mathcal{M}_b(\mathbb{R}^+)$, we can apply many of the results from the previous sections to construct the entire function $K(F, \nu_a, z)$, but the fact that $d\nu_a(\lambda) = \sin(a\lambda)\, d\lambda$ is a signed measure introduces additional difficulties when showing the sign changes of $f^+_{\nu_a} - K$.

To show this, we will require additional information about the function $g * d\nu_a$. Notice that

$$g * d\nu_a(w) = \int_0^\infty g(w-\lambda)\, d\nu_a(\lambda) = \int_0^\infty g'(w-\lambda) V_a(\lambda)\, d\lambda = \frac{1}{a}\int_0^\infty g'(w-\lambda)(1-\cos a\lambda)\, d\lambda. \tag{4.76}$$

Lemma 4.6.1. *Let $a > 0$. Let F be an odd $\mathcal{LP}$-function with a simple zero at the origin and at least one (smallest) positive zero τ_+. Assume that F is positive in $(0, \tau_+)$, and define $g = g_{\tau_+/2}$ by (4.18).*

1. *We have the representation*

$$g * d\nu_a(0) = \frac{g(0)}{a} - \frac{i}{2F(ia)}. \tag{4.77}$$

2. *For all real w we have*

$$g' * d\nu_a(w) - g' * d\nu_a(-w) = \sin(aw)\frac{ia}{F(ia)}. \tag{4.78}$$

Proof. Since F is odd with with at least three zeros we have that g and g' are non-negative on $\mathbb{R}$ and g' is even (by Lemma 4.3.4). To prove (4.77) we set $w = 0$ in (4.76) to get

$$g * d\nu_a(0) = \int_0^\infty g(-\lambda)\sin(a\lambda)\, d\lambda. \tag{4.79}$$

We perform an integration by parts, apply that g' is even, and use (4.36) to obtain

$$\begin{aligned} g * d\nu_a(0) &= \frac{g(0)}{a} - \frac{1}{a}\int_0^\infty g'(\lambda)\cos(a\lambda)\,d\lambda \\ &= \frac{g(0)}{a} - \frac{1}{2a}\int_{-\infty}^\infty g'(\lambda)\cos(a\lambda)\,d\lambda \\ &= \frac{g(0)}{a} - \frac{i}{2F(ia)} \end{aligned} \tag{4.80}$$

which finishes the proof of (4.77). Equations (4.36) and (4.37) and a change of variables give

$$\begin{aligned} g' * d\nu_a(w) &= \int_{-\infty}^\infty g'(w-\lambda)\sin(a\lambda)\,d\lambda - \int_{-\infty}^0 g'(w-\lambda)\sin(a\lambda)\,d\lambda \\ &= \sin(aw)\frac{ia}{F(ia)} + \int_0^\infty g'(w+\lambda)\sin(a\lambda)\,d\lambda \\ &= \sin(aw)\frac{ia}{F(ia)} + g' * d\nu_a(-w) \end{aligned} \tag{4.81}$$

which is (4.78). □

Theorem 4.6.2. *Let $a > 0$. Let F be an odd $\mathcal{LP}$-function with a simple zero at the origin and at least one (smallest) positive zero τ_+. Assume that F is positive in $(0, \tau_+)$, and define $g = g_{\tau_+/2}$ by (4.18). Then*

$$F(x)\left\{f^+_{\nu_a}(x) - K(F, \nu_a, x)\right\} \geq 0 \tag{4.82}$$

for all $x \in \mathbb{R}$, and

$$|K(F, \nu_a, x) - f^+_{\nu_a}(x)| = \mathcal{O}\left(\frac{F(x)}{1+x^2}\right) \tag{4.83}$$

for all real x.

Proof. Consider first $x < 0$. An expansion of the second term in (4.61) in a Laplace transform together with (4.54) gives

$$f^+_{\nu_a}(x) - K(F, \nu_a, x) = -F(x)\int_{-\infty}^0 \left(g * d\nu_a(w) - g * d\nu_a(0)\right)e^{-xw}\,dw, \tag{4.84}$$

By Lemma 4.4.2 we have $0 \leq g * d\nu_a(w) \leq Cg(w)$ for all real w. Inequality (4.25) with $\tau_2 = \tau_+$ applied to g implies that g has exponential decay on the negative real axis hence the integral in

(4.84) converges. Moreover, Lemma 4.4.2 gives $g' * d\nu_a(w) \geq 0$ for $w \leq 0$, and it follows that $g * d\nu_a(w) - g * d\nu_a(0) \leq 0$ for $w \leq 0$ which shows (4.82) for $x < 0$.

Let $x > 0$. Analogous to the case when $x < 0$, we expand the second term in (4.61) in a Laplace transform and use (4.54) to obtain

$$f^+_{\nu_a}(x) - K(F, \nu_a, x) = F(x) \int_0^\infty (g * d\nu_a(w) - g * d\nu_a(0))\, e^{-xw}\, dw. \tag{4.85}$$

Again, Lemma 4.4.2 along with inequality (4.25) with $\tau_1 = 0$ applied to g implies the integral in (4.85) converges. By an integration by parts we have

$$\int_0^\infty (g * d\nu_a(w) - g * d\nu_a(0))\, e^{-xw}\, dw = \frac{1}{x} \int_0^\infty g' * d\nu_a(w) e^{-xw}\, dw, \tag{4.86}$$

and inserting this into (4.85) gives

$$f^+_{\nu_a}(x) - K(F, \nu_a, x) = \frac{F(x)}{x} \int_0^\infty g' * d\nu_a(w) e^{-xw}\, dw. \tag{4.87}$$

In order to investigate the sign of the right hand side, we multiply (4.78) by e^{-xw} and integrate w over $[0, \infty)$ to obtain

$$\begin{aligned} \int_0^\infty (g' * d\nu_a(w) - g' * d\nu_a(-w)) e^{-xw}\, dw &= \frac{ia}{F(ia)} \int_0^\infty e^{-xw} \sin(aw)\, dw \\ &= \frac{i}{F(ia)} \frac{a^2}{x^2 + a^2}. \end{aligned} \tag{4.88}$$

Hence

$$\int_0^\infty g' * d\nu_a(w) e^{-xw}\, dw = \frac{i}{F(ia)} \frac{a^2}{x^2 + a^2} + \int_0^\infty g' * d\nu_a(-w) e^{-xw}\, dw \tag{4.89}$$

Since $g' * d\nu_a(-w) \geq 0$ for $w \geq 0$ and $iF(ia) < 0$ (by (4.38)), it follows that

$$\int_0^\infty g' * d\nu_a(w) e^{-xw}\, dw \geq 0. \tag{4.90}$$

Inserting this into (4.87) gives (4.82) for $x > 0$. Equations (4.84) and (4.85) give (4.83). $\square$

As in the case for the non-negative measures, we can easily construct interpolations for the odd extension of the truncated Poisson kernel

$$\widetilde{f_{\nu_a}}(x) = f_{\nu_a}^{+}(x) - f_{\nu_a}^{+}(-x) = \begin{cases} \frac{a}{x^2+a^2} & \text{if} \quad x > 0; \\ 0 & \text{if} \quad x = 0; \\ -\frac{a}{x^2+a^2}, & \text{if} \quad x < 0. \end{cases} \tag{4.91}$$

If F is an odd Laguerre-Pólya function that satisfies the assumptions of Theorem 4.6.2 define the entire function

$$\widetilde{K}(F, \nu_a, z) = K(F, \nu_a, z) - K(F, \nu_a, -z). \tag{4.92}$$

Analogous to Theorem 4.5.2 we have the following result.

Theorem 4.6.3. *Let $a > 0$. Let F be an odd $\mathcal{LP}$-function with a simple zero at the origin and at least one (smallest) positive zero τ_+. Assume that F is positive in $(0, \tau_+)$, and define $g = g_{\tau_+/2}$ by* (4.18)*. Then*

$$F(x)\left\{\widetilde{f_{\nu_a}}(x) - \widetilde{K}(F, \nu_a, x)\right\} \geq 0 \tag{4.93}$$

for all $x \in \mathbb{R}$.

4.7. Interpolations for the truncated Poisson kernel – even $\mathcal{LP}$ functions

We conclude this chapter by constructing interpolations for the truncated Poisson kernel using even $\mathcal{LP}$ functions. These interpolations will be used in Chapter 6 in the problem of finding optimal one-sided approximations with the added vanishing requirement.

As in the problem of construing interpolations to $f_{\nu_a}^{+}$ with odd Laguerre-Pólya functions, for the case of even $\mathcal{LP}$ functions we require additional information about the functions $g * d\nu_a$.

Lemma 4.7.1. *Let $a > 0$. Let F be an even $\mathcal{LP}$-function with a double zero at the origin and at least one (smallest) positive zero τ_+. Assume that F is positive in $(0, \tau_+)$, and define $g = g_{\tau_+/2}$ by* (4.18)*.*

1. *We have the representation*

$$g' * d\nu_a(0) = \frac{g'(0)}{a} + \frac{a}{2F(ia)}. \tag{4.94}$$

2. For all real w we have

$$g'' * d\nu_a(w) - g'' * d\nu_a(-w) = -\sin(aw)\frac{a^2}{F(ia)}. \tag{4.95}$$

Proof. The proof is analogous to Lemma 4.6.1, and we sketch out the differences. Since F is even with with at least four zeros we have that g'' is even (by Lemma 4.3.3). To prove (4.94) we differentiate (4.76) and set $w = 0$ to get

$$g' * d\nu_a(0) = \int_0^\infty g'(-\lambda)\sin(a\lambda)\,d\lambda. \tag{4.96}$$

We perform an integration by parts, apply that g'' is even, and use (4.26) to obtain

$$\begin{aligned} g' * d\nu_a(0) &= \frac{g'(0)}{a} - \frac{1}{a}\int_0^\infty g''(\lambda)\cos(a\lambda)\,d\lambda \\ &= \frac{g'(0)}{a} - \frac{1}{2a}\int_{-\infty}^\infty g''(\lambda)\cos(a\lambda)\,d\lambda \\ &= \frac{g'(0)}{a} + \frac{a}{2F(ia)} \end{aligned} \tag{4.97}$$

which finishes the proof of (4.94).

Equations (4.26) and (4.27) and a change of variables give

$$\begin{aligned} g'' * d\nu_a(w) &= \int_{-\infty}^\infty g''(w-\lambda)\sin(a\lambda)\,d\lambda - \int_{-\infty}^0 g''(w-\lambda)\sin(a\lambda)\,d\lambda \\ &= -\sin(aw)\frac{a^2}{F(ia)} + \int_0^\infty g''(w+\lambda)\sin(a\lambda)\,d\lambda \\ &= -\sin(aw)\frac{a^2}{F(ia)} + g' * d\nu_a(-w) \end{aligned} \tag{4.98}$$

which is (4.95). $\square$

Assume that $a > 0$, and that $F \in \mathcal{LP}$ and $\tau > 0$ satisfy the assumptions of Lemma 4.4.3 (for even F). We define $z \mapsto M^+(F, a, z)$ and $z \mapsto M^-(F, a, z)$ by

$$M^-(F, a, z) = A(F, \nu_a, z) + g * d\nu_a(0)\frac{F(z)}{z}, \tag{4.99}$$

$$M^+(F, a, z) = A(F, \nu_a, z) + g * d\nu_a(0)\frac{F(z)}{z} + \frac{2g'(0)}{a}\frac{F(z)}{z^2} \tag{4.100}$$

where $A_1(F, \nu_a, z)$ is defined in (4.54). Evidently M^+ and M^- are entire functions. Recall that $\mathcal{T}_F$ is the zero set of F. It is evident from the definitions that $M^{\pm}(F, a, \xi) = f^+_{\nu_a}(\xi)$ for all real $\xi \in \mathcal{T}_F \backslash \{0\}$. Since F has a double zero at the origin, we see that $M^-(F, a, 0) = 0$. Since g'' is non-negative and integrable on $\mathbb{R}$, (4.29) implies

$$\frac{2}{F''(0)} = \int_{-\infty}^{\infty} g''(w)\, dw. \tag{4.101}$$

As g'' is even and $g'(w)$ decays exponentially as $w \to -\infty$, we also have that $\int_{-\infty}^{\infty} g''(w)\, dw = 2g'(0)$. Hence, $z^{-2}F(z) \to 1/(2g'(0))$ as $z \to 0$ and $M^+(F, a, 0) = 1/a$. This means that

$$M^{\pm}(F, a, \xi) = f^+_{\nu_a}(\xi\pm) \tag{4.102}$$

for all real $\xi \in \mathcal{T}_F$, where $f^+_{\nu_a}(\xi\pm)$ denotes the one sided limits at ξ.

Theorem 4.7.2. *Let $a > 0$. Let F be an even $\mathcal{LP}$-function with a double zero at the origin and at least one (smallest) positive zero τ_+ and at least five zeros (counted with multiplicity). Assume that F is positive in $(0, \tau_+)$, and define $g = g_{\tau_+/2}$ by (4.18) and $g * d\nu_a$ by (4.76). Then*

$$F(x)\left\{M^+(F, a, x) - f^+_{\nu_a}(x)\right\} \geq 0 \tag{4.103}$$

holds for all real x.

Proof. Consider first $x < 0$. An expansion of the second term in (4.100) in a Laplace transform together with (4.54) gives

$$M^+(F, a, x) - f^+_{\nu_a}(x) = F(x) \int_{-\infty}^{0} (g * d\nu_a(w) - g * d\nu_a(0))e^{-xw}\, dw + \frac{F(x)}{x^2}\frac{2g'(0)}{a}. \tag{4.104}$$

Two integration by parts and (4.94) lead to

$$\begin{aligned} \int_{-\infty}^{0} (g * d\nu_a(w) - g * d\nu_a(0))e^{-xw}\, dw &= \frac{1}{x^2}\int_{-\infty}^{0} g'' * d\nu_a(w) e^{-wx} dx - \frac{g' * d\nu_a(0)}{x^2} \\ &= \frac{1}{x^2}\int_{-\infty}^{0} g'' * d\nu_a(w)e^{-xw}\, dw - \frac{g'(0)}{ax^2} - \frac{a}{2x^2F(ia)}, \end{aligned} \tag{4.105}$$

and inserting this in (4.104) gives

$$M^{+}(F,a,x)-f_{\nu_a}^{+}(x)=\frac{F(x)}{x^2}\left(\frac{g'(0)}{a}-\frac{a}{2F(ia)}+\int_{-\infty}^{0}g''*d\nu_a(w)e^{-xw}\,dw\right). \tag{4.106}$$

By assumption $-F(ia)>0$, and (4.49) implies $g''*d\nu_a(w)\geq 0$. Since by assumption $z^{-1}F(z)$ is a $\mathcal{LP}$-function that is positive in $(0,\tau_+)$, it follows from Lemma 4.3.3 that $g'(0)>0$. Hence (4.103) is shown for $x<0$.

Let $x>0$. From (4.54), (4.76), and (4.94) we get

$$\begin{aligned}M^{+}(F,a,x)-f_{\nu_a}^{+}(x)&=-F(x)\int_0^{\infty}g*d\nu_a(w)e^{-xw}\,dw+\frac{F(x)}{x}g*d\nu_a(0)+\frac{F(x)}{x^2}\frac{2g'(0)}{a}\\&=-F(x)\int_0^{\infty}(g*d\nu_a(w)-g*d\nu_a(0))e^{-xw}\,dw+\frac{F(x)}{x^2}\frac{2g'(0)}{a},\end{aligned} \tag{4.107}$$

and, analogously to (4.106), we obtain the representation

$$M^{+}(F,a,x)-f_{\nu_a}^{+}(x)=\frac{F(x)}{x^2}\left(\frac{g'(0)}{a}-\frac{a}{2F(ia)}-\int_0^{\infty}g''*d\nu_a(w)e^{-xw}\,dw\right) \tag{4.108}$$

for $x>0$. In order to investigate the sign of the right hand side, we multiply (4.95) by e^{-xw} and integrate w over $[0,\infty)$ to get with (4.58)

$$\begin{aligned}\int_0^{\infty}(g''*d\nu_a(w)-g''*d\nu_a(-w))e^{-xw}dw&=-\frac{a^2}{F(ia)}\int_0^{\infty}e^{-xw}\sin(aw)dw\\&=-\frac{a^2}{F(ia)}\frac{a}{x^2+a^2}.\end{aligned} \tag{4.109}$$

Hence

$$-\int_0^{\infty}g''*d\nu_a(w)e^{-xw}dw=\frac{a}{F(ia)}\frac{a^2}{x^2+a^2}-\int_0^{\infty}g''*d\nu_a(-w)e^{-xw}dw. \tag{4.110}$$

Since $g''*d\nu_a(-w)\geq 0$ for $w\geq 0$, we have from (4.94)

$$\int_0^{\infty}g''*d\nu_a(-w)e^{-xw}dw\leq\int_0^{\infty}g''*d\nu_a(-w)dw=\frac{g'(0)}{a}+\frac{a}{2F(ia)}. \tag{4.111}$$

We multiply (4.111) by -1 and insert the resulting inequality into (4.110) to get

$$-\int_0^\infty g'' * d\nu_a(w)e^{-xw}dw \geq \frac{a}{F(ia)}\frac{a^2}{x^2+a^2} - \frac{g'(0)}{a} - \frac{a}{2F(ia)}. \tag{4.112}$$

Inserting this into (4.108) gives

$$\frac{M^+(F,a,x) - f_{\nu_a}^+(x)}{F(x)} \geq \frac{a}{F(ia)}\frac{1}{x^2}\left(\frac{1}{(x/a)^2+1} - 1\right), \tag{4.113}$$

which is non-negative since $F(ia) < 0$ (by (4.28)). This proves (4.103) for $x > 0$. $\square$

Theorem 4.7.3. *Let F be an even $\mathcal{LP}$-function with a double zero at the origin and at least one (smallest) positive zero τ_+ and at least five zeros (counted with multiplicity). Assume that F is positive in $(0, \tau_+)$, and define $g = g_{\tau_+/2}$ by (4.18) and $g * d\nu_a$ by (4.76). Then*

$$F(x)\left\{M^-(F,a,x) - f_{\nu_a}^+(x)\right\} \leq 0 \tag{4.114}$$

holds for all real x.

Proof. From the definition of M^- and (4.104) we obtain for $x < 0$ the representation

$$M^-(F,a,x) - f_{\nu_a}^+(x) = F(x)\int_{-\infty}^0 (g * d\nu_a(w) - g * d\nu_a(0))e^{-xw}dw, \tag{4.115}$$

and since $g' * d\nu_a(w) \geq 0$ for real w, it follows that $g * d\nu_a(w) - g * d\nu_a(0) \leq 0$ for $w \leq 0$ which shows (4.114) for $x < 0$. Analogously, for $x > 0$

$$M^-(F,a,x) - f_{\nu_a}^+(x) = -F(x)\int_0^\infty (g * d\nu_a(w) - g * d\nu_a(0))e^{-xw}dw, \tag{4.116}$$

which gives (4.114) in this range. $\square$

Lemma 4.7.4. *The functions M^+ and M^- from Theorems 4.7.2 and 4.7.3 satisfy*

$$|M^\pm(F,a,x) - f_{\nu_a}^+(x)| = \mathcal{O}\left(\frac{F(x)}{1+x^2}\right) \tag{4.117}$$

for all real x.

Proof. Inequalities (4.104) and (4.108) yield (4.117) for M^+, while (4.115) and (4.116) imply (4.117) for M^-. $\square$

Recall that the odd extension of the truncated Poisson kernel is given by

$$\widetilde{f_{\nu_a}}(x) = f_{\nu_a}^+(x) - f_{\nu_a}^+(-x) = \begin{cases} \frac{a}{x^2+a^2} & \text{if } x > 0; \\ 0 & \text{if } x = 0; \\ -\frac{a}{x^2+a^2} & \text{if } x < 0. \end{cases} \tag{4.118}$$

By the above construction we can easily construct interpolations for this function. If $F \in \mathcal{LP}$ satisfies the assumptions of Theorem 4.7.2 (also Theorem 4.7.3), define the entire functions

$$\widetilde{M}^+(F, a, z) = M^+(F, a, z) - M^-(F, a, -z) \tag{4.119}$$

$$\widetilde{M}^-(F, a, z) = -\widetilde{M}^+(F, a, -z) \tag{4.120}$$

Applying Theorem 4.7.2 and Theorem 4.7.3 and the fact that F is even gives

$$\begin{aligned} &F(x)\left\{\widetilde{M}^+(F, a, x) - \widetilde{f_{\nu_a}}(x)\right\} \\ &\quad = F(x)\left\{M^+(F, a, x) - f_{\nu_a}^+(x)\right\} + F(-x)\left\{f_{\nu_a}^+(-x) - M^-(F, a, -x)\right\} \geq 0 \end{aligned} \tag{4.121}$$

for all real x. By symmetry, we obtain $F(x)\left\{\widetilde{f_{\nu_a}} - \widetilde{M}^-(F, a, x)\right\} \geq 0$ for all real x.

Theorem 4.7.5. *Let $a > 0$. Let F be an even $\mathcal{LP}$-function with a double zero at the origin and at least one (smallest) positive zero τ_+ and at least five zeros (counted with multiplicity). Assume that F is positive in $(0, \tau_+)$, and define $g = g_{\tau_+/2}$ by (4.18) and $g * d\nu_a$ by (4.76). Then*

$$F(x)\left\{\widetilde{M}^+(F, a, x) - \widetilde{f_{\nu_a}}(x)\right\} \geq 0 \tag{4.122}$$

and

$$F(x)\left\{\widetilde{f_{\nu_a}}(x) - \widetilde{M}^-(F, a, x)\right\} \geq 0 \tag{4.123}$$

holds for all real x and

$$\left|\widetilde{M}^{\pm}(F,a,x)-\widetilde{f_{\nu_a}}(x)\right|=\mathcal{O}\left(\frac{F(x)}{1+x^2}\right) \tag{4.124}$$

for all real x.

5. EXTREMAL SIGNATURES AND BEST APPROXIMATIONS IN $L^1(\mathbb{R}, \mu)$-NORM

5.1. Introduction

This chapter is primarily devoted to the study of Best Approximations in $L^1(\mu)$-norm. For more information on best approximations in $L^p(\mu)$ (or in general Banach spaces) see [61, Chapter 4] or [24, Chapter 3].

Definition 5.1.1. Let $\delta > 0$, μ be a Borel measure with polynomial growth, and $f : \mathbb{R} \to \mathbb{R}$ a measurable function. A function $K \in \mathcal{A}(\delta)$ is called a *best approximation* to f (of type δ) in $L^1(\mathbb{R}, \mu)$-norm if $f - K \in L^1(\mathbb{R}, \mu)$ and

$$\int_{-\infty}^{\infty} |f(x) - K(x)| \, d\mu(x) \leq \int_{-\infty}^{\infty} |f(x) - G(x)| \, d\mu(x) \tag{5.1}$$

for all $G \in \mathcal{A}(\delta)$.

A classical result in the theory of $L^1(\mathbb{R}, \mu)$-approximation is that a best approximation K from $\mathcal{A}(\delta)$ can be characterized by the sign change pattern of $f - K$.

Theorem 5.1.2 ([24, Theorem 2.10.1])**.** *Let μ be a Borel measure on $\mathbb{R}$ with polynomial growth. Let $\psi : \mathbb{R} \to \mathbb{C}$ be such that $|\psi(x)| = 1$ for almost every $x \in \mathbb{R}$ and*

$$\int_{-\infty}^{\infty} \psi(x) G(x) \, d\mu(x) = 0 \tag{5.2}$$

for every $G \in \mathcal{A}^1(\delta, \mu)$.

If $K \in \mathcal{A}(\delta)$ with $f - K \in L^1(\mathbb{R}, \mu)$ and $\psi = \operatorname{sgn}(f - K)$ a.e., then K is a best approximation to f in $L^1(\mathbb{R}, \mu)$-norm.

Proof. Let $G \in \mathcal{A}(\delta)$. We may assume that $f - G \in L^1(\mathbb{R}, \mu)$ (else there is nothing to prove). Since $f - K \in L^1(\mathbb{R}, \mu)$, we have $G - K \in \mathcal{A}^1(\delta, \mu)$. Using (5.2) it follows that

$$\|f - K\|_{L^1(\mu)} = \int_{-\infty}^{\infty} \psi(f - K) \, d\mu = \int_{-\infty}^{\infty} \psi(f - G) \, d\mu \leq \int_{-\infty}^{\infty} |f - G| \, d\mu = \|f - G\|_{L^1(\mu)}, \tag{5.3}$$

hence K is a best approximation to f in $L^1(\mathbb{R},\mu)$-norm. $\square$

Remark 5.1.3. By an application of the Hahn-Banach Theorem, we also have that if $K \in \mathcal{A}(\delta)$ is a best approximation to f in $L^1(\mathbb{R},\mu)$ and the set $\{x \in \mathbb{R} \mid f(x) = K(x)\}$ has measure zero, then $\operatorname{sgn}(f-K)$ is orthogonal to $\mathcal{A}^1(\delta,\mu)$ (see [24, Theorem 10.1] or [61, Theorem 4.2.2])

Given this connection, it is of interest to identify functions ψ that satisfy (5.2).

Definition 5.1.4. We say that a function $\psi \in L^\infty(\mathbb{R},\mu)$ is a *high-pass* function for $\mathcal{A}^1(\delta,\mu)$ if

$$\int_{\mathbb{R}} \psi(x) G(x)\, d\mu(x) = 0 \tag{5.4}$$

for all $G \in \mathcal{A}^1(\delta,\mu)$. We denote the class of all high-pass functions by $T_1(\delta,\mu)$. Furthermore, a function $\psi \in T_1(\delta,\mu)$ with $|\psi| = 1$ a.e. is called an *extremal signature* for μ.

The term high-pass originates from the special case where μ is the Lebesgue measure. In this case, an equivalent definition for a bounded function to be high-pass is that its (distributional) Fourier transform has no support inside $(-\delta/2\pi, \delta/2\pi)$ (see [61, Chapter 7]). For $\delta = \pi$, the simplest signatures with this property are of the form $\psi(x) := \operatorname{sgn}\sin\pi(x-\alpha)$ where α is any real number. Hence, for many functions it can be expected that a best approximation from $\mathcal{A}(\pi)$ in $L^1(\mathbb{R},dx)$-norm interpolates f at a translate of the integers and nowhere else on the real line. In his notable thesis [51], B. F. Logan gives necessary and sufficient characterizations (Theorem 5.1.5) of all sign change patterns which are high-pass for the Lebesgue measure. Using this, Logan is able to find best $L^1(\mathbb{R},dx)$-approximations of a class of so-called Krein kernels which are obtained by an application of Nagy's Criterion (see [51, Section 7.6] or [61, Section 7.4]).

Below we state Logan's result (written in the language of functions of bounded type and mean type).

Theorem 5.1.5 ([51, Section 7.6])**.** *Let ψ be a measurable function on $\mathbb{R}$ such that $\psi(x)^2 = 1$ a.e. Then $\psi \in T_1(\delta, dx)$ if and only if*

$$\psi(x) = \operatorname{sgn}\sin 2\varphi(x) \tag{5.5}$$

where $e^{2i\varphi(x)}$ is the boundary function of an inner function G, in the upper half-plane with mean type $v(G) \le -\delta$.

Proof. This is Theorem 6.5.1 of [51] see also Theorem 7.6.5.7 of [61]. □

In Sections 5.2 and 5.3 we determine extremal signatures for Hermite-Biehler weights (Theorem 5.2.1) and explore additional properties of these functions. In Section 5.4 we apply the general interpolation results from Sections 4.5 and 4.6 to construct best approximations to a variety of functions in $L^1(\mathbb{R}, |E(x)|^{-2}\,dx)$-norm. Lastly, in Section 5.5 we present the best-approximations of the Poisson and Conjugate Poisson kernel and give explicit values for the minimal error.

5.2. Extremal signatures for Hermite-Biehler weights

For a Hermite-Biehler function E, define the Hermite-Biehler weight (or de Branges measure) by

$$\mu_E(B) = \int_B \frac{dx}{|E(x)|^2}. \tag{5.6}$$

Recall that E^*/E is an inner function in the upper half-plane with continuous extension to the closed half-plane, hence by Theorem 2.6.5 there exists a continuous, increasing, real-valued function φ such that the identity

$$\frac{E^*(x)}{E(x)} = e^{2i\varphi(x)} \tag{5.7}$$

holds for all real x.

Theorem 5.2.1. *Let E be a Hermite-Biehler function with bounded type in the upper half-plane and mean type τ and no real zeros. If φ is a phase function of E, then*

$$\psi(x) = \operatorname{sgn} \sin 2\varphi(x) \tag{5.8}$$

belongs to $T_1(2\tau, |E(x)|^{-2}\,dx)$.

Remark 5.2.2. If α is a real number, then the entire function

$$E_\alpha(z) = e^{i\alpha}E(z) = A_\alpha(z) - iB_\alpha(z) \tag{5.9}$$

has phase $\varphi_\alpha = \varphi - \alpha$ and satisfies $|E_\alpha(z)| = |E(z)|$ for all z. Thus,

$$\psi_\alpha(x) = \operatorname{sgn} \sin(2(\varphi(x) - \alpha)) \tag{5.10}$$

belongs to $T_1(2\tau, |E(x)|^{-2}\,dx)$ for all real α.

Remark 5.2.3. Ultimately one would like a complete characterization (similar to Theorem 5.1.5) of the space $T_1(\delta, |E(x)|^{-2}\,dx)$.

Before we prove Theorem 5.2.1, we make the following useful observation.

Lemma 5.2.4. *Let E be a Hermite-Biehler function with no real zeros. If φ is a phase function of E, then the identity*

$$\operatorname{sgn}(A_\alpha B_\alpha) = \operatorname{sgn}\sin(2(\varphi-\alpha)) \tag{5.11}$$

is valid for all real α.

Proof. It follows from (5.7) that

$$\sin(2(\varphi-\alpha)) = \frac{i}{2}\left(\frac{E_\alpha}{E_\alpha^*} - \frac{E_\alpha^*}{E_\alpha}\right) = \frac{i(E_\alpha^2 - (E_\alpha^*)^2)}{2|E_\alpha|^2} = \frac{2A_\alpha B_\alpha}{|E_\alpha|^2} \tag{5.12}$$

on the real line, which gives (5.11). □

Remark 5.2.5. Equation (5.11) takes a particularly useful form when $\alpha = \pi/4$. Since,

$$A_{\pi/4}B_{\pi/4} = -\frac{1}{4}\left(E^2 + (E^*)^2\right) \tag{5.13}$$

we have that

$$\operatorname{sgn}\left(E^2 + (E^*)^2\right) = -\operatorname{sgn}(A_{\pi/4}B_{\pi/4}) = -\operatorname{sgn}\sin(2(\varphi-\pi/4)) = -\operatorname{sgn}\cos(2\varphi). \tag{5.14}$$

For the proof of Theorem 5.2.1 we make use of the following lemma.

Lemma 5.2.6. *Let E be a Hermite-Biehler function with bounded type in the upper half-plane and mean type τ and no real zeros. Let $N \in \mathbb{N}$ and define for $z \in \overline{\mathbb{C}^+}$*

$$F_N(z) = \frac{4}{\pi i}\sum_{k=0}^{N-1}\frac{1}{2k+1}\left(\frac{E^*(z)}{E(z)}\right)^{2k+1}. \tag{5.15}$$

Then

$$\int_{-\infty}^{\infty} F_N(x)F(x)\frac{dx}{|E(x)|^2} = 0 \tag{5.16}$$

for all $F \in \mathcal{A}^1(2\tau, |E(x)|^{-2}\,dx)$*.*

Proof. Let $F \in \mathcal{A}^1(2\tau, |E(x)|^{-2}\,dx)$. By Lemma 3.3.4 and Theorem 3.6.1 we have that F/E^2 belongs to $H^1(\mathbb{C}^+)$. Then

$$\int_{-\infty}^{\infty} F_N(x)F(x)\frac{dx}{|E(x)|^2} = \frac{4}{\pi i}\sum_{k=0}^{N-1}\frac{1}{2k+1}\int_{-\infty}^{\infty}\left(\frac{E^*(x)}{E(x)}\right)^{2k}\frac{F(x)}{E^2(x)}dx, \tag{5.17}$$

and since $E^*/E \in H^\infty(\mathbb{C}^+)$ it follows that the integrand on the right is an element of $H^1(\mathbb{R})$, and Corollary 2.3.4 implies that each integral on the right is equal to zero. □

Proof of Theorem 5.2.1. Let $F \in \mathcal{A}^1(2\tau, |E(x)|^{-2}\,dx)$ be real entire. Taking real parts in (5.16) leads to

$$\int_{-\infty}^{\infty}\frac{4}{\pi}\sum_{k=0}^{N-1}\frac{\sin(2(2k+1)\varphi(x))}{2k+1}F(x)\frac{dx}{|E(x)|^2} = 0. \tag{5.18}$$

The function $y \mapsto \operatorname{sgn}\sin y$ is 2π periodic on $\mathbb{R}$ and has the Fourier expansion

$$\operatorname{sgn}\sin y = \lim_{N\to\infty}\frac{4}{\pi}\sum_{k=0}^{N-1}\frac{\sin((2k+1)y)}{2k+1}. \tag{5.19}$$

Moreover, the partial sums on the right hand side of (5.19) are uniformally bounded in N and y, and it follows that

$$\int_{-\infty}^{\infty}\operatorname{sgn}\sin(2\varphi(x))F(x)\frac{dx}{|E(x)|^2} = 0 \tag{5.20}$$

for every real entire $F \in \mathcal{A}^1(2\tau, |E(x)|^{-2}\,dx)$. For general $F \in \mathcal{A}^1(2\tau, |E(x)|^{-2}\,dx)$ we have that $F = F_1 - iF_2$ for real entire functions F_1 and F_2 given by $F_1 = 2^{-1}(F + F^*)$ and $F_2 = i2^{-1}(F - F^*)$. As $F \in \mathcal{A}^1(2\tau, |E(x)|^{-2}\,dx)$ it follows that F^* and hence F_1 and F_2 belong to $\mathcal{A}^1(2\tau, |E(x)|^{-2}\,dx)$ and the result follows. □

Remark 5.2.7. We see in the proof of Lemma 5.2.6 (and hence of Theorem 5.2.1) that one can remove the assumption that E is of bounded type if we assume that F/E^2 and F^*/E^2 belong to $H^1(\mathbb{C}^+)$ for all $F \in \mathcal{A}^1(2\tau, |E(x)|^{-2}\,dx)$.

5.3. Sign changes of signatures

For the Lebesgue measure, Logan shows that high-pass functions cannot dwell at 1 (or -1) for too long (see Theorem 5.3.1). In Theorem 5.3.2 we show that for Hermite-Biehler weights a similar property holds.

Theorem 5.3.1 ([51, Theorem 7.3.1])**.** *If $\psi \in T_1(\delta, dx)$ such that $|\psi(x)| \leq 1$ for all real x and*

$$\psi(x) = 1 \tag{5.21}$$

for $a < x < b$ then

$$b - a \leq \pi/\delta, \tag{5.22}$$

with equality possible if and only if

$$\psi(x) = \operatorname{sgn}\sin(\delta(x-a)) \tag{5.23}$$

for almost every real x.

For a Hermite-Biehler function with phase φ and α a real number, define

$$\Gamma_\alpha := \left\{ t \in \mathbb{R} \mid \varphi(t) \equiv \alpha \bmod \frac{\pi}{2} \right\}. \tag{5.24}$$

Notice that if E does have have real zeros, we have that $\Gamma_\alpha = \mathcal{T}_{A_\alpha B_\alpha}$ (recall that $\mathcal{T}_F$ is the zero set of F). The following theorem implies that if $\psi \in T_1(2\tau, |E(x)|^{-2}dx)$ with $|\psi(x)| \leq 1$ on the real line, then for all real α we have $\psi \not\equiv 1$ (or -1) on any interval (a, b) that contains two elements in Γ_α.

Theorem 5.3.2. *Let E be a Hermite-Biehler function of bounded type in the upper half-plane and mean type τ with no real zeros. Let $\alpha \in \mathbb{R}$. If $\psi \in T_1(2\tau, |E(x)|^{-2}dx)$ is bounded on the real line and*

$$\psi(x) = 1 \tag{5.25}$$

on an interval (a,b) that contains two elements in Γ_α, then

$$\sup_{x\in\mathbb{R}} |\psi(x)| > 1. \tag{5.26}$$

Proof. Let $M = \sup_{x\in\mathbb{R}} |\psi(x)| < \infty$. Assume there exists a $G \in \mathcal{A}^1(2\tau, |E(x)|^{-2}\,dx)$ with $G(x) > 0$ on (a,b). (We will show that such a G exists below.) Since $\psi \in T_1(2\tau, |E(x)|^{-2}\,dx)$, we have that

$$\int_{-\infty}^{\infty} G(x)\psi(x)\frac{dx}{|E(x)|^2} = 0, \tag{5.27}$$

thus

$$\int_{(a,b)} G(x)\frac{dx}{|E(x)|^2} = -\int_{\mathbb{R}\setminus(a,b)} G(x)\psi(x)\frac{dx}{|E(x)|^2}. \tag{5.28}$$

Since $G(x) > 0$ on (a,b), (5.28) gives

$$\int_{(a,b)} |G(x)|\frac{dx}{|E(x)|^2} = \left|\int_{\mathbb{R}\setminus(a,b)} G(x)\psi(x)\frac{dx}{|E(x)|^2}\right| \le M\int_{\mathbb{R}\setminus(a,b)} |G(x)|\frac{dx}{|E(x)|^2} \tag{5.29}$$

which leads to

$$M \ge \frac{\int_{(a,b)} |G(x)|\frac{dx}{|E(x)|^2}}{\int_{\mathbb{R}\setminus(a,b)} |G(x)|\frac{dx}{|E(x)|^2}}. \tag{5.30}$$

Let γ_i and γ_{i+1} be two (consecutive) elements of Γ_α that belong to (a,b). If (a,b) contains more than two elements from Γ_α, choose a sub-interval that only contains two elements from Γ_α (notice that (5.30) holds with any sub-interval $(a',b') \subset (a,b)$).

In what follows, we construct a function $G_{\alpha,i} \in \mathcal{A}^1(2\tau, |E(x)|^{-2}\,dx)$ with $G_{\alpha,i}(x) > 0$ on (a,b) such that

$$\int_{\mathbb{R}\setminus(a,b)} |G_{\alpha,i}(x)|\frac{dx}{|E(x)|^2} < \int_{(a,b)} |G_{\alpha,i}(x)|\frac{dx}{|E(x)|^2}, \tag{5.31}$$

which (along with (5.30)) gives $M > 1$ and finishes the proof.

Define

$$G_{\alpha,i}(z) = C_{\alpha,i}\frac{A_\alpha(z)B_\alpha(z)}{(z-\gamma_i)(z-\gamma_{i+1})} \tag{5.32}$$

where

$$C_{\alpha,i} = -\mathrm{sgn}\left(A_\alpha\left(\frac{\gamma_i+\gamma_{i+1}}{2}\right)B_\alpha\left(\frac{\gamma_i+\gamma_{i+1}}{2}\right)\right). \tag{5.33}$$

Since E does not have any real zeros, it follows that the (real) zeros of $A_\alpha B_\alpha$ are simple and are given by the set Γ_α (see (5.12)). By construction, the function $G_{\alpha,i}$ is entire and positive on $(\gamma_{i-1}, \gamma_{i+2}) \supset (a, b)$ (if there is no γ_{i-1} or γ_{i+2} take these to be $-\infty$ or ∞, respectively). Since $G_{\alpha,i}$ is continuous on $\mathbb{R}$ and $A_\alpha B_\alpha/|E(x)|^2$ is bounded on the real line, it follows that $G_{\alpha,i} \in \mathcal{A}^1(2\tau, |E(x)|^{-2}\,dx)$.

By Theorem 5.2.1 and Lemma 5.2.4 the function

$$\psi_{\alpha,i} := \operatorname{sgn}\left(C_{\alpha,i} A_\alpha B_\alpha\right). \tag{5.34}$$

belongs to $T_1(2\tau, |E(x)|^{-2}\,dx)$. Since $G_{\alpha,i} \in \mathcal{A}^1(2\tau, |E(x)|^{-2}\,dx)$, we have that

$$\int_{-\infty}^{\infty} G_{\alpha,i}(x)\psi_{\alpha,i}(x)\frac{dx}{|E(x)|^2} = 0. \tag{5.35}$$

Notice that

$$\psi_{\alpha,i}(x) = \begin{cases} \operatorname{sgn} G_{\alpha,i}(x), & x \in \mathbb{R} \setminus (\gamma_i, \gamma_{i+1}) \\ -\operatorname{sgn} G_{\alpha,i}(x), & x \in (\gamma_i, \gamma_{i+1}), \end{cases} \tag{5.36}$$

hence (5.35) becomes

$$\int_{\mathbb{R}\setminus(\gamma_i,\gamma_{i+1})} |G_{\alpha,i}(x)|\frac{dx}{|E(x)|^2} = \int_{(\gamma_i,\gamma_{i+1})} |G_{\alpha,i}(x)|\frac{dx}{|E(x)|^2}. \tag{5.37}$$

Since $G_{\alpha,i}$ is continuous and positive on $(\gamma_i, \gamma_{i+1}) \subset (a, b)$, inequality (5.31) follows. □

5.4. Best approximations in $L^1(\mathbb{R}, \mu_E)$-norm

In this section we apply the results of Sections 4.5 and 4.6 to construct best approximations of the class of functions defined in Chapter 4.

Recall that if $\nu \in \mathcal{M}_b^+(\mathbb{R}^+)$, or the signed measure $d\nu_a(t) = \sin(at)dt$, and F is an an odd Laguerre-Pólya function then the function $z \mapsto K(F, \nu, z)$ (see (4.61) and Theorem 4.5.1) is an entire function such that

$$\operatorname{sgn}(F(x)) = f_\nu^+(x) - K(F, \nu, x) \tag{5.38}$$

and

$$|f_\nu^+(x) - K(F,\nu,x)| = \mathcal{O}\left(\frac{F(x)}{1+x^2}\right) \tag{5.39}$$

for all real x. Moreover, (5.38) and (5.39) also hold for the odd extension of f_ν^+, $\widetilde{f}_\nu$, with $z \mapsto \widetilde{K}(F,\nu,z)$ (see Theorem 4.5.2).

Theorem 5.4.1. *Let E be a Hermite-Biehler function of bounded type in the upper half-plane and exponential type $\tau(E)$ with no real zeros such that $E(-z) = E^*(z)$ for all z. The following propositions hold:*

1. *The entire function $z \mapsto K(AB,\nu,z)$ is of exponential type 2τ and is a best approximation to f_ν^+ from $\mathcal{A}(2\tau(E))$ in $L^1(\mathbb{R}, |E(x)|^{-2}\,dx)$-norm.*

2. *The entire function $z \mapsto \widetilde{K}(AB,\nu,z)$ is of exponential type 2τ and is a best approximation to $\widetilde{f}_\nu$ from $\mathcal{A}(2\tau(E))$ in $L^1(\mathbb{R}, |E(x)|^{-2}\,dx)$-norm.*

Moreover, if $AB \notin \mathcal{H}(E^2)$ and $\varphi'(x)$ is bounded on the real line, then the best approximations to f_ν^+ and $\widetilde{f}_\nu$ are unique.

Remark 5.4.2. Unfortunately, there does not seem to be an approach that gives the minimal $L^1(\mathbb{R}, |E(x)|^{-2}\,dx)$-error. However, in Section 5.5 we are able to give (via an application of the Residue Theorem) simple expressions for the minimal $L^1(\mathbb{R}, |E(x)|^{-2}\,dx)$-error for the Poisson Kernel and Conjugate Poisson Kernel.

Proof of Theorem 5.4.1. We show the result for the odd function $\widetilde{f}_\nu$. The proof for f_ν^+ is nearly identical, and we omit the details.

The Hermite-Biehler inequality (3.8) implies that A and B have only real zeros, and since E has no real zeros, it follows from Lemma 3.5.4 that the zeros of A and B are simple and interlace. As E is of bounded type and $E^*(z) = E(-z)$, Lemmas 3.3.11 and 4.2.8 applied to

$$E^2 = (A - iB)^2 = (A^2 + B^2) - i2AB \tag{5.40}$$

imply that AB is an odd $\mathcal{LP}$ function with a simple zero at the origin. Moreover, Lemma 3.3.11 implies that $z \mapsto E(iz)$ is real entire, hence $iAB(ia) = -4^{-1}(E^2(ia) - E^2(-ia))$ and since E is

Hermite-Biehler it follows that $iAB(ia) < 0$. Thus, the results of Sections 4.5 and 4.6 are applicable, and we have that

$$\operatorname{sgn}(AB(x)) = \widetilde{f}_\nu(x) - \widetilde{K}(AB, \nu, x) \tag{5.41}$$

for all real x. Since AB/E^2 is bounded on $\mathbb{R}$, it follows from (4.63) that

$$\int_{-\infty}^{\infty} \frac{|\widetilde{f}_\nu(x) - \widetilde{K}(AB, \nu, x)|}{|E(x)|^2}\, dx < \infty. \tag{5.42}$$

Since E has exponential type τ, it follows that AB is of exponential type 2τ and (4.55) implies that $z \mapsto \widetilde{K}(AB, \nu, z)$ of exponential type 2τ. Theorems 5.2.1 and 5.1.2 along with Lemma 5.2.4 prove that $\widetilde{K}$ is extremal.

Next we prove uniqueness. If $G \in \mathcal{A}(2\tau)$ such that

$$\begin{aligned} \int_{\infty}^{\infty} |\widetilde{f}_\nu(x) - G(x)| \frac{dx}{|E(x)|^2} &= \int_{\infty}^{\infty} |\widetilde{f}_\nu(x) - \widetilde{K}(AB, \nu, x)| \frac{dx}{|E(x)|^2} \\ &= \int_{\infty}^{\infty} \operatorname{sgn}(AB) \left(\widetilde{f}_\nu(x) - \widetilde{K}(AB, \nu, x)\right) \frac{dx}{|E(x)|^2}, \end{aligned} \tag{5.43}$$

then $\operatorname{sgn}(AB)\left(\widetilde{f}_\nu - G\right)$ does not change sign on the real line. Since G is continuous, we conclude that

$$G(\xi) = \widetilde{f}_\nu(\xi) = \widetilde{K}(AB, \nu, \xi) \tag{5.44}$$

for all $\xi \in \mathcal{T}_{AB} \setminus \{0\}$, and we easily deduce that $G(0) = 2^{-1}(\widetilde{f}_\nu(\xi+) + \widetilde{f}_\nu(\xi-)) = \widetilde{K}(AB, \nu, 0) = 0$. As $G - \widetilde{K} \in \mathcal{H}^1(E^2) \subset \mathcal{H}^\infty(E^2)$ (by Lemma 3.8.5) and $AB \notin \mathcal{H}(E^2)$ the interpolation formula for $\mathcal{H}(E^2)$ (Theorem 3.8.1), applied to $G - \widetilde{K}$, gives

$$G(z) - \widetilde{K}(AB, \nu, z) = \beta AB(z) \tag{5.45}$$

for some β. Since $G - \widetilde{K} \in L^1(\mathbb{R}, |E(x)|^{-2}dx)$ and $AB \notin L^1(\mathbb{R}, |E(x)|^{-2}\,dx)$ (see Lemma 5.4.3 below), we conclude that $\beta = 0$, hence $G = \widetilde{K}$. □

Lemma 5.4.3. *Let $E = A - iB$ be a Hermite-Biehler function. If $AB \notin \mathcal{H}(E^2)$, then $AB \notin \mathcal{H}^1(E^2)$*

Proof. Since $AB \notin \mathcal{H}(E)$ and AB/E^2 is bounded and hence of bounded type with non-positive mean type, we have that

$$\int_{-\infty}^{\infty} \frac{|AB|^2}{|E^2|^2}\, dx = \infty. \tag{5.46}$$

Using that AB/E^2 is bounded by 1 on the real line it follows that

$$\infty = \int_{-\infty}^{\infty} \frac{|AB|^2}{|E^2|^2}\, dx \leq \int_{-\infty}^{\infty} \frac{|AB|}{|E|^2}\, dx, \tag{5.47}$$

hence $AB \notin \mathcal{H}^1(E^2)$. $\square$

Remark 5.4.4. In the recent works [11, 13], Carneiro and Littmann consider the interpolation problem for classes of even functions given by

$$f_\nu(x) = \int_0^{\infty} e^{-\lambda|x|}\, d\nu(\lambda) \qquad \text{and} \qquad g_\nu(x) = \int_0^{\infty} e^{-x^2\lambda}\, d\nu(\lambda). \tag{5.48}$$

For a given even $\mathcal{LP}$-function, they construct interpolations $z \mapsto L(F,\nu,z)$ such that

$$\operatorname{sgn}(F(x)) = \{f_\nu(x) - L(F,\nu,x)\} \qquad \text{and} \qquad \operatorname{sgn}(F(x)) = \{g_\nu(x) - L(F,\nu,x)\} \tag{5.49}$$

and

$$|f_\nu(x) - L(F,\nu,x)| = \mathcal{O}\left(\frac{F(x)}{x^2+1}\right) \qquad \text{and} \qquad |g_\nu(x) - L(F,\nu,x)| = \mathcal{O}\left(\frac{F(x)}{x^2+1}\right) \tag{5.50}$$

hold for all real x. Proceeding as in the proof of Theorem 5.4.1, we can apply these results to the even $\mathcal{LP}$-function $A_{\pi/2}B_{\pi/2}$ to obtain best approximations of f_ν and g_ν in $L^1(\mathbb{R}, |E(x)|^{-2}dx)$-norm.

5.5. Best approximations to the Poisson and conjugate Poisson kernels in $L^1(\mathbb{R},\mu_E)$-norm

We conclude the discussion on Best Approximations by presenting two special cases of best approximations in $L^1(\mathbb{R},\mu_E)$-norm. For $\lambda > 0$, we define the Poisson kernel P_λ and conjugate Poisson kernel Q_λ by

$$P_\lambda(x) = \frac{1}{\pi}\frac{\lambda}{x^2+\lambda^2} \tag{5.51}$$

and

$$Q_\lambda(x) = \frac{1}{\pi}\frac{x}{x^2+\lambda^2}. \tag{5.52}$$

For a Hermite-Biehler function E such that $E(-z) = E^*(z)$, we define the functions $K_{P_\lambda,E}, K_{Q_\lambda,E} : \mathbb{C} \to \mathbb{C}$ by

$$K_{P_\lambda,E}(z) = \frac{1}{\pi}\frac{\lambda}{z^2+\lambda^2}\left(1 - \frac{E(z)^2+E^*(z)^2}{E(i\lambda)^2+E^*(i\lambda)^2}\right) = \frac{1}{\pi}\frac{\lambda}{z^2+\lambda^2}\left(1 - \frac{A_{\pi/4}(z)B_{\pi/4}(z)}{A_{\pi/4}(i\lambda)B_{\pi/4}(i\lambda)}\right) \tag{5.53}$$

and

$$K_{Q_\lambda,E}(z) = \frac{1}{\pi}\frac{1}{z^2+\lambda^2}\left(z - i\lambda\frac{E(z)^2-E^*(z)^2}{E(i\lambda)^2-E^*(i\lambda)^2}\right) = \frac{1}{\pi}\frac{1}{z^2+\lambda^2}\left(z - i\lambda\frac{A(z)B(z)}{A(i\lambda)B(i\lambda)}\right) \tag{5.54}$$

Since these functions have been fixed up to have no singularities at $\pm i\lambda$ it follows that they are entire functions. Moreover, if E is of bounded type in the upper half-plane with mean type τ (and hence of exponential type τ by Krein's Theorem), we have that these functions are of exponential type 2τ.

These functions turn out to be best approximations of the Poisson kernel and conjugate Poisson kernel, respectively, with remarkably explicit error bounds.

Theorem 5.5.1. *Let $\lambda > 0$. Let E be a Hermite-Biehler function of bounded type in $\mathbb{C}^+$ and exponential type $\tau(E)$ with no real zeros such that $E(-z) = E^*(z)$ for all z. If $F : \mathbb{C} \to \mathbb{C}$ is an entire function of exponential type $2\tau(E)$, then*

$$\int_{-\infty}^{\infty} |P_\lambda(x) - F(x)| \frac{dx}{|E(x)|^2} \geq \frac{4}{\pi}\frac{1}{E(i\lambda)E(-i\lambda)} \arctan\left(\frac{E(-i\lambda)}{E(i\lambda)}\right). \tag{5.55}$$

Moreover, there is equality in the inequality (5.55) *if $F(z) = K_{P_\lambda,E}(z)$.*

Proof. By (5.53) we have

$$P_\lambda(x) - K_{P_\lambda,E}(x) = \frac{1}{\pi}\frac{\lambda}{x^2+\lambda^2}\frac{E(x)^2+E^*(x)^2}{E(i\lambda)^2+E^*(i\lambda)^2}, \tag{5.56}$$

hence $P_\lambda - K_{P_\lambda,E} \in L^1(\mathbb{R}, |E(x)|^{-2}\,dx)$. As $E(-z) = E^*(z)$ for all z, it follows that $E(i\lambda)$ is real valued and

$$\operatorname{sgn}(P_\lambda(x) - K_{P_\lambda,E}(x)) = \operatorname{sgn}(E(x)^2 + E^*(x)^2) = -\operatorname{sgn}\sin(2(\varphi - \pi/4)) = \operatorname{sgn}\cos 2\varphi(x) \quad (5.57)$$

for all real x. By Theorem 5.1.2 and Theorem 5.2.1 we have that

$$\int_{-\infty}^{\infty} |P_\lambda(x) - F(x)| \frac{dx}{|E(x)|^2} \geq \int_{-\infty}^{\infty} |P_\lambda(x) - K_{P_\lambda,E}(x)| \frac{dx}{|E(x)|^2} \quad (5.58)$$

for every $F \in \mathcal{A}(2\tau)$. We prove that $K_{P_\lambda,E}$ gives the desired integral value (5.55) in Lemma 5.5.4 which completes the proof.

□

Theorem 5.5.2. *Let $\lambda > 0$. Let E be a Hermite-Biehler function of bounded type in $\mathbb{C}^+$ and exponential type $\tau(E)$ with no real zeros such that $E(-z) = E^*(z)$ for all z. If $F : \mathbb{C} \to \mathbb{C}$ is an entire function of exponential type $2\tau(E)$, then*

$$\int_{-\infty}^{\infty} |Q_\lambda(x) - F(x)| \frac{dx}{|E(x)|^2} \geq \frac{4}{\pi} \frac{1}{E(i\lambda)E(-i\lambda)} \operatorname{arctanh}\left(\frac{E(-i\lambda)}{E(i\lambda)}\right). \quad (5.59)$$

Moreover, there is equality in the inequality (5.59) *if $F(z) = K_{Q_\lambda,E}(z)$.*

Proof. By (5.54) we have

$$Q_\lambda(x) - K_{Q_\lambda,E}(x) = \frac{1}{\pi} \frac{i\lambda}{x^2 + \lambda^2} \frac{E(x)^2 - E^*(x)^2}{E(i\lambda)^2 - E^*(i\lambda)^2}. \quad (5.60)$$

hence $Q_\lambda - K_{Q_\lambda,E} \in L^1(\mathbb{R}, \mu_E)$. As $E(-z) = E^*(z)$ for all z, we have that $E(i\lambda)$ is real valued and since E is Hermite-Biehler it follows that $E(i\lambda)^2 \geq E^*(i\lambda)^2$. Thus,

$$\operatorname{sgn}(Q_\lambda(x) - K_{Q_\lambda,E}(x)) = \operatorname{sgn}(i(E(x)^2 - E^*(x)^2)) = \operatorname{sgn}\sin(2\varphi(x)) \quad (5.61)$$

for all real x. By Theorem 5.1.2 and Theorem 5.2.1 we have that

$$\int_{-\infty}^{\infty} |Q_\lambda(x) - F(x)| \frac{dx}{|E(x)|^2} \geq \int_{-\infty}^{\infty} |Q_\lambda(x) - K_{Q_\lambda,E}(x)| \frac{dx}{|E(x)|^2} \quad (5.62)$$

for every $F \in \mathcal{A}(2\tau)$. We prove that $K_{Q_\lambda,E}$ gives the desired integral value (5.59) in Lemma 5.5.5 which completes the proof. $\square$

The following lemma is used to show that $K_{P_\lambda,E}$ and $K_{Q_\lambda,E}$ give the desired integral values (5.55) and (5.59).

Lemma 5.5.3. *Let $\lambda > 0$ and $k \in \mathbb{Z}$. Let E be a Hermite-Biehler function. We have*

$$\int_{-\infty}^{\infty} \left(\frac{E^*(x)}{E(x)}\right)^k \frac{1}{x^2+\lambda^2}\, dx = \begin{cases} \frac{\pi}{\lambda}\left(\frac{E^*(i\lambda)}{E(i\lambda)}\right)^k & \text{if } k \geq 0 \\ \frac{\pi}{\lambda}\left(\frac{E^*(-i\lambda)}{E(-i\lambda)}\right)^k & \text{if } k < 0 \end{cases} \tag{5.63}$$

Moreover, if $E(-z) = E^*(z)$ for all z, then for $k \geq 0$ we have

$$\int_{-\infty}^{\infty} \left(\frac{E^*(x)}{E(x)}\right)^k \frac{1}{x^2+\lambda^2}\, dx = \int_{-\infty}^{\infty} \left(\frac{E^*(x)}{E(x)}\right)^{-k} \frac{1}{x^2+\lambda^2}\, dx = \frac{\pi}{\lambda}\left(\frac{E(-i\lambda)}{E(i\lambda)}\right)^k. \tag{5.64}$$

Proof. Since E is Hermite-Biehler, we have E^*/E is analytic and bounded by 1 in the upper half-plane and has continuous extension to the closed half-plane. Hence, for $k \geq 0$ closing the contour in the upper half-plane and applying the residue theorem gives (5.63). Similarly, for $k < 0$ we have that E/E^* is analytic and bounded by 1 in the lower half-plane, so closing the contour in the lower half-plane gives 5.63. Equation (5.64) follows directly.

$\square$

Lemma 5.5.4. *Let $\lambda > 0$. Let E be a Hermite-Biehler function with no real zeros such that $E(-z) = E^*(z)$ for all z. Then*

$$\int_{-\infty}^{\infty} |P_\lambda(x) - K_{P_\lambda,E}(x)| \frac{dx}{|E(x)|^2} = \frac{4}{\pi} \frac{1}{E(i\lambda)E(-i\lambda)} \arctan\left(\frac{E(-i\lambda)}{E(i\lambda)}\right) \tag{5.65}$$

Proof. As in the the proof of Theorem 5.55 we have that $\operatorname{sgn}(P_\lambda(x) - K_{P_\lambda,E}(x)) = \operatorname{sgn}\cos 2\varphi(x)$ for all real x. It follows that

$$\int_{-\infty}^{\infty} |P_\lambda(x) - K_{P_\lambda,E}(x)| \frac{dx}{|E(x)|^2} = \int_{-\infty}^{\infty} (P_\lambda(x) - K_{P_\lambda,E}(x)) \operatorname{sgn}\cos 2\varphi(x) \frac{dx}{|E(x)|^2} \tag{5.66}$$

Expanding $\operatorname{sgn}\cos 2\varphi(x)$ into its Fourier series and interchanging integration and summation (which is valid since the partial sums are uniformly bounded) gives

$$\begin{aligned}
&\int_{-\infty}^{\infty}(P_\lambda(x) - K_{P_\lambda,E}(x))\operatorname{sgn}\cos 2\varphi(x)\,\frac{dx}{|E(x)|^2} \\
&\quad = \frac{2}{\pi}\sum_{n\in\mathbb{Z}}\frac{(-1)^n}{2n+1}\int_{-\infty}^{\infty}(P_\lambda(x) - K_{P_\lambda,E}(x))\left(e^{2i\varphi(x)}\right)^{2n+1}\frac{dx}{|E(x)|^2} \\
&\quad = \frac{2}{\pi^2}\frac{\lambda}{E(i\lambda)^2+E(-i\lambda)^2}\sum_{n\in\mathbb{Z}}\frac{(-1)^n}{2n+1}\int_{-\infty}^{\infty}\frac{E(x)^2+E^*(x)^2}{x^2+\lambda^2}\left(\frac{E^*(x)}{E(x)}\right)^{2n+1}\frac{dx}{|E(x)|^2}
\end{aligned} \tag{5.67}$$

It remains to study the integrals on the right hand side of (5.67). For $n \geq 0$, we write $|E(x)|^2 = E(x)E^*(x)$ and apply Lemma 5.5.3 to obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty}\frac{E(x)^2+E^*(x)^2}{x^2+\lambda^2}\left(\frac{E^*(x)}{E(x)}\right)^{2n+1}\frac{dx}{|E(x)|^2} \\
&\qquad = \int_{-\infty}^{\infty}\frac{1}{x^2+\lambda^2}\left(1+\frac{E^*(x)^2}{E(x)^2}\right)\left(\frac{E^*(x)}{E(x)}\right)^{2n}dx \\
&\qquad = \int_{-\infty}^{\infty}\left(\frac{E^*(x)}{E(x)}\right)^{2n}\frac{1}{x^2+\lambda^2}\,dx + \int_{-\infty}^{\infty}\left(\frac{E^*(x)}{E(x)}\right)^{2n+2}\frac{1}{x^2+\lambda^2}\,dx \\
&\qquad = \frac{\pi}{\lambda}\left(\frac{E(-i\lambda)}{E(i\lambda)}\right)^{2n} + \frac{\pi}{\lambda}\left(\frac{E(-i\lambda)}{E(i\lambda)}\right)^{2n+2} \\
&\qquad = \frac{\pi}{\lambda}\left(\frac{E(-i\lambda)}{E(i\lambda)}\right)^{2n}\frac{E(i\lambda)^2+E(-i\lambda)^2}{E(i\lambda)^2}.
\end{aligned} \tag{5.68}$$

Similarly, for $n < 0$ we have

$$\begin{aligned}
&\int_{-\infty}^{\infty}\frac{E(x)^2+E^*(x)^2}{x^2+\lambda^2}\left(\frac{E^*(x)}{E(x)}\right)^{2n+1}\frac{dx}{|E(x)|^2} \\
&\qquad = \int_{-\infty}^{\infty}\frac{1}{x^2+\lambda^2}\left(\frac{E(x)^2}{E^*(x)^2}+1\right)\left(\frac{E^*(x)}{E(x)}\right)^{2n+2}dx \\
&\qquad = \int_{-\infty}^{\infty}\frac{1}{x^2+\lambda^2}\left(\frac{E^*(x)}{E(x)}\right)^{2n}dx + \int_{-\infty}^{\infty}\frac{1}{x^2+\lambda^2}\left(\frac{E^*(x)}{E(x)}\right)^{2n+2}dx \\
&\qquad = \frac{\pi}{\lambda}\left(\frac{E(i\lambda)}{E(-i\lambda)}\right)^{2n} + \frac{\pi}{\lambda}\left(\frac{E(i\lambda)}{E(-i\lambda)}\right)^{2n+2} \\
&\qquad = \frac{\pi}{\lambda}\left(\frac{E(i\lambda)}{E(-i\lambda)}\right)^{2n}\frac{E(i\lambda)^2+E(-i\lambda)^2}{E(-i\lambda)^2}.
\end{aligned} \tag{5.69}$$

Inserting these into (5.67) and applying the identities

$$\sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} z^{2n} = \frac{\arctan(z)}{z} \quad \text{for} \quad |z| < 1 \tag{5.70}$$

and

$$\sum_{n=-\infty}^{-1} \frac{(-1)^n}{2n+1} z^{2n} = \frac{\arctan(1/z)}{z} \quad \text{for} \quad |z| > 1 \tag{5.71}$$

gives

$$\begin{aligned}
&\int_{-\infty}^{\infty} (P_\lambda(x) - K_{P_\lambda,E}(x)) \operatorname{sgn} \cos 2\varphi(x) \, \frac{dx}{|E(x)|^2} \\
&\quad = \frac{2}{\pi} \left(\frac{1}{E(i\lambda)^2} \sum_{n=0}^{\infty} \frac{(-1)^n}{2n+1} \left(\frac{E(-i\lambda)}{E(i\lambda)} \right)^{2n} + \frac{1}{E(-i\lambda)^2} \sum_{n=-\infty}^{-1} \frac{(-1)^n}{2n+1} \left(\frac{E(i\lambda)}{E(-i\lambda)} \right)^{2n} \right) \\
&\quad = \frac{2}{\pi} \left(\frac{1}{E(i\lambda)^2} \frac{E(i\lambda)}{E(-i\lambda)} \arctan \left(\frac{E(-i\lambda)}{E(i\lambda)} \right) + \frac{1}{E(-i\lambda)^2} \frac{E(-i\lambda)}{E(i\lambda)} \arctan \left(\frac{E(-i\lambda)}{E(i\lambda)} \right) \right) \\
&\quad = \frac{4}{\pi} \frac{1}{E(i\lambda)E(-i\lambda)} \arctan \left(\frac{E(-i\lambda)}{E(i\lambda)} \right)
\end{aligned} \tag{5.72}$$

which is (5.65).

□

Lemma 5.5.5. *Let $\lambda > 0$. Let E be a Hermite-Biehler function with no real zeros such that $E(-z) = E^*(z)$ for all z. Then*

$$\int_{-\infty}^{\infty} |Q_\lambda(x) - K_{Q_\lambda,E}(x)| \, \frac{dx}{|E(x)|^2} = \frac{4}{\pi} \frac{1}{E(i\lambda)E(-i\lambda)} \operatorname{arctanh} \left(\frac{E(-i\lambda)}{E(i\lambda)} \right) \tag{5.73}$$

Proof. The proof is nearly identical to Lemma 5.59. We sketch out the minor differences.

As in the proof of Theorem 5.55 we have that $\operatorname{sgn}(Q_\lambda(x) - K_{Q_\lambda,E}(x)) = \operatorname{sgn} \sin 2\varphi(x)$ for all real x. It follows that

$$\int_{-\infty}^{\infty} |Q_\lambda(x) - K_{Q_\lambda,E}(x)| \, \frac{dx}{|E(x)|^2} = \int_{-\infty}^{\infty} (Q_\lambda(x) - K_{Q_\lambda,E}(x)) \operatorname{sgn} \sin 2\varphi(x) \, \frac{dx}{|E(x)|^2} \tag{5.74}$$

Expanding $\operatorname{sgn}\sin 2\varphi(x)$ into its Fourier series and interchanging integration and summation gives

$$\begin{aligned}\int_{-\infty}^{\infty} (Q_\lambda(x) - K_{Q_\lambda,E}(x))\operatorname{sgn}\sin 2\varphi(x) \, \frac{dx}{|E(x)|^2} \\ = \frac{2}{\pi^2} \frac{\lambda}{E(i\lambda)^2 - E(-i\lambda)^2} \sum_{n\in\mathbb{Z}} \frac{1}{2n+1} \int_{-\infty}^{\infty} \frac{E(x)^2 - E^*(x)^2}{x^2+\lambda^2} \left(\frac{E^*(x)}{E(x)}\right)^{2n+1} \frac{dx}{|E(x)|^2}.\end{aligned} \tag{5.75}$$

As in Lemma 5.5.4 we compute the integrals on the right hand side of (5.75). For $n \geq 0$ we have

$$\int_{-\infty}^{\infty} \frac{E(x)^2 - E^*(x)^2}{x^2+\lambda^2} \left(\frac{E^*(x)}{E(x)}\right)^{2n+1} \frac{dx}{|E(x)|^2} = \frac{\pi}{\lambda} \left(\frac{E(-i\lambda)}{E(i\lambda)}\right)^{2n} \frac{E(i\lambda)^2 - E(-i\lambda)^2}{E(i\lambda)^2} \tag{5.76}$$

and for $n < 0$ we have

$$\int_{-\infty}^{\infty} \frac{E(x)^2 - E^*(x)^2}{x^2+\lambda^2} \left(\frac{E^*(x)}{E(x)}\right)^{2n+1} \frac{dx}{|E(x)|^2} = -\frac{\pi}{\lambda} \left(\frac{E(i\lambda)}{E(-i\lambda)}\right)^{2n} \frac{E(i\lambda)^2 - E(-i\lambda)^2}{E(-i\lambda)^2}. \tag{5.77}$$

Inserting these into (5.75) and applying the identities

$$\sum_{n=0}^{\infty} \frac{1}{2n+1} z^{2n} = \frac{\operatorname{arctanh}(z)}{z} \quad \text{for} \quad |z| < 1 \tag{5.78}$$

and

$$\sum_{n=-\infty}^{-1} \frac{1}{2n+1} z^{2n} = -\frac{\operatorname{arctanh}(1/z)}{z} \quad \text{for} \quad |z| > 1 \tag{5.79}$$

gives the desired result. $\square$

6. EXTREMAL PROBLEMS WITH VANISHING CONDITION

6.1. Introduction

In this chapter, we consider the following one-sided $L^1(\mathbb{R}, dx)$-approximation problem with vanishing condition.

Problem 6.1.1. *Given $f : \mathbb{R} \to \mathbb{R}$, $\delta > 0$, and $\alpha = ib$ with $b > 0$, can we find an entire function $F : \mathbb{C} \to \mathbb{C}$ of exponential type $2\pi\delta$ that satisfies $F(x) \geq f(x)$ for all real x and $F(\alpha) = 0$ such that the integral*

$$\int_{-\infty}^{\infty} \{F(x) - f(x)\}\, dx \tag{6.1}$$

is minimized?

Remark 6.1.2. Let F be a real entire function, i.e., $F^* = F$. If $F(\alpha) = 0$, then $F(\overline{\alpha}) = \overline{F(\alpha)} = 0$, hence vanishing at α implies vanishing at $\overline{\alpha}$.

We approach this problem by modifying the function f and encoding the vanishing condition into a new measure (so that the vanishing condition may be dropped). This modification is done by noticing that any majorant $F \in \mathcal{A}(\delta)$ of f that vanishes at $ib \in i\mathbb{R}$ is necessarily of the form

$$F(z) = F_b(z)(z^2 + b^2) \tag{6.2}$$

where $F_b \in \mathcal{A}(\delta)$ is a majorant of $f_b(x) = f(x)(x^2 + b^2)^{-1}$. Hence, we seek to find $G^+ \in \mathcal{A}(\delta)$ such that $G^+(x) \geq f(x)(x^2 + b^2)^{-1}$ for all real x, and the integral

$$\int_{-\infty}^{\infty} \left\{ G^+(x) - \frac{f(x)}{x^2 + b^2} \right\} (x^2 + b^2)\, dx \tag{6.3}$$

is minimal.

In Sections 6.2, 6.3, and 6.4 we solve the more general weighted one-sided $L^1(\mathbb{R}, \mu)$ approximation problem for $x \mapsto f(x)(x^2 + b^2)^{-1}$ when f is taken to be the Heaviside function (equivalently the signum function), a monomial, or the Poisson kernel and μ_E is a Hermite-Biehler weight. In

Section 6.5 we define a Hermite-Biehler function E so that $\mathcal{A}^2(\delta, (x^2+b^2)\,dx) = \mathcal{H}(E)$ isometrically and explore various properties of this weighted Paley-Wiener space. Lastly, in Section 6.6 we combine these results to solve the vanishing problem for the signum function, monomials, and the Poisson kernel.

Before we state the results of this chapter we briefly review some of the notation from Chapter 3. Recall an entire function E is called a Hermite-Biehler function if

$$|E^*(z)| < |E(z)| \tag{6.4}$$

for every $z \in \mathbb{C}^+$, where $E^*(z) = \overline{E(\overline{z})}$. For a given Hermite-Biehler function E and $\alpha \in \mathbb{R}$, we define the functions

$$A_\alpha(z) = \frac{1}{2}\left(E(z)e^{i\alpha} + E^*(z)e^{-i\alpha}\right) \text{ and } B_\alpha(z) = \frac{i}{2}\left(E(z)e^{i\alpha} - E^*(z)e^{-i\alpha}\right). \tag{6.5}$$

For $\alpha = 0$, we set $A = A_0$ and $B = B_0$. If E is a Hermite-Biehler function, then the de Branges space $\mathcal{H}^2(E)$ is is a reproducing kernel Hilbert space (RKHS). The reproducing kernel is given by

$$K_E(w,z) = \frac{E(z)E^*(\overline{w}) - E^*(z)E(\overline{w})}{2\pi i(\overline{w}-z)} = \frac{B(z)A(\overline{w}) - A(z)B(\overline{w})}{\pi(z-\overline{w})} \tag{6.6}$$

for $z \neq \overline{w}$ and when $z = \overline{w}$ we find that

$$K_E(\overline{z},z) = \frac{E(z)\overline{E'(z)} - E^*(z)E'(z)}{2\pi i} = \frac{B'(z)A(z) - A'(z)B(z)}{\pi}. \tag{6.7}$$

6.2. One-sided approximations to the truncated Poisson kernel in $L^1(\mathbb{R}, \mu_E)$-norm

For $a > 0$, we define $t_a : \mathbb{R} \to \mathbb{R}$ by

$$t_a(x) = \begin{cases} \frac{1}{x^2+a^2} & \text{if } x > 0, \\ 0 & \text{else.} \end{cases} \tag{6.8}$$

Theorem 6.2.1. *Let E be a Hermite-Biehler function of bounded type in the upper half-plane and exponential type $\tau(E)$ with no real zeros such that $E(-z) = E^*(z)$ for all z. If $F^+, F^- \in \mathcal{A}(2\tau)$*

with

$$F^-(x) \le t_a(x) \le F^+(x) \tag{6.9}$$

for all real x, then

$$\int_{\mathbb{R}} (F^+(x) - F^-(x)) \frac{dx}{|E(x)|^2} \ge \frac{1}{a^2 K(0,0)}, \tag{6.10}$$

and there exist functions $T_a^\pm \in \mathcal{A}(2\tau)$ satisfying (6.9) *such that there is equality in* (6.10) *for $F^+ = T_a^+$ and $F^- = T_a^-$.*

Proof of Theorem 6.2.1. Inequality (3.8) implies that B has only real zeros, and since E has no real zeros, it follows from Lemma 3.5.4 that the zeros of B are simple. Since, E is of bounded type and $E^*(z) = E(-z)$, Lemmas 3.3.11 and 4.2.8 give that B is an odd $\mathcal{LP}$ function with a simple zero at the origin. Evidently, B^2 is an even $\mathcal{LP}$ function that has a double zero at the origin, and the results of Section 4.7 are applicable.

We define the entire functions T_a^+ and T_a^- by

$$T_a^+(z) = a^{-1} M^+(B^2, a, z) \tag{6.11}$$

$$T_a^-(z) = a^{-1} M^-(B^2, a, z) \tag{6.12}$$

with M^- and M^+ as in (4.99) and (4.100) (see Section 4.7). Since E has exponential type τ, it follows that B is also of exponential type τ and (4.55) implies that T_a^+ and T_a^- are of exponential type 2τ. Since $B^2 \ge 0$ on $\mathbb{R}$, inequality (4.103) implies

$$T_a^+(x) \ge t_a(x) \tag{6.13}$$

for all real x, and (4.102) implies that

$$T_a^+(\xi) = t_a(\xi+) \tag{6.14}$$

for all ξ with $B(\xi) = 0$. Since B^2/E^2 is bounded on $\mathbb{R}$, it follows from (4.117) that

$$\int_{-\infty}^{\infty} \frac{T_a^+(x) - t_a(x)}{|E(x)|^2} dx < \infty. \tag{6.15}$$

A similar argument gives the same statement for $t_a - T_a^-$. Since $T_a^- \leq t_a \leq T_a^+$ we obtain that $|T_a^+ - T_a^-|$ is integrable with respect to $\mu_E = |E(x)|^{-2}dx$. It follows from Lemma 3.7.1 that there exists $U \in \mathcal{H}(E)$ such that

$$T_a^+ - T_a^- = UU^*. \tag{6.16}$$

We prove next the optimality of T_a^+. Let F be a function of type 2τ with $F \geq t_a$ on $\mathbb{R}$. We may assume that

$$\int_{-\infty}^{\infty} \frac{F(x) - t_a(x)}{|E(x)|^2} dx < \infty, \tag{6.17}$$

(since otherwise there is nothing to show). The inequality $T_a^- \leq t_a \leq F$ gives

$$|F(x) - T_a^-(x)| \leq (F(x) - t_a(x)) + (t_a(x) - T_a^-(x)) \tag{6.18}$$

and hence $F - T_a^-$ is an entire function of exponential type 2τ that is integrable with respect to μ_E. Evidently, $F - T_a^- \geq F - t_a \geq 0$. Applying Lemma 3.7.1 again implies that there exists $V \in \mathcal{H}(E)$ such that

$$F - T_a^- = VV^*, \tag{6.19}$$

It follows from (6.16) and (6.19) that

$$F - T_a^+ = VV^* - UU^*. \tag{6.20}$$

An application of Theorem 3.5.5 to U and V together with $T_a^+(\xi) = t_a(\xi+)$ for all $\xi \in \mathbb{R}$ with $B(\xi) = 0$ implies

$$\int_{-\infty}^{\infty} \frac{F(x) - T_a^+(x)}{|E(x)|^2} dx = \sum_{B(\xi)=0} \frac{F(\xi) - t_a(\xi)}{K(\xi,\xi)} \geq 0, \tag{6.21}$$

hence T_a^+ is extremal.

An analogous calculation (which we omit) shows that T_a^- is an extremal minorant. It remains to prove that

$$\int_{-\infty}^{\infty} (T_a^+(x) - T_a^-(x)) \frac{dx}{|E(x)|^2} = \frac{1}{a^2 K(0,0)}. \tag{6.22}$$

It follows from (6.16) and Theorem 3.5.5 that

$$\int_{-\infty}^{\infty} (T_a^+(x) - T_a^-(x)) \frac{dx}{|E(x)|^2} = \sum_{B(\xi)=0} \frac{T_a^+(\xi) - T_a^-(\xi)}{K(\xi,\xi)}. \tag{6.23}$$

The only non-zero summand is the term for $\xi = 0$. Since $T_a^+(0) - T_a^-(0) = t_a(0+) - t_a(0-) = 1/a^2$, the proof is complete. $\square$

6.3. One-sided approximations to $x^n(x^2+\lambda^2)^{-1}$ in $L^1(\mathbb{R}, \mu_E)$-norm

As with the problem of best approximations, the optimal one-sided approximations to the Poisson kernel P_λ and conjugate Poisson kernel Q_λ are explicit and we obtain remarkably simple error bounds. In fact, this approach extends to a slightly larger class of functions. For $n \in \mathbb{N}_0$ and $\lambda > 0$, define $f_{n,\lambda} : \mathbb{R} \to \mathbb{R}$ by

$$f_{n,\lambda}(x) = \frac{x^n}{x^2 + \lambda^2}. \tag{6.24}$$

Notice that for $n = 0$ and $n = 1$, the functions $f_{1,\lambda}$ and $f_{2,\lambda}$ are essentially the Poisson kernel P_λ and conjugate Poisson kernel Q_λ, respectively.

For a positive λ and a Hermite-Biehler function E such that $E^*(z) = E(-z)$ for all complex z, we define the functions $M_{f_{n,\lambda},E}$, $L_{f_{n,\lambda},E} : \mathbb{C} \to \mathbb{C}$ by the following:

For $n \equiv 0 \bmod 4$,

$$M_{f_{n,\lambda},E}(z) = \frac{1}{z^2+\lambda^2}\left(z^n - (i\lambda)^n \left(\frac{B(z)}{B(i\lambda)}\right)^2\right), \tag{6.25}$$

$$L_{f_{n,\lambda},E}(z) = \frac{1}{z^2+\lambda^2}\left(z^n - (i\lambda)^n \left(\frac{A(z)}{A(i\lambda)}\right)^2\right); \tag{6.26}$$

and for $n \equiv 1 \bmod 4$

$$M_{f_{n,\lambda},E}(z) = \frac{1}{z^2+\lambda^2}\left(z^n - i\frac{(i\lambda)^n}{2} \left(i\frac{B(z)}{B(i\lambda)} - \frac{A(z)}{A(i\lambda)}\right)^2\right), \tag{6.27}$$

$$L_{f_{n,\lambda},E}(z) = -M_{f_{n,\lambda},E}(-z). \tag{6.28}$$

Notice that the functions $f_{n,\lambda}$ satisfy the following recursive relationship

$$f_{n,\lambda}(x) = x^{n-2} - \lambda^2 f_{n-2,\lambda}(x). \tag{6.29}$$

Using this we can easily define $M_{f_{n,\lambda},E}$ and $L_{f_{n,\lambda},E}$ for the remaining cases.
For $n \equiv 2 \bmod 4$,

$$M_{f_{n,\lambda},E}(z) = z^{n-2} - \lambda^2 L_{n-2,\lambda,E}(z) = \frac{1}{z^2+\lambda^2}\left(z^n - (i\lambda)^n \left(\frac{A(z)}{A(i\lambda)}\right)^2\right), \tag{6.30}$$

$$L_{f_{n,\lambda},E}(z) = z^{n-2} - \lambda^2 M_{n-2,\lambda,E}(z) = \frac{1}{z^2+\lambda^2}\left(z^n - (i\lambda)^n \left(\frac{B(z)}{B(i\lambda)}\right)^2\right); \tag{6.31}$$

and for $n \equiv 3 \bmod 4$,

$$M_{f_{n,\lambda},E}(z) = z^{n-2} - \lambda^2 L_{n-2,\lambda,E}(z) = \frac{1}{z^2+\lambda^2}\left(-z^n - i\frac{(i\lambda)^n}{2}\left(i\frac{B(z)}{B(i\lambda)} + \frac{A(z)}{A(i\lambda)}\right)^2\right), \tag{6.32}$$

$$L_{f_{n,\lambda},E}(z) = z^{n-2} - \lambda^2 M_{n-2,\lambda,E}(z) = -M_{f_{n,\lambda},E}(-z). \tag{6.33}$$

Since these functions have been fixed up to have no singularities at $\pm i\lambda$ it follows that they are entire functions. Moreover, if E is of bounded type in the upper half-plane with mean type τ (and hence of exponential type τ by Krein's Theorem), we have that these functions are of exponential type 2τ.

The results for $f_{n,\lambda}$ (and hence for the Poisson kernel and conjugate Poisson kernel) are the following:

Theorem 6.3.1. *Let $\lambda > 0$ and $n \in \mathbb{N}_0$ such that $n \equiv 0 \bmod 4$. Let E be a Hermite-Biehler function of bounded type in $\mathbb{C}^+$ and exponential type $\tau(E)$ with no real zeros such that $E^*(z) = E(-z)$ for all complex z. The following properties hold:*

1. *Assume $B \notin \mathcal{H}(E)$. If $M : \mathbb{C} \to \mathbb{C}$ is an entire function of exponential type $2\tau(E)$ with $M(x) \geq f_{n,\lambda}(x)$ for all real x, then*

$$\int_{-\infty}^{\infty} (M(x) - f_{n,\lambda}(x)) \frac{dx}{|E(x)|^2} \geq \frac{2\pi\lambda^{n-1}}{E(i\lambda)(E(i\lambda) - E(-i\lambda))}. \tag{6.34}$$

Moreover, there is equality in the inequality (6.34) if $M(z) = M_{f_{n,\lambda},E}(z)$.

2. *Assume $A \notin \mathcal{H}(E)$. If $L : \mathbb{C} \to \mathbb{C}$ is an entire function of exponential type $2\tau(E)$ with $P_\lambda(x) \geq L(x)$ for all real x, then*

$$\int_{-\infty}^{\infty} (f_{n,\lambda}(x) - L(x)) \frac{dx}{|E(x)|^2} \geq \frac{2\pi\lambda^{n-1}}{E(i\lambda)(E(i\lambda) + E(-i\lambda))}. \tag{6.35}$$

Moreover, there is equality in the inequality (6.35) if $L(z) = L_{f_{n,\lambda},E}(z)$.

Using the recursive relationships (6.29) and (6.31) the case when $n \equiv 2 \bmod 4$ follows directly from Theorem 6.3.1.

Theorem 6.3.2. *Let $\lambda > 0$ and $n \in \mathbb{N}_0$ such that $n \equiv 2 \bmod 4$. Let E be a Hermite-Biehler function of bounded type in $\mathbb{C}^+$ and exponential type $\tau(E)$ with no real zeros such that $E^*(z) = E(-z)$ for all complex z. The following properties hold:*

1. *Assume $A \notin \mathcal{H}(E)$. If $M : \mathbb{C} \to \mathbb{C}$ is an entire function of exponential type $2\tau(E)$ with $M(x) \geq f_{n,\lambda}(x)$ for all real x, then*

$$\int_{-\infty}^{\infty} (M(x) - f_{n,\lambda}(x)) \frac{dx}{|E(x)|^2} \geq \frac{2\pi\lambda^{n-1}}{E(i\lambda)(E(i\lambda) + E(-i\lambda))}. \tag{6.36}$$

Moreover, there is equality in the inequality (6.36) if $M(z) = M_{f_{n,\lambda},E}(z)$.

2. *Assume $B \notin \mathcal{H}(E)$. If $L : \mathbb{C} \to \mathbb{C}$ is an entire function of exponential type $2\tau(E)$ with $P_\lambda(x) \geq L(x)$ for all real x, then*

$$\int_{-\infty}^{\infty} (f_{n,\lambda}(x) - L(x)) \frac{dx}{|E(x)|^2} \geq \frac{2\pi\lambda^{n-1}}{E(i\lambda)(E(i\lambda) - E(-i\lambda))}. \tag{6.37}$$

Moreover, there is equality in the inequality (6.37) if $L(z) = L_{f_{n,\lambda},E}(z)$.

Theorem 6.3.3. *Let $\lambda > 0$ and n an odd positive integer. Let E be a Hermite-Biehler function of bounded type in $\mathbb{C}^+$ and exponential type $\tau(E)$ with no real zeros such that $E^*(z) = E(-z)$ for all complex z. For $\alpha_\lambda = \arg(A(i\lambda) + B(i\lambda))$, the following properties hold:*

1. *Assume $B_{\alpha_\lambda} \notin \mathcal{H}(E)$. If $M : \mathbb{C} \to \mathbb{C}$ is an entire function of exponential type $2\tau(E)$ with $M(x) \geq f_{n,\lambda}(x)$ for all real x, then*

$$\int_{-\infty}^{\infty} (M(x) - f_{n,\lambda}(x)) \frac{dx}{|E(x)|^2} \geq \frac{2\pi\lambda^{n-1}}{E^2(i\lambda) - E^2(-i\lambda)}. \tag{6.38}$$

   Moreover, there is equality in the inequality (6.38) *if $M(z) = M_{f_{n,\lambda},E}(z)$.*

2. *Assume $B_{\alpha_\lambda} \notin \mathcal{H}(E)$. If $L : \mathbb{C} \to \mathbb{C}$ is an entire function of exponential type $2\tau(E)$ with $f_{n,\lambda}(x) \geq L(x)$ for all real x, then*

$$\int_{-\infty}^{\infty} (f_{n,\lambda}(x) - L(x)) \frac{dx}{|E(x)|^2} \geq \frac{2\pi\lambda^{n-1}}{E^2(i\lambda) - E^2(-i\lambda)}. \tag{6.39}$$

   Moreover, there is equality in the inequality (6.39) *if $L(z) = L_{f_{n,\lambda},E}(z)$.*

Remark 6.3.4. In the above results we are able to give explicit $L^1(\mathbb{R}, |E(x)|^{-2}dx)$-errors of the difference between $f_{n,\lambda}$ and its majorant/minorant. It is worth mentioning that if we look at the integral value of the difference between majorant and minorant (as in Theorem 6.2.1), then for all $n \in \mathbb{N}_0$ the inequalities (6.34), (6.35),(6.36), (6.37), (6.38), and (6.39) read

$$\int_{\infty}^{\infty} (M(x) - L(x)) \frac{dx}{|E(x)|^2} \geq \frac{4\pi\lambda^{n-1}}{E^2(i\lambda) - E^2(-i\lambda)}. \tag{6.40}$$

Remark 6.3.5. The results for the conjugate Poisson kernel (i.e., $f_{1,\lambda}$) are especially interesting since they are the first for an odd continuous function in de Branges spaces. In fact, very little is known about optimal majorants and minorants for continuous odd functions even with respect to the Lebesgue measure. Using the results for the conjugate Poisson kernel, it may be possible to construct good, not necessarily optimal, majorants and minorants for other continuous odd functions.

In an effort to better understand extremal problems for other continuous odd functions it is instructive to consider the special case of the previous theorem for the Lebesgue measure. This is done by applying the previous theorem with the Hermite-Biehler function $E_\pi(z) = e^{-\pi i z}$. The needed assumptions for E_π are verified directly. By letting $\alpha_\lambda = \frac{1}{\pi}\arctan(\tanh(\pi\lambda))$, we see that $Q_\lambda^+(\gamma) = Q_\lambda(\gamma)$ for any $\gamma = \alpha_\lambda + \mathbb{Z}$. It follows that the extremal majorant for Q_λ of type 2π with respect to the Lebesgue measure can also be written as

$$Q_\lambda^+(z) = \frac{\sin^2(\pi(z-\alpha_\lambda))}{\pi^2} \sum_{n=-\infty}^{\infty} \left(\frac{Q_\lambda(n+\alpha_\lambda)}{(z-(n+\alpha_\lambda))^2} + \frac{Q_\lambda'(n+\alpha_\lambda)}{z-(n+\alpha_\lambda)} \right). \tag{6.41}$$

We turn to the proof of Theorem 6.3.1 (The case when $n \equiv 0 \bmod 4$).

Proof of Theorem 6.3.1. We divide the proof into two pieces. First, we show that the functions $M_{f_{n,\lambda},E}$ and $L_{f_{n,\lambda},E}$ satisfy

$$L_{f_{n,\lambda},E}(x) \le f_{n,\lambda}(x) \le M_{f_{n,\lambda},E}(x) \tag{6.42}$$

for all real x and give equality in (6.34) and (6.35), respectively.

Since $E^*(z) = E(-z)$ for all complex z, we have that $A(i\lambda), iB(i\lambda) \in \mathbb{R}$. It follows that

$$M_{f_{n,\lambda},E}(x) - f_{n,\lambda}(x) = -\frac{\lambda^n}{x^2+\lambda^2}\frac{B^2(x)}{B^2(i\lambda)} \ge 0 \tag{6.43}$$

and

$$f_{n,\lambda} - L_{f_{n,\lambda},E}(x) = \frac{\lambda^n}{x^2+\lambda^2}\frac{A^2(x)}{A^2(i\lambda)} \ge 0 \tag{6.44}$$

for all real x, which gives (6.42).

To show that $M_{f_{n,\lambda},E}$ and $L_{f_{n,\lambda},E}$ give equality in (6.34) and (6.35), respectively, we write $B = i/2(E - E^*)$ and $|E(x)|^2 = EE^*$, expand the square, and apply Lemma 5.5.3 to obtain

$$\begin{aligned}
\int_{-\infty}^{\infty} (M_{f_{n,\lambda},E}(x) - f_{n,\lambda}(x)) \frac{dx}{|E(x)|^2} &= -\frac{\lambda^n}{B^2(i\lambda)} \int_{-\infty}^{\infty} \frac{B^2(x)}{|E(x)|^2}\frac{1}{x^2+\lambda^2}\,dx \\
&= \frac{\lambda^n}{4B^2(i\lambda)} \int_{-\infty}^{\infty} \left(\frac{E(x)}{E^*(x)} + \frac{E^*(x)}{E(x)} - 2 \right) \frac{1}{x^2+\lambda^2}\,dx \\
&= \frac{1}{2}\frac{\pi\lambda^{n-1}}{B^2(i\lambda)} \left(\frac{E(-i\lambda)}{E(i\lambda)} - 1 \right) \\
&= \frac{2\pi\lambda^{n-1}}{E(i\lambda)(E(i\lambda) - E(-i\lambda))}.
\end{aligned} \tag{6.45}$$

In a similar way expanding A gives

$$\int_{-\infty}^{\infty}(f_{n,\lambda}(x)-L_{f_{n,\lambda},E}(x))\frac{dx}{|E(x)|^2}=\frac{1}{2}\frac{\pi\lambda^{n-1}}{A^2(i\lambda)}\left(\frac{E(-i\lambda)}{E(i\lambda)}+1\right)=\frac{2\pi\lambda^{n-1}}{E(i\lambda)(E(i\lambda)+E(-i\lambda))}. \tag{6.46}$$

Next we prove the optimality of $M_{f_{n,\lambda},E}$ and $L_{f_{n,\lambda},E}$. Let M be a real entire function of exponential type $2\tau(E)$ such that $M(x)\geq f_{n,\lambda}(x)$ for all real x. We may assume that

$$\int_{-\infty}^{\infty}(M(x)-f_{n,\lambda}(x))\frac{dx}{|E(x)|^2}<\infty. \tag{6.47}$$

Since $M_{f_{n,\lambda},E}-L_{f_{n,\lambda},E}$ and $M-L_{f_{n,\lambda},E}$ are entire functions of exponential type $2\tau(E)$ that are nonnegative on the real line and integrable with respect to $\mu_E=|E(x)|^{-2}\,dx$, it follows from Lemma 3.7.1 that there exists $U,V\in\mathcal{H}(E)$ such that

$$M_{f_{n,\lambda},E}-L_{f_{n,\lambda},E}=UU^* \tag{6.48}$$

$$M-L_{f_{n,\lambda},E}=VV^* \tag{6.49}$$

hence

$$M(z)-M_{f_{n,\lambda},E}(z)=V(z)V^*(z)-U(z)U^*(z) \tag{6.50}$$

for all $z\in\mathbb{C}$. An application of Theorem 3.5.5 with $\alpha=0$ applied to U and V together with the fact that $M_{f_{n,\lambda},E}(\gamma)=f_{n,\lambda}(\gamma)$ for all $\gamma\in\mathbb{R}$ with $B(\gamma)=0$ implies

$$\int_{-\infty}^{\infty}\frac{M(x)-M_{f_{n,\lambda},E}(x)}{|E(x)|^2}\,dx=\sum_{B(\gamma)=0}\frac{M(\gamma)-f_{n,\lambda}(\gamma)}{K(\gamma,\gamma)}\geq 0, \tag{6.51}$$

proving that $M_{f_{n,\lambda},E}$ is extremal. The proof of the extremality of the minorant $L_{f_{n,\lambda},E}$ is similar, and we omit the details. □

Next we prove the odd cases.

Proof of Theorem 6.3.3. The proof is similar to Theorem 6.3.1, and we highlight some of the differences.

First consider the case when $n\equiv 1 \bmod 4$. We begin by showing that $M_{f_{n,\lambda},E}$ and $L_{f_{n,\lambda},E}$ satisfy the desired properties. Since $E^*(z)=E(-z)$ for all complex z we have that $A(i\lambda),iB(i\lambda)\in$

$\mathbb{R}$, hence

$$M_{f_{n,\lambda},E}(x) - f_{n,\lambda}(x) = \frac{\lambda^n}{2\pi}\left(i\frac{B(x)}{B(i\lambda)} - \frac{A(x)}{A(i\lambda)}\right)^2 \frac{1}{x^2+\lambda^2} \geq 0 \tag{6.52}$$

for all real x.

To show (6.38) we expand the square and apply Lemma 5.5.3 to obtain

$$\begin{aligned}
&\int_{-\infty}^{\infty} (M_{f_{n,\lambda},E}(x) - f_{n,\lambda}(x))\frac{dx}{|E(x)|^2} \\
&\quad= \frac{\lambda^n}{2}\int_{-\infty}^{\infty}\left(i\frac{B(x)}{B(i\lambda)} - \frac{A(x)}{A(i\lambda)}\right)^2 \frac{1}{x^2+\lambda^2}\frac{dx}{|E(x)|^2} \\
&\quad= \frac{\lambda^n}{8}\left(\frac{1}{B^2(i\lambda)} + \frac{2}{B(i\lambda)A(i\lambda)} + \frac{1}{A^2(i\lambda)}\right)\int_{-\infty}^{\infty}\frac{E(x)}{E^*(x)}\frac{1}{x^2+\lambda^2}\,dx \\
&\qquad+ \frac{\lambda^n}{8}\left(\frac{1}{B^2(i\lambda)} - \frac{2}{B(i\lambda)A(i\lambda)} + \frac{1}{A^2(i\lambda)}\right)\int_{-\infty}^{\infty}\frac{E(x)}{E^*(x)}\frac{1}{x^2+\lambda^2}\,dx \\
&\qquad+ \frac{\lambda^n}{4}\left(-\frac{1}{B^2(i\lambda)} + \frac{1}{A^2(i\lambda)}\right)\int_{-\infty}^{\infty}\frac{1}{x^2+\lambda^2}\,dx \\
&\quad= \frac{\pi\lambda^{n-1}}{4}\left(\frac{1}{B^2(i\lambda)} + \frac{1}{A^2(i\lambda)}\right)\frac{E(-i\lambda)}{E(i\lambda)} + \frac{\pi\lambda^{n-1}}{4}\left(-\frac{1}{B^2(i\lambda)} + \frac{1}{A^2(i\lambda)}\right) \\
&\quad= \frac{2\pi\lambda^{n-1}}{E^2(i\lambda) - E^2(-i\lambda)}.
\end{aligned} \tag{6.53}$$

It remains to show that $M_{f_{n,\lambda},E}$ is extremal. Let $f(z) = iB(z)A(i\lambda) - A(z)B(i\lambda)$. Notice that for $\gamma \in \mathbb{R}$ with $f(\gamma) = 0$ we have $M_{f_{n,\lambda},E}(\gamma) = f_{n,\lambda}(\gamma)$. Let φ be a phase function associated to E and define $\alpha_\lambda = \arg(A(i\lambda) + B(i\lambda))$ and $r = |A(i\lambda) + B(i\lambda)|$. For real x we have

$$\begin{aligned}
f(x) &= \frac{1}{2}\left(-A(i\lambda)(E(x) - E^*(x)) - B(i\lambda)(E(x) + E^*(x))\right) \\
&= -\frac{r}{2}\left(E(x)e^{i\alpha_\lambda} - E^*(x)e^{-i\alpha_\lambda}\right) \\
&= -\frac{r}{2}B_{\alpha_\lambda}(x)
\end{aligned} \tag{6.54}$$

It follows that if $\gamma \in \mathcal{T}_{\alpha_\lambda}$ (recall $\mathcal{T}_\alpha = \{t \in \mathbb{R} \mid \varphi(t) \equiv \alpha \bmod \pi\}$), then $f(\gamma) = 0$ and hence $M_{f_{n,\lambda},E}(\gamma) = f_{n,\lambda}(\gamma)$. The proof now follows analogously to that of Theorem 6.3.1 by applying Theorem 3.5.5 with $\alpha = \alpha_\lambda$. By symmetry the result for the minorant follows. Lastly, the case when $n \equiv 3$ mod 4 is identical (or follows from the recursive relationship (6.29)), and we omit these details. □

6.4. One-sided approximations to $(x^2+\lambda^2)^{-1}(x^2+\beta^2)^{-1}$ in $L^1(\mathbb{R},\mu_E)$-norm

For the Poisson and Conjugate Poisson kernels we see that the extremal majorants and minorants are built up in a certain way to remove the poles at $\pm i\lambda$. We can push this idea a bit further to construct majorants and minorants for the function

$$h_{\lambda,\beta}(x) = \frac{f_{1,\lambda}(x)}{x^2+\beta^2} = \frac{1}{(x^2+\lambda^2)(x^2+\beta^2)} \tag{6.55}$$

where λ and β are positive.

Let E be a Hermite-Biehler function such that $E^*(z) = E(-z)$ for all complex z. Define for positive λ and β such that $\lambda \neq \beta$ the functions $M_{h_{\lambda,\beta},E}, L_{h_{\lambda,\beta},E} : \mathbb{C} \to \mathbb{C}$ by

$$M_{h_{\lambda,\beta},E}(z) = \frac{1}{(z^2+\lambda^2)(z^2+\beta^2)}\left(1 - B^2(z)\left(\frac{1}{\lambda^2-\beta^2}\left(\frac{\lambda^2+z^2}{B^2(i\beta)} - \frac{\beta^2+z^2}{B^2(i\lambda)}\right)\right)\right) \tag{6.56}$$

and

$$L_{h_{\lambda,\beta},E}(z) = \frac{1}{(z^2+\lambda^2)(z^2+\beta^2)}\left(1 - A^2(z)\left(\frac{1}{\lambda^2-\beta^2}\left(\frac{\lambda^2+z^2}{A^2(i\beta)} - \frac{\beta^2+z^2}{A^2(i\lambda)}\right)\right)\right). \tag{6.57}$$

Again, these functions have been fixed up to remove the singularities at $\pm i\lambda$ and $\pm i\beta$ and are therefore entire. Also, if E is of bounded type with exponential type $\tau(E)$ it follows that these functions are of exponential type $2\tau(E)$.

Theorem 6.4.1. *Let $\lambda > 0$ and $\beta > 0$ such that $\lambda \neq \beta$. Let E be a Hermite-Biehler function of bounded type in $\mathbb{C}^+$ and exponential type $\tau(E)$ with no real zeros such that $E^*(z) = E(-z)$ for all complex z. The following propositions hold:*

1. *Assume $B \notin \mathcal{H}(E)$. If $M : \mathbb{C} \to \mathbb{C}$ is an entire function of exponential type $2\tau(E)$ with $M(x) \geq h_{\lambda,\beta}(x)$ for all real x, then*

$$\begin{aligned}\int_{-\infty}^{\infty} (M(x) - h_{\lambda,\beta}(x))\frac{dx}{|E(x)|^2} \\ \geq \frac{2\pi}{\lambda^2-\beta^2}\left(\frac{1}{\beta E(i\beta)(E(i\beta)-E(-i\beta))} - \frac{1}{\lambda E(i\lambda)(E(i\lambda)-E(-i\lambda))}\right).\end{aligned} \tag{6.58}$$

Moreover, there is equality in the inequality (6.58) *if $M(z) = M_{h_{\lambda,\beta},E}(z)$.*

2. Assume $A \notin \mathcal{H}(E)$. If $L : \mathbb{C} \to \mathbb{C}$ is an entire function of exponential type $2\tau(E)$ with $h_{\lambda,\beta}(x) \geq L(x)$ for all real x, then

$$\begin{aligned}\int_{-\infty}^{\infty} (h_{\lambda,\beta}(x) - L(x)) \frac{dx}{|E(x)|^2} & \\ \geq \frac{2\pi}{\lambda^2 - \beta^2} & \left(\frac{1}{\beta E(i\beta)(E(i\beta) + E(-i\beta))} - \frac{1}{\lambda E(i\lambda)(E(i\lambda) + E(-i\lambda))} \right).\end{aligned} \tag{6.59}$$

Moreover, there is equality in the inequality (6.59) *if $L(z) = L_{h_{\lambda,\beta},E}(z)$.*

Remark 6.4.2. Under the assumptions of the previous theorem, (6.58) and (6.59) give

$$\begin{aligned}\int_{-\infty}^{\infty} \left(M_{h_{\lambda,\beta},E}(x) - L_{h_{\lambda,\beta},E}(x)\right) \frac{dx}{|E(x)|^2} & \\ = \frac{4\pi}{\lambda^2 - \beta^2} & \left(\frac{\beta^{-1}}{E^2(i\beta) - E^2(-i\beta)} - \frac{\lambda^{-1}}{E^2(i\lambda) - E^2(-i\lambda)} \right).\end{aligned} \tag{6.60}$$

As in the proof of Theorem 6.3.1, we require knowledge of certain integral values to assist in the computation of the explicit error values of Theorem 6.4.1.

Lemma 6.4.3. *Let $\lambda > 0$. Let E be a Hermite-Biehler function and q_2 a polynomial of degree at most 2. We have*

$$\int_{-\infty}^{\infty} \frac{E(x)}{E^*(x)} \frac{q_2(x)}{(x^2 + \lambda^2)(x^2 + \beta^2)}\, dx = \frac{\pi}{\lambda^2 - \beta^2} \left(\frac{E(-i\beta)}{E^*(-i\beta)} \frac{q_2(-i\beta)}{\beta} - \frac{E(-i\lambda)}{E^*(-i\lambda)} \frac{q_2(-i\lambda)}{\lambda} \right) \tag{6.61}$$

and

$$\int_{-\infty}^{\infty} \frac{E^*(x)}{E(x)} \frac{q_2(x)}{(x^2 + \lambda^2)(x^2 + \beta^2)}\, dx = \frac{\pi}{\lambda^2 - \beta^2} \left(\frac{E^*(i\beta)}{E(i\beta)} \frac{q_2(-i\beta)}{\beta} - \frac{E^*(i\lambda)}{E(i\lambda)} \frac{q_2(-i\lambda)}{\lambda} \right). \tag{6.62}$$

Moreover, if $E^*(z) = E(-z)$ for all complex z, then the two integrals are both equal to

$$\frac{\pi}{\lambda^2 - \beta^2} \left(\frac{E(-i\beta)}{E(i\beta)} \frac{q_2(-i\beta)}{\beta} - \frac{E(-i\lambda)}{E(i\lambda)} \frac{q_2(-i\lambda)}{\lambda} \right). \tag{6.63}$$

Proof. The proof is an application of the residue theorem and is analogous to Lemma 5.5.3, and we omit the details. $\square$

Proof of Theorem 6.4.1. We first show that the functions $M_{h_{\lambda,\beta},E}$ and $L_{h_{\lambda,\beta},E}$ satisfy the needed requirements. For real x we have

$$M_{h_{\lambda,\beta},E}(x) - h_{\lambda,\beta}(x) = -B^2(x)q(x)h_{\lambda,\beta}(x) \tag{6.64}$$

where

$$q(x) = \frac{1}{\lambda^2-\beta^2}\left(\frac{\lambda^2+x^2}{B^2(i\beta)} - \frac{\beta^2+x^2}{B^2(i\lambda)}\right). \tag{6.65}$$

To show that $M_{h_{\lambda,\beta},E}$ is a majorant of $h_{\lambda,\beta}$ it will be sufficient to show that $q(x) \le 0$ for all real x since B^2 and $h_{\lambda,a}$ are non-negative on $\mathbb{R}$. Without loss of generality assume that $\beta < \lambda$. Since $y \mapsto B^2(iy)$ is negative and decreasing for $y > 0$ (follows from the fact that B^2 is of Pólya-Class) we have that

$$q(0) = \frac{1}{\lambda^2-\beta^2}\left(\frac{\lambda^2}{B^2(i\beta)} - \frac{\beta^2}{B^2(i\lambda)}\right) \le \frac{1}{\lambda^2-\beta^2}\left(\frac{(\lambda B(i\lambda))^2-(\beta B(i\beta))^2}{B^2(i\beta)B^2(i\lambda)}\right) \le 0 \tag{6.66}$$

and

$$q''(x) = \frac{2}{\lambda^2-\beta^2}\left(\frac{1}{B^2(i\beta)} - \frac{1}{B^2(i\lambda)}\right) < 0, \tag{6.67}$$

and it follows that $q(x) \le 0$ for all real x.

The proof that $L_{h_{\lambda,\beta},E}$ is a minorant follows by a similar argument using that $y \mapsto A^2(iy)$ is positive and increasing for positive y.

Next, we show that $M_{h_{\lambda,\beta},E}$ gives the desired integral value (6.58). Using (6.64) we write $B = i/2(E-E^*)$ and $|E|^2 = EE^*$, apply Lemma 6.4.3, and use that

$$\begin{aligned}\int_{-\infty}^{\infty} &\frac{q(x)}{(x^2+\lambda^2)(x^2+\beta^2)}\,dx \\ &= \frac{1}{\lambda^2-\beta^2}\left(\frac{1}{A^2(i\beta)}\int_{-\infty}^{\infty}\frac{1}{x^2+\beta^2}\,dx - \frac{1}{A^2(i\lambda)}\int_{-\infty}^{\infty}\frac{1}{x^2+\lambda^2}\,dx\right) \\ &= \frac{\pi}{\lambda^2-\beta^2}\left(\frac{1}{\beta A^2(i\beta)} - \frac{1}{\lambda A^2(i\lambda)}\right)\end{aligned} \tag{6.68}$$

to obtain

$$
\begin{aligned}
\int_{-\infty}^{\infty} &(M_{h_{\lambda,\beta},E}(x) - h_{\lambda,\beta}(x)) \frac{dx}{|E(x)|^2} \\
&= -\int_{-\infty}^{\infty} \frac{B^2(x)}{|E(x)|^2} \frac{q(x)}{(x^2+\lambda^2)(x^2+\beta^2)}\, dx \\
&= \frac{1}{4}\int_{-\infty}^{\infty} \left(\frac{E(x)}{E^*(x)} + \frac{E^*(x)}{E(x)} - 2\right) \frac{q(x)}{(x^2+\lambda^2)(x^2+\beta^2)}\, dx \\
&= \frac{1}{2}\int_{-\infty}^{\infty} \left(\frac{E(x)}{E^*(x)} - 1\right) \frac{q(x)}{(x^2+\lambda^2)(x^2+\beta^2)}\, dx \\
&= \frac{\pi}{2(\lambda^2-\beta^2)} \left(\frac{1}{\beta B^2(i\beta)}\left(\frac{E(-i\beta)}{E(i\beta)} - 1\right) - \frac{1}{\lambda B^2(i\lambda)}\left(\frac{E(-i\lambda)}{E(i\lambda)} - 1\right)\right) \\
&= \frac{2\pi}{\lambda^2-\beta^2}\left(\frac{1}{\beta E(i\beta)(E(i\beta)-E(-i\beta))} - \frac{1}{\lambda E(i\lambda)(E(i\lambda)-E(-i\lambda))}\right).
\end{aligned} \tag{6.69}
$$

The computations (which we omit) for $L_{h_{\lambda,\beta},E}$ are nearly identical. Moreover, the proof that $M_{P_{\lambda,\beta},E}$ and $L_{P_{\lambda,\beta},E}$ are optimal is analogous to the proof of Theorem 6.3.1, and we omit these details. □

6.5. Properties of the vanishing measure

For $a > 0$ and $\delta > 0$ we define the entire function $E_{a,\delta} : \mathbb{C} \to \mathbb{C}$ by

$$
E_{a,\delta}(z) = \sqrt{\frac{2}{\sinh(2\delta a)}}\frac{\sin(\delta(z+ia))}{z+ia}. \tag{6.70}
$$

Theorem 6.5.1. *Let $a > 0$ and $\delta > 0$. The function $E_{a,\delta}$ is Hermite-Biehler with bounded type in the upper half-plane and exponential type δ, does not have real zeros, satisfies $E^*_{a,\delta}(z) = E_{a,\delta}(-z)$ for all z, and $e^{i\alpha}E_{a,\delta} - e^{-i\alpha}E^*_{a,\delta} \notin \mathcal{H}(E_{a,\delta})$ for all real α.*

Moreover, the space $\mathcal{A}^2(\delta,(x^2+a^2)\,dx)$ is isometrically equal to the de Branges space $\mathcal{H}(E_{a,\delta}) = \mathcal{A}^2(\delta, |E_{a,\delta}(x)|^{-2}\,dx)$.

Proof. Since $z \mapsto \sin\delta z$ is $\mathcal{LP}$ and hence of Pólya class, we have that $z \mapsto \sin(\delta(z+ia))$ is also of Pólya class. By [23, Section 7, Lemma 1] it follows that $E_{a,\delta}$ is of Pólya class. This implies

$$
|E_{a,\delta}(z)| \geq |E^*_{a,\delta}(z)| \tag{6.71}
$$

for all z with $\mathrm{Im}(z) > 0$. Since $E_{a,\delta}$ has no zeros in the upper half-plane, the function $E^*_{a,\delta}/E_{a,\delta}$ is analytic in the upper half-plane and has modulus bounded by 1. Since this quotient is not constant, the modulus is never equal to 1 by the maximum principle, hence $E_{a,\delta}$ satisfies (3.8) and therefore Hermite-Biehler.

It can be checked directly that $E_{a,\delta}$ has bounded type δ (or apply the reverse direction of Krein's theorem) and hence of exponential type δ. Evidently $E_{a,\delta}$ has no real zeros and $E^*_{a,\delta}(z) = E_{a,\delta}(-z)$ for all z. By Theorem 3.6.1 we have that $\mathcal{H}(E_{a,\delta}) = \mathcal{A}^2(\delta, |E_{a,\delta}(x)|^{-2}\,dx)$.

Notice that

$$|E_{a,\delta}(x)|^2 = \frac{\cosh(2\delta a) - \cos(2\delta x)}{\sinh(2\delta a)} \frac{1}{x^2 + a^2} \tag{6.72}$$

It follows that

$$\tanh(a\delta) \leq |E_{a,\delta}(x)|^2 (x^2 + a^2) \leq \coth(a\delta). \tag{6.73}$$

and

$$\tanh(a\delta)(x^2 + a^2) \leq \frac{1}{|E_{a,\delta}(x)|^2} \leq \coth(a\delta)(x^2 + a^2), \tag{6.74}$$

for all real x.

A direct calculation gives

$$A_{a,\delta}(z) = \sqrt{\frac{2}{\sinh(2\delta a)}} \frac{z\cosh(\delta a)\sin(\delta z) + a\sinh(\delta a)\cos(\delta z)}{z^2 + a^2}, \tag{6.75}$$

$$B_{a,\delta}(z) = \sqrt{\frac{2}{\sinh(2\delta a)}} \frac{a\cosh(\delta a)\sin(\delta z) - z\sinh(\delta a)\cos(\delta z)}{z^2 + a^2}, \tag{6.76}$$

and (6.74) implies $A_{a,\delta}, B_{a,\delta} \notin \mathcal{H}(E_{a,\delta})$. An analogous calculation gives $e^{i\alpha}E_{a,\delta} - e^{-i\alpha}E_{a,\delta} \notin \mathcal{H}(E_{a,\delta})$ for all real α. Moreover, (3.16) leads to the representation

$$K_{a,\delta}(x, x) = \frac{\delta(a^2 + x^2) - a\coth(2\delta a) + a\cos(2\delta x)\mathrm{csch}(2\delta a)}{\pi(a^2 + x^2)^2}. \tag{6.77}$$

It follows from (6.74) that $\mathcal{A}^2(\delta, (x^2 + a^2)\,dx)$ and $\mathcal{A}^2(\delta, |E_{a,\delta}(x)|^{-2}\,dx)$ are equal as sets with equivalent norms. The main statement to prove is the fact that the two norms are equal on the smaller spaces.

We show that $\mathcal{A}^2(\pi, (x^2+a^2)\,dx) = \mathcal{H}(E_{a,\pi})$ isometrically and sketch out an alternative proof (which relies on deep de Branges space theory) for the general statement.

We note first that

$$\frac{(z+ia)(z-ia)}{\sin(\pi(z+ia))\sin(\pi(z-ia))} = \frac{2(z^2+a^2)}{\cosh(2\pi a)-\cos(2\pi z)} \tag{6.78}$$

holds, in particular, the right hand side is 1-periodic after division by z^2+a^2. Furthermore,

$$\int_0^1 \frac{1}{\cosh(2\pi a)-\cos(2\pi x)}dx = \frac{1}{\sinh(2\pi a)}. \tag{6.79}$$

This means that p_a defined by $p_a(x) = \sinh(2\pi a)(\cosh(2\pi a)-\cos(2\pi x))^{-1}-1$ is 1-periodic and has mean value zero. Since $a>0$, this function is infinitely differentiable on the real line. It follows that the Fourier series of p_a converges absolutely and uniformly, and that it represents the function, i.e., there exists a sequence a_n so that

$$p_a(x) = \sum_{n\neq 0} a_n e^{2\pi i n x} \tag{6.80}$$

for all real x.

Let $H \in L^1(\mathbb{R}, dx)$ be an entire function of exponential type 2π. Since $H \in L^2(\mathbb{R}, dx)$ by Theorem 2.2.3, the Paley-Wiener theorem implies that the Fourier transform of H satisfies $\widehat{H}(t) = 0$ for $|t|>1$. Since $H\in L^1(\mathbb{R},dx)$ it follows that $\widehat{H}$ is continuous, hence $\widehat{H}(t)=0$ for $|t|\geq 1$. This implies

$$\int_{-\infty}^{\infty} H(x) \sum_{\substack{|n|\leq N\\ n\neq 0}} a_n e^{2\pi i n x} dx = \sum_{\substack{|n|\leq N\\ n\neq 0}} a_n \widehat{H}(-n) = 0. \tag{6.81}$$

Since the partial sums of the series in (6.80) converge uniformly, we obtain with an application of Lebesgue dominated convergence that

$$\int_{-\infty}^{\infty} H(x)\left(\frac{\sinh(2\pi a)}{\cosh(2\pi a)-\cos(2\pi x)}-1\right)dx = 0. \tag{6.82}$$

Let $F, G \in \mathcal{A}_2(\pi, (x^2+a^2)dx)$ and define H by $H(z) = F(z)G^*(z)(z^2+a^2)$. It follows from (6.78) that

$$\begin{aligned}\langle F, G\rangle_{\mathcal{H}(E_{a,\pi})} - \langle F, G\rangle_{L^2(\mathbb{R},\mu_a)} &= \int_{-\infty}^{\infty} F(x)G^*(x)\{|E_{a,\pi}(x)|^{-2} - (x^2+a^2)\}dx \\ &= \int_{-\infty}^{\infty} H(x)\left(\frac{\sinh(2\pi a)}{\cosh(2\pi a) - \cos(2\pi x)} - 1\right)dx,\end{aligned} \tag{6.83}$$

and since H is a Lebesgue integrable entire function of exponential type 2π, it follows from (6.82) that

$$\langle F, G\rangle_{\mathcal{H}(E_{a,\pi})} = \langle F, G\rangle_{L^2(\mathbb{R},(x^2+a^2)dx)} \tag{6.84}$$

as claimed. □

Sketch of Alternative Proof. Define for $a, \delta > 0$ the meromorphic function $W_{a,\delta}$ by

$$W_{a,\delta}(z) = -e^{-2a\delta}\frac{a+iz}{a-iz} \tag{6.85}$$

and note that $W_{a,\delta}$ is analytic and has modulus ≤ 1 in the upper half-plane. The identity

$$\frac{E_{a,\delta}(z) + E^*_{a,\delta}(z)W_{a,\delta}(z)}{E_{a,\delta}(z) - E^*_{a,\delta}(z)W_{a,\delta}(z)} = \coth(2a\delta) - e^{2\delta iz}\operatorname{csch}(2a\delta) \tag{6.86}$$

is valid for all $z \in \mathbb{C}$. Also, for real x and $y > 0$ we have

$$\begin{aligned}\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{(t^2+a^2)|E_{a,\delta}(t)|^2}{(x-t)^2+y^2}\,dt &= \frac{1}{\sinh(2a\delta)}\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{\cosh(2\delta a) - \cos(2\delta t)}{(x-t)^2+y^2}\,dt \\ &= \coth(2\delta a) - e^{-2\delta y}\cos(2\delta x)\operatorname{csch}(2\delta a) \\ &= \operatorname{Re}\left(\coth(2a\delta) - e^{2\delta iz}\operatorname{csch}(2a\delta)\right)\end{aligned} \tag{6.87}$$

Where we have used the fact that

$$\frac{y}{\pi}\int_{-\infty}^{\infty}\frac{\cos(2\delta t)}{(x-t)^2+y^2}\,dt = e^{-2\delta y}\cos(2\delta x) \tag{6.88}$$

holds for all real x and $y > 0$.

Theorem V.A. of [20] with $d\mu_{a,\delta}(t) = (t^2 + a^2)|E_{a,\delta}(t)|^2\, dt$ implies

$$\int_{-\infty}^{\infty} |F(x)|^2 (x^2 + a^2)\, dx = \int_{-\infty}^{\infty} \left| \frac{F(x)}{E_{a,\delta}(x)} \right|^2 dx \tag{6.89}$$

for every $F \in \mathcal{H}(E_{a,\delta})$. $\square$

A classical result of Plancherel and Pólya [55] is that the L^p Paley-Wiener spaces $\mathcal{A}^p(\delta, dx)$ are closed under differentiation i.e., if $F \in \mathcal{A}^p(\delta, dx)$, for some $1 \le p \le \infty$, then $F' \in \mathcal{A}^p(\delta, dx)$. The case $p = \infty$ is due to Bernstein [3]. For the space $\mathcal{A}^p(2\delta, |E_{a,\delta}(x)|^{-2p}\, dx)$ we have a similar result.

Theorem 6.5.2. *Let $a > 0$ and $\delta > 0$. Let $F \in \mathcal{A}^p(2\delta, |E_{a,\delta}(x)|^{-2p}\, dx)$, for some $1 \le p < \infty$. Then $F' \in \mathcal{A}^p(2\delta, |E_{a,\delta}(x)|^{-2p}\, dx)$.*

Remark 6.5.3. The results of the previous theorem implies that $\mathcal{H}(E^2_{a,\delta})$ is closed under differentiation. Lemma 6 of [32] gives that $\varphi'(x)$ is bounded hence $\mathcal{H}^1(E_{a,\delta}) \subset \mathcal{H}^\infty(E_{a,\delta})$ (by Lemma 3.8.5) making the interpolation formulas for $\mathcal{H}^\infty(E_{a,\delta})$ (Theorem 3.8.1) applicable when proving uniqueness of best approximations in $L^1(\mathbb{R}, \mu_{E_{a,\delta}})$-norm (see Theorem 5.4.1).

A similar result (to Theorem 6.5.2) is shown in [13, Theorem 20] for the family of homogeneous de Branges spaces which are used to treat the power weights $d\mu_\nu = |x|^{2\nu+1}dx$, $\nu > -1$. The following proof follows their approach.

Proof. By (6.74) we see that $F \in \mathcal{A}^p(2\delta, dx)$ and by Plancherel-Pólya it follows that F' has exponential type 2δ. It remains to show that $F' \in L^p(\mathbb{R}, |E_{a,\delta}(x)|^{-2p}\, dx)$.

Define the entire function $G : \mathbb{C} \to \mathbb{C}$ by

$$G(z) = (z^2 + a^2)^2 E_a(z) E_a^*(z) = \operatorname{csch}(2\pi a)(z^2 + a^2)(\cosh(2\pi a) - \cos(2\pi z)). \tag{6.90}$$

By (6.74) we have

$$\tanh(\pi a)(x^2 + a^2) \le G(x) \le \coth(\pi a)(x^2 + a^2) \tag{6.91}$$

for all $x \in \mathbb{R}$.

Since

$$G'(z) = 2\text{csch}(2\pi a)(z(\text{csch}(2\pi a) - \cos(2\pi z)) + \pi(z^2 + a^2)\sin(2\pi z)), \tag{6.92}$$

we obtain

$$|G'(x)| \leq c_a(x^2 + a^2) \tag{6.93}$$

for some constant $c_a > 0$ and all real x. Hence (6.74) implies that $FG' \in L^p(\mathbb{R}, dx)$.

By assumption and (6.74) we have that $FG \in L^p(\mathbb{R}, dx)$. Since FG is an entire function of exponential type Plancherel-Pólya implies that $(FG)' \in L^p(\mathbb{R}, dx)$.

Using that

$$|F'(x)G(x)|^p = |(FG)'(x) - F(x)G'(x)|^p \leq 2^p \left(|(FG)'(x)|^p + |F(x)G'(x)|^p\right) \tag{6.94}$$

we find that $F'G \in L^p(\mathbb{R}, dx)$. By (6.74) and (6.91) this is equivalent to

$$F' \in L^p(\mathbb{R}, |E_{a,\delta}(x)|^{-2p}\, dx) \tag{6.95}$$

which proves the theorem. □

Recall that in Lemma 3.7.1 we had that any function in $\mathcal{H}^1(E^2)$ that is non-negative on $\mathbb{R}$ can be written as a square of a function belonging to $\mathcal{H}^2(E)$. The fact that the space $\mathcal{H}^1(E^2_{a,\delta})$ is closed under differentiation plays a large role in showing that any real entire function in $\mathcal{H}^1(E^2_{a,\delta})$ can be written as a difference of squares belonging to $\mathcal{H}^2(E_{a,\delta})$.

Theorem 6.5.4. *Let $a > 0$ and $\delta > 0$. Let $F : \mathbb{C} \to \mathbb{C}$ be a real entire function of exponential type 2δ such that $F \in L^1(\mathbb{R}, |E_{a,\delta}(x)|^{-2}dx)$. Then there exist $U, V \in \mathcal{H}(E_{a,\delta})$ such that*

$$F = UU^* - VV^*. \tag{6.96}$$

With (6.74) and Theorem 6.5.2 the proof of Theorem 6.5.4 is identical to Corollary 23 of [13]. Since we do not explicitly need this result in this investigation, we omit the proof.

6.6. Extremal problems with vanishing condition

In the following section we combine the results from Chapter 4 and Sections 6.2, 6.3, 6.4, and 6.5 to study the one-sided approximation problem with additional vanishing condition. We start by stating the vanishing theorem for the Heaviside function[1], H. Notice that this problem is equivalent to the vanishing problems for the signum function using the relationship $\operatorname{sgn}(x) = H(x) - H(-x)$.

Theorem 6.6.1. *Let $a > 0$ and $\delta > 0$. If $S, T \in \mathcal{A}(2\pi\delta)$ with*

$$S(ia) = T(ia) = 0 \tag{6.97}$$

and

$$S(x) \le H(x) \le T(x) \tag{6.98}$$

for all real x, then

$$\int_{-\infty}^{\infty} \{S(x) - T(x)\}\, dx \ge \frac{\pi a}{\pi a\delta - \tanh(\pi a\delta)}, \tag{6.99}$$

and there exist $S^+_{a,2\pi\delta}, S^-_{a,2\pi\delta} \in \mathcal{A}(2\pi\delta)$ such that there is equality in (6.99) for $S = S^-_{a,2\pi\delta}$ and $T = S^+_{a,2\pi\delta}$.

Remark 6.6.2. This implies that for fixed a and $\delta \to \infty$ the integral is $\sim \delta^{-1}$, while for $\delta \to 0+$ the integral is $\sim 3(\pi b)^{-2}\delta^{-3}$. In (3) we see that the integral value for the corresponding extremal problem for the signum function (equivalently the Heaviside functions via $H = 2^{-1}(\operatorname{sgn}(x) - \operatorname{sgn}(-x))$) without the vanishing condition (i.e., Beurling's Problem) is equal to δ^{-1}. This shows that the prescribed vanishing at $\alpha = ia$ substantially affects the integral value for small values of δ, but the vanishing condition leads only to a small change if δ becomes large.

Proof of Theorem 6.6.1. We define $S^+_{a,2\pi\delta}$ and $S^-_{a,2\pi\delta}$ by

$$S^+_{a,2\pi\delta}(z) = a^{-1} M^+(B^2_{a,\pi\delta}, a, z)(z^2 + a^2), \tag{6.100}$$

$$S^-_{a,2\pi\delta}(z) = a^{-1} M^-(B^2_{a,\pi\delta}, a, z)(z^2 + a^2) \tag{6.101}$$

with M^- and M^+ as in (4.99) and (4.100) (see Section 4.7). By Theorem 6.5.1 along with Lemmas 3.3.11 and 4.2.8 we have that $B_{a,\pi\delta}$ is an odd $\mathcal{LP}$ function with a simple zero at the origin. Hence

[1]Here H is the Heaviside function defined by $H(x) = 1$ for $x > 0$ and $H(x) = 0$ for $x \le 0$.

$F = B^2_{a,\pi\delta}$ satisfies the assumptions of Theorems 4.7.2 and 4.7.3. For the remainder of the proof we set $M^{\pm}(z) = a^{-1}M^{\pm}(B^2_{a,\pi\delta}, a, z)$. Since $B^2_{a,\pi\delta} \in \mathcal{A}(2\pi\delta)$ we obtain that $M^+, M^- \in \mathcal{A}(2\pi\delta)$. Theorems 4.7.2 and 4.7.3 imply

$$M^-(x) \leq t_a(x) \leq M^+(x) \tag{6.102}$$

for all real x. It follows from (6.10) and (6.77) that

$$\int_{-\infty}^{\infty} (M^+(x) - M^-(x)) \frac{dx}{|E_{a,\pi\delta}(x)|^2} = \frac{1}{a^2 K_{a,\pi\delta}(0,0)} = \frac{\pi a}{\pi a\delta - \tanh(\pi a\delta)}. \tag{6.103}$$

By definition of $S^{\pm}_{a,2\pi\delta}$ we have

$$\int_{-\infty}^{\infty} \left\{ S^+_{a,2\pi\delta}(z) - S^-_{a,2\pi\delta}(x) \right\} dx = \int_{-\infty}^{\infty} (M^+(x) - M^-(x))(x^2 + a^2) dx. \tag{6.104}$$

Since $M^+ - M^- \in \mathcal{A}_1(2\pi\delta, |E_{a,\pi\delta}(x)|^{-2}dx)$, Lemma 3.7.1 implies that $M^+ - M^- = UU^*$ with $U \in \mathcal{H}(E_{a,\pi\delta})$. Theorem 6.5.1, (6.103), and (6.104) imply

$$\int_{-\infty}^{\infty} \left\{ S^+_{a,2\pi\delta}(x) - S^-_{a,2\pi\delta}(x) \right\} dx = \frac{\pi a}{\pi a\delta - \tanh(\pi a\delta)}, \tag{6.105}$$

which gives the case of equality in (6.99).

Let now $S, T \in \mathcal{A}(2\pi\delta)$ such that $S(ia) = T(ia) = 0$ and $S(x) \leq t_a(x) \leq T(x)$ on the real line. We may assume that $S - M^-$ and $T - M^+$ are integrable with respect to $(x^2 + a^2)dx$. Since S and T are real entire, it follows that $S(-ia) = T(-ia) = 0$, hence

$$S(z) = (z^2 + a^2)\sigma(z) \tag{6.106}$$

and

$$T(z) = (z^2 + a^2)\tau(z) \tag{6.107}$$

where σ, τ are entire and have exponential type $2\pi\delta$. Furthermore, $\sigma - t_a$ and $\tau - t_a$ are integrable and

$$\sigma(x) \leq t_a(x) \leq \tau(x) \tag{6.108}$$

for all real x. It follows from Theorem 6.2.1 that

$$\int_{-\infty}^{\infty}(\sigma(x)-\tau(x))(x^2+a^2)dx \geq \frac{1}{a^2K_{a,\pi\delta}(0,0)}, \tag{6.109}$$

which is (6.99). □

Applying Theorems 6.3.1, 6.3.2, and 6.3.3 with $E_{a,2\delta}$ and following the same argument as Theorem 6.6.1 gives the vanishing result for monomials.

Theorem 6.6.3. *Let a, $\delta > 0$ and $n \in \mathbb{N}_0$. If $S, T \in \mathcal{A}(2\pi\delta)$ with*

$$S(ia) = T(ia) = 0 \tag{6.110}$$

and

$$S(x) \leq x^n \leq T(x) \tag{6.111}$$

for all real x, then

$$\int_{-\infty}^{\infty}\{S(x)-T(x)\}\,dx \geq \frac{16\pi a^{n+1}\sinh(2\pi a\delta)}{\cosh(4\pi a\delta)-8(\pi a\delta)^2-1} \tag{6.112}$$

and there exist $S^{+}_{n,a,2\pi\delta}, S^{-}_{n,a,2\pi\delta} \in \mathcal{A}(2\pi\delta)$ such that there is equality in the inequality (6.112) *for $S = S^{-}_{n,a,2\pi\delta}$ and $T = S^{+}_{n,a,2\pi\delta}$.*

Remark 6.6.4. This implies that for fixed a and $\delta \to 0^+$ the integral is $\sim 3\pi^{-1}a^{n-2}\delta^{-3}$, while for $\delta \to \infty$ the integral is $\sim 16\pi a^{n+1}\exp(-2\pi a\delta)$. Notice that for all $n \in \mathbb{N}_0$, the monomimal $P_n(z) = z^n$ is an entire function of exponential type 0, i.e., it is its own extremal majorant and minorant with L^1-error of 0. Hence, prescribing vanishing at $\alpha = ia$ substantially affects the integral value for small values of δ, but the vanishing condition leads only to a small change if δ becomes large.

Remark 6.6.5. By Theorem 6.6.3 we can easily construct non-optimal majorants and minorants to any polynomial in $\mathbb{R}[x]$ which satisfy the vanishing condition at α. Let $p(x) = \sum_{n=0}^{N} a_n x^n$ belong to $\mathbb{R}[x]$ (i.e., $a_n \in \mathbb{R}$). For simplicity assume that $a_n \neq 0$ for $n = 0, 1, ..., N$.

We define

$$P^{+}(z) = \sum_{n=0}^{N} a_n S_{n,a,2\pi\delta}^{\mathrm{sign}(a_n)}(z) \tag{6.113}$$

and

$$P^{-}(z) = \sum_{n=0}^{N} a_n S_{n,a,2\pi\delta}^{\mathrm{sign}(-a_n)}(z), \tag{6.114}$$

where $\mathrm{sign}(a) = +$ for $a > 0$ and $\mathrm{sign}(a) = -$ for $a < 0$. By construction we have that $P^{\pm} \in \mathcal{A}(2\pi\delta)$,

$$P^{-}(x) \le p(x) \le P^{+}(x) \tag{6.115}$$

for all real x, and $P^{+}(ia) = P^{-}(ia) = 0$. Moreover, (6.112) gives

$$\int_{-\infty}^{\infty} \{P^{+}(x) - P^{-}(x)\}\, dx = \frac{16\pi \sinh(2\pi a\delta)}{\cosh(4\pi a\delta) - 8(\pi a\delta)^2 - 1} \sum_{n=0}^{N} |a_n| a^{n+1} \tag{6.116}$$

We turn to the proof of Theorem 6.6.3.

Proof. The proof is similar to Theorem 6.6.1 and we sketch out the minor differences.

Define $S_{n,a,2\pi\delta}^{+}$ and $S_{n,a,2\pi\delta}^{-}$ by

$$S_{n,a,2\pi\delta}^{+}(z) = M_{f_{n,a},E_{a,\pi\delta}}(z)(z^2 + a^2) \tag{6.117}$$

$$S_{n,a,2\pi\delta}^{-}(z) = L_{f_{n,a},E_{a,\pi\delta}}(z)(z^2 + a^2) \tag{6.118}$$

with M and L as in (6.25), (6.26), (6.27), (6.28), (6.30), (6.31), (6.32), and (6.33) (See Section 6.3). By construction $S_{n,a,2\pi\delta}^{\pm}(ia) = 0$ and Theorems 6.3.1, 6.3.2, and 6.3.3 imply $S_{n,a,2\pi\delta}^{\pm} \in \mathcal{A}(2\pi\delta)$ and

$$S_{n,a,2\pi\delta}^{-}(x) \le x^n \le S_{n,a,2\pi\delta}^{+}(x) \tag{6.119}$$

for all real x.

For the remainder of the proof we set $M = M_{f_{n,a},E_{a,\pi\delta}}$ and $L = L_{f_{n,a},E_{a,\pi\delta}}$. It follows from (6.34), (6.35), (6.36), (6.37), (6.38), and (6.39) (see also Remark 6.3.4) that

$$\int_{-\infty}^{\infty} (M(x) - L(x)) \frac{dx}{|E_{a,\pi\delta}(x)|^2} = \frac{4\pi a^{n-1}}{E_{a,\pi\delta}^2(ia) - E_{a,\pi\delta}^2(-ia)}. \tag{6.120}$$

By definition of $S^{\pm}_{n,a,2\pi\delta}$ we have

$$\int_{-\infty}^{\infty} \left\{ S^{+}_{n,a,2\pi\delta}(x) - S^{-}_{n,a,2\pi\delta}(x) \right\} dx = \int_{-\infty}^{\infty} (M(x) - L(x))\,(x^2 + a^2)\, dx. \tag{6.121}$$

Since $M - L \in \mathcal{A}_1(2\pi\delta, |E_{a,\pi\delta}(x)|^{-2}dx)$, Lemma 3.7.1 implies that $M - L = UU^*$ with $U \in \mathcal{H}(E_{a,\pi\delta})$. Theorem 6.5.1 and (6.120) imply

$$\begin{aligned} \int_{-\infty}^{\infty} \left\{ S^{+}_{n,a,2\pi\delta}(x) - S^{-}_{n,a,2\pi\delta}(x) \right\} dx &= \frac{4\pi a^{n-1}}{E^2_{a,\pi\delta}(ia) - E^2_{a,\pi\delta}(-ia)} \\ &= \frac{16\pi a^{n+1} \sinh(2\pi a\delta)}{\cosh(4\pi a\delta) - 8(\pi a\delta)^2 - 1} \end{aligned} \tag{6.122}$$

which gives the case of equality in (6.112). The proof that $S^{\pm}_{n,a,2\pi\delta}$ are optimal is analogous to Theorem 6.6.1, and we omit the details.

□

Similarly, we obtain the vanishing results for the Poisson kernel by using Theorem 6.4.1 applied with $E_{a,\pi\delta}$ and applying the same argument as Theorem 6.6.3. As the proof is nearly identical, we omit the details.

Theorem 6.6.6. *Let a, δ, $\lambda > 0$ with $a \neq \lambda$. If $S, T \in \mathcal{A}(2\pi\delta)$ with*

$$S(ia) = T(ia) = 0 \tag{6.123}$$

and

$$S(x) \leq P_\lambda(x) \leq T(x) \tag{6.124}$$

for all real x, then

$$\begin{aligned} \int_{-\infty}^{\infty} \{S(x) - T(x)\}\, dx \geq\ & \frac{8\lambda a}{(a^2 - \lambda^2)((2\pi a\delta)^2 \mathrm{csch}(2\pi a\delta) - \sinh(2\pi a\delta)} \\ &+ \frac{4(a^2 - \lambda^2)\sinh(2\pi a\delta)}{(a-\lambda)^2 \cosh(2\pi(a+\lambda)\delta) - (a+\lambda)^2\cosh(2\pi(a-\lambda)\delta) + 4\lambda a} \end{aligned} \tag{6.125}$$

and there exist $S^{+}_{\lambda,a,2\pi\delta}, S^{-}_{\lambda,a,2\pi\delta} \in \mathcal{A}(2\pi\delta)$ such that there is equality in (6.125) for $S = S^{-}_{\lambda,a,2\pi\delta}$ and $T = S^{+}_{\lambda,a,2\pi\delta}$.

Remark 6.6.7. This implies that for fixed a and $\delta \to 0^+$ the integral is $\sim 3\lambda^{-1}a^{-2}(\pi\delta)^{-3}$, while for $a < \lambda$ and $\delta \to \infty$ the integral is $\sim 16a\lambda(\lambda^2 - a^2)^{-1}\exp(-2\pi a\delta)$ and for $a > \lambda$ and $\delta \to \infty$ the integral is $\sim 4(a+\lambda)(a-\lambda)^{-1}\exp(-2\pi\lambda\delta)$.

Theorem 6.3.1 applied with the Hermite-Biehler function $E(z) = e^{-\pi i\delta z}$ gives the corresponding extremal problem for the Poisson kernel with out the vanishing condition, and applying E to (6.34) and (6.35) the integral value becomes $2\text{csch}(2\pi\lambda\delta)$. This again shows that the prescribed vanishing at $\alpha = ia$ substantially affects the integral value for small values of δ, but the vanishing condition only leads to a small change if δ becomes large.

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