A NEW GENERALIZATION OF COHEN-KAPLANSKY DOMAINS

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ABSTRACT

The goal of this thesis is to provide an new generalization of Cohen-Kaplansky domains, stemming from questions related to valuation domains. Recall that a Cohen-Kaplansky domain is an atomic integral domain that contains only a finite number of irreducible elements (up to units). In the new generalization presented in this thesis, we remove the atomic condition required in the definition of a Cohen-Kaplansky domain and add in the extra condition that our integral domain has finitely many irreducible elements, say $\pi_1, \pi_2, \cdots, \pi_n$, such that for every nonzero nonunit y in the domain there exists an irreducible element, say π_i with $1 \leq i \leq n$, such that $\pi_i \mid y$.

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1. COMMUTATIVE RING THEORY

Commutative ring theory emerged as a distinct field of research in mathematics only at the beginning of the twentieth century [8]. It has connections with algebraic number theory, algebraic geometry, and invariant theory [21]. Algebraic number theory and algebraic geometry provide many examples of commutative rings and has motivated the development of commutative ring theory.

The goal of this thesis is to give a new generalization of Cohen-Kaplansky domains, stemming from a question related to valuation domains. In 1946, I. S. Cohen and Irving Kaplansky wrote the seminal paper studying what is now referred to as Cohen-Kaplansky domains or CK domains [9]. In order to understand Cohen-Kaplansky domains and the generalizations provided in this thesis, we will require several definitions, examples, and results which would typically come from courses in commutative ring theory. A summary of this needed background material is provided in the following sections. In Section 1.1 we recall rings, integral domains and then identify some special types of elements that a ring may contain. In Section 1.2 we recall ideals, some special types of ideals a ring may contain, the quotient ring of a ring and Zorn's Lemma. In Section 1.3 we define polynomial rings as well as power series rings. In Section 1.4 we recall multiplicatively closed sets and localizations. In Section 1.5 we recall Noetherian rings and characterize them as rings where every (prime) ideal is finitely generated. In Section 1.6 we recall the notions of integrality and almost integrality. In Section 1.7 we recall many of the well studied classes of integral domains, including Euclidean domains, principal ideal domains, unique factorization domains, atomic domains, Dedekind domains, Prüfer domains, GCD domains and Bézout domains. Lastly, in Section 1.8 we recall valuation domains and look at many of their properties.

1.1. Rings and Integral Domains

Some sets are naturally endowed with two binary operations: addition and multiplication. The most familiar example of such a set is the integers which we denote by

$$\mathbb{Z} = \{\ldots, -3, -2, -1, 0, 1, 2, 3, \ldots\}.$$

Other familiar examples include the integers modulo n, the real numbers, matrices, and polynomials.

The notion of a ring originated in the mid-nineteenth century through the work of Richard Dedekind. The term ring was first applied in 1897 by David Hilbert; although its first formal abstract definition was not given until 1914 by Abraham Fraenkel [15]. However, Fraenkel's definition was marred by the inclusion of some ad hoc assumptions that are not appropriate for general theory. The concept of a ring, as seen below, is due to Emmy Noether who formulated it in a paper in 1921. Before this the term "Zahlring" had occurred in algebraic number theory [19]. Formally defining axioms for rings and fields took place in the nineteenth century, mainly as an extension of algebraic number theory [25].

Definition 1.1.1. A **ring** R is a set together with two binary operations $+$ and $\cdot$ (called addition and multiplication) satisfying the following conditions:

1. $(R, +)$ is an abelian group,
2. $\cdot$ is associative: $(a \cdot b) \cdot c = a \cdot (b \cdot c)$ for all $a, b, c \in R$,
3. the distributive laws hold in R: for all $a, b, c \in R$, $(a+b) \cdot c = a \cdot c + b \cdot c$ and $a \cdot (b+c) = a \cdot b + a \cdot c$.

We say that the ring R is **commutative** if the multiplication operation is commutative, that is, $a \cdot b = b \cdot a$ for all $a, b \in R$. The ring R is said to have an **identity** if there exists some element $1_R \in R$ such that $1_R \cdot a = a \cdot 1_R = a$ for all $a \in R$.

To a certain degree, the notion of a ring was invented in an attempt to put the algebraic properties of the integers into an abstract setting. A ring is not the appropriate abstraction of the integers, however, for too much is lost in the process. Integral domains are a class of rings which share the essential features of the integers, which rings in general do not enjoy: commutativity, existence of an identity, and the cancellation property. Integral domains play a prominent role in number theory and algebraic geometry [15].

Definition 1.1.2. If R is a commutative ring with identity, then R is an **integral domain** if $x, y \in R$ and $x \cdot y = 0$ implies that either $x = 0$ or $y = 0$.

Many of the more familiar rings are in fact integral domains. For example, $\mathbb{Z}$ is an integral domain under the usual addition and multiplication as are the integers modulo n where n is prime.

Dedekind gave the definition of a field in 1871 [25]. The rational numbers, denoted by $\mathbb{Q}$, the real numbers, denoted by $\mathbb{R}$, and the complex numbers, denoted by $\mathbb{C}$, are all examples of fields. But any field is an integral domain, so $\mathbb{Q}$, $\mathbb{R}$, and $\mathbb{C}$ are examples of integral domains. We remark that the complex numbers were introduced by Bombelli in 1572 [25]. The concept of field was implicit in the work of Abel and Galois in the theory of equations, but it became explicit when Dedekind introduced number fields of finite degree as the setting for algebraic number theory [25].

An example of an integral domain that "behaves like" the integers is $\mathbb{Z}[i]$, the set of numbers of the form $a + bi$, where $a, b \in \mathbb{Z}$. This domain is called the Gaussian integers, because Gauss, around 1832, was the first to study them and prove their basic properties. $\mathbb{Z}[i]$ is like $\mathbb{Z}$ in being closed under the operations $+, -, \times$, but also in having primes and unique prime factorization [25]. The Gaussian integers is another example of an integral domain.

Throughout this thesis, all rings will be considered to be commutative with identity, unless specified otherwise. Furthermore, we will write ab for $a \cdot b$.

A ring may contain several different types of elements. Irreducible elements, or atoms, are the basic building blocks of factorization theory. For integral domains, which we will be concerned with, the notion of prime is a specialization of irreducible.

Definition 1.1.3. Let R be an integral domain and $x \in R$.

1. We say that x is a **unit** if there exists some element $y \in R$ such that $xy = 1_R$. We note that such a y is usually denoted as x^{-1}.

2. We say that two elements a and b of R are **associate** if $a = ub$ for some unit $u \in R$.

3. We say that x is **idempotent** if $x^2 = x$.

4. We say that x is **irreducible** (or an atom) if x is a nonzero nonunit and $x = ab$ implies that a or b is a unit in R.

5. We say that x is **nilpotent** if $x^m = 0$ for some $m \in \mathbb{N}$.

6. We say that x is **prime** if x is a nonzero nonunit and $x \mid ab$ implies that $x \mid a$ or $x \mid b$.

7. We say that x is a **zero divisor** if there exists a nonzero element $y \in R$ such that $xy = 0$. We remark that an integral domain is a ring which contains no zero divisors other than 0.

Note that in the general setting, 0 is a prime element if and only if R is an integral domain. Furthermore, note that 0 is not an irreducible element.

We show in Proposition 1.2.4 that, in an integral domain R, one can determine whether the element x in R is a unit, a prime element or an irreducible element by simply looking at the ideal generated by x.

As the notion of prime versus irreducible will be of importance throughout the remainder of this thesis, we examine the relationship between these two types of elements.

Proposition 1.1.4. *Let R be an integral domain. Every nonzero prime element of R is also an irreducible element of R.*

Proof. Let R be a integral domain. Suppose that x is a nonzero prime element of R. To show that x is irreducible, suppose $x = ab$. Since $x = ab$, we have that $x|ab$. But x is prime, so we have that $x|a$ or $x|b$. Without loss of generality, we assume that $x|a$. Then we can write $a = rx$ for some $r \in R$. Plugging in for a we have $x = ab = rxb$, and so $x - rxb = 0$ which gives that $x(1 - rb) = 0$. Since R is an integral domain either x is zero or $1 - rb$ is zero. However x is nonzero by assumption, thus we have that $1 - rb = 0$, or $1 = rb$, making b a unit. Thus we have that x is irreducible. □

However, it is not true in general, that every irreducible element is a prime element. An example of such a ring is $\mathbb{Z}[\sqrt{-5}]$; in this ring we have that $2 \mid 6$ and $6 = (1 + \sqrt{-5})(1 - \sqrt{-5})$, but 2 does not divide $1 \pm \sqrt{-5}$. This shows that 2 is not prime in $\mathbb{Z}[\sqrt{-5}]$. By a norm argument, one can show that 2 is irreducible in $\mathbb{Z}[\sqrt{-5}]$.

It is the case that the prime elements and the irreducible elements coincide for a very specific type of integral domain, called an AP-domain, which we define in Section 1.7.

Next we give the definition of overring, as it will arise in several of the following sections.

Definition 1.1.5. Let R be an integral domain with quotient field K. An **overring** T of R is an integral domain such that $R \subseteq T \subseteq K$.

1.2. Ideal Theory

Ideals are important in the study of the structure of rings. One of the historical reasons for introducing the concept of ideal was to obtain some sort of unique factorization theorems (for ideals) in rings of algebraic integers in which factorization of elements was not necessarily unique [17].

Definition 1.2.1. Let R be a ring and $J \subseteq R$ a nonempty set. We say that J is an **ideal** of R if

1. $x, y \in J$ implies that $x - y \in J$, and
2. for every $x \in J$, $r \in R$, we have $rx \in J$.

As discussed in the previous section, a ring may contain several different types of elements. In a similar fashion, a ring may contain several different types of ideals.

Definition 1.2.2. Let R be a ring and $I \subsetneq R$ an ideal.

1. We say that I is **finitely generated** if I is generated by a finite set.
2. We say that I is **irreducible** if $I = J_1 \cap J_2$ implies that $J_1 = I$ or $J_2 = I$.
3. We say that I is **maximal** if given J such that $I \subseteq J \subsetneq R$, then $J = I$.
4. We say that I is **prime** if $ab \in I$ implies that $a \in I$ or $b \in I$.
5. We say that I is **principal** if I is generated by a single element.

Example 1.2.3.

1. The zero ideal in any integral domain is prime since $ab = 0$ if and only if $a = 0$ or $b = 0$ [17].
2. If p is a prime integer, then the principal ideal (p) in $\mathbb{Z}$ is prime since $ab \in (p)$ implies that $p \mid ab$ which implies that $p \mid a$ or $p \mid b$ which implies that $a \in (p)$ or $b \in (p)$ [17].

Proposition 1.2.4. *Let R be an integral domain and let $x \in R$.*

1. *The element x is a unit if and only if $(x) = R$.*
2. *The element x is prime if and only if (x) is a prime ideal in R.*
3. *The element x is irreducible if and only if (x) is maximal in the set S of all proper principal ideals of R.*

Proof. Let R be an integral domain and suppose that $x \in R$.

1. For the forward implication, we assume that x is a unit. Then there exists $x^{-1} \in R$ such that $xx^{-1} = 1_R$. We show that $(x) = R$ by demonstrating both set inclusions. For $\subseteq$: Let $a \in (x)$. Then we can write $a = rx$ for some $r \in R$. Since $r \in R$, $x \in R$, and R is closed under multiplication, we have that $rx \in R$. Thus $a = rx \in R$. For $\supseteq$: Let $b \in R$. We can write $b = b \cdot 1_R = bxx^{-1} = (bx^{-1})x$. Since $b \in R$, $x^{-1} \in R$, and R is closed under multiplication, we have that $bx^{-1} \in R$. Hence $b \in (x)$. For the reverse implication, we suppose that $(x) = R$. Since $1_R \in R$ and $R = (x)$, we have that $1_R \in (x)$. Thus $1_R = yx$ for some $y \in R$. Thus, by definition, x is a unit.

2. For the forward implication, we assume that x is a prime element. To show that (x) is a prime ideal in R, we suppose that $ab \in (x)$. Then $ab = vx$ for some $v \in R$. Notice that $x \mid vx$, so we have that $x \mid ab$. But x is a prime element, so either $x \mid a$ or $x \mid b$. Without loss of generality, we assume that $x \mid a$. Then $a = wx$ for some $w \in R$. Hence $a \in (x)$. Hence we've shown that (x) is a prime ideal. For the reverse implication, assume that (x) is a prime ideal in R. To show that x is a prime element, we suppose that $x \mid mn$. Then $mn = zx$ for some $z \in R$. Notice that $zx \in (x)$, hence $mn \in (x)$. But (x) is a prime ideal, so either $m \in (x)$ or $n \in (x)$. Without loss of generality, we assume that $m \in (x)$. Thus $m = sx$ for some $s \in R$. Notice that $x \mid sx$, hence $x \mid m$. Therefore x is a prime element.

3. For the forward implication, we assume that x is irreducible. Note that (x) is an ideal of R. If (x) is not a proper ideal, we have that $(x) = R$. Then, since $1_R \in R$, we have that $1_R \in (x)$. Thus there exists $\alpha \in R$ such that $\alpha x = 1_R$. Thus x is a unit in R, a contradiction to the definition of irreducible element. Thus (x) must be a proper ideal of R. Next we show that (x) is maximal in the set S of all proper principal ideals of R. If $(x) \subseteq (y)$, then there exists $\beta \in R$ such that $x = \beta y$. Since x is irreducible, either β is a unit in R or y is a unit in R. In the case that β is a unit, we have that $(x) = (y)$. In the case that y is a unit, we have that $(y) = R$. Hence (x) is maximal in S. For the reverse implication, we assume that (x) is maximal in the set S of all proper principal ideals of R. Since (x) is maximal in S, then x is a nonzero nonunit of R. If $x = ab$, then $(x) \subseteq (a)$. But since (x) is maximal in S, we have that either $(x) = (a)$ or $(a) = R$. If $(a) = R$, then a is a unit. If $(x) = (a)$, then $a = xm$ for some $m \in R$. Thus $x = ab = xmb$. Since R is an integral domain, $1_R = mb$, hence b is a unit. Therefore x is irreducible. $\square$

Of particular interest are maximal ideals and prime ideals. For commutative rings, we can nicely characterize maximal ideals and prime ideals by the structure of their quotient rings. Recall that if R is a ring and I is an ideal of R, then the ring R/I, the set of all cosets $r+I = \{r+a \mid a \in I\}$ for all $r \in R$, with addition given by $(r+I)+(r'+I) = (r+r')+I$ and multiplication given by $(r+I)\cdot(r'+I) = rr'+I$, is called the **quotient ring** of R by I.

Proposition 1.2.5. *Assume that R is a commutative ring. The ideal M is a maximal ideal of R if and only if the quotient ring R/M is a field.*

Proof. Assume that R is a commutative ring. For the forward implication, we suppose that M is a maximal ideal of R and let $b \in R$ but $b \notin M$. It suffices to show that $b+M$ has a multiplicative inverse. Consider $B = \{br+m \mid r \in R, m \in M\}$. This is an ideal of R that properly contains M. Since M is maximal, we must have that $B = R$. Thus, $1_R \in B$, say $1_R = bc+n$ where $n \in M$. Then $1_R + M = bc+n+M = bc+M = (b+M)(c+M)$. This shows that every nonzero element of R/M has a multiplicative inverse. Hence R/M is a field. For the reverse implication, we suppose that R/M is a field and B is an ideal of R that properly contains M. Let $b \in B$ but $b \notin M$. Then $b+M$ is a nonzero element of R/M and, therefore, there exists an element $c+M$ such that $(b+M)(c+M) = 1_R+M$, the multiplicative identity of R/M. Since $b \in B$, we have that $bc \in B$. Because $1_R+M = (b+M)(c+M) = bc+M$, we have that $1-bc \in M \subset B$. So, $1_R = (1-bc)+bc \in B$. This implies that $B = R$. Hence M is maximal. □

Proposition 1.2.6. *Assume that R is a commutative ring. The ideal P is a prime ideal of R if and only if the quotient ring R/P is an integral domain.*

Proof. Assume that R is a commutative ring. For the forward implication, we note that R/P is a commutative ring with identity for any proper ideal P. Thus, it suffices to show that when P is prime, R/P has no zero divisors. So, suppose that P is a prime ideal and $(a+P)(b+P) = 0+P = P$. Then $ab \in P$ and, since P is prime, we have that $a \in P$ or $b \in P$. Hence, one of $a+P$ or $b+P$ is the zero coset in R/P. Thus R/P is an integral domain. For the reverse implication, we suppose that R/P is an integral domain and $ab \in P$. Then $(a+P)(b+P) = ab+P = P$, the zero element of the ring R/P. So either $a+P = P$ or $b+P = P$. That is, either $a \in P$ or $b \in P$. Hence P is prime. □

For the integers, $\mathbb{Z}$, the maximal ideals and the nonzero prime ideals coincide. This is not true in general, but we do get that every maximal ideal is a prime ideal.

Proposition 1.2.7. *Assume that R is a commutative ring. Every maximal ideal of R is a prime ideal.*

Proof. We assume that R is a commutative ring and we let M be a maximal ideal of R. Since M is maximal, we have that R/M is a field. But a field is an integral domain. So R/M is an integral domain. Hence M is a prime ideal by Proposition 1.2.6. □

Recall that a nonempty set A is **partially ordered** by a relation $\leq$ if $\leq$ is reflexive, antisymmetric, and transitive. Let the nonempty set A be partially ordered by $\leq$. A subset B of A is called a **chain** if for all $x, y \in B$ either $x \leq y$ or $y \leq x$. An **upper bound** for a subset B of A is an element $u \in A$ such that $b \leq u$ for all $b \in B$. A **maximal element** of A is an element $m \in A$ such that if $m \leq x$ for any $x \in A$ then $m = x$. With these definitions at hand, we can recall Zorn's Lemma which states that if A is a nonempty partially ordered set in which every chain in A has an upper bound in A, then A contains a maximal element. Zorn's Lemma is a powerful tool and will be used in this thesis.

Proposition 1.2.8. *Let R be a commutative ring with identity. Then every proper ideal of R is contained in a maximal ideal of R.*

Proof. We mimic the proof given in [17]. Let R be a commutative ring with identity and let I be a proper ideal of R. Let Γ be the set of all proper ideals of R which contain I. Then Γ is nonempty, since $I \in \Gamma$, and is partially ordered by inclusion. If $\mathcal{C}$ is a chain in Γ, we define $J = \cup_{A \in \mathcal{C}} A$. We first show that J is an ideal. J is nonempty since $\mathcal{C}$ is nonempty, specifically $0 \in J$ since 0 is in every ideal A. If $a, b \in J$, then there are ideals $A, B \in \mathcal{C}$ such that $a \in A$ and $b \in B$. By definition of a chain, either $A \subseteq B$ or $B \subseteq A$. In either case, $a - b \in J$, so J is closed under subtraction. Since $A \in \mathcal{C}$ is closed under left and right multiplication by elements of R, so is J. Thus J is an ideal of R. If J is not a proper ideal, then $1_R \in J$. In this case, by definition of J, we must have that $1_R \in A$ for some $A \in \mathcal{C}$. This is a contradiction because each $A \in \mathcal{C} \subseteq \Gamma$ is a proper ideal. Thus J is a proper ideal of R, so $J \in \Gamma$. Hence each chain in Γ has an upper bound in Γ. By Zorn's Lemma, Γ has a maximal element, which is therefore a maximal (proper) ideal containing

I. Since I was an arbitrary proper ideal, we have that every proper ideal is contained in a maximal ideal. $\square$

Thus, in a commutative ring with identity, every proper ideal is contained in some maximal ideal, but each maximal ideal is prime, so we have that every proper ideal is contained in some prime ideal.

Corollary 1.2.9. *Any commutative ring with identity has a maximal ideal.*

Proof. Let R be a commutative ring with identity. Since $\{0\}$ is a proper ideal of R, by Proposition 1.2.8, there exists a maximal ideal M of R such that $\{0\} \subseteq M$. $\square$

Hence every commutative ring with identity has at least one maximal ideal. But one may wonder exactly how many maximal ideals a given commutative ring with identity may have: exactly one, finitely many, or infinitely many. This leads us to the following definitions.

Definition 1.2.10. An integral domain R is said to be

1. a **field** if every nonzero element of R is a unit.
2. a **local domain** if R is Noetherian and contains a single maximal ideal.
3. a **quasi-local domain** if R contains a single maximal ideal.
4. a **semi-local domain** if R is Noetherian and contains only finitely many maximal ideals.
5. a **semi-quasi-local domain** if R contains only finitely many maximal ideals.

Example 1.2.11.

1. If R is a field, then its only ideals are 0 and R. The ideal R cannot be maximal as it is not proper. But 0 is a maximal ideal. So a field has exactly one maximal ideal.
2. The ideal $n\mathbb{Z}$ of $\mathbb{Z}$ is a maximal ideal if and only if $\mathbb{Z}/n\mathbb{Z}$ is a field. We remark that $\mathbb{Z}/n\mathbb{Z}$ is a field if and only if n is a prime number. Since $\mathbb{Z}$ has infinitely many prime elements we have that $\mathbb{Z}$ has infinitely many maximal ideals.

1.3. Polynomial Rings and Power Series Rings

The study of polynomials dates back to 1650 B.C., when Egyptians were solving certain linear polynomial equations. In 600 B.C., Hindus had learned how to solve quadratic equations. However, polynomials, as we know them today, i.e., polynomials written in our notation, did not exist until approximately 1700 A.D. [22].

Polynomial rings and power series rings are structures of fundamental importance in ring theory. Along with localization, polynomial rings and power series rings are types of ring extensions. We begin this section by defining these and developing some notation.

Definition 1.3.1. Let R be a ring. The **power series ring**, denoted $R[[x]]$, is the set $\left\{\sum_{k=0}^{\infty} r_k x^k \,\middle|\, r_k \in R\right\}$ with addition given by

$$\left(\sum_{k=0}^{\infty} r_k x^k\right)+\left(\sum_{k=0}^{\infty} s_k x^k\right)=\sum_{k=0}^{\infty}(r_k+s_k)x^k$$

and multiplication given by

$$\left(\sum_{k=0}^{\infty} r_k x^k\right)\left(\sum_{k=0}^{\infty} s_k x^k\right)=\sum_{k=0}^{\infty} c_k x^k, \text{ where } c_k=\sum_{j=0}^{k} r_j s_{k-j}.$$

The **polynomial ring**, denoted by $R[x]$, is the subring of $R[[x]]$ consisting of all finite sums of the form $\sum_{k=0}^{n} r_k x^k$.

We remark here that we have the inclusions $R \subseteq R[x] \subseteq R[[x]]$. Furthermore, if R is commutative or has an identity, the so does $R[x]$ and $R[[x]]$.

Let $f = \sum_{i=0}^{n} a_i x^i$ be a polynomial in $R[x]$. Then the elements $a_i \in R$ are called the **coefficients** of f and the element a_0 is called the **constant term** of f. If $a_n \neq 0$, then a_n is called the **leading coefficient** of f and we say that f is a polynomial of **degree** n, which we denote by $\deg(f) = n$. If R has an identity and the leading coefficient of f is 1_R, then f is said to be **monic**.

Next we make a several observations. The polynomials of degree 0 in $R[x]$ are exactly those elements from $R \setminus \{0\}$. $0 \in R[x]$ has no degree. Let $f(x)$ and $g(x)$ be two nonzero polynomials in $R[x]$. If $f(x)g(x) \neq 0$, then $\deg(f(x)g(x)) \leq \deg(f(x)) + \deg(g(x))$. In the case that R is an integral domain, we get $\deg(f(x)g(x)) = \deg(f(x)) + \deg(g(x))$. If $f(x) + g(x) \neq 0$, then

$\deg(f(x)+g(x)) \leq \max\{\deg(f(x)), \deg(g(x))\}$. Also, if $f(x)$ and $g(x)$ are monic, then so is $f(x)g(x)$.

The following result is the division algorithm for polynomials.

Proposition 1.3.2. *Let R be an integral domain and let $f, g \in R[x]$ with $f \neq 0$. Assume that the leading coefficient of f is a unit in R. Then there exist $r, q \in R[x]$ such that $g = fq + r$ and either $r = 0$ or deg$(r) <$ deg(f).*

Proof. We mimic the proof given in [18]. Let R be an integral domain and let $f, g \in R[x]$ with $f \neq 0$. Assume that the leading coefficient of f is a unit in R. Write $n = \deg(f)$ and $f(x) = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$. If $g = 0$ or $\deg(g) < n$, we can take $q = 0$ and $r = g$, and there is nothing to prove. We assume, therefore, that $\deg(g) = m \geq n$, and working by induction on m, we assume that the result holds if g is replaced by any polynomial of degree less than m. Let b be the leading coefficient of g and write $h(x) = ba_n^{-1} f(x)x^{m-n}$. Observe that the degree and leading coefficient of the polynomial $h(x)$ match those of g. If follows that $g - h$ involves no power of x as high as x^m, and so either $g - h = 0$ or $\deg(g - h) < m$. Our result thus holds for $g - h$, and we can write $g - h = fq + r$ with $r = 0$ or $\deg(r) < n$. Thus we have that $g = h + fq + r$ and since h is a multiple of f, the result follows. $\square$

As in [7], we define the following. If $f(x) = \sum_{i=0}^{\infty} a_i x^i$ is a nonzero power series in $R[[x]]$, by the **order** of f we shall mean the nonnegative integer n such that $a_i = 0$ for $i < n$ and $a_n \neq 0$. Further, we write $\phi(f)$ for the order of the power series f and if f has order n, we call a_n the **initial coefficient** of f.

Next we make several observations. $\phi(f + g) \geq \min\{\phi(f), \phi(g)\}$ for all $f, g \in R[[x]]$ with equality assured if $\phi(f) \neq \phi(g)$. $\phi(fg) \geq \phi(f) + \phi(g)$ for all $f, g \in R[[x]]$. Suppose that $f = \sum_{i=m}^{\infty} a_i x^i$ and $g = \sum_{i=n}^{\infty} b_i x^i$ with $a_m \neq 0 \neq b_n$. If a_m or b_n is not a zero divisor in R, then $\phi(fg) = \phi(f) + \phi(g)$. Hence, in the case that R is an integral domain, we have that $\phi(fg) = \phi(f) + \phi(g)$ for all $f, g \in R[[x]]$ since R has no zero divisors.

Now we give some useful results for polynomial rings.

Proposition 1.3.3. *Let R be an integral domain. Then (x) is a prime ideal of $R[x]$.*

Proof. Let R be an integral domain. Since $R[x]/(x) \cong R$ and R is an integral domain, by Proposition 1.2.6, we have that (x) is prime in $R[x]$. $\square$

Proposition 1.3.4. *If R is an integral domain, then $R[x]$ is also an integral domain.*

Proof. Let R be an integral domain. Suppose that $fg = 0$ in $R[x]$ and that neither f nor g is the zero polynomial. If $\deg(f) = n > 0$, then $\deg(0) = \deg(fg) = \deg(f) + \deg(g) = n + \deg(g) > 0$, a contradiction. Hence the degrees of both f and g are 0 and hence they are both in R. So $fg = 0$ for two nonzero elements of R, which is a contradiction to R being an integral domain. Thus $R[x]$ is an integral domain. $\square$

The following result characterizes the units of a polynomial ring over an integral domain.

Proposition 1.3.5. *Let R be an integral domain and $U(R)$ be the set of units in R. Then $U(R) = U(R[x])$.*

Proof. Let R be an integral domain. We denote the set of units in R by $U(R)$ and we denote the set of units in $R[x]$ by $U(R[x])$. We demonstrate $U(R) = U(R[x])$ by showing both set inclusions. For $\subseteq$: Let $a \in U(R)$. Then there exists $b \in R$ such that $ab = 1_R$. But $R \subseteq R[x]$. So $a, b \in R[x]$ and we have that $ab = 1_R = 1_{R[x]}$. Hence $a \in U(R[x])$. For $\supseteq$: Suppose that $f(x) \in U(R[x])$. Then there exists $g(x) \in R[x]$ such that $f(x) \cdot g(x) = 1_{R[x]}$. Hence we must have that $\deg(f(x) \cdot g(x)) = \deg(f(x)) + \deg(g(x)) = \deg(1_{R[x]}) = 0$. The only way this is possible is that $\deg(f(x)) = \deg(g(x)) = 0$. Hence $f(x)$ and $g(x)$ are both in R, say $f(x) = f_0$ and $g(x) = g_0$. Then, since $f(x) \cdot g(x) = f_0 \cdot g_0 = 1_{R[x]} = 1_R$, we have that $f(x) = f_0 \in U(R)$. $\square$

In Section 1.7 we define a principal ideal domain to be an integral domain with the property that every ideal is principal. Next we show that the polynomial extenstion of a field turns out to be a principal ideal domain.

Proposition 1.3.6. *Let F be a field. Then $F[x]$ is a principal ideal domain.*

Proof. We mimic the proof given in [18]. Let F be a field. Since any field is an integral domain, by Proposition 1.3.4, we have that $F[x]$ is an integral domain. So it remains to show that every ideal in $F[x]$ is principal. Suppose that I is a nonzero ideal of $F[x]$. Then I contains some nonzero elements, and we can choose $f \in I$ with $\deg(f)$ as small as possible. Since $f \in I$ and I is an ideal, we have that $(f) \subseteq I$. Now let $g \in I$ and write $g = qf + r$, by the Division Algorithm for polynomials, where $r = 0$ or $\deg(r) < \deg(f)$. Then $r = g - fq$ is an element of I, and by our choice of f we cannot have $\deg(r) < \deg(f)$. Thus we have that $r = 0$. Hence $g = fq \in (f)$ and so $I \subseteq (f)$. Therefore $I = (f)$ and I is principal. Since I was an arbitrary nonzero ideal, every nonzero ideal of $F[x]$ is principal. Hence $F[x]$ is a principal ideal domain as desired. $\square$

There are similar useful results for power series rings.

Proposition 1.3.7. *If R is an integral domain, then (x) is a prime ideal of $R[[x]]$.*

Proof. Let R be an integral domain. Since $R[[x]]/(x) \cong R$ and R is an integral domain, by Proposition 1.2.6, we have that (x) is prime in $R[[x]]$. $\square$

Proposition 1.3.8. *If R is an integral domain, then $R[[x]]$ is also an integral domain.*

Proof. Let R be an integral domain. Suppose that $fg = 0$ in $R[[x]]$ and that neither f nor g is 0. If $\phi(f) = n > 0$, then since R is an integral domain we have that $\phi(0) = \phi(fg) = \phi(f) + \phi(g) = n + \phi(g) > 0$, a contradiction. Hence the order of both f and g are 0 and hence they are both in R. So $fg = 0$ for two nonzero elements of R, which is a contradiction to R being an integral domain. Thus $R[[x]]$ has no zero divisors and is an integral domain. $\square$

The following result characterizes the units of a power series ring over an integral domain; it states that a power series over an integral domain R is a unit if and only if the constant term of the power series is a unit in R.

Proposition 1.3.9. *Let R be an integral domain and $U(R)$ be the set of units in R. Then we have that $U(R[[x]]) = \{f \in R[[x]] \mid f(0) \in U(R)\}$.*

Proof. Let R be an integral domain and $U(R)$ the set of units in R. We demonstrate both set inclusions. For $\subseteq$: Let $h(x) \in U(R[[x]])$, say $h(x) = \sum_{i=0}^{\infty} a_i x^i$ with each $a_i \in R$. Then there exists $g(x) \in R[[x]]$, say $g(x) = \sum_{i=0}^{\infty} b_i x^i$ with each $b_i \in R$, such that $h(x)g(x) = 1_{R[[x]]}$. That is, $\left(\sum_{i=0}^{\infty} a_i x^i\right) \cdot \left(\sum_{i=0}^{\infty} b_i x^i\right) = a_0 b_0 + \sum_{i=1}^{\infty} c_i x^i = 1_{R[[x]]} = 1_R + \sum_{i=1}^{\infty} 0 \cdot x^i$, where the c_i's are the corresponding coefficients from multiplying $h(x)$ and $g(x)$. In order for $a_0 b_0 + \sum_{i=1}^{\infty} c_i x^i = 1_R + \sum_{i=1}^{\infty} 0 \cdot x^i$, we must have that $c_i = 0$ for every $i \geq 1$. Thus, $a_0 b_0 = 1_R$ and hence $a_0, b_0 \in U(R)$. Thus $h(0) = a_0 \in U(R)$. So $h(x) \in \{f \in R[[x]] \mid f(0) \in U(R)\}$. For $\supseteq$: Let $k(x) \in \{f \in R[[x]] \mid f(0) \in U(R)\}$. So $k(x) \in R[[x]]$, say $k(x) = \sum_{i=0}^{\infty} d_i x^i$ with each $d_i \in R$, and $k(0) = d_0 \in U(R)$.

Thus there exists $f \in R$ such that $d_0 f = 1_R$. We wish to construct $m(x) = \sum_{i=0}^{\infty} f_i x^i \in R[[x]]$ so that $k(x)m(x) = 1_{R[[x]]}$. Recall that the product of $k(x)$ and $m(x)$ is

$$k(x) \cdot m(x) = \left(\sum_{i=0}^{\infty} d_i x^i\right) \cdot \left(\sum_{i=0}^{\infty} f_i x^i\right) = \sum_{k=0}^{\infty}\left(\sum_{m=0}^{k} d_m f_{k-m} x^k\right).$$

Thus we have

$$\sum_{k=0}^{\infty}\left(\sum_{m=0}^{k} d_m f_{k-m} x^k\right) = 1_R + \sum_{j=1}^{\infty} 0 \cdot x^j. \tag{1.1}$$

Using this equation, we now construct the coefficients f_i of $m(x)$ by induction. For the base case, $n = 0$, from (1.1) we have that $d_0 f_0 = 1_R$. Multiplying both sides by f yields $f d_0 f_0 = f \cdot 1_R = f$, but $f d_0 = 1_R$, so we have $f_0 = f$. Suppose that for some n we have that $f_n = f(-d_1 f_{n-1} - d_2 f_{n-2} - \cdots - d_{n-1} f_1 - d_n f_0)$. From (1.1) we have that the $(n+1)^{\text{th}}$ coefficient of $k(x) \cdot m(x)$ is given by $\sum_{i=0}^{n+1} d_i f_{n+1-i} = d_0 f_{n+1} + d_1 f_n + \cdots + d_n f_1 + d_{n+1} f_0$. But this must be equal to 0. We solve for $d_0 f_{n+1}$ getting $d_0 f_{n+1} = -d_1 f_n - d_2 f_{n-1} - \cdots - d_n f_1 - d_{n+1} f_0$. Now multiplying each side by f and recalling that $f d_0 = 1_R$, we get that $f_{n+1} = f(-d_1 f_n - d_2 f_{n-1} - \cdots - d_n f_1 - d_{n+1} f_0)$, as desired. We remark here that f is a known value, each d_i is a known value, and f_j for $1 \leq j \leq n$ are also known values. Thus, we can construct all of the coefficients of $m(x)$. Therefore we have found $m(x)$ such that $k(x) \cdot m(x) = 1_{R[[x]]}$. Thus $k(x) \in U(R[[x]])$. Hence we've demonstrated that $U(R[[x]]) = \{f \in R[[x]] \mid f(0) \in U(R)\}$. □

It turns out that if the constant term of a power series is irreducible, then the power series is irreducible.

Proposition 1.3.10. *Let R be an integral domain and $f = \sum_{i=0}^{\infty} a_i x^i \in R[[x]]$. If a_0 is irreducible in R, then f is irreducible in $R[[x]]$.*

Proof. Let R be an integral domain and $f = \sum_{i=0}^{\infty} a_i x^i \in R[[x]]$. Suppose that a_0 is irreducible in R. Assume that we have that $f = gh$ in $R[[x]]$, where $g = \sum_{i=0}^{\infty} b_i x^i$ and $h = \sum_{i=0}^{\infty} c_i x^i$. Thus we have that $a_0 + a_1 x + a_2 x^2 + \cdots = f = gh = b_0 c_0 + (b_0 c_1 + b_1 c_0)x + (b_0 c_2 + b_1 c_1 + b_2 c_0)x^2 + \cdots$. Hence we have that $a_0 = b_0 c_0$ in R. But a_0 is irreducible, so either b_0 or c_0 is a unit. Without loss of generality, say b_0 is a unit in R. Thus, by Proposition 1.3.9, we have that g is a unit in $R[[x]]$. Therefore f is irreducible in $R[[x]]$ as desired. □

We remark here that if $f \in R[[x]]$ is actually a polynomial with an irreducible constant term then f need not be irreducible in the polynomial ring $R[x]$. To see this, consider $g = x^2 + 3x + 2$. g is irreducible in $\mathbb{Z}[[x]]$ by Proposition 1.3.10 since its constant term 2 is irreducible in $\mathbb{Z}$, but it is not irreducible in $\mathbb{Z}[x]$ since $x^2 + 3x + 2 = (x + 1)(x + 2)$ but neither $x + 1$ nor $x + 2$ is a unit in $\mathbb{Z}[x]$. To see that $x + 1$ is not a unit in $\mathbb{Z}[x]$, suppose for a contradiction there exists $h \in \mathbb{Z}[x]$, say $h = a_n x^n + a_{n-1}x^{n-1} + \cdots + a_1 x + a_0$, with $(x + 1)h = 1_{\mathbb{Z}[x]} = 1_{\mathbb{Z}}$. Multiplying out $(x + 1)h$ gives us $a_n x^{n+1} + (a_{n-1} + a_n)x^n + \cdots + (a_0 + a_1)x + a_0 = 1_{\mathbb{Z}}$. Equating coefficients on each side yields that we must have $a_n = 0$, then $a_{n-1} + a_n = 0$ and so $a_{n-1} = 0$, $\cdots$, then $a_0 + a_1 = 0$ and so $a_0 = 0$, then from equating the constant terms we have that $a_0 = 1_{\mathbb{Z}}$, a contradiction. So there does not exist $h \in \mathbb{Z}[x]$ such that $(x + 1)h = 1_{\mathbb{Z}[x]} = 1_{\mathbb{Z}}$. Therefore $x + 1$ is not a unit in $\mathbb{Z}[x]$. A similar argument will show that $x + 2$ is not a unit in $\mathbb{Z}[x]$.

One commonly asked question in commutative ring theory is given property "X" of a ring R, does $R[x]$ or $R[[x]]$ have property "X". In some cases we can answer this question positively. For example, if R is a unique factorization domain, then $R[x]$ is also a unique factorization domain. Another positive example is that if R is a commutative Noetherian ring with identity, then both $R[x]$ and $R[[x]]$ are also commutative Noetherian rings with identity [17]. Unfortunately, there are cases where the answer to the question is negative. For example, if R is a unique factorization domain, then $R[[x]]$ is not a unique factorization domain.

1.4. Multiplicative Sets and Localization

Multiplicative subsets of an integral domain reveal a lot about its multiplicative structure. Furthermore, the multiplicative subsets of an integral domain determine its various rings of fractions.

Definition 1.4.1. Let R be an integral domain. A nonempty subset $S \subseteq R$ is said to be **multiplicatively closed** if $s, t \in S$ implies that $st \in S$. We assume that if S is multiplicatively closed, then $0 \notin S$. A multiplicatively closed set $S \subseteq R$ is said to be **saturated** if $ab \in S$ implies that $a \in S$ and $b \in S$.

Example 1.4.2.

1. The set of all nonzero elements in an integral domain is multiplicatively closed.

2. The set of units in any ring with identity is multiplicatively closed.

3. If P is a prime ideal in a commutative ring R, then both P and $S = R\setminus P$ are multiplicatively closed.

Proposition 1.4.3. *Let R be a commutative ring with identity and $S \subseteq R$ a multiplicatively closed set. If I is a proper ideal of R such that $I \cap S = \emptyset$, then there exists a prime ideal $P \supseteq I$ maximal with respect to $P \cap S = \emptyset$.*

Proof. Let R be a commutative ring with identity and $S \subseteq R$ a multiplicatively closed set. We assume that I is a proper ideal of R such that $I \cap S = \emptyset$. Now consider $\Gamma = \{J \supseteq I \mid J \text{ is an ideal of } R \text{ and } J \cap S = \emptyset\}$. Observe that Γ is nonempty since $I \in \Gamma$. Γ is partially ordered by set inclusion. To apply Zorn's Lemma, we need that every chain in Γ has an upper bound in Γ. Suppose that $\mathcal{C} = \{J_\alpha\}_{\alpha\in\Lambda}$ is a chain in Γ and let $M = \bigcup_{\alpha\in\Lambda} J_\alpha$. Notice that M is an upper bound and we demonstrate that $M \in \Gamma$. Let $r \in R$ and $x \in M$. Then $x \in J_\beta$ for some $\beta \in \Lambda$. Since J_β is an ideal, we have that $rx \in J_\beta \subseteq \bigcup_{\alpha\in\Lambda} J_\alpha = M$. Hence $rx \in M$. Now let $y, z \in M$. Then $y \in J_\lambda$ and $z \in J_\omega$ for some $\lambda, \omega \in \Lambda$. Without loss of generality, we have that $J_\lambda \subseteq J_\omega$. So $y, z \in J_\omega$, and since J_ω is an ideal, we have that $y - z \in J_\omega \subseteq \bigcup_{\alpha\in\Lambda} J_\alpha = M$. Hence $y - z \in M$. Therefore M is an ideal of R. Now notice that since $J_\alpha \supseteq I$ for each α, we have that $M \supseteq I$. Lastly, suppose that $m \in M \cap S$; hence $m \in \left(\bigcup_{\alpha\in\Lambda} J_\alpha\right) \cap S$. Thus $m \in J_\alpha$ for some $\alpha \in \Lambda$ and $m \in S$. So $m \in J_\alpha \cap S$, a contradiction. Hence M is an upper bound for $\mathcal{C}$ with M in Γ. By Zorn's Lemma, there exists a maximal element of Γ; we call this maximal element P. Next we show that P is prime. Suppose $a, b \in R$ with $ab \in P$ but $a \notin P$ and $b \notin P$. Since a is not an element of P, we have that $(P, a) \supsetneq P$, and hence $(P, a) \cap S \neq \emptyset$. Similarly, since b is not an element of P, we have that $(P, b) \supsetneq P$, and hence $(P, b) \cap S \neq \emptyset$. Thus there exist $p_1, p_2 \in P$ and $r_1, r_2 \in R$ such that $p_1 + r_1 a = s_1 \in S$ and $p_2 + r_2 b = s_2 \in S$. Then, $(p_1 + r_1 a)(p_2 + r_2 b) = p_1 p_2 + p_1 r_2 b + p_2 r_1 a + r_1 a r_2 b = s_1 s_2$. But each of the four terms $p_1 p_2$, $p_1 r_2 b$, $p_2 r_1 a$, and $r_1 a r_2 b$ are in P. So $s_1 s_2 \in P$, yielding that $s_1 s_2 \in P \cap S = \emptyset$, a contradiction. Thus P must be prime. $\square$

Proposition 1.4.4. *Let $S \subseteq R$ be a multiplicatively closed set, so $0 \notin S$. The following are equivalent:*

1. S is saturated.

2. $S^c = \bigcup_{P\in\Gamma} P$*, where* Γ *is any collection of prime ideals of* R*.*

Proof. Let $S \subseteq R$ be a multiplicatively closed set, so $0 \notin S$. We first show that (1) implies (2). Let $x \in S^c$. Since S is saturated and $(x) \cap S = \emptyset$, by Zorn's Lemma, there exists a prime ideal $P_x \supseteq (x)$ such that $P_x \cap S = \emptyset$. Hence any $x \in S^c$ is a prime disjoint from S. Thus $S^c = \bigcup_{x\in S^c} P_x$. Next we show that (2) implies (1). Assume that $S^c = \bigcup_{P\in\Gamma} P$, where Γ is some collection of prime ideals of R, and $xy \in S$ with x not in S. Since x is not in S, we have that $x \in \bigcup_{P\in\Gamma} P$. Thus x is in one of the prime ideals in the union, say x is in P. But P is an ideal, so xy is in P. Thus $xy \in \bigcup_{P\in\Gamma} P = S^c$, a contradiction. Therefore S is saturated. □

Definition 1.4.5. Let R be an integral domain and $S \subseteq R$ a multiplicatively closed set. We define the **localization** of R at S to be

$$R_S = \left\{ \frac{r}{s} \,\middle|\, r \in R, s \in S \right\}$$

with addition given by

$$\frac{r_1}{s_1} + \frac{r_2}{s_2} = \frac{r_1 s_2 + r_2 s_1}{s_1 s_2}$$

and multiplication given by

$$\frac{r_1}{s_1}\frac{r_2}{s_2} = \frac{r_1 r_2}{s_1 s_2}.$$

We remark that if R is a domain with quotient field K, and $S \subseteq R$ is a multiplicative set, then we have that a localization is always an overring of R, that is $R \subseteq R_S \subseteq K$. However, it is not true in general that an overring is a localization.

The most frequent type of localization considered in this thesis is when the multiplicatively closed set is given by $S = R \setminus P$, where P is a prime ideal of R. As an abuse of notation, this is often denoted by $R_P = \{\frac{r}{s} \mid r \in R, s \notin P\}$. An important consequence of this construction is that R_P is a local ring with maximal ideal $P_P \subseteq R_P$ [17].

Proposition 1.4.6. *Let* R *be a ring and* $S \subseteq R$ *be a multiplicatively closed set. Then every ideal of* R_S *is of the form* $I_S = \{\frac{i}{s} \mid i \in I, s \in S\}$ *for some ideal* I *of* R*.*

Proof. Let R be a ring and $S \subseteq R$ be a multiplicatively closed set. Let J be an ideal of R_S. Fix $x \in S$ and define $f : R \to R_S$ via $f(r) = \frac{rx}{x}$. Let $I = f^{-1}(J) = \{r \in R \mid f(r) \in J\}$. We show, by demonstrating both set containments, that $I_S = J$. For $\subseteq$: Let $j \in J$. We write $j = \frac{r}{s}$ with $r \in R$ and $s \in S$. Note that we can write $sj = \frac{rx}{x}$ in R_S. Thus $sj = f(r)$, making $r \in f^{-1}(J) = I$. Hence $j = \frac{r}{s}$ with $r \in I$ and $s \in S$. Therefore $j \in I_S$. For $\supseteq$: Let $\frac{i}{s} \in I_S$. By definition, we have that $i \in f^{-1}(J)$, and so we have that $f(i) \in J$. Thus, $f(i) = \frac{ix}{x} \in J$. Since $\frac{ix}{x} \in J$ and J is an ideal of R_S, we have that $\left(\frac{ix}{x}\right)\left(\frac{x}{xs}\right) = \frac{i}{s} \in J$. Hence we have that $I_S = J$. Since J was an arbitrary ideal of R_S, we have that every ideal of R_S is of the form I_S for some ideal I of R. $\square$

1.5. Noetherian Rings

Noetherian rings are a class of rings which satisfy the ascending chain condition on ideals; this turns out to be equivalent to the class of rings in which every (prime) ideal is finitely generated. In this section, we recall Noetherian rings, identify several examples, and look at some of the nice properties of Noetherian rings.

Definition 1.5.1. Let R be a ring. We say that R is **Noetherian** if given any chain of ideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots \subseteq I_n \subseteq \cdots$ there exists $N \in \mathbb{N}$ such that $I_n = I_N$ for all $n \geq N$.

Example 1.5.2.

1. Every principal ideal domain is Noetherian by Proposition 1.5.3 since every ideal of a principal ideal domain is finitely generated. Thus both $\mathbb{Z}$ and $\mathbb{Z}[i]$ are Noetherian.

2. The polynomial ring $F[x]$ where F is a field is Noetherian [14].

3. The ring $\mathbb{Z}[x_1, x_2, \cdots]$ is not Noetherian since the ideal $(x_1, x_2, \cdots)$ cannot be generated by any finite set (any finite set of generators involves only finitely many of the x_i) [17].

Next we provide equivalent characterizations of Noetherian rings; we will frequently use the characterization that in a Noetherian ring every ideal is finitely generated.

Proposition 1.5.3. *Let R be a ring. The following are equivalent:*

1. *R is Noetherian.*

2. *Every ideal $I \subseteq R$ is finitely generated.*

3. *Every prime ideal $P \subseteq R$ is finitely generated.*

Proof. Let R be a ring. We first show that (1) implies (2). Assume that R is Noetherian and let I be an ideal of R. Select $x_1 \in I$. If $I = (x_1)$, we are done. If not, select $x_2 \in I \setminus (x_1)$. Now we have $(x_1) \subsetneq (x_1, x_2)$. If $(x_1, x_2) = I$, we are done. If not, select $x_3 \in I \setminus (x_1, x_2)$. Now we have $(x_1) \subsetneq (x_1, x_2) \subsetneq (x_1, x_2, x_3)$. If $I = (x_1, x_2, x_3)$, we are done. If not, continue this selection process. If this process terminates, we have at some step $I = (x_1, x_2, \cdots, x_n)$, and we are done. If not, we get an infinite ascending chain $(x_1) \subsetneq (x_1, x_2) \subsetneq (x_1, x_2, x_3) \subsetneq \cdots \subsetneq (x_1, x_2, \cdots, x_n) \subsetneq \cdots$, contradicting R being Noetherian. Hence I must be finitely generated. Since I was arbitrary, every ideal of R is finitely generated. Next, we show that (2) implies (1). Assume that every ideal of R is finitely generated. Suppose we have an ascending chain of ideals $I_1 \subseteq I_2 \subseteq I_3 \subseteq \cdots \subseteq I_n \subseteq I_{n+1} \subseteq \cdots$. Consider $I = \bigcup_{n=1}^{\infty} I_n$. I is an ideal of R and hence is finitely generated, say $I = (x_1, x_2, \cdots, x_m)$. So there exists $M \in \mathbb{N}$ such that I_M contains $x_1, x_2, \cdots, x_m$. So $I_M = I$. Thus R is Noetherian. Next we show that (2) implies (3). Assume that every ideal of R is finitely generated. Let P be an arbitrary prime ideal of R. Then, since P is an ideal of R, P must be finitely generated. Lastly, we show that (3) implies (2). Assume that every prime ideal P of R is finitely generated. Suppose R contains an ideal that is not finitely generated. Then, by Zorn's Lemma, R contains an ideal M that is maximal with respect to being not finitely generated. Since M is an ideal maximal with respect to being not finitely generated, M is prime. Hence M, an ideal maximal with respect to being not finitely generated, is prime, and hence finitely generated by assumption, a contradiction. Thus every ideal of R must be finitely generated. $\square$

One should note that the above proposition is not true if we replace prime by maximal in statement 3.

We next demonstrate that the localization of a Noetherian domain at a maximal ideal remains Noetherian.

Proposition 1.5.4. *If R is a Noetherian integral domain and M is a maximal ideal of R, then R_M is also Noetherian.*

Proof. Let R be a Noetherian integral domain and M be a maximal ideal of R. Since M is a maximal ideal, by Proposition 1.2.7, we have that M is a prime ideal and is hence multiplicatively closed. Then, by Proposition 1.4.6 we have that every ideal in R_M is of the form I_M where I is some ideal in R. Since I is an ideal in R, a Noetherian ring, I must be finitely generated. Since

I is finitely generated, I_M will also be finitely generated. Therefore every ideal of R_M is finitely generated making R_M Noetherian. $\square$

1.6. Integrality

We recall that if T is a commutative ring with identity and R is a subring of T containing 1_T, then T is said to be an **extension ring** of R.

Definition 1.6.1. Let $R \subseteq T$ be an extension of rings. An element $y \in T$ is said to be **integral** over R if y is the root of a monic polynomial $p(x) \in R[x]$, that is y satisfies $y^n + r_{n-1}y^{n-1} + \cdots + r_1 y + r_0 = 0$ where each $r_i \in R$.

Example 1.6.2.

1. Every element $r \in R$ is integral over R.

2. The element $\sqrt{2}$ is integral over $\mathbb{Z}$ since $\sqrt{2}$ is a root of $x^2 - 2 \in \mathbb{Z}[x]$.

3. Consider $R = \mathbb{Q}[x^2, x^3] = \{q_0 + q_2x^2 + q_3x^3 + \cdots + q_nx^n \mid q_i \in \mathbb{Q}\}$. Notice that $x \notin R$, however x is a root of $Y^2 - x^2 \in R[Y]$, so x is integral over R.

Definition 1.6.3. Let $R \subseteq T$ be an extension of rings.

1. We call the ring $\overline{R_T} = \{z \in T \mid z \text{ is integral over } R\}$ the **integral closure** of R in T.

2. If $T = K$, where K is the quotient field of domain R, then $\overline{R} = \overline{R_K}$ is the **integral closure** of R.

3. If $R = \overline{R}$, we say that R is **integrally closed**.

4. If every element of T is integral over R, we say that T is an **integral extension** of R.

Example 1.6.4.

1. The integral domain $\mathbb{Z}$ is integrally closed in $\mathbb{Q}$, however $\mathbb{Z}$ is not integrally closed in $\mathbb{C}$ since $i \in \mathbb{C}$ is integral over $\mathbb{Z}$.

2. Suppose d is a square-free integer. Then $\mathbb{Z}[d]$ is integrally closed if and only if $d \cong 2, 3(\text{mod } 4)$.

3. If R is a ring, then $\overline{R}$ is integrally closed.

4. Any unique factorization domain is integrally closed. See the proof of Proposition 1.7.13.

Proposition 1.6.5. *Suppose that $R \subseteq T$ is an extension of rings with T is integral over R. If $r \in R$ is a nonunit, then r is a nonunit of T.*

Proof. Suppose that $R \subseteq T$ is an extension of rings with T is integral over R and suppose that $r \in R$ is a nonunit. For a contradiction, assume that $r \in T$ is a unit. So there exists $t \in T$ such that $rt = 1_T$. Since $t \in T$ and T is integral over R, we have have that t is integral, so we can write $t^n + r_{n-1}t^{n-1} + \cdots + r_1 t + r_0 = 0$ with each $r_j \in R$. Now multiply both sides by r^n to get $(rt)^n + rr_{n-1}(rt)^{n-1} + \cdots + r^{n-1}r_1(rt) + r^n r_0 = 0$. Recalling that $rt = 1_T$, we have that $1_T + rr_{n-1}1_T + \cdots + r^{n-1}r_1 1_T + r^n r_0 = 0$. Solving for 1_T yields $1_T = r(-r_{n-1} - \cdots - r^{n-2}r_1 - r^{n-1}r_0)$. Since each $r_j \in R$ and $r \in R$, we have that $-r_{n-1} - \cdots - r^{n-2}r_1 - r^{n-1}r_0 \in R$. Then, since $R \subseteq T$ is an extension of rings, we have that $1_R = 1_T$, so $1_T = 1_R = r(-r_{n-1} - \cdots - r^{n-2}r_1 - r^{n-1}r_0)$ where each term on the right side is in R. Thus r is a unit in R, a contradiction. Thus r must be a unit in T. □

It turns out that any localization of an integrally closed domain is also integrally closed.

Proposition 1.6.6. *Let R be an integral domain with quotient field K. If R is integrally closed and $S \subseteq R$ is a multiplicatively closed set $(0 \notin S)$, then R_S is integrally closed.*

Proof. Let R be an integral domain with quotient field K. Suppose R is integrally closed and let $S \subseteq R$ be a multiplicatively closed set. We first remark that the quotient field of R_S is exactly the same as the quotient field of R, namely K. Now assume $\lambda \in K$ is integral over R_S. Then we have

$$\lambda^n + \left(\frac{r_{n-1}}{s_{n-1}}\right)\lambda^{n-1} + \cdots + \left(\frac{r_1}{s_1}\right)\lambda + \frac{r_0}{s_0} = 0$$

where each $r_i \in R$ and each $s_j \in S$. Let $s = s_0 s_1 \cdots s_{n-1}$ and $t_i = \frac{s}{s_i}$. Observe that $t_i \in S$ for each i. To clear denominators in the integrality equation we multiply both sides by s yielding

$$s\lambda^n + r_{n-1}t_{n-1}\lambda^{n-1} + \cdots + r_1 t_1 \lambda + r_0 t_0 = 0.$$

Next we multiply each side of this equation by s^{n-1} to get

$$(s\lambda)^n + r_{n-1}t_{n-1}(s\lambda)^{n-1} + \cdots + r_1t_1s^{n-2}(s\lambda) + r_0t_0s^{n-1} = 0.$$

Notice that $r_i \in R$, $s = s_0s_1\cdots s_{n-1} \in S \subseteq R$ and $t_i = \frac{s}{s_i} \in S \subseteq R$. So

$$(s\lambda)^n + r_{n-1}t_{n-1}(s\lambda)^{n-1} + \cdots + r_1t_1s^{n-2}(s\lambda) + r_0t_0s^{n-1} = 0$$

is the integrality equation (over R) for $s\lambda$. But R is integrally closed, so we have that $s\lambda = r \in R$. Hence, solving for λ, gives $\lambda = \frac{r}{s} \in R_S$. Therefore R_S is integrally closed. $\square$

We next examine almost integrality, a notion that is weaker than integrality.

Definition 1.6.7. Let R be an integral domain with quotient field K. An element $\alpha \in K$ is **almost integral** over R if there exists $r \in R \setminus \{0\}$ such that $r\alpha^n \in R$ for every $n \geq 0$.

Definition 1.6.8. Let R be a domain with quotient field K. If R contains all of the elements $\omega \in K$ that are almost integral over R, we say that R is **completely integrally closed**.

With these two definitions in hand, we look at some results relating to almost integrallity. First, we show that any integral element in an integral domain is also almost integral.

Proposition 1.6.9. *Let R be an integral domain with quotient field K. If $\alpha \in K$ is integral over R, then α is almost integral over R.*

Proof. Let R be an integral domain with quotient field K. Suppose that $\alpha = \frac{a}{b} \in K$ is integral over R. Then we have that $\alpha^n + r_{n-1}\alpha^{n-1} + \cdots + r_1\alpha + r_0 = 0$ where each r_i is an element of R. Solving for α^n, we obtain $\alpha^n = -r_{n-1}\alpha^{n-1} - \cdots - r_1\alpha - r_0$. We now show that $b^{n-1}\alpha^m \in R$ for every $m \geq 1$. If $m \leq n-1$, then we have that $b^{n-1}\alpha^m = b^{n-1}\left(\frac{a^m}{b^m}\right) = a^m b^{n-1-m}$. Since $m \leq n-1$ in this case, we have that $n-1-m \geq 0$. So $b^{n-1}\alpha^m = a^m b^{n-1-m} \in R$. We proceed by induction to show that $b^{n-1}\alpha^k$ for $k > n-1$. For the base case, when $k = n$, we have that

$$\begin{aligned} b^{n-1}\alpha^k &= b^{n-1}\alpha^n \\ &= b^{n-1}[-r_{n-1}\alpha^{n-1} - \cdots - r_1\alpha - r_0] \\ &= -r_{n-1}b^{n-1}\alpha^{n-1} - \cdots - r_1b^{n-1}\alpha - r_0b^{n-1}. \end{aligned}$$

Above we showed that $b^{n-1}\alpha^l \in R$ whenever $l \leq n-1$. Hence we have that $-b^{n-1}r_{n-1}\alpha^{n-1}, \cdots,$ $-r_1b^{n-1}\alpha$, and $-r_0b^{n-1}$ are each elements of R. Therefore we have that $b^{n-1}\alpha^k \in R$. For the inductive hypothesis, we suppose that there exists some $k > n-1$ such that for every $j \leq k$ we have that $b^{n-1}\alpha^j \in R$. Now consider $b^{n-1}\alpha^{k+1}$. We write

$$\begin{aligned}
b^{n-1}\alpha^{k+1} &= b^{n-1}[\alpha^{k+1-n}\alpha^n] \\
&= b^{n-1}[\alpha^{k+1-n}(-r_{n-1}\alpha^{n-1} - \cdots - r_1\alpha - r_0)] \\
&= b^{n-1}[-r_{n-1}\alpha^k - \cdots - r_1\alpha^{k-(n-2)} - r_0\alpha^{k-(n-1)}] \\
&= -r_{n-1}b^{n-1}\alpha^k - \cdots - r_1b^{n-1}\alpha^{k-(n-2)} - r_0b^{n-1}\alpha^{k-(n-1)}.
\end{aligned}$$

By the inductive hypothesis, we have that $-r_{n-1}b^{n-1}\alpha^k, \cdots, -r_1b^{n-1}\alpha^{k-(n-2)}$, and $-r_0b^{n-1}\alpha^{k-(n-1)}$ are each elements of R, making $b^{n-1}\alpha^{k+1} \in R$. Therefore, we have shown that $b^{n-1}\alpha^m \in R$ for every $m \geq 1$. Thus α is almost integral over R. $\square$

Example 1.6.10. Consider the ring $R = \mathbb{Q} + x\mathbb{R}[x]$. Consider π; $\pi = \frac{\pi x}{x}$ and $\pi x^n \in R$ for every $n \geq 1$. So π is almost integral over R, however it is not integral over R.

In the Noetherian case, the concepts of integrality and almost integrality coincide, as shown in the following result.

Proposition 1.6.11. *If R is a Noetherian integral domain with quotient field K, then $\omega \in K$ is integral over R if and only if ω is almost integral over R.*

Proof. Let R be Noetherian integral domain with quotient field K. The forward implication follows from Proposition 1.6.9 since R is an integral domain. For the reverse implication, assume that ω is almost integral over R. So there exists a nonzero element $r \in R$ such that $r\omega^n \in R$ for every $n \geq 1$. Consider $I_\omega = (r\omega, r\omega^2, r\omega^3, r\omega^4, \cdots)$. Since R is Noetherian, we must have that the ideal I_ω is finitely generated. So $I_\omega = (r\omega, r\omega^2, r\omega^3, \cdots, r\omega^m)$. Note that $r\omega^{m+1} \in I_\omega$, so we can write $r\omega^{m+1} = s_0r\omega + s_1r\omega^2 + \cdots + s_{m-1}r\omega^m$ where each $s_j \in R$. Thus we have that $\omega^{m+1} = s_0\omega + s_1\omega^2 + \cdots + s_{m-1}\omega^m$ or $\omega^{m+1} - s_{m-1}\omega^m - \cdots - s_1\omega^2 - s_0\omega = 0$. Thus ω is a root of $x^{m+1} - s_{m-1}x^m - \cdots - s_1x^2 - s_0x \in R[x]$. Hence ω is integral over R. $\square$

1.7. Classes of Domains

In this section we make note of several classes of rings which have more algebraic structure than generic rings.

Definition 1.7.1. An integral domain R is called **Euclidean** if there exists a function $f : R\backslash\{0\} \to \mathbb{N} \cup \{0\}$ such that

1. for every nonzero $x, y \in R$, $f(xy) \geq f(x)$, and
2. if $x, y \in R$ such that x is nonzero, then there exist $q, r \in R$ with $y = qx + r$ and $r = 0$ or $f(r) < f(x)$.

Example 1.7.2.

1. Fields are trivial examples of Euclidean domains with norm $N(a) = 0$ for all a.
2. $\mathbb{Z}$ is a Euclidean domain with norm given by $N(a) = |a|$, the usual absolute value.
3. If $\mathbb{F}$ is a field, then the polynomial ring $\mathbb{F}[x]$ is a Euclidean domain with norm given by $N(p(x)) = \deg(p(x))$.

It turns out that every ideal in a Euclidean domain must be principal. This condition is often used to show that some integral domains are not Euclidean by demonstrating the existence of nonprincipal ideals. For instance, in $\mathbb{Z}[x]$, the ideal $(2, x)$ is not principal, hence the ring $\mathbb{Z}[x]$ is not a Euclidean domain. Next, we look at the class of domains in which every ideal is principal.

Definition 1.7.3. Let R be an integral domain. We say that R is a **principal ideal domain (PID)** if every ideal of R is principal.

Example 1.7.4.

1. $\mathbb{Z}$ is a principal ideal domain.
2. $\mathbb{Z}[i]$ is a principal ideal domain.
3. The ring $\mathbb{Z}\left[\frac{1+\sqrt{-19}}{2}\right]$ is a principal ideal domain but not a Euclidean domain. See [14] for a proof.

Next we show that Euclidean domains are in fact principal ideal domains.

Proposition 1.7.5. *If R is a Euclidean domain, then R is a principal ideal domain.*

Proof. Suppose that R is a Euclidean domain with Euclidean function $f : R \setminus \{0\} \to \mathbb{N} \cup \{0\}$. Assume that I is a nonzero ideal of R. Now consider the set $S = \{f(x) \mid x \in I \setminus \{0\}\}$. We observe that $S \subseteq \mathbb{N} \cup \{0\}$. Let $y \in I$ be such that $f(y)$ is minimal in S. We show that $I = (y)$. By choice, $y \in I$, so $(y) \subseteq I$. Now choose $z \in I$. Then there exists $q, r \in R$ such that $z = qy + r$ where $f(r) < f(y)$ or $r = 0$. But $z - qy = r \in I$. Thus $f(r)$ cannot be less than $f(y)$ by the minimality of y. Thus $r = 0$ and hence $z = qy$. So $z \in (y)$, yielding that $I \subseteq (y)$. Hence $I = (y)$ and therefore I is a principal ideal. Since I was an arbitrary nonzero ideal of R, we have that all nonzero ideals of R are principal. □

The next result shows that in a principal ideal domain, the prime elements and the irreducible elements coincide, which again, is not true in general.

Proposition 1.7.6. *If R is a principal ideal domain, then p is prime if and only if p is irreducible.*

Proof. Let R be a principal ideal domain. For the forward implication, assume that $p \in R$ is a prime element, so p is a nonzero nonunit such that if $p \mid ab$ then either $p \mid a$ or $p \mid b$. To show that p is an irreducible element, assume that $p = xy$. Since $p = xy$ and $p \mid p$, we have that $p \mid xy$. But p is prime, so either $p \mid x$ or $p \mid y$. Without loss of generality, say that $p \mid x$. So $x = r_1 p$ for some $r_1 \in R$. Now plugging $x = r_1 p$ into $p = xy$ yields $p = r_1 p y$. Since R is an integral domain and p is nonzero, we have that $1_R = r_1 y$, making r_1 and y units in R. Hence $p = xy$ with y a unit, making p an irreducible element. For the reverse implication, assume that $p \in R$ is an irreducible element, so p is a nonzero nonunit such that if $p = ab$ then either a or b is a unit in R. By Proposition 1.2.4(3), we have that (p) is maximal in the set of all proper principal ideals of R. To show that p is a prime element, suppose that $p \mid cd$. Consider $J = (p, c)$, an ideal of R. Since R is a principal ideal domain, J must be principal. So we can write $J = (p, c) = (\alpha)$ for some $\alpha \in R$. Observe that $(p) \subseteq (p, c) = (\alpha)$. But (p) is maximal in the set of all proper principal ideals of R. So either $(p) = (p, c) = (\alpha)$ or $(p, c) = (\alpha) = R$. In the case that $(p) = (p, c)$, since $c \in (p, c)$, we have that $c \in (p)$. Hence $c = r_2 p$ for some $r_2 \in R$, so then $p \mid c$ and therefore p is a prime element. In the case that $(p, c) = R$, there exist $r_3, r_4 \in R$ such that $r_3 p + r_4 c = 1_R$. Now multiply through by d

to obtain $r_3pd + r_4cd = d$. But by assumption $p \mid cd$, so we can write $cd = rp$ for some $r \in R$. Plugging in $cd = rp$ into $r_3pd + r_4cd = d$ yields $p(r_3d + r_4r) = d$. Since p divides the left hand side of this equation, we must have that p divides the right hand side of the equation. So $p \mid d$, making p a prime element. Hence, in either case, p is a prime element of R. $\square$

We next examine unique factorization domains, which have been studied extensively and have many nice properties. For an integral domain R it is a classical result that R is a unique factorization domain if and only if $R[x]$, the polynomial extension of R, is a unique factorization domain. However, it has been shown in literature that if R is a unique factorization domain, then $R[[x]]$, the power series extension of R, need not be a unique factorization domain.

Definition 1.7.7. An integral domain R is a **unique factorization domain (UFD)** if every nonzero nonunit of R is a (finite) product of prime elements.

Classically, the more familiar definition of a UFD states that an integral domain R is a unique factorization domain provided that (i) every nonzero nonunit element x of R can be written $x = c_1c_2\cdots c_n$, with $c_1, c_2, \cdots, c_n$ irreducible and (ii) if $x = c_1c_2\cdots c_n$ and $x = d_1d_2\cdots d_m$, with each c_i, d_j irreducible, then $n = m$ and for some permutation σ of $\{1, 2, \cdots, n\}$, c_i and $d_{\sigma(i)}$ are associates for every i. Below we show that the compact definition above is equivalent to the classical definition.

Example 1.7.8.

1. Any principal ideal domain turns out to be a unique factorization domain. See the proof of Proposition 1.7.12.

2. $\mathbb{Z}[x]$ is a unique factorization domain.

3. Let $\mathbb{F}$ be a field. Then $\mathbb{F}[x]$ is a unique factorization domain.

Proposition 1.7.9. *Let R be an integral domain and $x \in R$ be a finite product of prime elements, say $x = p_1p_2\cdots p_n$. Then this factorization of x into irreducible elements is unique, up to order and units.*

Proof. Let R be an integral domain and let $x \in R$ be a finite product of prime elements, say $x = p_1p_2\cdots p_n$. Recall, by Proposition 1.1.4, every nonzero prime element is an irreducible element.

So $x = p_1p_2\cdots p_n$ is an irreducible factorization of x. Now suppose that $x = a_1a_2\cdots a_m$ is another irreducible factorization of x. Then we have that $a_1a_2\cdots a_m = p_1p_2\cdots p_n$. Since p_1 is prime and $p_1 \mid a_1a_2\cdots a_m$, we have that $p_1 \mid a_j$ for some $1 \leq j \leq m$. Without loss of generality, we assume that $p_1 \mid a_1$. Since a_1 is irreducible, we have that $a_1 = u_1p_1$ with $u_1 \in U(R)$. Plugging in $a_1 = u_1p_1$ and canceling p_1 from the above equation yields $u_1a_2\cdots a_m = p_2\cdots p_n$. Inductively, we obtain that $n = m$ and that each $a_i = u_ip_i$, again without loss of generality. Thus the factorization of x into irreducible elements is unique, up to order and units. □

Proposition 1.7.10. *Let R be an integral domain. The following are equivalent:*

1. *Every nonzero nonunit of R is a product of primes.*

2. *Every nonzero nonunit of R is a product of irreducible elements and this irreducible factorization is unique.*

Proof. Let R be an integral domain. Note that (1) implies (2) is exactly Proposition 1.7.9. For (2) implies (1), assume that every nonzero nonunit of R is a product of irreducible elements and this irreducible factorization is unique. It suffices to show that every irreducible is prime. Suppose that π is an irreducible and that $\pi \mid ab$, where a and b are nonunits. Then $ab = \pi c$ for some $c \in R$. Next factor a and b into irreducible elements, say $a = \alpha_1\alpha_2\cdots\alpha_u$ and $b = \beta_1\beta_2\cdots\beta_v$. Then we have that $\alpha_1\alpha_2\cdots\alpha_u\beta_1\beta_2\cdots\beta_v = \pi c$. The right hand side of this equation has irreducible factor π; by uniqueness, either one of the α_i's or one of the β_j's is an associate of π. Without loss of generality, say π is associated to α_i, and hence $\pi \mid a$. Hence π is prime. □

The above result demonstrates the equivalency of the compact definition of a unique factorization domain with the classical definition. Next we give a very useful characterization of unique factorization domains.

Proposition 1.7.11. *An integral domain R is a unique factorization domain if and only if every nonzero prime ideal contains a nonzero prime element.*

Proof. Let R be an integral domain. For the forward implication, we assume that R is a unique factorization domain. Let P be a nonzero prime ideal of R. Since P is nonzero, there exists $x \in P$ with $x \neq 0$. Since $x \in P \subseteq R$, a unique factorization domain, we can write x as a product of prime elements, say $x = p_1p_2\cdots p_n$ with each p_i a prime element. Since $x \in P$, we have $p_1p_2\cdots p_n \in P$, a prime ideal. Hence $p_j \in P$ for some $1 \leq j \leq n$. Therefore P contains a (nonzero) prime element. Since P was an arbitrary nonzero prime ideal of R, we have the desired result. For the reverse implication, we assume that every nonzero prime ideal of R contains a nonzero prime element. Let Γ be the set of all elements of R that can be written as a product of prime elements, so $\Gamma = \{up_1^{a_1}p_2^{a_2}\cdots p_k^{a_k} \mid u \in U(R), \text{each } p_i \text{ is prime and each } a_j \geq 0\}$. Note that Γ is multiplicatively closed and saturated. Suppose that there exists a nonzero $a \in R$ such that $a \notin \Gamma$; that is, a cannot be factored into prime elements. We will show that $(a) \cap \Gamma = \emptyset$. Let $x \in (a) \cap \Gamma$. Then $x \in (a)$ and $x \in \Gamma$. Since $x \in (a)$, we can write $x = ra$ for some $r \in R$. Thus we have $x = ra \in \Gamma$; but Γ is saturated, hence $r \in \Gamma$ and $a \in \Gamma$, a contradiction since $a \notin \Gamma$. Therefore $(a) \cap \Gamma = \emptyset$. Thus, by Proposition 1.4.3 there exists a prime ideal P_a maximal with respect to $P_a \cap \Gamma = \emptyset$. In particular, P_a contains no prime element, a contradiction to our assumption. Thus we must have that $a \in \Gamma$, making R a unique factorization domain. □

Next we show that any principal ideal domain is in fact a unique factorization domain.

Proposition 1.7.12. *Any principal ideal domain is a unique factorization domain.*

Proof. Suppose R is a principal ideal domain. Let $P \subseteq R$ be a nonzero prime ideal. Then, since R is a principal ideal domain, $P = (x)$ for some $x \in R$. Since x generates a nonzero prime ideal, we must have that x is a nonzero prime element of R by Proposition 1.2.4. So P contains a nonzero prime element. Since P was an arbitrary nonzero prime ideal, we have that every nonzero prime ideal contains a nonzero prime element. Hence, by Proposition 1.7.11, R is a unique factorization domain. □

The converse of the above result is false. For example the polynomial ring $\mathbb{Z}[x]$ can be shown to be a unique factorization domain, but $\mathbb{Z}[x]$ is not a principal ideal domain [17].

Another classical result states that any unique factorization domain is integrally closed.

Proposition 1.7.13. *Any unique factorization domain is integrally closed.*

Proof. Assume that R is a unique factorization domain with quotient field K. We show that R is integrally closed. Suppose that $\omega = \frac{a}{b} \in K$, with $\gcd(a, b) = 1$, is integral over R. Then ω is a root of $x^n + r_{n-1}x^{n-1} + \cdots + r_2x^2 + r_1x + r_0 \in R[x]$. Thus, plugging in $\omega = \frac{a}{b}$, we get

$$\frac{a^n}{b^n} + r_{n-1}\frac{a^{n-1}}{b^{n-1}} + \cdots + r_2\frac{a^2}{b^2} + r_1\frac{a}{b} + r_0 = 0.$$

Multiplying both sides by b^n we obtain that

$$a^n + r_{n-1}ba^{n-1} + \cdots + r_2b^{n-2}a^2 + r_1b^{n-1}a + r_0b^n = 0.$$

Now, suppose that $p \in R$ is a nonzero prime dividing b. Then we must have that $p \mid a^n$, which implies that $p \mid a$. Thus $p \mid \gcd(a, b)$, a contradiction. So there does not exist a prime p dividing b. Hence b is a unit of R and so $\omega = \frac{a}{b} = ab^{-1} \in R$. Therefore R is integrally closed. $\square$

We can use the above result to show that a ring is not a unique factorization domain. For example, $\mathbb{Z}[\sqrt{-3}]$ does not contain $\omega = \frac{-1+\sqrt{-3}}{2}$, which is a root of $x^2 + x + 1$; hence $\mathbb{Z}[\sqrt{-3}]$ is not integrally closed and thus cannot be a unique factorization domain.

Not only is a unique factorization domain integrally closed, but it is also completely integrally closed.

Proposition 1.7.14. *Let R be an integral domain with quotient field K. If R is a unique factorization domain, then R is completely integrally closed.*

Proof. Let R be an integral domain with quotient field K. Assume that R is a unique factorization domain. Suppose that $\omega \in K$ is almost integral over R. Then there exists a nonzero $r \in R$ such that $r\omega^n \in R$ for every $n \geq 0$. Since $\omega \in K$, we can write $\omega = \frac{a}{b}$ with $a, b \in R$ and, since R is a unique factorization domain, $\gcd(a, b) = 1$. Also since R is a unique factorization domain, we can write $a = p_1p_2 \cdots p_n$, $b = q_1q_2 \cdots q_m$ and $r = r_1r_2 \cdots r_t$ with each p_i, q_j, r_k prime elements of R. Then, for any $l \geq 0$, we have $r\omega^l \in R$, or $r_1r_2 \cdots r_t\left(\frac{p_1p_2\cdots p_n}{q_1q_2\cdots q_m}\right)^l = s_l \in R$. Clearing denominators, we obtain $r_1r_2 \cdots r_tp_1^lp_2^l \cdots p_n^l = s_lq_1^lq_2^l \cdots q_m^l$. By assumption $\gcd(a, b) = 1$, so no q_i divides any p_j. Thus we have that $q_1^lq_2^l \cdots q_m^l \mid r_1r_2 \cdots r_t$. This implies that $lm \leq t$ for any l; hence $m = 0$. So $b = q_1q_2 \cdots q_m$ and $m = 0$, which implies that b is a unit in R. Thus $\omega = \frac{a}{b} = ab^{-1} \in R$. Therefore every almost integral element over R is in R, making R completely integrally closed. $\square$

Thus we have the following inclusions among classes of commutative rings with identity:

$$\text{fields} \subset \text{Euclidean Domains} \subset \text{PIDs} \subset \text{UFDs} \subset \text{integral domains}$$

with all inclusions being proper [17].

Above we defined a unique factorization domain to be an integral domain in which every nonzero nonunit is a product of prime elements. If we replace "prime" by "irreducible", we obtain the definition of an atomic domain. The term "atomic" was introduced by P. M. Cohn in 1968, who called an irreducible element of an integral domain an "atom" [10].

Definition 1.7.15. An integral domain R is said to be **atomic** if every nonzero nonunit of R is a (finite) product of irreducible elements.

Example 1.7.16.

1. Any unique factorization domain is atomic since every prime element is irreducible by Proposition 1.1.4.

2. Any Noetherian domain is atomic. See the proof of Corollary 1.7.22.

3. Let $\mathbb{F}$ be a field. Then $\mathbb{F}[x, y, \frac{y}{x}, \frac{y}{x^2}, \frac{y}{x^3}, \cdots]$ is not atomic since we can write $y = (\frac{y}{x})x = (\frac{y}{x^2})xx = (\frac{y}{x^3})xxx = \cdots$.

As the distinction between irreducible elements and prime elements is essential for this thesis, we next look at a very specific type of integral domain, called an AP domain, where the definitions of irreducible element and prime element coincide. The "AP" in AP domain refers to the atoms (or irreducible elements) being prime.

Definition 1.7.17. An integral domain R is an **AP domain** if every irreducible element of R is also prime.

We observe that in the definition of AP domain, R was not required to be an atomic domain. In fact, if R is an atomic AP domain, then it is a unique factorization domain.

Example 1.7.18.

1. In the proof of Proposition 1.7.6 we showed that any principal ideal domain is an AP domain.

2. By Proposition 1.7.5, we have that any Eucidean domain is a principal ideal domain and is thus an AP domain.

3. Any GCD domain is an AP domain. See the proof of Proposition 1.7.42.

4. Any unique factorization domain is a GCD domain by Proposition 1.7.40. Then, by Proposition 1.7.42, any unique factorization domain is an AP domain.

5. Consider $R = \mathbb{Z}_{(2)} + x\mathbb{C}[[x]]$. The only atom in R is 2, which turns out to also be prime in R. Hence R is an AP domain. However R is not a GCD domain since x^2 and ix^2 have no greatest common divisor.

Next we give the definition of an ACCP domain, where ACCP is short for the ascending chain condition for principal ideals.

Definition 1.7.19. Let R be an integral domain. We say that R is **ACCP** if there does not exist a strictly increasing chain of principal ideals.

Example 1.7.20. Every principal ideal domain is an ACCP domain [22].

It turns out that any integral domain satisfying ACCP must be atomic.

Proposition 1.7.21. *If an integral domain R is ACCP, then R is atomic.*

Proof. Let R be an integral domain and assume that R is ACCP. We first show that any nonzero nonunit of R is divisible by an irreducible element. Suppose that x is a nonzero nonunit of R. If (x) is maximal with respect to being principal, then x is irreducible by Proposition 1.2.4. If (x) is not contained in an ideal that is maximal with respect to being principal, then for every $y \in R$ such that $(y) \supsetneq (x)$, we have that there exists $y_1 \in R$ such that $(y_1) \supsetneq (y) \supsetneq (x)$. Since the same is true for (y_1), we can construct an infinite chain of principal ideals $(x) \subsetneq (y) \subsetneq (y_1) \subsetneq (y_2) \subsetneq \cdots$, which violates ACCP. Hence any nonzero nonunit of R is divisible by an irreducible. Now, let x be a nonzero nonunit of R. Then x is divisible by an irreducible, say π_1. If $\frac{x}{\pi_1}$ is a unit, then x is associated to the irreducible element π_1. If not, then $\frac{x}{\pi_1}$ is divisible by an irreducible, say π_2. Continuing this process gives rise to the increasing chain of principal ideals $(x) \subsetneq (\frac{x}{\pi_1}) \subsetneq (\frac{x}{\pi_1\pi_2}) \subsetneq \cdots$. Since R is ACCP this chain must terminate. So, using the notation from above, we can find a unit $u \in R$ such that $u = \frac{x}{\pi_1\pi_2\cdots\pi_n}$ and hence $x = u\pi_1\pi_2\cdots\pi_n$. Therefore x, an arbitrary nonzero nonunit of R, can be expressed as product of irreducible elements, making R atomic. □

It turns out that the converse of Proposition 1.7.21 is false. An example of such a domain is given in [16].

Corollary 1.7.22. *Any Noetherian domain is ACCP and hence is atomic.*

Proof. Let R be a Noetherian domain. Suppose we have a strictly increasing chain of principal ideals, say $(\alpha_1) \subsetneq (\alpha_2) \subsetneq (\alpha_3) \subsetneq \cdots \subsetneq (\alpha_n) \subsetneq \cdots$. Then this is a strictly increasing chain of ideals in a Noetherian domain; hence the chain must stabilize. Thus there does not exist a strictly increasing chain of principal ideals. So R is ACCP. Now, by Proposition 1.7.21, R must be atomic. □

Beyond the realm of unique factorization domains, there is a large class of integral domains for which each nonunit can be factored as a product of irreducible elements, yet the factorization may not be unique. Classically Dedekind domains are such domains [8]. The class of Dedekind domains lies properly between the class of principal ideal domains and the class of Noetherian integral domains [17]. In order to define a Dedekind domain, we must first introduce the notions of fractional ideals and invertible ideals. For a refresher on modules, one is referred to Chapter IV of [17].

Definition 1.7.23. Let R be an integral domain with quotient field K.

1. An R-submodule $I \subseteq K$ is called a **fractional ideal** if there exists $a \in R$, $a \neq 0$, such that $aI \subseteq R$.

2. If I and J are fractional ideals of R, we define $IJ = \sum_{i=1}^{n} \alpha_i\beta_i$ where $\alpha_i \in I$ and $\beta_i \in J$.

3. If I is fractional ideal of R, we define $I^{-1} = \{x \in K \mid xI \subseteq R\}$.

4. We note that if I is a fractional ideal of R, then $II^{-1} \subseteq R$; if $II^{-1} = R$ we say that I is **invertible**.

Example 1.7.24.

1. If $I \subseteq R$ is an ideal, then I is a fractional ideal since $1_R I = I \subseteq R$.

2. $\frac{1}{2}\mathbb{Z} = \{\cdots, \frac{-3}{2}, -1, \frac{-1}{2}, 0, \frac{1}{2}, 1, \frac{3}{2}, \cdots\}$ is a fractional ideal of $\mathbb{Z}$ since $2(\frac{1}{2}\mathbb{Z}) = \mathbb{Z}$.

3. $\mathbb{Q}$ is not a fractional ideal of $\mathbb{Z}$ since for $n \neq 0$ we have that $\frac{1}{2n} \in \mathbb{Q}$ but $n(\frac{1}{2n}) = \frac{1}{2} \notin \mathbb{Z}$.

4. If R is a domain and $x \neq 0$ is an element of the quotient field, then $I = (x)$ is a fractional invertible ideal with $I^{-1} = (x^{-1})$.

Proposition 1.7.25. *Let R be an integral domain with quotient field K and let I be a nonzero fractional ideal of R. Then I^{-1} is a fractional ideal of R.*

Proof. We mimic the proof given in [14]. Let R be an integral domain with quotient field K and let I be a nonzero fractional ideal of R. Then there is some nonzero element $d \in R$ such that $dI \subseteq R$. We remark that this means that I contains nonzero elements of R. We first show, using the submodule criterion, that I^{-1} is an R-submodule of K. $I^{-1} \neq \emptyset$ since $d \in I^{-1}$. Now, let $r \in R$ and $x, y \in I^{-1}$. Since $x \in I^{-1}$, we have that $xI \subseteq R$. Similarly, since $y \in I^{-1}$, we have that $yI \subseteq R$. Thus we have that $(x + ry)I^{-1} = xI^{-1} + (ry)I^{-1} = xI^{-1} + r(yI^{-1}) \subseteq R$. Hence I^{-1} is an R-submodule of K. Next we show that I^{-1} is a fractional ideal of R. Let a be any nonzero element of I contained in R, which we know exist from above. Then by definition of I^{-1} we have that $aI^{-1} \subseteq R$, making I^{-1} a fractional ideal of R. $\square$

Next we show that any invertible fractional ideal of an integral domain must be finitely generated.

Proposition 1.7.26. *Let R be an integral domain and I a fractional ideal of R. If I is invertible, then I is finitely generated.*

Proof. Let R be an integral domain and I a fractional ideal of R. We further assume that I is invertible. Then $II^{-1} = R$. So there exists $a_1, a_2, \cdots, a_n \in I$ and $b_1, b_2, \cdots, b_n \in I^{-1}$ such that $a_1b_1 + a_2b_2 + \cdots + a_nb_n = 1_R$. We will show that $I = (a_1, a_2, \cdots, a_n)$. Note that each $a_i \in I$ and so $(a_1, a_2, \cdots, a_n) \subseteq I$. For the other inclusion, let $z \in I$. Then $a_1(zb_1) + a_2 + (zb_2) + \cdots + a_n(zb_n) = z$. But each $b_i \in I^{-1}$, so $zb_i = r_i \in R$. Thus we have $a_1r_1 + a_2r_2 + \cdots + a_nr_n = z$. Hence $z \in (a_1, a_2, \cdots, a_n)$. So $I \subseteq (a_1, a_2, \cdots, a_n)$. Therefore we have that $I = (a_1, a_2, \cdots, a_n)$ and hence is finitely generated. $\square$

Next we give an equivalent characterization for invertible fractional ideals of an integral domain.

Proposition 1.7.27. *Let I be a fractional ideal of an integral domain R. Then I is invertible if and only if there exists a fractional ideal J of R such that IJ is principal.*

Proof. Let I be a fractional ideal of an integral domain R. Then there exists $x \in R \setminus \{0\}$ such that $xI \subseteq R$. For the forward implication we assume that I is invertible. Then $II^{-1} = R = (1_R)$. Now we consider xII^{-1}; observe that $xII^{-1} = I(xI^{-1}) = xR = (x)$. Thus letting $J = xI^{-1}$, a fractional ideal of R, we have that IJ is principal. For the reverse implication, we assume that there exists a fractional ideal J of R such that IJ is principal. Thus, we have $IJ = (y) = yR$. Multiplying both sides by y^{-1} we obtain $I(Jy^{-1}) = R$. Hence $I^{-1} = Jy^{-1}$, making I invertible. □

Proposition 1.7.28. *Any invertible ideal in a quasi-local integral domain is principal.*

Proof. We mimic the proof given in [20]. Let R be a quasi-local integral domain and let I be an invertible ideal in R. Then $II^{-1} = R$. So we can write $\sum_{i=0}^{n} a_i b_1 = 1_R$ with each $a_i \in I$ and each $b_j \in I^{-1}$. Notice that the elements $a_i b_i$ lie in R and their sum is 1_R. Hence one of them, say $a_1 b_1$ is a unit. We deduce that $I = (a_1)$. So I is principal. But I was an arbitrary invertible ideal, so the desired result follows. □

We show next that a localization of an invertible ideal is invertible.

Proposition 1.7.29. *Let R be an integral domain and S a multiplicatively closed set. If I is an invertible ideal of R, then I_S is invertible in R_S.*

Proof. Let R be an integral domain and S a multiplicatively closed set. Assume that I is an invertible ideal of R. Then $II^{-1} = R$. We will show that $(I_S)^{-1} = (I^{-1})_S$. That is, we show that $I_S(I^{-1})_S = R_S$. Note that a typical element of $(I^{-1})_S$ looks like $\frac{z}{s}$ with $z \in I^{-1}$ and $s \in S$. If $\frac{y}{s'} \in I_S$, then $\frac{y}{s'} \cdot \frac{z}{s} = \frac{yz}{ss'} \in R_S$. Thus $I_S(I^{-1})_S \subseteq R_S$. Now since $II^{-1} = R$, there exists $a_1, a_2, \cdots, a_n \in I$ and $b_1, b_2, \cdots, b_n \in I^{-1}$ such that $a_1b_1 + a_2b_2 + \cdots + a_nb_n = 1_R$. We are done since $a_i \in I \subseteq I_S$ and $b_j \in I^{-1} \subseteq (I^{-1})_S$. □

Proposition 1.7.30. *Suppose R is an integral domain and I is a finitely generated ideal of R. Then I is invertible if and only if I_M is principal in R_M for every maximal ideal M of R.*

Proof. Let R be an integral domain with quotient field K and suppose I is a finitely generated ideal of R. For the forward implication, we assume that I is invertible. Let $S = R \setminus M$ where M is a maximal ideal of R. By Proposition 1.7.29, we have that $I_M = I_S$ in invertible in $R_M = R_S$. Since R_S is quasi-local, I_M is principal. For the reverse implication, we assume that I_M is principal in R_M for every maximal ideal M of R. Let $I^{-1} = \{x \in K \mid xI \subseteq R\}$ and let P be a maximal ideal of R. Then we have that $(I^{-1})_P = (I_P)^{-1}$. So $(II^{-1})_P = (I_P)(I^{-1})_P = (I_P)(I_P)^{-1}$. Since I_P is principal, we have that I_P is invertible. Hence we can write $(I_P)(I_P)^{-1} = R_P$ or $(II^{-1})_P = R_P$ and thus $II^{-1} = R$. □

Let R be a ring. A **saturated chain** of primes (of length n) is a chain of prime ideals $P_0 \subsetneq P_1 \subsetneq P_2 \subsetneq \cdots \subsetneq P_n$ such that there does not exist prime ideal Q that can be inserted into the chain. We say the **(Krull) dimension** of R, denoted by $\dim(R)$, is given by $\dim(R) = \sup\{n \mid P_0 \subsetneq P_1 \subsetneq P_2 \subsetneq \cdots \subsetneq P_n$ is a saturated chain of primes$\}$.

Proposition 1.7.31. *Let R be an integral domain. The following are equivalent:*

1. *Every nonzero fractional ideal of R is invertible.*
2. *Every nonzero ideal $I \subseteq R$ is invertible.*
3. *Every nonzero ideal of R is uniquely a (finite) product of prime ideals.*
4. *R is Noetherian, integrally closed, and dim$(R) \leq 1$.*

The proof of this can be found in Chapter VIII Section 6 of [17].

Definition 1.7.32. Any domain satisfying one, hence all, of the above conditions is called a **Dedekind domain**.

It was Emmy Noether who characterized abstract commutative rings in which every nonzero ideal is a unique product of prime ideals. Such rings are now called Dedekind domains [21].

Example 1.7.33.

1. Any principal ideal domain, R, is a Dedekind domain. Since every ideal of R is principal, the ideals are finitely generated, and so R is Noetherian. Since R is a PID, by Proposition 1.7.12, we have that R is a UFD. Then, by Proposition 1.7.13, we have that R is integrally closed.

Also, the nonzero prime ideals of R are maximal, so we have that $\dim(R) \leq 1$. Hence R is a Dedekind domain.

2. If F is a field, then the principal ideals (x_1) and (x_2) in the polynomial domain $F[x_1, x_2]$ are prime but not maximal since $(x_i) \subsetneq (x_1, x_2) \subsetneq F[x_1, x_2]$. Consequently, $F[x_1, x_2]$ is not Dedekind. However, $F[x_1, x_2]$ is Noetherian by the Hilbert Basis Theorem [17].

We can view Dedekind domains as being a slight generalization of principal ideal domains. One can prove that the ideals of Dedekind domains never require more than two generators; we omit this proof as it is outside the scope of what is needed for this thesis.

Next, we turn our attention to Prüfer domains, which are named after Heinz Prüfer.

Definition 1.7.34. A **Prüfer domain** is an integral domain in which every finitely generated ideal is invertible.

Example 1.7.35.

1. Any Dedekind domain is a Prüfer domain as shown in the proof of Proposition 1.7.37.
2. $\mathbb{Z}[i]$ is a principal ideal domain, hence a Dedekind domain, and hence a Prüfer domain.
3. Any Bézout domain is Prüfer since in a Bézout domain every finitely generated ideal is principal and hence by Example 1.7.24(4) invertible.
4. The ring of entire functions is a Prüfer domain [20].

There are numerous equivalent characterizations of Prüfer domains, we give two equivalent characterizations below, both related to localization.

Proposition 1.7.36. *Let R be an integral domain. The following are equivalent:*

1. *R is a Prüfer domain.*
2. *For every prime ideal P in R, R_P is a valuation domain.*
3. *For every maximal ideal M in R, R_M is a valuation domain.*

Proof. Let R be an integral domain. For (1) implies (2), assume that R is a Prüfer domain. Let P be a prime ideal of R and let J be a finitely generated nonzero ideal in R_P. If J is generated by $\frac{a_1}{s_1}, \frac{a_2}{s_2}, \cdots, \frac{a_n}{s_n}$ with $a_i, s_i \in R$ and $s_i \notin P$, then $J = I_P$ where $I = (a_1, a_2, \cdots, a_n)$. Since R is Prüfer and I is a finitely generated ideal of R, we have that I is invertible. By Proposition 1.7.29, we have that $I_P = J$ is invertible in R_P. Then, since R_P is quasi-local, by Proposition 1.7.28, we have that J is principal. Thus, in the quasi-local domain R_P, we have that every finitely generated ideal is principal. So R_P is quasi-local and Bézout. By Proposition 1.8.7 we have that R_P is a valuation domain. Since P was an arbitrary prime ideal of R, we have that for every prime ideal P in R, R_P is a valuation domain. For (2) implies (3), assume that for every prime ideal P in R we have that R_P is a valuation domain. Recall from Proposition 1.2.7 that every maximal ideal of R is also a prime ideal of R. Thus we have that for every maximal ideal M in R, M is a prime ideal in R, and hence by assumption we have that R_M is a valuation domain. For (3) implies (1), assume that for every maximal ideal M in R, R_M is a valuation domain. We show that every finitely generated ideal of R is invertible. Let $I = (a_1, a_2, \cdots, a_n)$ be an ideal of R. In R_M, I_M is principal since R_M is a valuation domain and hence a Bézout domain. Thus, for every maximal ideal M in R, I_M is principal. Hence, by Proposition 1.7.30, I is invertible. Thus R is Přufer. $\square$

Proposition 1.7.37. *A Prüfer domain is Dedekind if and only if it is Noetherian.*

Proof. Let R be a Prüfer domain. Then every finitely generated ideal of R is invertible. For the forward implication, assume that R is Dedekind. Then, by Proposition 1.7.31(4), R is Noetherian. For the reverse implication, assume that R is Noetherian. Let I be a nonzero ideal of R. Since R is Noetherian, I must be finitely generated. Then, since R is Prüfer, I is invertible. Hence every nonzero ideal of R is invertible. Therefore, by 1.7.31(2), R is Dedekind. $\square$

Next we define a GCD domain. The term "GCD domain" seems to have been popularized by Kaplansky. Bourbaki used the term "pseudo-Bezout" and Gilmer and Cohn used "HCF-ring." Earlier, Dribin used the term "complete" and Prüfer used "domains satisfying property $\mathcal{B}$A." Two other earlier works to consider GCD domains are Jaffard and Boccioni. See [8] for references to each.

Definition 1.7.38. Let R be an integral domain and $a, b \in R$.

1. We say that the **greatest common divisor** of a and b is a common divisor d with the property that if x is any other common divisor of a and b then x divides d; we usually denote d by $\gcd(a, b)$.

2. A **GCD domain** is an integral domain in which every two nonzero elements have a greatest common divisor.

It should be noted here that we are not assuming that the greatest common divisor is a linear combination of the two elements. This stronger assumption can be recast as saying that all finitely generated ideals are principal, and these domains have been called Bézout domains [20], which we will define shortly.

Example 1.7.39.

1. Any unique factorization domain is a GCD domain. See the proof of Proposition 1.7.40.

2. Any valuation domain V is a GCD domain. Since V is a valuation domain, by Proposition 1.8.7 we have that V is a Bézout domain. Then by Proposition 1.7.45 we have that V is a GCD domain.

Now we show that any unique factorization domain is a GCD domain.

Proposition 1.7.40. *Let R be a unique factorization domain, then R is a GCD domain.*

Proof. Let R be a unique factorization domain. Suppose that $a, b \in R \setminus \{0\}$. Then we can factor a and b into primes, say $a = p_1^{a_1} p_2^{a_2} \cdots p_k^{a_k}$ and $b = p_1^{b_1} p_2^{b_2} \cdots p_k^{b_k}$ with $a_i, b_i \geq 0$ for $1 \leq i \leq k$. Now, let $d = p_1^{c_1} p_2^{c_2} \cdots p_k^{c_k}$ where $c_j = \min\{a_j, b_j\}$ for $1 \leq j \leq k$. Then d is a common divisor of a and b since $a = d(p_1^{a_1 - c_1} p_2^{a_2 - c_2} \cdots p_k^{a_k - c_k})$ and $b = d(p_1^{b_1 - c_1} p_2^{b_2 - c_2} \cdots p_k^{b_k - c_k})$. Now suppose that t is another common divisor of a and b. So $t = p_1^{r_1} p_2^{r_2} \cdots p_k^{r_k}$ with $r_m \geq 0$ for $1 \leq m \leq k$. We must have that $r_n \leq c_n$ for $1 \leq n \leq k$, for otherwise t would not be a common divisor of a and b. Thus we must have that $t \mid d$, making d the greatest common divisor of a and b. Since a and b were arbitrary nonzero elements of R, we have that R is a GCD domain. □

Proposition 1.7.41. *Let R be a GCD domain. Then, for $a, b, x \in R \setminus \{0\}$, the following hold:*

1. *$x(gcd(a, b)) = gcd(xa, xb)$.*

2. If $gcd(x, a) = 1_R$ and $gcd(x, b) = 1_R$, then $gcd(x, ab) = 1_R$.

3. If $gcd(x, a) = 1_R$ and $x \mid ab$, then $x \mid b$.

Proof. Let R be a GCD domain. Then, given $r_1, r_2 \in R \setminus \{0\}$ we have that $\gcd(r_1, r_2)$ exists. Let $a, b, x \in R \setminus \{0\}$.

1. Since a and b are nonzero elements of R, a GCD domain, we have that $\gcd(a, b)$ exists; say $\gcd(a, b) = d$. Also, since x is a nonzero element of R, we have that both xa and xb are nonzero elements of R. Let $\gcd(xa, xb) = \alpha$. So $\alpha \mid xa$ and $\alpha \mid xb$. Thus we can write $xa = r_1\alpha$ and $xb = r_2\alpha$ for some $r_1, r_2 \in R$. Since $d = \gcd(a, b)$, we can write $a = k_1 d$ and $b = k_2 d$ for some $k_1, k_2 \in R$. Multiplying both of these equations by x, we get $xa = k_1 dx$ and $xb = k_2 dx$. Hence dx is a common divisor of xa and xb. Since $\alpha = \gcd(xa, xb)$, $dx \mid \alpha$. Thus $\alpha = tdx$ for some $t \in R$. Now we have $xa = r_1\alpha = r_1 tdx$, and so plugging in for a yields $xdk_1 = r_1 tdx$. Since R is an integral domain, we have that $k_1 = r_1 t$. We also have $xb = r_2\alpha = r_2 tdx$, and so plugging in for b yields $xdk_2 = r_2 tdx$. Since R is an integral domain, we have that $k_2 = r_2 t$. Now $a = dk_1$ implies that $a = dr_1 t$; also $b = dk_2$ implies that $b = dr_2 t$. Thus dt is a common divisor of a and b. Hence $dt \mid d$ and we can write $d = m(dt)$. Since R is an integral domain, we then have that $1_R = mt$, so t is unit in R. So $\alpha = t(dx)$ implies that α is a unit multiple of dx. Therefore $dx = \gcd(xa, xb)$ as desired.

2. Suppose that $\gcd(x, a) = 1_R$ and $\gcd(x, b) = 1_R$. Let $\gcd(x, ab) = \alpha$. Then $\alpha \mid x$ and $\alpha \mid ab$; so we can write $x = r\alpha$ and $ab = t\alpha$ for some $r, t \in R$. Since $\gcd(x, a) = 1$, by (1) we have that $\gcd(xb, ab) = 1_R b = b$. This implies that $\gcd(r\alpha b, t\alpha) = b$. Note that α is a common divisor of $r\alpha b$ and $t\alpha$, so $\alpha \mid b$. Thus α is a common divisor of x and b, since $\alpha \mid x$ and $\alpha \mid b$. Thus $\alpha \mid \gcd(x, b)$ or $\alpha \mid 1_R$. Thus α is a unit. Therefore $\gcd(x, ab) = \alpha$ where α is a unit. So $\gcd(x, ab) = 1_R$ as desired.

3. Suppose that $\gcd(x, a) = 1_R$ and $x \mid ab$. Then we can write $ab = rx$ for some $r \in R$. By (1), since $\gcd(x, a) = 1_R$, we have that $\gcd(xb, ab) = 1_R b = b$. Thus $\gcd(xb, rx) = b$. Note that x is a common divisor of xb and rx, so $x \mid \gcd(xb, rx)$ and hence $x \mid b$, as desired. □

With these properties in hand, we show that every GCD domain is also an AP domain, meaning that the prime elements and the irreducible elements coincide in a GCD domain.

Proposition 1.7.42. *Let R be a GCD domain, then R is an AP domain.*

Proof. Let R be a GCD domain. Then, given $a, b \in R \setminus \{0\}$ we have that $\gcd(a, b)$ exists. Let $\pi \in R$ be an irreducible. We show that π is prime. Suppose that $\pi \mid ab$. Since a and π are nonzero elements of R we have that $\gcd(a, \pi)$ exists; either $\gcd(a, \pi) = 1$ or $\gcd(a, \pi) \neq 1$. In the case that $\gcd(a, \pi) = 1$, we have that $\pi \mid b$ by Proposition 1.7.41(3). In the case that $\gcd(a, \pi) \neq 1$, say $\gcd(a, \pi) = d$. Then $d \mid \pi$ but π is irreducible, so $\pi = ud$ for some unit $u \in R$. But $d \mid a$, so $a = md$ for some $m \in R$. Thus we have that $a = md = m(u^{-1}\pi) = (mu^{-1})\pi$. Hence $\pi \mid a$. In either case, π is prime. Therefore every irreducible of R is prime, making R an AP domain. □

Next we define a Bézout domain, which is slightly more special than a Prüfer domain since every finitely generated ideal is required to be principal, instead of just invertible.

Definition 1.7.43. A **Bézout domain** is an integral domain where every finitely generated ideal is principal.

Example 1.7.44.

1. Any principal ideal domain is a Bézout domain.

2. Any valuation domain is a Bézout domain. See the proof of Proposition 1.8.7.

3. The ring of entire functions is a Bézout domain [20].

Next we show that any Bézout domain is a GCD domain.

Proposition 1.7.45. *Let R be a Bézout domain, then R is a GCD domain.*

Proof. Let R be a Bézout domain. Suppose that $a, b \in R \setminus \{0\}$. Consider the ideal $J = (a, b)$. Since J is finitely generated and R is a Bézout domain, we have that J must be principal. Say $J = (a, b) = (c)$ for some $c \in R$. Note that since $(a, b) = (c)$, we have that $a = r_1c$ and $b = r_2c$ for some $r_1, r_2 \in R$. So c is a common divisor of a and b. Suppose that d is another common divisor of a and b. Then $a = s_1d$ and $b = s_2d$ for some $s_1, s_2 \in R$. But we also have $c \in (c) = (a, b)$, so we can write $c = ya + zb$ for some $y, z \in R$. Thus, $c = ya + zb = y(s_1d) + z(s_2d) = (ys_1 + zs_2)d$. Hence $d \mid c$. Hence $c = \gcd(a, b)$. Now, since a and b were arbitrary, we have that R is a GCD domain. □

The above result shows that any Bézout domain is a GCD domain and, by Proposition 1.7.42, we thus have that any Bézout domain is an AP domain. Hence the prime elements and the irreducible elements coincide in a Bézout domain.

In this section we have seen that Euclidean domains, principal ideal domains, unique factorization domains, GCD domains and Bézout domains are all examples of AP domains, that is, domains where the prime elements and the irreducible elements coincide. In the next section, we will also see that valuation domains turn out be another example of AP domains. Despite these familiar domains being examples of AP domains, it is the case that integral domains in general do not have prime elements and irreducible elements coinciding.

1.8. Valuation Domains

The theory of valuations and valuation domains is an important contribution to the field of commutative ring theory by W. Krull. Valuation theory provides a link between commutative algebra and the theory of partially ordered abelian groups. Also, valuation domains are important as they determine integral closure.

Definition 1.8.1. An integral domain V is a **valuation domain** if for every $a, b \in V \setminus \{0\}$ either $a \mid b$ or $b \mid a$.

There is an equivalent characterization of a valuation domain, which we provide next.

Proposition 1.8.2. *Let V be an integral domain with quotient field K. V is a valuation domain if and only if for every $x \in K \setminus \{0\}$ either $x \in V$ or $x^{-1} \in V$.*

Proof. Let V be an integral domain with quotient field K. For the forward implication, assume that V is a valuation domain. Suppose that $x \in K \setminus \{0\}$. Then we can write $x = \frac{a}{b}$ with $a, b \in V$ and $a, b \neq 0$. Since $a, b \in V \setminus \{0\}$, we have that either $a \mid b$ or $b \mid a$. If $a \mid b$, then $b = v_1 a$ for some $v_1 \in V$. Hence $x^{-1} = \frac{1}{x} = \frac{b}{a} = \frac{v_1 a}{a} = v_1$, which is in V. If $b \mid a$, then $a = v_2 b$ for some $v_2 \in V$. Hence $x = \frac{a}{b} = \frac{v_2 b}{b} = v_2$, which is in V. Thus either x or x^{-1} is in V. For the reverse implication, assume that for every $x \in K \setminus \{0\}$ either $x \in V$ or $x^{-1} \in V$. Suppose $c, d \in V \setminus \{0\}$ and consider $y = \frac{c}{d}$. Notice that $y \in K \setminus \{0\}$, so either y or y^{-1} is in V. If $y \in V$, then $y = \frac{c}{d} = v_3 \in V$. Thus, we have that $c = v_3 d$ and so $d \mid c$. If $y^{-1} \in V$, then $y^{-1} = \frac{1}{y} = \frac{d}{c} = v_4 \in V$. Thus, we have that $d = v_4 c$ and so $c \mid d$. Therefore either $c \mid d$ or $d \mid c$. Hence V is a valuation domain. $\square$

Example 1.8.3.

1. Any field is a valuation domain.

2. Let $\mathbb{F}$ be a field. Then $\mathbb{F}[[x]]$ is a valuation domain.

3. The ring $\mathbb{Z}_p = \{\frac{a}{b} \mid a \in \mathbb{Z} \text{ and } \gcd(b,p) = 1\}$ is a valuation domain.

4. Let $\mathbb{F}$ be a field, $R = \mathbb{F}[x, y, \frac{y}{x}, \frac{y}{x^2}, \cdots]$ and M be the maximal ideal generated by $\{x, y, \frac{y}{x}, \frac{y}{x^2}, \cdots\}$. Then R_M is a valuation domain.

We now explore many of the properties of valuation domains, which will be applied to our generalization of Cohen-Kaplansky domains.

First, we show that in a valuation domain every finitely generated ideal must be principal.

Proposition 1.8.4. *Let V be a valuation domain. Every finitely generated ideal $I \subseteq V$ is principal.*

Proof. Let V be a valuation domain. Let $I \subseteq V$ be a finitely generated ideal, say $I = (\alpha_1, \alpha_2, \cdots, \alpha_n)$. Note that $\alpha_i \in I$ implies that $\alpha_i \in V$ for every $1 \leq i \leq n$. Since V is a valuation domain, either $\alpha_1 \mid \alpha_2$ or $\alpha_2 \mid \alpha_1$. Without loss of generality, assume that $\alpha_1 \mid \alpha_2$ (otherwise we reindex). So $\alpha_2 = r_1\alpha_1$. Hence we can write $I = (\alpha_1, \alpha_3, \cdots, \alpha_n)$. Also we have that $\alpha_1 \mid \alpha_3$ or $\alpha_3 \mid \alpha_1$. Without loss of generality, assume that $\alpha_1 \mid \alpha_3$ (otherwise we reindex). So $\alpha_3 = r_2\alpha_1$ and we can write $I = (\alpha_1, \alpha_4, \cdots, \alpha_n)$. Continuing in this fashion, at each step assuming without loss of generality that $\alpha_1 \mid \alpha_j$, we come to $I = (\alpha_1, \alpha_n)$. We now have that either $\alpha_1 \mid \alpha_n$ or $\alpha_n \mid \alpha_1$. Without loss of generality we assume that $\alpha_1 \mid \alpha_n$. So we can write $I = (\alpha_1)$. Hence I is principal. Since I was arbitrary, we have that every finitely generated ideal of V is principal. □

Proposition 1.8.5. *Let V be a valuation domain. Then V is quasi-local and all of its ideals are linearly ordered.*

Proof. Let V be a valuation domain. We first show that V is quasi-local. Suppose that M and N are maximal ideals of V. We choose $x \in M \setminus N$ and notice that $(x, N) = R$ by maximality of N. So there exists $r \in V$ and $n \in N$ such that $rx + n = 1$. Since x and n are elements of V, a valuation domain, either $x \mid n$ or $n \mid x$. Observe that n does not divide x for otherwise $\frac{x}{n} \in V$, which implies that $x \in nV \subseteq N$, a contradiction. Thus $x \mid n$. Hence we have that $r + \frac{n}{x} = \frac{1}{x}$ and thus $\frac{1}{x} \in V$.

But $x \in M$, so $M = V$. Hence V can only have one maximal ideal, so V is quasi-local. Next we need to show that all of the ideals of V are linearly ordered. Let I and J be ideals of V. Suppose, for a contradiction, that $I \not\subseteq J$ and $J \not\subseteq I$. Choose $x \in I \setminus J$ and $y \in J \setminus I$. Since V is a valuation domain and $x, y \in V$, either $x \mid y$ or $y \mid x$. If $x \mid y$, then $y = rx$ for some $r \in V$. Thus $y \in (x) \subseteq I$, a contradiction to $y \in J \setminus I$. If $y \mid x$, then $x = sy$ for some $s \in V$. Thus $x \in (y) \subseteq J$, a contradiction to $x \in I \setminus J$. Thus either $I \subseteq J$ or $J \subseteq I$. Since I and J were arbitrary, the ideals of V are linearly ordered. $\square$

Next we show that every overring of a valuation domain is a localization and must be again a valuation domain; further we show that a valuation domain is integrally closed.

Proposition 1.8.6. *Let V be a valuation domain. Then*

1. *every overring of V is a valuation domain,*
2. *every overring of V is a localization of V, and*
3. *V is integrally closed.*

Proof. Let V be a valuation domain with quotient field K.

1. Suppose that T is an overring of V, so $V \subseteq T \subseteq K$. We show that T is a valuation domain. Let $x, y \in T \setminus \{0\}$ and consider $\lambda = \frac{x}{y}$. Notice that $\lambda \in K \setminus \{0\}$. Since V is a valuation domain and K is its quotient field, we have that either $\lambda \in V$ or $\lambda^{-1} \in V$. If $\lambda \in V$, then $\lambda = \frac{x}{y} = v_1 \in V \subseteq T$. So $x = v_1 y$ and thus $y \mid x$ in T. If $\lambda^{-1} \in V$, then $\lambda^{-1} = \frac{1}{\lambda} = \frac{y}{x} = v_2 \in V \subseteq T$. So $y = v_2 x$ and thus $x \mid y$ in T. Hence either $x \mid y$ or $y \mid x$ in T, showing that T a valuation domain.

2. Suppose $V \subseteq T \subseteq K$ where T is an overring of V. Suppose $t \in T \setminus V$. Since V is a valuation domain, either t or t^{-1} is in V, and $t \notin V$ by choice, thus we have that $t^{-1} \in V$. Let $S = \{\frac{1}{t} \mid t \in T \setminus V\}$. Observe that $S \subseteq V$. We show that S is multiplicatively closed. Suppose $\frac{1}{x}, \frac{1}{y} \in S$, so $x, y \in T \setminus V$. For a contradiction, assume that $\frac{1}{xy} \notin S$. Since $\frac{1}{xy} \notin S$, we have that $xy \in V$. Hence $\frac{1}{x}, xy \in V$. So $xy(\frac{1}{x}) = y \in V$, a contradiction to $y \in T \setminus V$. So S must be multiplicatively closed. We now show that $T = V_S$. Let $t \in T$. If $t \in V$, then $t \in V \subseteq V_S$. If $t \notin V$, we have that $\frac{1}{t} \in S$. Thus $\frac{1}{t} \in V \subseteq V_S$ and $\frac{1}{\frac{1}{t}} = t \in V_S$. Thus $T \subseteq V_S$. Now suppose $\frac{v}{s} \in V_S$. By definition $S = \{\frac{1}{t} \mid t \in T \setminus V\}$. If we let $s = \frac{1}{t}$ with $t \in T \setminus V$, we can then write $\frac{v}{s} = vt \in T$. Hence $V_S = T$. Hence, since T was an arbitrary overring, we have that every overring of V is a localization of V.

3. Let $y \in K$ be integral over V. Thus y is a root of $x^n + v_{n-1}x^{n-1} + v_{n-2}x^{n-2} + \cdots + v_1 x + v_0$, or $y^n + v_{n-1}y^{n-1} + v_{n-2}y^{n-2} + \cdots + v_1 y + v_0 = 0$. Since V is a valuation domain, we have that either $y \in V$ or $y^{-1} \in V$. If $y \in V$, then we are done. So assume that $y^{-1} \in V$. We multiply the integrality equation by $(y^{-1})^{n-1} = y^{1-n} \in V$ to get

$$y^{1-n}[y^n + v_{n-1}y^{n-1} + v_{n-2}y^{n-2} + \cdots + v_1 y + v_0] = y^{1-n}[0],$$

$$\text{or } y + v_{n-1} + v_{n-2}y^{-1} + \cdots + v_1 y^{2-n} + v_0 y^{1-n} = 0,$$

$$\text{or } y = -v_{n-1} - v_{n-2}y^{-1} - \cdots - v_1 y^{2-n} - v_0 y^{1-n},$$

$$\text{or } y = -v_{n-1} - v_{n-2}(y^{-1}) - \cdots - v_1 (y^{-1})^{n-2} - v_0 (y^{-1})^{n-1}.$$

Notice that each term on the right hand side is an element of V, which implies that $y \in V$, a contradiction. Thus, since $y \in K$ was an arbitrary integral element over V, we have that every integral element over V is in V, making V integrally closed. $\square$

Next we give an equivalent characterization of a valuation domain.

Proposition 1.8.7. *V is a valuation domain if and only if V is a quasi-local Bézout domain.*

Proof. For the forward implication, we suppose that V is a valuation domain. From Proposition 1.8.5, we have that V is quasi-local. From Proposition 1.8.4, we have that every finitely generated ideal of V is principal. Hence V is a Bézout domain. For the reverse implication, we suppose that V is a quasi-local Bézout domain. Let M be the unique maximal ideal of V. Let $a, b \in V \setminus \{0\}$. If a is a unit in V, then $b = a(a^{-1}b)$, and hence $a \mid b$. If b is a unit in V, then $a = b(b^{-1}a)$, and hence $b \mid a$. So assume that a and b are not units. Now consider $J = (a, b)$. Since V is a Bézout domain and $J \subseteq V$ is a finitely generated ideal, we have that $J = (a, b) = (m)$. Thus $m = ra + sb$ for some $r, s \in V$. Also, $a = km$ for some $k \in V$ and $b = pm$ for some $p \in V$. Hence $m = ra + sb$ gives $m = rkm + spm$ or $m = m(rk + sp)$. Since $a, b \neq 0$, $m \neq 0$, and since V is a domain, $1_V = rk + sp$. Now if k and p are nonunits, then $k, p \in M$ which gives that $rk + sp \in M$ or $1_V = rk + sp \in M$, a contradiction to M being a maximal ideal. So either k is a unit or p is a unit. Without loss of generality, assume that k is a unit. Then $a = km$ or $m = k^{-1}a$ and so $b = pk^{-1}a$. Thus $a \mid b$. Therefore, since a and b were arbitrary, V is a valuation domain. $\square$

An **antimatter domain** is defined to be an integral domain which does not have any irreducible elements (or atoms). The term "antimatter" was coined by Jim Coykendall and first appeared in [11]. The following result, from [11], provides a useful dichotomy for valuation domains.

Proposition 1.8.8. *Let V be a valuation domain. Then either V is an anitmatter domain or V contains (up to associates) exactly one atom. In the latter case, this atom is, in fact, a prime element of V which generates the unique maximal ideal of V.*

Proof. We mimic the proof given in [11]. Let V be a valuation domain. Suppose that V is not an antimatter domain. Then V contains at least one atom, say r. We claim that each nonunit s of V is divisible by r. Since V is a valuation domain, either $r \mid s$ or $s \mid r$. If $r \mid s$ we have that s is divisible by r. If $r \nmid s$, then we have that $s \mid r$ or $r = as$ for some $a \in V$. But r is irreducible, so either a is a unit of V or s is a unit of V. By assumption, s is a nonunit, so we must have that a is a unit. Hence a^{-1} exists in V, so we have that $r = as$ or $ra^{-1} = s$, giving that $r \mid s$, and hence s is divisible by r. Thus, if r is an atom of V and M is the maximal ideal of V, we have that $M = Vr$. Since r generates the maximal ideal of V, and any maximal ideal is prime by Proposition 1.2.7, r generates a nonzero prime ideal. Thus, by Proposition 1.2.4(2), r is a prime element of V. Finally, if t is another atom of V, the above reasoning gives $M = Vt$, implying that $M = Vr = Vt$. Thus t and r are associated in V. Hence either V an antimatter domain or contains (up to associates) exactly one atom, where the atom is a prime element of V which generates the unique maximal ideal of V. $\square$

Proposition 1.8.9. *Let V be a valuation domain. The following are equivalent:*

1. *V is atomic.*
2. *V is Noetherian.*
3. *V is a principal ideal domain with unique maximal ideal.*

Proof. Let V be a valuation domain. We first show that (1) implies (2). Assume that V is atomic. Without loss of generality, V is not a field, and so, by Proposition 1.8.8, V has a unique atom up to associates, say r. Since V is atomic, each element $s \in V \setminus \{0\}$ can be written as $s = ur^n$ for some $u \in U(V)$ and some uniquely determined nonnegative integer $n = n(s)$. It follows easily that if I is any nonzero proper ideal of V, then $I = Vr^k$, where $k = \min\{n(s) \mid 0 \neq s \in I\}$. Thus, V is a principal ideal domain and hence, by Proposition 1.5.3, Noetherian. Next we show that (2) implies (3). Assume that V is Noetherian. Then, by Proposition 1.5.3, every ideal $I \subseteq V$ is finitely generated. By Proposition 1.8.4 every finitely generated ideal is principal. Hence every ideal of V is principal, making V a principal ideal domain. By Proposition 1.8.7 V is quasi-local, meaning V has a unique maximal ideal. Lastly, we show that (3) implies (1). Assume that V is a principal ideal domain with unique maximal ideal. From [22], we have that every principal ideal domain is an ACCP domain. Then, by Proposition 1.7.21, we have that V is atomic. □

The first part of the next result shows that any valuation domain is an AP domain, meaning that the prime elements and the irreducible elements coincide in a valuation domain.

Proposition 1.8.10. *Let V be a valuation domain.*

1. *V is an AP domain.*

2. *V has atoms if and only if the maximal ideal of V is principal.*

Proof. Let V be a valuation domain.

1. We recall, from Proposition 1.8.7, that any valuation domain is a Bézout domain. Then, from Proposition 1.7.45, we have that any Bézout domain is a GCD domain. Finally, from Proposition 1.7.42, we have that any GCD domain is an AP domain. Hence V is an AP domain.

2. For the forward implication, we assume that V has atoms. Let $\alpha \in V$ be an atom and let M be the unique maximal ideal of V. Suppose M is not principal. Then M must have infinitely many generators, for otherwise $M = (x_1, \cdots, x_n)$ will yield that $M = (x)$ since V is a valuation domain and hence a Bézout domain. Now, let $A = \{\alpha_i\}_{i \in I}$ be the set of generators of M. We write $A = \Gamma_1 \cup \Gamma_2$, where $\Gamma_1 = \{x \in A : x \mid \alpha\}$ and $\Gamma_2 = \{x \in A : \alpha \mid x \text{ and } x \nmid \alpha\}$. If $y \in \Gamma_1$, then $y \mid \alpha$ implies that $\alpha = yk$ for some $k \in V$. But since α is an atom, either y is a unit or k is a unit. y cannot be a unit since it is a generator of our maximal ideal M. Thus we have that k is a unit. So $\alpha k^{-1} = y$ implies that $y \in (\alpha)$. Now let $x \in \Gamma_2$. Then $\alpha \mid x$ which implies that $x = \alpha t$ for some $t \in V$, and hence $x \in (\alpha)$. So $\Gamma_1 \subseteq (\alpha)$ and $\Gamma_2 \subseteq (\alpha)$. Thus $A = \{\alpha_i\}_{i \in I} \subseteq (\alpha)$. But A generates M, the maximal ideal. Therefore, since α is an atom, $M = (\{\alpha_i\}_{i \in I}) = (\alpha)$. Thus M is principal. For the reverse implication, we assume that the maximal ideal, M, of V is principal,

say $M = (\lambda)$. We recall that any maximal ideal is prime, hence $M = (\lambda)$ is a prime ideal. Since λ generates a prime ideal, we have that λ is a prime element of V. But in a domain, prime elements are irreducible. So λ is irreducible, or an atom. Therefore V has at least one atom, so V has atoms. $\square$

2. COHEN-KAPLANSKY DOMAINS AND NEW GENERALIZATIONS

As previously stated, Cohen-Kaplansky domains were first introduced by I. S. Cohen and Irving Kaplansky in 1946. The terminology "Cohen-Kaplansky domains" (or CK domains) was applied by Anderson and Mott in the 1980's and since that time a number of authors have studied CK domains. In Section 2.1, we state the definition of a Cohen-Kaplansky domain and then we provide some examples along with some properties of Cohen-Kaplansky domains. In Section 2.2, we examine generalizations of Cohen-Kaplansky domains that have already been made. We remark that these generalizations differ drastically from the generalizations made in this thesis. In Sections 2.3 and 2.4, we arrive at the goal of this thesis, to provide a new generalization of Cohen-Kaplansky domains. These new generalizations stemmed from the following question related to valuation domains: in a valuation domain V, is it possible to have one element that divides all the other elements? The answer to this question is an immediate yes since any unit u in V will divide all of the elements of V. Revising the question, we then asked the following question: in a valuation domain V, is it possible to have a nonzero nonunit element that divides all of the other nonzero nonunits? The answer to this question also turns out to be yes. In examining such types of domains, we are actually examining domains which are a lot like Cohen-Kaplansky domains. In the generalizations presented in this thesis, we take a Cohen-Kaplansky domain and remove the condition that it must be atomic, we then add the condition that every nonzero nonunit element of the domain must be divisible by at least one of the irreducible elements.

2.1. Cohen-Kaplansky Domains

Recall that an integral domain R is atomic if each nonzero nonunit element of R can be written as a (finite) product of irreducible elements. We note here that the factorization of an element in an atomic domain into irreducible elements need not be unique (up to units). In fact, the factorization is unique precisely when R is a unique factorization domain or equivalent when each irreducible is prime [4].

Definition 2.1.1. An integral domain R is said to be a **Cohen-Kaplansky domain (CK domain)** if it is atomic and contains only a finite number of irreducible elements (up to associates). More generally, we have the following definition. We define a **CK-n domain** to be a CK domain containing exactly n irreducible elements (up to associates).

In [13], they introduce a **CK*-n domain** which is defined to be a CK domain in which every irreducible element is nonprime.

Example 2.1.2.

1. Let $p_1, \cdots, p_n$ be n distinct prime integers. If we consider the ideals (p_1), $\cdots$, (p_n), then the set complement of the union of these ideals forms a multiplicatively closed set. We can thus consider the localization $\mathbb{Z}_{[(p_1)\cup\cdots\cup(p_n)]^c} = \{\frac{a}{b} \mid a \in \mathbb{Z}, b \notin (p_1) \cup \cdots \cup (p_n)\}$. In the localization, any element relatively prime to $p_1, \cdots, p_n$ becomes a unit and so the only irreducible elements are $p_1, \cdots, p_n$. That is, no new irreducible elements could have been created in the localization process. However, since each of these irreducible elements was in fact a prime element in $\mathbb{Z}$, they remain prime in $\mathbb{Z}_{[(p_1)\cup\cdots\cup(p_n)]^c}$. This construction can create a CK-n domain for every natural number n, but we remark that all n irreducible elements are in fact prime elements [24].

2. Let $\mathbb{F}$ be a field. Then $\mathbb{F}[[x]]$ is a CK-1 domain. Here x is the unique irreducible element, but it is also prime since $\mathbb{F}[[x]]$ is a valuation domain and is hence, by Proposition 1.8.10, an AP domain.

3. $\mathbb{Z}[\sqrt{-3}]_P$ where $P = (2, 1+\sqrt{-3})$ is a CK-3 domain. The irreducible elements in $\mathbb{Z}[\sqrt{-3}]_P$ are 2, 2ω, and $2\overline{\omega}$, where $\omega = \frac{-1+\sqrt{-3}}{2}$.

4. $\mathbb{F}_2 + x\mathbb{F}_4[[x]]$ is a CK-3 domain with maximal ideal $M = x\mathbb{F}[[x]]$ and three distinct nonprime irreducible elements $x, x\omega, x\omega^2$, where $\mathbb{F}_4 = \{0, 1, \omega, \omega^2\}$ [24].

5. Rings of the form $\mathbb{A}[[x^{a_1}, x^{a_2}, \cdots, x^{a_n}]] = \mathbb{A}[[\{x^{a_i}\}_{i=1}^n]]$, where $\mathbb{A}$ is a finite field and $\gcd(a_1, a_2, \cdots, a_n) = 1$, are CK domains [24].

6. An example of a local CK domain without unique factorization is $K + xL[[x]]$, where $K \subsetneq L$ are finite fields [4].

7. Let $k \subseteq K$ be a pair of finite fields and $n \geq 1$. Then $R = k + x^n K[[x]]$ is a local CK domain with maximal ideal $M = x^n K[[x]]$. The irreducible elements of R have the form ux^i where u is a unit in $K[[x]]$ and $n \leq i \leq 2n-1$. Hence R has $n|\frac{K^*}{k^*}||K|^{n-1}$ nonassociate irreducible elements [4].

It is interesting to note that [24] shows that, assuming the strengthened Goldbach Conjecture, there exists a CK-n domain for every positive integer n.

Next we demonstrate that any CK-1 domain and any CK-2 domain is a unique factorization domain.

Proposition 2.1.3. *Any CK-1 domain is a unique factorization domain.*

Proof. We mimic the proof given in [24]. Let R be a CK-1 domain; then R is an atomic integral domain with exactly one irreducible element, say p. Since R is atomic, every nonzero nonunit is a product of irreducible elements. Hence every nonzero nonunit $r \in R$ can be written in the form $r = up^n$ where u is a unit in R. If R is not a unique factorization domain, then there would exist two different irreducible factorization of some element x, say we can write $u_1 p^{n_1} = x = u_2 p^{n_2}$, where u_1 and u_2 are both units in R. If $n_1 \neq n_2$, then we have that $p^{|n_1 - n_2|} = u_3$ for some unit $u_3 \in R$. But this implies that p is a unit, which is a contradiction to p being irreducible. Thus we must have that $n_1 = n_2$ and every factorization is unique up to units. Hence R is a unique factorization domain. □

Proposition 2.1.4. *Any CK-2 domain is a unique factorization domain.*

Proof. We mimic the proof given in [24]. Let R be a CK-2 domain; then R is an atomic integral domain with two nonassociate irreducibles, say p and q. Since R is atomic, as it is a CK-2 domain, we have that every nonzero nonunit can be factored into a product of irreducible elements. If R was not a unique factorization domain, it follows that there would have to exist some element x that could be factored into two different irreducible factorizations, say $u_1 p^{n_1} q^{m_1} = x = u_2 p^{n_2} q^{m_2}$ with u_1 and u_2 units of R. Canceling the appropriate powers of p and q, which is allowed since R is an integral domain, yields an equation of the form $p^n = uq^m$ for some unit u in R and n, m natural numbers. Notice that if $n \leq 0$, then q would be a unit, which is a contradiction to q being irreducible. Similarly, if $m \leq 0$, then p would be a unit, which is a contradiction to p being irreducible. So it follows that both m and n are positive integers. Since p is a nonzero nonunit of R, by Proposition 1.2.8, there must exist some maximal ideal M that contains p. Now, since $q^m = u^{-1}p^n$, it follows that q^m is also an element of M. But since M is a maximal ideal, by Proposition 1.2.7, we have that M is also a prime ideal, and hence we have that $q \in M$ as well. So now consider the element $p + q$. Since both p and q are elements of M it follows that $p + q$ is also in M. In particular, this means that $p+q$ must either be 0 or a nonzero nonunit of R. If $p+q = 0$, then $q = -p$ which would imply that p and q were associates, a contradiction. Thus $p + q$ must be a nonzero nonunit, hence it must be able to be factored into a product of irreducible elements, say $p+q = vp^aq^b$ for some unit v in R and $a, b \geq 0$ with at least one of a and b being positive. Without loss of generality, suppose $a > 0$. Thus it follows that since p divides vp^aq^b, we must have that p divides $p + q$ and thus we must have that p divides q, a contradiction. Therefore there cannot be two different irreducible factorizations of x. Hence R must be a unique factorization domain. □

We remark that if any maximal ideal of a CK domain has only one or two irreducible elements, from [24], it follows that those irreducible elements would also have to be prime. Thus any maximal ideal in a CK domain must contain at least three irreducible elements in order for each irreducible to be nonprime [24].

Once we include more than two irreducible elements, factorization no longer needs to be unique in a CK domain. We mimic the explanation from [24] with regards to how to deal with three irreducible elements. Suppose that R is a CK-3 domain in which each irreducible is nonprime; then R is an atomic integral domain with three nonassociate nonprime irreducibles, say p, q, and r. We note that p, q, and r must all lie in the same maximal ideal, say M. Now consider the element $qr + p$, which must be an element of M, so it is either zero or a nonzero nonunit. $qr + p$ cannot be zero, so it must be a nonzero nonunit and hence it can be written as a product of irreducible elements. Notice that q and r cannot divide $qr + p$ for otherwise they would divide p. Thus it follows that $qr + p$ must be divisible by p and therefore we can write $qr = u_1p^a$ where u_1 is a unit and $a \geq 2$. Notice that if $a = 0$, then q and r would be units, a contradiction. Also notice that if $a = 1$, it would follow that either q or r would be a unit since p is an irreducible element, which is also a contradiction. Similarly, we find that $qp = u_2r^b$ and $pr = u_3q^c$ where u_2 and u_3 are units and $b, c \geq 2$. Now consider the product $(qr)(qp)(pr) = (u_1p^a)(u_2r^b)(u_3q^c)$ or $p^2q^2r^2 = u_1u_2u_3p^aq^cr^b$.

In order for this equality to hold, we must have that $a = b = c = 2$ in order for a non-UFD CK-3 domain to be possible. Such a domain with this multiplicative structure can be constructed, see Example 2.1.2(4) and let $p = (x)$, $q = (x\omega)$ and $r = (x\omega^2)$.

Recall that a semi-local domain is a Noetherian integral domain which contains only finitely many maximal ideals. It turns out that any CK domain is a semi-local domain.

Proposition 2.1.5. *If R is a CK domain, then R is a semi-local domain.*

Proof. We mimic the proof given in [24]. Let R be a CK domain. Then R is atomic and contains only a finite number of irreducible elements (up to associates), say $p_1, p_2, \cdots, p_n$. Notice that if P is any nonzero prime ideal of R, then P has a basis consisting of a subset of $\{p_1, p_2, \cdots, p_n\}$. From this we get that every prime ideal of R is finitely generated and that R has only finitely many prime ideals. By Proposition 1.2.7, we know that every maximal ideal is prime, hence there can only exist finitely many maximal ideals. The fact that R is Noetherian follows from the fact that every prime ideal is finitely generated, hence by Proposition 1.5.3, every ideal is finitely generated. Hence R is a Noetherian integral domain with only finitely many maximal ideals, making R a semi-local domain. □

Proposition 2.1.6. *Two maximal ideals of a CK domain R cannot have a prime element in common.*

Proof. We mimic the proof given in [9]. Let R be a CK domain; then R is an atomic integral domain and contains only a finite number or irreducible elements (up to associates). Notice that if P is any nonzero prime ideal of R it has a basis of prime elements. Thus it follows that there can only be finitely many prime ideals. By Proposition 1.2.7, we know that every maximal ideal is prime, hence there can only exist finitely many maximal ideals, say $M_1, M_2, \cdots M_t$. We suppose that there is a prime element in common to two maximal ideals. Without loss of generality, we suppose the prime element r is in $M_1 \cap M_2$. Since M_1 is not contained in any of the other maximal ideals, it contains an element, and hence also a prime element not in any of the other maximal ideals. Let $p_1, p_2, \cdots, p_k$ with $k > 0$ be those prime elements in M_1 which are not in any of $M_2, \cdots, M_t$. Now let $a_i \in M_i \setminus M_1$ for $2 \leq i \leq t$ and set $a = a_2 a_3 \cdots a_t$. Then a is in $M_2 \cap \cdots \cap M_t$ but not in M_1. The element $a + p_1 p_2 \cdots p_k$ is in none of the maximal ideals, and so must be a unit in R, call it

α. Since r is not one of the p_i, we have that $r\alpha = r\prod p_i + ra$ is not divisible by any p_i, hence neither is ra, hence neither is $c = \prod p_i + ra$. But $c \in M_1$, and so must be divisible by some prime, necessarily in M_j with $j \neq 1$. This implies that $\prod p_i \in M_j$ with $j \neq 1$, which is impossible. Hence two maximal ideals cannot have a prime element in common. □

Let R be a CK domain. Cohen and Kaplansky showed that R is a one-dimensional semi-local domain, that an irreducible element of R is contained in a unique maximal ideal, and that if M is a maximal ideal of R, then R_M is a CK domain and the irreducible elements of R_M are precisely those irreducible elements of R that are contained in M [4]. Hence if R is a CK domain, then R_M is also a CK domain, where M is a maximal ideal of R.

In [4], they show that any overring of a CK domain is also a CK domain.

The following equivalent characterization of a CK domain can be found in [6]. An integral domain R is a CK domain if and only if R is an intersection of a finite number of local CK overrings.

2.2. Known Generalizations of Cohen-Kaplansky Domains

In this section results are given without proof, as our intent is to present known generalizations of Cohen-Kaplansky domains before presenting our new generalization in the subsequent sections.

In the previous section we defined a CK domain; however, one could extend this notion to the general case of a commutative ring with identity. We will say that a commutative ring with identity is called a **Cohen-Kaplansky ring (CK ring)** if it is an atomic ring with only a finite number of nonassociate atoms.

The first generalization we examine was introduced in [2] in 1992. We begin by defining what they called a generalized CK ring.

Definition 2.2.1. Let R be a commutative ring with identity. We say that R is a **generalized Cohen-Kaplansky ring (generalized CK ring)** if R is an atomic ring with almost all atoms prime.

Examples of generalized CK domains besides unique factorization domains include $\mathbb{Z}[2i]$, $k + xK[[x]]$, and $k + xK[x]$ where $k \subseteq K$ are finite fields [1].

It turns out that a finite direct product of CK rings is a CK ring. The characterization of generalized CK domains given in [2] is incomplete. In [1], it is shown that R is a generalized CK ring if and only if R is a finite direct product of CK rings and generalized CK domains.

In [5], they define a universal set as follows. Let D be an atomic integral domain. A subset S of D is **universal** if each $s \in S$ is divisible by each atom of D.

Suppose that (D, M) is a CK domain. In [9], Cohen and Kaplansky showed that if D has exactly n nonassociate atoms, then M^{n-1} is universal and that if n is prime, M^2 is universal. Thus if (D, M) is a CK domain with exactly three nonassociate atoms, M^2 is universal. Also, if (D, M) is an atomic domain with M^2 universal, then for atoms $a_1, \cdots, a_n \in D$, $a_1 \cdots a_n M = M^{n+1}$. In particular, if a and b are atoms of D, then $aM = M^2 = bM$.

This notion of a universal set will be a set of particular importance for the new generalization of Cohen-Kaplansky domains presented in this thesis.

Another generalization of Cohen-Kaplansky domains can be found in [3]. Let R be a commutative ring with identity. R is called a **weak Cohen-Kaplansky ring (weak CK ring)** if R is atomic and each maximal ideal of R contains only finitely many nonassociate atoms. In [3], they show several results, including the following.

Proposition 2.2.2. *For a commutative ring (with identity) R the following conditions are equivalent:*

1. *Every ideal of R is a finite union of principal ideals;*
2. *Every prime ideal of R is a finite union of principal ideals;*
3. *R is Noetherian and every maximal ideal of R is a finite union of principal ideals;*
4. *R is atomic and every maximal ideal of R is a finite union of principal ideals;*
5. *R is a finite direct product of finite local rings, SPIRs, and (one-dimensional) Noetherian domains in which every maximal ideal is a finite union of principal ideals;*
6. *R is a weak CK ring.*

Hence a weak CK ring R has dim $R \leq 1$.

In attempting to show that if R is a weak CK ring and M is a maximal ideal of R, then R_M is a CK ring, in [3] they actually proved a stronger result.

Proposition 2.2.3. *Let R be an (commutative with identity) atomic ring. If M is a maximal ideal of R that is a finite union of principal ideals, then R_M is a CK ring.*

From which they obtain the desired result.

Corollary 2.2.4. *If R (commutative with identity) is a weak CK ring and M is a maximal ideal of R, then R_M is a CK ring.*

In [3] they also show the following result.

Proposition 2.2.5. *A Dedekind domain is a weak CK domain if and only if it is a PID.*

In [3] they too define a generalized CK domain as an atomic domain in which almost all atoms are prime. They relate weak CK domains and generalized CK domains as follows. For an integral domain R the following are equivalent: (a) R is a weak CK domain and a generalized CK domain; (b) R is a one-dimensional generalized CK domain. They end by considering the existence of weak CK domains. They remark that of course a principal ideal domain is a weak CK domain. They claim that a weak CK domain is semi-local if and only if it is a CK domain. They also prove the following construction yields a weak CK domain.

Proposition 2.2.6. *Let B be an integral domain with quotient field K and let A be a subring of B. Let $R = A + xB[x]$. Then the following conditions are equivalent:*

1. *R is a one-dimensional generalized CK domain;*
2. *R is a weak CK domain;*
3. *$A = B = K$ or $B = K$ is a finite field.*

2.3. Unrestricted Cohen-Kaplansky Domains

In this section we begin the process of introducing our new generalization of Cohen-Kaplansky domains. The first step toward this new generalization is to take a CK domain and remove the condition that it must be atomic, which we do next.

Definition 2.3.1. We say that the integral domain R is an **unrestricted CK-n domain**, denoted uCK-n domain, if R has exactly $n < \infty$ irreducible elements (up to associates).

Example 2.3.2.

1. Let $\mathbb{F}$ be a field. $\mathbb{F}[[x]]$ is a valuation domain. We remark that since $\mathbb{F}[[x]]$ is a valuation domain, by Proposition 1.8.7, we have that $\mathbb{F}[[x]]$ is quasi-local. Notice that every nonzero elements of $\mathbb{F}[[x]]$ is of the form $x^n \cdot u(x)$ with $n \geq 0$ and $u(x)$ a unit in $\mathbb{F}[[x]]$. Then, for $x^n \cdot u(x)$ to be a nonunit, we must have that $n \geq 1$. So x divides all the nonzero nonunits of $\mathbb{F}$; hence the unique maximal ideal of $\mathbb{F}[[x]]$ is (x). Thus $\mathbb{F}[[x]]$ is a uCK-1 domain, but it also an atomic domain.

2. Consider $R = \mathbb{Z}_{(2)} + x\mathbb{Q}[[x]]$. Then R is a uCK-1 domain with unique irreducible element 2 and R is not atomic since x cannot be factored into a finite product of atoms.

3. Let $p_1, \cdots, p_n$ be n distinct prime integers. If we consider the ideals (p_1), $\cdots$, (p_n), then the set complement of the union of these ideals forms a multiplicatively closed set, say $S = [(p_1) \cup \cdots \cup (p_n)]^c$. Then consider $R = \mathbb{Z}_S + x\mathbb{Q}[[x]]$. R is a uCK-n domain since it is an AP domain with exactly n irreducible elements.

The following result from [12] guarantees us the existence of a uCK-n domain for any natural number n.

Proposition 2.3.3. *Given any natural number n, there exists a non-atomic CK-n domain.*

A proof of this result can be found in [12]; we omit it here as it uses a monoid construction which is outside the scope of this thesis.

Let R be an integral domain. We show, in the proof of Proposition 2.4.6, that R is a CK-n domain if and only if R is an atomic uCK-n domain.

2.4. Quasi Unrestricted Cohen-Kaplansky Domains

In this section we are finally able to fully introducing our new generalization of Cohen-Kaplansky domains. We remark again that the idea for this generalization came out of studying valuation domains. In a valuation domain we have that given any two nonzero elements of the domain one of the elements must divide the other element. The question the author asked was "in an integral domain is it possible to have a single nonzero nonunit divide all of the other nonzero nonunits?" It turns out the answer to this question was yes and the author argued that the nonzero nonunit that divides all the other nonzero nonunits must be a prime element and hence, by Proposition 1.1.4, must be an irreducible element. In the new generalization presented below, we extend this notion from one irreducible to finitely many irreducible elements in the integral domain.

The first step toward this new generalization was to take a CK domain and remove the condition that it must be atomic, which we defined as a uCK-n domain in Section 2.3. Now we take a uCK-n domain and add in the condition that every nonzero nonunit in the integral domain must be divisible by at least one of the irreducible elements.

Definition 2.4.1. We say that the integral domain R is a **quasi-CK-n domain** if R is a uCK-n domain with irreducible elements $\pi_1, \pi_2, \cdots, \pi_n$ such that for every nonzero nonunit $y \in R$ there exists an irreducible element, say π_i with $1 \leq i \leq n$, such that $\pi_i \mid y$.

Example 2.4.2.

1. $\mathbb{Z}_{(p)}$, where $p \in \mathbb{Z}$ is a prime element, is a quasi-CK-1 domain that is also a valuation domain.

2. Let $\mathbb{F}$ be a field. Then $\mathbb{F}[[x]]$ is a quasi-CK-1 domain that is also a valuation domain.

3. Consider $R = \mathbb{Z}_{(2)} + x\mathbb{Q}(y)[[x]]$. R is not a valuation domain since $x \nmid xy$ and $xy \nmid x$. Let $g = c + x\sum_{i=0}^{\infty} f_i(y)x^i \in R$. Notice that if $2 \mid c$, we will have that g is not a unit. Also notice that $x = 2(\frac{x}{2})$, so $2 \mid x$. So $2 \mid c$ and $2 \mid x$; thus $2 \mid g$. Hence 2 is a nonzero nonunit that divides all nonzero nonunits of R. Therefore R is a quasi-CK-1 domain that is not a valuation domain.

4. The following construction will yield a quasi-CK-n domain for any natural number n. Let $p_1, \cdots, p_n$ be n distinct prime integers. If we consider the ideals $(p_1), \cdots, (p_n)$, then the set complement of the union of these ideals forms a multiplicatively closed set, say $S = [(p_1) \cup \cdots \cup (p_n)]^c$. Now consider $R = \mathbb{Z}_S + x\mathbb{Q}[[x]]$. Then R is a quasi-CK-n domain with irreducible elements $p_1, \cdots, p_n$. Further, we remark that R is not atomic since x cannot be factored into a finite product of atoms.

Before giving some results about a general quasi-CK-n domains, we look at at some properties of quasi-CK-1 domains.

Let R be a quasi-CK-1 domain that is also a valuation domain. We make several observations. Note that R is Prüfer since

$$R_M = \{\frac{r}{m} \mid r \in R \text{ and } m \notin M\} = \{ur \mid r \in R \text{ and } u \in U(R)\} = R$$

is a valuation domain, where M is the unique maximal ideal of R. Also, R is integrally closed; the proof of this follows immediately from the fact that any valuation domain is integrally closed, which was shown in the proof of Proposition 1.8.6. Also we see that R is quasi-local, all of its ideals are linearly ordered, and it is a Bézout domain; the proof of this follows immediately from R being a valuation domain, Proposition 1.8.5 and Proposition 1.8.7. We also have that R is an AP domain meaning that our unique irreducible element must also be prime; the proof of this follows from Proposition 1.8.10.

Proposition 2.4.3. *Let a domain R be the intersection $V_1 \cap V_2 \cap \cdots \cap V_n$, where the V_i's are quasi-CK-1 valuation domains between R and its quotient field. Then R is a Bézout domain.*

The proof of this result follows from the fact that the V_i's are valuation domains between R and its quotient field and Theorem 107 of [20].

If R is a quasi-CK-1 domain that is not a valuation domain, then we note that R is not Prüfer since $R_M = R$ is not a valuation domain, where M is the unique maximal ideal of R.

Next we give an equivalent characterization for a quasi-CK-1 domain.

Proposition 2.4.4. *Let R be an integral domain. R is a quasi-CK-1 domain if and only if R is quasi-local and the maximal ideal is principal.*

Proof. Let R be an integral domain. For the forward implication, assume that R is a quasi-CK-1 domain with unique irreducible element $x \in R$. Then we have that $x \mid z$ for every nonzero nonunit $z \in R$. Let M be any maximal ideal of R. Since x is a nonzero nonunit, $x \in M$ implies that $(x) \subseteq M$. Now let $m \in M$. If $m = 0$, then $m = 0 = 0 \cdot x$. Thus $x \mid m$. If $m \neq 0$, m is a nonzero nonunit. Thus $x \mid m$ by assumption, so $m = rx$. Thus $m \in (x)$ and hence $M \subseteq (x)$. Therefore, $M = (x)$. Thus every maximal ideal of R is principal and equal to (x). So R is quasilocal. For the reverse implication, assume that R is quasilocal, with maximal ideal M, and further assume that $M = (y)$. Let z be any nonzero nonunit of R, then $z \in M = (y)$ implies that $z = ty$ for some $t \in R$. Thus $y \mid z$. So y divides all nonzero nonunits. Hence R is a quasi-CK-1 domain. □

Next we show that any quasi-CK-1 domain turns out to be an AP domain, meaning that the prime elements and the irreducible elements of a quasi-CK-1 domain coincide.

Proposition 2.4.5. *If R is a quasi-CK-1 domain, then R is an AP domain.*

Proof. Let R be a quasi-CK-1 domain with unique irreducible element $y \in R$. Then we have that $y \mid z$ for every nonzero nonunit $z \in R$. By the previous result, we have that R is quasilocal and its maximal ideal, say M, is principal. Further, we must have that $M = (y)$. Since M is a maximal ideal, we have that M is a prime ideal. Since y generates a prime ideal, y is a prime element. Next we demonstrate that y is irreducible. Suppose that y reduces, say $y = ab$ where a and b are not units in R. Then, $y \mid a$ implies that $a = r_1 y$ for some $r_1 \in R$. Hence, $y = ab = r_1 yb$, and so $y - r_1 yb = 0$ or $y(1_R - r_1 b) = 0$. Thus either $y = 0$ or $1_R - r_1 b = 0$. But $y \neq 0$, so we have that $1_R - r_1 b = 0$ or $1_R = r_1 b$, which implies that b is a unit, a contradiction to b being a nonunit. Hence y is irreducible. Next we demonstrate that y is the only irreducible up to units. Suppose that $\pi \in R$ is also irreducible. Then π is a nonzero nonunit, so $y \mid \pi$, which gives that $\pi = r_2 y$ for some $r_2 \in R$. But π is irreducible, so either r_2 is a unit or y is a unit. But y is not a unit, so r_2 must be a unit. Thus $\pi = u \cdot y$, so π is an associate to y. Thus R is an AP domain. □

Recall that a subset S of an atomic integral domain D is universal if each $s \in S$ is divisible by each atom of D. We remark that

1. for a quasi-CK-1 domain R with unique irreducible element π, although R is not necessarily atomic, we do have that the set of nonzero nonunits of R is a universal subset of R since each nonzero nonunit is divisible by π.

2. for a quasi-CK-n domain R with irreducible elements $\pi_1, \pi_2, \cdots, \pi_n$, although R is not necessarily atomic, we have that the set $S = \{u\pi_1^{a_1}\pi_2^{a_2} \cdots \pi_n^{a_n} \mid u \in U(R) \text{ and } a_i > 0 \text{ for every } 1 \leq i \leq n\}$ is a universal subset of R since each element of S is divisible by every atom of R.

The second remark above may lead one to the following definition. We say that a subset S of an integral domain R is **almost-universal** if each $s \in S$ is divisible by at least one atom of R. With this definition, one could then say that for a quasi-CK-n domain R, the set of nonzero nonunits of R is an almost-universal subset of R.

Now we look at some more properties of a quasi-CK-n domain.

Recall that an element a of a ring R is said to be **idempotent** if $a^2 = a$. We remark here that if R is a quasi-CK-n domain, the only idempotent elements in R are 0 and 1_R. To see why this is the case, let R be a quasi-CK-n domain and suppose that $x \in R$ is an idempotent element. Then we have that $x^2 = x$, that is the same as having $x^2 - x = 0$, which is the same as having $x(x - 1_R) = 0$. Since R is an integral domain, we have that either $x = 0$ or $x - 1_R = 0$. Hence we have that either $x = 0$ or $x = 1_R$.

Let R be a quasi-CK-n domain with $n \geq 2$. One might hope that R could possibly be a valuation domain, but unfortunately it cannot. To see why, suppose that R is also a valuation domain. We will name the n irreducible elements of R as $\pi_1, \pi_2, \cdots, \pi_n$. Since each irreducible element π_i is a nonzero nonunit, we then have that either $\pi_1 \mid \pi_2$ or $\pi_2 \mid \pi_1$. In either case, this would imply that π_1 and π_2 are associates, which is a contradiction. Hence a quasi-CK-n domain with $n \geq 2$ cannot be a valuation domain.

Next we demonstrate the relationship between CK-n domains, uCK-n domains, and quasi-CK-n domains. We recall here that an integral domain R is said to be **Archimedean** if $\cap Rr^n = 0$ for each nonunit r of R.

Proposition 2.4.6. *Let R be an integral domain. The following are equivalent:*

1. *R is a CK-n domain.*
2. *R is an atomic uCK-n domain.*
3. *R is an atomic quasi-CK-n domain.*
4. *R is a one-dimensional quasi-CK-n domain.*

Proof. Let R be an integral domain. For (1) implies (2), we assume that R is a CK-n domain. Then, by definition, R is atomic and contains exactly n irreducible elements (up to associates). Hence R is atomic and R is an integral domain which has exactly n irreducible elements (up to associates), making R an atomic uCK-n domain. For (2) implies (3), we assume that R is an atomic uCK-n domain. Then R is atomic and R is an integral domain which has exactly n irreducible elements (up to associates), say $\pi_1, \pi_2, \cdots, \pi_n$ are the n irreducible elements of R. We need to show that if r is any nonzero nonunit of R there exists an irreducible element that divides it. So let r be a nonzero nonunit of R; since R is atomic, r can be written as a finite product of irreducible elements, say $r = u\pi_1^{a_1}\pi_2^{a_2}\cdots\pi_n^{a_n}$ where u is a unit in R and $a_i \geq 0$ for each $0 \leq i \leq n$. Since r is a nonunit at least one of the a_i's needs to be strictly greater than 0. Without loss of generality, say $a_1 > 0$. Then we have that π_1 divides $u\pi_1^{a_1}\pi_2^{a_2}\cdots\pi_n^{a_n}$ and thus π_1 divides r. Therefore we have that R is an atomic quasi-CK-n domain. For (3) implies (4), we assume that R is an atomic quasi-CK-n domain. Since R is already a quasi-Ck-n domain, we only need to show that it is one-dimensional. For a contradiction, suppose that there exist prime ideals $0 \neq P \subsetneq Q$. Let p be a nonzero prime element in P; we remark by Proposition 1.1.4, that p must be an irreducible element. Now list all of the irreducible elements in $Q \setminus P$, of which there can only be finitely many since R has exactly n irreducible elements, say $q_1, q_2, \cdots, q_k$ are all of the irreducible elements in $Q \setminus P$. Now consider the element $q_1q_2\cdots q_k + p$. Since each q_j is in $Q \setminus P \subseteq Q$ and p is in $P \subsetneq Q$, we have that $q_1q_2\cdots q_k + p$ is an element of Q. We note that if $q_1q_2\cdots q_k + p$ is an element of P, then we would have that $q_1q_2\cdots q_k$ is an element of P, but since P is a prime ideal of R, we have that $q_l \in P$ for some $1 \leq l \leq k$, which is a contradiction. Hence we have that $q_1q_2\cdots q_k + p$ is an element of $Q \setminus P$. Therefore $q_1q_2\cdots q_k + p$ must be divisible by an irreducible element from $Q \setminus P$, which would mean that there exists q_m for some $1 \leq m \leq k$ such that $q_m \mid p$, which is a contradiction. Thus we cannot have prime ideals $0 \neq P \subsetneq Q$, making R one-dimensional, as desired. For (4) implies (1), we assume that R is a one-dimensional quasi-CK-n domain. Then R is an integral domain with exactly n irreducible elements (up to associates), say $\pi_1, \pi_2, \cdots, \pi_n$. It remains to show that R is atomic. For a contradiction, we assume that R is not atomic. Then there exists a nonzero element $x \in R$ such that x cannot be factored into a finite product of atoms. Thus we must have that one of the irreducible elements, say π_i divides x infinitely often. Thus we have that $0 \neq x \in \cap_{n=1}^{\infty} R(\pi_i)^n$. Hence R is not Archimedean; hence, by Corollary 1.4 of [23], we have that R is not one-dimensional, a contradiction. Therefore R must be atomic. $\square$

Next we remark that one might hope that any atomic quasi-CK-n domain is a unique factorization domain, but this is not always the case. To see why, let R be an atomic quasi-CK-n domain. If $n = 1$, then by Proposition 2.4.6, we have that R is a CK-1 domain which was shown to be a unique factorization domain in Proposition 2.1.3. If $n = 2$, then by Proposition 2.4.6, we have that R is a CK-2 domain which was shown to be a unique factorization domain in Proposition 2.1.4. For $n \geq 3$, by Proposition 2.4.6, we have that R is a CK-n domain which need not be a unique factorization domain.

In the following example we show that a polynomial ring extension and a power series extension of a quasi-CK-n domain need not be a quasi-CK-n domain.

Example 2.4.7. Let R be a quasi-CK-n domain with irreducible elements $\pi_1, \pi_2, \cdots, \pi_n$. Then we have the following:

1. $R[y]$ is not a quasi-CK-n domain. Since R is an integral domain, by Proposition 1.3.5, we have that $U(R) = U(R[y])$. Hence $R[y]$ has exactly n irreducible elements and those irreducible elements are $\pi_1, \pi_2, \cdots, \pi_n$. Now consider the polynomial $f(y) = \pi_1 + \pi_2 y \in R[y]$. Suppose that $f(y)$ is divisible by one of the irreducible elements of $R[y]$, say $f(y)$ is divisible by π_i. So we have that $\pi_i \mid f(y)$ or $\pi_i \mid (\pi_1 + \pi_2 y)$ or $\pi_1 + \pi_2 y = g(y) \cdot \pi_i$ for some $g(y) \in R[y]$. Since $\pi_i \mid g(y) \cdot \pi_i$ we have that $\pi_i \mid (\pi_1 + \pi_2 y)$ which would mean that $\pi_i \mid \pi_1$ and $\pi_i \mid \pi_2$, which is impossible. Thus $R[y]$ has at least one nonzero nonunit that is not divisible by any of the irreducible elements of $R[y]$; hence $R[y]$ cannot be a quasi-CK-n domain.

2. $R[[y]]$ is not a quasi-CK-n domain. Let $f(y) = \sum_{i=0}^{\infty} r_i y^i \in R[[y]]$. Since R is an integral domain, by Proposition 1.3.10, if r_0 is irreducible in R, then $f(y)$ is irreducible in $R[[y]]$. Then the following collection of power series

$$\pi_1 + y + y^2 + y^3 + y^4 + \cdots$$

$$\pi_2 + y + y^2 + y^3 + y^4 + \cdots$$

$$\vdots$$

$$\pi_n + y + y^2 + y^3 + y^4 + \cdots$$

is a collection of n irreducible elements in $R[[y]]$. Now let $r \in R$ be a nonzero nonunit such that $\pi_1 \nmid r$. Then the power series $\pi_1 + ry + ry^2 + ry^3 + ry^4 + \cdots$ is another irreducible element in $R[[y]]$. Hence, since $R[[y]]$ has more than n irreducible elements, $R[[y]]$ cannot be a quasi-CK-n domain.

Next we illustrate how to construct a new quasi-CK-n domain from a given quasi-CK-n domain.

Proposition 2.4.8. *If R is a quasi-CK-n domain with quotient field K, then $R + xK[[x]]$ is a quasi-CK-n domain.*

Proof. Suppose R is a quasi-CK-n domain with quotient field K. We identify the n irreducible elements of R by $\pi_1, \pi_2, \cdots, \pi_n$. Now consider $R + xK[[x]]$; we remark that a general element of $R + xK[[x]]$ looks like

$$f(x) = r + x \cdot g(x) = r + x[k_0 + k_1x + k_2x^2 + k_3x^3 + \cdots] = r + k_0x + k_1x^2 + k_2x^3 + k_3x^4 + \cdots$$

where $r \in R$, $g(x) = k_0+k_1x+k_2x^2+k_3x^3+\cdots$ and each k_j is in K. We first claim that every nonzero element $f(x)$ of $R+xK[[x]]$ such that $f(0) \neq 0$ is associated to an element of R. To see this, we note that we can write our general element $f(x)$ from above as $f(x) = r[1_R + \frac{k_0}{r}x + \frac{k_1}{r}x^2 + \frac{k_2}{r}x^3 + \cdots]$ provided that $r \neq 0$. Hence we can associate $f(x)$ to its constant term r provided $f(0) = r \neq 0$. If $f(0) = 0$, then $f(x)$ is divisible by all of the irreducible elements of R. Thus the only irreducible elements in $R + xK[[x]]$ are the irreducible elements of R, namely $\pi_1, \cdots, \pi_n$, and every nonzero nonunit of $R+xK[[x]]$ is divisible by at least one of these irreducible elements. Therefore $R+xK[[x]]$ is a quasi-CK-n domain. □

We remark here that Proposition 2.4.8 is not true if we replace "quasi-CK-n domain" by "CK-n domain". To see why this is the case, suppose that R is a CK-n domain with quotient field K. Since R is not a field, it must contain some nonzero nonunit, say y. Then in $R + xK[[x]]$ we can write $x = y(\frac{x}{y}) = y^2(\frac{x}{y^2}) = \cdots$. Thus $R + xK[[x]]$ is not a CK-n domain.

Recall that an integral domain R is said to be a **fragmented** if, for each nonzero nonunit $r \in R$, there exists a nonzero nonunit $s \in R$ such that $r \in \cap_{n=0}^{\infty} Rs^n$. From [11] we have that any fragmented domain is an antimatter domain. Therefore, we have that if R is a quasi-CK-n domain, then R cannot be fragmented, since fragmented domains have no atoms (anitmatter).

We conclude by giving some areas that could be further investigated. We remarked above that if R is a quasi-CK-1 domain which is also a valuation domain, then R was integrally closed. We also remarked above that if R is a quasi-CK-n domain with $n > 1$, then R cannot be a valuation domain. So what can one say about the integral closure of a quasi-CK-n domain with $n \geq 2$? Let R be a quasi-CK-n domain and let K be the quotient field of R. In Proposition 2.4.8 we saw that $R + xK[[x]]$ is a quasi-CK-n domain; one could investigate the validity of the statement that $R+xK[[x]]$ is integrally closed if and only if R is integrally closed in K. Further one could investigate the validity of the statement that $R + xK[[x]]$ is never completely integrally closed unless $R = K$. For the case of a CK domain, as shown in [4], every overring of a CK domain is a CK domain. One could investigate whether the overrings of a quasi-CK-n domain are necessarily quasi-CK-n domains. One could also investigate whether or not there exist other equivalent characterizations of quasi-CK-n domains.

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