MODELING FINANCIAL SWAPS AND GEOPHYSICAL DATA USING THE BARNDORFF-NIELSEN AND SHEPHARD MODEL

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The supervisory committee certifies that this dissertation complies with North Dakota State University's regulations and meets the accepted standards for the degree of

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ABSTRACT

This dissertation uses Barndoff-Nielsen and Shephard (BN-S) models to model swap, a type of financial derivative, and analyze geophysical data for estimation of major earthquakes. From empirical observation of the stock market activity and earthquake occurrence, we observe that the distributions have high kurtosis and right skewness. Consequently, such data cannot be captured by stochastic models driven by a Wiener process. Non-Gaussian processes of Ornstein-Uhlenbeck type are one of the most significant candidates for being the building blocks of models of financial economics. These models offer the possibility of capturing important distributional deviations from Gaussianity and thus these are more practical models of dependence structures. Introduced by Barndorff-Nielsen and Shephard these processes are not only convenient to model volatility in financial market, but have an independent interest for modeling stationary time series of various kinds. For the financial data we use BN-S models to price the arbitrage-free value of volatility, variance, covariance, and correlation swap. Such swaps are used in financial markets for volatility hedging and speculation. We use the $S\&P500$ and $NASDAQ$ index for parameter estimation and numerical analysis. We show that with this model the error estimation in fitting the delivery price is much less than the existing models with comparable parameters.

For any given time interval, the earthquake magnitude data have three main properties: (1) magnitude is a non-negative stationary stochastic process; (2) for any finite interval of time there are only finite number of jumps; (3) the sample path of the magnitude of an earthquake consists of upward jumps (significant earthquake) and a gradual decrease (aftershocks). For such geophysical data we specifically use Gamma Ornstein Uhlenbeck processes driven by a Lévy process to estimate a major earthquake in a certain region in California. Rigorous regression analysis is provided, and based on that, first-passage times are computed for different sets of data. Both applications demonstrate the significance of BN-S models to phenomena that follow non-Gaussian distributions.

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DEDICATION

I would like to dedicate this doctoral dissertation to my loving parents, brothers and sisters, and of course, my wife and son. Thanks so much for their never-ending love, support, and spirit, during this tedious period.

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1. INTRODUCTION

A financial *security* is a financial contract whose value at expiration is determined by the price process of the underlying assets. A *derivative* is a financial security whose price depends upon or is derived from one or more underlying assets. There are two types of derivitatives: options and forwards. An option is a contract to buy or sell a financial product at a designated date, over a fixed period of time. An option gives the buyer (the owner or holder) the right, but not an obligation, to buy or sell an underlying asset or financial instrument at a price specified in the contract, which is called strike price, at a fixed period. Option pricing in stochastic volatility models of the Ornstein-Uhlenbeck (OU) type is discussed in detail by Barndorff-Nielsen and Shephard (BN-S) in 1997. This research has changed how financial data is modeled, including option pricing. Moreover, BN-S models have an interesting feature that captures stock volatility and stationary distributions. A forward contract is similar to an option that obliges the buyer to exercise the contract at the specified period of time. A swap is a financial derivative in which two counter parties agree to exchange future cash flow, where at the begining of the contract the size of the cash flow is determined. Swap is first introduced in the 1980s, and is an agreement between two financial sectors to exchange cash flow at one or several future dates as defined in [65]. Swap is used to hedge and speculate on stock price. For example, volatility swap gives traders an exposure to profit from the risk of increase or decrease in the volatility of the stock, or hedge against these volatility risks. The four types of swaps, in order of their quantitative importance are listed below.

- ***An interest rate swap*** allows two parties to exchange a fixed and floating cash flow on investments or loans held by either parties. The most popular type of interest rate swap is the plain vanilla swap that allows two companies to exchange cash flow based on interest rate of the same currency: i.e., fixed versus floating interest rate on a fixed date.

- ***A currency swap*** is a contract based on two currencies. An example can be an American company that wants to expand in Europe, and a European company that wants to expand in America. Moreover, assume the interest rate to buy a currency for domestic and international is different. At this point, these two companies agree to exchange the interst rate to buy currency.

- ***A commodity swap*** is commonly used among companies or people that use finished products or raw materials. The commodity swap is usually used to hedge against the price of a commodity. The most common commodity swap is observed in the oil market.

- ***A credit default swap*** deals with insurance for a third-party borrower.

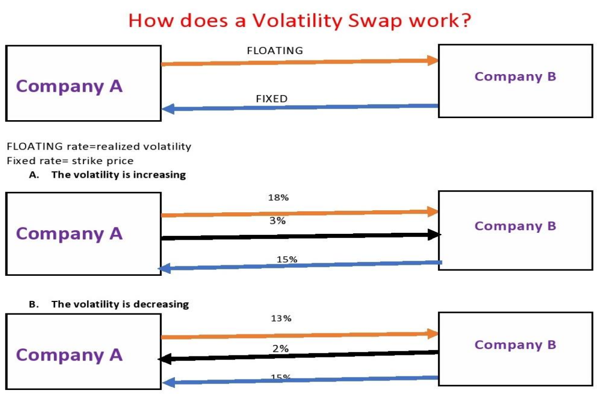

Figure 1.1. How does swap work.

In Figure 1.1, we see how swap works between two companies (company A versus company B), to exchange cash flow based on fixed and floating interest rate. It literally means that, at the end of the contract, if the volatility of the market surges, then company A is obliged to pay company B the difference between fixed and floating interest rate. Likewise, if the volatility of the market falls, then company A is going to get back the difference between fixed and floating interest rate.

There are many other types of swaps including volatility, variance, covariance, and correlation. These are forward contracts whose value is determined at the begining of the contract.

Variance, volatility, covariance, and correlation swap have been an active research area within quantitative finance since the publication of Black-Scholes (BS) equation in [10]. Researchers devote a lot of time in expanding the Black-Scholes equation for pricing call and put options of a financial market. One of the drawbacks of Black-Scholes formula is the assumption of normality of the stock return which commonly has a right tailed distribution. This was remedied in the late

1980s and early 1990s using a class of infinitely divisible distributions known as Levy processes. Levy processes have a higher kurtosis and skewness than that of the Normal distribution. Since then they have been refined to take into account different variations of the Black-Scholes model and different models of the market. The Black-Scholes model also assumes that the volatility is constant, which is unrealistic given the empirical observations of the log-return. These problems have been addressed in several models. Volatility is assumed to be a deterministic function of time ($\sigma = \sigma(t)$). Volatiltiy is assumed to be a function of time and current level of the stock, which mathematically means, $\sigma = \sigma(t, S_t)$ which is known as a local volatility. The volatility of the log-retun of the market can be also made more realistic by incorporating stock movement from a designated period of time, which means $\sigma = \sigma(t, S(t-\tau))_{\tau \in [-\theta, 0]}, \quad \theta > 0$. These and others are natural extensions of a model in which volatility is a function of stochastic process.

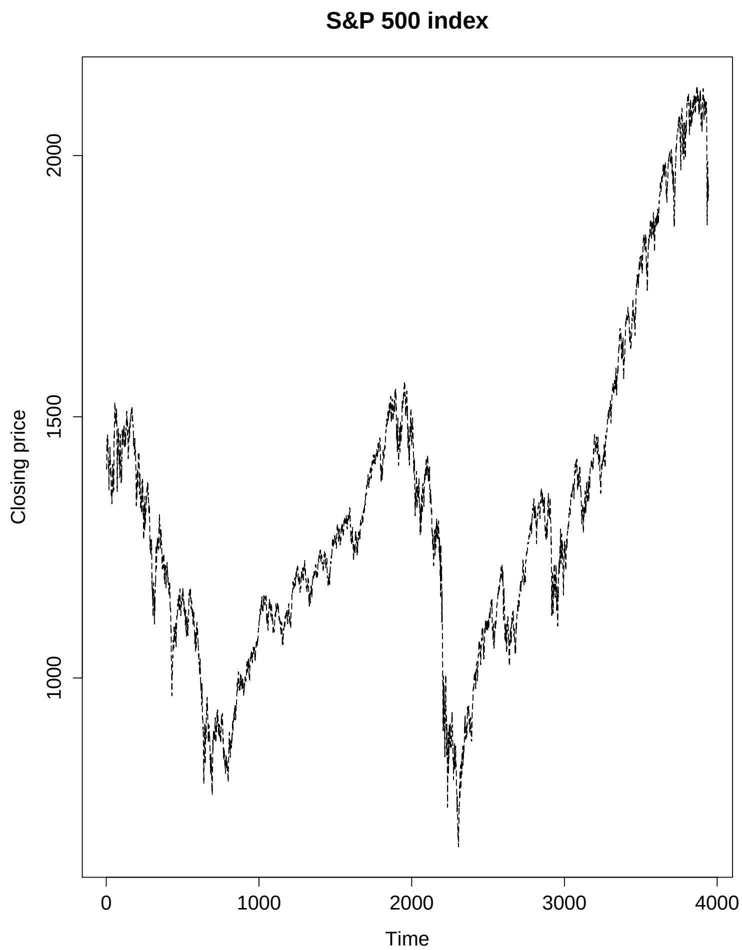

Figure 1.2. Closing Price of the S&P500 Stock Price from 2010 to 2015.

From Figure 1.2, we can observe a pattern of the stock price movement. First, we can clearly see that the stock price has a mean reverting property. Second, given any finite time interval, the distribution of the stock movement is stationary and makes big jumps followed by an exponential decay. Stationary distributions driven by a subordinator have the ability of capturing the big jumps. In this dissertation we focus on modeling realized variance, volatility, covariance and correlation of the above stock dynamics. Volatility is an important element in determining the stock movement. The higher the volatility is the riskier the market. Volatility swap gives investors an exposure of profiting from the increase or decrease in the stocks movement or to hedge against these risks. Here we define two types of volatility swap and conclude this chapter by reviewing literature.

1.1. Types of Volatility

1.1.1. Historical Volatility

Historical volatility is the volatility that is calculated based on the underlying stock, as it is determined in terms of the annualized standard deviation of the stock price. It is also called statistical volatility, as it calculates how much the volatility is moving during a certain time interval. It is the standard deviation calculated using historical price data (daily, weekly, monthly, quarterly, and yearly). The log returns over a one-year period is called the annualized volatility. Historical volatiltiy is important in comparing the volatiltiy of one stock with the volatility of other stocks. We can mathematically represent historical volatility as

$$\sigma = \sqrt{\frac{1}{n-1}\sum_{i=1}^{n}(R_i - \bar{R})^2}, \tag{1.1}$$

where the log-return $R_i = \log\left(\frac{(S_t)_i}{(S_t)_{i-1}}\right)$, and $(S_t)_i$ is the closing stock price at time t_i for $i = 1, 2, \ldots, n$, and $\bar{R}$ is the mean of the log-return. Log-return of the stock is assumed to be standard normal distribution and therefore $\bar{R} = 0$.

1.1.2. Implied Volatility

Despite the fact that future volatility is not directly observable in the market, it is possible to extract the market expectation of future volatility from the options traded on public exchange (or over the counter). Such estimation is called *Implied volatility.* The Implied and historical volatility have two major differences. Historical volatility is directly measured by the recent movement of the price of the stock over a given time (day, weeks, or yearly). Implied volatility, on the other hand, is set by the market price of the derivative contract. BS equation gives the implied volatility of the option.

The BS frame-work is first introduced in [10]. Since then, it has become an important model in option pricing. The BS model assumes that the stock price S_t follows a Geometric Brownian motion which is given by

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{1.2}$$

where the diffusion μ is the annualized expected return on the stock, σ measures the annualized stock volatility and W_t is the standard Wiener process. In this case, we call S_t is driven by a Wiener process.

1.2. Recent Developments in Swaps

Recent literature on valuing volatility and variance swap is growing fast. We outline a brief overview of recent developments in this area. In [8] the authors investigate swaps written on powers of realized volatility in the stochastic BN-S models. In that paper, a formula for the realized variance is derived and the swap price dynamics is represented in terms of Laplace transforms. In [68] the author gives analytic approach for pricing discretely sampled generalized variance swaps under the stochastic volatility models with simultaneous jumps in the asset price and variance processes. An analytical approximation for the valuation of volatility swaps and analyze other options with the provided analytic estimation is given in [16]. In [29] the authors have discussed the valuation and hedging of volatility swaps within the frame of a GARCH(1,1) stochastic volatility model. A general partial differential equation approach is utilized to determine the first two moments of the realized variance in a continuous or discrete context. This information is used to approximate the expected realized volatility via a convexity adjustment.

A new probabilistic approach using the Heston model to study variance, volatility, covariance and correlation swaps for financial markets is given in the work by [60]. As an application, the authors provide a numerical example using S&P60 Canada Index to price swap on the volatility. In [61, 59] variance swaps for financial markets with underlying asset and stochastic volatility with delay are considered. They provide some analytical closed form expressions for expectation and variance of the realized continuously sampled variance. The variance swap is evaluated with delay both in a risk-neutral world and in the physical world. An upper bound for delay as a measure of risk is obtained and applications two numerical examples using S&P60 Canada Index (1998-2002) and S&P500 Index (1990-1993) are provided to price variance swaps with delay. As observed in [61], variance swap for stochastic volatility with delay is similar (but with more parameters) to variance swaps for stochastic volatility in Heston model.

In [62] the authors present a variance drift-adjusted version of the Heston model which leads to a significant improvement of the market volatility surface fitting compared to Heston model. This model has two additional parameters compared to the Heston model and, thus, it can be implemented very easily. The main idea of the proposed model is to take into account some past history of the variance process. They used the change of time method for continuous local martingales to derive a closed formula for the approximation of the volatility swap price. In [15] the authors investigated the effect of discrete sampling and asset price jumps on fair variance and volatility swap strikes. Fair discrete volatility and variance delivery prices (strikes) are derived in different models such as the Black-Scholes model, the Heston stochastic volatility model, the Merton jump-diffusion model and the Bates and Scott stochastic volatility and jump model. The fair discrete and continuous variance and volatility strikes for these models are determined analytically using variance reduction and numerical integration techniques. It was found that the effect of discrete sampling is typically small while the effect of jumps can be significant.

In [32] a model-independent lower bound on variance swap is derived. In [23] a theory of robust pricing and hedging of a weighted variance swap is developed given market prices for a finite number of co-maturing put options. Assuming no arbitrage for the put option prices, no arbitrage bounds on the weighted variance swap is deduced along with super and sub replicating strategies that enforce them. It is shown in that paper that the market quotes for variance swaps are very close to the model-free lower bounds. The main tool that is used in [23] is Föllmer's path wise stochastic calculus. In [17] the authors develop strategies for pricing and hedging options on realized variance and volatility. These combined strategies have nice features such as readily available inputs, comprehensive and readily computable outputs, accuracy and robustness, and easy modification to price and hedge options on implied volatility. In [25] the author used Barndorff-Nielsen and Shephard model to value variance, volatiliy, covariance, and correlation swap. In [19] it is proved that a multiple of a log contract prices a variance swap, under arbitrary exponential Lévy dynamics, stochastically time-changed by an arbitrary continuous clock having arbitrary correlation with the driving Lévy process, subject to integrability conditions. The valuations in some cases admit enforcement by hedging strategies which perfectly replicate variance swaps by holding log contracts and trading the underlying assets. In a further note [18], the work in [19] is extended with G-variation, which generalizes power variation. In [18] quadratic variation is generalized to G-variation, and the share-weighted payoff problems are solved. Also, the tools developed in [18] are used to analyze and minimize the risk in a family of hedging strategies for G-variation.

Covariance swaps are a generalization of the variance swap. Covariance and correlation swaps for financial markets with Markov-modulated volatility are analyzed in [50]. Stochastic volatility driven by two-state continuous Markov chain are considered and numerical examples are presented for two volatility indexes, VIX and VXN, for that case. In [33] pricing of derivatives written on the discretely sampled realized variance of an underlying security is considered. Two new methods are proposed to evaluate the prices of options on the discretely sampled realized variance. The first method is based on correcting prices of options on quadratic variation by asymptotic results and the other method is exact that uses Fourier-Laplace techniques. In [56, 57] analytical methodology is developed for pricing and hedging options on the realized variance under the Heston model augmented with jumps in asset returns and variance. Moreover they analyzed the effect of the discrete sampling is analyzed on the valuation of options on the realized variance in the Heston model. A method of mixing is proposed and accurately approximates the distribution of discrete variance in the Heston model. Semi-analytical Fourier transform methods are applied for pricing shorter-term options on the realized variance. In [27] a forward characteristic function approach is implemented to price variance and volatility swaps and options on swaps that are defined via contingent claims whose payoffs depend on the terminal level of a discretely monitored version of the quadratic variation of some observable reference process.

One of the main challenges in the research on volatility, variance or covariance swaps is to obtain a closed form pricing expression that can be accurately computed. The model should not incorporate a large number of parameters to slightly improve the existing results. On the contrary, if the major improvement over the existing results is possible with almost the same number of parameters as in the existing models, then it would be a significant improvement. This will also demonstrate the superiority of the new model. To this end, in this dissertation we consider Non-

Gaussian processes driven by Lévy process. These processes have significant potential as building blocks for stochastic models of time series in finance. Such models are mathematically tractable and it is possible to build compelling stochastic volatility models using Ornstein-Uhlenbeck processes to represent volatility. It is also well-known that log-returns from these types of models share many common properties with familiar discrete time GARCH models. In our work we use the Barndorff-Nielsen and Shephard model for stock and volatility dynamics and implement that to obtain the arbitrage free pricing of variance, volatility, covariance, and correlation swaps. For the model we obtain closed form expressions for the arbitrage-free pricing of variance, volatility, and covariance swap and an approximation solution for correlation swap. Moreover, we show that such expressions depend only on various cumulants of the driving Lévy process. This model has the same (or in some cases one more) number of parameters as the Heston model and, thus, this model can be implemented very easily. Moreover, it is shown that the error estimation for this model in fitting the fair delivery price is much less than existing models. Thus, the models and pricing formulas proposed in this dissertation are simple to compute and more accurate than similar models and hence can be efficiently used in practical applications.

Finally, we have extended the Gamma-Ornstein-Uhlenbeck process, to model and analyze geophysical data. Such non-Gaussian Ornstein–Uhlenbeck processes offer the possibility of capturing important distributional deviations from Gaussianity and make the model flexible of dependence structures. It is shown that the Gamma-Ornstein–Uhlenbeck process is a possible candidate for earthquake data modeling. Rigorous regression analysis is provided and based on that the first-passage times are computed for different sets of data. It is shown that this model may be used to estimate parameters related to some major events namely major earthquakes. A detailed introduction and literature review toward modeling major earthquakes is given in the last chapter.

The structure of this dissertation is as follows. In Chapter 2 we give brief introduction to the variance, volatility, covariance and correlation swap using the Hull-White Model when the stock price follows Geometric Brownian motion. In addition, we discuss overview of pricing procedure swap when the market and volatility dynamics are driven by Gaussian processes. In Chapter 3 the Barndorff-Nielsen and Shephard (BN-S) model for stock and volatility dynamics is introduced and derived the log cumulant function and characteristic function when the volatility is driven by BDLP or subordinator. In Chapter 4, we derived the main result of variance and volatility swap if the stock dynamics follows BN-S model and also used the S&P500 index to estimate model parameters and compared our model with others such as Hull-White model and Heston model. In Chapter 5, we have extended the BN-S model to find the covariance and correlation swap for two assets. Finally in Chapter 6, we derived a model to estimate a major earthquake using Ornstein-Uhlenbeck (OU) process for a certain regions in California.

2. THE HULL -WHITE MODEL FOR DERIVING SWAP

2.1. Introduction

In mathematical finance, the Hull-White model is used to model future interest rate. The Hull–White model is introduced by John C. Hull and Alan White in 1990, and it is still one of the popular models in capturing interest rates. The Hull-White model extends the Vasicek and Cox-Ingersoll-Ross (CIR) models. It has a short term mean revertion (mean reversion is a theory suggesting that prices and returns eventually move back towards the long term mean or average). In this chapter we consider that a stock dynamic follows a Geometric Brownian motion and it is given by equation (2.1) as described below. The volatility square of the log-return of the stock dynamics follows the Hull-White model given in equation (2.2). Since the Hull-White model treats the log-return of the stock price as a standard normal distiribution, its short-term average is zero. This model is demonstrated to value the price of variance, volatility, covariance, and correlation swap.

$$dS_t = S_t(rdt + \sigma_t dW_t^1). \tag{2.1}$$

Where r is the risk-free interest rate, W_t^1 is the standard Wiener process, and σ_t is the volatility of the stock at a given time t, this is given by

$$d\sigma_t^2 = \kappa\sigma_t^2 dt + \zeta\sigma_t^2 dW_t^2, \qquad \kappa < 0. \tag{2.2}$$

Where κ and ζ are real numbers, and W_t^1 and W_t^2 are independent Wiener processes. The instantaneous variance of the log-return is found by taking the variance of (2.1) which is given by

$$Var(rdt + \sigma_t dW_t^1) = \sigma_t^2 dt. \tag{2.3}$$

2.2. Variance and Volatility Swap

2.2.1. Variance Swap

Definition 2.2.1. *Variance swap is a forward contract in which two counter parties exchange cash-flow on future realized price which is set at the initiation of the contract. The payoff of a variance swap at maturity is given by*

$$N(\sigma_R^2(S) - K_{var}), \tag{2.4}$$

where N is the notional amount of the swap in dollar per annualized volatility point. The holder of the variance swap receives N dollars for every time where the stock realized variance exceeds the variance delivery price. $\sigma_R^2(S)$ is the realized variance is the average of the instantaneous variance which is given by

$$\sigma_R^2(S) := \frac{1}{T}\int_0^T \sigma_s^2 ds, \tag{2.5}$$

and K_{var} is the delivery or exercise price for the variance swap. Valuing a variance forward price is the same as that of other derivatives. The value of a variance swap price P at expiry is given by the expected present value of a future payoff in the risk-neutral world.

$$P_{\text{var}} = E(e^{-rT}(\sigma_R^2(S) - K_{var})). \tag{2.6}$$

In the above expression r is the risk free interest rate and T is the exercise or expiry time. Here we assume the notional amount to be one for convenience purposes.

To value the price of the variance swap, we need to calculate the expected value of the realized variance. Moreover, we need to solve equation (2.2) completely. Notice that the variance of the market varies with the variance of the stock price and if we divide equation (2.2) by σ_t^2 and integrate from 0 to t, then

$$\int_0^t \frac{d\sigma_s^2}{\sigma_s^2} = \int_0^t \kappa ds + \int_0^t \zeta dW_t^2$$
$$\sigma_t^2 = \sigma_0^2 \exp\left((\kappa - \frac{1}{2}\zeta^2)t + \zeta W_t^2\right). \tag{2.7}$$

The exponential part of the above equation indicates a shifted Gaussian distribution. Using

Itô's lemma, $E(e^{X_t}) = \exp(E(X_t) + \frac{1}{2}Var(X_t))$ where X_t is a Brownian motion, we get the expected value of equation (2.7)

$$E(\sigma_t^2) = \sigma_0^2 e^{\kappa t}. \tag{2.8}$$

The expected value of the realized variance is

$$\begin{aligned} E(\sigma_R^2(S)) &= \frac{1}{T}\int_0^T E(\sigma_s^2)ds \\ &= \frac{1}{T}\sigma_0^2 \int_0^T e^{\kappa t}dt \\ &= \frac{\sigma_0^2}{\kappa T}\left(e^{\kappa T} - 1\right). \end{aligned} \tag{2.9}$$

Theorem 2.2.2. *The arbitrage free price of the variance swap for the Hull-White Model is given by*

$$P_{var} = e^{-rT}\left(\frac{\sigma_0^2}{\kappa T}\left(e^{\kappa T} - 1\right) - K_{var}\right). \tag{2.10}$$

2.2.2. Volatility Swap

Definition 2.2.3. *Volatility swap is a forward contract on the future realized volatility of a given underlying asset.*

Volatility is a statistical term which is a standard deviation of the stock return. From basic statistics definition we know that volatility is square root of variance, hence the realized volatility is given by

$$\sigma_R(S) := \sqrt{\frac{1}{T}\int_0^T \sigma_s^2 ds}. \tag{2.11}$$

Volatility is an important element in determining whether one stock is risky or not. A high volatility implies the security is risky and we may not want to invest in such security. On the other hand, if the volatility is low, this means the stock is less risky, so it is good to invest in such security. Volatility swap allows investors to trade the volatility of an asset directly, as much as they would trade in price index. The payoff of a volatility swap at the trading date or maturity time T is given by

$$N(\sigma_{\mathrm{R}} - \mathrm{K}_{\mathrm{vol}}), \tag{2.12}$$

where N is the notional amount in dollar per annualized volatility point, σ_R is the realized volatility of the stock which is given by equation (2.11), and $\mathrm{K_{vol}}$ is the annualized volatility delivery price. The holder of the volatility swap at expiration receives N dollars for every point by which the stock's realized volatility σ_R has exceeded the exercise price $\mathrm{K_{vol}}$. The price of the volatility swap is the expected value of the present payoff in the risk neutral-world and is given by

$$P_{\mathrm{vol}} = E\left[e^{-rt}(\sigma_{\mathrm{R}} - \mathrm{K_{Vol}})\right], \tag{2.13}$$

where r is the risk free interest rate and $E(.)$ is the expectation with respect to some risk-neutral measure. To find the price of the volatility swap in a risk neutral world, we need to find the expected value of the realized volatility σ_R as it is the only random in equation (2.13). We observe

$$E(\sigma_R) = E(\sqrt{\sigma_R^2}). \tag{2.14}$$

It is not usually easy to find the expected value of a square root function. However, using second degree Taylor series approximation around its mean is given by

$$E(\sqrt{\sigma_R^2}) \approx \sqrt{E(\sigma_R^2)} - \frac{\mathrm{Var}(\sigma_R^2)}{8(E(\sigma_R^2)^{3/2}}. \tag{2.15}$$

Basic statistical definition of variance gives

$$\mathrm{Var}(\sigma_R^2) = E((\sigma_R^2)^2) - (E(\sigma_R^2))^2. \tag{2.16}$$

To evaluate equation (2.15) for Hull-White model, it remains to find the expected value of $(\sigma_R^2)^2$, which is given by the lemma below.

Lemma 2.2.4. *For any given time s , t and given variance of the stock by equation* (2.7)*,*

$$E(\sigma_t^2\sigma_s^2) = (\sigma_0^2)^2 exp(\kappa(s+t) + \zeta^2(t \wedge s)), \tag{2.17}$$

where $t \wedge s = \min(t, s)$.

Proof. Without loss of generality we can assume that $s < t$. Then using equation (2.7) we have

$$\begin{aligned} E(\sigma_t^2\sigma_s^2) &= (\sigma_0^2)^2 exp\left[(\kappa - \frac{1}{2}\zeta^2)(s+t)\right] E\left[e^{\zeta(W_t^2+W_s^2)}\right] \\ &= (\sigma_0^2)^2 exp\left[(\kappa - \frac{1}{2}\zeta^2)(s+t)\right] E\left[e^{\zeta(W_t^2-W_s^2)}\right] E\left[e^{2\zeta W_s^2}\right] \\ &= (\sigma_0^2)^2 exp(\kappa(s+t) + \zeta^2 s). \end{aligned} \tag{2.18}$$

Hence the lemma is proved . □

Lemma 2.2.5. *The expected value of the square of the realized variance is given by*

$$E\left[(\sigma_R^2)^2\right] = \frac{2(\sigma_0^2)^2}{T^2(\kappa+\zeta^2)}\left[\frac{1-e^{\kappa T}}{\kappa} - \frac{1-e^{(2\kappa+\zeta^2)T}}{2\kappa+\zeta^2}\right]. \tag{2.19}$$

Proof. We know that

$$\sigma_R^2 = \frac{1}{T}\int_0^T \sigma_t^2 dt, \tag{2.20}$$

which gives

$$(\sigma_R^2)^2 = \frac{1}{T^2}\int_0^T\int_0^T \sigma_t^2\sigma_s^2 dsdt. \tag{2.21}$$

Using Lemma (2.2.4) and the fact that the double integral is invariant while the variables s and t are interchangeable, we obtain

$$\begin{aligned} E\left[(\sigma_R^2)^2\right] &= E\left[\frac{1}{T^2}\int_0^T\int_0^T \sigma_t^2\sigma_s^2 dsdt\right] \\ &= \frac{1}{T^2}\int_0^T\int_0^T E(\sigma_t^2\sigma_s^2)dsdt \\ &= \frac{2(\sigma_0^2)^2}{T^2}\int_0^T\int_{s=0}^t exp(\kappa(s+t)+\zeta^2 s)dsdt \\ &= \frac{2(\sigma_0^2)^2}{T^2(\kappa+\zeta^2)}\left[\frac{1-e^{\kappa T}}{\kappa} - \frac{1-e^{(2\kappa+\zeta^2)T}}{2\kappa+\zeta^2}\right], \end{aligned} \tag{2.22}$$

which gives as desired. □

Using equation (2.16) it can be easily shown that the realized variance of the underlying asset is given by

$$\mathrm{Var}(\sigma_R^2) = \frac{2(\sigma_0^2)^2}{T^2(\kappa+\zeta^2)}\left(\frac{1-e^{\kappa T}}{\kappa} - \frac{1-e^{(2\kappa-\zeta^2)T}}{2\kappa+\zeta^2}\right) - \left[\frac{\sigma_0^2}{\kappa T}(e^{\kappa T}-1)\right]^2. \tag{2.23}$$

Theorem 2.2.6. *The arbitrage free price of volatility swap for the stock dynamics* (2.1) *and volatility dynamics* (2.2) *is given by*

$$P_{vol} \approx e^{-rT}\left(\left(\sqrt{\frac{\sigma_0^2}{\kappa T}(e^{\kappa T}-1)} - \frac{Var(\sigma_R^2)}{8(\frac{\sigma_0^2}{\kappa T}(e^{\kappa T}-1))^{3/2}}\right) - K_{Vol}\right), \tag{2.24}$$

where $Var(\sigma_R^2)$ *can be obtained from* (2.23).

2.3. Covariance and Correlation Swap

Covariance and correlation swaps are among the recent financial derivatives used to hedge and speculate using two different financial underlying assets. For example, options dependent on the movement of exchange rate, such as those who pays different currency other than the underlying currency. Such exposure to currency swaps lead to a correlation between the assets and exchange rate. This risk can be eliminated by using a covariance swap. Covariance (Correlation) swap is a forward contract on the stocks realized covariance (correlation) respectively. Covariance (correlation) swap pays the difference between an implied covariance (correlation) respectively and the realized pairwise covariance (correlation) stock prices.

2.3.1. Covariance Swap

Definition 2.3.1. *A covariance swap is a forward contract on the underlying stocks* S^1 *and* S^2 *in which the payoff at the maturity is given by the formula*

$$N(cov_R(S^1, S^2) - K_{cov}). \tag{2.25}$$

The value of the covariance forward swap price P on a future realized covariance with a strike price $\mathrm{K_{cov}}$ is the expected value of the future payoff in the risk-neutral world, is given by

$$P = E\{e^{-rT}(\mathrm{cov_R}(S^1, S^2) - \mathrm{K_{cov}})\}, \tag{2.26}$$

where r is the risk free interest rate, $\mathrm{K_{cov}}$ is the strike or exercise price of covariance swap and

$\text{cov}_{\text{R}}(S^1, S^2)$ is the realized covariance of the two stock prices. From the above equation all other terms are constant except $\text{cov}_R(S^1, S^2)$ which can be calculated using the definition of quadratic variation. If the stocks dynamics of the two assets follow an exponential Brownian motion

$$dS_t^i = S_t^i (r^i dt + \sigma_t^i d(W_t^1)^i) \quad i = 1, 2, \tag{2.27}$$

where σ_t^i is the volatility of the log-return which follows the Hull-White model as

$$d(\sigma_t^2)^1 = \kappa^1 (\sigma_t^2)^1 dt + (\zeta)^1 (\sigma_t^2)^1 dW_t, \quad \kappa^1 < 0, \tag{2.28}$$

and

$$d(\sigma_t^2)^2 = \kappa^2 (\sigma_t^2)^2 dt + (\zeta)^2 (\sigma_t^2)^2 d\hat{W}_t, \quad \kappa^2 < 0. \tag{2.29}$$

and the two driving Wiener process are related by

$$d\hat{W}_t = \rho^2 dW_t + \sqrt{1-\rho^2} d\tilde{W}_t, \tag{2.30}$$

where r^i is the fixed interest rate of the i^{th} stock, and ζ is constant real number and $\hat{W}_t$, W_t and $\tilde{W}_t$ are related by the above equation (2.30), and W_t and $\tilde{W}_t$ are independent standard Wiener processes. Then one can calculate the realized covariance as

$$\begin{aligned} \text{cov}_{\text{R}}(S_T^1, S_T^2) &= \frac{1}{T} \left[\ln S_T^1, \ln S_T^2 \right] \\ &= \frac{1}{T} \int_0^T \sigma_t^1 \sigma_t^2 dt. \end{aligned} \tag{2.31}$$

The square bracket $[,]$ is the quadratic covariation of the two stocks and the solution of the above model is given by equation (2.28) is

$$(\sigma_t^2)^1 = (\sigma_0^2)^1 \exp\left((\kappa^1 - \frac{1}{2}(\zeta^2)^1)t + \zeta^1 W_t \right). \tag{2.32}$$

The solution of equation (2.29) and using (2.30) which is given by

$$(\sigma_t^2)^2 = (\sigma_0^2)^2 \exp\left((\kappa^2 - \frac{1}{2}(\zeta^2)^2)t + (\zeta^2)(\rho^2 W_t + \sqrt{1-\rho^2}\tilde{W}_t) \right). \tag{2.33}$$

Now if we multiply the two above volatility square models we get

$$(\sigma_t^2)^1(\sigma_t^2)^2 = (\sigma_0^2)^1(\sigma_0^2)^2 e^{\phi(t)}, \tag{2.34}$$

where

$$\phi(t) = (\kappa^1 + \kappa^2)t - \frac{1}{2}((\zeta^2)^1 + (\zeta^2)^2)t + (\zeta^1 + \zeta^2\rho^2)W_t + \zeta^2\sqrt{1-\rho^2}\tilde{W}_t, \tag{2.35}$$

and $\phi(t)$ is a shifted Brownian motion and remember that W_t and $\tilde{W}_t$ are independent, which makes it easier to calculate the mean and variance of $\phi(t)$ which is given by

$$E(\phi(t)) = (\kappa^1 + \kappa^2)t - \frac{1}{2}(\zeta^1 + \zeta^2)t, \tag{2.36}$$

and

$$Var(\phi(t)) = \left((\zeta^1 + \zeta^2\rho^2)^2 + (\zeta^2(1-\rho^2))^2\right)t. \tag{2.37}$$

Applying Itô's lemma and let

$$g(\kappa, \zeta) = (\kappa^1 + \kappa^2) - \frac{1}{2}(\zeta^1 + \zeta^2) \tag{2.38}$$

and

$$h(\zeta, \rho) = \left((\zeta^1 + \zeta^2\rho^2)^2 + (\zeta^2(1-\rho^2))^2\right), \tag{2.39}$$

the expected value of equation (2.35) is

$$\begin{aligned} E((\sigma_t^2)^1(\sigma_t^2)^2) &= (\sigma_0^2)^1(\sigma_0^2)^2 \exp\left(E(\phi(t)) + \frac{1}{2}Var(\phi(t))\right) \\ &= (\sigma_0^2)^1(\sigma_0^2)^2 \exp\left(g(\kappa,\zeta)t + \frac{1}{2}h(\zeta,\rho)t\right). \end{aligned} \tag{2.40}$$

Hence the expected covariance is given by

$$\begin{aligned} E(\text{cov}_R(S_T^1, S_T^2)) &= \frac{(\sigma_0^2)^1(\sigma_0^2)^2}{T}\int_0^T \exp\left(g(\kappa,\zeta)t + \frac{1}{2}h(\zeta,\rho)t\right)dt \\ &= \frac{(\sigma_0^2)^1(\sigma_0^2)^2}{T(g(\kappa,\zeta) + 0.5h(\zeta,\rho))}\left(e^{(g(\kappa,\zeta)+.5h(\zeta,\rho))T} - 1\right). \end{aligned} \tag{2.41}$$

Theorem 2.3.2. *The arbitrage free covariance swap is given by*

$$P_{cov} = e^{-rT}\left(\frac{(\sigma_0^2)^1(\sigma_0^2)^2}{T(g(\kappa,\zeta)+0.5h(\zeta,\rho))}\left(e^{(g(\kappa,\zeta)+.5h(\zeta,\rho))T}-1\right)-K_{cov}\right). \tag{2.42}$$

2.3.2. Correlation Swap

Definition 2.3.3. *Correlation swap is a correlation forward contract of the underlying rates S^1 and S^2 which payoff at the maturity equal to*

$$N(corr_R(S^1,S^2)-K_{Corr}). \tag{2.43}$$

Where $\mathrm{K_{corr}}$ is the strike or exercise price, N is the notional amount, $corr_R(S^1,S^2)$ is a correlation between two assets S^1 and S^2.

The correlations of the two asset from the basic statistics formula is given by

$$\mathrm{corr_R}(S^1,S^2) = \frac{\mathrm{cov}_R(S^1,S^2)}{\sqrt{(\mathrm{Var}_R(S^1))}\sqrt{(\mathrm{Var_R}(S^2))}}. \tag{2.44}$$

To value the correlation swap in the risk neutral world we need to find

$$P = e^{-rT}(E(corr_R(S^1,S^2))-K_{Corr}). \tag{2.45}$$

To find the expected value of a realized correlation is a little bit challenging, here we are going to use an approximation of squre roots which is given by ([59],p. 200) and we give the result without proof,

$$E(corr_R(S^1,S^2)) \approx \frac{E(cov_R(S^1,S^2))}{\sqrt{E((\sigma_R^2)^1)}\sqrt{(E(\sigma_R^2)^2)}}. \tag{2.46}$$

Now since the denominator of the above equation is deterministic it can be factored to find the expected value of the realized correlation. We will close this chapter by giving the final theorem of the correlation swap

Theorem 2.3.4. *The arbitrage free correlation swap price in the risk-neutral world is given by*

$$P_{corr} \approx e^{-rT}\left[\frac{(\sigma_0^2)^1(\sigma_0^2)^2}{T(g(\kappa,\zeta)+0.5h(\zeta,\rho))\sqrt{(\sigma_0^2)^1(e^{\kappa^1 T}-1)/\kappa T}\sqrt{(\sigma_0^2)^2(e^{\kappa^2 T}-1)/\kappa^2 T}}\left(\left(e^{(g(\kappa,\zeta)+.5h(\zeta,\rho))T}-1\right)-K_{corr}\right)\right]. \tag{2.47}$$

3. BARNDORFF-NIELSEN AND SHEPHARD MODEL FOR STOCK AND VOLATILITY DYNAMICS

Consider a financial market without a transaction costs where a risk free asset with constant return rate r and a stock are traded up to a fixed exercise date T. Barndorff-Nielsen and Shephard (see [6, 5]) assumed that the price process of the stock $S = (S_t)_{t\geq 0}$ is defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P)$ and is given by:

$$S_t = S_0 \exp(X_t), \tag{3.1}$$

$$dX_t = (\mu + \beta\sigma_t^2)\, dt + \sigma_t\, dW_t + \rho\, dZ_{\lambda t}, \tag{3.2}$$

$$d\sigma_t^2 = -\lambda\sigma_t^2\, dt + dZ_{\lambda t}, \quad \sigma_0^2 > 0, \tag{3.3}$$

where the constants $\mu, \beta, \rho, \lambda \in \mathbb{R}$ with $\lambda > 0$ and $\rho \leq 0$ is the leverage effect. $W = (W_t)$ is a Brownian motion and the process $Z = (Z_{\lambda t})$ is a subordinator (subordinator is a real-valued Lévy process with no Gaussian component and non decreasing sample paths). Poisson process, Variance gamma and inverse Gaussian are some examples of a subordinator Lévy process. Barndorff-Nielsen and Shephard refer to Z as the *background driving Lévy process* (BDLP). Also W and Z are assumed to be independent and $(\mathcal{F}_t)$ is assumed to be the usual augmentation of the filtration generated by the pair (W, Z). This model is known in literature as Barndorff-Nielsen and Shephard model (BN-S model) and in this dissertation is referred as *classical* BN-S model. The log-return of the stock dynamic process of equation (3.2) is a linear process as it appeared to be a linear combination of a Brownian motion and the Lévy process. Also, the negative sign appearing in (3.3) makes the associated process mean-reverting. We want to mention that equation(3.3) is a non-Gaussian process as it is driven by Z (instead of W) .

Non-Gaussian processes of OU type have considerable potential as building blocks for stochastic models of observational series from a wide range of fields. They offer the possibility of capturing important distributional deviations from Gaussianity and for flexible modeling of dependence structures. It is been well studied that financial time series of different assets has many common features such as heavy tailed distribution and log-return, aggregational Gaussianity, quasi long range dependence. Such properties of the stock dynamics are successfully modeled by Ornstein-Uhlenbeck(OU) type stationary stochastic process driven by a subordinators.

This model is first introduced by Barndorff Nielsen and Shephard. Since then, it becomes one of an important model in generalizing the BS model. For further reading about the BN-S model we refer the reader to existing literature (see [3, 6, 5]). Since the BN-S model involves a new idea in option pricing, it can be used in Ecomonetric analysis of realised variance and estimating stochastic volatility model. This model has been used in different literature such as [42, 52, 53, 36, 9]. Moreover, BN-S model and its generalized version are also used in pricing exotic options (see [54, 55, 21]).

The formal definition of Lévy process is given below.

Definition 3.0.5. ***([63],p68)*** Lévy process: *A* cadlag *stochastic process* $(X_t)_{t\geq 0}$ *on* $(\Omega, \mathfrak{F}, \mathbf{P})$ *with values in* $\mathbb{R}$ *such that* $X_0 = 0$ *is called a Lévy process if it possesses the following properties:*

- Independent Increment: *for every increasing sequence of times* $t_0, t_1, \ldots, t_n$*, the random variables* $X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ *are independent.*
- Stationary increaments: *the law of* $X_{t+h} - X_t$ *does not depend on* t*.*
- Stochastic Continuity: $\forall \epsilon > 0$, $\lim_{h\to 0} \mathbf{P}(|X_{t+h} - X_t| \geq \epsilon) = 0$.

A large family of mean reverting processes can be constructed using a Lévy process as a driving noise. Positiveness and the choice of marginal distribution can be urged on those Lévy process. These Lévy-driven processes are known as non-Gaussian Ornstein-Uhlenbeck processes or simply Ornstein-Uhlenbeck processes. One of the most significant candidate for being the building block of financial economics is Non-Gaussian processes of OU. The deviation from Gaussianity can be captured by those models.

As it is well established by Barndorff-Nielsen and Shephard (see [6, 5]), these processes are not only appropriate to model *volatility* in financial market, but have also an independent interest for modeling stationary time series of different kinds. In this Chapter, we define properties of stationary disrtibution driven by Lévy processes and Show how to solve them. Log-laplace transform and characteristic function of a stationary process is also derived.

We assume that the BDLP Z satisfies the assumptions as described in [42]. The assumptions are as follows.

Assumption 1. *Z has no deterministic drift and its Lévy measures have densities $w(x)$. Thus by [51] (Theorem 19.3) the cumulant transform*

$$\kappa(\theta) = \log E[e^{\theta Z_1}], \tag{3.4}$$

where it exists, takes the form $\kappa(\theta) = \int_{\mathbb{R}_+} (e^{\theta x} - 1) w(x)\, dx$.

Assumption 2. *Letting $\hat{\theta} = \sup\{\theta \in \mathbb{R} : \kappa(\theta) < +\infty\}$, then $\hat{\theta} > 0$.*

Assumption 3. $\lim_{\theta \to \hat{\theta}} \kappa(\theta) = +\infty$.

Then it is shown in ([42] Theorem 3.2) that there exists an equivalent martingale measure (EMM) under which the equations (3.2) and (3.3) are transformed into the following equations.

$$dX_t = b_t\, dt + \sigma_t\, dW_t + \rho\, dZ_{\lambda t} \tag{3.5}$$

$$d\sigma_t^2 = -\lambda \sigma_t^2\, dt + dZ_{\lambda t}, \quad \sigma_0^2 > 0, \tag{3.6}$$

where

$$b_t = (r - \lambda \kappa(\rho) - \frac{1}{2}\sigma_t^2), \tag{3.7}$$

where W_t and $Z_{\lambda t}$ are is Brownian motion and Lévy process respectively with respect to the equivalent martingale measure, b_t is the appreciation rate, and $\kappa(\theta)$ is the cumulant transform for Z_1 under the new measure. The heuristic derivation of the above EMM is given in ([41], ch 6). For the rest of this section we assume that the risk-neutral dynamics of the stock price and volatility are given by (3.1), (3.5) and (3.6) and we derive the formula for the price of variance, volatility, covariance, and correlatin swap using this model.

Equation (3.6) is a linear stochastic differential equation and can be solved easly using an integrating factor as it is given by

$$\sigma_t^2 = e^{-\lambda t}\sigma_0^2 + \int_0^t e^{-\lambda(t-s)}\, dZ_{\lambda s}. \tag{3.8}$$

Since the process $\sigma^2 = (\sigma_t^2)$ is driven by a subordinator, it is strictly positive and bounded from below by the deterministic function $\sigma_0^2 \exp(-\lambda t)$. Moreover, the stationary distribution σ_t^2 jumps at the same time point of the subordinator but tailed off due to the negative sign. The instantaneous variance the log return of the stock dynamics is given by calculating the variance of equation (3.5) which is $(\sigma_t^2 + \rho^2\lambda \mathrm{Var}[Z_1])\, dt$. Therefore the continuous realized variance in the interval $[0, T]$ is the average of the instantaneous variance given by

$$\sigma_R^2 = \frac{1}{T}\int_0^T \sigma_t^2\, dt + \rho^2\lambda \mathrm{Var}[Z_1]. \tag{3.9}$$

Lemma 3.0.6. *The realized variance is given by*

$$\sigma_R^2 = \frac{1}{T}\left(\lambda^{-1}(1-e^{-\lambda T})\sigma_0^2 + \lambda^{-1}\int_0^T \left(1-e^{-\lambda(T-s)}\right) dZ_{\lambda s}\right) + \rho^2\lambda\, Var[Z_1]. \tag{3.10}$$

Proof. Substituting equation (3.8) and using the integration by parts letting $U = \int_0^{\lambda t} e^s dZ_s$ and $dV = e^{-\lambda t}dt$ which gives

$$\begin{aligned}
\sigma_R^2 &= \frac{1}{T}\int_0^T \left(e^{-\lambda t}\sigma_0^2 + e^{-\lambda t}\int_0^{\lambda t} e^s dZ_s dt\right) dt + \rho^2\lambda \mathrm{Var}[Z_1] \\
&= \frac{1}{T}\left(\lambda^{-1}(1-e^{-\lambda T})\sigma_0^2 + \int_0^T e^{-\lambda t}\left(\int_0^{\lambda t} e^s dZ_s\right) dt\right) + \rho^2\lambda \mathrm{Var}[Z_1] \\
&= \frac{1}{T}\left(\lambda^{-1}(1-e^{-\lambda T})\sigma_0^2 + \frac{-1}{\lambda}e^{-\lambda t}\int_0^{\lambda t} e^s dZ_s|_0^T + \frac{1}{T}\int_0^T dZ_{\lambda t}\right) + \rho^2\lambda \mathrm{Var}[Z_1] \\
&= \frac{1}{\lambda T}\left((1-e^{-\lambda T})\sigma_0^2 + \int_0^T \left(1-e^{-\lambda(T-s)}\right) dZ_{\lambda s}\right) + \rho^2\lambda \mathrm{Var}[Z_1].
\end{aligned} \tag{3.11}$$

□

The purpose of this dissertation is to investigate the variance, volatility, covariance, and correlation swap for a BN-S type model. For that we need to find the mean and variance of the realized stock variance. In the above lemma all the terms are deterministic except the subordinator.

We investiigate this in more details for the rest of this section. Below we are going to state some definitions and theorems which help us to find the mean and varaince.

Definition 3.0.7. ***(Self-decomposability, [5])*** *A probability measure P on $\mathbb{R}$ is said to be* self-decomposable *or to belong to the Lévy class L, if for each $l > 0$ there exists a probability measure Q_l on $\mathbb{R}$ such that*

$$\Phi(\zeta) = \Phi(e^{-l}\zeta)\Phi_l(\zeta), \tag{3.12}$$

where Φ and Φ_l denote the characteristic functions of P and Q_l, respectively. A random variable X with law in L is also called self-decomposable.

The following two theorems give a relation between self-decomposability and Lévy processes. For the proofs see [4, 67].

Theorem 3.0.8. ***(Stationarity, [67])*** *If X is self-decomposable then there exists a* stationary stochastic process *$\{\sigma^2(t)\}_{t\geq 0}$, and a Lévy process $\{Z_t\}_{t\geq 0}$, independent of σ_0^2, such that $\sigma_t^2 \stackrel{d}{=} X$ for all $t \geq 0$ and*

$$\sigma_t^2 = \exp(-\lambda t)\sigma_0^2 + \int_0^t \exp\left(-\lambda(t-s)\right) dZ_{\lambda s}, \quad \textit{for all } \lambda > 0. \tag{3.13}$$

Conversely, if $\{\sigma_t^2\}_{t\geq 0}$, is a stationary stochastic process *and $\{Z_t\}_{t\geq 0}$ is a Lévy process independent of σ_0^2, such that $\{\sigma_t^2\}$ and $\{Z_t\}$ satisfy*

$$d\sigma_t^2 = -\lambda\sigma_t^2\, dt + dZ_{\lambda t}, \quad \sigma_0^2 > 0, \tag{3.14}$$

for all $\lambda > 0$, then σ_t^2 is self-decomposable.

Theorem 3.0.9. ***(Jurek and Vervaat,[31])*** *A random variable X has law in L if and only if X has a representation of the form $X = \int_0^\infty e^{-t}\, dZ_t$, where Z_t is a Lévy process. In this case the Lévy measure U and W of X and Z_1 are related by $U(dx) = \int_0^\infty W(e^t\, dx)\, dt$. In addition, if $u(x)$, the Lévy density of U is differentiable, then the Lévy measure W has a density w, and u and w are related by*

$$w(x) = -u(x) - xu'(x). \tag{3.15}$$

It is clear from ([51] Theorem 17.5(ii)) that for any self-decomposable law D there exists a

Lévy process Z such that the process of OU type driven by Z has invariant distribution given by D.

It is well known that inverse Gaussian (IG) distributions and variance gamma distributions are self-decomposable. Suppose that the stationary distribution of σ_t^2 is given by IG(δ, γ) law. IG(δ, γ) is concentrated on $\mathbb{R}_+$ and has probability density

$$p(x) = \frac{1}{\sqrt{2\pi}} \delta e^{\delta\gamma} x^{-3/2} \exp\left(-\frac{\delta^2 x^{-1} + \gamma^2 x}{2} \right), \quad \gamma \geq 0, \quad \delta > 0. \tag{3.16}$$

Since IG is infinitely divisible, for all $n \geq 2$ the sum of $(\sigma_t^2)_n$ with distribution as $IG(\delta, \frac{\gamma}{n})$ have the same distribution as of (σ_t^2). Mathematically this is the same as

$$\sigma_t^2 \overset{d}{=} \sum_{i=0}^{n} (\sigma_t^2)_i. \tag{3.17}$$

Now it is easy to see that the Lévy density of (σ_t^2) as the limit of

$$\lim_{n \to \infty} n {\sigma_t^2}_n(dx) = \nu d(x). \tag{3.18}$$

Here we give the result of an Inverse Gaussian Lévy density without proof, for detailed proof we refer to [41].

$$u(x) = \frac{1}{\sqrt{2\pi}} \delta x^{-3/2} \exp(-\gamma^2 x/2), \quad x > 0. \tag{3.19}$$

Applying theorem (3.0.9), Lévy density of Z_1 is given by $w(x) = \frac{\delta}{2\sqrt{2\pi}} x^{-\frac{3}{2}} (1 + \gamma^2 x) e^{-\frac{1}{2}\gamma^2 x}$. Likewise if the stationary distribution of σ_t^2 is given by gamma law $\Gamma(\nu, \alpha)$ and knowing gamma density as an infinitely divisible Lévy process, its Lévy density of $\Gamma(\nu, \alpha)$ is given by $u(x) = \nu x^{-1} e^{-\alpha x}$, $x > 0$, then once again by (3.0.9) we obtain the Lévy density of Z_1 as $w(x) = \nu\alpha e^{-\alpha x}$, $x > 0$.

Here we define the (unconditional) cumulant generating functions or the log Laplace transforms of σ_t^2 and Z_1 (if they exist) by

$$\bar{\kappa}^u(\theta) = \log(E^u[\exp(\theta\sigma_t^2)]), \tag{3.20}$$

and

$$\kappa^u(\theta) = \log(E^u[\exp(\theta Z_1)]), \tag{3.21}$$

where the superscript "u" stands for unconditional. Moreover when we say unconditional which means the distribution is independent of σ_0^2. The relation of equations (3.20) and (3.21) are based on the following key result from Lévy processes. We state this as a lemma and the proof can be found in [41].

Lemma 3.0.10. ***(Key,Formula,Barndorff-Nielsen and Shephard [2], p.5)*** *Let f denote a continuous function, Z a Lévy process and set $Y = \int_{\Re_+} f(t)dZ_t$. Then*

$$\kappa_Y(\theta) = \int_{\Re_+} \kappa_{(Z_1)}(\theta f(t))dt. \tag{3.22}$$

Using the above lemma it is easy to see the relation as $\bar{\kappa}^u(\theta) = \int_0^\infty \kappa^u(\theta \exp(-s))ds$. This relation can be expressed using derivative as it is proved in equation (6.4) and is given by $\kappa^u(\theta) = \theta \frac{d(\bar{\kappa}^u(\theta))}{d\theta}$. The expected value and variance is the first and second derivative of the cumulant generating function evaluated at $\theta = 0$ when they exist. It follows that if we write the cumulant of Z_1 and σ_t^2 (when they exist) as κ_m and $\bar{\kappa}_m$ ($m = 1, 2, 3, \ldots$) respectively where m denotes the m^{th} derivative, then

$$\kappa_m^u = m\bar{\kappa}_m^u. \quad \text{for} \quad m = 1, 2, 3 \ldots. \tag{3.23}$$

Therefore, $E^u(Z_1) = E^u(\sigma_t^2)$ and $\text{Var}^u(Z_1) = 2\text{Var}^u(\sigma_t^2)$. However, since Z is independent of σ_0^2 this relations can also be written as

$$E(Z_1) := E(Z_1|\sigma_0^2) = E^u(Z_1) = E^u(\sigma_t^2), \tag{3.24}$$

and

$$\text{Var}(Z_1) := \text{Var}(Z_1|\sigma_0^2) = \text{Var}^u(Z_1) = 2\text{Var}^u(\sigma_t^2). \tag{3.25}$$

In general

$$\kappa_m = m\bar{\kappa}_m^u, \quad \text{for} \quad m = 1, 2, 3 \ldots. \tag{3.26}$$

Note that in present setting though the unconditional distribution of σ_t^2 is stationary, the conditional distribution of σ_t^2 given σ_0^2 is *not* stationary. However due to the independence of the subordinator Z and σ_0^2 and the relation (3.8), there is a tractable way of dealing with such conditional distribution in terms of parameters of corresponding unconditional distribution. We conclude this section with derivations of first two cumulants for Z_1 when the unconditional stationary distribution of σ_t^2 is given by IG, Γ, or positive tempered stable (PTS) processes. In the next section we use these results to compute the conditional cumulants of σ_t^2.

Lemma 3.0.11. *If the stationary distribution of σ_t^2 is given by an IG(δ,γ) then its log cumulant function is given by*

$$\bar{\kappa}^u(\theta) = \log(E^u(e^{\theta\sigma_t^2})) = \delta(\gamma - \sqrt{\gamma^2 - 2\theta}). \tag{3.27}$$

Proof. Since σ_t^2 follows inverse Gaussian distribution and using the definition of expected value we can deduce

$$\begin{aligned}
\bar{\kappa}^u(\theta) &= \log(E^u(e^{\theta\sigma_t^2})) \\
&= \log\left(\int_0^\infty \frac{1}{\sqrt{2\pi}}\delta e^{\theta x} e^{\delta\gamma} x^{-3/2} e^{-\frac{\delta^2 x^{-1}+\gamma^2 x}{2}} dx\right) \\
&= \log\left(\int_0^\infty \frac{1}{\sqrt{2\pi}}\delta e^{\delta\sqrt{\gamma^2-2\theta}-\delta(\sqrt{\gamma^2-2\theta}-\gamma)} x^{-3/2} e^{-\frac{\delta^2 x^{-1}+(\gamma^2-2\theta)x}{2}} dx\right) \\
&= \log\left(e^{-\delta(\sqrt{\gamma^2-2\theta}-\gamma)} \int_0^\infty \frac{1}{\sqrt{2\pi}}\delta e^{\delta\sqrt{\gamma-2\theta}} x^{-3/2} e^{-\frac{\delta^2 x^{-1}+(\gamma^2-2\theta)x}{2}} dx\right) \\
&= -\delta(\sqrt{\gamma^2-2\theta}-\gamma).
\end{aligned} \tag{3.28}$$

□

Therefore Using the above relation or from equation (6.4) the log cumulant function of Z_1 can be deduced as follows

$$\kappa(\theta) = \theta\frac{d}{d\theta}\left(\bar{\kappa}^u(\theta)\right) = \theta\left(\frac{d}{d\theta}\left[\delta(\gamma - \sqrt{\gamma^2-2\theta})\right]\right) = \frac{\delta\theta}{\sqrt{\gamma^2-2\theta}}. \tag{3.29}$$

Since the expected value of Z_1 is the first derivative of the log cumulant function at $\theta = 0$

which is given by

$$E(Z_1) = \frac{d}{d\theta}(\frac{\delta\theta}{\sqrt{\gamma^2 - 2\theta}})\Big|_{\theta=0} = \delta\gamma^{-1}. \tag{3.30}$$

Similarly the variance of Z_1 can be obtained by

$$\text{Var}(Z_1) = \frac{d^2}{d\theta^2}\left(\frac{\delta\theta}{\sqrt{\gamma^2 - 2\theta}}\right)\Big|_{\theta=0} = 2\delta\gamma^{-3}. \tag{3.31}$$

Lemma 3.0.12. *If the stationary distribution of σ_t^2 is given by an $\Gamma(\nu, \alpha)$ law then its log cumulant function is given by*

$$\bar{\kappa}^u(\theta) = \log(E^u(e^{\theta\sigma_t^2})) = -\nu\log(1 - \theta\alpha^{-1}). \tag{3.32}$$

Proof. If σ_t^2 follows a $\Gamma(\nu, \alpha)$ with its density function is given by $p(x) = \frac{\alpha^\nu}{\Gamma(\nu)}x^{\nu-1}e^{-\alpha x}$ then the log cumulant function is given by

$$\begin{aligned}
\bar{\kappa}^u(\theta) &= \log(E^u(e^{\theta\sigma_t^2})) \\
&= \log\left(\int_0^\infty e^{\theta x}\frac{\alpha^\nu}{\Gamma(\nu)}x^{\nu-1}e^{-\alpha x}dx\right) \\
&= \log\left(\int_0^\infty \frac{\alpha^\nu(\alpha-\theta)^\nu}{\Gamma(\nu)(\alpha-\theta)^\nu}x^{\nu-1}e^{-(\alpha-\theta)x}dx\right) \\
&= \log\left(\left(\frac{\alpha}{\alpha-\theta}\right)^\nu \int_0^\infty \frac{(\alpha-\theta)^\nu}{\Gamma(\nu)}x^{\nu-1}e^{-(\alpha-\theta)x}dx\right) \\
&= \nu\log\left(\frac{\alpha}{\alpha-\theta}\right).
\end{aligned} \tag{3.33}$$

□

Therefore using the above lemma the log cumulant function of Z_1 can be also found as

$$\kappa(\theta) = \theta\frac{d}{d\theta}\left(\bar{\kappa}^u(\theta)\right) = \theta\left(\frac{d}{d\theta}\left[-\nu\log(1-\theta\alpha^{-1})\right]\right) = \frac{\nu\theta}{\alpha-\theta}. \tag{3.34}$$

Similarly the expected and variance of Z_1 can be calculated using the log cumulant function which is given by

$$E(Z_1) = \frac{d}{d\theta}(\frac{\nu\theta}{\alpha-\theta})\Big|_{\theta=0} = \nu\alpha^{-1}, \tag{3.35}$$

and

$$\mathrm{Var}(Z_1) = \frac{d^2}{d\theta^2}\left(\frac{\nu\theta}{\alpha-\theta}\right)\Big|_{\theta=0} = 2\nu\alpha^{-2}. \tag{3.36}$$

So far we have seen a self-decomposable process having two parameters. Here we are going to add one more self-decomposable Lévy process which has three parameters and is called Positive Tempered Stable (PTS). For further reading regarding PTS Lévy process we refer to (see [35, 49, 66, 13, 14]). Now consider a positive stable $\mathrm{PS}(\kappa,\delta)$ process whose its Lévy density for such a process is given by $u(y) = \delta 2^k \frac{\kappa}{\Gamma(1-\kappa)} y^{-1-\kappa}$. Unfortunately, the probability density function $p_S(y;\kappa,\delta)$ of $\mathrm{PS}(\kappa,\delta)$ of Positive stable process is unknown in general. However the Probability density function of $\mathrm{PTS}(\kappa,\delta,\gamma)$ family is obtained by exponentially tilting the probability density function $p_S(y;\kappa,\delta)$ and is given by:

$$p(y;\kappa,\delta,\gamma) = e^{\delta\gamma}\exp\left(-\frac{1}{2}\gamma^2 y\right)p_S(y;\kappa,\delta), \quad y>0. \tag{3.37}$$

This is not in general known in simple form. However, for $\mathrm{PTS}(\kappa,\delta,\gamma)$ process and its Lévy density is given by (see [7])

$$u(x) = \delta\gamma^{-2\kappa}\frac{\kappa}{\Gamma(\kappa)\Gamma(1-\kappa)} y^{-\kappa-1}\exp\left(-\frac{1}{2}\gamma^2 y\right), \quad y,\delta>0, 0<\kappa<1, \gamma\geq 0. \tag{3.38}$$

Now again if you assume the stationary distribution of σ_t^2 follows a $\mathrm{PTS}(\kappa,\delta,\gamma)$ law. Then $\bar{\kappa}^u(\theta) = \delta\gamma - \delta(\gamma^{1/\kappa} - 2\theta)^\kappa$, and $E^u(\sigma_t^2) = 2\kappa\delta\gamma^{\frac{\kappa-1}{\kappa}}$ and $\mathrm{Var}^u(\sigma_t^2) = 4\kappa(1-\kappa)\delta\gamma^{\frac{\kappa-2}{\kappa}}$. Hence in this case

$$E(Z_1) = 2\kappa\delta\gamma^{\frac{\kappa-1}{\kappa}}, \tag{3.39}$$

and

$$\mathrm{Var}(Z_1) = 8\kappa(1-\kappa)\delta\gamma^{\frac{\kappa-2}{\kappa}}. \tag{3.40}$$

Now we have seen enough background to be able to come up to our result of valuing price of the variance, volatility, covariance and correlation swap if the stock dynamics follows a BN-S model. The rest of this dissertation discusses particularly on our main result and numerical analysis.

4. PRICING VARIANCE AND VOLATILITY SWAP FOR BN-S MODEL

4.1. Variance Swap

Definition 4.1.1. *A variance swap is a forward contract on the realized variance. The payoff of a variance swap at expiry is given by*

$$N(\sigma_R^2 - K_{var}). \tag{4.1}$$

Furnished with the BN-S model and the understanding of the variance process, in this chapter we are going to prove the main result related to the arbitrage-free pricing of variance and volatility swap. Since variance swap is easy to implement, we are going to derive the price of the variance swap first, then using equation (2.15) we showed an approximate estimate for volatility swap price. However, the Taylor series approximation may not converge if the stock market is not stable, and it might not be appropriate to use equation (2.15) when there is highly volatile market and, in this section we also derive a closed form solution for volatility swap pricing for general case.

Theorem 4.1.2. *The arbitrage free price of the variance swap is given by*

$$P_{Var} = e^{-rT}\left[\frac{1}{T}\left(\lambda^{-1}\left(1-e^{-\lambda T}\right)\left(\sigma_0^2-\kappa_1\right)+\kappa_1 T\right)+\rho^2\lambda\kappa_2 - K_{Var}\right], \tag{4.2}$$

where κ_1 and κ_2 are the first cumulant (i.e., the expected value) and the second cumulant (i.e., the variance) of Z_1 respectively.

Proof. The (conditional given σ_0^2) expected value of equation (3.10) gives the value

$$\begin{aligned} E(\sigma_R^2) &= \frac{1}{T}\left(\lambda^{-1}(1-e^{-\lambda T})\sigma_0^2 + \lambda^{-1}\kappa_1\int_0^T\left(1-e^{-\lambda(T-s)}\right)\lambda\, ds\right) + \rho^2\lambda \mathrm{Var}[Z_1] \\ &= \frac{1}{T}\left(\lambda^{-1}(1-e^{-\lambda T})\sigma_0^2 + \kappa_1(T-\lambda^{-1}(1-e^{-\lambda T}))\right) + \rho^2\lambda\kappa_2. \end{aligned} \tag{4.3}$$

Hence the theorem follows from simplification of (4.3). □

4.2. Volatility Swap

Definition 4.2.1. *A volatility swap is a forward contract on future realized volatility of a given underlying asset. The payoff of a volatility swap at the maturity T is given by*

$$N(\sigma_R - K_{vol}), \tag{4.4}$$

where N is the notional amount in dollar, σ_R is the realized volatility and K_{Vol} is the annualized volatility delivery price.

Theorem 4.2.2. *The arbitrage free value of the volatility swap is given by*

$$\begin{aligned} P_{Vol} \approx e^{-rT} \Bigg(& \sqrt{\frac{1}{T}\left(\lambda^{-1}\left(1-e^{-\lambda T}\right)\left(\sigma_0^2-\kappa_1\right)+\kappa_1 T\right)+\rho^2\lambda\kappa_2} \\ & - \frac{\frac{\lambda^{-2}}{T^2}\kappa_2(2e^{-\lambda T}-\frac{3}{2}-\frac{1}{2}e^{-2\lambda T}+\lambda T)}{8\left(\frac{1}{T}\left(\lambda^{-1}\left(1-e^{-\lambda T}\right)\left(\sigma_0^2-\kappa_1\right)+\kappa_1 T\right)+\rho^2\lambda\kappa_2\right)^{3/2}} - K_{Vol}\Bigg), \end{aligned} \tag{4.5}$$

where κ_2 is the second cumulant (i.e., the variance) of Z_1.

Proof. The (conditional given σ_0^2) variance of σ_R^2 can be obtained from the following computation.

$$\begin{aligned} \operatorname{Var}(\sigma_R^2) &= \operatorname{Var}\left(\frac{\lambda^{-1}}{T}\int_0^T \left(1-e^{-\lambda(T-s)}\right) dZ_{\lambda s}\right) \\ &= \frac{\lambda^{-2}}{T^2}\kappa_2 \int_0^{\lambda T} \left(1-e^{-s}\right)^2 ds \\ &= \frac{\lambda^{-2}}{T^2}\kappa_2(2e^{-\lambda T}-\frac{3}{2}-\frac{1}{2}e^{-2\lambda T}+\lambda T). \end{aligned} \tag{4.6}$$

Hence the theorem follows from (2.15) with the substitution of $E(\sigma_R^2)$ from (4.3) and $\operatorname{Var}(\sigma_R^2)$ from the above expression. □

4.3. Closed Form Solution of Volatility Swap

Since equation (2.15) uses Taylor series expansion around the mean of the realized variance. It assumes that, the long-term expected value of the realized variance, $E(\sigma_R^2) < 1$, which is not usually the case, as the market might be highly volatile. For example during the market crash 2008 where the stock price was fluctuating and unpredictable. Since it is sufficient to know $E(\sigma_R)$ to find the price of the volatility swap, in this section we find an analytical formula for $E(\sigma_R)$ where the terms can be computed using the parameters of unconditional distribution of σ_t^2.

Let us assume A_t, be any stochastic process defined on $0 \leq t \leq T$ which is independent of σ_0^2. Assume that the characteristic function and the cumulant generating function of A_t is given by $\Phi_{A_t}(\theta) = E(\exp(i\theta A_t))$ and $\kappa_{A_t}(\theta) = \log E(\exp(\theta A_t))$ respectively. Since A_t is independent of σ_0^2 there is no any difference between superscript u and without u, for that case we omit the subscript. The relation between the characteristic function and the cumulant generating function of A_t is given by

$$\Phi_{A_t}(\theta) = \exp[\kappa_{A_t}(i\theta)]. \tag{4.7}$$

It is easy to see the above equality as

$$\kappa_{A_t}(i\theta) = \log(E(exp(i\theta A_t))) = \log(\Phi_{A_t}(\theta)). \tag{4.8}$$

Lemma 4.3.1. *The moments of A_t can be obtained from $\Phi_{A_t}(\theta)$ by*

$$E(A_t^k) = (-i)^k \frac{d^k \Phi_{A_t}(\theta)}{d\theta^k}\bigg|_{\theta=0}, \quad k = 1, 2, \ldots. \tag{4.9}$$

Proof. From the definition of expected value for a continuous functions and assume P is some distribution of A_t we have

$$\begin{aligned}
\frac{d^k}{d\theta^k}(\Phi_{A_t}(\theta)) &= \frac{d^k}{d\theta^k}\left(\int_0^\infty (\exp(i\theta A_t))d(P(A_t))\right) \\
&= \left(\int_0^\infty \frac{d^k}{d\theta^k}(\exp(i\theta A_t))d(P(A_t)\right) \\
&= \left(\int_0^\infty ((iA_t)^k \exp(i\theta A_t))d(P(A_t)\right) \\
&= \left(\int_0^\infty ((iA_t)^k d(P(A_t)\right) \\
&= i^k E(A_t^k).
\end{aligned} \tag{4.10}$$

The fourth step is found by substituting $\theta = 0$. ☐

Lemma 4.3.2. *Suppose that $A_t = \alpha + \int_0^{\lambda t} (1 - e^{-s})\, dZ_s$, where $\alpha \in \mathbb{R}$ is a constant, and $0 \leq t \leq T$.*

Then

$$\Phi_{A_t}(\theta) = \exp\left(i\theta\alpha + \int_0^{\lambda t} \kappa(i\theta(1-e^{-s}))\,ds\right), \tag{4.11}$$

where $\kappa(\cdot)$ is the cumulant generating function for Z_1. The moments of A_t are given by

$$E(A_t^k) = (-i)^k g_k(0), \quad k = 1, 2, \ldots, \tag{4.12}$$

where

$$g_1(\theta) = i\left(\alpha + \int_0^{\lambda t} (1-e^{-s})\kappa'(i\theta(1-e^{-s}))\,ds\right), \tag{4.13}$$

and

$$g_{k+1}(\theta) = g_1(\theta)g_k(\theta) + g_k'(\theta), \quad k = 1, 2, \ldots. \tag{4.14}$$

In the above formulas prime represents the derivative with respect to the parameter in parenthesis.

Proof. We have

$$\begin{aligned}
\kappa_{A_t}(i\theta) &= \log E(\exp(i\theta A_t)) \\
&= \log E\left(\exp(i\theta\alpha)\exp(i\theta\int_0^{\lambda t}(1-e^{-s})dZ_s)\right) \\
&= i\theta\alpha + \log E\left(exp\left(i\theta\sum_{i=1}^{n}(1-e^{-s_i})(Z_{s_i}-Z_{s_{i-1}})\right)\right) \\
&= i\theta\alpha + \sum_{i=1}^{n}\log\left(E\left(exp(i\theta(1-e^{s_i})Z_{(s_i-s_{i-1})})\right)\right) \\
&= i\theta\alpha + \int_0^{\lambda t}\log E\left(\exp(i\theta(1-e^{-s})Z_1)\right)\,ds \\
&= i\theta\alpha + \int_0^{\lambda t}\kappa_{Z_1}(i\theta(1-e^{-s}))\,ds.
\end{aligned} \tag{4.15}$$

Step three is using the fact that the BDLP Z has a finite variation on any closed interval and using (4.7) we obtain (4.11).

To prove the formula for moments we observe $\kappa(0) = 0$. By differentiation (4.11) and using (4.9) we obtain $E(A_t) = -ig_1(0)$. The results related to $E(A_t^k)$ for $k = 2, 3, \ldots$ follows from induction. □

Note that $\kappa^{(k)}(0) = \kappa_k$, for $k = 1, 2, \dots$. Using Lemma 4.3.2 we can compute any moment for $A_t = \alpha + \int_0^{\lambda t}(1 - e^{-s})\, dZ_s$ in terms of cumulants of Z. For example

$$E(A_t) = -ig_1(0) = \alpha + \int_0^{\lambda t} \kappa_1(1 - e^{-s})\, ds = \alpha + \kappa_1(\lambda t - 1 + e^{-\lambda t}). \tag{4.16}$$

$$\begin{aligned} E(A_t^2) = (-i)^2 g_2(0) &= -(g_1(0)^2 + g_1'(0)) \\ &= \left(\alpha + \int_0^{\lambda t} \kappa_1(1 - e^{-s})\, ds\right)^2 + \int_0^{\lambda t} \kappa_2(1 - e^{-s})^2\, ds \\ &= \left(\alpha + \kappa_1(\lambda t - 1 + e^{-\lambda t})\right)^2 + \kappa_2\left(2e^{-\lambda t} - \frac{3}{2} - \frac{1}{2}e^{-2\lambda t} + \lambda t\right). \end{aligned} \tag{4.17}$$

Since the realized variance is a finite number, it is necessary to bound by a constant real number from above. Assume that $\sigma_R^2 < \beta^2$, for some $\beta > 0$. For example, since σ_t is expressed in percentage, for a stable market, where it does not have "crash-like fluctuations" $\beta = 1$ is a very reasonable assumption. We also note that for $|x| < 1$, $\sqrt{1+x}$ can be represented by the convergent series

$$\sqrt{1+x} = \sum_{k=0}^{\infty} \frac{(-1)^{k+1}(2k)!}{4^k(k!)^2(2k-1)} x^k. \tag{4.18}$$

The theorem which is given below gives an analytic formula for the arbitrage free value of the volatility swap. The theorem gives the arbitrage free value of the volatility swap in terms of a convergent infinite series. As the series converges very fast it is reasonable to take the first few terms for analysis purpose.

Theorem 4.3.3. *Assume that $\sigma_R^2 < \beta^2$, for some $\beta > 0$. Then the arbitrage free value of the volatility swap is given by*

$$P_{vol} = e^{-rT}\left(\sum_{k=0}^{\infty} \frac{(-1)^{k+1}(2k)!}{4^k(k!)^2(2k-1)} \frac{1}{\beta^{2k-1}\lambda^k T^k} E(A_T^k) - K_{Vol}\right), \tag{4.19}$$

where $c_1 = \sqrt{\frac{\sigma_0^2}{\lambda T}(1 - e^{-\lambda T}) + \rho^2\lambda\kappa_2}$ and $A_T = \alpha + \int_0^{\lambda T}(1 - e^{-s})\, dZ_s$, with $\alpha = \lambda T(c_1^2 - \beta^2)$. The quantities $E(A_T^k)$, $k = 1, 2, \dots$ can be computed using Lemma 4.3.2.

Proof. We obtain from (3.10)

$$
\begin{aligned}
\sigma_R^2 &= \frac{\sigma_0^2}{\lambda T}(1-e^{-\lambda T}) + \rho^2\lambda\kappa_2 + \frac{1}{\lambda T}\int_0^T \left(1-e^{-\lambda(T-s)}\right) dZ_{\lambda s} \\
&= \frac{\sigma_0^2}{\lambda T}(1-e^{-\lambda T}) + \rho^2\lambda\kappa_2 + \frac{1}{\lambda T}\int_0^{\lambda T} \left(1-e^{-s}\right) dZ_s \\
&= \beta^2\left(\frac{\frac{\sigma_0^2}{\lambda T}(1-e^{-\lambda T}) + \rho^2\lambda\kappa_2 + \frac{1}{\lambda T}\int_0^{\lambda T}(1-e^{-s})\,dZ_s}{\beta^2}\right) \\
&= \beta^2\left(1+\frac{\lambda T(c_1^2-\beta^2)+\int_0^{\lambda T}(1-e^{-s})\,dZ_s}{\beta^2\lambda T}\right) \\
&= \beta^2\left(1+\frac{\alpha+\int_0^{\lambda T}(1-e^{-s})\,dZ_s}{\beta^2\lambda T}\right), \qquad (4.20)
\end{aligned}
$$

where $c_1^2 = \frac{\sigma_0^2}{\lambda T}(1-e^{-\lambda T}) + \rho^2\lambda\kappa_2$, and $\alpha = \lambda T(c_1^2-\beta^2)$. By the construction we have

$$
\left|\frac{\alpha+\int_0^{\lambda T}(1-e^{-s})\,dZ_s}{\beta^2\lambda T}\right| < 1. \qquad (4.21)
$$

Therefore we obtain

$$
\begin{aligned}
\sigma_R &= \beta\left(1+\frac{\alpha+\int_0^{\lambda T}(1-e^{-s})\,dZ_s}{\beta^2\lambda T}\right)^{1/2} \\
&= \sum_{k=0}^{\infty}\frac{(-1)^{k+1}(2k)!}{4^k(k!)^2(2k-1)}\frac{1}{\beta^{2k-1}\lambda^kT^k}\left(\alpha+\int_0^{\lambda T}\left(1-e^{-s}\right)dZ_s\right)^k. \qquad (4.22)
\end{aligned}
$$

Therefore

$$
\begin{aligned}
E(\sigma_R) &= \sum_{k=0}^{\infty}\frac{(-1)^{k+1}(2k)!}{4^k(k!)^2(2k-1)}\frac{1}{\beta^{2k-1}\lambda^kT^k}E\left(\alpha+\int_0^{\lambda T}\left(1-e^{-s}\right)dZ_s\right)^k \\
&= \sum_{k=0}^{\infty}\frac{(-1)^{k+1}(2k)!}{4^k(k!)^2(2k-1)}\frac{1}{\beta^{2k-1}\lambda^kT^k}E(A_T^k). \qquad (4.23)
\end{aligned}
$$

It is intuitive that the infinite series in (4.23) converges from (4.18) and (4.21). Thus (4.23) gives an analytical formula for computation of $E(\sigma_R)$ where the quantities $E(A_T^k)$, $k = 1, 2, \ldots$ can be computed using Lemma 4.3.2. Thus the theorem follows from the fact that the price of a volatility swap in a risk-neutral world is $P_{\text{vol}} = e^{-rT}(E(\sigma_R) - K_{\text{Vol}})$. □

Next we consider the infinite series $\sum_{k=0}^{\infty} \frac{(-1)^{k+1}(2k)!}{4^k(k!)^2(2k-1)} \frac{1}{\beta^{2k-1}\lambda^k T^k} E(A_T^k)$ of Theorem 4.3.3. It is clear from the proof of that theorem that $A_T/\beta^2\lambda T < 0$. It is also clear from (4.21) that $|A_T/\beta^2\lambda T| < 1$.

Theorem 4.3.4. *Suppose that A_T is given by Theorem 4.3.3. Then the quantity $E(\sigma_R)$ can be approximated by n-th partial sum*

$$\sum_{k=0}^{n-1} \frac{(-1)^{k+1}(2k)!}{4^k(k!)^2(2k-1)} \frac{1}{\beta^{2k-1}\lambda^k T^k} E(A_T^k), \tag{4.24}$$

with the absolute error of approximation less than the quantity $\beta \frac{1}{(2n-1)\sqrt{3n+1}}$, for $n \geq 1$.

Proof. The infinite series representation of $E(\sigma_R)$ is an alternating series and therefore

$$\begin{aligned} &\left| E(\sigma_R) - \sum_{k=0}^{n-1} \frac{(-1)^{k+1}(2k)!}{4^k(k!)^2(2k-1)} \frac{1}{\beta^{2k-1}\lambda^k T^k} E(A_T^k) \right| \\ &\leq \frac{\beta}{4^n(2n-1)} \binom{2n}{n} E \left| \frac{A_T}{\beta^2 \lambda T} \right|^n \\ &< \frac{\beta}{4^n(2n-1)} \binom{2n}{n}. \end{aligned} \tag{4.25}$$

It can be proved by induction the lower and upper bound of the following inequality which is given by

$$\frac{4^n\sqrt{n}}{2n} \leq \binom{2n}{n} \leq \frac{4^n}{\sqrt{3n+1}}, \tag{4.26}$$

for all $n \geq 1$. Therefore we obtain

$$\left| E(\sigma_R) - \sum_{k=0}^{n-1} \frac{(-1)^{k+1}(2k)!}{4^k(k!)^2(2k-1)} \frac{1}{\beta^{2k-1}\lambda^k T^k} E(A_T^k) \right| < \beta \frac{1}{(2n-1)\sqrt{3n+1}}. \tag{4.27}$$

□

The constant β can be used as a "control parameter" that improves the rate of convergence of the infinite series $\sum_{k=0}^{\infty} \frac{(-1)^{k+1}(2k)!}{4^k(k!)^2(2k-1)} \frac{1}{\beta^{2k-1}\lambda^k T^k} E(A_T^k)$. This is shown in the next theorem.

Theorem 4.3.5. *Suppose that A_T is given by Theorem 4.3.3 and it is possible to choose β so that $|A_T/\beta^2\lambda T| < \frac{1}{2+\epsilon}$, for some $\epsilon > 0$. Then the quantity $E(\sigma_R)$ can be approximated by the n-th* *partial sum of the infinite series*

$$\sum_{k=0}^{\infty} \frac{(-1)^{k+1}(2k)!}{4^k(k!)^2(2k-1)} \frac{1}{\beta^{2k-1}\lambda^k T^k} E(A_T^k), \tag{4.28}$$

with the absolute error of approximation less than the quantity $\beta_1 \frac{1}{(2n-1)\sqrt{3n+1}} \frac{1}{(1+\epsilon)^n}$, *for* $n \geq 1$, *where* β_1 *is a constant and is equal to* $\beta \left(\frac{1+\epsilon}{2+\epsilon}\right)^{\frac{1}{2}}$.

Proof. It is easy to obtain that

$$\sigma_R = \beta \left(\sum_{k=0}^{n-1} \frac{(-1)^{k+1}(2k)!}{4^k(k!)^2(2k-1)} \left(\frac{A_T}{\beta^2 \lambda T}\right)^k + \frac{(-1)^{n+1}(2n)!}{4^n(n!)^2(2n-1)} \left(\frac{A_T}{\beta^2 \lambda T}\right)^n \frac{1}{(1+\mu)^{n-\frac{1}{2}}} \right), \tag{4.29}$$

for some μ between $\dfrac{A_T}{\beta^2 \lambda T}$ and 0. Thus the assumption $|A_T/\beta^2 \lambda T| < \frac{1}{2+\epsilon}$ gives $-\frac{1}{2+\epsilon} < \mu$. We have

$$E(\sigma_R) = \sum_{k=0}^{n-1} \frac{(-1)^{k+1}(2k)!}{4^k(k!)^2(2k-1)} \frac{1}{\beta^{2k-1}\lambda^k T^k} E(A_T^k) + E\left[R_n\right], \tag{4.30}$$

where we denote the error term by

$$R_n = \beta \frac{(-1)^{n+1}(2n)!}{4^n(n!)^2(2n-1)} \left(\frac{A_T}{\beta^2 \lambda T}\right)^n \frac{1}{(1+\mu)^{n-\frac{1}{2}}}. \tag{4.31}$$

We obtain

$$\begin{aligned} |R_n| &= \frac{\beta}{4^n(2n-1)} \binom{2n}{n} \left|\frac{A_T}{\beta^2 \lambda T}\right|^n \frac{1}{|1+\mu|^{n-\frac{1}{2}}} \\ &< \frac{\beta}{4^n(2n-1)} \binom{2n}{n} \left(\frac{1}{2+\epsilon}\right)^n \frac{1}{(1-\frac{1}{2+\epsilon})^{n-\frac{1}{2}}} \\ &= \frac{\beta}{4^n(2n-1)} \binom{2n}{n} \left(\frac{1+\epsilon}{2+\epsilon}\right)^{\frac{1}{2}} \frac{1}{(1+\epsilon)^n}. \end{aligned} \tag{4.32}$$

Therefore using (4.26) we obtain

$$|R_n| < \beta_1 \frac{1}{(2n-1)\sqrt{3n+1}} \frac{1}{(1+\epsilon)^n}, \tag{4.33}$$

and consequently,

$$|E(R_n)| \le E(|R_n|) < \beta_1 \frac{1}{(2n-1)\sqrt{3n+1}} \frac{1}{(1+\epsilon)^n}, \tag{4.34}$$

where $\beta_1 = \beta \left(\frac{1+\epsilon}{2+\epsilon}\right)^{\frac{1}{2}}$. □

The next theorem gives a control on estimation of regression parameters based on partial sum approximation of $E(\sigma_R)$. For convenience we denote

$$S_n = \sum_{k=0}^{n-1} \frac{(-1)^{k+1}(2k)!}{4^k (k!)^2 (2k-1)} \frac{1}{\beta^{2k-1}\lambda^k T^k} E(A_T^k). \tag{4.35}$$

We note that S_n depends on λ and various parameters of the stochastic process A_T. To emphasis this dependence, in the next theorem, we write S_n as $S_n(p)$, where p stands for all parameters that govern S_n. Also, the value of $E(\sigma_R)$ computed based on the set of parameter p is denoted as $E(\sigma_R, p)$.

Theorem 4.3.6. *Let D_T be a finite set of empirical data for volatility delivery prices with various maturity days T such that $0 < T \le T_{\max}$, for some $T_{\max} > 0$. Suppose for $\epsilon > 0$, there exists a set of parameters $p^{(n)}$ such that $\max_{0<T\le T_{\max}} |D_T - S_n(p^{(n)})| < \frac{\epsilon}{6}$, for some $n > 0$. If errors $\max_{0<T\le T_{\max}} |D_T - S_n(p^{(k)})|$ are decreasing as k increases, then $\max_{0<T\le T_{\max}} |E(\sigma_R, p^{(n_1)}) - E(\sigma_R, p^{(n_2)})| < \epsilon$ for sufficiently large $n_1, n_2 \ge n$.*

Proof. It is clear that for $n_1, n_2 \ge n$,

$$\begin{aligned} &\max_{0<T\le T_{\max}} |S_{n_1}(p^{(n_1)}) - S_{n_2}(p^{(n_2)})| \\ &< \max_{0<T\le T_{\max}} |D_T - S_{n_1}(p^{(n_1)})| + \max_{0<T\le T_{\max}} |D_T - S_{n_2}(p^{(n_2)})| \\ &< \frac{\epsilon}{6} + \frac{\epsilon}{6} = \frac{\epsilon}{3}. \end{aligned} \tag{4.36}$$

By Theorem 4.3.4 it is clear that for for a given $\epsilon > 0$, sufficiently large n_1 and n_2 can be chosen such that

$$\max_{0<T\le T_{\max}} |E(\sigma_R, p^{(n_1)}) - S_{n_1}(p^{(n_1)})| < \frac{\epsilon}{3}, \tag{4.37}$$

and

$$\max_{0<T\leq T_{\max}} |E(\sigma_R, p^{(n_2)}) - S_{n_2}(p^{(n_2)})| < \frac{\epsilon}{3}. \tag{4.38}$$

Hence the result $\max_{0<T\leq T_{\max}} |E(\sigma_R, p^{(n_1)}) - E(\sigma_R, p^{(n_2)})| < \epsilon$ follows from (4.36), (4.37), and (4.38). □

4.4. Model Fitting and Parameter Estimate

In this chapter we demonstrate the theoretical result using numerical data of the stock price. We also show that the performances of our results agree with the empirical data better (with respect to various measures of goodness of fit as described below), than the existing comparable models. We use Theorems 4.1.2, 4.2.2 and 4.3.3 to calibrate fair delivery price. We use closing stock prices of the S&P 500 index for 943 trading dates from 12/05/2011 to 09/04/2015. Once calibration is performed over the described historical data set, we obtain the model parameters that can be used to price the fixed leg (fair delivery price) of the variance or volatility swap. For *goodness of fit* of the calibration of fair delivery price, we use the absolute percentage error (APE), the average absolute error (AAE), the average relative percentage error (ARPE) and the root-mean-square error (RMSE) given by the following formulas.

$$APE = \frac{1}{\text{mean price}} \sum_{\text{data points}} \frac{|\text{market price} - \text{model price}|}{\text{data points}}, \tag{4.39}$$

$$AAE = \sum_{\text{data points}} \frac{|\text{market price} - \text{model price}|}{\text{data points}}, \tag{4.40}$$

$$ARPE = \frac{1}{\text{data points}} \sum_{\text{data points}} \frac{|\text{market price} - \text{model price}|}{\text{data points}}, \tag{4.41}$$

$$RMSE = \sqrt{\sum_{\text{data points}} \frac{(\text{market price} - \text{model price})^2}{\text{data points}}}. \tag{4.42}$$

We also use Residual standard Error (RSE) for the goodness of fit analysis. This is a statistical measure that is used to describe standard deviation of a point estimate around the fitted function, and this is an estimate of the accuracy of the dependent variable being measured. Mathematically, $RSE = \sqrt{\frac{SSE}{n-k}}$ where n is the number of observations, k is the number of parameters to be estimated, and SSE is the sum of square error.

For the calibration we consider the BN-S model with $\rho = -1$ in (3.2), so that the Γ and inverse Gaussian models have the same number of parameters as in the Heston model. For the analysis, σ_0 is taken to be 0.01. The calibration results for various cases, with the application of Theorem 4.1.2 for variance swap, are shown in Table 1,2, and 3. The corresponding fittings are shown in Figure 1.

The t-value and the probability of each table explains the rejection region under any α level for the significance of the variable. In the tables, "$Pr(>|t|) < 2e-16$" means that the probability of that parameter being zero is less than $2e^{-16}$. The column marked "Standard Error" displays the estimated standard errors of these parameter estimate.

Table 4 gives goodness-of-fit comparison for different models. It is clear that the BN-S model with $\rho = -1$ is producing significant improvement in the root mean square error, than the Heston model with the same parameter. From table (4.3), we can see that the PTS has four parameters, which is one more parameter than Inverse Gaussian, Variance Gamma or Heston model. The improvement of the error is even significant shown in table (4.4).

Table 4.1. Parameter estimate of Gamma distribution for variance swap

Parameters	Estimate	Standard Error	t Value	$Pr(>\|t\|)$
ν	0.0065283	0.0005311	12.29	$< 2e-16$
λ	0.1092836	0.0052738	20.72	$< 2e-16$
α	0.1603521	0.0087384	-18.35	$< 2e-16$

Table 4.2. Parameter estimate of Inverse Gaussian for variance swap

Parameters	Parameter estimate	Standard Error	t Value	$Pr(>\|t\|)$
λ	0.109283	0.005274	20.72	$< 2e-16$
γ	0.023807	0.0010678	-54.90	$< 2e-16$
δ	0.586191	0.01039	22.90	$< 2e-16$

We use analysis of variance to establish the best model used to estimate the variance swap.

Table 4.3. Parameter estimate of Positive tempered stable for variance swap

Parameters	Estimate	Standard Error	t Value	$Pr(>\|t\|)$
λ	0.109283	0.005274	20.7	$< 2e - 16$
δ	0.023807	0.001039	22.86	$< 2e - 16$
κ	.456078	0.005483	24.98	$< 2e - 16$
γ	0.586191	0.010678	54.90	$< 2e - 16$

For this, R^2 is calculated for each model. The quantity R^2 is used to estimate the percentage of the given data that can be explained by the model. It is found that the IG and Γ model explain 81.05% and 82.65% of the data respectively, and all parameters in these cases turn out to be highly significant. On the other hand, the PTS model uses one more parameter. The R^2 value of the PTS is found to be around 85.05%. Heston model and Hull-White model explain about 72.85% and 65.28% of the given data respectively.

Table 4.4. Comparing errors of different models for variance swap

Model	RMSE	RSE	APE	AAE	ARPE
Hull-White	0.9610284	0.518945	0.00854231	0.0296473	1.9821059
Heston Model	0.0588036	0.0791205	0.000785167	0.59817601	1.0516412
Variance Gamma	0.00229305	0.00105	0.000005095391	0.002303117	0.1287677
Inverse Gaussian	0.00221305	0.00163	0.000005095203	0.002303032	0.1287595
Positive tempered stable	0.001002722	0.0001006	0.0000005106	0.0001350163	0.013285

Again, for the calibration of the volatility swap, we consider the BN-S model with $\rho = -1$ in (3.2), so that the Γ and inverse Gaussian models have same number of parameters as in the Heston model. The calibration results for various cases with the application of Theorem 4.2.2 for approximate volatility swap, are shown in Table 5, 6, and 7. Their corresponding fittings are shown in Figure 2. Table 8 gives a comparison of goodness-of-fit for different models. It is clear that the BN-S models with $\rho = -1$ are producing better result than the Heston and Hull-White models, which both models are driven by a Brownian motion.

Next, we compare the results for the volatility swap for different models and estimate the parameters along with their standard errors. Using statistical analysis at the α level of 0.00001, it turns out that all the model parameters are highly significant to predict the true value of the

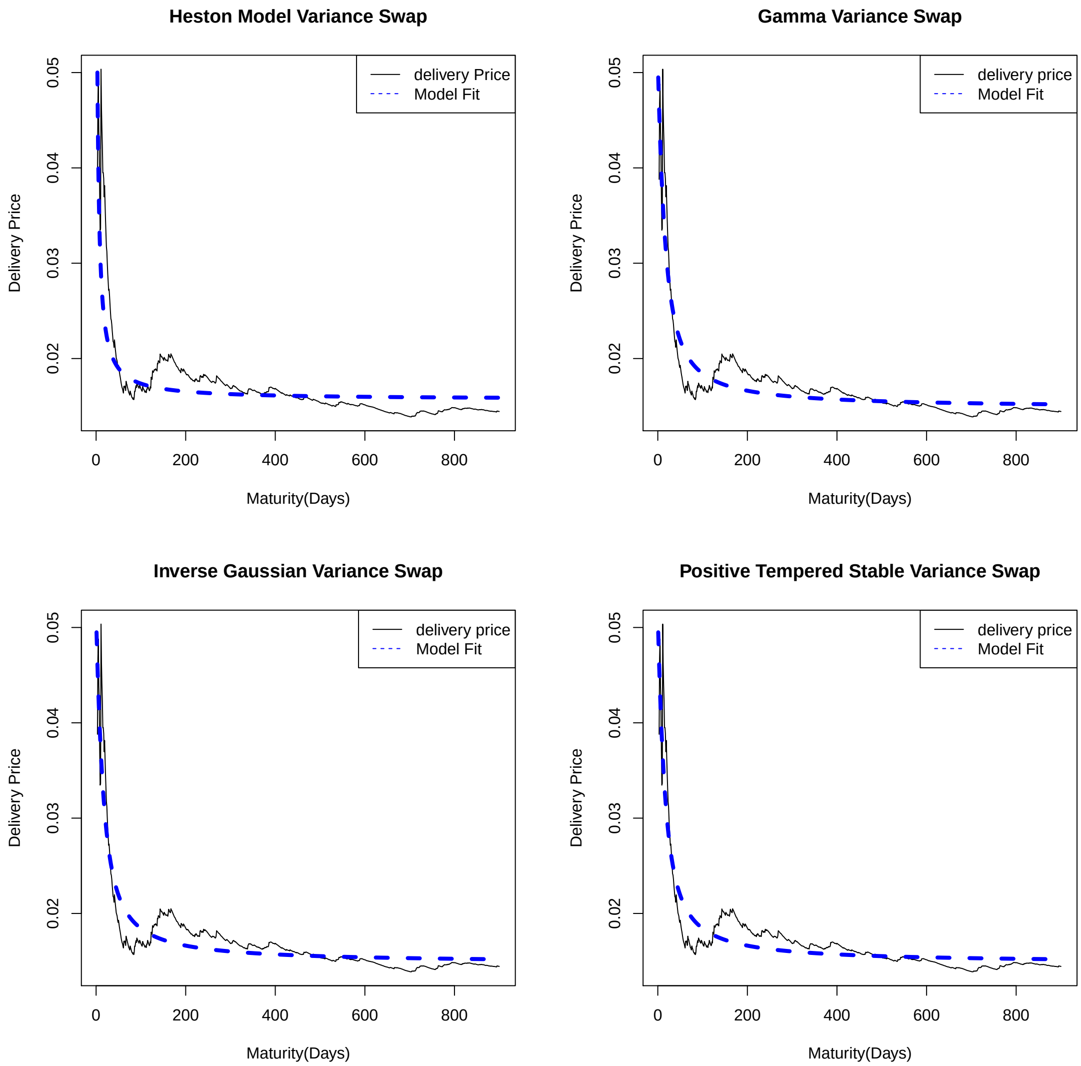

Figure 4.1. Fitting of variance swap

realized volatility. Analysis with R^2 reveals that more than 83.79% of the data can be explained by Γ model, and more than 83.60% of the data can be explained by IG model. For both cases all the parameters are highly significant. On the other hand, the PTS model uses one additional parameter and as a consequence gives a better goodness-of-fit estimate. From the R^2 value it is found that around 87.54% of the given data is explained by PTS model. The Heston model and the Hull-White model explain about 68.83% and 63.45% of the given data respectively.

Finally, we consider he calibration for the BN-S model with $\rho = -1$ using Theorem 4.3.3. The calibration parameters for fair delivery price of volatility swaps are shown in Table 9, 10, and 11. The corresponding fittings are shown in Figure 3. Table 12 gives a comparison of goodness-of-fit

Table 4.5. Parameter estimates for Γ volatility swap with Theorem 4.2.2

Parameters	Estimate	Standard Error	t Value	$Pr(>\|t\|)$
λ	0.1845832	0.0085634	21.55	$< 2e-16$
ν	0.0093435	0.0002426	38.511	$< 2e-16$
α	0.8902156	0.0202934	43.87	$< 2e-16$

Table 4.6. Parameter estimates for IG volatility swap with Theorem 4.2.2

Parameters	Estimate	Standard Error	t Value	$Pr(>\|t\|)$
λ	0.483033	0.027331	17.649	$< 2e-16$
δ	0.038029	0.001266	30.052	$< 2e-16$
γ	1.577838	0.027993	56.38	$< 2e-16$

for different models.

It is clear that the expression in Theorem 4.3.3 is a formal asymptotic expansion. However, Theorem 4.3.4 and Theorem 4.3.5 gives control over the convergence of the infinite series. Theorem 4.3.6 can be used in practice to demonstrate the reliability of the approximation. It is clear from the above results and the proof of Theorem 4.3.6, that in order to have rapid convergence of the infinite series for $E(\sigma_R)$, the quantity $\max_{0<T\leq T_{\max}} |S_{n_1}(p^{(n_1)}) - S_{n_2}(p^{(n_2)})|$ must be reasonably small as n_1 and n_2 increase. This reliability analysis is important when the method is completely wrong, but the calibration procedure gives good fit (see [12]). In the following Figure 4, we take $n_2 = N$ and $n_1 = N+1$, to demonstrate the rapid convergence of the expression $\max_{0<T\leq T_{\max}} |S_{N+1}(p^{(N+1)}) - S_N(p^{(N)})|$ for various N. Along with Theorem 4.3.6, these results show the numerical evidence for reliability of the procedure.

4.5. Conclusion

In this Chapter we have presented a new approach based on the BN-S model to obtain the arbitrage-free pricing for variance and volatility swaps for financial markets. The stochastic volatility models used for analysis are empirically reasonable,and the many appealing features from a finance perspective. The results derived in this dissertation are potentially important as this means that stochastic volatility models built out of OU processes with gamma or inverse Gaussian or positive tempered stable marginals have excellent numerical accuracy in obtaining the fair delivery prices for various swaps. Further, we get closed form pricing formulas depending on various cumulants of the BDLP Lévy process Z. In this dissertation we have also derived an

Table 4.7. Parameter estimates for PTS volatility swap with Theorem 4.2.2

Parameters	Estimate	Standard Error	t Value	$Pr(>\|t\|)$
λ	0.483033	0.027406	17.64	$< 2e - 16$
δ	0.038029	0.001266	30.05	$< 2e - 16$
κ	0.61204	0.0054186	40.05	$< 2e - 16$
γ	1.577838	0.0027993	56.38	$< 2e - 16$

Table 4.8. Comparing errors of different models for volatility swap

Model	RMSE	RSE	APE	AAE	ARPE
Hull-White	0.8245997	0.721024	0.0001446324	0.076537383	0.5889565
Heston Model	0.245611	0.1746	0.0005610508	0.253595	1.887359
Variance Gamma	0.007656574	0.005441	0.00001886869	0.008528647	0.06435461
Inverse Gaussian	0.0077152225	0.005483	0.00001867513	0.008441158	0.06352462
Positive tempered stable	0.00021903	0.000163	0.00001867513	0.00844115	0.006352462

algorithmic process to compute the cumulants of Z. The improvement of numerical results in the analysis is very significant over the existing models with a similar number of parameters, such as the Heston model.

More generally, we also used numerical approximations and statistical analysis for model adequecy and it turns out that Ou Prcess captured most of the variance and volatiltiy jumps as suppose to Heston and Hull-white model. Those models can be also extended for commodity swap of natural gass and crude oil. The approach towards this model can be considered in future works.

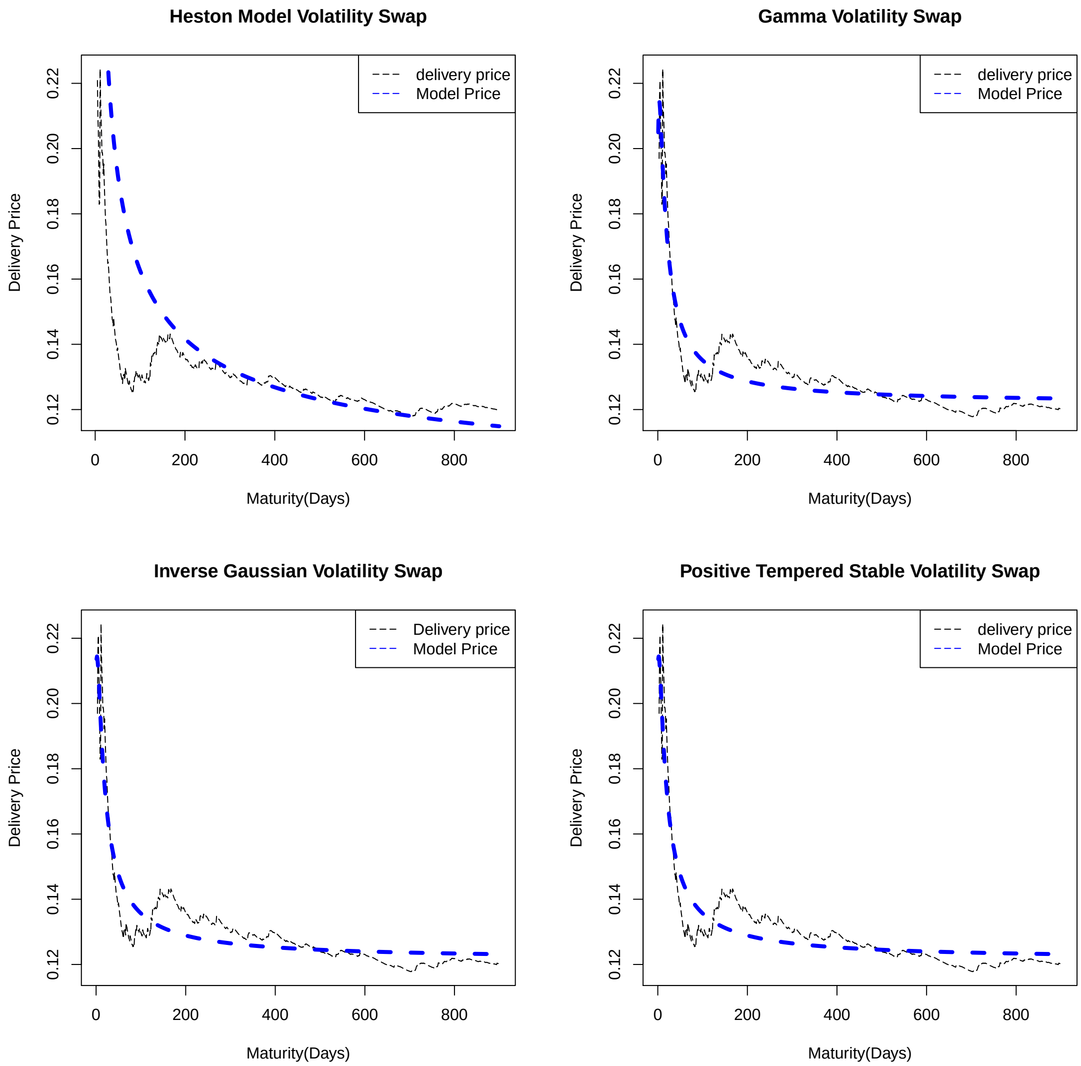

Figure 4.2. Fitting of volatility swap

Table 4.9. Parameter estimates for Γ volatility swap with Theorem 4.3.3

Parameters	Estimate	Standard Error	t Value	$Pr(>\|t\|)$
λ	0.204673	0.003423	59.80	$< 2e-16$
ν	0.443597	0.013299	33.35	$< 2e-16$
α	1.386975	0.041721	33.24	$< 2e-16$

Table 4.10. Parameter estimates for IG volatility swap with Theorem 4.3.3

Parameters	Estimate	Standard Error	t Value	$Pr(>\|t\|)$
λ	0.199502	0.003372	59.16	$< 2e-16$
δ	0.385977	0.005965	64.70	$< 2e-16$
γ	1.206734	0.08757	64.33	$< 2e-16$

Table 4.11. Parameter estimates for PTS volatility swap with Theorem 4.3.3

Parameters	Estimate	Standard Error	t Value	$Pr(>\|t\|)$
λ	0.199554	0.003373	59.17	$< 2e-16$
δ	0.385881	0.005962	64.72	$< 2e-16$
κ	0.614802	0.001289	64.78	$< 2e-16$
γ	1.206434	0.018746	64.36	$< 2e-16$

Table 4.12. Comparing errors of different models for volatility swap

Model	RMSE	RSE	APE	AAE	ARPE
Variance Gamma	0.007674167	0.005123	0.00001907175	0.009620432	0.0513133
Inverse Gaussian	0.007663909	0.005446	0.00001905593	0.008613282	0.06508503
Positive tempered stable	0.000137236	0.00054102	0.00000127545	0.008620456	0.06513158

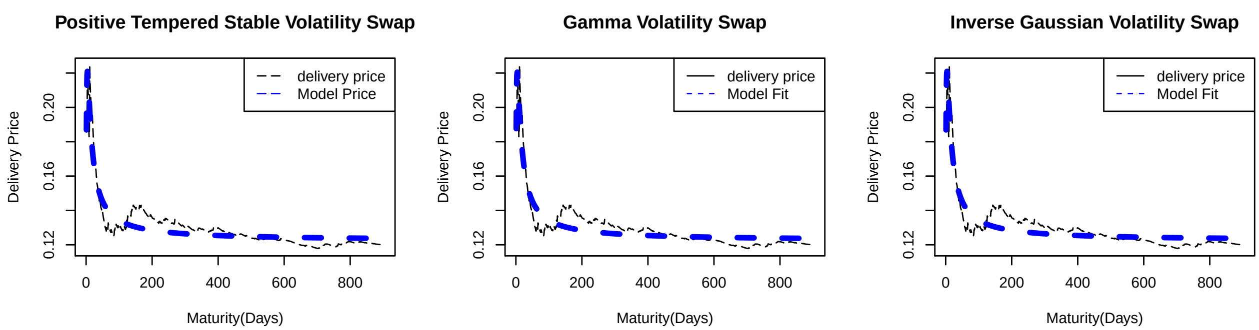

Figure 4.3. Fitting of volatility swap with improved model

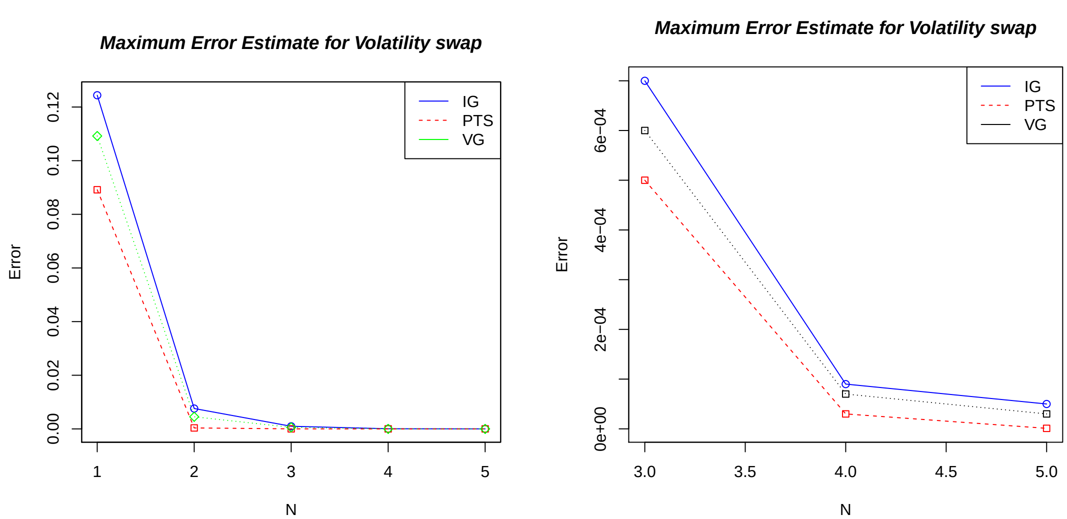

Figure 4.4. (Left) Reliability of fitting with truncated infinite series for volatility swap. (Right) Zoomed in version of the left picture.

5. PRICING COVARIANCE AND CORRELATION SWAP FOR TWO ASSETS WITH A STOCHASTIC VOLATILITY

In this chapter we price covariance swaps for financial market when it is governed by the BN-S model. Covariance swaps are recent financial products that are useful for volatility hedging and speculation using two different financial underlying assets. Introduced in [20] this contract pays the excess of the realized covariance between two derivatives over a constant specified at the outset of the contract. Such a contract may serve as a useful complement for the variance contracts that trade over the counter on several currencies. This makes covariance swap an over the counter financial derivative that allows one to speculate on or hedge risks with the magnitude of the movement, i.e. volatility of the underlying assets like exchange rate, interest rate, or stock index. For example, options dependent on exchange rate movements have an exposure to movements of the correlation between the asset and the exchange rate. This risk may be hedged by using covariance swap. The analysis of a contract paying the realized covariance is a necessary precursor for the analysis of further derivatives written on covariance.

5.1. Pricing Covariance Swap

Definition 5.1.1. *The covariance swap is a covariance forward contract between two assets (S_t^1 and S_t^2, $0 \leq t \leq T$) of a realized covariance, and its payoff at maturity is given by*

$$N(Cov_R(S^1, S^2) - K_{Cov}), \tag{5.1}$$

where N is a notional amount, $Cov_R(S^1, S^2)$ is the realized covariance of two given assets and K_{Cov} is the strike price.

The arbitrage free price of the covariance swap (over the period $0 \leq t \leq T$) is given by

$$P_{\text{Cov}} = E\left[e^{-rT}(\text{Cov}_R(S^1, S^2) - K_{\text{Cov}})\right], \tag{5.2}$$

where $E(\cdot)$ is the expectation with respect to some equivalent martingale measure. In this section we implement a generalized version of the Barndorff-Nielsen and Shephard model to model one of the two assets for covariance and correlation swaps. Let Z_t and Z_t^* be two *independent* Lévy subordinators. Here the *independence* of the Lévy processes Z_t and Z_t^* is understood in the sense of [63] (Proposition 5.3). That is, if (X_t, Y_t) is a Lévy process with Lévy measure $\nu_{(X,Y)}$ and without Gaussian part, then its components are independent if and only if the support of $\nu_{(X,Y)}$ is contained in the set $\{(x, y) : xy = 0\}$, that is, if and only if they *never* jump together. In this case $\nu_{(X,Y)}(A) = \nu_X(A_X) + \nu_Y(A_Y)$, where $A_X = \{x : (x, 0) \in A\}$ and $A_Y = \{y : (0, y) \in A\}$, and ν_X and ν_Y are Lévy measures of X_t and Y_t.

In this case, we define a subordinator which a linear combination of the above to Lévy processes as

$$d\tilde{Z}_t = \rho' \, dZ_t + \sqrt{1-\rho'^2} \, dZ_t^*, \tag{5.3}$$

provided $0 \le \rho' \le 1$. Thus, for $0 \le \rho' \le 1$, Z_t and $\tilde{Z}_t$ are positively correlated Lévy subordinators (see [55]). It is clear that $\mathrm{Var}(Z_t) = t\kappa_2$, $\mathrm{Var}(\tilde{Z}_t) = t\tilde{\kappa}_2$, and using the linearity of covariance and $\tilde{Z}_t = \rho' Z_t + \sqrt{1-\rho'^2} Z_t^*$ gives

$$\begin{aligned}
\mathrm{Cov}(Z_t, \tilde{Z}_t) &= \mathrm{Cov}(Z_t, \rho' Z_t + \sqrt{1-\rho'^2} Z_t^*) \\
&= (Cov)(Z_t, \rho' Z_t) + \mathrm{Cov}(Z_t, \sqrt{1-\rho'^2} Z_t^*) \\
&= \rho'(Cov)(Z_t, \ Z_t) \\
&= \rho' t\kappa_2.
\end{aligned} \tag{5.4}$$

The last step is from independence and κ_2 and $\tilde{\kappa}_2$ are the variances (second cumulant) of Z_1 and $\tilde{Z}_1$ respectively. Therefore the correlation coefficient between Z_t and $\tilde{Z}_t$ is independent of time t and is given by

$$\begin{aligned}
\mathrm{Corr}(Z_t, \tilde{Z}_t) &= \frac{\mathrm{Cov}(Z_t, \tilde{Z}_t)}{\sqrt{\mathrm{Var}(Z_t)}\sqrt{\mathrm{Var}(\tilde{Z}_t)}} \\
&= \frac{\rho' t\kappa_2}{\sqrt{t\kappa_2}\sqrt{t\tilde{\kappa}_2}} \\
&= \rho' \sqrt{\frac{\kappa_2}{\tilde{\kappa}_2}}.
\end{aligned} \tag{5.5}$$

Assume that the risk-neutral dynamics of the two assets are given by $S_t^i = e^{X_t^i}$, with $i = 1, 2$, where the stochastic processes X_t^i are driven by a linear combination of a Weiner process and a Lévy process as given by:

$$dX_t^i = b_t^i \, dt + (\sigma^i)_t \, dW_t^i + \rho^{(i)} \, dZ_{\lambda t}, \quad i = 1, 2, \tag{5.6}$$

and

$$d(\sigma^1)_t^2 = -\lambda(\sigma^1)_t^2 \, dt + dZ_{\lambda t}, \quad (\sigma^1)_0^2 > 0, \tag{5.7}$$

and

$$d(\sigma^2)_t^2 = -\lambda(\sigma^2)_t^2 \, dt + d\tilde{Z}_{\lambda t}, \quad (\sigma^2)_0^2 > 0, \tag{5.8}$$

where W_t^1 and W_t^2 are correlated Wiener processes with $\mathrm{Cov}(W_t^1, W_t^2) = \bar{\rho}t$, and b_t^1 and b_t^2 are deterministic functions of $(\sigma^1)_t^2$ and $(\sigma^2)_t^2$ respectively. In (5.6) $\rho^{(1)}$ and $\rho^{(2)}$ are leverage parameters corresponding to S^1 and S^2 respectively.

From the definition of $\mathrm{Cov}_{\mathrm{R}}(S^1, S^2)$ we have

$$\mathrm{Cov}_{\mathrm{R}}(S^1, S^2) = \frac{1}{T}[\ln S_T^1, \ln S_T^2] = \frac{1}{T}[X_T^1, X_T^2], \tag{5.9}$$

where $[\cdot, \cdot]$ represents the quadratic covariation.

We make the following assumption for the Lévy subordinator Z_t.

Assumption 4. *Z_t is a Lévy subordinator with finite variation and no deterministic drift. Therefore, if J_Z is the random measure describing jumps of Z then*

$$Z_t = \int_{s=0}^{t} \int_0^{\infty} y J_Z(ds, dy). \tag{5.10}$$

Remark 5.1.2. *If the Lévy measures of Z and Z^* are ν_Z and ν_{Z^*} respectively, then by Assumption 1 and [63] (Theorem 4.1), the characteristic triplet of $\tilde{Z}$ is given by $(\tilde{A}, \tilde{\gamma}, \nu_{\tilde{Z}})$, where $\tilde{A} = 0$, $\nu_{\tilde{Z}}(B) = \nu_Z\left(\frac{B}{\rho'}\right) + \nu_{Z^*}\left(\frac{B}{\sqrt{1-\rho'^2}}\right)$, for $B \in \mathcal{B}(\mathbb{R})$ and*

$$\tilde{\gamma} = \rho'\gamma + \sqrt{1-\rho'^2}\gamma^* - \int_{\mathbb{R}} y \left(1_{|y|\le 1}(y) - 1_{S_1}(y)\right) \nu_{\tilde{Z}}(dy) = \rho'\gamma + \sqrt{1-\rho'^2}\gamma^*, \tag{5.11}$$

where S_1 is given by $S_1 = \{\rho' x_1 + \sqrt{1-\rho'^2} x_2 | x_1^2 + x_2^2 \le 1, x_1, x_2 \in \mathbb{R}\}$. Therefore in general

$\tilde{Z}$ *has a* drift *component. However, if both* Z *and* Z^* *satisfy Assumption 4, i.e., if both* Z *and* Z^* *are processes of finite variation and* $\gamma = \int_{|x|\leq 1} x\nu_Z(dx)$ *and* $\gamma^* = \int_{|x|\leq 1} x\nu_{Z^*}(dx)$*, then* $\tilde{\gamma} = \int_{|x|\leq 1} x\nu_{\tilde{Z}}(dx)$ *and hence the deterministic drift (in the sense of Corollary 3.1 in [63]) for* Z*,* Z^* *and* $\tilde{Z}$ *is zero. Therefore, if* $J_{\tilde{Z}}$ *is the random measure describing jumps of* $\tilde{Z}$ *then*

$$\tilde{Z}_t = \int_{s=0}^{t} \int_0^\infty y J_{\tilde{Z}}(ds, dy). \tag{5.12}$$

From (5.6) we obtain

$$X_T^i = \int_0^T b_t^i \, dt + \int_0^T (\sigma^i)_t \, dW_t^i + \int_0^{\lambda T} \int_0^\infty \rho^{(i)} y J_Z(ds, dy), \quad i = 1, 2. \tag{5.13}$$

Therefore the quadratic covariation of X_T^1 and X_T^2 is given by (see [63], Section 8.2.2)

$$[X_T^1, X_T^2] = \int_0^T \bar{\rho}(\sigma^1)_t(\sigma^2)_t \, dt + \rho^{(1)}\rho^{(2)} \int_0^{\lambda T} \int_0^\infty y^2 J_Z(ds, dy). \tag{5.14}$$

It is clear from (5.2) and (5.9) that to find the arbitrage free price of covariance swap it is sufficient to compute $E(\mathrm{Cov}_R(S^1, S^2)) = \frac{1}{T} E[X_T^1, X_T^2]$. We proceed to find this in the following results. The first lemma is similar to Lemma 4.3.2

Lemma 5.1.3. *Suppose that* $B_t = \alpha_1 + \int_0^{\lambda t} e^s \, dV_s$*, where* $\lambda > 0$*,* $\alpha_1, \lambda \in \mathbb{R}$ *are constants, and* $0 \leq t \leq T$*, and* V *is a Lévy subordinator with no deterministic drift. Then*

$$\Phi_{B_t}(\theta) = \exp\left(i\theta\alpha_1 + \int_0^{\lambda t} \kappa_{V_1}(i\theta e^s) \, ds \right), \tag{5.15}$$

where $\kappa_{V_1}(\cdot)$ *is the cumulant generating function for* V_1*. The moments of* B_t *are given by*

$$E(B_t^k) = (-i)^k \tilde{g}_k(0), \quad k = 1, 2, \ldots, \tag{5.16}$$

where

$$\tilde{g}_1(\theta) = i \left(\alpha + \int_0^{\lambda t} e^s \kappa_{V_1}'(i\theta e^s) \, ds \right), \tag{5.17}$$

and

$$\tilde{g}_{k+1}(\theta) = \tilde{g}_1(\theta)\tilde{g}_k(\theta) + \tilde{g}_k'(\theta), \quad k = 1, 2, \ldots. \tag{5.18}$$

In the above formulas prime represents the derivative with respect to the parameter in parenthesis.

Proof. The proof is similar to that of Lemma 4.3.2 and we omit the details here. □

When $V_t = Z_t$, or $V_t = \tilde{Z}_t$, any moment for $B_t = \alpha_1 + \int_0^{\lambda t} e^s \, dZ_s$ (or, $B_t = \alpha_1 + \int_0^{\lambda t} e^s \, d\tilde{Z}_s$) can be obtained in terms of κ_m, $m = 1, 2, \ldots$, which are cumulants of Z_1 (or, in terms of $\tilde{\kappa}_m$, $m = 1, 2, \ldots$, which are cumulants of $\tilde{Z}_1$). For example with $B_t = \alpha_1 + \int_0^{\lambda t} e^s \, dZ_s$,

$$E(B_t) = -i\tilde{g}_1(0) = \alpha_1 + \int_0^{\lambda t} \kappa_1 e^s \, ds = \alpha_1 + \kappa_1(e^{\lambda t} - 1), \tag{5.19}$$

and

$$\begin{aligned} E(B_t^2) = (-i)^2\tilde{g}_2(0) &= -(\tilde{g}_1(0)^2 + \tilde{g}_1'(0)) \\ &= \left(\alpha + \int_0^{\lambda t} \kappa_1 e^s \, ds\right)^2 + \int_0^{\lambda t} \kappa_2 e^{2s} \, ds \\ &= \left(\alpha + \kappa_1(e^{\lambda t} - 1)\right)^2 + \frac{\kappa_2}{2}(e^{2\lambda t} - 1). \end{aligned} \tag{5.20}$$

Similar results hold for $B_t = \alpha_1 + \int_0^{\lambda t} e^s \, d\tilde{Z}_s$, with κ_1 and κ_2 replaced by $\tilde{\kappa}_1$ and $\tilde{\kappa}_2$ respectively.

Before proceeding further we prove a result related to the correlation coefficient of $(\sigma^1)_t^2$ and $(\sigma^2)_t^2$. We show that if Z, $\tilde{Z}$, and Z^* are related by (5.3) then the correlation coefficient of $(\sigma^1)_t^2$ and $(\sigma^2)_t^2$ is independent of t.

Theorem 5.1.4. *Suppose Z, $\tilde{Z}$, and Z^* are related by (5.3). Then the correlation coefficient of $(\sigma^1)_t^2$ and $(\sigma^2)_t^2$ is independent of time and is given by $\rho'\sqrt{\frac{\kappa_2}{\tilde{\kappa}_2}}$, where κ_2 and $\tilde{\kappa}_2$ are second cumulant of Z_1 and $\tilde{Z}_1$ respectively.*

Proof. It is clear from (5.7) and (5.8) that

$$(\sigma^1)_t^2 = e^{-\lambda t}\left((\sigma^1)_0^2 + \int_0^{\lambda t} e^s dZ_s\right), \tag{5.21}$$

and

$$(\sigma^2)_t^2 = e^{-\lambda t}\left((\sigma^2)_0^2 + \int_0^{\lambda t} e^s d\tilde{Z}_s\right). \tag{5.22}$$

Therefore by Lemma 5.1.3, we can obtain

$$\text{Var}((\sigma^1)_t^2) = e^{-2\lambda t}\text{Var}\left(\int_0^{\lambda t} e^s dZ_s\right) = e^{-2\lambda t}\frac{\kappa_2}{2}(e^{2\lambda t}-1), \tag{5.23}$$

and

$$\text{Var}((\sigma^2)_t^2) = e^{-2\lambda t}\text{Var}\left(\int_0^{\lambda t} e^s d\tilde{Z}_s\right) = e^{-2\lambda t}\frac{\tilde{\kappa}_2}{2}(e^{2\lambda t}-1). \tag{5.24}$$

Note that

$$(\sigma^2)_t^2 = \rho'(\sigma^1)_t^2 + e^{-\lambda t}((\sigma^2)_0^2 - \rho'(\sigma^1)_0^2) + \sqrt{1-\rho'^2}\int_0^{\lambda t} e^s\, dZ_s^*. \tag{5.25}$$

Therefore using the independence of Z and Z^*, we obtain

$$\text{Cov}((\sigma^1)_t^2, (\sigma^2)_t^2) = \rho'\text{Var}((\sigma^1)_t^2) = e^{-2\lambda t}\frac{\rho'\kappa_2}{2}(e^{2\lambda t}-1). \tag{5.26}$$

Hence the theorem follows from (5.23), (5.24), and (5.26). □

For proceeding further with the arbitrage free pricing for covariance swap, we assume that for $0 \le t \le T$, $(\sigma^1)_t^2 < \beta_1^2$ and $(\sigma^2)_t^2 < \beta_2^2$, for some $\beta_1, \beta_2 > 0$. We take $\beta = \max\{\beta_1, \beta_2\}$. This is a very reasonable assumption. For example, since $(\sigma^1)_t$ and $(\sigma^2)_t$ are expressed in percentages, for a normal market, where it does not have "crash-like fluctuations" $\beta = 1$ can be assumed. With this definition of β we state the next lemma.

Lemma 5.1.5. *Suppose that $(\sigma^1)_t^2$ and $(\sigma^2)_t^2$ are given by (5.7) and (5.8) respectively, and Z, $\tilde{Z}$ and Z^* are related by (5.3). Let there exists $\beta > 0$ such that for $0 \le t \le T$, $(\sigma^1)_t^2, (\sigma^2)_t^2 < \beta^2$. Then*

$$\begin{aligned} &E\left((\sigma^1)_t(\sigma^2)_t\right) \\ &= e^{-\lambda t}\sum_{k=0}^{\infty}\sum_{p=0}^{k}\sum_{u=0}^{p}\frac{(-1)^{p+1}(2k)!}{4^k(k!)^2(2k-1)}\binom{k}{p}\binom{p}{u}(\beta)^{-4p+2}\rho'^u(1-\rho'^2)^{\frac{(p-u)}{2}}N_t(p,u), \end{aligned} \tag{5.27}$$

where

$$N_t(p,u) = E\left((\sigma^1)_0^2 + \int_0^{\lambda t} e^s dZ_s\right)^{p+u} E\left(\frac{(\sigma^2)_0^2 - \rho'(\sigma^1)_0^2}{\sqrt{1-\rho'^2}} + \int_0^{\lambda t} e^s dZ_s^*\right)^{p-u}, \tag{5.28}$$

can be computed using Lemma 5.1.3.

Proof. It is clear from (5.7) and (5.8) that

$$(\sigma^1)_t^2 = e^{-\lambda t}(\sigma^1)_0^2 + e^{-\lambda t}\int_0^t e^{\lambda s} dZ_{\lambda s} = e^{-\lambda t}\beta^2\left(\frac{(\sigma^1)_0^2 + \int_0^{\lambda t} e^s dZ_s}{\beta^2}\right), \tag{5.29}$$

and

$$\begin{aligned}(\sigma^2)_t^2 &= e^{-\lambda t}(\sigma^2)_0^2 + e^{-\lambda t}\int_0^t e^{\lambda s} d\tilde{Z}_{\lambda s} \\ &= e^{-\lambda t}\beta^2\left(\frac{(\sigma^2)_0^2 + \rho'\int_0^{\lambda t} e^s dZ_s + \sqrt{1-\rho'^2}\int_0^{\lambda t} e^s dZ_s^*}{\beta^2}\right). \end{aligned} \tag{5.30}$$

We denote $F_t = (\sigma^1)_0^2 + \int_0^{\lambda t} e^s dZ_s$, $c' = (\sigma^2)_0^2 - \rho'(\sigma^1)_0^2$ and $G_t = \int_0^{\lambda t} e^s dZ_s^*$ and Therefore we obtain

$$\begin{aligned}(\sigma^1)_t(\sigma^2)_t &= e^{-\lambda t}\beta^2\sqrt{\frac{\rho' F_t^2 + c'F_t + \sqrt{1-\rho'^2}F_tG_t}{\beta^4}} \\ &= e^{-\lambda t}\beta^2\sqrt{1 + \frac{\rho' F_t^2 + c'F_t + \sqrt{1-\rho'^2}F_tG_t - \beta^4}{\beta^4}}. \end{aligned} \tag{5.31}$$

By the construction we have

$$\left|\frac{\rho' F_t^2 + c'F_t + \sqrt{1-\rho'^2}F_tG_t - \beta^4}{\beta^4}\right| < 1. \tag{5.32}$$

Using (4.18) we obtain

$$
\begin{aligned}
(\sigma^1)_t(\sigma^2)_t &= e^{-\lambda t}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}(2k)!}{4^k(k!)^2(2k-1)}\frac{1}{\beta^{4k-2}}\left(\rho' F_t^2 + c'F_t + \sqrt{1-\rho'^2}F_tG_t - \beta^4\right)^k \\
&= e^{-\lambda t}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}(2k)!}{4^k(k!)^2(2k-1)\beta^{4k-2}}\sum_{p=0}^{k}\binom{k}{p}F_t^p\left(\rho' F_t + c' + \sqrt{1-\rho'^2}G_t\right)^p(-\beta)^{4k-4p} \\
&= e^{-\lambda t}\sum_{k=0}^{\infty}\frac{(-1)^{k+1}(2k)!}{4^k(k!)^2(2k-1)}\sum_{p=0}^{k}\binom{k}{p}F_t^p\sum_{u=0}^{p}\binom{p}{u}\rho'^u F_t^u(c' + \sqrt{1-\rho'^2}G_t)^{p-u}(-\beta)^{4k-4p} \\
&= e^{-\lambda t}\sum_{k=0}^{\infty}\sum_{p=0}^{k}\sum_{u=0}^{p}\frac{(-1)^{p+1}(2k)!}{4^k(k!)^2(2k-1)}\binom{k}{p}\binom{p}{u}(\beta)^{-4p+2}\rho'^u(1-\rho'^2)^{\frac{(p-u)}{2}}F_t^{p+u}\tilde{G}_t^{p-u} \qquad (5.33)
\end{aligned}
$$

where $\tilde{G}_t = G_t + \frac{c'}{\sqrt{1-\rho'^2}}$. Since Z and Z^* are independent, we note that F_t and $\tilde{G}_t$ are independent. Therefore

$$
\begin{aligned}
&E\left((\sigma^1)_t(\sigma^2)_t\right) = \\
&= e^{-\lambda t}\sum_{k=0}^{\infty}\sum_{p=0}^{k}\sum_{u=0}^{p}\frac{(-1)^{p+1}(2k)!}{4^k(k!)^2(2k-1)}\binom{k}{p}\binom{p}{u}(\beta)^{-4p+2}\rho'^u(1-\rho'^2)^{\frac{(p-u)}{2}}E\left(F_t^{p+u}\right)E\left(\tilde{G}_t^{p-u}\right). \qquad (5.34)
\end{aligned}
$$

We note that $E(F_t^{p+u})$ in (5.34) can be computed by Lemma 5.1.3 where $\alpha_1 = (\sigma^1)_0^2$ and $V_t = Z_t$. Similarly $E(\tilde{G}_t^{p-u})$ in (5.34) can be computed by Lemma 5.1.3 where $\alpha_1 = \frac{(\sigma^2)_0^2 - \rho'(\sigma^1)_0^2}{\sqrt{1-\rho'^2}}$ and $V_t = Z_t^*$. □

Next we consider the infinite series

$$
e^{-\lambda t}\sum_{k=0}^{\infty}\sum_{p=0}^{k}\sum_{u=0}^{p}\frac{(-1)^{p+1}(2k)!}{4^k(k!)^2(2k-1)}\binom{k}{p}\binom{p}{u}(\beta)^{-4p+2}\rho'^u(1-\rho'^2)^{\frac{(p-u)}{2}}N_t(p,u) \qquad (5.35)
$$

of Lemma 5.1.5. It is clear from the proof of that Lemma that $\frac{\rho' F_t^2 + c'F_t + \sqrt{1-\rho'^2}F_tG_t - \beta^4}{\beta^4} < 0$. It is also clear that $\left|\frac{\rho' F_t^2 + c'F_t + \sqrt{1-\rho'^2}F_tG_t - \beta^4}{\beta^4}\right| < 1$.

Lemma 5.1.6. *The quantity $E((\sigma^1)_t(\sigma^2)_t)$ can be approximated by the n-th partial sum*

$$
e^{-\lambda t}\sum_{k=0}^{n-1}\sum_{p=0}^{k}\sum_{u=0}^{p}\frac{(-1)^{p+1}(2k)!}{4^k(k!)^2(2k-1)}\binom{k}{p}\binom{p}{u}(\beta)^{-4p+2}\rho'^u(1-\rho'^2)^{\frac{(p-u)}{2}}N_t(p,u), \qquad (5.36)
$$

with the absolute error of approximation less than the quantity $e^{-\lambda t}\beta^2 \frac{1}{(2n-1)\sqrt{3n+1}}$, *for* $n \geq 1$.

Proof. Observe that

$$\left| \frac{\rho' F_t^2 + c' F_t + \sqrt{1-\rho'^2} F_t G_t - \beta^4}{\beta^4} \right| < 1, \tag{5.37}$$

and

$$(\sigma^1)_t(\sigma^2)_t = e^{-\lambda t}\beta^2 \sum_{k=0}^{\infty} \frac{(-1)^{k+1}(2k)!}{4^k (k!)^2 (2k-1)} \frac{1}{\beta^{4k}} \left(\rho' F_t^2 + c' F_t + \sqrt{1-\rho'^2} F_t G_t - \beta^4 \right)^k. \tag{5.38}$$

The rest of the proof is similar to the proof of Theorem 4.3.4. □

Remark 5.1.7. *The quantity* β *can be used as a "control parameter" that improves the rate of convergence of the infinite series*

$$e^{-\lambda t} \sum_{k=0}^{\infty} \sum_{p=0}^{k} \sum_{u=0}^{p} \frac{(-1)^{p+1}(2k)!}{4^k (k!)^2 (2k-1)} \binom{k}{p} \binom{p}{u} (\beta)^{-4p+2} \rho'^u (1-\rho'^2)^{\frac{(p-u)}{2}} N_t(p,u). \tag{5.39}$$

Suppose that it is possible to choose β *so that* $\left| \frac{\rho' F_t^2 + c' F_t + \sqrt{1-\rho'^2} F_t G_t - \beta^4}{\beta^4} \right| < \frac{1}{2+\epsilon}$, *for some* $\epsilon > 0$ *and* $0 \leq t \leq T$. *Then a procedure analogous to the proof of Theorem 4.3.5 can be used to show that the quantity* $E((\sigma^1)_t(\sigma^2)_t)$ *can be approximated by the n-th partial sum of the infinite series* (5.39), *with the absolute error of approximation less than the quantity*

$$e^{-\lambda t}\beta^2 \left(\frac{1+\epsilon}{2+\epsilon} \right)^{\frac{1}{2}} \frac{1}{(2n-1)\sqrt{3n+1}} \frac{1}{(1+\epsilon)^n}, \quad \text{for} \quad n \geq 1. \tag{5.40}$$

Also, we note that in this context it is possible to derive a result analogous to Theorem 4.3.6.

Now we prove the main theorem for this section. As in Remark 5.1.2 we denote the Lévy measure for Z by ν_Z.

Theorem 5.1.8. *The arbitrage-free value of a covariance swap is given by*

$$P_{Cov} = e^{-rT} \left(g_1(T) + g_2 - K_{Cov} \right), \tag{5.41}$$

where

$$g_1(T) = \frac{\bar{\rho}}{T}\sum_{k=0}^{\infty}\sum_{p=0}^{k}\sum_{u=0}^{p}\frac{(-1)^{p+1}(2k)!}{4^k(k!)^2(2k-1)}\binom{k}{p}\binom{p}{u}(\beta)^{-4p+2}\rho'^u(1-\rho'^2)^{\frac{(p-u)}{2}}L(p,u), \tag{5.42}$$

where $L(p,u) = \int_0^T e^{-\lambda t}N_t(p,u)\,dt$ *with* $N_t(p,u)$ *given by* (5.28), *and*

$$g_2 = \rho^{(1)}\rho^{(2)}\lambda\kappa_2. \tag{5.43}$$

Proof. It is sufficient to show that under the equivalent martingale measure

$$E(\mathrm{Cov}(S_T^1, S_T^2) = g_1(\lambda, T) + g_2(\lambda, T). \tag{5.44}$$

From (5.14) we obtain

$$\begin{aligned} E(\mathrm{Cov}_R(S^1, S^2)) &= \frac{1}{T}E[X_T^1, X_T^2] \\ &= \frac{1}{T}\int_0^T \bar{\rho}E\left((\sigma^1)_t(\sigma^2)_t\right)\,dt + \frac{\rho^{(1)}\rho^{(2)}}{T}E\left(\int_0^{\lambda T}\int_0^{\infty} y^2 J_Z(ds,dy)\right). \end{aligned} \tag{5.45}$$

Using Lemma 5.1.5 we obtain

$$\begin{aligned} &\frac{1}{T}\int_0^T \bar{\rho}E\left((\sigma^1)_t(\sigma^2)_t\right)\,dt \\ &= \frac{\bar{\rho}}{T}\sum_{k=0}^{\infty}\sum_{p=0}^{k}\sum_{u=0}^{p}\frac{(-1)^{p+1}(2k)!}{4^k(k!)^2(2k-1)}\binom{k}{p}\binom{p}{u}(\beta)^{-4p+2}\rho'^u(1-\rho'^2)^{\frac{(p-u)}{2}}\int_0^T e^{-\lambda t}N_t(p,u)\,dt, \end{aligned} \tag{5.46}$$

where $N_t(p,u)$ is given by (5.28).

Observing $E(J_Z(ds,dy)) = \nu_Z(dy)\,ds$, we obtain

$$\frac{\rho^{(1)}\rho^{(2)}}{T}E\left(\int_0^{\lambda T}\int_{\mathbb{R}} y^2 J_Z(ds,dy)\right) = \rho^{(1)}\rho^{(2)}\lambda\int_0^{\infty} y^2\nu_Z(dy). \tag{5.47}$$

However since Z_t is a subordinator, therefore (see [63], Proposition 3.13) we obtain that $\kappa_2 = \mathrm{Var}(Z_1) = \int_0^{\infty} y^2\nu_Z(dy)$. Hence the theorem is proved. □

5.2. Pricing Correlation Swap

Definition 5.2.1.

We can immediately derive a corollary of Theorem 5.1.8 related to the *correlation swap*. A correlation swap is a forward contract on the correlation between the underlying assets S^1 and S^2 for which payoff at maturity is equal to

$$N(\mathrm{Corr_R}(S^1, S^2) - K_{\mathrm{Corr}}), \tag{5.48}$$

where $\mathrm{K_{Corr}}$ is the strike price, N is the notional amount and $\mathrm{Corr_R}(S^1, S^2)$ is the realized correlation defined by

$$\mathrm{Corr_R}(S^1, S^2) = \frac{\mathrm{Cov_R}(S^1, S^2)}{\sqrt{(\sigma^1)^2_R}\sqrt{(\sigma^2)^2_R}}. \tag{5.49}$$

The arbitrage free value of the correlation swap is given by

$$P_{\mathrm{Corr}} = e^{-rT} E\left(\mathrm{Corr_R}(S^1, S^2) - \mathrm{K_{Corr}}\right). \tag{5.50}$$

Corollary 5.2.2. *Suppose that $(\sigma^1)^2_R$ and $(\sigma^2)^2_R$ are realized variances of S^1 and S^2 respectively over the time interval $[0, T]$. Then the arbitrage-free value of a correlation swap can be approximated by*

$$P_{Corr} \approx e^{-rT}\left(\frac{g_1(T) + g_2}{\sqrt{E(\sigma^1)^2_R}\sqrt{E(\sigma^2)^2_R}} - K_{Corr}\right), \tag{5.51}$$

where $g_1(T)$ and g_2 can be obtained from Theorem 5.1.8, and

$$E(\sigma^1)^2_R = \frac{1}{T}\left(\lambda^{-1}(1 - e^{-\lambda T})(\sigma^1)^2_0 + \kappa_1(T - \lambda^{-1}(1 - e^{-\lambda T}))\right) + \left(\rho^{(1)}\right)^2 \lambda\kappa_2, \tag{5.52}$$

$$E(\sigma^2)^2_R = \frac{1}{T}\left(\lambda^{-1}(1 - e^{-\lambda T})(\sigma^2)^2_0 + \tilde{\kappa}_1(T - \lambda^{-1}(1 - e^{-\lambda T}))\right) + \left(\rho^{(2)}\right)^2 \lambda\kappa_2. \tag{5.53}$$

Proof. As both X_t^1 and X_t^2 are driven by Z, the realized volatility are given by

$$(\sigma^1)^2_R = \frac{1}{T}\int_0^T (\sigma^1)^2_t\, dt + \left(\rho^{(1)}\right)^2 \lambda\kappa_2, \tag{5.54}$$

and

$$(\sigma^2)_R^2 = \frac{1}{T}\int_0^T (\sigma^2)_t^2\, dt + \left(\rho^{(2)}\right)^2 \lambda\kappa_2. \tag{5.55}$$

Thus the expressions of $E(\sigma^1)_R^2$ and $E(\sigma^2)_R^2$ can be obtained from (4.3).

It follows from [50] that

$$E(\mathrm{Corr}_R(S^1, S^2)) \approx \frac{E\left(\mathrm{Cov}_R(S^1, S^2)\right)}{\sqrt{E(\sigma^1)_R^2}\sqrt{E(\sigma^2)_R^2}}. \tag{5.56}$$

Hence the corollary follows from Theorem 5.1.8. □

We conclude the theoretical part of this section by an alternative approximate version of Theorem 5.1.8. For that we need the following lemmas.

Lemma 5.2.3. *Suppose Z, $\tilde{Z}$, and Z^* are related by* (5.3). *For the volatility dynamics* (5.7) *of two assets S_t^1 and S_t^2,*

$$\begin{aligned}
&E\left[(\sigma^1)_t^2(\sigma^2)_t^2\right] \\
&= e^{-2\lambda t}(\sigma^1)_0^2(\sigma^2)_0^2 + \left((\sigma^1)_0^2\sqrt{1-\rho'^2}\kappa_1^* + (\rho'(\sigma^1)_0^2 + (\sigma^2)_0^2)\kappa_1\right) e^{-2\lambda t}(e^{\lambda t}-1) \\
&+ e^{-2\lambda t}\left(\rho'\left(\kappa_1^2(e^{\lambda t}-1)^2 + \frac{\kappa_2}{2}(e^{2\lambda t}-1)\right) + \sqrt{1-\rho'^2}\kappa_1\kappa_1^*(e^{\lambda t}-1)^2\right),
\end{aligned} \tag{5.57}$$

and where κ_1 and κ_2 are first two cumulants of Z_1, and κ_1^* is the first cumulant of Z_1^*.

Proof. It is clear from (5.7) and (5.8) that

$$\begin{aligned}
(\sigma^1)_t^2(\sigma^2)_t^2 = e^{-2\lambda t}(\sigma^1)_0^2(\sigma^2)_0^2 + e^{-2\lambda t}\left((\sigma^1)_0^2\int_0^t e^{\lambda s} d\tilde{Z}_{\lambda s} + (\sigma^2)_0^2\int_0^t e^{\lambda s} dZ_{\lambda s}\right) \\
+ e^{-2\lambda t}\int_0^t e^{\lambda s} dZ_{\lambda s}\int_0^t e^{\lambda s} d\tilde{Z}_{\lambda s}.
\end{aligned} \tag{5.58}$$

Note that the first cumulant of $\tilde{Z}_1$ (denoted as $\tilde{\kappa}_1$) is related to those of Z_1 and Z_1^* by the simple relation $\tilde{\kappa}_1 = \rho'\kappa_1 + \sqrt{1-\rho'^2}\kappa_1^*$. Therefore we observe

$$E\left(\int_0^t e^{\lambda s} dZ_{\lambda s}\right) = \kappa_1(e^{\lambda t}-1), \tag{5.59}$$

and

$$E\left(\int_0^t e^{\lambda s} d\tilde{Z}_{\lambda s}\right) = (\rho' \kappa_1 + \sqrt{1-\rho'^2}\kappa_1^*)(e^{\lambda t} - 1). \tag{5.60}$$

Using (5.3) and the independence of Lévy processes Z and Z^*, we obtain

$$\begin{aligned}
&E\left(\int_0^t e^{\lambda s} dZ_{\lambda s} \int_0^t e^{\lambda s} d\tilde{Z}_{\lambda s}\right) \\
&= \rho' E\left(\int_0^t e^{\lambda s} dZ_{\lambda s}\right)^2 + \sqrt{1-\rho'^2} E\left(\int_0^t e^{\lambda s} dZ_{\lambda s}\right) E\left(\int_0^t e^{\lambda s} dZ^*_{\lambda s}\right) \\
&= \rho' \left(\kappa_1^2 (e^{\lambda t} - 1)^2 + \frac{\kappa_2}{2}(e^{2\lambda t} - 1)\right) + \sqrt{1-\rho'^2}\kappa_1 \kappa_1^* (e^{\lambda t} - 1)^2. \qquad (5.61)
\end{aligned}$$

Hence (5.57) follows from (5.58) and the above results. □

To keep track of constants for the next computations, we define the followings:

$$a_1 = (\sigma^1)_0^2 (\sigma^2)_0^2, \tag{5.62}$$

$$a_2 = (\sigma^1)_0^2 \sqrt{1-\rho'^2}, \tag{5.63}$$

$$a_3 = (\sigma^2)_0^2 + \rho' (\sigma^1)_0^2, \tag{5.64}$$

$$a_4 = \rho', \tag{5.65}$$

and

$$a_5 = \sqrt{1-\rho'^2}. \tag{5.66}$$

Using Lemma 5.1.3 we can compute the following:

$$E\left(\int_0^t e^{\lambda s} dZ_{\lambda s}\right)^3 = \kappa_1^3 (e^{\lambda t} - 1)^3 + \frac{3\kappa_1\kappa_2}{2}(e^{\lambda t} - 1)(e^{2\lambda t} - 1) + \frac{\kappa_3}{3}(e^{3\lambda t} - 1), \tag{5.67}$$

and

$$E\left(\int_0^t e^{\lambda s} dZ_{\lambda s}\right)^4 = \kappa_1^4(e^{\lambda t}-1)^4 + 3\kappa_1^2\kappa_2(e^{\lambda t}-1)^2(e^{2\lambda t}-1)$$
$$+\frac{3}{4}\kappa_2^2(e^{2\lambda t}-1)^2 + \frac{4}{3}\kappa_1\kappa_3(e^{\lambda t}-1)(e^{3\lambda t}-1) + \frac{\kappa_4}{4}(e^{4\lambda t}-1). \tag{5.68}$$

Let us define $X_{1t} = \int_0^t e^{\lambda s} dZ_{\lambda s}$ and $X_{2t} = \int_0^t e^{\lambda s} dZ^*_{\lambda s}$. For notational convenience we will simply write $X_{1t} = X_1$ and $X_{2t} = X_2$. Clearly X_1 and X_2 are independent. Thus from (5.58) it can be easily shown that

$$\begin{aligned}\mathrm{Var}\left[(\sigma^1)_t^2(\sigma^2)_t^2\right] &= \mathrm{Var}(a_1 + a_2X_2 + a_3X_1 + a_4X_1^2 + a_5X_1X_2)\\ &= a_2^2\mathrm{Var}(X_2) + a_3^2\mathrm{Var}(X_1) + a_4^2\mathrm{Var}(X_1^2) + a_5^2\mathrm{Var}(X_1X_2)\\ &+ 2a_2a_5\mathrm{Cov}(X_2, X_1X_2) + 2a_3a_4\mathrm{Cov}(X_1, X_1^2)\\ &+ 2a_3a_5\mathrm{Cov}(X_1, X_1X_2) + 2a_4a_5\mathrm{Cov}(X_1^2, X_1X_2).\end{aligned} \tag{5.69}$$

Using the independence of X_1 and X_2, it is easy to show

$$\mathrm{Var}(X_1^2) = E(X_1^4) - (E(X_1^2))^2, \tag{5.70}$$

$$\mathrm{Var}(X_1X_2) = E(X_1^2)E(X_2^2) - (E(X_1))^2(E(X_2))^2, \tag{5.71}$$

$$\mathrm{Cov}(X_2, X_1X_2) = E(X_1)\mathrm{Var}(X_2), \tag{5.72}$$

$$\mathrm{Cov}(X_1, X_1^2) = E(X_1^3) - E(X_1^2)E(X_1), \tag{5.73}$$

$$\mathrm{Cov}(X_1, X_1X_2) = E(X_2)\mathrm{Var}(X_1), \tag{5.74}$$

$$\mathrm{Cov}(X_1^2, X_1X_2) = E(X_2)(E(X_1^3) - E(X_1^2)E(X_1)). \tag{5.75}$$

Clearly,

$$E(X_1) = \kappa_1(e^{\lambda t} - 1), \quad E(X_2) = \kappa_1^*(e^{\lambda t} - 1), \tag{5.76}$$

$$E(X_1^2) = \kappa_1^2(e^{\lambda t} - 1)^2 + \frac{\kappa_2}{2}(e^{2\lambda t} - 1), \quad E(X_2^2) = (\kappa_1^*)^2(e^{\lambda t} - 1)^2 + \frac{\kappa_2^*}{2}(e^{2\lambda t} - 1), \tag{5.77}$$

and

$$\mathrm{Var}(X_1) = \frac{\kappa_2}{2}(e^{2\lambda t} - 1), \quad \mathrm{Var}(X_2) = \frac{\kappa_2^*}{2}(e^{2\lambda t} - 1). \tag{5.78}$$

The quantities $E(X_1^3)$ and $E(X_1^4)$ can be computed using (5.67) and (5.68) respectively. Thus (5.69) can be used to construct the variance of $(\sigma^1)_t^2(\sigma^2)_t^2$.

We conclude the section with following approximation theorem.

Theorem 5.2.4. *The arbitrage-free value of a covariance swap is given by*

$$P_{Cov} \approx e^{-rT}\left(g_3(T) + g_2 - K_{Cov}\right), \tag{5.79}$$

where $g_2 = \rho^{(1)}\rho^{(2)}\lambda\kappa_2$, and

$$g_3(T) \approx \frac{\bar{\rho}}{T}\int_0^T \left(\sqrt{E[(\sigma^1)_t^2(\sigma^2)_t^2]} - \frac{Var[(\sigma^1)_t^2(\sigma^2)_t^2]}{8(E[(\sigma^1)_t^2(\sigma^2)_t^2])^{3/2}}\right) dt, \tag{5.80}$$

where $E[(\sigma^1)_t^2(\sigma^2)_t^2)]$ can be computed using Lemma 5.2.3 and $Var[(\sigma^1)_t^2(\sigma^2)_t^2)]$ can be obtained using (5.69).

Proof. We observe that,

$$E\left((\sigma^1)_t(\sigma^2)_t\right) \approx \sqrt{E[(\sigma^1)_t^2(\sigma^2)_t^2]} - \frac{\mathrm{Var}[(\sigma^1)_t^2(\sigma^2)_t^2]}{8(E[(\sigma^1)_t^2(\sigma^2)_t^2])^{3/2}}. \tag{5.81}$$

The rest of the proof follows from Theorem 5.1.8. □

5.3. Model Fitting and Parameter Estimate

We use the stock prices of S&P500 index and NASDAQ during the time period 10/01/2010 through 01/15/2015. The data set has 1080 closing daily stock prices, and these are used for the computation of realized covariance. For the numerical simulation we take $\rho^{(1)} = \rho^{(2)} = -1$. From the regression fit model for non linear least square estimate we find that the α level of the parameters for all cases are significantly less than 0.05. Therefore this tells us all the parameters are significantly important to estimate the realized covariance swap price. Once calibration is performed over the described historical data set, we obtain the model parameters and these parameters can be used to price the fixed leg (fair delivery price) of the covariance swap. The calibration results with the application of Theorem 5.1.8 for covariance swap for various cases are shown in Tables 5.1, 5.2 and 5.3 and the corresponding fittings are shown in Figure 5.1. Moreover Figure 5.2 deals with the error and rate of convergence for theorem (5.1.8) and it clearly shows how fast the convergence existst as $n \to \infty$.

For Table 5.1 it is assumed that for the unconditional distribution $(\sigma^1)_t^2 \sim \Gamma(\nu_1, \alpha_1)$ and $(\sigma^2)_t^2 \sim \Gamma(\nu_2, \alpha_2)$. For Table 5.2 it is assumed that for the unconditional distribution $(\sigma^1)_t^2 \sim IG(\delta_1, \gamma_1)$ and $(\sigma^2)_t^2 \sim IG(\delta_2, \gamma_2)$. Finally for Table 5.3 it is assumed that for the unconditional distribution $(\sigma^1)_t^2 \sim PTS(\kappa_1, \delta_1, \gamma_1)$ and $(\sigma^2)_t^2 \sim PTS(\kappa_2, \delta_2, \gamma_2)$.

It is clear that the expression in Theorem 5.1.8 is a formal asymptotic expansion. However, Remark 5.1.7 gives a control over the convergence of the infinite series. This can also be used in practice to demonstrate the reliability of the approximation. A similar argument as in the case of volatility swap and Remark 5.1.7 show that in order to have rapid convergence of the infinite series in context, the quantity $\max_{0<T\leq T_{\max}} |n_1 \text{ th partial sum} - n_2 \text{ th partial sum}|$ must be reasonably small as n_1 and n_2 increase. In the following Figure 6, we take $n_2 = N$ and $n_1 = N + 1$, to demonstrate the rapid convergence of the expression $\max_{0<T\leq T_{\max}} |(N+1)\text{th partial sum} - N \text{ th partial sum}|$ for various N. These results show the numerical evidence of reliability of the procedure.

5.4. Conclusion

In this Chapter we have presented a new extended approach based on the BN-S model to obtain the arbitrage-free pricing for covariance and correlation swaps for financial markets. We used the S&P500 index and NASDAQ for numerical purposes and model fitting. Covariance and correlation swap are important in minimizing the risk that might come from different currencies. We use an OU process driven by a subordinator.

The stochastic volatility models that are used for analysis are empirically reasonable as well as having many appealing features from a theoretical finance perspective. The results derived in this dissertation are potentially important as this means that stochastic volatility models built out of OU processes with gamma or inverse Gaussian or positive tempered stable marginals have excellent numerical accuracy in obtaining the fair delivery prices for various swaps. Further, we get closed form pricing formulas depending on various cumulants of the BDLP Lévy process Z. In this dissertation we have also derived an algorithmic process to compute the cumulants of Z. The improvement of numerical results in the analysis is very significant over the existing models with a similar number of parameters, such as the Heston model.

More generally, the results obtained in this dissertation have important implications for their use in, for example, energy markets. Energy is the most important commodity sector. Crude oil and natural gas are one of the most liquid option markets among all commodities. Since Crude oil and natural gas have the properties of mean reverting,which means that they tend to return over time to the long term average.So it is important to estimate or model the direction of price of those types of commodites. Varianceor volatility risk premia for energy commodities, crude oil and natural gas, is becoming increasingly popular and the approach considered in this dissertation can be further developed to analyze such markets. Moreover, the idea used in this dissertation can be generalized for the analysis of covariance and correlation risk of two commodities of an energy market. These aspects will be developed in future works.

Table 5.1. Parameter estimates for Variance Gamma Covariance swap

Parameters	Estimate	Standard Error	t Value	$Pr(>\|t\|)$
λ	0.15247	0.03418	4.460	9.05e-06
ν_1	65.01301	25.08776	2.591	0.009688
ν_2	65.01301	25.08776	2.591	0.009688
$\bar{\rho}$	-0.1415	0.0215	3.219	$2e-8$
ρ'	0.038901	0.0014	214.913	$< 2e-16$
α_1	33.99290	10.25798	3.314	0.000951
α_2	19.72202	4.64716	4.244	$2.39e-05$

Table 5.2. Parameter estimates for Inverse Gaussian covariance swap

Parameters	Estimate	Standard Error	t Value	$Pr(>\|t\|)$
λ	0.15247	0.03418	4.460	$9.05e-06$
δ_1	11.15092	2.62873	4.242	$2.41e-05$
δ_2	11.15092	2.62873	4.242	$2.41e-05$
$\bar{\rho}$	-0.38901	0.0215	3.219	$2e-8$
ρ'	0.38901	0.0014	214.913	$< 2e-16$
γ_1	5.83038	0.87971	6.628	$5.39e-11$
γ_2	3.38266	0.30425	11.118	$< 2e-16$

Table 5.3. Parameter estimates for Positive Tempered Stable covariance swap

Parameters	Estimate	Standard Error	t Value	$Pr(>\|t\|)$
inserts single horizontal line λ	0.000644975	$6.954e-6$	-92.75	$< 2e-16$
$\bar{\rho}$	91.9824	$1.4e-6$	214.913	$< 2e-16$
ρ'	2.46078	$9.708e-2$	-17.77	$< 2e-16$
κ_1	0.508837	2.62873	4.242	$< 2e-16$
κ_2	0.508837	$2.53e-03$	1166.18	$2e-16$
δ_1	3.06699	$2.53e-03$	1166.18	$2e-16$
δ_2	3.06699	$2.53e-03$	1166.18	$2e-16$
γ_1	1.5000	$2.148e-03$	793.16	$< 2e-16$
γ_2	1.98114	0.00030425	212.861	$< 2e-16$

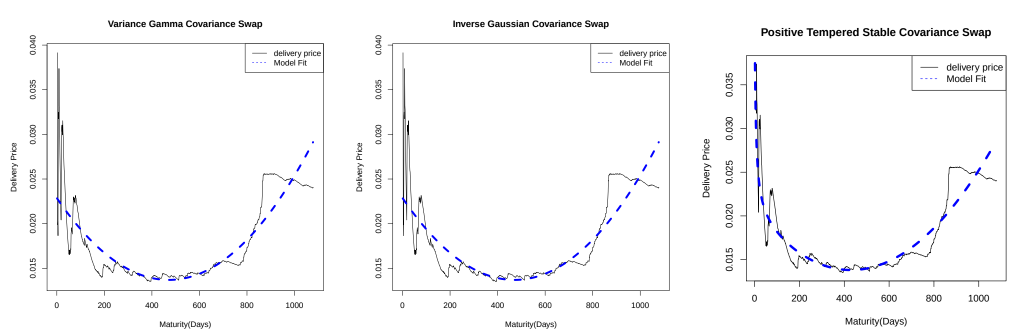

Figure 5.1. Fitting of covariance swap

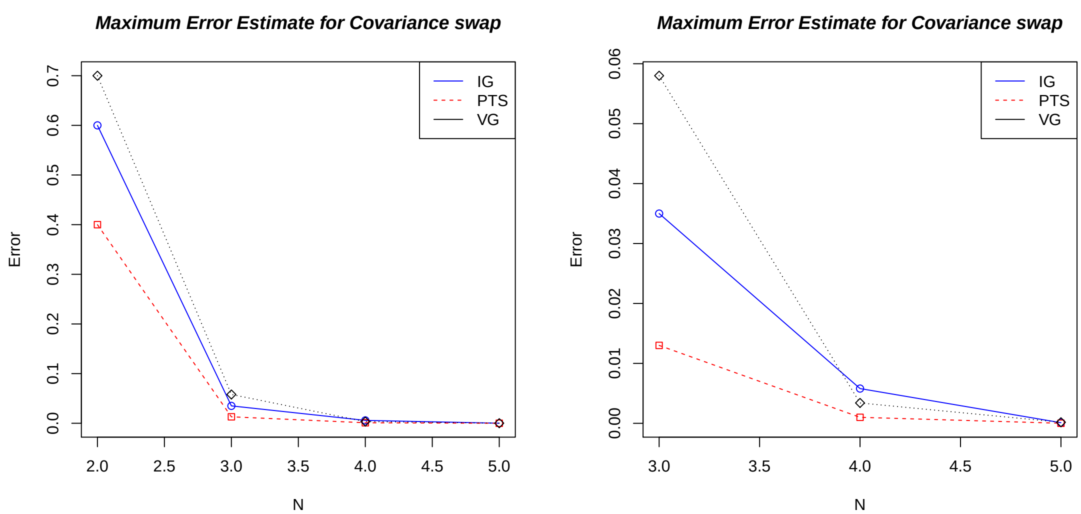

Figure 5.2. (Left) Reliability of fitting with truncated infinite series for covariance swap. (Right) Zoomed in version of the left picture.

6. ORNSTEIN-UHLENBECK PROCESS FOR GEOPHYSICAL DATA ANALYSIS

6.1. Introduction

Earthquake occurs in a region where there is a multiscale networks or systems that are driven by an external forces arising from plate tectonic motions. In the study of the relations between seismic dynamics and underlying physical processes, a major role is played by the question of how the magnitude rates evolve with time and if it is at all possible to have a priori estimate of big jumps in the magnitude rate. Recent researchers showed a renewed interest in modeling the earthquake. But still there is an argument whether earthquake is a deterministic or stochastic. OU Process has an interesting futures in detecting random jumps followed by a mean reverting. As the historical data reveals that earthquake is a stochastic which is not random, here we are going to use one of a financial model to estimate the future major earthquakes. The main objective is to explore and develop mathematical and computation related to estimation of earthquake. The models are complex and new numerical methods need to be devised for solving these problems. We have been working on related subjects like critical phenomena modeling, and more recently we have embarked to work in seismologic events modeling. In earthquake studies, the primary tool which describes the earthquake signal is the ground acceleration signal recorded using a seismometer. Seismometers placed at different locations will record the signal differently, depending on the distance from the epicenter of the earthquake and the soil composition between the epicenter and the location of recording. The earthquake signal exhibited clear stochastic volatility behavior. This led us to believe that the tools we develop in this area will have great importance for the field.

In [34] the authors analyze the signal recorded for the Parkfield county California earthquake of September 28, 2004. The magnitude of this earthquake was 6.0 on the Richter scale. The author analyzed data from two different stations: Red Hills and Donna Lee, situated approximately 20 miles apart. In the top left of Figure 6.1 a plot of the raw acceleration signal recorded is provided. An application of the technique of estimating the variability of the signal using a Markov chain gives the variability (stochastic volatility) estimate in bottom left of Figure 6.1. The two signals are plotted using the same scales and clearly the one recorded further away bears little resemblance with the one recorded closer to the epicenter. An application of the technique of estimating the variability of the signal using a Markov chain gives the variability (stochastic volatility) estimate in the right side of Figure 6.1.

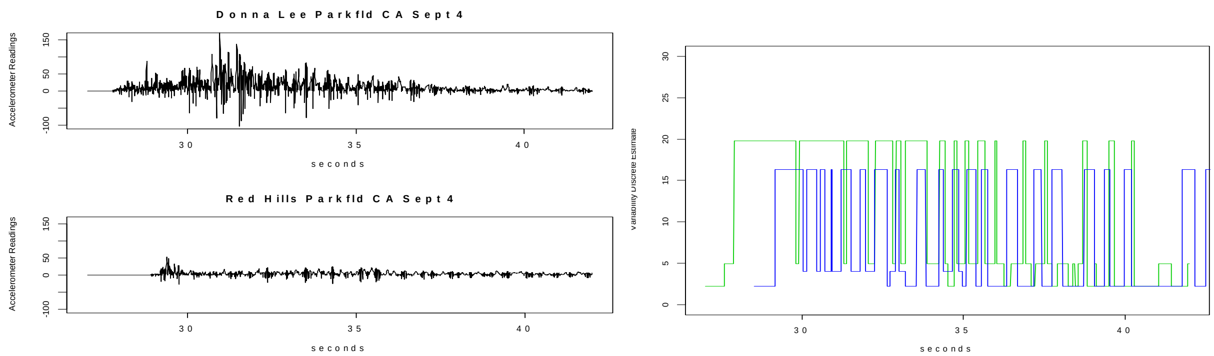

Figure 6.1. Images comparing the raw signal and the variability estimates for two locations. The Red Hills variability is trailing the Donna Lee variability by about 1.3 seconds

A further extension is to use more general Lévy flight dynamics for the earthquake signal. In the study of relations between seismic dynamics and underlying physical processes, a major role is played by the question of how the *magnitude* rates evolve with time and if it is at all possible to have an a priori estimate of big jumps in the magnitude rate. This is highlighted in the fault interaction model [58] which is based on the hypothesis that small and sudden stress changes cause large changes in seismic rate. Several statistical methods have been developed to serve this purpose. The general framework assumes that the temporal dynamics on a closed time interval of geophysical processes is fully described by the expected numbers of events in the time period. In [24] the author used Gamma-Ornstein–Uhlenbeck process to capture major earthquakes in certain region of California.

In a recent work [38] scale invariant functions and stochastic Lévy models are applied to geophysical data and it is shown that a pattern arises from the scale invariance property and Lévy flight models that may be used to estimate parameters related to some major earthquakes. Modern literature also uses the generalized *Omori law* and *ETAS* model (Epidemic-Type aftershock sequence) for quantitative statistic modeling of seismic regime [40, 46, 48]. However, it appears that there is a major drawback with the procedure described in the previous works. Firstly, it is now well accepted that an earthquake model is *not* a deterministic one and hence the deterministic models such as scale-invariance technique may not be a suitable model.

However, it appears that there is a major drawback with the procedure described in the previous works. Firstly, it is now well accepted that an earthquake model is *not* a deterministic one and hence the deterministic models such as scale-invariance technique may not be a suitable model. For the existing stochastic models there is no concrete theory that models the data point and estimate the earthquake data from a separate estimation depending on the model parameters. For example, in [37, 38] earthquake estimation is solely dependent on the data itself and the first *outlier* data gives an estimation. This method is good for some geographical regions but fails significantly for most of the other regions. The possible reasons are: (i) those models do not take into account the *physical* behavior of the earthquake data, and (ii) all the previous models are not completely stochastic.

In this proposal we propose a model of earthquake data based on a completely stochastic process. The model process will depend on some parameters which will be dependent on a particular time frame and a specific geographical location. One of the most important features of such modeling is the *stationary*. This means the magnitude process of a geophysical event has invariant statistical distributions for different temporal non-overlapping ranges of the same size. A stochastic process $(X_t)_{t\geq 0}$ has *stationary increments* if the *law* of $X_{t+h}-X_t$, with $h>0$, does not depend on t. This means that the statistical description of the process over a closed interval of time is invariant with respect to shifts of the starting time provided the same length of time interval is considered. Stationary is used in literature for modeling different geophysical events (see [26, 40, 46, 64] and references therein). For a particular earthquake prone region if the time series of data points (magnitude) of an earthquake are joined with lines then the following properties are clear for any earthquake data:

1. Magnitude is a non-negative stationary stochastic process.
2. For any finite interval of time there are only finite number of jumps.
3. The sample path of magnitude of earthquake consists of upward jumps (significant earthquake) and gradual decrease (aftershocks).

The purpose of this proposal is to use non-Gaussian Ornstein-Uhlenbeck (OU) processes to model the magnitude process of earthquake. Non-Gaussian processes of OU type have considerable potential as building-blocks for different stochastic models of observational time series from a variety of fields. They offer the possibility of capturing important distributional deviations from Gaussianity and provide flexibility in modeling dependence structure.

Most common Non-Gaussian OU processes in literature are Gamma-OU and Inverse Gaussian OU processes. For Inverse Gaussian OU process, it jumps infinitely often in every interval of time and hence it may not be a good candidate for modeling geophysical data. However, Gamma OU process satisfies all the three aforementioned criteria. This proposal aims at developing Gamma-OU process and its modifications in relation with the geophysical data, drawing on and extending powerful results from probability theory for applications in stochastic computations. This analysis will eventually lead to the estimation of future major earthquakes.

6.2. Ornstein-Uhlenbeck Processes for Geophysical Modeling

When Lévy processes are used as driving noise it is possible to construct a large family of *mean-reverting* jump processes with linear dynamics on which various properties such as positiveness or the choice of marginal distribution, can be imposed. We consider continuous time stationary and non-negative processes which are defined by the following stochastic differential equation

$$dM_t = -\lambda M_t \, dt + dZ_{\lambda t}, \quad M_0 > 0, \tag{6.1}$$

where the process Z_t is a subordinator- that is, it is a Lévy process with no Gaussian component and positive increments. The rate parameter λ is arbitrary positive and Z_t is called the *Background Driving Lévy Process* (BDLP). The unusual timing $Z_{\lambda t}$ is deliberately chosen so that it will turn out that whatever the value of λ the marginal distribution of (M_t) will be unchanged. We call the process $M = (M_t)_{t\geq 0}$ to be *Ornstein-Uhlenbeck* (OU) process.

The solution of (6.1) can be explicitly written as

$$M_t = e^{(-\lambda t)} M_0 + \int_0^t \exp\left(-\lambda(t-s)\right) \, dZ_{\lambda s}, \tag{6.2}$$

which can also be written as $M_t = e^{-\lambda t} M_0 + e^{-\lambda t} \int_0^{\lambda t} e^s \, dZ_s$. As Z is an increasing process and

$M_0 > 0$, the process (M_t) is strictly positive and it is bounded from below by the deterministic function $\exp(-\lambda t)M_0$. Since the process Z_t has positive increments and no drift, (M_t) moves up entirely by jumps and then tails off exponentially. Since equation (6.2) is a driven by a subordinator we can apply the definition of (3.0.7) and theorems of (3.0.8) and (3.0.9). We can find the log cumulant function of M_t and Z_1 respectively by the identity we saw on chapter 3.

It is clear from [51] (Theorem 17.5(ii)) that for any self-decomposable law D there exists a Lévy process Z such that the process of OU type driven by Z has invariant distribution given by D. If $\kappa^D(\theta) = \log E[e^{\theta D}]$, and $\kappa(\theta)$ is the cumulant transform for Z_1, then it is well known (see [3, 6]) that they are related by the fundamental equality

$$\kappa^D(\theta) = \int_0^\infty \kappa(\theta \exp(-s))ds. \tag{6.3}$$

This can be expressed as

$$\kappa(\theta) = \theta \frac{d\kappa^D}{d\theta}(\theta). \tag{6.4}$$

To prove the above relation let $u = \theta e^{-s}$ then $du = -uds$ hence the above equation (6.4) can be derived as

$$\begin{aligned}
\kappa^D(\theta) &= \int_0^\infty \kappa(\theta \exp(-s))ds \\
&= -\int_\theta^0 \frac{\kappa(u)}{u} du, \\
\kappa(\theta) &= \theta \frac{d\kappa^D}{d\theta}(\theta)
\end{aligned} \tag{6.5}$$

Lemma 6.2.1. *If M_t follows a $\Gamma(\nu, \alpha)$ then its cumulative generating function is given by*

$$\kappa^{M_t}(\theta) = \nu \log\left(\frac{\alpha}{\alpha - \theta}\right) \tag{6.6}$$

Proof. We refer you to (3.0.12).

□

From the above lemma it is easy to see that the cumulative function of Z_t is given by

$$\kappa(\theta) = \frac{\nu\theta}{\alpha - \theta}. \tag{6.7}$$

It has been know that $\Gamma(\nu, \alpha)$ is an infinite divisible which makes it easier to find its Lévy density which is given by $u(x) = \nu x^{-1} e^{-\alpha x}$. Using theorem (3.0.9), we can calculate the Lévy density function of the BDLP Z and is

$$w(x) = \alpha\nu e^{-\alpha x}. \tag{6.8}$$

It is clear from (Schoutens [52]) the BDLP for the $\Gamma(\nu, \alpha)$-OU process is a *compound Poisson process*

$$Z_t = \sum_{n=1}^{N_t} x_n, \tag{6.9}$$

where $(N_t)_{t\geq 0}$ is a Poisson process with $E(N_t) = \nu t$ and $\{x_n, n = 1, 2, \dots\}$ is an independent and identically distributed sequence with each x_n has $\Gamma(1, \alpha)$ (that is $Exp(\alpha)$) law. From the properties of the compound Poisson process it is clear that the stochastic process $(M_t)_{t\geq 0}$ has many desired properties such as (see [63]):

- The process Z (which is the BDLP of M) has cádlág piece wise constant functions.
- The jump times of Z (and thus for M) have the same law as the jump times of the Poisson process N_t.
- Z_t has the characteristic function given by

$$E(e^{iuZ_t}) = \exp\left(t\nu \int_0^\infty (e^{iux} - 1)\alpha e^{-\alpha x}\, dx\right), \quad u \in \mathbb{R}. \tag{6.10}$$

The stochastic process (M_t) which is defined above BDLP is called the Gamma-OU process. Since the BDLP is compound Poisson process, on any compact time interval it jumps a finite number of times.Immediately this will follow that the Gamma-OU process (M_t) also jumps a *finite* number of times in every compact time-interval as it is driven by BDLP. Since a selected region of interest have finite number of major earthquakes, We need a model which has a finite number of jumps on a certain compact interval. That is why we choose Gamma-OU process over Inverse Gaussian-OU process as the later jumps infinitely over any compact time interval.

6.3. Computation of Characteristic Function

Assume that the magnitude of the earthquake data is given by $M = (M_t)_{t\geq 0}$ (defined in (6.1) or (6.2)) on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F})_{0\leq t\leq T}, P)$. Estimation of a major event in a future time depends on the analysis of the conditional distribution of the magnitude process M_T given the information up to a certain time t_0. In practice t_0 is taken to be the time till which the magnitude data for the geophysical process is available and T is taken to be some reasonable time after t_0 on which the estimation of future *major event* is computed. The characteristic function for M_T given the information up to time $t_0 \leq T$ is given by $E[\exp(iuM_T)|\mathcal{F}_{t_0}]$, where $u \in \mathbb{R}$. The conditional distribution of M_T given the information up to time $t_0 \leq T$ will be denoted by $M_{T|t_0}$. The characteristic function of a stochastic process completely characterizes its law. That is, two stochastic processes with the same characteristic function are identically distributed. A characteristic function is always continuous and its value is unity when $u = 0$. Additional smoothness properties of the characteristic function depend on the existence of the statistical *moments* of the stochastic process.

We need the following theorem to attain the main result of this section. We quote this result from [42].

Theorem 6.3.1. *Let Z be a subordinator with cumulant transform κ and let $f : \mathbb{R}^+ \to \mathbb{C}$ be a complex-valued, left continuous function such that $\Re(f) \leq 0$. Then*

$$E\left[\exp\left(\int_0^t f(s)\,dZ_{\lambda s}\right)\right] = \exp\left(\lambda\int_0^t \kappa(f(s))\,ds\right). \tag{6.11}$$

We can now compute the characteristic function of the stochastic process (M_t) given by (6.1) or equivalently by (6.2).

Theorem 6.3.2. *Consider the OU-process given by* (6.1). *Then the characteristic function $\phi(u) =$*

$E[\exp(iuM_T)|\mathcal{F}_{t_0}]$ *of* $M_{T|t_0}$ *is given by*

$$\phi(u) = \exp\left[iuM_{t_0}e^{-\lambda T} + \lambda \int_{t_0}^{T} \kappa\left(iue^{-\lambda(T-s)}\right) ds\right], \tag{6.12}$$

where $\kappa(\cdot)$ *is the cumulant transform for* Z*.*

Proof. Substituting equation (6.2) and using the above theorem (6.3.1) we get and $\Re\left(iue^{-\lambda(T-s)}\right) = 0$ we get

$$\begin{aligned}
\phi(u) &= E[\exp(iuM_T)|\mathcal{F}_{t_0}] \\
&= E\left[\exp\left(iu\left(\exp(-\lambda T)M_{t_0} + \int_{t_0}^{T} \exp\left(-\lambda(T-s)\right) dZ_{\lambda s}\right)\right)|\mathcal{F}_{t_0}\right] \\
&= \exp\left(iue^{-\lambda T}M_{t_0}\right) E\left[\exp\left(\int_{t_0}^{T} iue^{-\lambda(T-s)}\, dZ_{\lambda s}\right)\right] \\
&= \exp\left[iuM_{t_0}e^{-\lambda T} + \lambda \int_{t_0}^{T} \kappa\left(iue^{-\lambda(T-s)}\right) ds\right]. \qquad (6.13)
\end{aligned}$$

□

It is easy to see that equation (6.12) is true for $u = 0$. Thus, statistically, all moments of $M_{T|t_0}$ exist. Also, the probability density of the process $M_{T|t_0}$ can be recovered from a straightforward generalization of $\phi(u)$ to *Laplace transform*. This is similar to the computations in [42]. We now compute the characteristic function for M when it is Gamma-OU process. Using (6.7) we obtain

$$\begin{aligned}
\int_{t_0}^{T} \kappa\left(iue^{-\lambda(T-s)}\right) ds &= \int_{t_0}^{T} \frac{i\nu ue^{-\lambda(T-s)}}{\alpha - iue^{-\lambda(T-s)}} ds \\
&= -\frac{\nu}{2\lambda}[2\lambda(T-t_0) + \log(\alpha^2 + u^2) - \log\left(\alpha^2 e^{2\lambda(T-t_0)} + u^2\right) \\
&\quad + 2i\left(\arctan\left(\frac{\alpha}{u}\right) - \arctan\left(\frac{\alpha e^{\lambda(T-t_0)}}{u}\right)\right)]. \qquad (6.14)
\end{aligned}$$

Thus for Gamma-OU process substituting (6.14) into theorem (6.12) gives a closed form solution of the characteristic function which is given by

$$\phi(u) = \exp[-\frac{\nu}{2}\left[2\lambda(T-t_0) + \log(\alpha^2+u^2) - \log\left(\alpha^2 e^{2\lambda(T-t_0)} + u^2\right)\right]$$
$$+ i\left[uM_{t_0}e^{-\lambda T} - \nu\arctan\left(\frac{\alpha}{u}\right) + \nu\arctan\left(\frac{\alpha e^{\lambda(T-t_0)}}{u}\right)\right]]. \tag{6.15}$$

Thus

$$|\phi(u)| = \exp\left[-\frac{\nu}{2}\left[2\lambda(T-t_0) + \log(\alpha^2+u^2) - \log\left(\alpha^2 e^{2\lambda(T-t_0)} + u^2\right)\right]\right]$$
$$= e^{-\nu\lambda(T-t_0)}(\alpha^2+u^2)^{-\frac{\nu}{2}}\left(\alpha^2 e^{2\lambda(T-t_0)} + u^2\right)^{\frac{\nu}{2}}$$
$$= \left(\frac{\alpha^2 + u^2 e^{-2\lambda(T-t_0)}}{\alpha^2+u^2}\right)^{\frac{\nu}{2}}. \tag{6.16}$$

For the regression analysis in Section 6.5 this theoretical $|\phi(u)| \in \mathbb{R}$ for Gamma-OU processes are fitted with the absolute value of the characteristic function computed from the observed data to estimate the appropriate values of α, ν and λ.

6.4. Analysis of the First-Passage Time

In this section we will give some definition of the exit time that will help us in the earthquake data modeling with a Gamma-OU process. The first passage time or exit time has many interesting applications in different area such as mathematical finance, dam theory reliability analysis, etc. Here we give the formal definition stopping time.

Definition 6.4.1. *If U is an open subset of $\mathbb{R}$ then the first passage time of a stochastic process $(X_t)_{t\geq 0}$ is defined as*

$$\tau_U = \inf\{t > 0 : X_t \notin U\}. \tag{6.17}$$

Thus for a given geophysical data which is modeled by $(M_t)_{t\geq 0}$, it is important to approximate the distribution of the first passage time or exit time of a major earthquake. The first passage time has some interesting futures, for example the first passage time is *stopping time* (or *Markov time*) with respect to the filtration of the given stochastic process. Also exit time are also used *harmonic measure* and *hitting distribution*,for detailed we refer the reader to [22, 47].

If b is a threshold of the magnitude above which earthquakes can be attributed as "major" or "devastating" for a certain geographical region, then it is important to understand the exit time

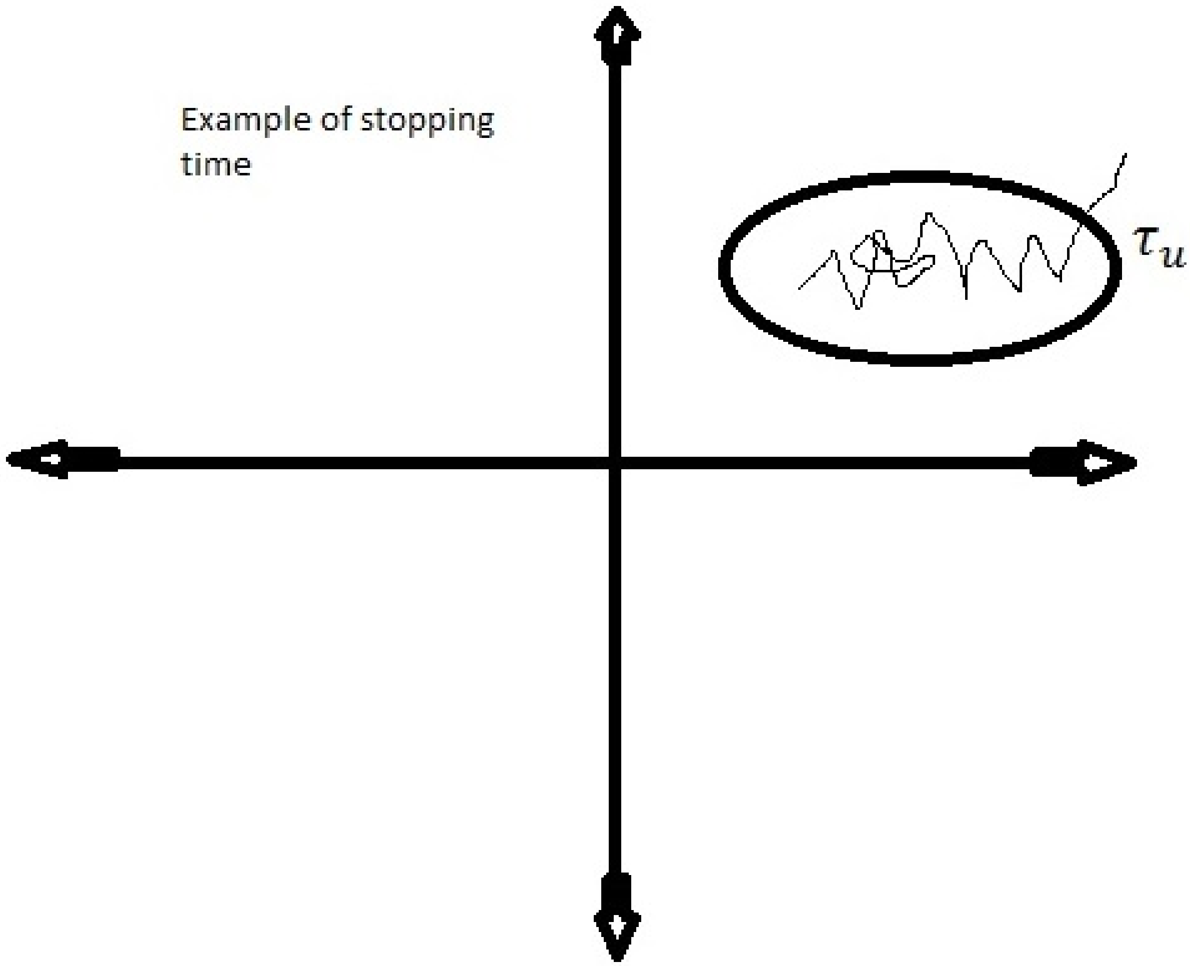

Figure 6.2. Graphical example of exit time.

given by

$$\tau_b = \inf\{t > 0 : M_t \geq b\}, \tag{6.18}$$

for some given value $b > M_0$. The distribution of $\tau_b$using the transition density of a process is is well understood for the Gaussian OU processes from the pioneering work [22]. However, this problem is significantly complicated for non-Gaussian OU processes.

The distribution of the first passage time is determined through its Laplace transform which is found by exploiting certain stopped martingales derived from using bounded partial eigenfunctions for the infinitesimal generator for the stochastic processes. There are several works in literature which deal with approximation, asymptotic expansions, integral equations, and implicit expressions related to the exit time of non-Gaussian OU processes (see [44, 43, 45, 28, 11]). However, an explicit expression of the distribution of the first-passage time for a general non-Gaussian OU process is still an area of active research interests.

For the analysis of the geophysical data we can reasonably assume

$$E(\log(1+|Z_\lambda|)) < \infty, \tag{6.19}$$

which is a necessary and sufficient condition of convergence of (M_t) in distribution to a proper limit (see [67]). When M is a Gamma-OU process, clearly $Z_{\lambda t} = \sum_{n=1}^{N_{\lambda t}} x_n$, is a compund poisson process , where $E(N_\lambda) = \nu\lambda$ and thus the moment generating function for Z_λ can be obtained from (6.10) as

$$E(e^{vZ_\lambda}) = \exp\left(\lambda\nu \int_0^\infty (e^{vx}-1)\alpha e^{-\alpha x}\, dx\right). \tag{6.20}$$

As used in [11], set

$$K = \sup\{u \geq 0 : E(e^{uZ_\lambda}) < \infty\}, \tag{6.21}$$

and

$$\Phi(u) = \frac{1}{\lambda}\int_0^u \frac{\log(E(e^{vZ_\lambda}))}{v}\, dv, \quad 0 \leq u < K. \tag{6.22}$$

We need the following simple Lemma to have an estimate of the expectation of the first passage time for Gamma-OU processes.

Lemma 6.4.2. *When M is a Gamma-OU process $K = \alpha$ and*

$$\Phi(u) = \nu \log\left(\frac{\alpha}{\alpha - u}\right), \quad 0 \leq u < \alpha. \tag{6.23}$$

Proof. It follows directly from (6.20) that $K = \alpha$. Moreover, simple calculation shows

$$\begin{aligned}\log(E(e^{vZ_\lambda})) &= \lambda\nu \int_0^\infty (e^{vx}-1)\alpha e^{-\alpha x}\, dx \\ &= \lambda\nu\alpha\left(\frac{e^{(v-\alpha)x}}{v-\alpha} + \frac{e^{-\alpha x}}{\alpha}\right)\Big|_{x=0}^{\infty} \\ &= \frac{\lambda\nu v}{\alpha - v}. \end{aligned}\tag{6.24}$$

The last statement comes from the fact that $v < \alpha$. Thus for $0 \leq u < \alpha$, $\Phi(u) = \nu \log\left(\frac{\alpha}{\alpha-u}\right)$. □

It is clear from Lemma 6.4.2 that when M is a Gamma-OU process $\Phi(K) = \Phi(\alpha) = +\infty$. Also, for this case, it is evident from [43] that if $G(z,\mu) = \int_0^\alpha e^{uz-\Phi(u)} u^{\mu-1}\, du$, with $\mu > 0$ and $\Phi(u)$ defined as in (6.23), then

$$E(e^{-\mu\lambda\tau_b} G(M_{\tau_b}, \mu)) = G(M_{t_0}, \mu), \quad \mu > 0. \tag{6.25}$$

We now state the main result for estimation of the first passage time of the process M.

Theorem 6.4.3. *Let M be a $\Gamma(\nu,\alpha)$-OU process given by (6.1). Then the expected value of τ_b satisfies the following relation*

$$E(\tau_b) \geq \frac{1}{\lambda} \int_0^\alpha \frac{(e^{ub} - e^{uM_0})}{u} \left(\frac{\alpha - u}{\alpha} \right)^\nu du. \tag{6.26}$$

Proof. We observe that (6.19) is satisfied and $0 < K = \alpha < \infty$ and $\Phi(K) = +\infty$. Thus all conditions of [11] (Theorem 2) are satisfied. Hence

$$\begin{aligned} E(\tau_b) &= \frac{1}{\lambda} E\left[\int_0^K \frac{(e^{uM_{\tau_b}} - e^{uM_0})}{u} e^{-\Phi(u)}\, du \right] \\ &\geq \frac{1}{\lambda} \int_0^\alpha \frac{(e^{ub} - e^{uM_0})}{u} e^{-\Phi(u)}\, du \\ &= \frac{1}{\lambda} \int_0^\alpha \frac{(e^{ub} - e^{uM_0})}{u} \left(\frac{\alpha - u}{\alpha} \right)^\nu du, \end{aligned} \tag{6.27}$$

where in the last step result for $\Phi(u)$ from Lemma 6.4.2 is used □

Theorem 6.4.3 plays a key role in Section 6.5. The parameters α, ν and λ will be found using regression analysis. Thus for the Gamma-OU model we can estimate the expected value of the major events using (6.26). This gives the estimated time after which the awareness level should be raised significantly that a major event will follow.

6.5. Statistical Analysis of the Data

In this section we analyzed the earthquake data using method of non-linear least square (nls) estimation regression. This nls use the given data and try and finds the given parameter which minimzes the least square error. We used a statistical software which is called R to do the analysis and get the root mean square error (RMSE). the first part of this section (Section 6.5.1) we describe the geophysical data set used for the computation. In Section 6.5.2 we give a detailed account of our results with comparison with existing results.

6.5.1. Geophysical (Earthquake) Data

The geophysical data was obtained from U.S. Geological Survey (USGS) from January 1, 1973 to November 9, 2010 for a certain region in *California*. This data contains information about the date, longitude, latitude, and magnitude of each recorded earthquake in the region.

This study will be characterized by the geographical region in a neighborhood of location of the major earthquake. The choice of this region should be done carefully. This area can not be too small because in that case there will be not enough data to run the regression analysis. The area can not be too big because in that case the magnitude data set is distorted due to *noise* from unrelated events. As in [38] the data is obtained using a *square* centered at the coordinates of the major event. The sides of the square are usually chosen as $\pm 0.1^\circ$–0.5° in latitude and $\pm 0.2^\circ$–0.5° in longitude. A segment 0.1° of latitude at the equator is $\approx$ 6.9 miles $\approx$ 11.11 km in length.

The earthquake magnitude is the recorded data used for 5 different square regions and the model is fitted to get the parameter estimate. The policy of the USGS regarding recorded magnitude is the following [1]: (i) Magnitude is a dimensionless number between 1 and 12. (ii) The reported magnitude should be moment magnitude, if available. (iii) The least complicated, and probably most accurate, terminology is to just use the term "*magnitude*" and to use the symbol M for it.

The magnitude is recorded in the data used and where available moment magnitude is used. A *major* earthquake event is defined as an earthquake with magnitude greater than 7. For more information we refer to the specific USGS documentation available at
http://earthquake.usgs.gov/aboutus/docs/020204mag_policy.php.

6.5.2. Regression Analysis and Estimation of Major Events

For the regression analysis we use data prior to a *major* earthquake event. The time series is shifted in such a way that the initial time is 0 and the final time (up to which data is available to estimate future *major* event) is t_0. For the model t_0 is taken to be the time before a *major* earthquake for which the magnitude is actually recorded. Consequently, for a particular geographical location, M_0 and M_{t_0} denote the magnitude of the data for the recorded initial and final times respectively. The time T can be taken to be any day between t_0 and the day of the major earthquake. Choice of T is on users' discretion as long as the time series for the magnitude process of earthquake in a geographical region is provided up to time $t_0 < T$. However, the two important aspects to keep in mind are: (i) T can not be too large. The geophysical data is not provided for small magnitudes which potentially distort the probability distribution of the magnitude process from the data. If too long is waited after t_0 it is very much possible that the probability distribution from the computed data set is already changed quite significantly. (ii) T can not be too small. Because, in that case there may not be any jump in the Lévy process Z and thus the contribution from the stochastic term in (6.12) is zero (or close to zero). So the model itself becomes insignificant.

Hence it is important to choose T carefully. For example, in our regression analysis T is taken to be $t_0 + 1$.

Based on the data of the magnitude process $(M_t)_{0 \leq t \leq t_0}$ it is possible to compute the characteristic function $\phi_O(u) = E[\exp(iuM_T)|\mathcal{F}_{t_0}]$, where the suffix '$O$' of ϕ stands for "observed" characteristic function based on the data set up to time t_0. We numerically compute the probability distributions and then use *Fast Fourier Transform* (FFT) to compute $\phi_O(u)$. We take modulus of these complex function to get $|\phi_O(u)|$, which is a real function of u. For the Gamma-OU model, we fit (6.16) to the observed $|\phi_O(u)|$ using non-linear regression. The fitting minimizes *root-mean square error* (RMSE). The software which is used for regression is R. This gives the best fit values of the parameters α, ν and λ. We find for each specific region, p-value is much less than 0.05 for all the parameters which shows that *all* the three parameters (α, ν and λ) are very significant for the model. Finally (6.26) is used to compute the lower bound of the expected time of a *major* earthquake. At that time the awareness level should be raised significantly in anticipation of a *major* earthquake. We note that (6.26) does *not* give a lower bound for τ_b. Instead it gives the lower bound for $E(\tau_b)$ which is a statistical property of τ_b. We acknowledge the fact that in some exceptional cases it is still possible to get a *major* earthquake even before the lower bound time of $E(\tau_b)$. However, statistically those incidences are very less probable. Partial compensation of the lower bound of $E(\tau_b)$ can be attained by little flexibility in the values of b in the definition of *major* earthquake. For our calculation b is chosen from a range 6.2 to 6.7.

Table 6.1 presents the estimated parameters from regression analysis. Table 6.2 presents the estimations from the regression model. In Table 6.2, t_r is the estimated date on which alert should be raised about possible earthquake in near future, and t_m is the actual known *major* earthquake date.

The following Figures (Figure 6.3 to Figure 6.7) present the fitting of absolute value of the characteristic function. In each of the figures, circles denote values of $|\phi_O(u)|$ which are computed from actual data points and the solid line is the regression fitting.

It is clear from Table 6.2 that the Gamma-OU model can be very effective in modeling magnitude process of earthquake in certain geographical locations. At the time t_r a *near-future* earthquake warning should be issued. Observe that t_r is always less than t_m which is why this model is reasonable for estimating earthquake date. For the previous works where the Ising model, scale invariance or Lévy flights are used [30, 39, 37, 38] the *estimated date* is typically but not always preceding the major event. For different deterministic models the estimation is based on finding some *accumulation point* of a sequence of numbers. However, there is no reason that the limit point is *before* the actual date of *major* earthquake. As shown in the following tables and figures, Gamma-OU works surprisingly good in that respect. On the other hand, for the stochastic Lévy flight models in literature, the estimated date was found from the given data set and some points in that data set may be after a major earthquake. For example, for the Lévy flight model with unit variance the estimation is based on the deviation of the data set from the cumulative distribution of the model [38]. However, that deviation can very well occur after the *major* earthquake. The model proposed in this study do not take into account the data *after* major earthquake. Gamma-OU model only takes the historical data and thus more realistic for practical applications.

6.6. Conclusion

In this chapter we use Gamma-OU process to model magnitude process of earthquake data. The predicted time is typically preceding the time of the major event. This methodology can be used in real time by looking at the minor earthquakes up to a certain date to estimate a major earthquake in future. However, there are some difficulties with this method in terms of *false alarm*. When this model is used to analyze some other data sets most of the time it works reasonably. However, sometime the *predicted* time is within some days after the first data. That is of course correct (in the sense that the estimated time precedes the actual time) but sort of *false alarm* to the situation as no *major* earthquake was reported near to that point. There are two ways two improve those situations. Firstly, we can modify the value of the *major* earthquake b so that t_r becomes bigger than t_0. Such t_r (with reasonable b) will be the new estimation. Secondly, we can modify the Gamma-OU model to a superimposition of several OU processes. This will allow the model to be the sum, or superimposition, of independent OU processes. As the processes do *not* need to be identically distributed, this offer some extra flexibility in the model. Modified processes are potential candidates for modeling long-range dependence and self-similarity (see for example [3, 6] and references therein). Additional work is in progress on this aspect in order to improve the present model.

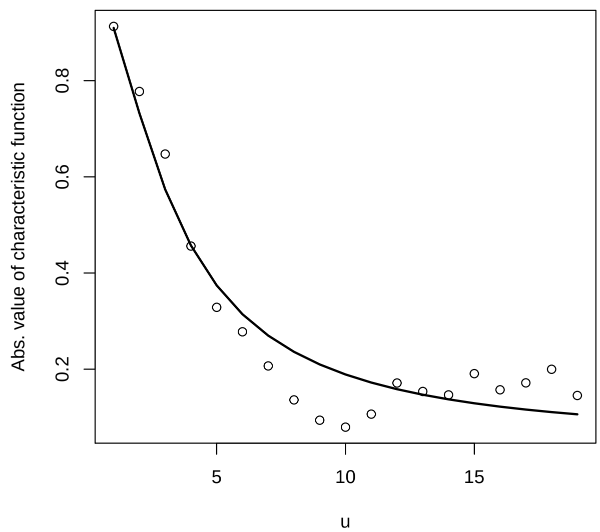

Figure 6.3. Fitting for Latitude $40.37^\circ \pm 0.1^\circ$ and Longitude $-124.32^\circ \pm 0.2^\circ$.

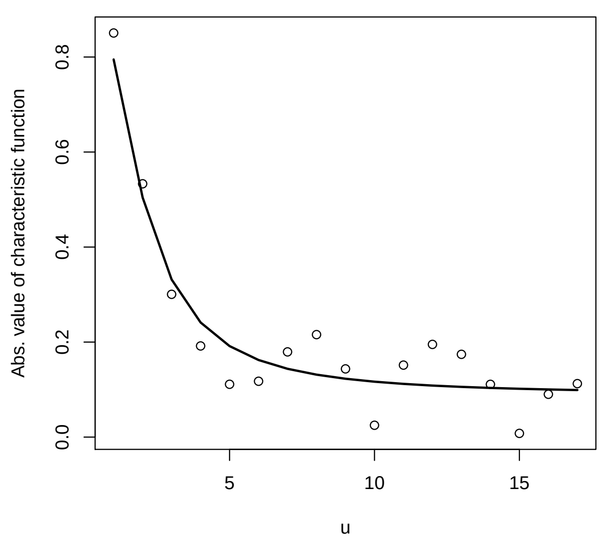

Figure 6.4. Fitting for Latitude $-23.34^\circ \pm 0.5^\circ$ and Longitude $-70.30^\circ \pm 0.5^\circ$.

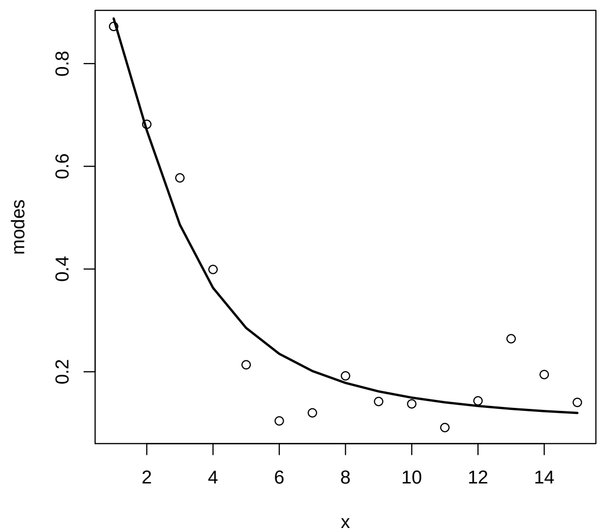

Figure 6.5. Fitting for Latitude $63.52^\circ \pm 0.17^\circ$ and Longitude $-147.44^\circ \pm 0.34^\circ$.

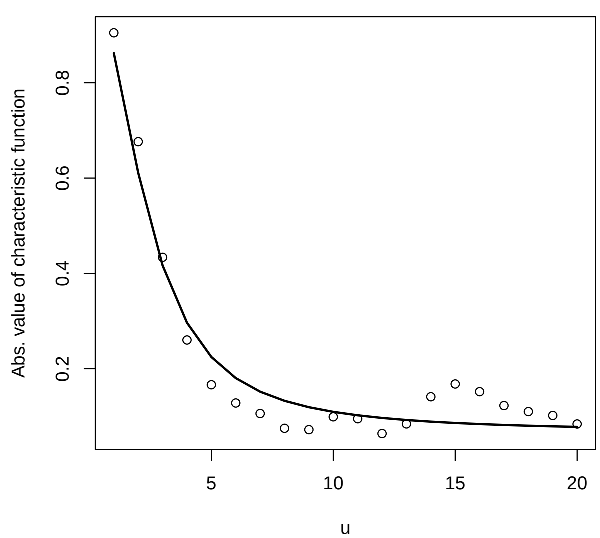

Figure 6.6. Fitting for Latitude $-17.66^\circ \pm 0.03^\circ$ and Longitude $-178.76^\circ \pm 0.06^\circ$.

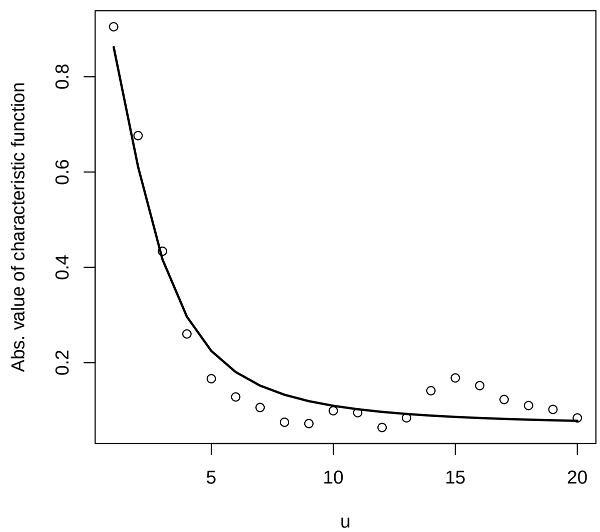

Figure 6.7. Fitting for Latitude $-19.93° \pm 0.05°$ and Longitude $-178.18° \pm 0.1°$.

Table 6.1. Results from regression analysis

Latitude	Longitude	M_0	t_0	α	ν	λ	RMSE
$40.37° \pm 0.1°$	$-124.32° \pm 0.2°$	3.9	12/20/1991	2.4113	1.2045	2.3879	0.06346
$-23.34° \pm 0.5°$	$-70.30° \pm 0.5°$	4.6	06/01/1995	2.0007	0.4244	0.9951	0.05913
$63.52° \pm 0.17°$	$-147.44° \pm 0.34°$	4.2	11/03/2002	2.8355	2.3713	0.99289	0.06875
$-17.66° \pm 0.03°$	$-178.76° \pm 0.06°$	4.1	07/15/2004	2.7123	2.7231	0.99052	0.04582
$-19.93° \pm 0.05°$	$-178.18° \pm 0.1°$	4.7	08/25/2005	3.2601	1.6960	0.99552	0.03451

Table 6.2. Estimation from the regression model.

Latitude	Longitude	$t = 0$	t_r	t_m
$40.37° \pm 0.1°$	$-124.32° \pm 0.2°$	09/30/1973	6572	6782
$-23.34° \pm 0.5°$	$-70.30° \pm 0.5°$	07/13/1973	7759	8052
$63.52° \pm 0.17°$	$-147.44° \pm 0.34°$	03/26/1976	10632	10753
$-17.66° \pm 0.03°$	$-178.76° \pm 0.06°$	01/17/1973	11394	11501
$-19.93° \pm 0.05°$	$-178.18° \pm 0.1°$	04/04/1976	10454	10865

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