INTEGRAL CLOSURE AND THE GENERALIZED MULTIPLICITY SEQUENCE

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ABSTRACT

For an arbitrary ideal I in a local ring R and a finitely generated R-module M, Achilles and Manaresi introduced the sequence of generalized multiplicities $c_k(I, M)$ $(k = 0, ..., \dim M)$ as a generalization of the classical Hilbert-Samuel multiplicity $e(I, M)$ of an $\mathfrak{m}$-primary ideal I. We prove a formula expressing each generalized multiplicity $c_k(I, M)$ as a linear combination of certain local multiplicities $e(IR_{\mathfrak{p}}, I^n(M_{\mathfrak{p}}/(x_2, ..., x_k)M_{\mathfrak{p}}))$, where $x_2, ..., x_k$ is a sequence of sufficiently general elements in I. As a consequence, when M is formally equidimensional, if $I \subseteq J$ have the same asymptotic primes and $c_k(I, M) = c_k(J, M)$ for all $k = 0, ..., \dim M$ then I is a reduction of (J, M). The converse of this statement is also known to be true by a result of Ciupercă. This theorem gives a complete numerical characterization of the integral closure, generalizing a well known theorem of Rees.

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DEDICATION

Dedicated to the memory of Timothy "Bowtie" Piasecki. I promised to dedicate a book to you all those years ago.

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1. INTRODUCTION

In this dissertation we discuss the relationship between the integral closure of an ideal and a finite sequence of invariants associated with it, the so-called generalized multiplicity sequence. Given a pair of ideals $I \subseteq J$ in a local noetherian ring $(R, \mathfrak{m})$, a central theme in multiplicity theory in commutative algebra is the search of numerical invariants associated with these ideals that can detect whether or not J is contained in the integral closure of I. In the case when both ideals are of finite colength, the classical Hilbert-Samuel multiplicity provides such a characterization. For formally equidimensional local rings (a rather weak, but necessary constraint on the ring), a well known result of Rees shows that J is contained in the integral closure of I if and only if the ideals I and J have the same Hilbert-Samuel multiplicity. In the case of ideals that are not necessarily of finite colength, a situation in which the classical Hilbert-Samuel multiplicity is no longer defined, there have been many attempts to generalize this numerical characterization by using more general invariants. An important chapter was initiated by Achilles and Manaresi who introduced the so-called j-multiplicity and the multiplicity sequence of an ideal. Both concepts are incremental generalizations of the classical Hilbert-Samuel multiplicity that are defined for arbitrary ideals in local rings. It is already known that if J is contained in the integral closure of I, then the multiplicity sequences of I and J are the same. The main goal of this dissertation is to obtain a converse of this result. We are able to obtain such a converse under some additional assumptions on the ideals I and J.

Chapters 2, 3, and 4 provide background material and give a survey of the main known results that will be later used in the dissertation. In Chapter 5 we discuss and prove several results about a series of technical constructions related to the concept of superficial element. The existence and properties of these special elements will provide the tools used to prove our central results. In Chapters 6 and 7 we prove the main results of the dissertation. Chapter 6 contains a formula that expresses each element of the generalized multiplicity sequence as a linear combination of certain local j-multiplicities. As a consequence of this, in Chapter 7 we obtain a generalization of the theorem of Rees for ideals that have the same asymptotic prime ideals.

We begin by discussing in the second chapter the classical Hilbert-Samuel multiplicity. Let $(R, \mathfrak{m})$ be a local ring, M a finitely generated R-module, and I an ideal of R such that the length $\lambda(M/IM)$ is finite. We consider the asymptotic growth of the length of the homogeneous components of the associated graded module $G_I(M)$. That is, for n large enough the length $\lambda(I^nM/I^{n+1}M)$ becomes a polynomial function in n of degree $d - 1 = \dim M - 1$ whose leading coefficient can be written in the form $\frac{e(I,M)}{(d-1)!}$, where $e(I, M)$ is the Hilbert-Samuel multiplicity of I on M. This multiplicity is a very important invariant in ideal theory. Among other things, it gives a numerical characterization of the integral closure of an ideal, or equivalently, of reduction ideals. For $I \subseteq J$ ideals, I is said to be a reduction of (J, M) if there exists some k with $J^{k+1}M = IJ^kM$. For $M = R$, the ideal J is contained in the integral closure of I if and only if I is a reduction of J. The following theorem was proved by Rees in 1961 for formally equidimensional rings, that is for rings where the dimension of the completion modulo its minimal primes are all equal.

Theorem 2.6.5. *Let $(R, \mathfrak{m})$ be a formally equidimensional local noetherian ring and let $I \subseteq J$ be $\mathfrak{m}$-primary ideals. Then I is a reduction of J if and only if $e(I, R) = e(J, R)$.*

In the case when $\lambda(M/IM)$ is not necessarily finite, we consider the asymptotic growth of the length of the homogeneous components of the bigraded module $G_{\mathfrak{m}}(G_I(M))$. We present a construction of Achilles and Manaresi that produces a sequence of numerical invariants $c_k(I, M)$ ($k = 0, \ldots, \dim M$). The first element of this sequence, the multiplicity $c_0(I, M)$, recovers the so-called the j-multiplicity $j(I, M)$ previously defined by Achilles and Manaresi [1] with different methods. The j-multiplicity is of particular interest for ideals of maximal analytic spread, which is the only case when it is nonzero. Moreover, in the case when $\lambda(M/IM)$ is finite, the j-multiplicity coincides with the classical Hilbert-Samuel multiplicity $e(I, M)$ and all the other elements $c_k(I, M)$ ($k \geq 1$) are zero.

There have been many attempts at proving generalizations of the theorem of Rees that give numerical characterizations of reduction ideals for arbitrary ideals. By using the above mentioned j-multiplicity, Flenner and Manaresi proved the following in 2001.

Theorem 3.2.5. *Let $(R, \mathfrak{m})$ be a local noetherian ring, $I \subseteq J$ ideals, and M a formally equidimensional finitely generated R-module. The following are equivalent:*

i. I is a reduction of (J, M);

ii. $j(IR_\mathfrak{p}, M_\mathfrak{p}) = j(JR_\mathfrak{p}, M_\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Supp}(M)$;

iii. $j(IR_\mathfrak{p}, M_\mathfrak{p}) \leq j(JR_\mathfrak{p}, M_\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Supp}(M)$.

While this result does give a numerical characterization of the integral closure for arbitrary ideals, it has the disadvantage that involves numerical invariants in localizations of the ring R. A much better characterization would be one that involves invariants that can be computed by considering only the ring R, and not all of its localizations. From a computational point of view, this would be an essential feature.

The j-multiplicity is tied to the analytic spread $\ell_M(I)$. The analytic spread can be computed as a dimension of the fiber cone. It is also the minimal number of generators of any minimal reduction. It is known that the j-multiplicity is nonzero if and only if the analytic spread is maximal, that is $\ell_M(I) = \dim M$. The set of asymptotic primes $\operatorname{Asym}(I, M)$ are the primes $\mathfrak{p}$ that have the property that the analytic spread of the localization $\ell_{M_\mathfrak{p}}(IR_\mathfrak{p})$ is maximal.

The sequence of multiplicities $c_0(I, M), \ldots, c_d(I, M)$ is a natural candidate for obtaining such a characterization. We first note a result of Achilles and Manaresi (1997) that gives several important properties of this sequence.

Theorem 4.1.6. *Let $(R, \mathfrak{m})$ be a local noetherian ring, I an ideal, and M a finitely generated R-module of dimension d. Let $\ell = \ell_M(I)$ and $q = \dim M/IM$. Then*

i. $c_k(I, M) = 0$ *for* $k < d - \ell$ *and* $k > q$*;*

ii. $c_{d-\ell}(I, M) = \sum_\mathfrak{B} e(\mathfrak{m}G_I(R)_\mathfrak{B}, G_I(M)_\mathfrak{B})e(G_I(R)/\mathfrak{B})$ *where* $\mathfrak{B}$ *runs through all highest dimensional associated prime ideals of* $G_I(M)/\mathfrak{m}G_I(M)$ *such that* $\dim G_I(R)/\mathfrak{B} + \dim G_I(M)_\mathfrak{B} = \dim G_I(M)$*;*

iii. $c_q(I, M) = \sum_\mathfrak{p} e(IR_\mathfrak{p}, M_\mathfrak{p})e(R/\mathfrak{p})$ *where* $\mathfrak{p}$ *runs through all highest dimensional associated prime ideals of* M/IM *such that* $\dim R/\mathfrak{p} + M_\mathfrak{p} = \dim M$.

We note here that one of the main results of this dissertation (Theorem 6.2.1) gives a formula that generalizes part (iii) of this theorem.

In 2003, Ciupercă proved the following result involving the generalized multiplicity sequence.

Theorem 4.1.8. Let $(R, \mathfrak{m})$ be a local noetherian ring, $I \subseteq R$ an ideal, and M a finitely generated R-module. If I is a reduction of (J, M), then $c_i(I, M) = c_i(J, M)$ for $i = 0, ..., d$.

Superficial elements in various incarnations have a long history in multiplicity theory. They have been used to prove various results, including Theorem 4.1.8 as well as an alternate proof to the theorem of Rees. In general, a fundamental property of these elements is that various concepts of multiplicity are preserved when modding out by such a superficial element. In the particular case of the above mentioned generalized multiplicity sequence $c_k(I, M)$ $(k = 0, \ldots, d)$, we have the following result of Ciupercă (2001).

Proposition 5.2.8. *Let $(R, \mathfrak{m})$ be a local noetherian ring, $I \subset R$ an ideal, and M a finitely generated R-module. Let $x \in I \setminus \mathfrak{m}I$ be a superficial element for I with respect to $G_{\mathfrak{m}}(G_I(M))$ such that x is a nonzero divisor on M. Then $c_i(I, M) = c_i(I, M/xM)$ for $i \leq d - 2$.*

A superficial element for I is a sufficiently general element, that is, it is obtained by avoiding finitely many proper subspaces of the $R/\mathfrak{m}$-vector space $I/\mathfrak{m}I$, thus superficiality is a Zariski open property. We have the following proposition, as proved by Swanson and Huneke, that shows the existence of such an element.

Proposition 5.1.6. *Let $(R, \mathfrak{m})$ be a local noetherian ring with infinite residue field, M a finitely generated R-module, and $I \subset R$ an ideal. Then there exists $x \in I \setminus \mathfrak{m}I$ such that x is superficial for I with respect to $G_I(M)$.*

A similar result shows the existence of superficial elements with respect to $G_{\mathfrak{m}}(G_I(M)))$. We also prove that superficial elements can be extended to case of $\mathbb{Z}^3$-graded algebras. As a consequence, we show the existence of superficial elements with respect to a certain infinite family of modules.

Proposition 5.3.3. *Let $(R, \mathfrak{m})$ be a local noetherian ring with infinite residue field, I and J ideals of R, and M a finitely generated R-module. Then there exists $x \in I$ such that x is superficial for I with respect to $G_{\mathfrak{m}}(G_I(J^jM))$ for all $j \geq 0$.*

Further, we show that we can find an element that is superficial for finitely many localizations.

Proposition 5.3.6. *Let $(R,\mathfrak{m})$ be a local noetherian ring with infinite residue field k, $I \subset R$ an ideal, and M a finitely generated module. Then there exists $x \in I$ such that x is superficial for I with respect to $G_{\mathfrak{m}}(G_I(M))$ and $\frac{x}{1} \in IR_{\mathfrak{p}}$ is superficial for $IR_{\mathfrak{p}}$ with respect to $G_{\mathfrak{p}R_{\mathfrak{p}}}(G_{IR_{\mathfrak{p}}}(M_{\mathfrak{p}}))$ for all the prime ideals $\mathfrak{p}$ such that $\ell_{M_{\mathfrak{p}}}(IR_{\mathfrak{p}}) = \dim M_{\mathfrak{p}}$.*

In the sixth chapter, we build a sequence of sufficiently general elements that satisfy certain properties. This construction allows us to prove the main theorem.

Theorem 6.2.1. *Let $(R,\mathfrak{m})$ be a local ring, $I \subseteq R$ an ideal, and M a finitely generated R-module of dimension d. Let $r = \min\{i \mid c_i(I,M) \neq 0\}$ and let $x_2, ..., x_{d-r} \in I$ be a sufficiently general sequence. Assume that $r < d$ and for each k denote*

$$\Lambda_k(I,M) = \{\mathfrak{p} \in \operatorname{Supp}(M/IM) \mid \dim R/\mathfrak{p} = k \text{ and } \dim I^n(M_{\mathfrak{p}}/(x_2,...,x_{d-k})M_{\mathfrak{p}}) = 1 \text{ for } n \gg 0\}.$$

Then, for $n \gg 0$ and k=r,...,$d-1$, we have

$$c_k(I,M) = \sum_{\mathfrak{p} \in \Lambda_k} e(IR_{\mathfrak{p}}, I^n(M_{\mathfrak{p}}/(x_2,...,x_{d-k})M_{\mathfrak{p}}))e(R/\mathfrak{p}).$$

In the last chapter we prove a partial converse of Ciupercă's result and generalization of Theorem 2.6.5. We achieve this by building two sequences of sufficiently general elements that satisfy certain extra conditions.

Corollary 7.2.5. *Let $(R,\mathfrak{m})$ be a local noetherian ring, M a finitely generated formally equidimensional R-module of dimension d, and $I \subseteq J$ ideals of R such that $\operatorname{Asym}(I,M) = \operatorname{Asym}(J,M)$. If $c_i(I,M) = c_i(J,M)$ for all $i = 0,...,d$, then I is a reduction of (J,M).*

2. THE HILBERT-SAMUEL MULTIPLICITY

In this chapter we develop the theory of multiplicities for graded modules. We start with defining the Hilbert function and multiplicity of a graded module over a homogeneous ring. We then naturally consider these concepts for the associated graded ring of an ideal. This will give us a way to define the Hilbert-Samuel multiplicity of an ideal with respect to a module. We further generalize this concept to the case of two ideals by introducing the so-called mixed multiplicity. Finally, we present a well known theorem of Rees that connects the integral closure of an ideal with its Hilbert-Samuel multiplicity.

2.1. Artinian modules

Throughout this section we assume that R is a local noetherian ring with maximal ideal $\mathfrak{m}$ and M is a finitely generated R-module.

Definition 2.1.1. A module $M \neq 0$ is said to be simple if (0) is the only proper submodule of M.

Definition 2.1.2. A composition series of a module M is a chain of submodules of M

$$M = M_0 \supset M_1 \supset M_2 \supset ... \supset M_n = 0$$

such that M_i/M_{i+1} is simple for all $i = 0, ..., n-1$. A module for which such a composition series exists is called a module of finite length.

Definition 2.1.3. A non-zero module M is said to be artinian if any descending chain of submodules

$$M \supseteq M_1 \supseteq M_2 \supseteq ... \supseteq M_i \supseteq ...$$

eventually stabilizes.

For a finitely generated artinian module over a noetherian ring, any descending chain of submodules stabilizes. If we consider the descending chain $M_i = \mathfrak{m}^i M$ we get that $\mathfrak{m}^n M = \mathfrak{m}^{n+1} M$ for $n \gg 0$. This implies that $\mathfrak{m}^n M = 0$ for $n \gg 0$ by Nakayama's Lemma. We summarize in the following proposition several equivalent definitions for artinian modules.

Proposition 2.1.4. *Let $(R, \mathfrak{m})$ be a local noetherian ring and M a finitely generated R-module. The following are equivalent.*

i. The module M is artinian.

ii. For some n we have $\mathfrak{m}^n M = 0$.

iii. The module M has finite length.

Definition 2.1.5. Let M be a finitely generated artinian module. The length of M, denoted by $\lambda(M)$, is the length of the longest chain of submodules of M

$$M = M_0 \supset M_1 \supset ... \supset M_n = 0$$

where $M_i \neq M_{i+1}$ for all i.

By the Jordan-Hölder theorem we know that every composition series of M have the same length $\lambda(M)$. As an observation, if N is a submodule of a finitely generated artinian module M, then N is artinian and $\lambda(N) \leq \lambda(M)$. Similarly, the quotient module M/N is artinian and further, if we consider the exact sequence

$$0 \to N \to M \to M/N \to 0$$

we have $\lambda(M) = \lambda(N) + \lambda(M/N)$.

Example 2.1.6. Let $R = \mathbb{Q}[x, y]_{(x,y)}$ and $M = R/(x^3, xy, y^2)$. The module M is artinian since $\mathfrak{m}^3 M = 0$ and the length $\lambda(M) = 4$.

2.2. Ideals of definition

Let R be a local noetherian ring with maximal ideal $\mathfrak{m}$ and M a finitely generated R-module.

Definition 2.2.1. The radical of an ideal I, denoted $\sqrt{I}$, is defined as

$$\sqrt{I} = \{x \in R \mid \text{there exists } n \text{ such that } x^n \in I\}.$$

Definition 2.2.2. We say that an ideal I of R is an ideal of definition on M if M/IM is artinian.

As an immediate consequence of Proposition 2.1.4 we have the following.

Proposition 2.2.3. *Let $(R, \mathfrak{m})$ be a local noetherian ring, M a finitely generated R-module, and I an ideal of R. The following are equivalent.*

i. The ideal I is an ideal of definition on M.

ii. For some n we have $\mathfrak{m}^n(M/IM) = 0$.

iii. The module M/IM has finite length.

Note that for $M \neq 0$ the second condition of Proposition 2.2.3 is equivalent to the condition $\mathfrak{m} = \sqrt{I + \operatorname{Ann}(M)}$. As an immediate consequence, for every i we have $\sqrt{I^i + \operatorname{Ann} M} = \mathfrak{m}$ and therefore M/I^iM is artinian for all i. If we consider the short exact sequence

$$0 \to I^nM/I^{n+1}M \to M/I^{n+1}M \to M/I^nM \to 0,$$

as before, we have $\lambda(M/I^{n+1}M) = \lambda(M/I^nM) + \lambda(I^nM/I^{n+1}M)$.

2.3. Hilbert functions and multiplicities of graded modules

We now present the basic theory of graded rings and modules.

Definition 2.3.1. A ring R is said to be graded if

$$R = \bigoplus_{i \geq 0} R_i$$

where each R_i is an abelian group and for all m, n we have $R_mR_n \subseteq R_{m+n}$. Further, if $R = R_0[R_1]$, then R is said to be a homogeneous ring.

Definition 2.3.2. Let $R = \bigoplus_{i \geq 0} R_i$ be a graded ring. An R-module M is called a graded R-module if

$$M = \bigoplus_{i \geq 0} M_i$$

where each M_i is an abelian group and for all m, n we have $R_mM_n \subseteq M_{m+n}$.

If M is finitely generated over R and R is noetherian, note that each component M_i is a finitely generated R_0-module. Further, if R_0 is an artinian ring, then each M_i is artinian, since each M_i is finitely generated over R_0. In this case, we can define the Hilbert function $h_M(n)$ as follows.

Definition 2.3.3. Let R be a homogeneous noetherian ring such that R_0 is artinian and M is a finitely generated graded R-module. The Hilbert function $h_M(n)$ is defined by

$$h_M(n) = \lambda(M_n)$$

for all n.

Of particular interest is the asymptotic behavior of the Hilbert function. We have the following theorem.

Theorem 2.3.4. [3, Theorem 4.1.3] *Let R be a homogeneous noetherian graded ring with R_0 artinian. Let M be a finitely generated graded R-module of dimension d. Then there exists a polynomial P_M with rational coefficients and degree $d-1$ such that $h_M(n) = P_M(n)$ for $n \gg 0$.*

We have the following Lemma.

Lemma 2.3.5. [3, Lemma 4.1.4] *Let $P(X) \in \mathbb{Q}[X]$ be a polynomial of degree $d-1$. Then the following conditions are equivalent:*

i. $P(n) \in \mathbb{Z}$ for all $n \in \mathbb{Z}$.

ii. There exist integers $a_0, ..., a_{d-1}$ such that

$$P(X) = \sum_{i=0}^{d-1} a_i \binom{X+i}{i}.$$

If $P(x)$ is a polynomial of degree $d-1$, it takes $d-1$ successive differences to become constant. Therefore, if $P(n)$ is an integer for d consecutive integers, then $P(n)$ is an integer for all integers. Therefore, is it sufficient for $P(n) \in \mathbb{Z}$ for $n \gg 0$ for Lemma 2.3.5 to hold. In particular, for $n \gg 0$ we can write $h_M(n) = \sum_{i=0}^{d-1} e_i \binom{n+i}{i}$ for some integers e_i.

Definition 2.3.6. Let $d = \dim M$. The multiplicity $e(M)$ is defined by

$$e(M) = e_{d-1} = \lim_{n \to \infty} \frac{h_M(n)(d-1)!}{n^{d-1}}$$

Let R be a homogeneous noetherian ring with R_0 artinian. If $N \subseteq M$ are finitely generated graded modules over a R with compatible gradings, then so is M/N. Consider the short exact sequence

$$0 \to N \to M \to M/N \to 0.$$

If we consider the multiplicity as the normalized coefficient e_{d-1} of the Hilbert polynomial, since $\lambda(M_i) = \lambda(N_i) + \lambda(M_i/N_i)$, there are three possibilities. If $\dim N < \dim M$, then we must have $\dim M = \dim M/N$ and $e(M) = e(M/N)$. If $\dim N = \dim M$ and $\dim M/N < \dim M$, then $e(N) = e(M)$. If $\dim N = \dim M = \dim M/N$, then $e(M) = e(N) + e(M/N)$. We next define the concept of multiplicity of an ideal.

2.4. Associated graded modules

We begin by constructing a graded ring associated with an ideal I in a noetherian ring.

Definition 2.4.1. Let R be a noetherian ring and I an ideal. The associated graded ring $G_I(R)$ is defined by

$$G_I(R) = \bigoplus_{n \geq 0} I^n/I^{n+1}.$$

Note that $G_I(R)$ is a homogeneous ring. Further, since R is noetherian, so is $G_I(R)$.

Definition 2.4.2. Let R be a local noetherian ring, M a finitely generated R-module, and I an ideal. The associated graded module $G_I(M)$ is defined by

$$G_I(M) = \bigoplus_{n \geq 0} I^n M/I^{n+1} M.$$

Note that $G_i(M)$ is a finitely generated graded $G_I(R)$-module.

Theorem 2.4.3. [8, Proposition 5.1.6] *Let R be a noetherian ring, M a finitely generated R-module, and I an ideal. Then*

$$\dim G_I(M) = \dim M.$$

If we assume that I is an ideal of definition on M, we may consider $R' = R/\operatorname{Ann}(M)$ and $I' = (I + \operatorname{Ann} M)/\operatorname{Ann} M$. The module M is both an R-module and an R'-module and I' is an ideal of R' and an ideal of definition on M. But we now have that R'/I' is artinian.

We now consider the graded ring $G_{I'}(R')$ and the graded module $G_{I'}(M)$. By Theorem 2.3.4, the Hilbert function $h_{G_{I'}(M)}(n)$ is a polynomial function for $n \gg 0$. Since $(I')^nM/(I')^{n+1}M = I^nM/I^{n+1}M$, we define $h_{G_I(M)}(n) = h_{G_{I'}(M)}(n)$ and therefore $h_{G_I(M)}(n) = \lambda(I^nM/I^{n+1}M)$ is a polynomial function for $n \gg 0$. The degree of this polynomial is $\dim G_I(M) - 1 = \dim M - 1$.

Definition 2.4.4. Let R be a local noetherian ring, M a finitely generated R-module, and I an ideal of definition on M. The Hilbert-Samuel multiplicity of I with respect to M, denoted $e(I, M)$, is defined by

$$e(I, M) = e(G_I(M)).$$

Example 2.4.5. Let $R = M = \mathbb{Q}[x, y]_{(x,y)}$ and $I = (x^3, xy, y^2)$. The Hilbert polynomial $h(n) = \lambda(I^nM/I^{n+1}M) = 5n + 4$. Since $\dim M = 2$, the Hilbert-Samuel multiplicity $e(I, M) = 5$.

Since $\lambda(I^nM/I^{n+1}M)$ is a polynomial function with rational coefficients and

$$\lambda(I^nM/I^{n+1}M) + \lambda(M/I^nM) = \lambda(M/I^{n+1}M)$$

we can conclude that for n large enough $\lambda(M/I^{n+1}M)$ is a polynomial function of degree $d = \dim M$ whose degree d coefficient is

$$\frac{e(I, M)}{d!}.$$

Let R be a local noetherian ring, M a finitely generated R-module, and $I \subseteq J$ ideals of definition on M. From the short exact sequence

$$0 \to J^nM/I^nM \to M/I^nM \to M/J^nM \to 0$$

we have $\lambda(M/I^nM) \geq \lambda(M/J^nM)$ for all n and thus $e(I, M) \geq e(J, M)$.

The following is known as the linearity or associativity formula and relates the multiplicity of an ideal with respect to a module to multiplicities over a family of integral domains.

Theorem 2.4.6. [3, Corollary 4.7.8] *Let $(R, \mathfrak{m})$ be a local noetherian ring, M a finitely generated R-module of dimension d, and I an ideal of definition on M. Then*

$$e(I, M) = \sum_{\mathfrak{p}} e(I, R/\mathfrak{p})\lambda(M_{\mathfrak{p}})$$

where the sum is taken over all prime ideals $\mathfrak{p}$ with $\dim R/\mathfrak{p} = d$.

Note that only the primes minimal over $\operatorname{Ann}(M)$ contribute to the sum and therefore we have finitely many terms.

2.5. Mixed multiplicities

Similar to the Hilbert-Samuel multiplicity, we can construct the mixed multiplicity for two ideals.

Theorem 2.5.1. [8, Theorem 17.4.2] *Let R be a local noetherian ring, M a finitely generated R-module with $\dim M = d$, and $I \subseteq J$ ideals of definition on M. Then there exists a polynomial $P(n_1, n_2)$ with rational coefficients of total degree d such that for $n_1, n_2 \gg 0$ we have*

$$\lambda(M/I^{n_1}J^{n_2}M) = P(n_1, n_2).$$

We next define the mixed multiplicities as the normalized coefficients of highest degree.

Definition 2.5.2. Let R be a local noetherian ring, M a finitely generated R-module with $\dim M = d$, and $I \subseteq J$ ideals of definition on M. Let $P_d(n_1, n_2)$ be the homogeneous part of degree d of $P(n_1, n_2)$ and write

$$P_d(n_1, n_2) = \sum_{i=0}^{d} \frac{a_i}{i!(d-i)!} n_1^{d-i} n_2^{i}$$

for a_i nonnegative integers. For $i = 0, ..., d$, we define the ith mixed multiplicity to be $e_i(I, J; M) = a_i$.

We have the following relation between the mixed multiplicities and the Hilbert-Samuel multiplicities.

Theorem 2.5.3. [10, Lemma 2.4] *Let $(R, \mathfrak{m})$ be a local noetherian ring, M a finitely generated R-module, and $I \subseteq J$ be ideals of definition on M. Then $e_0(I, J; M) = e(I, M)$ and $e_d(I, J; M) = e(J, M)$.*

2.6. Integral closure and reductions

The multiplicity of an ideal only depends on its asymptotic behavior, that is the behavior of I^n for large values of n. For a pair of ideals $I \subseteq J$, if $I^k J^n$ and J^{n+k} are the same, then the ideals have similar asymptotic behavior.

Definition 2.6.1. Let R be a ring, $I \subseteq J$ be ideals, and M an R-module. The ideal I is said to be a reduction of (J, M) if there exists an integer n such that

$$IJ^nM = J^{n+1}M.$$

If $M = R$, then I is said to be a reduction of J.

Note that if $I \subseteq J$ is a reduction, then $\sqrt{I} = \sqrt{J}$. However, if R is noetherian, $I \subseteq J$ and $\sqrt{I} = \sqrt{J}$, then there exists an integer n such that $J^n \subseteq I$. This is not the same as I being a reduction of J.

Definition 2.6.2. Let R be a noetherian ring and I be an ideal. An element x is said to be integral over I if x satisfies an equation

$$x^n + a_1x^{n-1} + ... + a_{n-1}x + a_n = 0$$

where $a_i \in I^i$ for $i = 1, ..., n$. The elements integral over I form an ideal $\overline{I}$ which is called the integral closure of I.

Proposition 2.6.3. [8, Corollary 1.2.5] *Let R be a noetherian ring and $I \subseteq J$. Then I is a reduction of J if and only if $J \subseteq \overline{I}$, or equivalently, $\overline{I} = \overline{J}$.*

Definition 2.6.4. Let R be a local noetherian ring. The ring R is formally equidimensional if for every minimal prime $\mathfrak{p}$ of the completion $\widehat{R}$, we have $\dim \widehat{R}/\mathfrak{p} = \dim \widehat{R}$. A finitely generated R-module M is said to be formally equidimensional if $R/\operatorname{Ann}(M)$ is formally equidimensional.

We note that being formally equidimensional is a fairly weak constraint.

The following theorem was originally proven by Rees in 1961.

Theorem 2.6.5. [10, Theorem 3.2] *Let $(R, \mathfrak{m})$ be a formally equidimensional local noetherian ring and let $I \subseteq J$ be $\mathfrak{m}$-primary ideals. Then I is a reduction of J if and only if $e(I, R) = e(J, R)$.*

Our main goal is to obtain a generalization of this theorem that can be applied to arbitrary ideals.

3. THE J-MULTIPLICITY

In this section we consider the case when I is not necessarily an ideal of definition on M. In this situation $I^nM/I^{n+1}M$ does not have finite length and thus the classical Hilbert-Samuel multiplicity is not defined. Instead, we consider the largest submodule of $I^nM/I^{n+1}M$ that has finite length and proceed as before.

3.1. The $\mathfrak{m}$-torsion module

Definition 3.1.1. Let $(R, \mathfrak{m})$ be a local noetherian ring and M be a finitely generated R-module. The $\mathfrak{m}$-torsion module $\Gamma_{\mathfrak{m}}(M)$ is the submodule of M defined by

$$\Gamma_{\mathfrak{m}}(M) = \{x \in M \mid \mathfrak{m}^n x = 0 \text{ for some } n\}.$$

We have $\lambda(\Gamma_{\mathfrak{m}}(M)) < \infty$.

We conclude that $\Gamma_{\mathfrak{m}}(M) \subseteq M$ is a submodule. In particular, note that if M is artinian then $\Gamma_{\mathfrak{m}}(M) = M$. We consider the graded submodule $\Gamma_{\mathfrak{m}}(G_I(M)) \subseteq G_I(M)$.

Since R is noetherian and M is a finitely generated R-module of dimension d, $G_I(M)$ is a finitely generated module over the noetherian ring $G_I(R)$. Therefore the submodule $\Gamma_{\mathfrak{m}}(G_I(M))$ is finitely generated as a $G_I(R)$-module and thus there exists k such that $\mathfrak{m}^k\Gamma_{\mathfrak{m}}(G_I(M)) = 0$. This implies that $\Gamma_{\mathfrak{m}}(G_I(M))$ is a finitely generated graded module over the homogeneous ring $G_I(R)/\mathfrak{m}^kG_I(R)$ whose degree zero component is an artinian ring. In particular, for $n \gg 0$ the length $\lambda(\Gamma_{\mathfrak{m}}(I^nM/I^{n+1}M))$ is a polynomial function of degree $\dim_{G_I(R)} \Gamma_{\mathfrak{m}}(G_I(M)) - 1 \leq \dim M - 1$.

3.2. j-multiplicities

We now introduce the j-multiplicity, originally defined in [2, Definition 1.2].

Definition 3.2.1. Let $(R, \mathfrak{m})$ be a local noetherian ring, I an ideal, and M a finitely generated R-module of dimension d. The j-multiplicity is defined by

$$j(I, M) = \begin{cases} e(\Gamma_{\mathfrak{m}}(G_I(M)) & \text{if } \dim_{G_I(R)} \Gamma_{\mathfrak{m}}(G_I(M)) = d \\ 0 & \text{if } \dim_{G_I(R)} \Gamma_{\mathfrak{m}}(G_I(M)) < d \end{cases}.$$

Equivalently, we may define the j-multiplicity as

$$j(I, M) = \lim_{n\to\infty} \frac{\lambda(\Gamma_{\mathfrak{m}}(I^n M/I^{n+1}M))(d-1)!}{n^{d-1}}.$$

If I is an ideal of definition on M, then $e(I, M) = j(I, M)$ since $\Gamma_{\mathfrak{m}}(G_I(M)) = G_I(M)$. While we have $e(I, M) > 0$, we only have $j(I, M) \geq 0$. To be able to describe conditions under which $j(I, M) > 0$, we need to define the analytic spread of an ideal.

Definition 3.2.2. Let $(R, \mathfrak{m})$ be a local noetherian ring and I an ideal. The fiber cone $\mathcal{F}$ is defined by

$$\mathcal{F} = \bigoplus_{n\geq 0} I^n/\mathfrak{m}I^n$$

Definition 3.2.3. The analytic spread of I over M is defined by

$$\ell_M(I) = \dim_{\mathcal{F}} \left(\bigoplus_{n\geq 0} I^n M/\mathfrak{m}I^n M \right).$$

Note that

$$\ell_M(I) \leq \dim_{G_I(R)} \left(\bigoplus_{n\geq 0} I^n M/I^{n+1}M \right) = \dim_R M.$$

In the event that $\ell_M(I) = \dim_R M$, I is said to have maximal analytic spread.

Theorem 3.2.4. [7, Remark 6.1.6] *The j-multiplicity $j(I, M)$ is nonzero if and only if I has maximal analytic spread over M.*

We also note that if I is a reduction of (J, M) then $j(I, M) = j(J, M)$. However, the converse is not necessarily true. By considering all the localized j-multiplicities, Flenner and Manaresi proved the following.

Theorem 3.2.5. [6, Theorem 3.3] *Let $(R, \mathfrak{m})$ be a local noetherian ring, $I \subseteq J$ ideals, and M a formally equidimensional finitely generated R-module. The following are equivalent:*

i. I is a reduction of (J, M);

ii. $j(IR_{\mathfrak{p}}, M_{\mathfrak{p}}) = j(JR_{\mathfrak{p}}, M_{\mathfrak{p}})$ for all $\mathfrak{p} \in \mathrm{Supp}(M)$;

iii. $j(IR_{\mathfrak{p}}, M_{\mathfrak{p}}) \leq j(JR_{\mathfrak{p}}, M_{\mathfrak{p}})$ for all $\mathfrak{p} \in \mathrm{Supp}(M)$.

Note that the second condition requires checking equality of the j-multiplicities localized at every prime $\mathfrak{p} \in \operatorname{Supp}(M)$. In general, there may be infinitely many primes in $\operatorname{Supp}(M)$.

Definition 3.2.6. [9, Proposition 3.9] Let R be a noetherian ring, I be an ideal, and M be a finitely generated R-module. Then the sequence $\operatorname{Ass}(M/\overline{I^n}M)$ is nondecreasing and stabilizes. The set of asymptotic primes is defined by

$$\operatorname{Asym}(I, M) = \bigcup_{n \geq 1} \operatorname{Ass}(M/\overline{I^n}M).$$

Theorem 3.2.7. [9, Proposition 4.1] *Let $(R, \mathfrak{m})$ be a local ring, I an ideal, and M a finitely generated module. For a prime ideal $\mathfrak{p} \in \operatorname{Supp}(M)$, if $\ell_{M_\mathfrak{p}}(IR_\mathfrak{p}) = \dim M_\mathfrak{p}$, then $\mathfrak{p} \in \operatorname{Asym}(I, M)$. If M is formally equidimensional, the converse is also true.*

Remark 3.2.8. Assume that R is a noetherian ring, I is an ideal, and M is a finitely generated R-module. Since $\operatorname{Ass}(M/\overline{I^n}M)$ is a nondecreasing set that eventually stabilizes, we have

$$\bigcup_{n \geq 1} \operatorname{Ass}(M/\overline{I^n}M) = \operatorname{Ass}(M/\overline{I^k}M)$$

for some k large enough. Since $\operatorname{Ass}(M/\overline{I^k}M)$ is a finite set, so is $\operatorname{Asym}(I, M)$. Therefore, there are only finitely many primes $\mathfrak{p}$ such that $\ell_{M_\mathfrak{p}}(IR_\mathfrak{p}) = \dim M_\mathfrak{p}$.

Thus in Theorem 3.2.5 we only need to test that the j-multiplicities are equal when localized at finitely many primes. However, determining which primes are in $\operatorname{Asym}(I, M)$ may be difficult.

4. GENERALIZED MULTIPLICITY SEQUENCE

In this section we consider again the case where I is not necessarily an ideal of definition on M. Instead of considering the largest submodule of $I^nM/I^{n+1}M$ with finite length, we consider the associated bigraded module $G_{\mathfrak{m}}(G_I(M))$. Each graded component has finite length. Rather than polynomial growth in one variable, it has polynomial growth in two variables. The highest degree components will give us the generalized multiplicity sequence.

4.1. The generalized multiplicity sequence

We begin by defining the associated bigraded module.

Definition 4.1.1. Let $(R, \mathfrak{m})$ be a local noetherian ring, I an ideal, and M a finitely generated R-module. The associated bigraded module $G_{\mathfrak{m}}(G_I(M))$ is

$$G_{\mathfrak{m}}(G_I(M)) = \bigoplus_{i,j\geq 0} \mathfrak{m}^i \left(\frac{I^jM}{I^{j+1}M}\right) / \mathfrak{m}^{i+1} \left(\frac{I^jM}{I^{j+1}M}\right) = \bigoplus_{i,j\geq 0} \frac{\mathfrak{m}^i I^j M + I^{j+1}M}{\mathfrak{m}^{i+1} I^j M + I^{j+1}M}$$

The following proposition is a direct result from the general theory of bigraded rings [11, Theorem 7].

Proposition 4.1.2. *Let $(R, \mathfrak{m})$ be a local noetherian ring, I an ideal, and M a finitely generated R-module of dimension d. Then for $i, j \gg 0$ there exist integers $a_{k,l}$ such that*

$$\lambda\left(\frac{\mathfrak{m}^i I^j M + I^{j+1}M}{\mathfrak{m}^{i+1} I^j M + I^{j+1}M}\right) = \sum_{k+l\leq d-2} a_{k,l} \binom{i+k}{k}\binom{j+l}{l}.$$

Since this length is eventually a polynomial in i, j of degree at most $d-2$, by taking the double sum with i from 0 to u and j from 0 to v, we have the following proposition.

Proposition 4.1.3. *Let the Hilbert function $h(u, v)$ be defined by*

$$h(u,v) = \sum_{i=0}^{u}\sum_{j=0}^{v} \lambda\left(\frac{\mathfrak{m}^i I^j M + I^{j+1}M}{\mathfrak{m}^{i+1} I^j M + I^{j+1}M}\right).$$

The function $h(u, v)$ is polynomial $p(u, v)$ of total degree d with rational coefficients for $u, v \gg 0$.

The degree d terms are

$$\frac{c_k}{k!(d-k)!}u^k v^{d-k}$$

for $k = 0, ..., d$, where each $c_k \geq 0$.

Definition 4.1.4. Let $(R, \mathfrak{m})$ be a local noetherian ring, I an ideal, and M a finitely generated R-module and let $p(u, v)$ be the Hilbert polynomial defined above. The generalized multiplicity sequence $\{c_k(I, M)\}_{k=0}^d$ consists of the coefficients c_k introduced above.

Example 4.1.5. Let $R = M = \mathbb{Q}[x, y, z]_{(x,y,z)}$ and $I = (xy, yz)$. The homogeneous degree 3 part of the Hilbert polynomial is $\frac{1}{2}u^2v + uv^2$. So we have $c_0(I, M) = 0$, $c_1(I, M) = 2$, $c_2(I, M) = 3$, and $c_3(I, M) = 0$.

The following theorem gives sufficient conditions for when $c_k(I, M)$ is zero. It also relates the Hilbert-Samuel multiplicity to the generalized multiplicity sequence. A version of it for rings was proved by Achilles and Manaresi in [1, Theorem 2.3]. Following the same proof one can prove a similar result for modules.

Theorem 4.1.6. *Let $(R, \mathfrak{m})$ be a local noetherian ring, I an ideal, and M a finitely generated R-module of dimension d. Let $\ell = \ell_M(I)$ and $q = \dim M/IM$. Then*

i. $c_k(I, M) = 0$ *for* $k < d - \ell$ *and* $k > q$;

ii. $c_{d-\ell}(I, M) = \sum_{\mathfrak{B}} e(\mathfrak{m}G_I(R)_{\mathfrak{B}}, G_I(M)_{\mathfrak{B}})e(G_I(R)/\mathfrak{B})$ *where $\mathfrak{B}$ runs through all the highest dimensional associated prime ideals of $G_I(M)/\mathfrak{m}G_I(M)$ such that* $\dim G_I(R)/\mathfrak{B} + \dim G_I(M)_{\mathfrak{B}} = \dim G_I(M)$;

iii. $c_q(I, M) = \sum_{\mathfrak{p}} e(IR_{\mathfrak{p}}, M_{\mathfrak{p}})e(R/\mathfrak{p})$ *where $\mathfrak{p}$ runs through all the highest dimensional associated prime ideals of M/IM such that* $\dim R/\mathfrak{p} + M_{\mathfrak{p}} = \dim M$.

Example 4.1.7. Let $R = M = \mathbb{Q}[x, y, z]_{(x,y,z)}$ and $I = (xy, yz)$. We have $\dim M = 3$, $\dim M/IM = 2$, and $\ell_M(I) = 2$. Therefore $c_0(I, M) = c_3(I, M) = 0$.

The minimal primes over I are (y) and (x, z). We have $\dim R/(y) = 2$ and $\dim R/(x, z) = 1$. So $\mathrm{Assh}(M/IM) = \{(y)\}$.

The ring $R_{(y)}$ is a regular local ring with maximal ideal (y). Therefore $e(IR_{(y)}, M_{(y)}) = 1$. Similarly, $R/(y)$ is a regular local ring and thus $e(R/(y)) = 1$. Therefore $c_2(I, M) = 1$.

The set of highest dimensional associated prime ideals of M/IM, denoted by $\operatorname{Assh}(M/IM)$, consists of all the prime ideals $\mathfrak{p}$ minimal over $I+\operatorname{Ann}(M)$ with the property $\dim R/\mathfrak{p} = \dim M/IM$.

By Theorem 4.1.6, if I is an ideal of definition on M, then $c_0(I, M) = e(I, M)$ and $c_i(I, M) = 0$ for all $i > 0$. Further, if $\ell_M(I) = \dim M$, then $c_0(I, M) = j(I, M)$. This shows that the multiplicity sequence is indeed a generalization of the Hilbert-Samuel multiplicity and the j-multiplicity.

The following theorem shows that the multiplicity sequence is preserved when passing to a reduction of an ideal.

Theorem 4.1.8. [5, Proposition 2.7] *Let $(R, \mathfrak{m})$ be a local noetherian ring, $I \subseteq R$ an ideal, and M a finitely generated R-module. If I is a reduction of (J, M), then $c_i(I, M) = c_i(J, M)$ for $i = 0, ..., d$.*

5. SUFFICIENTLY GENERAL ELEMENTS

In this section, we prove several technical results about so-called superficial elements in I. These elements can always be chosen to be "sufficiently general." Our goal is to identify such elements $x \in I$ that preserve the multiplicity sequence when passing to the ring R/xR.

5.1. The single graded case

Definition 5.1.1. Let $(R, \mathfrak{m})$ be a local noetherian ring and I an ideal. For a given $r \in I \setminus \{0\}$, let m be the largest integer such that $r \in I^m$. The initial form r^* is defined as the image of r in I^m/I^{m+1}. Then take the largest number n such that $r^* \in \mathfrak{m}^n(I^m/I^{m+1}) \subseteq G_I(R)$. The initial form r' is defined as the image of r^* in $(\mathfrak{m}^n I^m + I^{m+1})/(\mathfrak{m}^{n+1}I^m + I^{m+1}) \subseteq G_{\mathfrak{m}}(G_I(R))$. The initial form I^* of an ideal I of R is defined to be the ideal generated by the initial forms $r^* \in G_I(R)$ of all the elements $r \in I$. The initial form I' of an ideal I of R is defined to be the ideal generated by the initial forms $r' \in G_{\mathfrak{m}}(G_I(R))$ of all the elements $r \in I$.

Remark 5.1.2. Note that $I^* = (I/I^2)G_I(R) = \bigoplus_{n \geq 1} I^n/I^{n+1}$ and $I' = (I/\mathfrak{m}I)G_{\mathfrak{m}}(G_I(R)) = \bigoplus_{i \geq 0} \bigoplus_{j \geq 1} \dfrac{\mathfrak{m}^i I^j + I^{j+1}}{\mathfrak{m}^{i+1} I^j + I^{j+1}}$.

We define what it means for an element to be superficial with respect to a single graded module.

Definition 5.1.3. Let $(R, \mathfrak{m})$ be a local noetherian ring, $I \subset R$ an ideal, M a finitely generated module. Let A be the $G_I(R)$-module $G_I(M)$. An element $x \in I$ is said to be superficial for I with respect to $G_I(M)$ if for some k, $(I^*)^k A \cap (0 :_A x^*) = 0$.

With this definition, a superficial element is one where the inital form is a nonzero divisor for large degrees of the associated graded module. That is, x is superficial for I with respect to $G_I(M)$ if and only if there exists an integer c such that $(I^{n+1}M :_M x) \cap I^c M = I^n M$ for $n \geq c$.

Proposition 5.1.4. *Let $(R, \mathfrak{m})$ be a local noetherian ring with infinite residue field, $I \subset R$ an ideal, and M a finitely generated R-module. If x is superficial for I with respect to $G_I(M)$, then for $n \gg 0$ we have $xM \cap I^n M = xI^{n-1}M$.*

Proof. By the Artin-Rees Lemma there exists k such that for all $n \geq k$ we have $xM \cap I^n M \subseteq xI^{n-k}M$. Further, we have $xM \cap I^n M = x(I^n M :_M x)$. Therefore we have

$$x(I^n M :_M x) \subseteq xI^{n-k}M.$$

We can rewrite the previous equation as

$$(I^n M :_M x) \subseteq I^{n-k}M + (0 :_M x).$$

Since x is superficial for x with respect to $G_I(M)$, there exists c such that for $n \geq c$ we have $(I^n M :_M x) \cup I^c M = I^{n-1}M$. For $n \geq c + k$, we have $n - k \geq c$ and thus $I^{n-k}M \subseteq I^c M$. Therefore, for $n \geq c + k$

$$(I^n M :_M x) \subseteq I^c M + (0 :_M x).$$

Intersecting with $(I^n M :_M x)$, we have the following equality

$$(I^n M :_M x) = (I^c M + (0 :_M x)) \cap (I^n M :_M x)$$

Since $(0 :_M x) \subseteq (I^n M :_M x)$, we can rewrite this as

$$(I^n M :_M x) = (I^c M \cap (I^n M :_M x)) + (0 :_M x).$$

Using the fact that x is superficial for I with respect to $G_I M$, we have

$$(I^n M :_M x) = I^{n-1}M + (0 :_M x).$$

Multiplying by x yields

$$x(I^n M :_M x) = xI^{n-1}M.$$

Therefore $xM \cap I^n M = xI^{n-1}M$. $\square$

Theorem 5.1.5. [8, Proposition 11.1.9] *Let $(R, \mathfrak{m})$ be a local noetherian ring, M a finitely generated module of dimension $d \geq 2$, $I \subseteq R$ an ideal of definition on M, and $x \in I$ a superficial element for*

I with respect to $G_I(M)$ that is not contained in any minimal prime ideal of M. Then $e(I, M) = e(I, M/xM)$.

In the above theorem, it is sufficient to assume that x is a nonzero divisor on M since x would not be in any minimal prime, but not necessary.

We have the following result on the existence of superficial elements.

Proposition 5.1.6. [8, Proposition 8.5.7] *Let $(R, \mathfrak{m})$ be a local noetherian ring with infinite residue field, M a finitely generated R-module, and $I \subset R$ an ideal. Then there exists $x \in I \setminus \mathfrak{m}I$ such that x is a superficial element for I with respect to $G_I(M)$.*

5.2. The bigraded case

Definition 5.2.1. Let $(R, \mathfrak{m})$ be a local noetherian ring, $I \subset R$ an ideal, M a finitely generated R-module, and $B = G_{\mathfrak{m}}(G_I(M))$. An element $x \in I$ is said to be superficial for I with respect to $G_{\mathfrak{m}}(G_I(M))$ if there exists n such that $(I')^n B \cap (0' :_B x') = 0$.

Lemma 5.2.2. *Let R be a noetherian ring, I an ideal, and M a finitely generated R-module. An element x is superficial for I with respect to $G_I(M)$, if and only if x is also superficial for I with respect to $G_I(I^k M)$ for any $k \geq 0$. Similarly, x is superficial for I with respect to $G_{\mathfrak{m}}(G_I(M))$, if and only if x is also superficial for I with respect to $G_{\mathfrak{m}}(G_I(I^k M))$ for any $k \geq 0$.*

Proof. Indeed, if x is superficial for I with respect to $G_I(M)$, then there exists c such that

$$(I^{n+1}M :_M x) \cap I^c M = I^n M$$

for all $n \geq c$. For $k \gg 0$, we also have

$$(I^{n+1}I^k M :_{I^k M} x) = (I^{n+k+1}M :_M x) \cap I^k M.$$

Intersecting with $I^c M$, for $n \geq c$, we obtain

$$(I^{n+1}I^k M :_{I^k M} x) \cap I^c M = (I^{n+k+1}M :_M x) \cap I^c M \cap I^k M = I^{n+k}M \cap I^k M = I^{n+k}M.$$

Therefore, x is superficial for I with respect to $I^k M$ for all $k \geq 0$.

If x is superficial for I with respect to I^kM for some $k \geq 0$, then there exists c such that for $n \geq c$

$$(I^{n+1}I^kM :_{I^kM} x) \cap I^cM = I^{n+k}M$$

Rewriting and intersecting with $I^{c+k}M$ we have

$$(I^{n+1}I^kM :_{I^kM} x) \cap I^cM \cap I^{c+k}M = (I^{n+k+1}M : x) \cap I^cM \cap I^{c+k}M \cap I^kM = I^{n+k}M.$$

Since $I^cM \subseteq I^{c+k}M$ and $I^kM \subseteq I^{c+k}M$, we have

$$(I^{n+k+1}M : x) \cap I^{c+k}M = I^{n+k}M$$

for $n \geq c$. Then for $c' = c + k$ and $n' = n + k$, we have for $n' \geq c'$

$$(I^{n'+1}M : x) \cap I^{c'}M = I^{n'}.$$

Therefore, x is a superficial element for I with respect to $G_I(M)$.

If x is superficial for I with respect to $G_{\mathfrak{m}}(G_I(M))$, then there exists n such that

$$(I')^nB \cap (0' :_B x') = 0.$$

Using the notation as in Definition 5.2.1, $G_{\mathfrak{m}}(G_I(I^kM)) = (I')^kB$. Then we have

$$(I')^n(I')^kB \cap (0' :_{(I')^kB} x') = (I')^{n+k}B \cap (0' :_B x') = 0.$$

Therefore x is superficial for I with respect to $G_{\mathfrak{m}}(G_I(M))$.

If x is superficial for I with respect to $G_{\mathfrak{m}}(G_I(I^kM))$ for some k, then

$$(I')^n(I')^kB \cap (0' :_{(I')^kB} x') = 0.$$

Rewriting, we have

$$(I')^{n+k}B \cap (0' : x') \cap (I')^kB = 0.$$

Since $(I')^{n+k}B \subseteq (I')^k B$ and letting $n' = n + k$, we have

$$(I')^{n'} B \cap (0' : x') = 0.$$

Therefore x is a superficial element for I with respect to $G_{\mathfrak{m}}(G_I(M))$. $\square$

Proposition 5.2.3. [5, Definition 2.8, Remark 2.9] *Let $(R, \mathfrak{m})$ be a local noetherian ring, $I \subset R$ an ideal, M a finitely generated module, $S = G_{\mathfrak{m}}(G_I(R))$ and $N = G_{\mathfrak{m}}(G_I(M))$. Let $(0) = \bigcap_{i=0}^{t} N_i$ be an irredundant primary decomposition of the 0 submodule of N. Denote $P_i = \sqrt{(N_i :_S N)}$ for $i = 0, ..., t$. Assume that $I' \subseteq P_{r+1}, ..., P_t$ and $I' \nsubseteq P_1, ..., P_r$. If $x \in I$ such that $x' \notin P_1, ..., P_r$, then x is superficial for I with respect to $G_{\mathfrak{m}}(G_I(M))$.*

Proposition 5.2.4. *Let $(R, \mathfrak{m})$ be a local noetherian ring with infinite residue field, I an idead, and M a finitely generated R-module. Then there exists $x \in I \setminus \mathfrak{m}I$ such that x is a superficial element for I with respect to $G_{\mathfrak{m}}(G_I(M))$.*

Proof. By Proposition 5.2.3, if x is superficial with respect to $G_{\mathfrak{m}}(G_I(M))$ then the initial form x' avoids the primes P_i $(i = 1, ..., r)$ in the bigraded ring $G_{\mathfrak{m}}(G_I(R))$. Since $I' \nsubseteq P_i$ for all $i \leq r$, we define $Q_i = P_i \cap (I/\mathfrak{m}I) \subsetneq I/\mathfrak{m}I$; that is, each Q_i is a proper subspace of the $(R/\mathfrak{m})$-vector space $I/\mathfrak{m}I$. Since $R/\mathfrak{m}$ is an infinite field, $(I/\mathfrak{m}I) \setminus \bigcup_{i=1}^{r} Q_i$ is nonempty. Then every element $x \in I \setminus \mathfrak{m}I$ whose image in $(I/\mathfrak{m}I)$ is not in $\bigcup_{i=1}^{r} Q_i$ is a superficial element with respect to $G_{\mathfrak{m}}(G_I(M))$. $\square$

We can choose a superficial element for I with respect to $G_{\mathfrak{m}}(G_I(M))$ that avoids all the $R/\mathfrak{m}$-vector subspaces of the form $I/\mathfrak{m}I \cap \mathfrak{p}'$, where $\mathfrak{p}$ is a prime ideal that does not contain I.

Proposition 5.2.5. *Let $(R, \mathfrak{m})$ be a local noetherian ring with infinite residue field, I an idead, and M a finitely generated R-module. If $\operatorname{depth}_I(M) > 0$, then there exists $x \in I \setminus \mathfrak{m}I$ superficial element that is a nonzero divisor on M.*

Proof. Since $\operatorname{depth}_I(M) > 0$, the set $I \setminus \bigcup_{\mathfrak{p} \in \operatorname{Ass}(M)} \mathfrak{p}$ is nonempty. Therefore, the subspace $((I \cap \bigcup_{\mathfrak{p} \in \operatorname{Ass}(M)} \mathfrak{p}) + \mathfrak{m}I)/\mathfrak{m}I$ is a proper subspace of $I/\mathfrak{m}I$. We may then set $Q_0 = (I \cap \bigcup_{\mathfrak{p} \in \operatorname{Ass}(M)}) + \mathfrak{m}I)/\mathfrak{m}I$, so $(I/\mathfrak{m}I) \setminus \bigcup_{i=0}^{r} Q_i$ is nonempty and proceed as above. $\square$

Proposition 5.2.6. *Let $(R, \mathfrak{m})$ be a local noetherian ring, M a finitely generated R-module, and $I \subseteq R$ an ideal. If $\ell_M(I) > 0$, then there exists c such that $\operatorname{depth}_I(I^c M) > 0$.*

Proof. Let $0 = N_1 \cap ... \cap N_t$ be an irredundant primary decomposition of the 0 submodule of M and let $\mathfrak{p}_i = \sqrt{(N_i :_R M)}$. Further, assume that $I \not\subseteq \mathfrak{p}_i$ for $i = 1, ..., r$ and $I \subseteq \mathfrak{p}_i$ for $i = r+1, ..., t$. There exists c such that $I^c M \subseteq \bigcap_{i=r+1}^t N_i$. Let $x \in I$ be an element such that $x \notin \mathfrak{p}_i$ for $i = 1, ..., r$. We have $(0 :_M x) = \bigcap_{i=1}^t (N_i : x) \subseteq \bigcap_{i=1}^r N_i$, hence $I^c M \cap (0 :_M x) = 0$. Since $\ell_M(I) > 0$, $I^c M \neq 0$. Therefore, $x \in I$ is a nonzero divisor on $I^c M$, that is $\operatorname{depth}_I(I^c M) > 0$. □

Proposition 5.2.7. *Let $(R, \mathfrak{m})$ be a local noetherian ring with infinite residue field, M a finitely generated R-module, and $I \subset R$ an ideal. If $\ell_M(I) > 0$, then there exists an integer c and element $x \in I$ such that x is a superficial element for I with respect to $G_{\mathfrak{m}}(G_I(M))$ and a nonzero divisor on $I^c M$.*

Proof. This is a consequence of Lemma 5.2.2, Proposition 5.2.5, and Proposition 5.2.6. By Proposition 5.2.6, there exists c such that $\operatorname{depth}_I(I^c M) > 0$. By Proposition 5.2.5, there exists x such that x is superficial with respect to $G_{\mathfrak{m}}(G_I(I^c M))$ and a nonzero divisor on $I^c M$. By Lemma 5.2.2, x is also superficial with respect to $G_{\mathfrak{m}}(G_I(M))$. □

The following result shows that this choice of superficial element preserves the multiplicity sequence.

Proposition 5.2.8. [5, Theorem 2.11] *Let $(R, \mathfrak{m})$ be a local noetherian ring, $I \subset R$ an ideal, and M a finitely generated R-module. Let $x \in I \setminus \mathfrak{m}I$ be a superficial element for I with respect to $G_{\mathfrak{m}}(G_I(M))$ such that x is a nonzero divisor on M. Then $c_i(I, M) = c_i(I, M/xM)$ for $i \leq d-2$.*

Since we can always choose a superficial element that is a nonzero divisor on $I^n M$ for $n \gg 0$ provided $\ell_M(I) > 0$, we now show that the multiplicity sequence does not significantly change when we replace M with $I^n M$ for n large enough.

Proposition 5.2.9. *Let $(R, \mathfrak{m})$ be a local noetherian ring, M a finitely generated module with $\dim M = d$, and $I \subset R$ an ideal. Assume that $\dim M = \dim I^n M$ for all n. Then, for $n \gg 0$ $c_i(I, M) = c_i(I, I^n M)$ for $i \leq d-1$ and $c_d(I, I^n M) = 0$.*

Proof. Consider the Hilbert function

$$h_{I,M}(u, v) = \sum_{i=0}^{u} \sum_{j=0}^{v} \lambda \left(\frac{\mathfrak{m}^i I^j M + I^{j+1} M}{\mathfrak{m}^{i+1} I^j M + I^{j+1} M} \right)$$

which is eventually a polynomial $p_{I,M}(u,v)$ for $u,v \gg 0$. The degree d part of this polynomial $p_{I,M}(u,v)$ is:

$$\sum_{k+l=d} \frac{c_k(I,M)}{k!l!} u^k v^l.$$

We can now write $h_{I,I^nM}(u,v)$ as

$$\begin{aligned} h_{I,I^nM}(u,v) &= \sum_{i=0}^{u}\sum_{j=0}^{v} \lambda(\mathfrak{m}^i I^{j+n}M + I^{j+n+1}M/\mathfrak{m}^{i+1}I^{j+n}M + I^{j+n+1}M) \\ &= h_{I,M}(u,v+n) - h_{I,M}(u,n-1) \end{aligned}$$

which implies that for $u,v,n \gg 0$

$$p_{I,I^nM}(u,v) = p_{I,M}(u,v+n) - p_{I,M}(u,n-1).$$

Since the polynomials $p_{I,I^nM}(u,v)$ and $p_{I,M}(u,v+n) - p_{I,M}(u,n-1)$ are equal, we have an equality on the degree d parts. For a fixed n large enough, the degree d part of $p_{I,M}(u,v+n)$ as a polynomial in u and v is the same as the degree d part of $p_{I,M}(u,v)$ and the degree d part of $p_{I,M}(u,n-1)$ is

$$\frac{c_d(I,M)}{d!} u^d.$$

Since $\dim I^nM = \dim M$ we can conclude that $c_k(I,M) = c_k(I,I^nM)$ for $k \le d-1$ and $c_d(I,I^nM) = 0$. $\square$

If the degree of $p_{I,I^nM}(u,v)$ is $d = \dim M$, then $d \le \dim I^nM \le \dim M = d$ and so we get the following remark.

Remark 5.2.10. Let $(R,\mathfrak{m})$ be a local noetherian ring, M a finitely generated R-module of dimension d, and $I \subseteq R$ an ideal. If $c_k(I,M) \neq 0$ for some $k \neq d$, then $\dim M = \dim I^nM$ for all n.

5.3. Superficial elements with respect to multiple modules

Remark 5.3.1. Since a superficial element for I with respect to $G_I(M)$ or $G_{\mathfrak{m}}(G_I(M))$ can be obtained by avoiding finitely many proper subspaces of the $(R/\mathfrak{m})$-vector space $I/\mathfrak{m}I$, we may repeat the process finitely many times. In this way we can find a sufficiently general element that is superficial for I with respect to finitely many modules. So long as the residue field $R/\mathfrak{m}$ is infinite, such an element will always exist.

Proposition 5.3.2. [8, Proposition 17.2.2] *Let $(R, \mathfrak{m})$ be a local noetherian ring with infinite residue field, I and J ideals of R, and M a finitely generated R-module. Then there exists $x \in I$ such that x is superficial for I with respect to $G_I(J^j M)$ for all $j \geq 0$.*

Proposition 5.3.3. *Let $(R, \mathfrak{m})$ be a local noetherian ring with infinite residue field, I and J ideals of R, and M a finitely generated R-module. Then there exists $x \in I$ such that x is superficial for I with respect to $G_{\mathfrak{m}}(G_I(J^j M))$ for all $j \geq 0$.*

Proof. Let S be the finitely generated R-algebra

$$\bigoplus_{i,j,n \geq 0} \frac{\mathfrak{m}^n J^j I^i + J^j I^{i+1}}{\mathfrak{m}^{n+1} J^j I^i + J^j I^{i+1}}$$

and N be the finitely generated S-module

$$\bigoplus_{i,j,n \geq 0} \frac{\mathfrak{m}^n I^i J^j M + I^{i+1} J^j M}{\mathfrak{m}^{n+1} I^i J^j M + I^{i+1} J^j M}.$$

Note that S is a $\mathbb{Z}^3$-graded ring generated in degrees $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ and N is a $\mathbb{Z}^3$-graded S-module. Denote

$$S_k = \bigoplus_{j,n \geq 0, i \geq k} \frac{\mathfrak{m}^n J^j I^i + J^j I^{i+1}}{\mathfrak{m}^{n+1} J^j I^i + J^j I^{i+1}}.$$

Let $\bigcap_{i=1}^t N_i = 0$ be an irredundant primary decomposition of the 0 submodule of N. Let $P_i = \sqrt{N_i :_S N}$ and assume that $S_{(0,1,0)} = I/\mathfrak{m}I$ is not contained in $P_1, ..., P_r$ and $S_{(0,1,0)}$ is contained in $P_{r+1}, ..., P_t$. Since $R/\mathfrak{m}$ is an infinite field and P_i does not contain $I/\mathfrak{m}I$ for $i = 1, ..., r$, the set

$$(I/\mathfrak{m}I) \setminus \bigcup_{i=1}^r ((I/\mathfrak{m}I) \cap P_i)$$

is not empty.

Let $x \in I$ such that $\overline{x} \in (I/\mathfrak{m}I) \setminus \bigcup_{i=1}^t (I/\mathfrak{m}I \cap P_i)$. Note that $I/\mathfrak{m}I \subseteq S_1$ and $I/\mathfrak{m}I$ generates S_1 as an ideal. For $i = r+1, ..., t$ we have $I/\mathfrak{m}I \subseteq P_i$, therefore $S_1 \subseteq P_i$ and so there exists some c

such that $S_1^c N = S_c N \subseteq N_i$ for $i = r+1, ..., t$.

Now consider $(0 :_N \overline{x})$. Using the irredundant primary decomposition of the 0 submodule, we have $(0 :_N \overline{x}) = \bigcap_{i=1}^t (N_i :_N \overline{x})$. For $i = 1, ..., r$, we have $\overline{x} \notin P_i$ thus $(N_i :_N \overline{x}) = N_i$, which implies that $(0 :_N \overline{x}) \subseteq \bigcap_{i=1}^r N_i$.

For $c \gg 0$ we have $(0 :_N \overline{x}) \cap S_c N \subseteq \bigcap_{i=1}^r N_i \cap \bigcap_{i=r+1}^t N_i = 0$. Therefore $(0 :_N \overline{x}) \cap S_c N = 0$ for $c \gg 0$.

For each j let $N'_j = G_{\mathfrak{m}}(G_I(J^j M)) \subseteq N$ and $S'_j = G_{\mathfrak{m}}(G_I(J^j)) \subseteq S$. Note that

$$S_c \cap S'_0 = \bigoplus_{i \geq c, n \geq 0} \frac{\mathfrak{m}^n I^i + I^{i+1}}{\mathfrak{m}^{n+1} I^i + I^{i+1}} = I^c G_{\mathfrak{m}}(G_I(R)).$$

We have

$$(0 :_{N'_j} \overline{x}) \cap (S_c \cap S'_0) N'_j \subseteq (0 :_N \overline{x}) \cap S_c N = 0.$$

Therefore

$$(0 :_{G_{\mathfrak{m}}(G_I(J^j M)} \overline{x})) \cap I^c G_{\mathfrak{m}}(G_I(J^j M)) = 0.$$

By Definition 5.2.1, x is a superficial element for I with respect to $G_{\mathfrak{m}}(G_I(J^j M))$ for all j. $\square$

We will now show that we can find a superficial element that is compatible with various localizations.

Lemma 5.3.4. *Let $(R, \mathfrak{m})$ be a local ring with infinite residue field k, $\mathfrak{p} \subseteq R$ a prime ideal, $(S, \mathfrak{n})$ the local domain $R/\mathfrak{p}$, $Q(S)$ the fraction field of S. Let $\pi : S^n \twoheadrightarrow k^n$ be the natural surjection. Then for a subspace $V \subseteq Q(S)^n$, we have $\pi(V \cap S^n) = k^n$ if and only if $V = Q(S)^n$.*

Proof. Note that it is clear that if $V = Q(S)^n$, then $\pi(V \cap S^n) = k^n$. Suppose $\pi(V \cap S^n) = k^n$. Then $\pi(V \cap S^n) = (V \cap S^n + \mathfrak{n}S^n)/\mathfrak{n}S^n = S^n/\mathfrak{n}S^n$; therefore, by Nakayama's Lemma, we have $(V \cap S^n) + \mathfrak{n}S^n = S^n$ and $V \cap S^n = S^n$. Thus $V \supseteq S^n$. Since V is a vector space that contains S^n, in particular it contains the standard basis, therefore $V = Q(S)^n$. $\square$

Proposition 5.3.5. *Let $(R, \mathfrak{m})$ be a local noetherian ring with infinite residue field k and $I \subset R$ an ideal, $I = (x_1, ..., x_n)$. Let $T : k^n \to I/\mathfrak{m}I$ defined by $T(\overline{a_1}, ..., \overline{a_n}) = \overline{a_1 x_1} + ... + \overline{a_n x_n}$. Then for finitely many proper subspaces V_i $(i = 1, ..., r)$ of $I/\mathfrak{m}I$, there exists $(\overline{a_1}, ..., \overline{a_n})$ such that $T(\overline{a_1}, ..., \overline{a_n}) \notin \bigcup_{i=1}^r V_i$.*

Proof. Since $I/\mathfrak{m}I \cong k^s$ for some $s \le n$, there exists $x \in I/\mathfrak{m}I$ that avoids $\bigcup_{i=1}^r V_i$. By the surjectivity of T, there exists $(\overline{a_1}, ..., \overline{a_n}) \in k^n$ such that $T(\overline{a_1}, ..., \overline{a_n}) = x \notin \bigcup V_i$. $\square$

Proposition 5.3.6. *Let $(R, \mathfrak{m})$ be a local noetherian ring with infinite residue field k, $I \subset R$ an ideal, and M a finitely generated module. Then there exists $x \in I$ such that x is superficial for I with respect to $G_{\mathfrak{m}}(G_I(M))$ and $\frac{x}{1} \in IR_{\mathfrak{p}}$ is superficial for $IR_{\mathfrak{p}}$ with respect to $G_{\mathfrak{p}R_{\mathfrak{p}}}(G_{IR_{\mathfrak{p}}}(M_{\mathfrak{p}}))$ for all prime ideals $\mathfrak{p}$ such that $\ell_{M_{\mathfrak{p}}}(IR_{\mathfrak{p}}) = \dim M_{\mathfrak{p}}$.*

Proof. Let $I = (x_1, ..., x_n)$. By Remark 3.2.8 there are only finitely many primes $\mathfrak{p}$ with $\ell_{M_{\mathfrak{p}}}(IR_{\mathfrak{p}}) = \dim M_{\mathfrak{p}}$. Let $\{\mathfrak{p}_1, ..., \mathfrak{p}_t\}$ be the finite set of primes with this property, denote $\mathfrak{p}_0 = \mathfrak{m}$ and let $k_i = Q(R/\mathfrak{p}_i)$ the field of fractions of $R/\mathfrak{p}_i$. Let $\pi_i : (R/\mathfrak{p}_i)^n \twoheadrightarrow k^n$ be the natural surjection. By Proposition 5.3.5 we may consider subspaces of k_i^n instead of subspaces of $IR_{\mathfrak{p}_i}/\mathfrak{p}_i IR_{\mathfrak{p}_i}$. For each i, let $V_{(i,j)} \subsetneq k_i^n$ be the finitely many subspaces of k_i^n such that $a_1x_1 + ... + a_nx_n \in IR_{\mathfrak{p}_i} \setminus \mathfrak{p}_i IR_{\mathfrak{p}_i}$ is superficial whenever $(\overline{a_1}, ..., \overline{a_n}) \in k_i^n \setminus \bigcup_j V_{(i,j)}$.

By Lemma 5.3.4, for every i, j we have that $\pi_i(V_{(i,j)} \cap (R/\mathfrak{p}_i)^n)$ is a proper subspace of k^n and therefore $\bigcup_{i,j} \pi_i(V_{(i,j)} \cap (R/\mathfrak{p}_i)^n)$ is a union of finitely many proper subspaces of k^n. If $(\overline{a_1}, ..., \overline{a_n}) \in k^n \setminus \bigcup_{i,j} \pi_i(V_{(i,j)} \cap (R/\mathfrak{p}_i)^n)$, then $a_1x_1 + ... + a_nx_n \in IR_{\mathfrak{p}_i} \setminus \mathfrak{p}_i IR_{\mathfrak{p}_i}$ is superficial for $(IR_{\mathfrak{p}_i}, M_{\mathfrak{p}_i})$ for all $i = 0, ..., t$. $\square$

6. THE MAIN RESULT

In this section we prove a result relating the multiplicities $c_i(I,M)$ to local multiplicities that is similar to Proposition 4.1.6 (*iii*). We will use induction on the dimension of the module M and the idea of sufficiently general elements discussed in the previous section. We will use sufficiently general element to mean an element that satifies certain Zariski open properties, including being superficial with respect to some modules and being a nonzero divisor.

6.1. Sequences of sufficiently general elements

First we prove several properties of a single sufficiently general element. We will then established the existence of a set of element that were defined sequentially that have certain properties. This sequence will be referred to as a sufficiently general sequence.

Lemma 6.1.1. *Let $(R,\mathfrak{m})$ be a local noetherian ring with infinite residue field, $I \subset R$ an ideal, M a finitely generated R-module such that $\dim M = d \geq 2$ and $c_i(I,M) \neq 0$ for some $i \leq d-1$. Let $x \in I$ be a sufficiently general element. Then $I^k(I^nM/xI^nM) \cong I^{n+k}(M/xM)$ for $n,k \gg 0$.*

Moreover, if $c_i(I,M) \neq 0$ for some $i \leq d-2$, then $\dim I^n(M/xM) = d-1$ and for $n \gg 0$

$$c_i(I,M) = c_i(I,I^nM) = c_i(I,I^nM/xI^nM) = c_i(I,I^n(M/xM))$$

for all $i \leq d-2$.

Proof. Since $c_i(I,M) \neq 0$ for some $i \leq d-1$, by Remark 5.2.10 we have that $\dim M = \dim I^nM$ for every n. By Proposition 4.1.6 part i, we have $\ell_M(I) > 0$ and by Proposition 5.2.6 there exists c such that $\operatorname{depth}_I(I^cM) \neq 0$. By Propositions 5.1.6 and 5.2.7 there exists $x \in I$ superficial for I with respect to both $G_I(M)$ and $G_{\mathfrak{m}}(G_I(M))$ that is a nonzero divisor on I^cM. By Lemma 5.2.2, x is also superficial for I with respect to $G_{\mathfrak{m}}(G_I(I^nM))$ for all n.

Consider $I^{n+k}(M/xM)$ and $I^k(I^nM/xI^nM)$ for $n \geq c$ and $k \gg 0$. We have

$$I^k\left(\frac{I^nM}{xI^nM}\right) = \frac{I^{n+k}M + xI^nM}{xI^nM} \cong \frac{I^{n+k}M}{xI^nM \cap I^{n+k}M} = \frac{I^{n+k}M}{x(I^{n+k}M :_{I^nM} x)}.$$

Since $x \in I$ is a superficial element for I with respect to $G_I(M)$, for $n \geq c$ and $k \gg 0$ we have $(I^{n+k}M :_{I^nM} x) = (I^{n+k}M :_M x) \cap I^nM = I^{n+k-1}M$. Therefore, for $n \geq c$ and $k \gg 0$, we have

$$\frac{I^{n+k}M}{x(I^{n+k}M :_{I^nM} x)} = \frac{I^{n+k}M}{xI^{n+k-1}M}.$$

We also have

$$I^{n+k}\left(\frac{M}{xM}\right) = \frac{I^{n+k}M + xM}{xM} \cong \frac{I^{n+k}M}{xM \cap I^{n+k}M} = \frac{I^{n+k}M}{xI^{n+k-1}M}$$

for $n \geq c$ and $k \gg 0$.

The last equality follows from Proposition 5.1.4. Therefore $I^k(I^nM/xI^nM) \cong I^{n+k}(M/xM)$ for $n \geq c$ and $k \gg 0$.

By Proposition 5.2.9 we have

$$c_i(I, M) = c_i(I, I^nM) = c_i(I, I^{n+k}M)$$

for all $i \leq d-1$. Fix $n \geq c$ so that the above equation is true and the isomorphism from the first part of the Lemma holds. Since x is a nonzero divisor on I^nM, and thus a nonzero divisor on $I^{n+k}M$, and superficial for I with respect to $G_{\mathfrak{m}}(G_I(M))$ and $\dim I^nM = d$, we have

$$c_i(I, I^nM) = c_i(I, I^nM/xI^nM) = c_i(I, I^{n+k}M/xI^{n+k}M)$$

for all $i \leq d-2$ (Proposition 5.2.8). Note that $\dim I^nM/xI^nM = d-1$ since x is a nonzero divisor on I^nM. Therefore $c_i(I, I^nM/xI^nM) \neq 0$ for some $i \leq d-2$ and thus, we have $\dim I^k(I^nM/xI^nM) = d-1$ for all k by Remark 5.2.10 and

$$c_i(I, I^nM/xI^nM) = c_i(I, I^k(I^nM/xI^nM))$$

for all $i \leq d-2$ by Proposition 5.2.9.

Since $I^k(I^nM/xI^nM) \cong I^{n+k}(M/xM)$, we have $\dim I^{n+k}(M/xM) = d-1$ and

$$c_i(I, I^k(I^nM/xI^nM)) = c_i(I, I^{n+k}(M/xM))$$

for all $i \leq d-2$ and $k \gg 0$.

Therefore we have

$$c_i(I, M) = c_i(I, I^{n+k}M) = c_i(I, I^{n+k}M/xI^{n+k}M) = c_i(I, I^{n+k}(M/xM))$$

for $k \gg 0$.

We may replace $n+k$ with just n for $n \gg 0$ and we get

$$c_i(I, M) = c_i(I, I^nM) = c_i(I, I^nM/xI^nM) = c_i(I, I^n(M/xM)).$$

□

Corollary 6.1.2. *Let $(R, \mathfrak{m})$ be a local noetherian ring with infinite residue field, $I \subset R$ an ideal, and M a finitely generated R-module such that* $\dim M = d \geq 2$. *Assume that* $\operatorname{depth}_I(I^nM) > 0$ *and $c_i(I, M) \neq 0$ for some $i \leq d-2$. Let $x \in I$ be a sufficiently general element. Then*

$$c_i(I, M) = c_i(I, M/xM)$$

for all $i \leq d-2$.

Proof. We can choose x to be a superficial for I with respect to $G_{\mathfrak{m}}(G_I(M))$ and a nonzero divisor on M by Proposition 5.2.5. From Lemma 6.1.1 we have $c_i(I, M) = c_i(I, I^n(M/xM))$ and $\dim I^n(M/xM) = d-1$ for $n \gg 0$. In this case, we have $\dim M/xM = \dim I^n(M/xM) = d-1$ for $n \gg 0$ by Lemma 6.1.1. For all $k = 1, ..., n-1$, we have $I^n(M/xM) \subseteq I^k(M/xM) \subseteq M/xM$ and thus $\dim I^n(M/xM) = \dim M/xM$ for all n. Therefore by Proposition 5.2.9, for $n \gg 0$ we have $c_i(I, M/xM) = c_i(I, I^n(M/xM)) = c_i(I, M)$ for $i = 0, ..., d-2$. □

Proposition 6.1.3. *Let $(R, \mathfrak{m})$ be a local ring with infinite residue field, $I \subset R$, and M a finitely generated R-module with* $\dim M = d$. *Assume $c_i(I, M) \neq 0$ for some $i \leq d-1$ and let $r =$*

$\min\{i \mid c_i(I,M) \neq 0\}$. *Then there exists a sufficiently general sequence of elements* $x_2, ..., x_{d-r} \in I$ *such that the following properties hold for all* $k \leq d-r$ *and* $n \gg 0$*:*

i. $\dim I^n(M/(x_2, ..., x_k)M) = d - (k-1)$*;*

ii. $\dim I^n(M_{\mathfrak{p}}/(x_2, ..., x_k)M_{\mathfrak{p}}) = \dim M_{\mathfrak{p}} - (k-1)$ *for all* $\mathfrak{p}$ *such that* $c_0(IR_{\mathfrak{p}}, M_{\mathfrak{p}}) \neq 0$ *and* $\dim M_{\mathfrak{p}} \geq k$*;*

iii. $c_i(I,M) = c_i(I, I^n(M/(x_2, ..., x_k)M))$ *for* $i \leq d-k$*;*

iv. $c_0(IR_{\mathfrak{p}}, M_{\mathfrak{p}}) = c_0(IR_{\mathfrak{p}}, I^n(M_{\mathfrak{p}}/(x_2, ..., x_k)M_{\mathfrak{p}}))$ *for all* $\mathfrak{p}$ *such that* $c_0(IR_{\mathfrak{p}}, M_{\mathfrak{p}}) \neq 0$ *and* $\dim M_{\mathfrak{p}} \geq k$*.*

Proof. If $\dim M \leq 1$ or $r = d-1$, then the sequence of elements is empty, so we may assume $r \leq d-2$. Note that $\ell_M(I) > 0$ since $c_i(I,M) \neq 0$ for some $i \leq d-1$ by Theorem 4.1.6.

Choose x_2 to be a superficial element for I with respect to $G_I(M)$ and $G_{\mathfrak{m}}(G_I(M))$ that is a nonzero divisor on I^cM for some c. Such an element exists by Proposition 5.2.7, Proposition 5.1.6 and Remark 5.3.1. Further by 5.3.6, we may also require that $\frac{x_2}{1}$ is superficial for $IR_{\mathfrak{p}}$ with respect to $G_{\mathfrak{p}R_{\mathfrak{p}}}(G_{IR_{\mathfrak{p}}}(M_{\mathfrak{p}}))$ for all $\mathfrak{p} \in \operatorname{Asym}(I,M)$. Since x_2 is a nonzero divisor on I^cM we have $\frac{x_2}{1}$ is a nonzero divisor on $I^cM_{\mathfrak{p}}$.

By Lemma 6.1.1, we have $\dim I^n(M/x_2M) = d-1$ and $\dim I^n(M_{\mathfrak{p}}/x_2M_{\mathfrak{p}}) = \dim M_{\mathfrak{p}} - 1$ for all $\mathfrak{p}$ where $c_0(IR_{\mathfrak{p}}, M_{\mathfrak{p}}) \neq 0$ and $\dim M_{\mathfrak{p}} \geq 2$ for $n \gg 0$, so (*i*) and (*ii*) are satisified. Further, by Lemma 6.1.1 we have $c_i(I,M) = c_i(I, I^n(M/x_2M))$ for $i \leq d-2$, hence (*iii*) holds. Similarly, since $\dim M_{\mathfrak{p}} \geq 2$, we have $c_0(IR_{\mathfrak{p}}, M_{\mathfrak{p}}) = c_0(IR_{\mathfrak{p}}, I^n(M_{\mathfrak{p}}/x_2M_{\mathfrak{p}}))$, so (*iv*) is satisfied as well.

Note that $c_i(I, I^n(M/x_2M)) = 0$ for all $i < r$ and $c_r(I, I^n(M/x_2M)) \neq 0$. We can repeat this process if $r \leq d-3$ and $d \geq 3$.

Let $k \leq d-r$ and suppose there exist $x_2, ..., x_{k-1} \in I$ such that the conditions (*i*) – (*iv*) are satisfied. Denote the module $M/(x_2, ..., x_{k-1})M$ by M'. We will choose x_k in the same manner as x_2, by replacing the module M with I^nM' for some $n \gg 0$, that is, x_k is superficial for I with respect to $G_I(M')$ and $G_{\mathfrak{m}}(G_I(M'))$; a nonzero divisor for $I^n(M')$ for some $n \gg 0$. Further, we may assume that $\frac{x_k}{1}$ is superficial for $IR_{\mathfrak{p}}$ with respect to $G_{\mathfrak{p}R_{\mathfrak{p}}}(G_{IR_{\mathfrak{p}}}(M'_{\mathfrak{p}})$ for all $\mathfrak{p} \in \operatorname{Asym}(I, M')$.

The element x_k is superficial for I with respect to $G_I(M')$ and $G_{\mathfrak{m}}(G_I(M'))$ and a nonzero divisor on I^nM' for $n \gg 0$. Since $k \leq d-r$ and $\dim I^nM' = d-k+2$, we have

$$c_r(I, M) = c_r(I, I^nM')$$

for $n \gg 0$. Note that $r \leq d-k = \dim I^nM' - 2$. Therefore, by Lemma 6.1.1 we have

$$\dim I^{n'}(I^nM'/x_kI^nM') = \dim I^nM' - 1$$

for $n' \gg 0$. Further, by Lemma 6.1.1 we have

$$I^{n'}(I^nM'/x_kI^nM') \cong I^{n+n'}(M'/x_kM').$$

We can replace $n+n'$ with n for $n \gg 0$. Note that $M'/x_kM' \cong M/(x_2, ..., x_k)M$ and by assumption $\dim I^nM' = d-k+2$. Therefore, for $n \gg 0$ we have

$$\dim I^n(M/x_2, ..., x_k)M = d-k+1 = d-(k-1)$$

and thus (i) is proved.

In a similar fashion, if $\dim I^n(M'_{\mathfrak{p}}) \geq 2$, then $\dim I^n(M'_{\mathfrak{p}}/x_kM'_{\mathfrak{p}}) = \dim I^n(M'_{\mathfrak{p}}) - 1$. Since $\dim I^n(M'_{\mathfrak{p}}) \geq 2$, we have $\dim M_{\mathfrak{p}} \geq k$ and $\dim I^n(M_{\mathfrak{p}}/(x_2, ..., x_k)M_{\mathfrak{p}}) = \dim M_{\mathfrak{p}} - (k-1)$.

Since $\dim I^nM' = d-(k-2)$ and x_k is superficial for I with respect to $G_{\mathfrak{m}}(G_I(M'))$ by Lemma 5.2.2 it is also superficial for I with respect to $G_{\mathfrak{m}}(G_I(I^nM'))$. By assumption, we have $c_r(I, I^nM') \neq 0$ and $r \leq d-k = \dim I^nM - 2$. Therefore by Lemma 6.1.1, we have

$$c_i(I, I^nM') = c_i(I, I^{n'}(I^nM'/x_kI^nM)) = c_i(I, I^{n'+n}(M/(x_2, ..., x_k)M))$$

for $i \leq \dim I^nM' - 2 = d-k$ and for $n' \gg 0$. Again, we can replace $n'+n$ with n for $n \gg 0$ and (iii) is proved. for $i \leq \dim I^nM' - 2 = d-k$ and for $n' \gg 0$. Again, we can replace $n'+n$ with n for $n \gg 0$ and (iii) is proved.

By assumption $\dim I^n M'_{\mathfrak{p}} = d-(k-2) \geq 2$ and $c_0(IR_{\mathfrak{p}}, I^n M'_{\mathfrak{p}}) \neq 0$. Since $\frac{x_2}{1}$ is superficial for $IR_{\mathfrak{p}}$ with respect to $G_{\mathfrak{p}R_{\mathfrak{p}}}(G_{IR_{\mathfrak{p}}}(M_{\mathfrak{p}}))$, we have

$$c_0(IR_{\mathfrak{p}}, I^n M'_{\mathfrak{p}}) = c_0(IR_{\mathfrak{p}}, I^n(M_{\mathfrak{p}}/(x_2,...,x_k)M_{\mathfrak{p}})$$

and thus (iv) is proved. $\square$

6.2. A formula for the generalized multiplicity sequence

We we use the previously defined sequence of sufficiently general elements to give a generalization of the formula given by Theorem 4.1.6(3).

Theorem 6.2.1. *Let $(R,\mathfrak{m})$ be a local noetherian ring with infinite residue field, $I \subseteq R$ an ideal and, M a finitely generated R-module of dimension d. Let $r = \min\{i \mid c_i(I,M) \neq 0\}$ and let $x_2,...,x_{d-r} \in I$ be a sufficiently general sequence. Assume that $r < d$. For each $k = r,...,d$ denote Λ_k as*

$$\Lambda_k(I,M) = \{\mathfrak{p} \in \operatorname{Supp}(M/IM) \mid \dim R/\mathfrak{p} = k \text{ and } \dim I^n(M_{\mathfrak{p}}/(x_2,...,x_{d-k})M_{\mathfrak{p}}) = 1 \text{ for } n \gg 0\}.$$

Then for $n \gg 0$ and k=r,...,$d-1$

$$c_k(I,M) = \sum_{\mathfrak{p}\in\Lambda_k} e(IR_{\mathfrak{p}}, I^n(M_{\mathfrak{p}}/(x_2,...,x_{d-k})M_{\mathfrak{p}}))e(R/\mathfrak{p}).$$

Proof. By assumption, $r < d$, and thus $\dim I^n M = \dim M$ for all n by Remark 5.2.10 and $\ell_M(I) > 0$ by Proposition 4.1.6. Note that by Proposition 6.1.3(i), for $k = r,...,d-2$ and $\mathfrak{p} \in \Lambda_k$ we have $\dim I^n(M/(x_2,...,x_{d-k})M) = k+1 = \dim R/\mathfrak{p} + \dim I^n(M_{\mathfrak{p}}/(x_2,...,x_{d-k})M_{\mathfrak{p}})$.

Next, note that for $n \gg 0$

$$\Lambda_k(I,M) \subseteq \operatorname{Assh}\left(\frac{I^n(M/(x_2,...,x_{d-k})M)}{I^{n+1}(M/(x_2,...,x_{d-k})M)}\right).$$

Thus $\Lambda_k(I,M)$ is finite. Indeed, if $\mathfrak{p} \in \Lambda_k(I,M)$, then $\mathfrak{p} \supseteq I$ since $\mathfrak{p} \in \operatorname{Supp}(M/IM)$. Further

$\operatorname{Ann}(I^n(M/(x_2,...,x_{d-k})M)) \subseteq \mathfrak{p}$, and thus by Nakayama's Lemma

$$\mathfrak{p} \in \operatorname{Supp}\left(\frac{I^n(M/(x_2,...,x_{d-k})M)}{I^{n+1}(M/(x_2,...,x_{d-k})M)}\right).$$

We have $\mathfrak{p} \supseteq I + \operatorname{Ann}(I^n(M/(x_2,...,x_{d-k})M))$. By Proposition 6.1.3 $\dim I^n(M/(x_2,...,x_{d-k})M) = k+1$, we have $\dim R/\operatorname{Ann}(I^n(M/(x_2,...,x_{d-k})M)) = k+1$. Because $I^n(M/(x_2,...,x_{d-k})M) \neq 0$, I contains a nonzero divisor on $I^n(M/(x_2,...,x_{d-k})M) \neq 0$ for $n \gg 0$. Thus I is not contained in any minimal prime $\operatorname{Ann}(I^n(M/(x_2,...,x_{d-k})M))$ and we have $\dim R/(I+\operatorname{Ann}(I^nM/(x_2,...,x_{d-k})M)) \leq k$. Since $\mathfrak{p} \supseteq I + \operatorname{Ann}(I^n(M/(x_2,...,x_{d-k})M))$ and by assumption we have $\dim R/\mathfrak{p} = k$, we must have $\dim R/(I + \operatorname{Ann}(I^nM/(x_2,...,x_{d-k})M)) = k$. Therefore,

$$\mathfrak{p} \in \operatorname{Assh}\left(\frac{I^n(M/(x_2,...,x_{d-k})M)}{I^{n+1}(M/(x_2,...,x_{d-k})M)}\right).$$

Next, note that $\Lambda_{d-1}(I,M) = \Lambda_{d-1}(I,I^nM)$ for $n \gg 0$. Indeed, if $\mathfrak{p} \in \operatorname{Supp}(I^nM/I^{n+1}M)$, then $\mathfrak{p} \supseteq I + \operatorname{Ann}(I^nM)$. Since $\operatorname{Ann}(M) \subseteq \operatorname{Ann}(I^nM)$, we have $\mathfrak{p} \supseteq I + \operatorname{Ann}(M)$ and thus $\mathfrak{p} \in \operatorname{Supp}(M/IM)$. Therefore $\Lambda_{d-1}(I,I^nM) \subseteq \Lambda_{d-1}(I,M)$. Now suppose $\mathfrak{p} \in \Lambda_{d-1}(I,M)$. Then $\mathfrak{p} \in \operatorname{Supp}(M/IM)$ and so $\mathfrak{p} \supseteq I + \operatorname{Ann}(M)$. Furthermore, $\dim I^nM_{\mathfrak{p}} = 1$ for $n \gg 0$ thus $\mathfrak{p} \in \operatorname{Supp}(I^nM)$ and so $\mathfrak{p} \supseteq \operatorname{Ann}(I^nM)$. Since $\mathfrak{p} \supseteq I + \operatorname{Ann}(I^nM)$, we have $\mathfrak{p} \in \operatorname{Supp}(I^n/I^{n+1}M)$ and thus $\Lambda_{d-1}(I,M) = \Lambda_{d-1}(I,I^nM)$.

Consider $\Lambda_{d-1}(I,M)$. If $\Lambda_{d-1}(I,M)$ is not empty, then there exists $\mathfrak{p} \in \operatorname{Supp} M/IM$ such that $\dim R/\mathfrak{p} = d-1$ and $\dim I^nM_{\mathfrak{p}} = 1$. Since $\mathfrak{p} \in \operatorname{Supp} I^nM/I^{n+1}M$, we have $\dim I^nM/I^{n+1}M \geq \dim R/\mathfrak{p} = d-1$. By Proposition 5.2.6, for $n \gg 0$ we have $\operatorname{depth}_I(I^nM) > 0$. Therefore we have $\dim I^nM/I^{n+1}M \leq d-1$ for $n \gg 0$. Therefore we must have $\dim I^nM/I^{n+1}M = d-1$ for $n \gg 0$. Similarly, we have $\dim I^nM_{\mathfrak{p}}/I^{n+1}M_{\mathfrak{p}} = 0$ for $n \gg 0$. Therefore by Theorem 4.1.6(*iii*) we have

$$c_{d-1}(I,I^nM) \geq e(IR_{\mathfrak{p}},I^nM_{\mathfrak{p}})e(R/\mathfrak{p}) \geq e(IR_{\mathfrak{p}},I^nM_{\mathfrak{p}})$$

and thus $c_{d-1}(I,I^nM) \neq 0$. Therefore, if $c_{d-1}(I,I^nM) = 0$ then $\Lambda_{d-1}(I,M)$ is empty. If $c_{d-1}(I,I^nM) \neq 0$, note that $\Lambda_{d-1}(I,M) = \operatorname{Assh}(I^nM/I^{n+1}M)$ for $n \gg 0$.

Then by Proposition 4.1.6(iii) we have

$$c_{d-1}(I, I^n M) = \sum_{\mathfrak{p} \in \Lambda_{d-1}(I,M)} e(IR_{\mathfrak{p}}, I^n M_{\mathfrak{p}}) e(R/\mathfrak{p}).$$

By Proposition 5.2.9 $c_d(I, I^n M) = 0$ and $c_i(I, M) = c_i(I, I^n M)$ for $i \leq d-1$ for $n \gg 0$. And so, we have

$$c_{d-1}(I, M) = \sum_{\mathfrak{p} \in \Lambda_{d-1}(I,M)} e(IR_{\mathfrak{p}}, I^n M_{\mathfrak{p}}) e(R/\mathfrak{p}).$$

If $r = d-1$, we are done. From now on we assume $r \leq k \leq d-2$.

If $c_k(I, M) = 0$, then $c_k(I, I^n(M/(x_2, ..., x_{d-k})M)) = 0$ for $n \gg 0$ by Proposition 6.1.3. Further, we have $\dim I^n(M/(x_2, ..., x_{d-k})M) = k+1$, and the set $\Lambda_k(I, M)$ is empty as before, by Theorem 4.1.6.

Suppose that $c_k(I, M) \neq 0$ and consider $I^n(M/(x_2, ..., x_{d-k})M)$ for $n \gg 0$. By Lemma 6.1.3 we have $\dim I^n(M/(x_2, ..., x_{d-k})M) = k+1$ and for $i \leq k$ we also have

$$c_i(I, I^n(M/(x_2, ..., x_{d-k})M)) = c_i(I, M).$$

Further, by Proposition 5.2.9 we have

$$c_{k+1}(I, I^n(M/(x_2, ..., x_{d-k})M)) = 0.$$

By Remark 5.2.6, we have $\operatorname{depth}_I(I^n(M/(x_2, ..., x_{d-k})M)) > 0$ and

$$\dim I^n(M/(x_2, ..., x_{d-k})M)/I^{n+1}(M/(x_2, ..., x_{d-k})M) \leq k$$

for $n \gg 0$. Since $c_k(I, I^n(M/(x_2, ..., x_{d-k})M)) \neq 0$ by Theorem 4.1.6 (iii), we must have

$$\dim I^n(M/(x_2, ..., x_{d-k})M)/I^{n+1}(M/(x_2, ..., x_{d-k})M) = k.$$

Therefore, by Theorem 4.1.6 (iii) we have

$$c_k(I, I^n(M/(x_2, ..., x_{d-k})M)) \sum_{\mathfrak{p} \in \Lambda_k(I,M)} e(IR_\mathfrak{p}, I^n(M_\mathfrak{p}/(x_2, ..., x_{d-k})M_\mathfrak{p}))e(R/\mathfrak{p}).$$

Since $c_k(I, M) = c_k(I, I^n(M/(x_2, ..., x_{d-k})M))$ by Proposition 6.1.3, we are done. $\square$

Example 6.2.2. Let $R = M = \mathbb{Q}[x, y, z]_{(x,y,z)}$ and $I = (xy, yz)$. Since I is not contained in any minimal prime of M, we have $\Lambda_3(I, M)$ is empty and thus $c_3(I, M) = 0$.

The ideal I contains a nonzero divisor on M, thus is has positive depth and we do not need to replace M with $I^n M$. The only prime $\mathfrak{p} \in \Lambda_2(I, M)$ is (y). This is the minimal prime over I with $\dim R/\mathfrak{p} = 2$. As in Example 4.1.7, we have $c_2(I, M) = 2$.

The element $xy - yz$ is a sufficiently general element that satisfies all properties of Proposition 6.1.3. We now set $M = R/(xy - yz)$. But now I has no nonzero divisors on M since I is contained in a minimal prime of M. By replacing M with $IM = (xy, yz)M$, we will have $\mathrm{depth}_I(IM) > 0$.

Now we have $\Lambda_1(I, M) = \{(x, z), (y, x - z)\}$. For $\mathfrak{p} = (x, z)$, we have $e(IR_\mathfrak{p}, M_\mathfrak{p}) = 1$ and $e(R/\mathfrak{p}) = 1$. For $\mathfrak{p} = (y, x - z)$, we have $e(IR_\mathfrak{p}, M_\mathfrak{p}) = 1$ and $e(R/\mathfrak{p}) = 1$. Therefore $c_1(I, M) = 2$.

7. MULTIPLICITY AND REDUCTION

The primary motivation the formula of Theorem 6.2.1 is to prove the converse of Theorem 4.1.8. First we need a few preliminary results.

7.1. Mixed multiplicities and $\mathbb{Z}^3$-graded results

Proposition 7.1.1. *Let $(R, \mathfrak{m})$ be a local noetherian ring, M a finitely generated module, and $I \subseteq J \subset R$ ideals. Then I is a reduction of $(J, J^n M)$ for some n if and only if I is a reduction of (J, M).*

Proof. If I is a reduction of $(J, J^n M)$, then $IJ^{k+n}M = J^{k+n+1}M$ for some k. Similarly, if I is a reduction of (J, M), then $IJ^k M = J^{k+1}M$ for all $k \gg 0$. In particular, for $k \geq n$, $IJ^{k-n}(J^n M) = J^{k-n+1}(J^n M)$. □

Proposition 7.1.2. [4, Proposition 2.3] *Let $(R, \mathfrak{m})$ be a local noetherian ring and I an ideal. Let*

$$0 \to N \to M \to M/N \to 0$$

be a short exact sequence of finitely generated R-modules. If $\dim N = \dim M = \dim M/N$, then $c_i(I, M) = c_i(I, N) + c_i(I, M/N)$, for $i = 0, ..., \dim M$. If $\dim M/N < \dim M$, then $c_i(I, M) = c_i(I, N)$ for all $i = 0, ..., \dim M$.

Proposition 7.1.3. *Let $(R, \mathfrak{m})$ be a local noetherian ring, M a finitely generated module of dimension $d \geq 1$, and I and J ideals of definition on M. Then $e(I, J^j M) = e(I, M)$ for $j \gg 0$.*

Proof. Consider the mixed multiplicities $e_k(I, J; M)$ which come from the normalized degree d coefficients of the polynomial given by $\lambda(M/I^i J^j M)$. For $i, j \gg 0$, we have

$$\lambda(M/I^i J^j M) = \sum_{k+n=d} \frac{e_n(I, J; M)}{k!n!} i^k j^n + \text{ lower degree terms.}$$

We know that $e_0(I, J; M) = e(I, M)$ and $e_d(I, J; M) = e(J, M)$. The following sequence is exact:

$$0 \to \frac{J^j M}{I^i J^j M} \to \frac{M}{I^i J^j M} \to \frac{M}{J^j M} \to 0.$$

Since I and J are ideals of definition on M, each component has finite length. So we have $\lambda(M/I^iJ^jM) - \lambda(M/J^jM) = \lambda(J^jM/I^iJ^jM)$. These lengths are polynomial in i, j of degree d for $i, j \gg 0$. The degree $(d, 0)$ component of $\lambda(M/J^jM)$ is 0, since there is no dependence on i, and the degree $(d, 0)$ component of $\lambda(M/I^iJ^jM)$ is $\frac{e_0(I,J;M)}{d!}i^d = \frac{e(I,M)}{d!}i^d$. Consider $\lambda(J^jM/I^iJ^jM)$. For a fixed j, this length is given by a polynomial in i of degree d with degree d term $\frac{e(I,J^jM)}{d!}i^d$. And so we have $e(I, J^jM) = e(I, M)$ for $j \gg 0$. $\square$

Lemma 7.1.4. *Let $(R, \mathfrak{m})$ be a local noetherian ring, M a finitely generated R-module, and $I \subseteq J$ ideals of R such that $\sqrt{I} = \sqrt{J}$. Then for $n, k \gg 0$ we have $\dim I^kM = \dim J^nM = d^*$ and*

$$c_i(I, I^kM) = c_i(I, J^nM)$$

for all $i \leq d^*$.

Proof. If $\ell_M(I) = 0$ then $I^kM = 0$ for some k. Since $\sqrt{I} = \sqrt{J}$, we have $J^j \subseteq I$ for some j, we have $J^{jk}M = 0$. Therefore for $n = jk$ we have $c_i(I, I^kM) = c_i(I, J^nM) = 0$ for all i. So we may assume $\ell_M(I) > 0$.

Consider $\mathrm{Ann}(I^kM)$. For all k we have $\mathrm{Ann}(I^kM) \subseteq \mathrm{Ann}(I^{k+1}M)$ since $I^{k+1}M \subseteq I^kM$. Since R is noetherian, there exists k such that $\mathrm{Ann}(I^kM) = \mathrm{Ann}(I^nM)$ for all $n \geq k$. Therefore $\dim I^kM$ stabilizes for $k \gg 0$. Let $d^* = \dim I^kM$ for $k \gg 0$. Similarly, $\dim J^nM$ stabilizes for $n \gg 0$. Since $J^j \subseteq I$, we have $\dim J^{jk}M = \dim I^kM$ for $k \gg 0$ and thus $\dim I^kM = \dim J^nM$ for $n, k \gg 0$.

By Proposition 5.2.9 we have $c_{d^*}(I, I^kM) = 0$ for $k \gg 0$. Since $J^{jk}M \subseteq I^kM$ and $\dim J^{jk}M = \dim I^jM$, by Proposition 7.1.2 we have $c_{d^*}(I, J^{jk}M) \leq c_{d^*}(I, I^kM) = 0$. Therefore $c_{d^*}(I, J^nM) = 0$ for $n \gg 0$.

Since $I^{n+k}M \subseteq I^kJ^nM \subseteq J^{k+n}M$, we have $\dim I^kJ^nM = \dim I^kM = \dim J^nM$ for $k, n \gg 0$. By Proposition 5.2.6 we have that $\mathrm{depth}_I(I^kM) > 0$ and therefore $\mathrm{depth}_J(I^kM) > 0$. Thus there exists $x \in J^n$ that is a nonzero divisor on I^kM and so $\dim I^kM/J^nI^kM < d^*$.

Consider the short exact sequence

$$0 \to I^kJ^nM \to I^kM \to I^kM/I^kJ^nM \to 0.$$

By Proposition 7.1.2 we have $c_i(I, I^k J^n M) = c_i(I, I^k M)$ for $i \leq d^*$.

Similarly, since $J^{jk} \subseteq I^k$, we have $\operatorname{depth}_I(J^n M) > 0$ for $n \gg 0$. Therefore, there exists $x \in I^k$ that is a nonzero divisor on $J^n M$ and thus $\dim J^n M/I^k J^n M < d^*$. We have the short exact sequence

$$0 \to I^k J^n M \to J^n M \to J^n M/I^k J^n M \to 0.$$

Again, by Proposition 7.1.2 we have $c_i(I, J^n M) = c_i(I, I^k J^n M)$ for all $i \leq d^*$. Therefore

$$c_i(I, I^k M) = c_i(I, J^n M)$$

for $k, n \gg 0$ and for all $i \leq d^*$. $\square$

Remark 7.1.5. Since both the dimension and the multiplicity sequence stabilize asymptotically, we may take $n = k$ for $n \gg 0$ and thus we have

$$c_i(I, I^n M) = c_i(I, J^n M)$$

for $n \gg 0$ and for all $i \leq d^*$.

Proposition 7.1.6. *Let $(R, \mathfrak{m})$ be a local noetherian ring with infinite residue field, I a nonzero ideal, J and K ideals of R, and M a finitely generated R-module. Then there exists $x \in I$ such that x is superficial for I with respect to $G_{\mathfrak{m}}(G_I(J^j M/J^j KM))$ for all $j \geq 0$.*

Proof. Let S be the finitely generated R-algebra

$$\bigoplus_{i,j,n \geq 0} \frac{\mathfrak{m}^n I^i J^j + I^{i+1} J^j}{\mathfrak{m}^{n+1} I^i J^j + I^{i+1} J^j}$$

and N be the finitely generated S-module

$$\bigoplus_{i,j,n \geq 0} \frac{\mathfrak{m}^n I^i J^j M + I^{i+1} J^j M}{\mathfrak{m}^{n+1} I^i J^j M + I^{i+1} J^j M}.$$

Note that S is a $\mathbb{Z}^3$-graded ring generated in degrees $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$ and N is a

$\mathbb{Z}^3$-graded S-module. Let

$$S_k = \bigoplus_{j,n\geq 0, i\geq k} \frac{\mathfrak{m}^n I^i J^j + I^{i+1}J^j}{\mathfrak{m}^{n+1} I^i J^j + I^{i+1}J^j}.$$

Let $\overline{N}$ be the S-module

$$\bigoplus_{i,j,n\geq 0} \frac{\mathfrak{m}^n I^i J^j M + I^{i+1}J^j M + J^j K M}{\mathfrak{m}^{n+1} I^i J^j M + I^{i+1}J^j M + J^j K M}.$$

There is a natural surjection $N \twoheadrightarrow \overline{N}$, so $\overline{N}$ is also finitely generated as an S-module.

Let $\bigcap_{i=1}^t \overline{N}_i = 0$ be an irredundant primary decomposition of the 0 submodule of $\overline{N}$. Let $P_i = \sqrt{\overline{N}_i :_S \overline{N}}$ such that $S_{(0,1,0)} = I/\mathfrak{m}I$ is not contained in $P_1, ..., P_r$ and $S_{(0,1,0)}$ is contained in $P_{r+1}, ..., P_t$. Since $R/\mathfrak{m}$ is an infinite field and P_i does not contain $I/\mathfrak{m}I$ for $i = 1, ..., r$, the set

$$(I/\mathfrak{m}I) \setminus \bigcup_{i=1}^{r} ((I/\mathfrak{m}I) \cap P_i)$$

is not empty.

Let $x \in I$ such that $\overline{x} \in (I/\mathfrak{m}I) \setminus \bigcup_{i=1}^r (I/\mathfrak{m}I \cap P_i)$. Note that $I/\mathfrak{m}I \subseteq S_1$ and $I/\mathfrak{m}I$ generates S_1 as an ideal of S. For $i = r+1, ..., t$ we have $I/\mathfrak{m}I \subseteq P_i$, therefore $S_1 \subseteq P_i$ and so there exists some c such that $S_1^c \overline{N} = S_c \overline{N} \subseteq \overline{N}_i$ for $i = r+1, ..., t$.

Now consider $(0 :_{\overline{N}} \overline{x})$. Using the irredundant primary decomposition of the 0 submodule, we have $(0 :_{\overline{N}} \overline{x}) = \bigcap_{i=1}^r (\overline{N}_i :_{\overline{N}} \overline{x})$. For $i = 1, ..., r$, we have $\overline{x} \notin P_i$ thus $(\overline{N}_i :_{\overline{N}} \overline{x}) = \overline{N}_i$, which implies that $(0 :_{\overline{N}} \overline{x}) \subseteq \bigcap_{i=1}^r \overline{N}_i$.

For $c \gg 0$ we have $(0 :_{\overline{N}} \overline{x}) \cap S_c \overline{N} \subseteq \bigcap_{i=1}^r \overline{N}_i \cap \bigcap_{i=r+1}^t \overline{N}_i = 0$. Therefore $(0 :_{\overline{N}} \overline{x}) \cap S_c \overline{N} = 0$ for $c \gg 0$.

For each j let $N'_j = G_{\mathfrak{m}}(G_I(J^j M / J^j K M)) \subseteq N$ and $S'_j = G_{\mathfrak{m}}(G_I(J^j)) \subseteq S$. Note that

$$S_c \cap S'_0 = \bigoplus_{i\geq c, n\geq 0} \frac{\mathfrak{m}^n I^i + I^{i+1}}{\mathfrak{m}^{n+1} I^i + I^{i+1}} = I^c G_{\mathfrak{m}}(G_I(R)).$$

We have

$$(0 :_{N'_j} \overline{x}) \cap (S_c \cap S'_0) N'_j \subseteq (0 :_{\overline{N}} \overline{x}) \cap S_c \overline{N} = 0.$$

Therefore

$$(0:_{G_{\mathfrak{m}}(G_I(J^jM/J^jKM)} \overline{x})) \cap I^c G_{\mathfrak{m}}(G_I(J^jM/J^jKM)) = 0.$$

By Definition 5.2.1, x is a superficial element for I with respect to $G_{\mathfrak{m}}(G_I(J^jM/J^jKM))$ for all j. $\square$

Proposition 7.1.7. *Let $(R, \mathfrak{m})$ be a local noetherian ring with infinite residue field k, $I \subset R$ an ideal, and M a finitely generated R-module. Then there exists $x \in I$ such that x is superficial for I with respect to $G_{\mathfrak{m}}(G_I(J^nM))$ for all n and $\frac{x}{1} \in IR_{\mathfrak{p}}$ is superficial for $IR_{\mathfrak{p}}$ with respect to $G_{\mathfrak{p}R_{\mathfrak{p}}}(G_{IR_{\mathfrak{p}}}((JR_{\mathfrak{p}})^nM_{\mathfrak{p}}))$ for all n and all prime ideal $\mathfrak{p}$ with $\ell_{M_{\mathfrak{p}}}(IR_{\mathfrak{p}}) = \dim M_{\mathfrak{p}}$.*

Proof. A superficial element for I with respect to $\bigoplus_{n\geq 0} G_{\mathfrak{m}}(G_I(J^nM))$ is obtained by considering elements of the $R/\mathfrak{m}$ vector space $I/\mathfrak{m}I$ (here in degree (0,1,0)) that avoids finitely many subspaces, as in Proposition 5.3.3. Since there are only finitely many primes where $\ell_{M_{\mathfrak{p}}}(IR_{\mathfrak{p}}) = \dim M_{\mathfrak{p}}$, the argument follows as Proposition 5.3.6. $\square$

7.2. The reduction result

Proposition 7.2.1. *Let $(R, \mathfrak{m})$ be a local noetherian ring, $I \subseteq J \subset R$ ideals, and M a finitely generated formally equidimensional R-module. Let $d = \dim M$ and $q = \dim(M/IM)$. If $c_k(I, M) = c_k(J, M)$ for all $k = 0, ..., d$, then $\Lambda_q(I, M) = \Lambda_q(J, M)$ and $IR_{\mathfrak{p}}$ is a reduction of $(JR_{\mathfrak{p}}, M_{\mathfrak{p}})$ for all $\mathfrak{p} \in \Lambda_q$ where $\Lambda_q = \Lambda_q(I, M) = \Lambda_q(J, M)$.*

Proof. Since $\dim M/IM = q$, we have $\dim R/\mathfrak{p} \leq q$ for all $\mathfrak{p}\,\mathrm{Ass}(M/IM)$ and there must be at least one prime where equality holds. Since M is formally equidimensional, $\dim M_{\mathfrak{p}} = d - q$ and thus $\dim I^n(M_{\mathfrak{p}}/(x_2, ..., x_{d-q})M_{\mathfrak{p}}) = 1$. Thus $\Lambda_q(I, M) = \mathrm{Assh}(I, M)$. Similarly, since $c_q(J, M) \neq 0$ and $c_k(I, M) = 0$ for all $k > q$, we have $\dim M/JM = q$ by Proposition 4.1.6 and $\Lambda_q(J, M) = \mathrm{Assh}(J, M)$. Therefore, $\Lambda_q(I, M) \supset \Lambda_q(J, M)$. Further, since M is formally equidimensional, $c_q(I, M) \neq 0$. Since $c_k(I, M) = c_k(J, M)$, for $q = \dim(M/IM)$, $c_q(I, M) = c_q(J, M)$ and $c_k(J, M) = 0$ for all $k > q$. By Theorem 4.1.6 (*iii*), we have

$$\sum_{\mathfrak{p}\in\Lambda_q(I,M)} e(IR_{\mathfrak{p}}, M_{\mathfrak{p}})e(R/\mathfrak{p}) = \sum_{\mathfrak{p}\in\Lambda_q(J,M)} e(JR_{\mathfrak{p}}, M_{\mathfrak{p}})e(R/\mathfrak{p}).$$

For every $\mathfrak{p} \in \Lambda_q(J, M)$, we have $IR_{\mathfrak{p}} \subseteq JR_{\mathfrak{p}}$ and so $e(IR_{\mathfrak{p}}, M_{\mathfrak{p}}) \geq e(JR_{\mathfrak{p}}, M_{\mathfrak{p}})$. Since the left hand side of the equality has at least as many terms as the right hand side and $e(IR_{\mathfrak{p}}, M_{\mathfrak{p}}) \geq e(JR_{\mathfrak{p}}, M_{\mathfrak{p}}) > 0$, for each $\mathfrak{p}$ we must have $e(IR_{\mathfrak{p}}, M_{\mathfrak{p}}) = e(JR_{\mathfrak{p}}, M_{\mathfrak{p}})$. Therefore, we have $\Lambda_q(I, M) = \Lambda_q(J, M)$. Since these multiplicities are the typical Hilbert-Samuel multiplicities, $IR_{\mathfrak{p}}$ is a reduction of $(JR_{\mathfrak{p}}, M_{\mathfrak{p}})$ for all $\mathfrak{p} \in \Lambda_q$. □

Proposition 7.2.2. *Let $(R, \mathfrak{m})$ be a local noetherian ring with infinite residue field, $I \subseteq J$ ideals of R with $\sqrt{I} = \sqrt{J}$, and M a finitely generated R-module with $\dim M = d \geq 2$. Let x be a superficial element for I with respect to $G_I(J^n M)$ and with respect to $G_{\mathfrak{m}}(G_I(J^n M))$ for all n and a nonzero divisor for $I^k M$ for some k. Then for $n \gg 0$ we have $\dim J^n(M/xM) = d - 1$. Further if $c_i(I, M) \neq 0$ for some $i < d-1$, then $c_i(I, M) = c_i(I, J^n(M/xM))$ for all $i \leq d-2$ and $c_{d-1}(I, J^n M) = 0$.*

Proof. By Lemma 6.1.1 for $n \gg 0$, we have $\dim I^n(M/xM) = d-1$ and $c_i(I, M) = c_i(I, I^n(M/xM))$ for $i \leq d-2$. By Lemma 7.1.4 for $n \gg 0$ we have $\dim J^n M = \dim I^n M$ and $c_i(I, I^n(M/xM)) = c_i(I, J^n(M/xM))$ for $i \leq d-1$. By Proposition 5.2.9 we also have $c_{d-1}(I, J^n(M/xM)) = 0$ for $n \gg 0$. □

Proposition 7.2.3. *Let $(R, \mathfrak{m})$ be a local noetherian ring with infinite residue field, $I \subseteq J$ ideals of R, and M a finitely generated R-module of dimension $d \geq 2$ with $c_i(I, M) = c_i(J, M)$ for all $i = 0, ..., d$. Assume that $c_d(I, M) = 0$ and $c_i(I, M) \neq 0$ for some $i \leq d-2$ and let $r = \min\{i \mid c_i(I, M) \neq 0\}$. Then there exist sufficiently general sequences of elements $x_2, ..., x_{d-r} \in I$ and $y_2, ..., y_{d-r} \subseteq J$ such that the following properties are true for $2 \leq k \leq d-r$:*

i. For $n \gg 0$ we have $\dim J^n(M/(x_2, ..., x_k)M) = d - (k-1)$ and for $i \leq d-k$

$$c_i(I, J^n M) = c_i(I, J^n(M/(x_2, ..., x_k)M)).$$

Moreover

$$c_0(IR_{\mathfrak{p}}, J^n(M_{\mathfrak{p}}/(x_2, ..., x_{k-1})M_{\mathfrak{p}})) = c_0(IR_{\mathfrak{p}}, J^n(M_{\mathfrak{p}}/(x_2, ..., x_k)M_{\mathfrak{p}}))$$

for all $\mathfrak{p}$ *such that*

$$c_0(IR_{\mathfrak{p}}, J^n(M_{\mathfrak{p}}/(x_2, ..., x_{k-1})M_{\mathfrak{p}})) \neq 0 \ \textit{and} \ \dim J^n(M_{\mathfrak{p}}/(x_2, ..., x_{k-1})M_{\mathfrak{p}}) \geq 2.$$

ii. For $n \gg 0$ *we have* $\dim J^n(M/(y_2, ..., y_k)M) = d - (k-1)$ *and for* $i \leq d - k$

$$c_i(J, M) = c_i(J, J^n(M/(y_2, ..., y_k)M)).$$

Moreover

$$c_0(JR_{\mathfrak{p}}, J^n(M_{\mathfrak{p}}/(y_2, ..., y_{k-1})M_{\mathfrak{p}})) = c_0(JR_{\mathfrak{p}}, J^n(M_{\mathfrak{p}}/(y_2, ..., y_k)M_{\mathfrak{p}}))$$

for all $\mathfrak{p}$ *such that*

$$c_0(JR_{\mathfrak{p}}, J^n(M_{\mathfrak{p}}/(y_2, ..., y_{k-1})M_{\mathfrak{p}})) \neq 0 \ \textit{and} \ \dim J^n(M_{\mathfrak{p}}/(y_2, ..., y_{k-1})M_{\mathfrak{p}}) \geq 2.$$

iii. For $n \gg 0$ *and all* $j \leq k$ *we have* $\dim J^n(M/(y_2, ..., y_j, x_{j+1}, ...x_k)M) = d - (k-1)$ *and for* $i \leq d - k$

$$c_i(J, M/(x_{j+1}, ..., x_k)M) = c_i(J, M/(y_2, ..., y_j, x_{j+1}, ..., x_k)M).$$

Moreover

$$c_0(JR_{\mathfrak{p}}, J^n(M_{\mathfrak{p}}/(y_2, ..., y_{j-1}, x_{j+1}, ..., x_k)M_{\mathfrak{p}})) = c_0(JR_{\mathfrak{p}}, J^n(M_{\mathfrak{p}}/(y_2, ..., y_j, x_{j+1}, ..., x_k)M_{\mathfrak{p}}))$$

for all $\mathfrak{p}$ *such that*

$$c_0(JR_{\mathfrak{p}}, J^n(M_{\mathfrak{p}}/(y_2, ..., y_{j-1}, x_{j+1}, ..., x_k)M_{\mathfrak{p}})) \neq 0 \ \textit{and}$$

$$\dim J^n(M_{\mathfrak{p}}/(y_2, ..., y_{j-1}, x_{j+1}, ..., x_k)M_{\mathfrak{p}}) \geq 2.$$

iv. For $n \gg 0$ *and all* $j \leq k$ *we have* $\dim J^n(M/(y_2, ..., y_j, x_j, ...x_k)M) = d-k$ *and for* $i \leq d-k-1$

$$c_i(J, J^n(M/(x_j, ..., x_k)M)) = c_i(J, J^n(M/(y_2, ..., y_j, x_j, ..., x_k)M)).$$

Moreover

$$c_0(JR_{\mathfrak{p}}, J^n(M_{\mathfrak{p}}/(y_2, ..., y_{j-1}, x_j, ..., x_k)M_{\mathfrak{p}})) = c_0(JR_{\mathfrak{p}}, J^n(M_{\mathfrak{p}}/(y_2, ..., y_j, x_j, ..., x_k)M_{\mathfrak{p}}))$$

for all $\mathfrak{p}$ *such that*

$$c_0(JR_{\mathfrak{p}}, J^n(M_{\mathfrak{p}}/(y_2, ..., y_{j-1}, x_j, ..., x_k)M_{\mathfrak{p}})) \neq 0 \textit{ and}$$

$$\dim J^n(M_{\mathfrak{p}}/(y_2, ..., y_{j-1}, x_j, ..., x_k)M_{\mathfrak{p}}) \geq 1.$$

Proof. First, we choose the sequence $x_2, ..., x_{d-r}$. We choose x_2 to be a nonzero divisor on I^cM for some c (and thus a nonzero divisor on I^cJ^nM for all n), superficial for I with respect to $G_I(J^nM)$ and $G_{\mathfrak{m}}(G_I(J^nM))$ for all n, and superficial for $IR_{\mathfrak{p}}$ with respect to $G_{IR_{\mathfrak{p}}}(J^nM_{\mathfrak{p}})$ and $G_{\mathfrak{p}R_{\mathfrak{p}}}(G_{IR_{\mathfrak{p}}}(J^nM_{\mathfrak{p}}))$ for all n and for all $\mathfrak{p}$ such that $c_0(IR_{\mathfrak{p}}, M_{\mathfrak{p}}) \neq 0$. Such an element exists by Propositions 5.3.2, 5.3.3, and 5.3.6.

For $k \leq d-r$ we choose x_k to be a nonzero divisor on $I^c(M/(x_2, ..., x_{k-1})M)$ superficial for I with respect to $G_{\mathfrak{m}}(G_I(J^nM/(x_2, ...x_{k-1})J^nM)$ for all $n \geq 0$, and superficial for $IR_{\mathfrak{p}}$ with respect to $G_{\mathfrak{p}R_{\mathfrak{p}}}(G_{IR_{\mathfrak{p}}}(J^n(M_{\mathfrak{p}}/(x_2, ..., x_{k-1})(JR_{\mathfrak{p}})^nM_{\mathfrak{p}})$ for all $n \geq 0$ and for all $\mathfrak{p}$ such that $c_0(IR_{\mathfrak{p}}, M_{\mathfrak{p}}) \neq 0$. Such an element exists by Propositions 7.1.7. These elements satisfy (i) by Proposition 7.2.2.

Let $M_k = M/(x_3, ..., x_k)M$. We now choose y_2 to be a nonzero divisor on J^cM for some c, superficial for J with respect to

$$G_J(M) \text{ and } G_{\mathfrak{m}}(G_J(M))$$

and superficial for $JR_{\mathfrak{p}}$ with respect to

$$G_{JR_{\mathfrak{p}}}(M_{\mathfrak{p}}) \text{ and } G_{\mathfrak{p}R_{\mathfrak{p}}}(G_{JR_{\mathfrak{p}}}(M_{\mathfrak{p}}))$$

for all $\mathfrak{p}$ such that $c_0(JR_{\mathfrak{p}}, M_{\mathfrak{p}}) \neq 0$, satisfying (ii) by Proposition 7.2.2. Further, y_2 can be chosen to be superficial for J with respect to

$$G_J(M_k) \text{ and } G_{\mathfrak{m}}(G_J(M_k))$$

and for $JR_{\mathfrak{p}}$ with respect to $G_{\mathfrak{p}R_{\mathfrak{p}}}(G_{JR_{\mathfrak{p}}}((M_k)_{\mathfrak{p}})$ for all $\mathfrak{p}$ such that $c_0(JR_{\mathfrak{p}},(M_k)_{\mathfrak{p}}) \neq 0$ and $\dim(M_k)_{\mathfrak{p}} \geq 2$ and for $3 \leq k \leq d-r$, satisfying (iii) by Proposition 7.2.2. Finally, y_2 can also be chosen to be superficial for J with respect to

$$G_J(M_k/x_2M_k) \text{ and } G_{\mathfrak{m}}(G_J(M_k/x_2M_k))$$

for all $k \in \{2,...,d-r\}$ and superficial for $JR_{\mathfrak{p}}$ with respect to $G_{\mathfrak{p}R_{\mathfrak{p}}}(G_{JR_{\mathfrak{p}}}((M_k)_{\mathfrak{p}}/x_2(M_k)_{\mathfrak{p}}))$ for all $\mathfrak{p}$ such that $c_0(JR_{\mathfrak{p}},(M_k)_{\mathfrak{p}}/x_2(M_k)_{\mathfrak{p}}) \neq 0$, satisfying (iv) by Proposition 7.2.2. Since these are finitely many different modules, by Remark 5.3.1, such an element exists.

Let $M' = M/(y_2,...,y_{n-1})$. We now choose $y_n \in J$ to be a nonzero divisor on the modules J^cM', $J^c(M'/(x_{n+1},...,x_k)M)$, and $J^c(M'/(x_n,...,x_k)M)$ for some c and for all $k \leq d-r$ and all $n \leq k$, superficial for J with respect to

$$G_J(M/(y_2,...,y_{n-1})M) \text{ and } G_{\mathfrak{m}}(G_J(M/(y_2,...,y_{n-1})M))$$

and superficial for $JR_{\mathfrak{p}}$ with respect to

$$G_{JR_{\mathfrak{p}}}(M_{\mathfrak{p}}/(y_2,...,y_{n-1})M_{\mathfrak{p}}) \text{ and } G_{\mathfrak{p}R_{\mathfrak{p}}}(G_{JR_{\mathfrak{p}}}(M_{\mathfrak{p}}/(y_2,...,y_{n-1})M_{\mathfrak{p}}))$$

for all $\mathfrak{p}$ such that $c_0(J,M) \neq 0$, satisfying (ii) by Proposition 7.2.2. Further, y_n may be chosen to be superficial for J with respect to

$$G_J(M/(y_2,...,y_{n-1},x_{n+1},...,x_k)M) \text{ and } G_{\mathfrak{m}}(G_J(M/(y_2,...,y_{n-1},x_{n+1},...,x_k)M))$$

and for $JR_{\mathfrak{p}}$ with respect to

$$G_{\mathfrak{p}R_{\mathfrak{p}}}(G_{JR_{\mathfrak{p}}}(M_{\mathfrak{p}}/(y_2,...,y_{n-1}x_{n+1},...,x_k)M_{\mathfrak{p}}))$$

for all $\mathfrak{p}$ such that

$$c_0(JR_{\mathfrak{p}},J^n(M_{\mathfrak{p}}/(y_2,...,y_{n-1},x_{n+1},...,x_k)M_{\mathfrak{p}})) \neq 0$$

and $\dim M_{\mathfrak{p}}/(y_2,...,y_{n-1},x_{n+1},...,x_k)M_{\mathfrak{p}} \geq 2$ and for $3 \leq k \leq d-r$ and all $n \leq k$, satisfying (iii)

by Proposition 7.2.2. Finally, y_n is chosen to be superficial for J with respect to

$$G_J(M/(y_2,...,y_{n-1},x_n,...,x_k)M) \text{ and } G_{\mathfrak{m}}(G_J(M/(y_2,...,y_{n-1},x_n,...,x_k)M))$$

for all k such that $2 \leq k \leq d-r$ and superficial for $JR_{\mathfrak{p}}$ with respect to

$$G_{\mathfrak{p}R_{\mathfrak{p}}}(G_{JR_{\mathfrak{p}}}(M_{\mathfrak{p}}/(y_2,...,y_{n-1},x_n,...,x_k)M_{\mathfrak{p}}))$$

for all $\mathfrak{p}$ such that

$$c_0(JR_{\mathfrak{p}}, J^n(M_{\mathfrak{p}}/(y_2,...,y_{n-1},x_n,...,x_k)M_{\mathfrak{p}})) \neq 0$$

and for $3 \leq k \leq d-r$ and all $n \leq k$, satisfying (iv) by Proposition 7.2.2. Since these are finitely many different modules, by Remark 5.3.1, such an element exists. □

Corollary 7.2.4. *Let $(R, \mathfrak{m})$ be a local noetherian ring, M a finitely generated R-module of dimension d, and $I \subseteq J$ ideals of R such that $\sqrt{I} = \sqrt{J}$ and $c_i(I,M) = c_i(J,M)$ for all $i = 0,...,d$, and let $r = \min\{i \mid c_i(I,M) \neq 0\}$. Suppose $r < d$. Then there exist sufficiently general sequences of elements $x_2,...,x_{d-r} \in I$ and $y_2,...,y_{d-r} \in J$ such that for $n \gg 0$ $e(IR_{\mathfrak{p}}, I^n(M_{\mathfrak{p}}/(x_2,...,x_{d-k})M_{\mathfrak{p}})) = e(JR_{\mathfrak{p}}, J^n(M_{\mathfrak{p}}/(y_2,...,y_{d-k})M_{\mathfrak{p}}))$ for all $k \leq r$ and all $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $\dim R/\mathfrak{p} = k$ and $\dim J^n(M_{\mathfrak{p}}/(y_2,...,y_{d-k})M_{\mathfrak{p}}) = 1$.*

Proof. Let $(x_2,...,x_r)$ and $(y_2,...,y_r)$ be as defined in Proposition 7.2.3. Consider $c_i(J,M)$ for $i \leq d-2$. By Theorem 6.2.1 we have

$$c_i(J,M) = \sum_{\mathfrak{p} \in \Lambda_i(J,M)} e(JR_{\mathfrak{p}}, J^n(M_{\mathfrak{p}}/(y_2,...,y_{d-i})M_{\mathfrak{p}}))e(R/\mathfrak{p}).$$

Let $\mathfrak{p} \in \Lambda_i(J,M)$. We have $\dim R/\mathfrak{p} = i$ and $\dim J^n(M_{\mathfrak{p}}/(y_2,...,y_{d-i})M_{\mathfrak{p}}) = 1$. Suppose $c_0(JR_{\mathfrak{p}}, J^n(M_{\mathfrak{p}}/(y_2,...,y_{d-i})M_{\mathfrak{p}})) \neq 0$.

In particular, we have that J is an ideal of definition on $J^n(M_{\mathfrak{p}}/(y_2,...,y_{d-i})M_{\mathfrak{p}})$. Again, note that $\dim J^n(M_{\mathfrak{p}}/(y_2,...,y_{d-i})M_{\mathfrak{p}}) = 1$. Consider the exact sequence

$$0 \to ((y_2,...,y_{d-i}) \cap J^n)M_{\mathfrak{p}}/(y_2,...,y_{d-i})J^nM_{\mathfrak{p}} \to J^nM_{\mathfrak{p}}/(y_2,...,y_{d-i})J^nM_{\mathfrak{p}} \to ...$$

$$... \to J^n(M_{\mathfrak{p}}/(y_2,...,y_{d-i})M_{\mathfrak{p}}) \to 0.$$

Since J is an ideal of definition on $J^n(M_{\mathfrak{p}}/(y_2,...,y_{d-i})M_{\mathfrak{p}})$, we have

$$\mathfrak{p}^kR_{\mathfrak{p}} \subseteq J + \mathrm{Ann}(J^nM) + (y_2,...,y_{d-i})$$

for some k. Since $J^n((y_2,...,y_{d-i}) \cap J^n)M_{\mathfrak{p}}/(y_2,...,y_{d-i})J^nM_{\mathfrak{p}} = 0$ we have

$$\mathfrak{p}^{kn}((y_2,...,y_{d-i}) \cap J^n)M_{\mathfrak{p}}/(y_2,...,y_{d-i})J^nM_{\mathfrak{p}} = 0$$

and therefore $\dim((y_2,...,y_{d-i}) \cap J^n)M_{\mathfrak{p}}/(y_2,...,y_{d-i})J^nM_{\mathfrak{p}} = 0$. And thus we have

$$e(J, J^nM_{\mathfrak{p}}/(y_2,...,y_{d-i})J^nM_{\mathfrak{p}}) = e(J, J^n(M_{\mathfrak{p}}/(y_2,...,y_{d-i})M_{\mathfrak{p}}).$$

By [2, Lemma 11.1.7] we have

$$e(J, J^nM_{\mathfrak{p}}/(y_2,...,y_{d-i})J^nM_{\mathfrak{p}}) \leq e(J, J^nM_{\mathfrak{p}}/(y_2,...,y_{d-i},x_{d-i})J^nM_{\mathfrak{p}}).$$

By assumption, we have

$$e(J, J^nM_{\mathfrak{p}}/(y_2,...,y_{d-i},x_{d-i})J^nM_{\mathfrak{p}}) = e(J, J^nM_{\mathfrak{p}}/(y_2,...,y_{d-i-1},x_{d-i})J^nM_{\mathfrak{p}}).$$

We may repeat these arguments to get

$$e(J, J^nM_{\mathfrak{p}}/(y_2,...,y_{d-i})J^nM_{\mathfrak{p}}) \leq e(J, J^nM_{\mathfrak{p}}/(x_2,...,x_{d-i})J^nM_{\mathfrak{p}}) = e(J, J^n(M_{\mathfrak{p}}/(x_2,...,x_{d-i})M_{\mathfrak{p}})).$$

Since $\sqrt{I} = \sqrt{J}$, we have $e(I, J^n(M_\mathfrak{p}/(x_2, ..., x_{d-i})M_\mathfrak{p}))$ is defined and since $I \subseteq J$ we have

$$e(J, J^n(M_\mathfrak{p}/(x_2, ..., x_{d-i})M_\mathfrak{p})) \le e(I, J^n(M_\mathfrak{p}/(x_2, ..., x_{d-i})M_\mathfrak{p})).$$

By Proposition 7.1.4 we have

$$e(I, J^n(M_\mathfrak{p}/(x_2, ..., x_{d-i})M_\mathfrak{p})) = e(I, I^n(M_\mathfrak{p}/(x_2, ..., x_{d-i})M_\mathfrak{p})).$$

Therefore, for every $\mathfrak{p} \in \Lambda_i(J, M)$ we have

$$e(J, J^n(M_\mathfrak{p}/(y_2, ..., y_{d-i})M_\mathfrak{p})) \le e(I, I^n(M_\mathfrak{p}/(x_2, ..., x_{d-i})M_\mathfrak{p})).$$

Since $e(I, I^n(M_\mathfrak{p}/(x_2, ..., x_{d-i})M_\mathfrak{p})) \neq 0$ whenever $e(J, J^n(M_\mathfrak{p}/(y_2, ..., y_{d-i})M_\mathfrak{p})) \neq 0$ and $\dim J^n(M_\mathfrak{p}/(y_2, ..., y_{d-i})M_\mathfrak{p})) = I^n(M_\mathfrak{p}/(x_2, ..., x_{d-i})M_\mathfrak{p})) = 1$, we must also have $\mathfrak{p} \in \Lambda_i(I, M)$. Therefore, $\Lambda_i(J, M) \subseteq \Lambda_i(I, M)$. Since we have

$$\sum_{\mathfrak{p} \in \Lambda_i(J,M)} e(JR_\mathfrak{p}, J^n(M_\mathfrak{p}/(y_2, ..., y_{d-i})M_\mathfrak{p}))e(R/\mathfrak{p}) = \sum_{\mathfrak{p} \in \Lambda_i(I,M)} e(IR_\mathfrak{p}, I^n(M_\mathfrak{p}/(x_2, ..., x_{d-i})M_\mathfrak{p}))e(R/\mathfrak{p})$$

we must also have the equality

$$e(I, I^n(M_\mathfrak{p}/(x_2, ..., x_{d-i})M_\mathfrak{p})) = e(J, J^n(M_\mathfrak{p}/(y_2, ..., y_{d-i})M_\mathfrak{p})).$$

□

Corollary 7.2.5. *Let $(R, \mathfrak{m})$ be a local noetherian ring, M a finitely generated formally equidimensional R-module of dimension d, and $I \subseteq J$ ideals of R such that* $\operatorname{Asym}(I, M) = \operatorname{Asym}(J, M)$. *If $c_i(I, M) = c_i(J, M)$ for all $i = 0, ..., d$, then I is a reduction of (J, M).*

Proof. Let $r = \min\{i \mid c_i(I, M) \neq 0\}$. If $r = d$ then by Theorem 6.2.1(3) we have

$$\sum_{\mathfrak{p} \in \Lambda_d(I,M)} e(IR_\mathfrak{p}, M_\mathfrak{p})e(R/\mathfrak{p}) = \sum_{\mathfrak{p} \in \Lambda_d(J,M)} e(JR_\mathfrak{p}, M_\mathfrak{p})e(R/\mathfrak{p}).$$

Since $I \subseteq J$, we have $\Lambda_0(J,M) \subseteq \Lambda_0(I,M)$ and $e(IR_\mathfrak{p}, M_\mathfrak{p}) \geq e(JR_\mathfrak{p}, M_\mathfrak{p})$. Therefore we have $e(IR_\mathfrak{p}, M_\mathfrak{p}) = e(JR_\mathfrak{p}, M_\mathfrak{p})$ for all $\mathfrak{p} \in \Lambda_d(I,M)$. If $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $\dim R/\mathfrak{p} = d$ and $\mathfrak{p} \notin \Lambda_d(I,M)$ then $c_0(IR_\mathfrak{p}, M_\mathfrak{p}) = c_0(JR_\mathfrak{p}, M_\mathfrak{p}) = 0$.

Suppose $\mathfrak{p} \in \operatorname{Spec}(R)$ such that $\dim R/\mathfrak{p} = k$ for some $k < d$. If $c_0(IR_\mathfrak{p}, M_\mathfrak{p}) \neq 0$ then $\dim I^n M_\mathfrak{p} = \dim M_\mathfrak{p}$. We have $\dim I^n M \geq \dim R/\mathfrak{p} + \dim I^n M_\mathfrak{p} = \dim R/\mathfrak{p} + \dim M_\mathfrak{p}$. Since M is formally equidimensional, we have $\dim R/\mathfrak{p} + \dim M_\mathfrak{p} = \dim M$. Then $\dim I^n M \geq \dim M$ and so we have equality.

On the other hand, since $c_i(I,M) = 0$ for all $i < d$ we have $\dim I^n M < \dim M$. Therefore $c_0(IR_\mathfrak{p}, M_\mathfrak{p}) = 0$ for all $\mathfrak{p} \notin \Lambda_d(I,M)$. Similarly, $c_0(JR_\mathfrak{p}, M_\mathfrak{p}) = 0$ for all $\mathfrak{p} \notin \Lambda_d(I,M)$. We have $c_0(IR_\mathfrak{p}, M_\mathfrak{p}) = c_0(JR_\mathfrak{p}, M_\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Spec}(R)$ and therefore I is a reduction of (J,M).

Now suppose $r < d$ and let $k \geq r$. Let $x_2, ..., x_{d-r}$ and $y_2, ..., y_{d-r}$ be a sufficiently general sequences as in Proposition 7.2.3. Note that since $\operatorname{Asym}(I,M) = \operatorname{Asym}(J,M)$, we have $\sqrt{I} = \sqrt{J}$.

Suppose $\mathfrak{p} \notin \operatorname{Asym}(I,M)$, that is, $c_0(IR_\mathfrak{p}, M_\mathfrak{p}) = 0$. Since $\operatorname{Asym}(I,M) = \operatorname{Asym}(J,M)$ and M is formally equidimensional, we have $c_0(JR_\mathfrak{p}, M_\mathfrak{p}) = 0$. Similarly, if $c_0(IR_\mathfrak{p}, M_\mathfrak{p}/M_\mathfrak{p}) \neq 0$, then $c_0(JR_\mathfrak{p}, M_\mathfrak{p}/M_\mathfrak{p}) \neq 0$.

Suppose $c_0(IR_\mathfrak{p}, M_\mathfrak{p}/M_\mathfrak{p}) \neq 0$. Then by Proposition 6.1.3, for $n \gg 0$ we have

$$c_0(IR_\mathfrak{p}, M_\mathfrak{p}) = e(IR_\mathfrak{p}, I^n(M_\mathfrak{p}/(x_2, ..., x_{d-k})M_\mathfrak{p}))$$

and

$$c_0(JR_\mathfrak{p}, M_\mathfrak{p}) = e(IR_\mathfrak{p}, J^n(M_\mathfrak{p}/(y_2, ..., y_{d-k})M_\mathfrak{p})).$$

By Corollary 7.2.4 we have

$$e(IR_\mathfrak{p}, I^n(M_\mathfrak{p}/(x_2, ..., x_{d-k})M_\mathfrak{p})) = e(IR_\mathfrak{p}, I^n(M_\mathfrak{p}/(x_2, ..., x_{d-k})M_\mathfrak{p})).$$

Therefore, $c_0(IR_\mathfrak{p}, M_\mathfrak{p}) = c_0(JR_\mathfrak{p}, M_\mathfrak{p})$ for all $\mathfrak{p} \in \operatorname{Asym}(I,M)$.

Then $c_0(IR_\mathfrak{p}, M_\mathfrak{p}) = c_0(JR_\mathfrak{p}, M_\mathfrak{p})$ for every $\mathfrak{p} \in \operatorname{Spec}(R)$ and thus I is a reduction of (J,M)

$\square$

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