INVESTMENT BEHAVIOR ANALYSIS BASED ON TAIL RISK MANAGEMENT

A Paper
Submitted to the Graduate Faculty
of the
North Dakota State University
of Agriculture and Applied Science

By

Yu Sun

In Partial Fulfillment of the Requirements
for the Degree of
MASTER OF SCIENCE

Major Department:
Statistics

May 2018

Fargo, North Dakota

NORTH DAKOTA STATE UNIVERSITY

Graduate School

Title

INVESTMENT BEHAVIOR ANALYSIS BASED ON TAIL RISK MANAGEMENT

By

Yu Sun

The supervisory committee certifies that this paper complies with North Dakota State University's regulations and meets the accepted standards for the degree of

MASTER OF SCIENCE

SUPERVISORY COMMITTEE:

Dr. Rhonda Magel

Chair

Dr. Ruilin Tian

Co-Chair

Dr. Simone Ludwig

Approved:

11 May 2018

Date

Dr. Rhonda Magel

Department Chair

ABSTRACT

As behavioral finance is becoming more prevalent in academic area, a study is worth conducting to pinpoint investors' preference through managing tail risk of asset portfolios. This study investigates investors' investment behaviors by modeling their investment personalities based on tail risk management. We incorporate CVaR approach to model traditional and non-traditional investment behaviors by reshaping the tails of portfolio return. To be specific, we build model to maximize left-tail CVaR, minimize right-tail CVaR, minimize left-tail CVaR models, and a mixed model that maximize left-tail CVaR and minimize right-tail CVaR simultaneously based on various group of rational and irrational investors. Our work incorporates empirical historical data and Monte Carlo simulation to compare these models with the classical Markowitz approach via different dimensions. We make contributions to fill the gap by making a more comprehensively study that incorporates investors' psychological factors and exploring economic information regarding asset pricing puzzle and long-run risk.

ACKNOWLEDGEMENTS

I would like to show my gratitude to my advisors, Dr. Rhonda Magel and Dr. Ruilin Tian, who always guided me, encouraged me, and helped me in my research. Their excellent talents and strict attitude for research inspire me to be a qualified researcher.

I want to appreciate my committee members, Dr. Simone Ludwig, for her suggestions, comments, and supports during my graduate study. I also want to give my sincere thanks to Ryan Niemann and all other staffs and faculty members in the department of Statistics, who always ready to offer help when I need.

I would like to appreciate my mom, Shan Li, and my husband, Xiaoguang Zhang. Without their encouragement and concern, I cannot reach this step.

I also want to thank my friends, Peishan Merrick and Ying Lin, for their support and encouragement.

Finally, I want to give my thanks to the Department of Statistics and North Dakota State University.

TABLE OF CONTENTS

LIST OF TABLES

LIST OF FIGURES

Figure Page

1. INTRODUCTION

Traditional portfolio theories typically target rational investors who pursue optimal portfolio investment that balances the trade-off between risk and return. However, a more comprehensive study should incorporate investors' psychological factors, especially those of pessimistic and irrational investors'.

There has been a growing interest in behavioural finance that studies the investors' preference and the decision making based on their investment personality (Charles (2014)). Some investors have especially low risk tolerance to the lowest possible return who are interested in limiting their downside risk. Some would like to sacrifice the highest possible return in exchange for more chances of getting a higher return on average. Some even irrationally response to the market under a "worst-case" process by maximizing rather than minimizing the portfolio downside risk.

While the existing literature in portfolio optimization has provided a spectrum of studies of rational investors, there are very few analyses that consider investors' bias and irrational portfolio selection. However, most behavioral finance studies focus on survey analyses that have no connection with portfolio optimization. To fill the gap, this paper is to model investors' investment behaviors in response to the market frictions by managing the tail risk of asset portfolio. We employ the Conditional Value-at-Risk (CVaR) as the measure of tail risk. Specifically, we build four models that respectively (1) maximize the left-tail CVaR, (2) minimize the right-tail CVaR, (3) minimize the left-tail CVaR, and (4) jointly maximize the left-tail CVaR & minimizes the right-tail CVaR. We contribute to the literature by applying portfolio optimization approaches to analyze various groups of rational and irrational investors. Through portfolio optimization models that manage portfolio tail risk, we investigate behavioral-based investment strategies to accommodate investors' needs/concerns. This paper also sheds light on exploring economic information regarding asset pricing puzzles and long-run risk based on the investment behaviors of traditional and non-traditional investors.

The rest of this paper is organized as follows. Section 2 reviews the prevailing literature on pessimistic investment strategies that focus on “worst-case" scenarios, as well as portfolio optimization approaches on tail-risk controlling. In Section 3, we present the frameworks of our optimization models. Section 4 describes the data for empirical analysis. We present the results of empirical experiments in Section 5. Rather than analyze portfolio selection based on historical data, in Section 6, we re-examine the portfolio characteristics under different portfolio optimization models based on simulated data. We first estimate the assets parameters through the maximum likelihood method assuming the returns of assets are correlated Geometric Brownian motions. We show how the results from various tail-risk management models coincide under the normality assumption. Finally, we conclude the paper in Section 7.

2. LITERATURE REVIEW

Markowitz (1952)'s seminal paper serves as the foundation of modern portfolio theory. After that, Value at Risk (VaR) (see J.P.Morgan and Reuters (2006)) sought a new way to manage investment risk by targeting the tail of portfolio return distribution. VaR is commonly used as one of the major measures of downside risk (Jorion (2007),Consigli (2002), and Alexander and Baptista (2002)). Although VaR is popularly used, it has undesirable mathematical characteristics such as lack of sub-additivity and convexity (Artzner et al. (1996)), the limitation of coherency, and difficulties in optimization (Rockafellar and Uryasev (2000)). Moreover, VaR does not consider the crucial aspects about the extreme loss exceeding it. Introduced as a new measure of downside risk, Conditional Value at Risk (CVaR) provides expected value exceeding VaR. CVaR is a coherent risk measure with significant advantages over VaR since it is convex, transition-equivalent, and sub-additive. Rockafellar and Uryasev (2000) and Rockafellar and Uryasev (2002) propose practical techniques to model CVaR in optimization problems and calculate VaR at the same time.

As Rockafellar and Uryasev (2000), Acerbi and Simonetti (2002), and Bassett et al. (2004) have paved the way for efficiently achieving portfolio risk management through quadratic or linear programming, there are a great number of theoretical contributions based on these approaches. For example, Tian et al. (2010) build a MV+CVaR model to reshape the tail distribution by adding a CVaR-like constraint. We share a similar idea to manage the tail risk of portfolio through the Conditional Value-at-Risk (CVaR). But rather than count an approach to manipulate the left-tail downside risk portfolios, we propose a set of tail-risk management models to investigate the behaviors of irrational investors.

There has been a growing interest in behavioural finance that studies the investors' preference and the decision making based on their investment personality (Charles (2014)). We also see thorough analysis towards the portfolio optimization approaches. However, these two aspects are barely connected since most behavioural finance analysis depend on the statistic results from survey while the portfolio optimization approaches are typically based on homogeneous investment assumption that cannot address investors' personal "taste" in portfolio decision.

Bassett et al. (2004) proposes the idea of pessimistic Choquet preferences under risk of linear utility and provides methods to conduct pessimistic portfolio optimization based on Choquet return. We share a similar idea that controls the downside risk, but in different approaches. Bassett et al. (2004) challenges the traditional portfolio optimization by accentuating the probability of least favorable outcomes. Specifically, by adding another weight to adjust the pessimistic level through the concaveness of distortion function, one may obtain a better portfolio return density. This provides an evidence that a pessimistic investment approach sometimes could be superior to the optimal choice of rational investors, which is consistent with our results.

According to Coval and Thakor (2004), optimistic investors would take the highest risk, rational investors choose to take the second highest risk, while pessimistic investors take the least risk. In our study, we compare the tail-risk management models that characterize the non-traditional investment strategies with the traditional Markowitz method that optimizes the mean-variance trade-off of rational investors. Specifically, we consider four types of non-traditional investors: (i) conservative investors who have the least risk tolerance for the highest loss (*i.e.*, the lowest return on the left-tail of the return distribution); (ii) pessimistic investors who minimize the possible return for the worst case (minimum gain) scenario; (iii) moderate investors who would like to obtain a higher average return by scarifying part of the highest return (the right-tail of the return distribution); (iv) and mixed type investors who have both features of the first two groups. All these four types fill in the gap that is not covered by the traditional portfolio theories that focus on rational investors' asset management strategies.

Furthermore in the paper, we further analyze the hidden market information behind the investment choices of pessimistic investors. According to Bidder and Dew-Becker (2016), investors respond to the market based on their pessimistic predictions of the worst-case scenarios, which provides insights to explain some asset pricing puzzles. That means, "investors try to limit their downside risk over a set of models, rather than try to identify optimal behavior for a single one". This pessimistic approach is characteristic of various methods of decision making and forecasting in ambiguous situations (Gilboa and Schmeidler (1989) and Hansen and Sargent (2007)).

From the economic point of view, Bidder and Dew-Becker (2016) model people's decision making choices under a "worst-case" process by minimizing lifetime utility of consumption. Following similar logic, we investigate investors' portfolio construction strategies by maximizing instead of minimizing the portfolio downside risk. Specifically, we minimize the left-tail CVaR of portfolio returns, replicating the same pessimistic rational that incorporates the consideration of long-run risk. Zhu and Fukushima (2009) show the coherence of the worst-case problem and present the same logic by applying the conditional value at risk to the worst case. But they focus on the robust portfolio optimization for the cases when only partial information is available. Moreover, they follow Rockafellar and Uryasev (2000)'s model that considers the highest possible loss, while we study the lowest possible return.

Our paper complements the literature by analyzing investors' investment behaviors under ambiguity from a "brand-new" angle that analyzes the hidden information of pessimistic investors' behavior by modeling the minimized worst outcome (maximizing left-tail CVaR), minimized the best outcome (minimizing right-tail CVaR), and jointly modeling the left-tail and right-tail CVaR. In addition, we consider a pessimistic case that maximizes the risk in the worst-case scenarios (minimizing left-tail CVaR). As far as we know, this study has never been done by any existing works in the economic-, business-, or finance-related areas.

3. MODEL DESCRIPTION

3.1. Left-Tail VaR and CVaR

Let $r(x, y)$ be the portfolio return associated with the decision vector x, to be chosen from a certain subset $X \in \mathbb{R}^n$, and the random vector $Y \in \mathbb{R}^m$. In our portfolio selection model, $x = [x_1, x_2, \ldots, x_n]$ decides the asset investment strategy, where $x_i (i = 1, \ldots, n)$ is the proportion of wealth invested in asset i. The vector y stands for the uncertainties, e.g., the market-related variables that affect the portfolio return. For each x, the portfolio return $r(x, y)$ is a random variable having a distribution in R based on the probability distribution of y, $f(y)$.

For a given investment strategy x, the cumulative distribution function $F(x, \alpha)$ is given by

$$F(x, \alpha) = \int_{r(x,y) \leq \alpha} f(y) dy. \tag{3.1}$$

$F(x, \alpha)$ is non-decreasing with respect to α and continuous from the right, but not necessarily from the left because of the probability of jumps. Given the significance level β, the left-tail β-VaR is calculated as

$$\alpha_\beta(x) = \max\{\alpha \in \mathbb{R} : \mathrm{F}(x, \alpha) \leq \beta\}. \tag{3.2}$$

As a result, when cumulative probability of return is less than or equal to the significant level of β, the left-tail VaR is defined as the maximum value of α. In other words, within limited uncertainty, $\alpha_\beta(x)$ is the maximum return one can obtain at probability β. Different from Rockafellar and Uryasev (2000) that analyzes a loss random variable, in our study, we focus on the return of a portfolio. Specifically, our left-tail β-CVaR provides the conditional expectation of portfolio return that is less than or equal to the left-tail β-VaR:

$$\phi_\beta(x) = \frac{1}{\beta} \int_{r(x,y) \leq \alpha_\beta(x)} r(x, y) f(y) dy = E[r(x, y) | r(x, y) \leq \alpha_\beta(x)]. \tag{3.3}$$

Define $\psi_\beta(x,\alpha)$ on $X \times R$ as follows:

$$\psi_\beta(x,\alpha) = \alpha - \frac{1}{\beta}\int_{y\in R^m} [\alpha - r(x,y)]^+ f(y)dy$$
$$\text{where } [t]^+ = \begin{cases} t, \text{when } t > 0 \\ 0, \text{when } t \leq 0. \end{cases} \tag{3.4}$$

In our study, we get an approximated value of $\psi_\beta(x,\alpha)$ by sampling the probability distribution in y based on the sample set $y_1, y_2, \dots, y_m$ as follows:

$$\psi_\beta(x,\alpha) \approx \alpha - \frac{1}{m\beta}\sum_{k=1}^{m} [\alpha - r(x,y_k)]^+. \tag{3.5}$$

Rockafellar and Uryasev (2000)'s Theorem 1 and Theorem 2 with respect to a loss random variable can be adjusted to fit into our return variable as follows: **Theorem 1**: As a function of α, $\psi_\beta(x,\alpha)$ is convex and continuously differentiable. The left-tail β-CVaR of the return associated with any $x \in X$ can be determined from the formula:

$$\phi_\beta(x) = \max_{\alpha\in R} \psi_\beta(x,\alpha). \tag{3.6}$$

Theorem 2: Maximize the left-tail β-CVaR of the return associated with x over all $x \in X$ is equivalent to maximizing $\psi_\beta(x,\alpha)$ over all $(x,\alpha) \in X \times R$, in the sense that

$$\max_{x\in X} \phi_\beta(x) = \max_{(x,\alpha)\in X\times R} \psi_\beta(x,\alpha), \tag{3.7}$$

where in the pair (x^*, α^*), x^* maximizes the left-tail β-CVaR and $\alpha^* = \underset{\alpha\in R}{\operatorname{argmax}}\, \psi_\beta(x,\alpha)$ gives the corresponding left-tail β-VaR.

3.2. Right-Tail VaR and CVaR

If we instead define the right-tail CVaR, we modify the formulas in (3.1) as follows: For a given investment strategy x, the survival function $S(x,\gamma)$ is given by

$$S(x,\gamma) = \int_{r(x,y)\geq\gamma} f(y)dy. \tag{3.8}$$

$S(x, \gamma)$ is non-increasing with respect to γ and continuous from the left, but not necessarily from the right because of the probability of jumps. Given the significance level β, the right-tail β-VaR is calculated as

$$\gamma_\beta(x) = \max\{\gamma \in \mathbb{R} : \mathrm{S}(x, \gamma) \geq \beta\}. \tag{3.9}$$

As a result, when survival probability of return is greater than or equal to the significant level of β, the right-tail VaR is defined as the maximum value of γ. In other words, within limited uncertainty, $\gamma_\beta(x)$ is the maximum return we can obtain at accumulative probability $1 - \beta$. The right-tail β-CVaR provides the conditional expectation of portfolio return that is greater than or equal to the right-tail β-VaR:

$$\zeta_\beta(x) = \frac{1}{\beta} \int_{r(x,y) \geq \gamma_\beta(x)} r(x, y) f(y) dy = E[r(x, y) | r(x, y) \geq \gamma_\beta(x)]. \tag{3.10}$$

Define $\kappa_\beta(x, \gamma)$ on $X \times R$ as follows:

$$\kappa_\beta(x, \gamma) = \gamma + \frac{1}{\beta} \int_{y \in R^m} [r(x, y) - \gamma]^+ f(y) dy. \tag{3.11}$$

In our study, we get an approximated value of $\zeta_\beta(x, \gamma)$ by sampling the probability distribution in y based on the sample set $y_1, y_2, \ldots, y_m$ as follows:

$$\kappa_\beta(x, \gamma) \approx \gamma + \frac{1}{m\beta} \sum_{k=1}^{m} [r(x, y_k) - \gamma]^+. \tag{3.12}$$

A modified version of Theorem 1 shows the right-tail β-CVaR of the return can be determined as follows:

$$\zeta_\beta(x) = \min_{\gamma \in R} \kappa_\beta(x, \gamma). \tag{3.13}$$

The optimization (3.13) actually minimizes the highest possible return. Mathematically problem (3.13) is solved in the same way as (Rockafellar and Uryasev, 2000), which minimizes the most serious loss by minimizing the right-tail CVaR of a loss random variable. However, economically problem (3.13) that minimizes the right-tail CVaR of a return random variable is rational but conservative investment behavior.

3.3. Maximize Left-tail CVaR and Minimize Right-Tail CVaR Simultaneously

If we consider to maximize the left-tail CVaR and minimize the right-tail CVaR at the same time with given weight τ and $(1-\tau)$ respectively, we can simply combine the left-tail MaxCVaR and the right-tail MinCVaR model and minimize the following objective function. The sample version of the model can be represented as follows:

$$\eta_\beta(x,\tau,\alpha,\gamma) = \tau(-\alpha + \frac{1}{m\beta}\sum_{k=1}^{m}[\alpha - r(x,y)]^+) + (1-\tau)(\gamma + \frac{1}{m\beta}\sum_{k=1}^{m}[r(x,y)-\gamma]^+). \tag{3.14}$$

Problem (3.14) maximizes the lowest possible and minimizes the highest possible return at the same time. Mathematically the problem (3.14) can be solved linearly. Problem (3.14) captures investment preference of conservative investors who have especially low tolerance to the lowest possible return while would sacrifice the highest possible return in exchange for more chances of getting a higher return on average. In other words, the strategy (3.14) "truncates" both left and right tails of the return distribution.

3.4. Portfolio Optimization Models

Conservative but optimistic investors who have low tolerance to the lowest possible return generally make decision to minimize downside risk, *i.e.*, maximize the left-tail CVaR of portfolio return (problem (3.15) below). This model could be considered as an extension of the model proposed by Krokhmal et al. (2002). They suggest minimizing the right-tail CVaR of loss portfolios. Therefore, for return portfolios, we maximize the left-tail CVaR to increase the probability of getting a higher expected return on the left-tail of portfolio distribution. While irrational pessimistic investors instead maximize the downside risk (problem (3.17) below) by minimizing the left-tail CVaR of return in the worst scenarios. By further extending the idea of the pessimistic investment strategy, we come up with another model which minimizes the right-tail CVaR by solving problem (3.16). Mathematically it is achieved in the same way as Rockafellar and Uryasev (2000). But we minimize the highest portion of possible return which is rational but again conservative strategy from economical aspect. The left-tail MaxCVaR model, right-tail MinCVaR, left-tail MinCVaR, and a mixed model that jointly maximize left-tail CVaR and minimize right-tail CVaR are presented below respectively:

Maximize Left-tail β-CVaR Model:

$$\max_{x\in X} \phi_\beta(x) \tag{3.15}$$

Minimize Right-tail γ-CVaR Model:

$$\min_{x \in X} \zeta_\gamma(x) \tag{3.16}$$

Minimize Left-tail β-CVaR Model:

$$\min_{x \in X} \phi_\beta(x) \tag{3.17}$$

Maximize Left-tail β-CVaR $\&$ Minimize Right-tail γ-CVaR Model:

$$\min_{x \in X} -\tau\phi_\beta(x) + (1-\tau)\zeta_\gamma(x) \tag{3.18}$$

According to Theorem 2, maximizing $\beta-$CVaR is achieve by maximizing $\psi_\beta(x, \alpha)$ over all $(x, \alpha) \in X \times R$. With the linearizion technique proposed by Rockafellar and Uryasev (2000), the convexity of problem (3.15) and (3.16) are satisfied so their global optimum is guaranteed. Also we consider problem (3.18) that combining problem (3.15) and (3.16), which can be solve linearly as well. However, problem (3.17) actually solves a min-max problem, which is more challenging and could be a non-convex problem. Therefore, there's no guarantee to find global optimal solution for problem (3.17).

In our study, we will compare the portfolio construction strategies of (3.15), (3.17), (3.16) and (3.18), recognizing the impact of the long-run risk component embedded in the models. We will also compare these models with the traditional Markovitz model that achieves the mean-variance efficiency.

3.4.1. Traditional Markowitz

The traditional Markowitz model is presented in (3.19). The problem find the portfolio $x = [x_1, x_2, \ldots, x_n]$ that has the lowest variance and achieves the preset target return μ_0. Note that the sum of asset weight x_i must be equal to 1. Moreover, all x_i must greater than or equal to zero. In other words, we don't allow short selling of any asset.

$$
\begin{aligned}
\min \ & \sum_{i=1}^{n}\sum_{j=1}^{n}\sigma_{ij}x_ix_j \\
\text{subject to } & \sum_{i=1}^{n}x_i = 1, \\
& \sum_{i=1}^{n}\mu_ix_i = \mu_0, \\
& \text{and } x_i \geq 0 \text{ for } i = 1, 2, \ldots, n.
\end{aligned}
\tag{3.19}
$$

3.4.2. Maximize Left-Tail CVaR

As states in (3.7), maximizing left-tail CVaR can be achieved by maximizing $\psi_\beta(x, \alpha)$ over all $(x, \alpha) \in X \times R$. To achieve the optimization more conveniently, according to the Theorem 4 in Krokhmal et al. (2002), we linearize $[\alpha - r(x, y)]^+$ through m auxiliary variables $\omega = [\omega_1, \omega_2, \ldots, \omega_m]$ as illustrated in (3.20) below.

$$
\begin{aligned}
& \max \ \alpha - \frac{1}{\beta m}\sum_{l=1}^{m}\omega_l \\
& \text{subject to } \omega_l \geq \alpha - \sum_{i=1}^{n} r_{il}x_i \text{ for } l = 1, 2, \ldots, m, \\
& \omega_l \geq 0, \\
& \sum_{i=1}^{n}x_i = 1, \\
& \sum_{i=1}^{n}\mu_ix_i = \mu_0, \\
& \text{and } x_i \geq 0 \text{ for } i = 1, 2, \ldots, n.
\end{aligned}
\tag{3.20}
$$

3.4.3. Minimize Right-Tail CVaR

To minimize the highest possible return by minimizing the right-tail CVaR, we solve the following problem (3.21) after linearizing $[r(x,y)-\gamma]^+$ through m auxiliary variables $\upsilon = [\upsilon_1, \upsilon_2, \ldots, \upsilon_m]$.

$$\begin{aligned}
&\min \gamma + \frac{1}{\beta m}\sum_{l=1}^{m}\upsilon_l \\
&\text{subject to } \upsilon_l \geq \sum_{i=1}^{n} r_{il}x_i - \gamma \text{ for } l = 1,2,...m, \\
&\upsilon_l \geq 0, \\
&\sum_{i=1}^{n} x_i = 1, \\
&\sum_{i=1}^{n} \mu_i x_i = \mu_0, \\
&\text{and } x_i \geq 0 \text{ for } i = 1,2,\ldots,n.
\end{aligned} \tag{3.21}$$

3.4.4. Minimize Left-Tail CVaR

Though the function looks similar to the maximizing left-tail CVaR problem, solving it is totally different and much more challenging since it switches to a minimax problem rather than a linear one. We solve this optimization problem through ‘fminimax’ function in MATLAB. Due to the non-convex feature, not all points can be solved. But we can still see the general trend as expected, which we will discuss in section 5 based on the empirical experiments.

$$\begin{aligned}
&\min \max_{x\in X} \psi_\beta(x) \\
&\text{subject to } \sum_{i=1}^{n} x_i = 1, \\
&\sum_{i=1}^{n} \mu_i x_i = \mu_0, \\
&\text{and } x_i \geq 0 \text{ for } i = 1,2,\ldots,n.
\end{aligned} \tag{3.22}$$

3.4.5. Maximize Left-Tail CVaR and Minimize Right-Tail CVaR Simultaneously

Here we propose a mixed model that maximizes left-tail CVaR and minimizes right-tail CVaR simultaneously. Combining the two function by adding a negative sign to the left-tail MaxCVaR part, this new function can be solved in the similar way as before. In the new function, τ and $1-\tau$ are the weights for the left-tail MaxCVaR and right-tail MinCVaR functions, respectively. We can adjust $\tau \in [0,1]$ to change the importance of these two functions in the mixed model. We replace $[t]^+$ with a variable with constrains as we done in previous sections. The difference is, here we have two $[t]^+$ with different t inside, then we replace them with two variables ω and υ respectively. Therefore, we transfer the mixed problem following the steps below:

$$
\begin{aligned}
&\min \; -\tau\text{MaxCVaR(Left)} + (1-\tau)\text{MinCVaR(Right)} \\
\Longleftrightarrow \quad & -\tau(\alpha - \frac{1}{m\beta}\sum_{k=1}^{m}[\alpha - r(x,y)]^+) + (1-\tau)(\gamma + \frac{1}{m\beta}\sum_{k=1}^{m}[r(x,y)-\gamma]^+) \\
\Longleftrightarrow \quad & -\tau\alpha + \frac{\tau}{\beta m}\sum_{l=1}^{m}\omega_l + (1-\tau)\gamma + \frac{1-\tau}{\beta m}\sum_{l=1}^{m}\upsilon_l \\
\text{subject to } & \omega_l \geq \alpha - \sum_{i=1}^{n} r_{il}x_i \text{ for } l = 1,2,\ldots,m, \\
& \upsilon_l \geq \sum_{i=1}^{n} r_{il}x_i - \gamma \text{ for } l = 1,2,\ldots,m, \\
& \omega_l \geq 0, \\
& \upsilon_l \geq 0, \\
& \sum_{i=1}^{n} x_i = 1, \\
& \sum_{i=1}^{n} \mu_i x_i = \mu_0, \\
& \text{and } x_i \geq 0 \text{ for } i = 1,2,\ldots,n.
\end{aligned} \tag{3.23}
$$

4. DATA DESCRIPTION

Our empirical analysis is based on monthly historical data from three asset classes, i.e, stocks, bonds, and a risk-free asset ranging from Jan 1999 to Dec 2017. The three asset classes are represented, respectively, by S&P 500 index[1], BofA Merrill Lynch US Corp Bonds Index[2], and the three-month US Treasure Bill[3]. The monthly returns of S&P 500 index and BofA Merrill Lynch US Corp Bonds Index are calculated as the log of the ratio of the index values of two consecutive month. The monthly return of T-Bill is the annual discount rate divided by twelve.

Table 4.1. Data Statistics Table For the Monthly Returns of the Three Assets

Assets	Mean	Std. Deviation	Max	Min	Sharpe Ratio
Stocks	0.0250	0.0415	0.1245	−0.1525	0.5656
Bonds	0.0046	0.0135	0.0400	−0.0994	0.2326
T-Bill	0.0015	0.0016	0.0051	0.00001	

Table 4.1 describes the three assets and summarizes their statistics. Among the three assets, S&P 500 has the highest mean of return, as well as the highest standard deviation. It reflects the riskier and high-yield feature of stocks compared to the bonds and T-Bill. Correspondingly, as a risk-free asset, Treasure Bill provides the lowest mean and standard deviation, as well as relatively narrower return range than the other two assets. We also provide the Sharpe Ratio (Sharpe (1966)) for each asset.

$$S_i = \frac{E[R_i - R_f]}{\sigma_i}, \tag{4.1}$$

where R_i is the asset return, R_f is the risk free rate, and σ_i is the standard deviation of the asset excess return. $E[R_i - R_f]$ is the expected value of the excess asset return over the risk-free return. We notice the Sharp Ratio of stocks is higher than that of the bonds. The Sharpe ratio measures how well the return of an

[1]Data source: YAHOO Finance, url: https://finance.yahoo.com/quote/%5EGSPC/history?p=%5EGSPC.
[2]Data source: Federal Reserve Economic Data, url: https://fred.stlouisfed.org/series/BAMLCC0A0CMTRIV#0.
[3]Data source: Federal Reserve Economic Data, url: https://fred.stlouisfed.org/series/TB3MS.

asset compensates the investors for the risk taken. We notice that stocks provide a higher extra return per unit of risk.

Table 4.2. Correlation Coefficient of the Monthly Return

	Stocks	Bonds	T-bill
Stocks	1	0.1568	−0.1014
Bonds	0.1568	1	−0.0395
T-Bill	−0.1014	−0.0395	1

Table 4.2 shows the correlation coefficient matrix of the three assets. In this matrix, stocks and bonds are positively correlated, while T-Bill is negatively correlated with both stocks and bonds. It makes sense because T-Bill reversely influences the markets when the T-Bill rate in the money market, the cost of the loan increases, so do the prices of stocks and bonds. As the prices of stocks and bonds rise, their rates of return decrease.

Table 4.3. Data Statistics For the monthly Return of Bonds Category By Credit Rating

Assets	Mean	Std. Deviation	Max	Min	Sharpe Ratio
Corp AAA	0.0040	0.0128	0.0588	−0.0577	0.1999
Corp AA	0.0041	0.0110	0.0468	−0.0608	0.2392
Corp A	0.0043	0.0141	0.0476	−0.1133	0.2030
Corp BBB	0.0050	0.0151	0.0527	−0.1093	0.2349
Corp BB	0.0066	0.0262	0.0899	−0.2620	0.1959
Corp B	0.0048	0.0259	0.1215	−0.2029	0.1307
Corp CCC or Below	0.0062	0.0404	0.1762	−0.2671	0.1176

In addition to consider the three asset classes with the above-mentioned index, we further consider more categories in the bond class, differentiating in maturities[4] or credit ratings[5]. The statistics summary for bond returns by credit rating are reported in Table 4.3. As shown in Table 4.3, it generally follows the trend that the higher credit rating is, the lower the mean return and standard deviation is. The Sharpe ratios of bonds for investment grades (credit rating BBB or above) are about the same — around 0.2. Besides,

[4]We will not discuss this category because the results of it is similar to the ones of category by credit rating.
[5]Data source: Federal Reserve Economic Data, url: https://fred.stlouisfed.org/categories/32413.

corporation bonds with credit rating AA and BBB have relative higher Sharpe ratio, which indicates they have a more attractive trade-off between risk and return. All these information may give us clues to explain the results in Section 5.

5. EMPIRICAL STUDY

5.1. Static Optimization Experiments

5.1.1. Three-Asset Experiments

In the three-asset experiment, we consider 3 assets from three asset classes, *i.e.*., stocks, bonds, and a risk-free asset, with around 20-year monthly data ranging from Jan 1999 to Dec 2017 to construct "optimal portfolios". We compare the optimal portfolios constructed through Markowitz, left-tail MaxCVaR, right-tail MinCVaR, left-tail MinCVaR model, and a mixed model that maximizes left-tail CVaR and minimizes right-tail CVaR at the same time with weight $\tau = 0.6$. To understand our models better, we compare results from various dimensions[1] based on the solved nine portfolios on the efficient frontiers.[2]

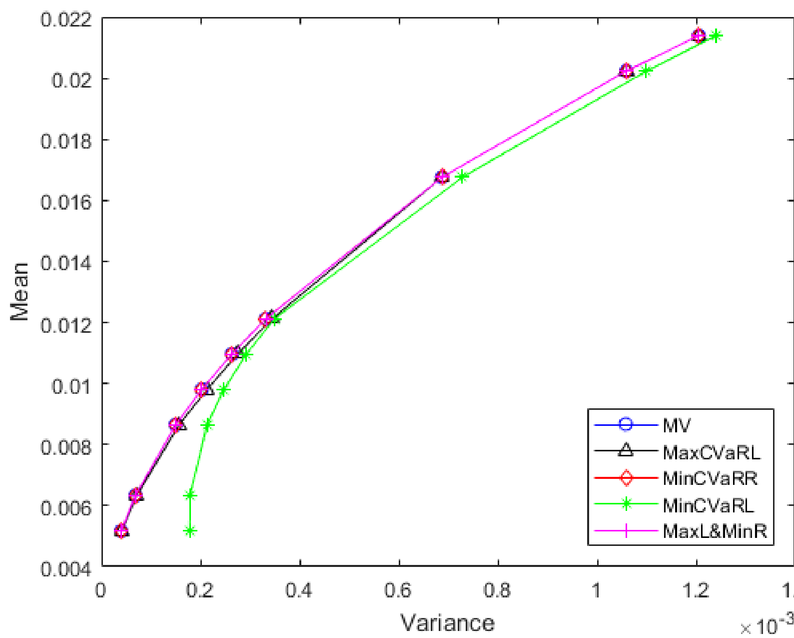

Figure 5.1. Mean-variance frontier of the three-asset portfolios with monthly data ranging from Jan 1999 to Dec 2017 from the Markowitz (MV), left-tail MaxCVaR, right-tail MinCVaR, left-tail MinCVaR, and the mixed models respectively. In the graph, the expected mean of return is on the vertical axis and the variance is on the horizontal axis.

[1]We don't include the skewness in this paper because the results are mixed and our models mainly target tail-risk management through CVaR.

[2]The points are not evenly chosen on the efficient frontier because some points are not solved. The non-solved cases are typically problem, which is a minimax problem and may not be convex. In the next a few sessions, there could be more or less than nine points. The number of points depends on how many points can be solved by the left-tail MinCVaR.

The plots in Figure 5.1 illustrate the mean-variance frontiers of the three-asset portfolios obtained from the Markowitz, left-tail MaxCVaR, right-tail MinCVaR, left-tail MinCVaR, and the mixed model. In the plot, the mean and variance are positively correlated as the mean-variance trade-off rule indicates. The trade-off lines almost coincide for all the models except the left-tail MinCVaR. The strategy of Left-tail MaxCVaR, right-tail MinCVaR and mixed model, though conservative, still hold a mean-variance frontier as good as that of the Markowitz model whose strategy target an optimal portfolio. When the mean and variance are relatively low, the left-tail MinCVaR portfolio provides a higher variance for a fixed level of portfolio return compared to the portfolio obtained by the other approaches. This is consistent with the intention of the model which represents irrational investors' strategies whose decision making is irrationally distorted, since irrational investors' choices are inferior to those of the rational investors' who target either lower variance (MV model) or lower downside risk (MaxCVaR). However, the imperfect investment strategies suggested by the left-tail MinCVaR model may carry long-term information and sometimes help us to resolve the asset pricing puzzles.

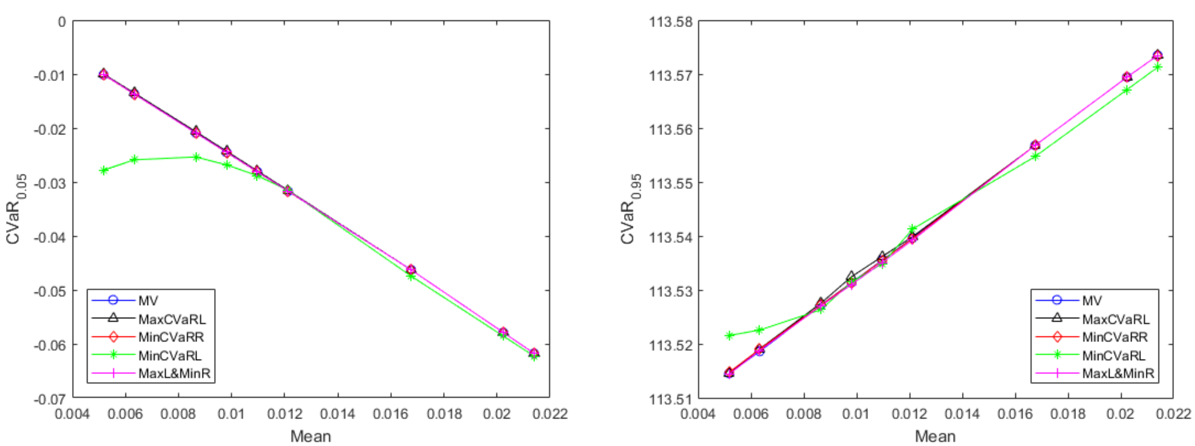

Figure 5.2. Mean-CVaR$_{5\%}$ frontier and mean-CVaR$_{95\%}$ of the three-asset portfolios with monthly data ranging from Jan 1999 to Dec 2017 from the Markowitz (MV), left-tail MaxCVaR, right-tail MinCVaR, left-tail MinCVaR, and the mixed model respectively. In the left graph, the expected mean of return is on the horizontal axis and CVaR$_{5\%}$ is on the vertical axis. The right graph shows the mean on the horizontal axis and CVaR$_{95\%}$ on the vertical axis.

The left and right plots in Figure 5.2 illustrate mean-CVaR$_{5\%}$ frontier and mean-CVaR$_{95\%}$ frontier of the three-asset portfolios obtained from the MV, left-tail MaxCVaR, right-tail MinCVaR, left-tail MinCVaR, and the mixed models, respectively. As shown in both plots, the results for the MV, left-tail MaxCVaR, right-tail MinCVaR, and the mixed model coincide. Unlike other models, the result from the left-tail MinCVaR model shows lower left-tail CVaR especially when mean is at a lower level. Though we design it to have a minimized left-tail CVaR, it only shows significant low left-tail CVaR when the target portfolio return is low. In other words, the model is more sensitive when the target return is low. Moreover, the $\text{CVaR}_{95\%}$ for the low-return portfolios from the left-tail MinCVaR model is higher than the $\text{CVaR}_{95\%}$ from the other models. This might because of the reallocation of probability massed after the left-tail MinCVaR is achieved. As the left-tail CVaR (i.e, $\text{CVaR}_{5\%}$) decreases to maintain the same level of target return μ_0, the portfolio has to increase right-tail CVaR (i.e, -$\text{CVaR}_{95\%}$).

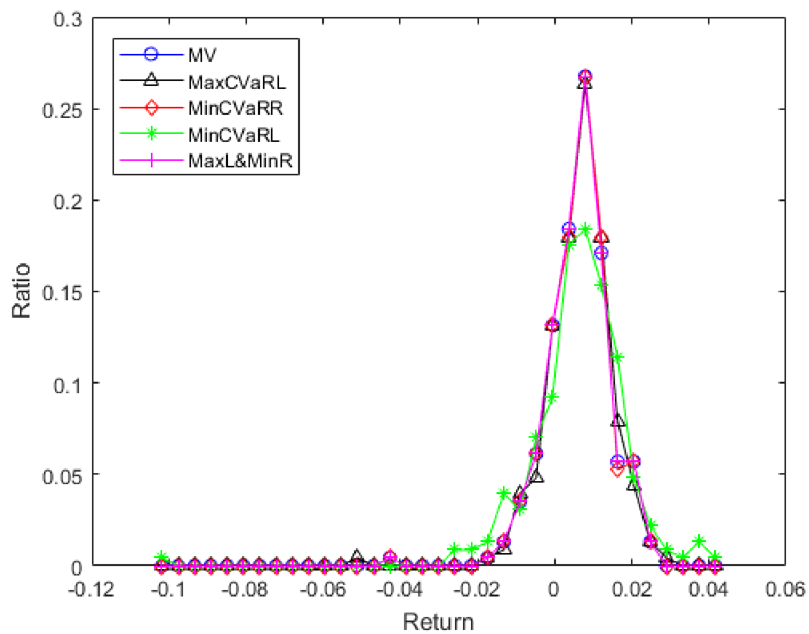

Figure 5.3. Distributions of three-asset portfolios of point two with monthly data ranging from Jan 1999 to Dec 2017 from the Markowitz (MV), left-tail MaxCVaR, right-tail MinCVaR, left-tail MinCVaR, and the mixed model respectively. Each curve represents the ratio of observations over total number of observations at its corresponding return.

Figure 5.3 shows the distributions of three-asset portfolios at point two with monthly data ranging from Jan 1999 to Dec 2017 from the Markowitz (MV), left-tail MaxCVaR, right-tail MinCVaR, left-tail MinCVaR and the mixed model respectively. Overall, all the distributions are significantly left skewed. While the distribution of left-tail MinCVaR are less concentrated and have fatter right tail compared with the other models. It is consistent with what we got in Figure 5.2 — the left-tail MinCVaR portfolio with the mean around 0.0065 (point #2) has higher $CVaR_{5\%}$ and $CVaR_{95\%}$.

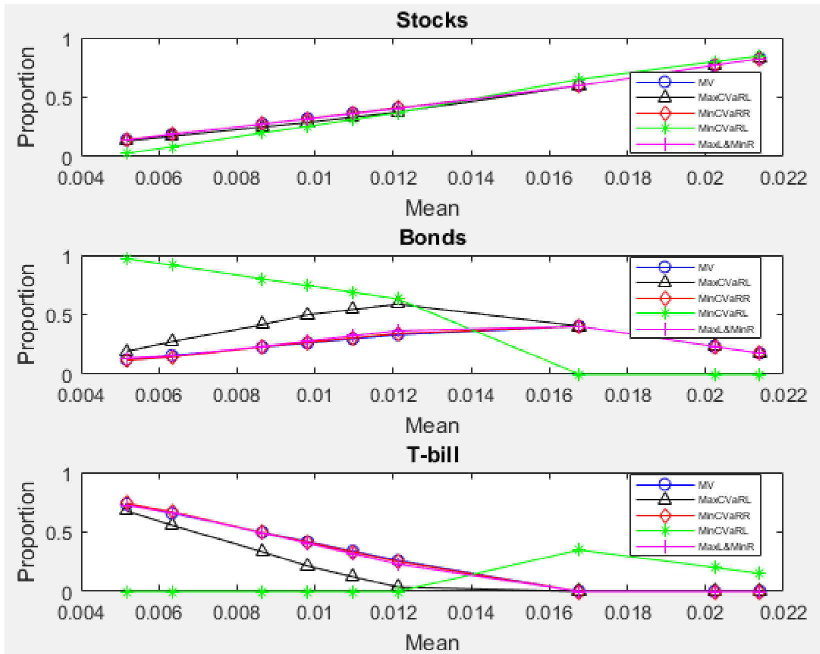

Figure 5.4. Asset allocation of three-asset portfolios with monthly data ranging from Jan 1999 to Dec 2017 from Markowitz (MV), left-tail MaxCVaR, right-tail MinCVaR, left-tail MinCVaR, and the mixed model, respectively.

In Figure 5.4, we compare the asset allocation of MV, left-tail MaxCVaR, right-tail MinCVaR, left-tail MinCVaR, and the mixed models. As shown in the graph, the allocation ratio of stock increases as the mean of return increases in all the models. Also, we notice that the mixed model has almost the same asset allocation as the one of right-tail MinCVaR model. It has a similar trend as the left-tail MaxCVaR but has approximately 10 to 20 percent differences in the choice of bonds and T-Bill. The left-tail MinCVaR shows an opposite trend in bonds and T-Bill to the other models: it invests more in bonds and gradually decreases the investment in bonds as the mean increases, while increasing the allocation in T-Bill instead. The differences in asset allocation helps to explain the differences on both the left and right tails in Figure 5.1 and Figure 5.2.

The results demonstrated in Figures 5.4 for the the left-tail MinCVaR approach actually show the long-term investment strategies. The asset allocation trend in the left-tail MinCVaR portfolios are consistent to that of a retirement fund, one of the most representative long-term investment funds which are shown

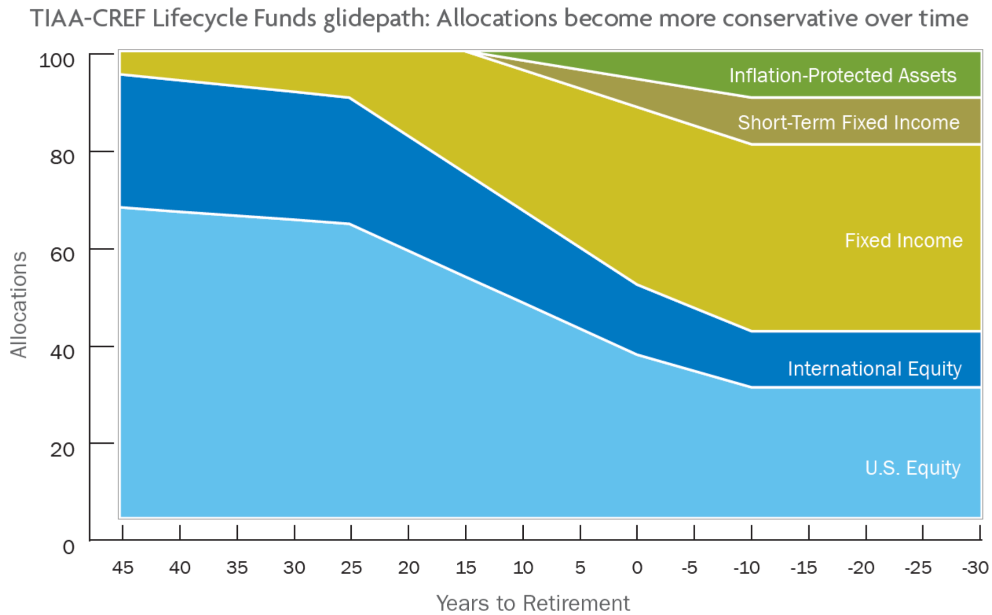

Figure 5.5. Source: 2016 TIAA-CREF Lifecycle Funds Report.

in Figure 5.5 — the composition of the TIAA-CREF lifecycle funds. As the cohort approaches to the retirement, the fund investments more in bonds.

5.1.2. By Credit Rating

In this section, we still analyze portfolios consisting of three asset classes, but we consider more tranches in the bond class. Specifically, rather than consider one bond index in the portfolio, we represent the bond class with four investment-grade bond indexes (AAA, AA, A, and BBB) and three speculative-grade bond indexes (BB, B, and CCC or below)[3]. Our experiments in this section are still based on 20 years monthly data. We are interested in comparing Markowitz (MV), left-tail MaxCVaR, right-tail MinCVaR, left-tail MinCVaR, and the mixed model. We want to know if we could grasp more information about investors' long-term strategies after considering multiple tranches in the bond class.

The results of mean-variance frontier, mean-CVaR frontier, and the shape of the histogram are similar to those we got in three-asset experiments. We will not provide the result details here.

In Figure 5.6, we compare the asset allocation of the MV, left-tail MaxCVaR, right-tail MinCVaR, left-tail MinCVaR, and the mixed model. We find that the asset allocation ratio of stock increases as mean of return increases in all models. We also notice that the proportion of allocation for the left-tail MaxCVaR

[3]We also did experiments that consider three asset classes, in which the bond class with three short-term bond indexes (1-3 years, 3-5 years, and 5-7 years) and three long-term bond indexes (7-10 years, 10-15 years, and more than 15 years). The general trends in all plots are similar to the ones in 3-asset and this section, thus we don't put them here.

and the right-tail MinCVaR are very close to each other in this case. Correspondingly their mixed model has very similar results as well. However, the left-tail MinCVaR invests much more in bonds and about ten percent less in stocks than other models, especially when the target portfolio mean is low (the first three points).

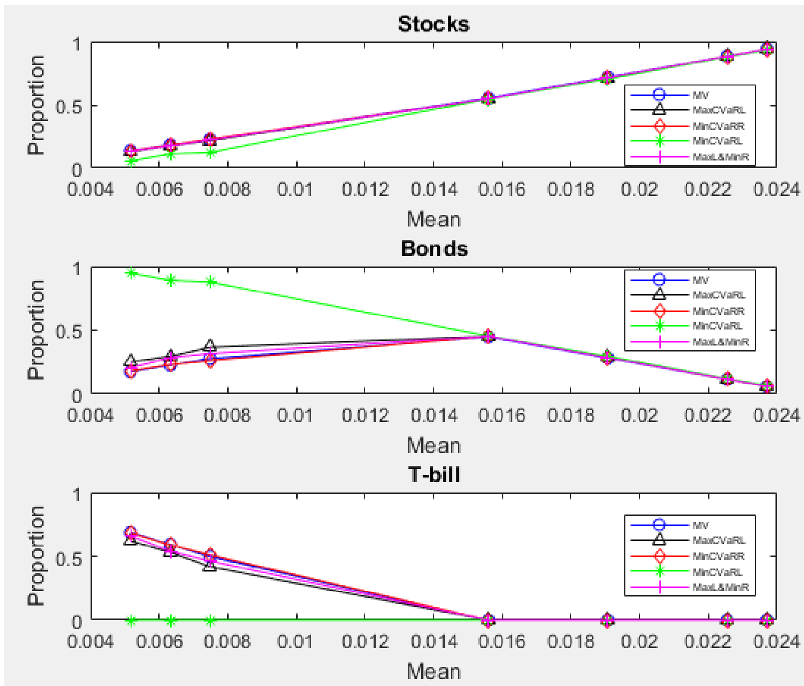

Figure 5.6. Asset allocation considering S&P500, T-Bill, and seven bond tranches for the Markowitz (MV), left-tail MaxCVaR, right-tail MinCVaR, left-tail MinCVaR, and the mixed model respectively.

After we compare the asset allocation of all the models for the three asset classes, we further study what kind of bonds are chosen in bond class for each model. In Figure 5.7, we found all models only invest in investment grade bonds, which may be due to the lower Sharpe ratio of the three speculative bonds in Table 4.1.

As we know all the models choose investment grade bonds in bond allocation, we further study which investment grade bonds are invested. As shown in Figure 5.8, MV invests all in AAA bonds. The left-tail MaxCVaR, right-tail MinCVaR and the mixed model give priority to AAA bonds, while they also invest some in AA bonds when the mean is low, and some in BBB bonds when the mean is high. The trends of the three models are similar, but not exactly the same: the mixed model is more like the left-tail MaxCVaR but invests a little more in AA bonds. The right-tail MinCVaR invests less in stocks and more

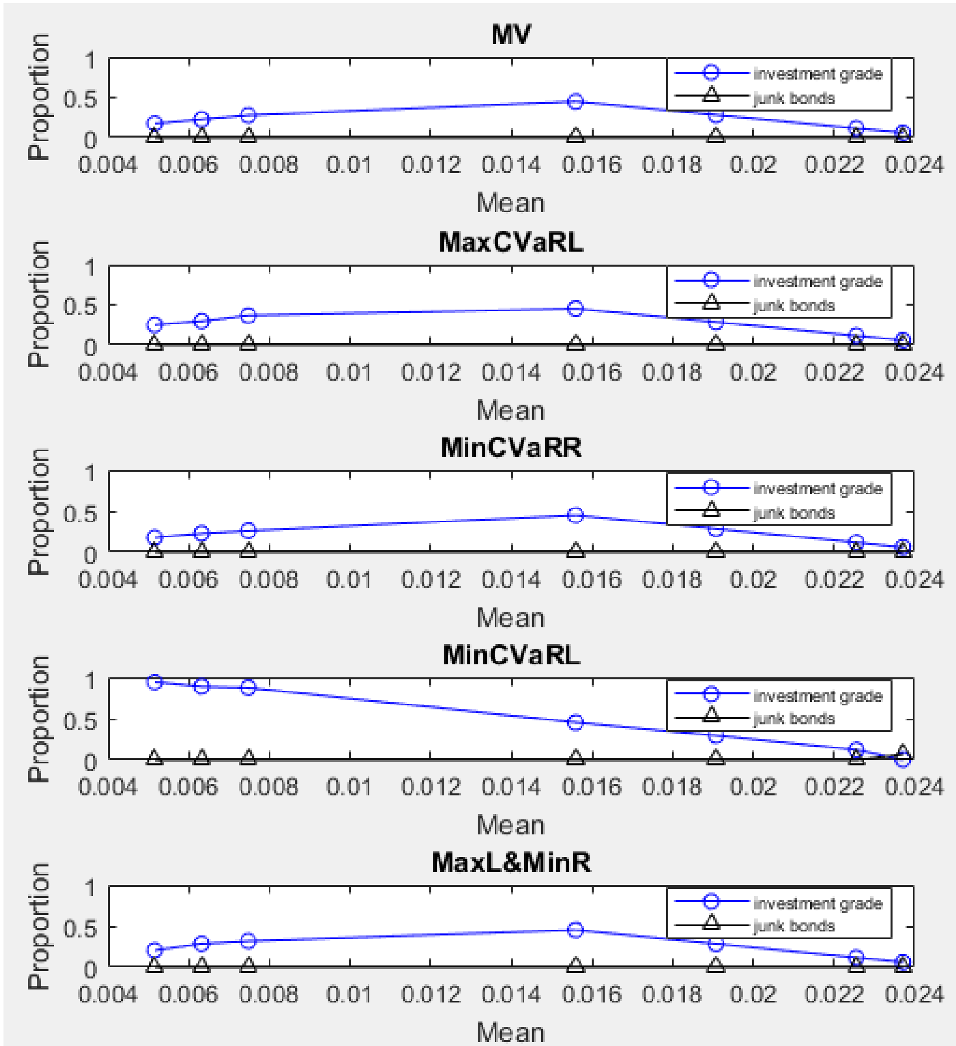

Figure 5.7. Bond allocation considering S&P500, T-Bill, and seven bond tranches for Markowitz (MV), left-tail MaxCVaR, right-tail MinCVaR, left-tail MinCVaR, and the mixed model respectively.

in AA and BBB bonds. The left-tail MinCVaR invests in only one type of investment grade bonds at each target mean. But it jumps so much that shows little trend.

At this point, we can conclude that the left-tail MinCVaR model is interior to the mean-variance frontier because it is the selection of irrational investors who are pessimistic. Interior though it is, it carries crucial information regarding long-run risk. In other words, the worst-case model is painful but plausible: it features the worst case and tries to capture the long-run risk which is very difficult to detect. On the other hand, the left-tail MaxCVaR, right-tail MinCVaR and the mixed model are based on rational but conservative investment attitude and they show similar portfolio allocation and results. As their features are to control the down-side risk, to sacrifice the highest possible return to pursue higher average return, or the combination of

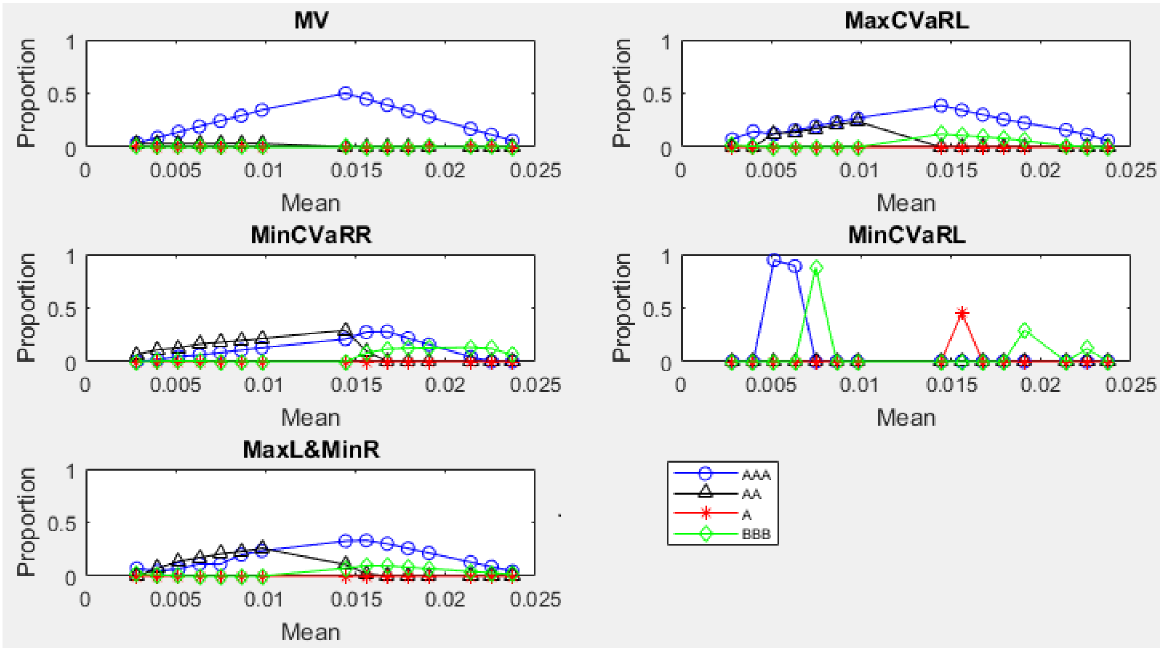

Figure 5.8. Investment grade bond allocation considering S&P500, T-Bill, and seven bond tranches for Markowitz (MV), left-tail MaxCVaR, right-tail MinCVaR and left-tail MinCVaR, and the mixed model respectively.

the two to truncate the return distribution on both tails. Their mean-variance frontiers are not always better but always close to that of MV model.

5.1.3. Equity Premium Puzzle

In this section, we first explain the equity premium puzzle and then provide a new way to explain it using our empirical results. The equity premium puzzle is a financial enigma and it raises great concern in the financial academic area. According to King (2013), "The equity premium puzzle refers to the phenomenon that observed returns on stocks over the past century are much higher than returns on government bonds. The equity premium, which is defined as equity returns minus bond returns, has been about 6% on average for the past century. The intuitive notion that stocks are much riskier than bonds is not a sufficient explanation of the observation that the magnitude of the disparity between the two returns." This is also verified by the Sharp ratios reported in Table 4.1. In Figure 5.4 and 5.6, the left-tail MinCVaR portfolios invest less in stocks and more in bonds. That may indicate, compared to the optimistic investors, the pessimistic investors feel the stocks are not as attractive as their risk-return trade-off indicates. In other words, even quantitatively the risk premium (stocks return minus risk-free rate) from stocks is much higher than that (bonds return minus risk-free rate), it is not as attractive as suggested by the risk premium difference. This could somehow explain the asset pricing puzzle: pessimistic investors may prefer less riskier portfolios and assuming they may face the worst case in the investment. To be specific, they invest more in bonds and less in stocks, though stocks has quantitatively high-risk premium.

5.2. Dynamic Optimization Experiment

We run the MV model for 3-year (benchmark), 3-year+3-month, 3-year+3-year time horizon. Then we compare these results with 3-year portfolio obtained by the left-tail MaxCVaR, right-tail MinCVaR, left-tail MinCVaR, and the mixed model that maximize left-tail CVaR and minimize right-tail CVaR approach ($\tau = 0.6$). We return to the simple three-asset case, *i.e.*, the bond class is represent by one index. We would like to capture more features of our models by comparing them in a non-recession period (Jan 2001 - Dec 2006) as well as a period transiting from stability to great recession (Jan 2004 - Dec 2009). Specifically, the 3-year benchmark period for the non-recession and transition cases are Jan 2001 – Dec 2003 and Jan 2004 – Dec 2006, respectively. Moreover, by adding a rolling window — 3 year + 3 month, 3 year + 3 year, we would like to evaluate the abilities of our models to detect long-term information.

5.2.1. Non-Recession Period

Figure 5.9 shows the mean-variance frontier of the three-asset portfolios from the 3-year MV, 3-year + 3-month MV, 3-year + 3-year MV, 3-year left-tail MaxCVaR, right-tail MinCVaR, 3-year left-tail MinCVaR, and the 3-year mixed models with weight $\tau = 0.6$. We notice that the MV results with 3-year + 3-month data almost coincide to the MV results with 3-year data (benchmark); while the MV results with 3-year + 3-year data deviate from the benchmark results. This makes sense as we do not expect the market has dramatically changed in one season while such changes are more likely to happen in a long horizon. The frontier of the 3-year + 3-year MV portfolios locate on the top left of the frontier of the 3-year MV portfolios as the market in the period of 2004 to 2006 is more favorable. We also notice the 3-year left-tail MaxCVaR, 3-year right-tail MinCVaR and the 3-year mixed model are very close to each other. Both of them locate between the 3-year MV frontier and left-tail MinCVaR frontier. This is also reasonable because these two models are based on investing strategy that are rational but kind of pessimistic and moderate.

The plot in Figure 5.10 illustrates the mean-CVaR$_{5\%}$ frontier of the three-asset portfolios by the 3-year MV, 3-year + 3-month MV, 3-year + 3-year MV, 3-year left-tail MaxCVaR, 3-year right-tail MinCVaR, 3-year left-tail MinCVaR, and the 3-year mixed model. The 3-year left-tail MinCVaR has the lowest CVaR$_{5\%}$; while the 3-year + 3-year MV has a frontier on the top left. The 3-year mixed models and the 3-year left-tail MaxCVaR is almost coincide. They are close but a little higher than the frontier of the 3-year

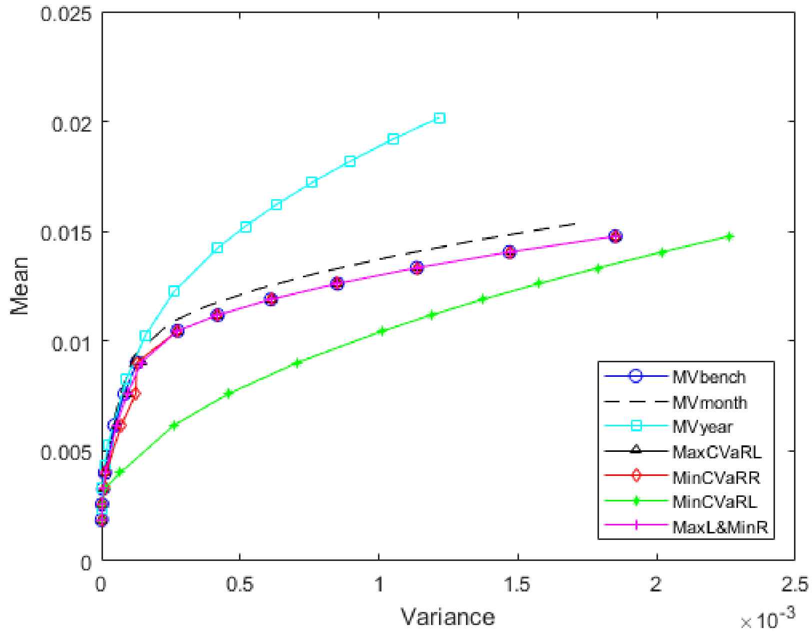

Figure 5.9. Mean-variance frontier of the three-asset portfolios at a non-recession period by the 3-year MV (MVbench), 3-year + 3-month MV (MVmonth), 3-year + 3-year MV (MVyear), 3-year left-tail MaxCVaR (MaxCVaRL), 3-year right-tail MinCVaR (MinCVaRR), 3-year left-tail MinCVaR (MinCVaRL), and the 3-year mixed models (MaxL&MinR). In the graph, the expected mean of return is on the vertical axis and variance is on the horizontal axis.

MV. Most points in the 3-year right-tail MinCVaR are close to the 3-year MV, but two points drop down and make the frontier a little twisted. The results are highly consistent with the mean-variance frontier in Figure 5.9 and the nature of our model design in which the left-tail MinCVaR has the lowest $\text{CVaR}_{5\%}$[4].

Figure 5.11 describes the distributions of three-asset portfolios by the 3-year MV, 3-year + 3-month MV, 3-year + 3-year MV, 3-year left-tail MaxCVaR, 3-year right-tail MinCVaR, 3-year left-tail MinCVaR, and the 3-year mixed models at point #2 on the frontier for each individual case. In the figure, the MV results given 3-year + 3-month data coincide with the MV results given 3-year data (benchmark); while the one given 3-year + 3-year data is more spread out with fatter right tail and left tail, which indicates it is riskier and more profitable. The distribution of the 3-year mixed portfolio coincides with that of the 3-year left-tail MaxCVaR portfolio. They shift a little to the left, which is counter-intuitive. The distributions of the 3-year right-tail MinCVaR and the 3-year left-tail MinCVaR also show a little spread out with a fatter left tail.

[4] We don't repeat the figure of $\text{CVaR}_{95\%}$ because the right-tail CVaR of 3-year + 3-month MV is not in the same scale as those of other models.

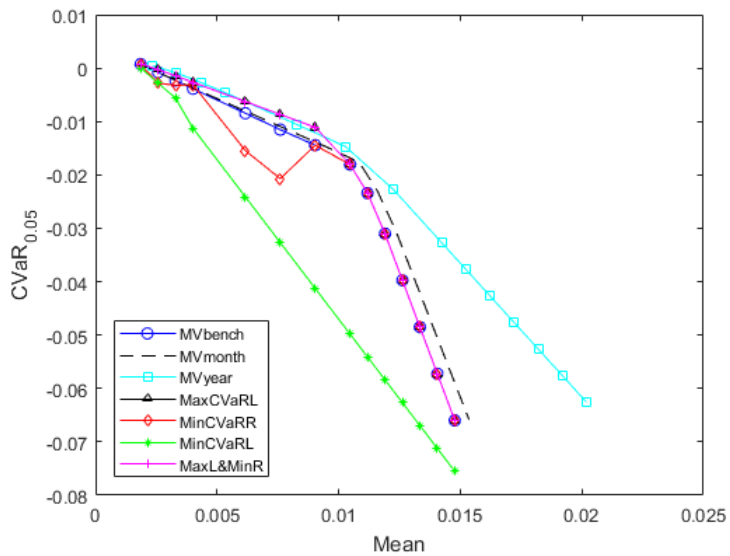

Figure 5.10. Mean-CVaR$_{5\%}$ frontier of the three-asset portfolios at the non-recession period by the 3-year MV (MVbench), 3-year + 3-month MV (MVmonth), 3-year + 3-year MV (MVyear), 3-year left-tail Max-CVaR (MaxCVaRL), 3-year right-tail MinCVaR (MinCVaRR), 3-year left-tail MinCVaR (MinCVaRL), and the 3-year mixed models (MaxL&MinR). In the graph, the expected mean of return is on the horizontal axis and CVaR$_{5\%}$ is on the vertical axis.

Figure 5.12 shows the asset allocation strategies of the 3-year MV, 3-year + 3-month MV, 3-year + 3-year MV, 3-year left-tail MaxCVaR, 3-year right-tail MinCVaR, 3-year left-tail MinCVaR, and the 3-year mixed models. All portfolios have a similar trend with minor difference except the 3-year left-tail MinC-VaR. The portfolios obtained by the 3-year left-tail MinCVaR allocates more in stocks and T-Bill. This is inconsistent to the results in Section 5.1 where the left-tail MinCVaR model invests more in bonds but less in stocks.

5.2.2. Recession Period

In this section, we investigate a time period that crosses a recession with dramatic investment environmental changes. Specifically, we consider a period from Jan 2004 to Dec 2009. The first three years are a period of steady economy; while the US economy experienced the Great Recession during Dec 2007 and June 2009. Setting the period of Jan 2004 to Dec 2006 as the 3-year benchmark period, we analyze the 3-year MV, 3-year + 3-month MV, 3-year + 3-year MV, 3-year left-tail MaxCVaR, 3-year right-tail MinCVaR, 3-year left-tail MinCVaR, and the 3-year mixed models ($\tau = 0.6$).

Figure 5.13 shows the mean-variance frontier of the three-asset portfolios from the 3-year MV, 3-year + 3-month MV, 3-year + 3-year MV, and 3-year left-tail MaxCVaR, 3-year right-tail MinCVaR,

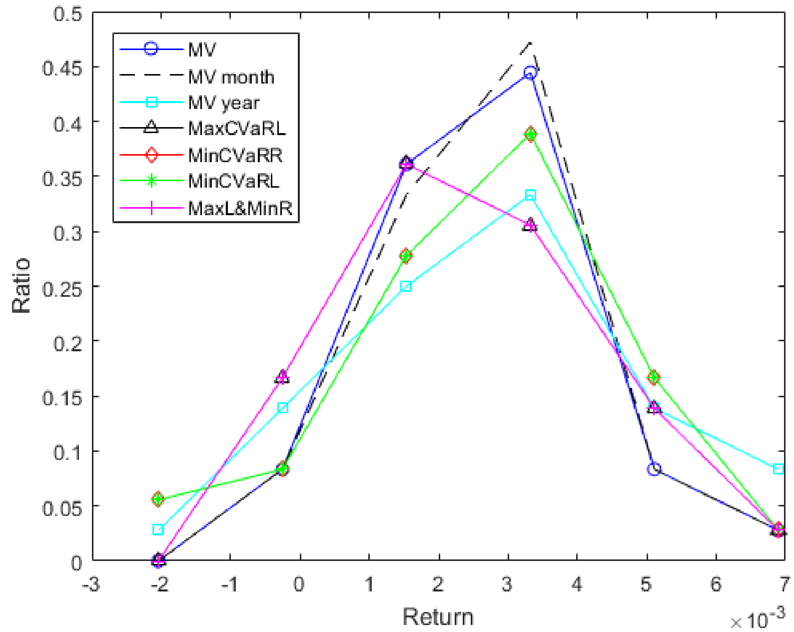

Figure 5.11. Distributions of three-asset portfolios for a non-recession period at point two from the 3-year MV (MVbench), 3-year + 3-month MV (MVmonth), 3-year + 3-year MV (MVyear), 3-year left-tail Max-CVaR (MaxCVaRL), 3-year right-tail MinCVaR (MinCVaRR), 3-year left-tail MinCVaR (MinCVaRL), and the 3-year mixed models (MaxL&MinR). Each curve represents the ratio of observations over total number of observations at its corresponding return.

3-year left-tail MinCVaR, and the 3-year mixed model. Similar as the non-recession case, MV results with 3-year + 3-month data coincide with the MV results with 3-year data (benchmark); while the MV results with 3-year + 3-year data deviate from the benchmark results. However, it deviates a lot towards the opposite direction compared with the non-recession period, which exactly meets our expectation because the second 3-year period is in the recession which dramatically deteriorates the overall performance of the MV portfolios. Moreover, we notice under the recession period, the frontier of 3-year + 3-year MV is relatively close to 3-year left-tail MinCVaR, which confirms our prior expectation that the left-tail MinCVaR model carries long-term information.

Figure 5.14 illustrates mean-$CVaR_{5\%}$ frontier of the three-asset portfolios obtained from the 3-year MV, 3-year + 3-month MV, 3-year + 3-year MV, 3-year left-tail MaxCVaR, 3-year right-tail MinCVaR, 3-year left-tail MinCVaR, and the 3-year mixed models. The results are highly consistent with the results in Figure 5.13. The 3-year+3-year MV has quite low left-tail CVaR, which indicates the recession dramatically decreases the lowest 5% average return. At the same time we find that its mean-$CVaR_{5\%}$ frontier is close to that of the 3-year left-tail MinCVaR. Taking one step further, the left-tail MinCVaR model could somehow

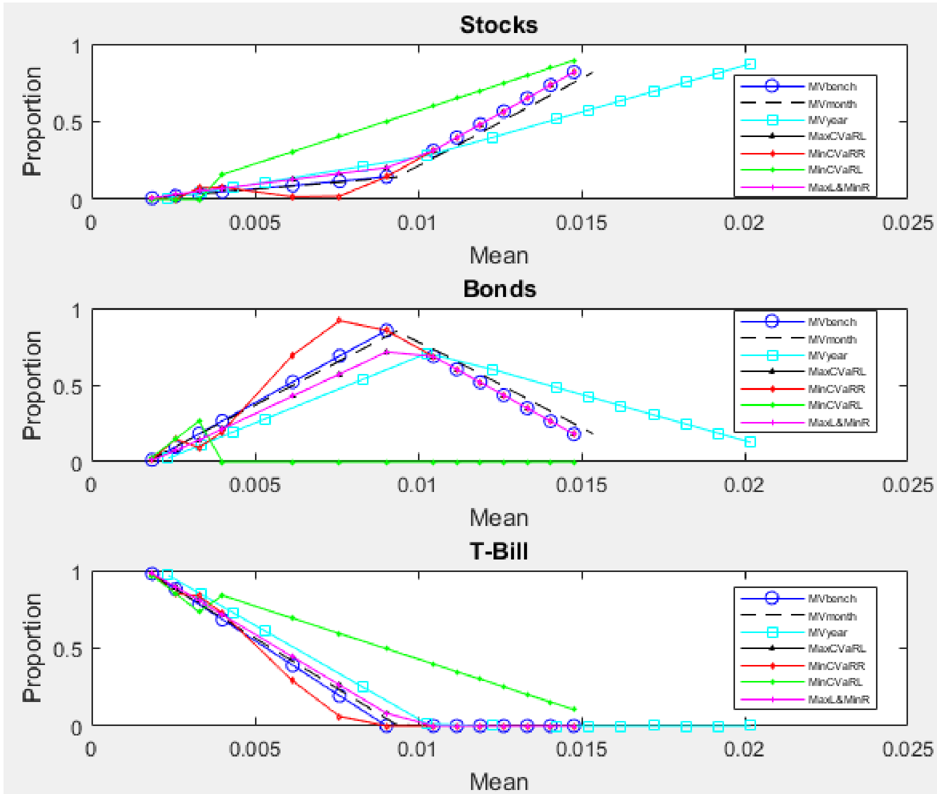

Figure 5.12. Asset allocation of three-asset portfolios a non-recession period by the 3-year MV (MVbench), 3-year + 3-month MV (MVmonth), 3-year + 3-year MV (MVyear), 3-year left-tail MaxCVaR (MaxCVaRL), 3-year right-tail MinCVaR (MinCVaRR), 3-year left-tail MinCVaR (MinCVaRL), and the 3-year mixed models (MaxL&MinR).

illustrate the long-term investment strategy during a worst time. I suspect that the differences between the two depend heavily on the severity of the recession.

Figure 5.15 describes the distributions of three-asset portfolios by the 3-year MV, 3-year + 3-month MV, 3-year + 3-year MV, 3-year left-tail MaxCVaR, 3-year right-tail MinCVaR, 3-year left-tail MinCVaR, and the 3-year mixed models at point #2 on the frontier for each individual case. In the plot, the MV results with 3-year + 3-month data, MV results with 3-year + 3-year, and the 3-year left-tail MaxCVaR coincide to the MV results with the 3-year MV data (benchmark). The distributions of the 3-year right-tail MinCVaR and the 3-year mixed model both shift a little to the right. The 3-year left-tail MinCVaR has a very spread out distribution with fat left tail and right tail, which corresponds to its design that maximizes the worst case measured by the left-tail CVaR.

Figure 5.16 shows asset allocation strategies of the 3-year MV, 3-year+3-month MV, 3-year+3-year MV, and 3-year left-tail MaxCVaR, 3-year right-tail MinCVaR, 3-year left-tail MinCVaR, and the 3-year

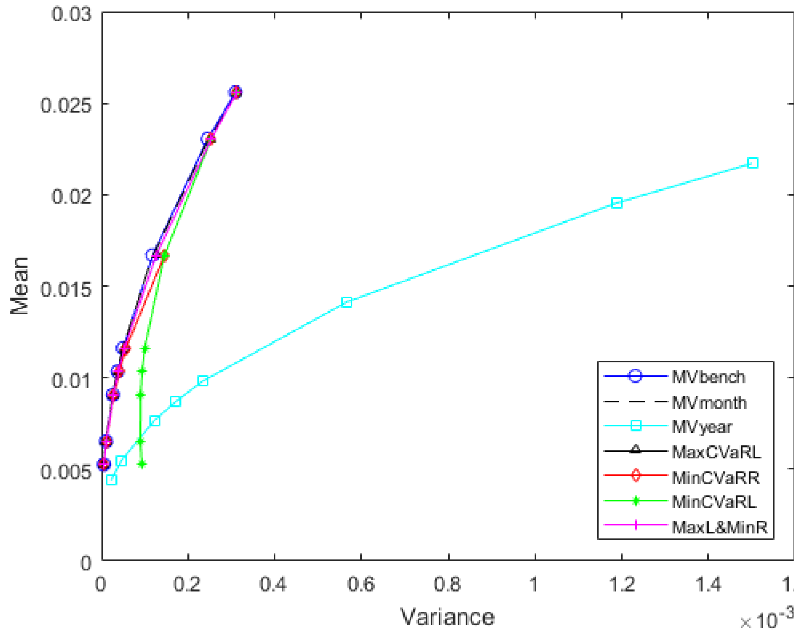

Figure 5.13. The mean-variance frontier of three-asset portfolios cross a recession period by the 3-year MV (MVbench), 3-year + 3-month MV (MVmonth), 3-year + 3-year MV (MVyear), 3-year left-tail MaxCVaR (MaxCVaRL), 3-year right-tail MinCVaR (MinCVaRR), 3-year left-tail MinCVaR (MinCVaRL), and the 3-year mixed model (MaxL&MinR). In the graph, the expected mean of return is on the vertical axis and variance is on the horizontal axis.

mixed models. All the MV portfolios have similar trends with minor differences. As the mean increases, the 3-year MV, the 3-year + 3-month MV, and the 3-year + 3-year MV models invest more in stocks, less in T-Bill and no bonds. While the portfolio from the 3-year left-tail MinCVaR allocates more in bonds. During the recession, stocks are mostly influenced among these three assets. Therefore, investing less in stocks could decrease the loss to some extent.

Table 5.1. Turnover Rate

Model	Turnover
MV	0.0927
MaxCVaR(L)	0.2207
MinCVaR(R)	0.0816
MinCVaR(L)	0.4514
MaxCVaR(L)&MinCVaR(R)	0.1099

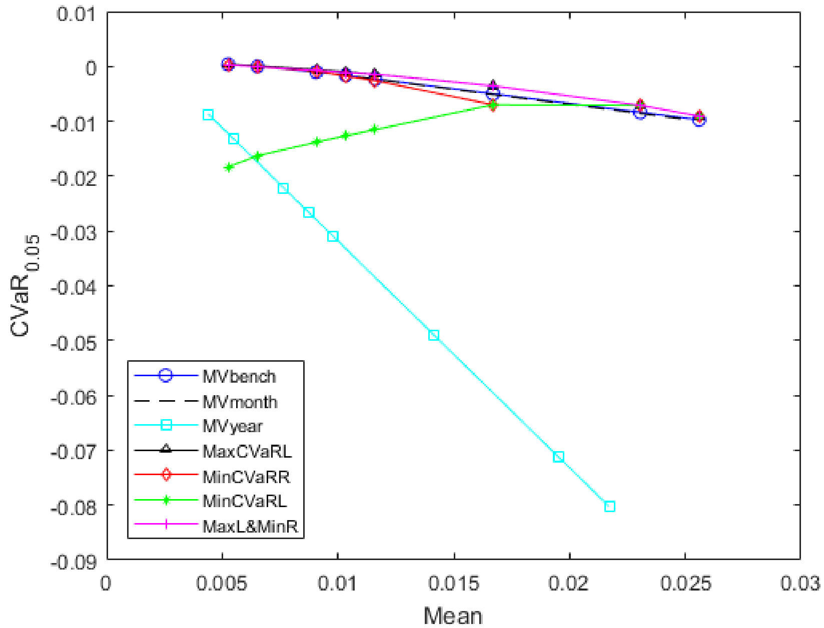

Figure 5.14. The mean-CVaR$_{5\%}$ frontier of the three-asset portfolios cross a recession period by the 3-year MV (MVbench), 3-year + 3-month MV (MVmonth), 3-year + 3-year MV (MVyear), 3-year left-tail MaxCVaR (MaxCVaRL), 3-year right-tail MinCVaR (MinCVaRR), 3-year left-tail MinCVaR (MinCVaRL), and the 3-year mixed models(MaxL&MinR). In the graph, the expected mean of return is on the horizontal axis and expected CVaR$_{5\%}$ is on the vertical axis.

Next, we analyze the turnover rate which the total value of assets traded during a specific period of time.

$$\boldsymbol{T} = \frac{1}{2}\sum_{i=1}^{n} |T_i^t| \quad \text{where } T_i^t = x_i^t - x_i^{t-1}. \tag{5.1}$$

Table 5.1 illustrates turnover rate for each model (t_{i-1} and t_i are from Jan 2000 to Dec 2006, Jan 2000 to Dec 2010, respectively). Generally, turnover measures the total proportion of assets traded during a period of time. The higher turnover is, the more adjustments are applied to the asset allocation. The formula is given as follows:

In (5.1), T_i^t is the differences of the proportion invested in asset i between period t and $t-1$. $\boldsymbol{T}$ is the total turnover that measures the absolute difference over all asset. Since buying and selling assets will make duplicated counts, the total turnover is obtained using the sum divide by two.

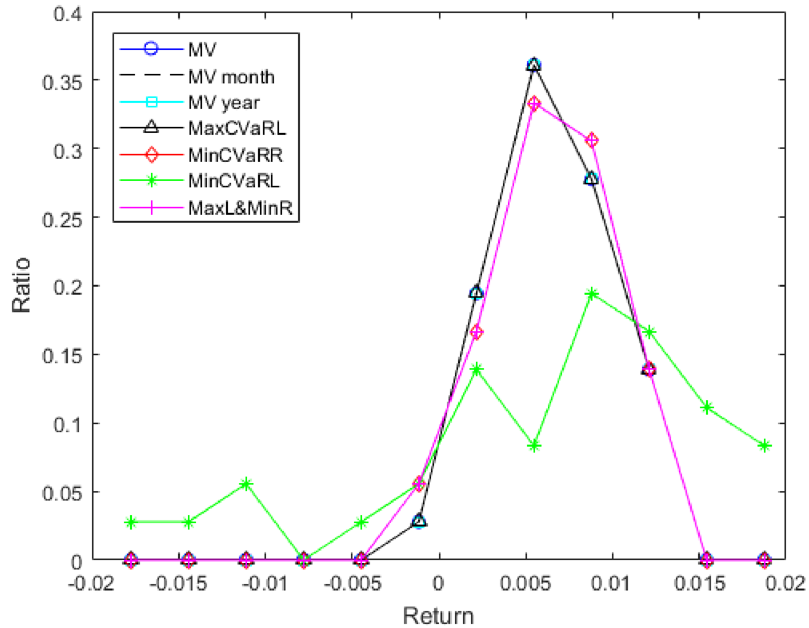

Figure 5.15. Distribution of three-asset portfolios cross a recession period at point two with monthly data ranging from Jan 2004 to Dec 2009 from the 3-year MV (MVbench), 3-year + 3-month MV (MVmonth), 3-year + 3-year MV (MVyear), 3-year left-tail MaxCVaR (MaxCVaRL), 3-year right-tail MinCVaR (MinCVaRR), 3-year left-tail MinCVaR (MinCVaRL), and the 3-year mixed models(MaxL&MinR). Each curve represents the ratio of observations over total number of observations at its corresponding return.

We notice that left-tail MinCVaR has the highest turnover rate since it may detect economic recession earlier and adjust the investment strategy accordingly. We also find the left-tail MaxCVaR has the second highest turnover rate. Since the left-tail CVaR is a model intends to make portfolio rationally, it will try to decrease loss for the worst cases.

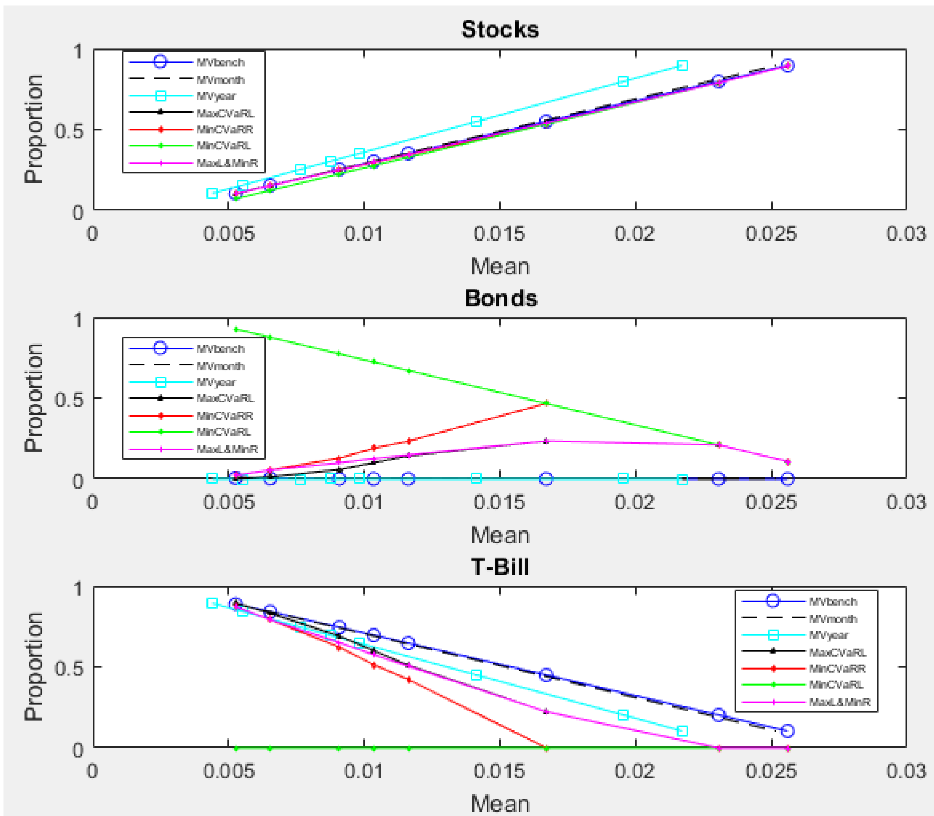

Figure 5.16. Asset allocation of three-asset portfolios cross a recession period by the 3-year MV (MVbench), 3-year+3-month MV (MVmonth), 3-year+3-year MV (MVyear), 3-year left-tail MaxCVaR (MaxCVaRL), 3-year right-tail MinCVaR (MinCVaRR), 3-year left-tail MinCVaR (MinCVaRL), and the 3-year mixed models(MaxL&MinR).

6. SIMULATION STUDY

Though historical data provide enriched information about market changes, we are also interested in the performance of our models in a simulated world. Specifically, we run Monte Carlo simulation to project the future evolution of asset returns based on the calibrated parameter and analyze the portfolio selection strategies based on the simulated data.

6.1. Distribution Assumption and Parameter Estimation

We assume the return of each asset follows geometric Brownian motion (Ross (2014)) following a stochastic differential equation:

$$dS_t = \mu S_t dt + \sigma S_t dW_t, \tag{6.1}$$

where S_t is the asset price at time period t where $0 \leq t \leq T$. W_t is the Wiener process that follows $W_t \sim N(0, \sqrt{t})$. For an arbitrary initial price S_0, one can obtain the asset price S_t from:

$$S_t = S_0 exp((\mu - \frac{1}{2}\sigma^2)t + \sigma W_t). \tag{6.2}$$

To be consistent with our previous study, we transfer the asset price to log return:

$$r_t = lnS_t - lnS_{t-1}. \tag{6.3}$$

The return $r_t = [r_{1t}, r_{2t}, \ldots, r_{nt}]$ follows a multivariate normal distribution with mean μ and variance $\sigma^2 = x'\sigma_{n\times n}x$ where $x_{n\times 1}$ is the portfolio decision vector. Let $\theta = (\mu, \sigma)$, the probability density function are given below:

$$f_\theta(r_t) = \frac{1}{S_t\sigma\sqrt{2\pi t}} exp(-\frac{(\frac{S_t}{S_0} - (\mu - \frac{1}{2}\sigma^2)t)^2}{2\sigma^2 t}). \tag{6.4}$$

Therefore, the likelihood function can be denoted as:

$$L(\theta) = f_\theta(r_1, r_2, \ldots, r_m) = \prod_{t=1}^{m} f_\theta(r_t) = \prod_{t=1}^{m} f(r_t|\theta). \tag{6.5}$$

The method of maximum likelihood find the optimal $\theta(\hat{\mu}, \hat{\sigma})$ by differentiating the likelihood function and making it equal to zero. According to Sypkens (2010), $\hat{\mu}$ and $\hat{\sigma}$ can be solved as follows:

$$\begin{aligned}
\hat{\sigma} &= \frac{\hat{v}}{\Delta t} \\
\hat{\mu} &= \frac{1}{2}\hat{\sigma} + \frac{\hat{u}}{\Delta t}. \\
\text{where } \hat{u} &= \sum_{t=1}^{m} \frac{r_t}{m} \\
\hat{v} &= \sum_{t=1}^{m} \frac{(r_t - \hat{u})^2}{m}.
\end{aligned} \tag{6.6}$$

We use the estimated $\hat{\mu}$ and $\hat{\sigma}$ for the simulation. In the process, we first generate correlated standard normal distribution according to the correlation coefficient matrix of historical data. Then we transfer the correlated Geometric Brownian motions, based on the estimated parameters $\hat{\mu}$ and $\hat{\sigma}$ obtained by maximum likelihood method.

Based on the approach above, we generate 10-year monthly data for each asset as one set of experiment, and repeat the experiments for 1,000 times with total 12000 simulated data. By applying the MV, left-tail MaxCVaR, right-tail MinCVaR, left-tail MinCVaR, and the mixed model (τ=0.6), we obtain the optimal portfolios for each model and take the average of the 1,000 experiments. Technically, since the left-tail MinCVaR takes much longer time complete one trial of optimization, we generate 10-year monthly data based on the average of 10 experiments.

6.2. Optimization Based on Simulation

As described in Section 6.1, we run simulation to project the future evolution of asset returns based on the calibrated parameters. In this section we analyze the portfolio selection strategies based on the simulated data. Specifically, we simulate 12,000 data for three asset classes, *i.e.*., stocks, bonds, and a risk-free asset. Again we run the Markowitz (MV), left-tail MaxCVaR, right-tail MinCVaR, left-tail MinCVaR, and a mixed model that maximizes left-tail CVaR and minimizes right-tail CVaR at the same time with weight $\tau = 0.6$.

The plot in Figure 6.1 illustrates the mean-variance frontiers of the simulated data obtained from the MV, left-tail MaxCVaR, right-tail MinCVaR, left-tail MinCVaR, and the mixed model. As shown in the graph, the mean-variance frontier for all the models almost coincide because our data is very neatly simulated. But we still can see departure of left-tail MinCVaR, especially when the mean of return is low.

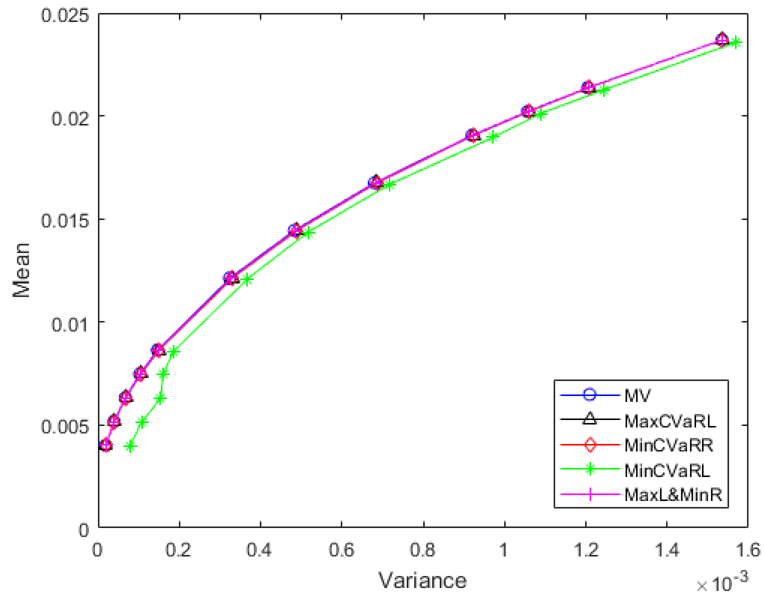

Figure 6.1. Mean-variance frontier of the simulated data from the Markowitz (MV), left-tail MaxCVaR, right-tail MinCVaR, left-tail MinCVaR, and the mixed model respectively. In the graph, the expected mean of return is on the vertical axis and the variance is on the horizontal axis.

This is consistent with the results we got in Section 5.1, which verify our thinking that left-tail MinCVaR is more sensitive when the mean of return is low. We also confirm that the left-tail MaxCVaR, the right-tail MinCVaR and the mixed model, though representing pessimistic investment strategies, are still rational and show close performance to the classical MV model which pursuing the optimal portfolio.

The left and right plots in Figure 6.2 illustrates $\text{CVaR}_{5\%}$ frontier and mean-$\text{CVaR}_{95\%}$ frontier of the simulated data for three-asset portfolios obtained from the MV, left-tail MaxCVaR, right-tail MinCVaR, left-tail MinCVaR, and the mixed model respectively. The left and right CVaR plots are also very close to each other for all the models.

Figure 6.3 describes the distribution of three-asset portfolios by the MV, left-tail MaxCVaR, right-tail MinCVaR, left-tail MinCVaR, and the mixed model respectively at point two with simulated monthly data. Consistent with our previous results, the distribution for the MV, left-tail MaxCVaR, right-tail MinCVaR, and the mixed model appear to be very close to each other. The left-tail MinCVaR shows an obvious spread out distribution where it distributes more on the two side compared with the distributions of the other models. This indicates the left-tail CVaR model tend to treat the whole portfolio distribution as a riskier one.

Figure 6.4 shows the asset allocation of three-asset portfolios by the MV, left-tail MaxCVaR, right-tail MinCVaR, left-tail MinCVaR, and the mixed model respectively with the simulated monthly data. The

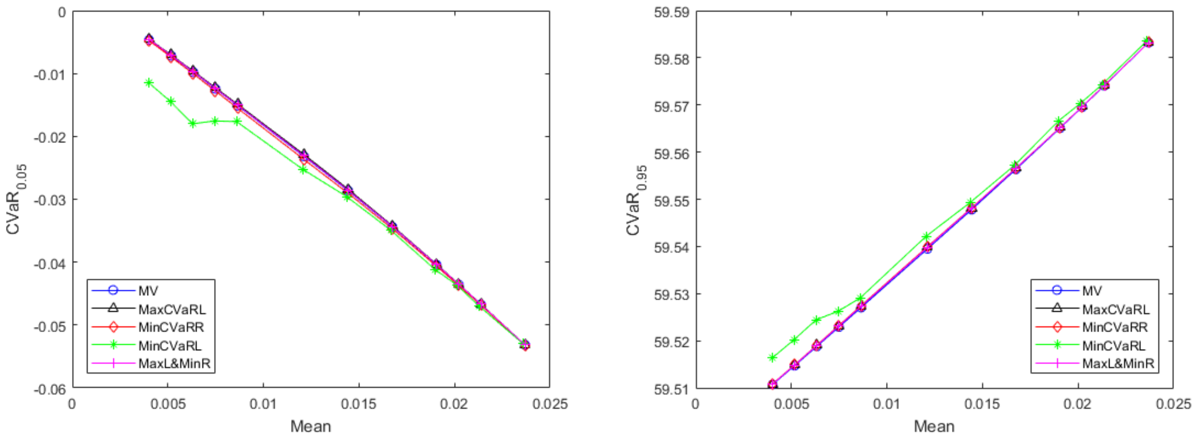

Figure 6.2. Mean-CVaR$_{5\%}$ frontier and mean-CVaR$_{95\%}$ frontier of simulated data for the three-asset portfolios from the Markowitz (MV), left-tail MaxCVaR, right-tail MinCVaR, left-tail MinCVaR and the mixed model respectively. In the left graph, the expected mean of return is on the horizontal axis and CVaR$_{5\%}$ is on the vertical axis. The right graph shows the mean on the horizontal axis and CVaR$_{95\%}$ on the vertical axis.

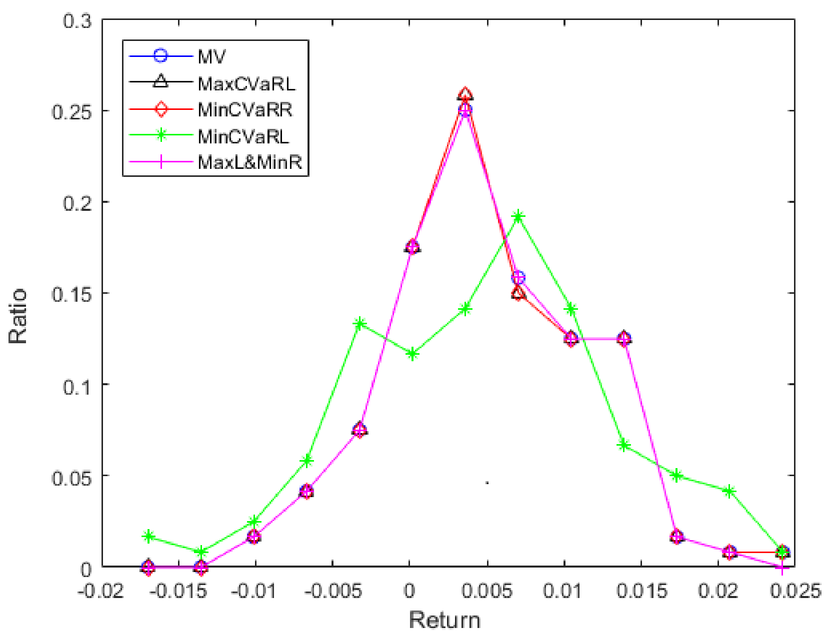

Figure 6.3. Distribution of three-asset portfolios with simulated data from the Markowitz (MV), left-tail MaxCVaR, right-tail MinCVaR, left-tail MinCVaR and the mixed model respectively. Each curve represent the ratio of observations at a given return over total number of observations.

results for investment allocation of the MV, left-tail MaxCVaR, right-tail MinCVaR, and the mixed model are also highly consistent. As mean increase, the investments in stocks keeps increasing, the investment in T-Bill keeps decreasing, while the investment in bonds increases first and then decreases again. This

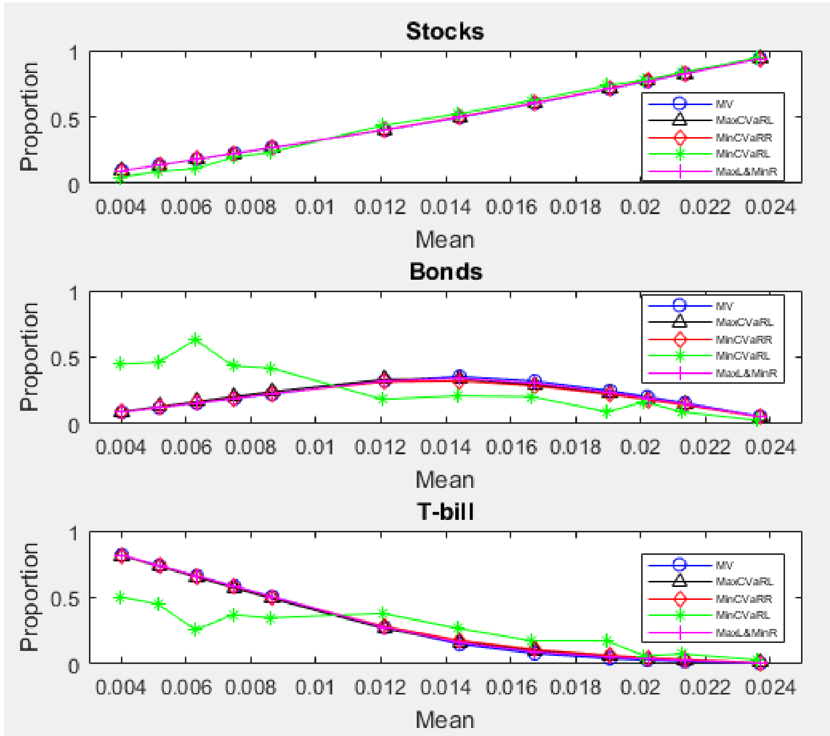

Figure 6.4. Asset allocation of three-asset portfolios with simulated data from the the Markowitz (MV), left-tail MaxCVaR, right-tail MinCVaR, left-tail MinCVaR, and the mixed model respectively.

allocation trend is quite reasonable because as a higher portfolio is targeted, one needs to choose to invest more in high-return assets such as stocks, and the proportions invested in the other two asset classes decrease correspondingly. Compared with other models, the left-tail MinCVaR invests more in bonds and less in T-Bill, especially when mean is low.

Since there is no jump in the simulated model, the results are very neat without considering economic shocks or crisis. We would like to add some jumps into the model with compounded jump-diffusion models in the future study.

7. CONCLUSION

This paper studies the investors' investment behaviors by modeling their investing personalities through managing tail risk. Taking traditional and nontraditional investors into consideration, we build four models to represent four different investment behaviors. Specifically, by extending the linear techniques of Rockafellar and Uryasev (2000), we optimize the portfolios linearly from left-tail MaxCVaR, right-tail MinCVaR, and the mixed model considering both left-tail MaxCVaR and right-tail MinCVaR, respectively. Besides, we solve a minimax problem to achieve the portfolio optimization for the left-tail MinCVaR model. In result analysis part, we compare results of those models with the classical Markowitz approach. We find that the models designed for rational but conservative investors (left-tail MaxCVaR, right-tail MinCVaR and the mixed model) show return-risk trade-off lines close to the one of MV, sometimes even better than that of MV during the economic turmoil. On the other hand, the results from the model designs for irrational investors (left-tail MinCVaR model) are significantly different from those of the other models. As expected, it shows inferior mean-variance trade-off line and lower mean-CVaR frontier, especially when the target return is low. Moreover, we illustrate the equity premium puzzle and long-run risk based on asset allocations analysis. Finally, we verify our results by simulating 10-year monthly data from Monte Carlo simulation.

REFERENCES

Acerbi, C. and P. Simonetti (2002). Portfolio optimization with spectral measures of risk. Working paper, Abaxbank, Italy.

Alexander, G. J. and A. M. Baptista (2002). Economic implications of using a mean-var model for portfolio selection: A comparison with mean-variance analysis. *Journal of Economic Dynamics & Control 26*(7), 1159–1193.

Artzner, P., F. Delbaen, J.-M. Eber, and D. Heath (1996). A characterization of measures of risk. Technical Report NO. 1186.

Bassett, G., R. Koenker, and G. Kordas (2004). Pessimistic portfolio allocation and choquet expected utility. *Journal of Financial Economitrics 2*(4), 477–492.

Bidder, R. and I. Dew-Becker (2016). Long-run risk is the worst-case scenario. *American economic Review 106*(9), 2494–2527.

Charles, A. (2014). Do investors emotions determine their investment personality? *Journal of Management 10*(2), 45–60.

Consigli, G. (2002). Tail estimation and mean-var portfolio selection in markets subject to financial instability. *Journal of Banking & Finance 26*(7), 1355–1382.

Coval, J. D. and A. V. Thakor (2004). Financial intermediation as a beliefs-bridge between optimistis and pessimists. *Journal of Financial Economics 75*(2005), 535–569.

Gilboa, I. and D. Schmeidler (1989). Maxmin expected utility with non-unique prior. *Journal of Mathematical Economics 18*, 141–153.

Hansen, L. P. and T. J. Sargent (2007). *Robustness*. Princeton, NJ: Princeton University Press.

Jorion, P. (2007). *Value at Risk–The New Benchmark for Managing Financial Risk*. New York: McGraw-Hill.

J.P.Morgan and Reuters (2006). *RiskMetrics*^{TM}. Morgan Guaranty Trust Company of New York.

King, N. A. S. (2013). Equity premium puzzle, is it still a puzzle? evidence from tanaznia. *International Journal of marketing, financial services and management research 2*(8), 111–119.

Krokhmal, P., J. Palmquist, and S. Uryasev (2002). Portfolio optimization with conditional value-at-risk objective and constraints. *Journal of risk 4*(2), 43–68.

Markowitz, H. M. (1952). Portfolio selection. *The Journey of Finance 7*(1), 77–91.

Rockafellar, T. and S. Uryasev (2000). Optimization of conditional value-at-risk. *Journal of risk 2*(3), 21–42.

Rockafellar, T. and S. Uryasev (2002). Conditional value-at-risk for general loss distributions. *Journal of Banking & Finance 26*(7), 1443–1471.

Ross, S. M. (2014). *Introduction To Probability Models*. Elsevier Inc.

Sharpe, W. F. (1966). Mutual fund performance. *The Journal of Business 39*(1), 199–138.

Sypkens, R. (2010, September). *Risk Properties and Parameter Estimation on Mean Reversion and Garch Models*. Ph. D. thesis, University of South Africa, Pretoria, South Africa.

Tian, R., S. H. Cox, Y. Lin, and L. F. Zuluaga (2010). Portfolio risk ranagement with cvar-like constraints. *North American Actuarial Journal 14*(1), 86–106.

Zhu, S. and M. Fukushima (2009). Worst-case conditional value-at-risk with application to robust portfolio management. *Operations Research 57*(5), 1155–1168.