MULTIDIMENSIONAL TOGGLE DYNAMICS

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DOCTOR OF PHILOSOPHY

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ABSTRACT

J. Propp and T. Roby isolated a phenomenon in which a statistic on a set has the same average value over any orbit as its global average, naming it homomesy. One set they investigated was order ideals of partially ordered sets (posets). They proved that the cardinality statistic on order ideals of the product of two chains poset under rowmotion or promotion exhibits homomesy. We prove an analogous result in the case of the product of three chains where one chain has two elements. In order to prove this result, we generalize from two to n dimensions the recombination technique that D. Einstein and Propp developed to study homomesy. We see that our main homomesy result does not fully generalize to an arbitrary product of three chains, nor to larger products of chains; however, we have a partial generalization to an arbitrary product of three chains. Additional corollaries include refined homomesy results in the product of three chains and a new result on increasing tableaux. We also generalize recombination to any ranked poset and from this, obtain a homomesy result for a type B minuscule poset cross a two-element chain. We conclude by extending the definition of promotion to infinite posets, exploring homomesy, recombination, and a connection to monomial ideals.

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1. INTRODUCTION

Homomesy is a surprisingly ubiquitous phenomenon, isolated by J. Propp and T. Roby [30], that occurs when a statistic on a combinatorial set has the same average value over orbits of that action as its global average. Homomesy has been found in actions on tableaux [4, 30], actions on binary strings [31], rotations on permutation matrices [31], toggles on noncrossing partitions [15], Suter's action on Young diagrams [30] (with proof due to D. Einstein), linear maps acting on vector spaces [30], a phase-shift action on simple harmonic motion [30], and others. A motivating instance of this phenomenon is the action of *rowmotion* on order ideals of a partially ordered set, or poset. Rowmotion on an order ideal is defined as the order ideal generated by the minimal poset elements that are not in the order ideal; this action has generated significant interest in recent algebraic combinatorics, giving rise to many beautiful results [5, 6, 8, 9, 11, 12, 13, 17, 18, 20, 21, 22, 24, 26, 30, 32, 33, 34, 42, 44]. For a survey of recent homomesy results, see [31]; for an introduction to dynamical algebraic combinatorics, including rowmotion, see [43]. Our initial motivation for this dissertation was Propp and Roby's result that the cardinality statistic on order ideals of the product of two chains poset $[a] \times [b]$ under rowmotion exhibits homomesy [30]. D. Rush and K. Wang generalized this result by showing all *minuscule posets* exhibit homomesy under rowmotion using the cardinality statistic [34]; the product of two chains poset is the type A case of this result.

We investigate homomesy in the product of three chains, or equivalently, a type A minuscule poset cross a chain. More specifically, we show order ideals of $[2] \times [a] \times [b]$ exhibit homomesy under rowmotion with cardinality statistic. However, we observe such a homomesy result does not hold for a general product of three chains. We also obtain a homomesy result on order ideals of a type B minuscule poset cross a chain of size two. To prove these results, we generalize the *recombination* technique of Einstein and Propp [17] from two to n dimensions. Recombination is a tool that Einstein and Propp developed to translate homomesy results between rowmotion and a related action called *promotion* by J. Striker and N. Williams in [44]. Striker and Williams showed that there is an equivariant bijection between order ideals of any ranked poset under promotion and rowmotion. This means that the orbit structure is the same under rowmotion and promotion, so if we want to study the orbits of rowmotion, we could instead study the orbits of promotion, or vice versa. K. Dilks, O. Pechenik, and Striker [11] generalized promotion to higher dimensions. Furthermore, they showed that for a given poset, there is an equivariant bijection between any of the multidimensional promotions they defined. Underlying all these results is the *toggle group* of P. Cameron and D. Fon-der-Flaass [8], who provided access to the tools of group theory by exhibiting rowmotion as a toggle group action.

Our first main theorem, Theorem 2.0.1, says that the order ideals of a product of three chains where one chain is of size two exhibits homomesy with average value ab under promotion when using the order ideal cardinality statistic. To prove this theorem, we generalize recombination to n dimensions in our second main theorem, Theorem 2.2.4, then translate a homomesy result on increasing tableaux to the product of chains setting. We also prove the following additional results. In Propositions 2.4.1 and 2.4.2, we show that our homomesy result does not generalize to arbitrary products of three chains, nor to a product of n chains where all chains are of size two. Although our result does not generalize fully to products of three chains, using Pechenik's result on frames of increasing tableaux [28], in Corollary 3.2.4, we partially generalize a homomesy result on a product of three chains where we consider the "outside" of the poset. Additionally, Corollaries 3.2.1 and 3.2.4 include refinements of our main homomesy result and this partial generalization, respectively. Theorem 3.3.4 shows an additional refinement of our homomesy result, generalizing a result of Propp and Roby. In Corollary 3.1.1 we also use our main result to show a new homomesy result on increasing tableaux under K-promotion. In Theorem 4.1.9, we generalize the recombination result of Theorem 2.2.4 from a product of chains to any ranked poset. We use this for Corollary 4.2.1, a homomesy result on the type B minuscule poset cross a two-element chain. Theorem 4.1.5 explicitly states the bijection between different n-dimensional promotions by presenting a conjugating toggle group element. With Definition 5.1.2, we generalize the toggle definition of promotion to infinite posets. With Lemma 5.2.7, we generalize a result of Striker and Williams from a two-dimensional product of finite chains to $\mathbb{N}^2$. In Theorems 5.2.9 and 5.2.10, we investigate how applying promotion or rowmotion to an order ideal of $\mathbb{N}^2$ affects the number of generators of a corresponding monomial ideal. From this, we obtain a homomesy result on order ideals of $\mathbb{N}^2$ in Theorem 5.3.2. Finally, we generalize recombination to order ideals of $\mathbb{N}^n$ in Theorem 5.3.4.

In Chapter 1, we begin with introductory definitions and background material, with Section 1.1 covering posets and Young tableaux, Section 1.2 introducing promotion and rowmotion, and

Section 1.3 introducing homomesy. In Chapter 2, we present our main homomesy result. To do this, we begin with the two-dimensional recombination technique of Einstein and Propp [17] in Section 2.1, generalize recombination to n-dimensions in Section 2.2, prove the main homomesy result in Section 2.3, and investigate homomesy in a general product of three chains in Section 2.4. In Chapter 3, we present several corollaries related to our results from Chapter 2. In Section 3.1, we obtain a new homomesy on increasing tableaux. In Section 3.2, we obtain refined homomesy results on columns in the product of three chains, whereas in Section 3.3, we obtain a refined homomesy results on antipodal elements. In Chapter 4, we generalize recombination. More specifically, in Section 4.1, we generalize recombination to any ranked poset. We then obtain a corollary involving the type B minuscule poset in Section 4.2. In Chapter 5, we investigate toggle group actions on infinite posets. In Section 5.1, we introduce a definition for promotion on an infinite poset. In Section 5.2, we define the boundary path for an order ideal of $\mathbb{N}^2$, along with connecting toggle actions to monomial ideals. In Section 5.3, we explore homomesy on order ideals of $\mathbb{N}^2$ and recombination on order ideals of $\mathbb{N}^n$. Lastly, in Chapter 6, we present future avenues of research.

1.1. Partially ordered sets and Young tableaux

In this section, we give some background on *partially ordered sets* (*posets*) and *Young tableaux*. For a more thorough background on posets, see Chapter 3 of [37]. For further background on Young tableaux, see [19]. We begin with posets.

Definition 1.1.1. A *poset* P is a set with a binary relation, denoted $\leq$, that is reflexive, weakly antisymmetric, and transitive. In other words, if a, b and c are elements of P, then:

1. $a \leq a$.

2. If $a \leq b$ and $b \leq a$, then $a = b$.

3. If $a \leq b$ and $b \leq c$, then $a \leq c$.

Note that by abuse of notation we refer to both the set and the poset as P. Two poset elements a and b are said to be *comparable* if $a \leq b$ or $b \leq a$. Additionally, we will use the notation $a < b$ to indicate $a \leq b$ and $a \neq b$.

Posets are general mathematical objects, with examples including the natural numbers under the relation of divisibility and the powerset of a set under the relation of inclusion. Totally ordered sets, also called *chains*, are posets as well.

Definition 1.1.2. A *chain* is a poset P in which every two elements of P are comparable.

We will frequently refer to chains that are subsets of $\mathbb{N}$. As a result, we establish the following notation.

Definition 1.1.3. Let $a \in \mathbb{N}$ and let $[a]$ denote the poset $\{1, 2, \ldots, a\}$ with the usual less than or equal to $\leq$. This is the chain with a elements.

If we have two posets, we can form a new poset using the usual *Cartesian product.*

Definition 1.1.4. Let P and Q be posets. The *Cartesian product* is $P \times Q = \{(p, q) \mid p \in P, q \in Q\}$ where $(p, q) \leq (p', q')$ if $p \leq p'$ in P and $q \leq q'$ in Q.

Many of our results will involve a *product of chains* poset.

Definition 1.1.5. A *product of chains* is a poset of the form $[a_1] \times [a_2] \times \cdots \times [a_n]$ where $a_1, a_2, \ldots, a_n \in \mathbb{N}$.

It is useful to be able to reference comparable poset elements that have no elements between them.

Definition 1.1.6. Given $s, t \in P$, t *covers* s if $s < t$ and there is no element $x \in P$ such that $s < x < t$.

With the definition of a cover, we can describe a representation of a finite poset called a *Hasse diagram.*

Definition 1.1.7. A *Hasse diagram* is a visual representation of the poset using vertices and edges. Vertices in the diagram represent poset elements, while an edge between two vertices represents a covering relation. In this case, the vertex that is vertically higher in the diagram covers the lower vertex.

Figure 1.1 shows the Hasse diagrams of two different posets.

Previously mentioned, the Cartesian product is one method of combining two posets to make a new poset. Another is by taking the *ordinal sum* of two posets.

(a) This is the product of chains poset $[3] \times [2]$.

(b) This poset is an example of a Tamari lattice.

Figure 1.1. Hasse diagrams of two posets.

Definition 1.1.8. If P and Q are posets, the *disjoint sum* of P and Q, denoted $P + Q$, is a poset with elements $P \cup Q$ such that $s \leq t$ in $P + Q$ if either the following hold:

1. $s, t \in P$ and $s \leq t$ in P.

2. $s, t \in Q$ and $s \leq t$ in Q.

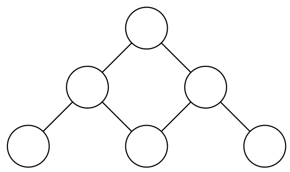
We also introduce several classes of posets derived from Lie theory. For further background, see [23] and [39].

Definition 1.1.9. The *type A_{n-1} positive root poset* $\Phi^+(A_{n-1})$ is a poset with elements $\{a_{i,j} \mid i, j \in \{1, \ldots, n-1\}$ and $i \geq j\}$ and covering relations $a_{i,j} \geq a_{i+1,j}$ and $a_{i,j} \geq a_{i+1,j+1}$. A *type A minuscule poset* is a product of chains poset $[a] \times [b]$. A *type B minuscule poset* is a poset of the form $([a] \times [a])/S_2$. In other words, $(x, y) \sim (y, x)$ in this poset. The poset $([a] \times [a])/S_2$ can be viewed as the left half of the Hasse diagram of $[a] \times [a]$.

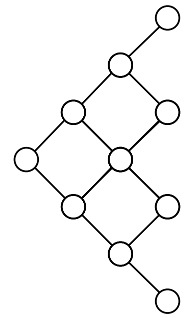
We give an example of $\Phi^+(A_3)$ in Figure 1.2a. The product of chains poset $[3] \times [2]$ from Figure 1.1a is an example of a type A minuscule poset. The poset $([4] \times [4])/S_2$ in Figure 1.2b is a type B minuscule poset.

Using a special subset of a poset P, we obtain an object called an *order ideal*. We also introduce *order filters*, which are dual to order ideals.

Definition 1.1.10. A subset I of P is called an *order ideal* if for any $t \in I$ and $s \leq t$ in P, then $s \in I$. Let $J(P)$ denote the set of order ideals of P. A subset I of P is called an *order filter* if for any $t \in I$ and $s \geq t$ in P, then $s \in I$.



(a) The positive root poset $\Phi^+(A_3)$



(b) The type B minuscule poset $([4] \times [4])/S_2$

Figure 1.2. The left figure is an example of the type A_3 positive root poset $\Phi^+(A_3)$ while the right figure is an example of a type B minuscule poset.

The majority of our results will be concerned with order ideals of particular posets. We show the 6 order ideals of $J([2] \times [2])$ in Figure 1.3. The elements shaded black represent the elements of the order ideal.

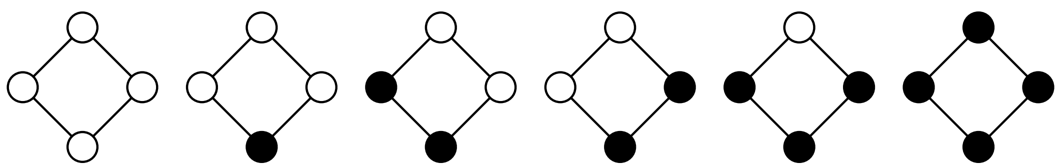


Figure 1.3. In this example, we show the 6 order ideals in $J([2] \times [2])$.

We conclude with several additional definitions relevant to posets.

Definition 1.1.11. A *rank function of* P is a function $\mathrm{rk} : P \to \mathbb{N}$ such that $\mathrm{rk}(f) = \mathrm{rk}(e) + 1$ if f covers e. We will also use the convention that if e is a minimum element of P, $\mathrm{rk}(e) = 0$. If P has such a function, we will call P a *ranked poset.*

Figure 1.1a is an example of a ranked poset, whereas Figure 1.1b is not a ranked poset.

Definition 1.1.12. An *interval* $[a, b]$ of P is the set of all elements $x \in P$ such that $a \leq x \leq b$. P is *locally finite* if for all $a, b \in P$, the interval $[a, b]$ is finite.

Definition 1.1.13. A *linear extension* of a poset P is a bijective function $f : P \to \{1, \ldots, n\}$ where $|P| = n$ such that if $p_1 < p_2$ in P then $f(p_1) < f(p_2)$. Let $\mathcal{L}(P)$ denote the set of linear extensions of P.

We can label the elements in a Hasse diagram to represent a linear extension. See Figure 1.4 for an example.

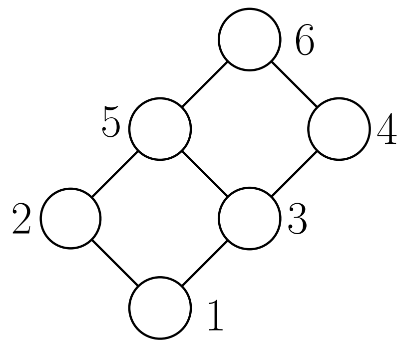


Figure 1.4. An example of a linear extension of the poset $[3] \times [2]$.

We now introduce standard Young tableaux.

Definition 1.1.14. A *partition* $\lambda = (\lambda_1, \ldots, \lambda_k)$ is a sequence of positive integers where $\lambda_j \geq \lambda_{j+1}$. We denote $|\lambda| = \sum_{i=1}^{k} \lambda_i$. A *Young diagram* is a configuration of boxes in left-justified rows, where the length of the rows are weakly decreasing. The Young diagram is said to have *partition shape*, as the length of the rows written as a vector form a partition.

Definition 1.1.15. A *standard Young tableau* of shape λ is a filling of the $|\lambda| = n$ boxes of a Young diagram with the integers $\{1, 2, \ldots, n\}$ such that the entries strictly increase from left to right across rows, strictly increase from top to bottom along columns, and each integer is used exactly once.

We give an example of a standard Young tableau of shape $\lambda = (4, 3, 2)$ in Figure 1.5.

1	2	4	7
3	5	6	
8	9		

Figure 1.5. A standard Young tableau of shape $\lambda = (4, 3, 2)$.

Standard Young tableaux are a special subset of *semistandard Young tableaux*.

Definition 1.1.16. A *semistandard Young tableau* of shape λ with entries at most q is a filling of boxes of partition shape λ with positive integers $\{1, 2, \ldots, q\}$ such that the entries weakly increase from left to right across rows and strictly increase from top to bottom along columns. When filling boxes, integers may be used more than once or not at all.

1.2. A brief history of promotion and rowmotion

In the previous section, we defined order ideals of posets, linear extensions of posets, and tableaux. In this section, we introduce actions on these objects.

1.2.1. Promotion on standard Young tableaux and linear extensions

Promotion is a natural action defined by M.-P. Schützenberger on standard Young tableaux and, more generally, linear extensions of finite poset [35], arising from study of evacuation and the RSK correspondence. We give the definition on linear extensions and then give an alternate definition.

Definition 1.2.1. Suppose P is a poset with n elements and $f \in \mathcal{L}(P)$, then the *promotion* of f, denoted $\mathrm{Pro}(f)$, is found as follows. We begin by deleting the label 1. We then slide down the smallest label of all covers of the now unlabeled element to replace the removed label 1; this is called a *jeu de taquin* slide. This jeu de taquin sliding process continues with the new unlabeled element until the unlabeled element is maximal; we then label this with $n + 1$. By subtracting 1 from every label, we obtain a new linear extension, which is $\mathrm{Pro}(f)$.

This is not the only way to view promotion on a linear extension; it can also be defined using a sequence of involutions. These involutions are a special case of involutions introduced by E. Bender and D. Knuth on semistandard Young tableaux [2].

Definition 1.2.2. Let the action of the *ith Bender-Knuth involution* ρ_i on $f \in \mathcal{L}(P)$ be as follows: swap the labels i and $i + 1$ if the result is a linear extension, otherwise do nothing.

Theorem 1.2.3 ([36])**.** *Suppose P is a poset with n elements and $f \in \mathcal{L}(P)$.* $\mathrm{Pro}(f) = \rho_{n-1} \circ \cdots \circ \rho_2 \circ \rho_1$.

Promotion has many beautiful properties and significant applications in representation theory. See R. Stanley's survey [36] for many of these properties, including further history and details on promotion via Bender-Knuth involutions. This survey also discusses evacuation, which is an action defined using a series of (partial) promotions. As we reference evacuation in Chapter 3, we define it here.

Definition 1.2.4 ([35])**.** Let P be a poset with linear extension $f \in \mathcal{L}(P)$. To perform *evacuation*, first apply promotion to f. Next, freeze the largest label in f and apply promotion to the unfrozen labels. Continue this process of freezing the next largest label and applying promotion to the unfrozen labels until all labels are frozen. The result is the evacuation $\epsilon(f)$. In terms of Bender-Knuth involutions, this is $\epsilon(f) = \rho_1 \circ (\rho_2 \circ \rho_1) \circ \cdots \circ (\rho_{n-3} \circ \cdots \circ \rho_2 \circ \rho_1) \circ (\rho_{n-2} \circ \cdots \circ \rho_2 \circ \rho_1) \circ (\rho_{n-1} \circ \cdots \circ \rho_2 \circ \rho_1)(f)$.

1.2.2. Rowmotion and the toggle group

Rowmotion is an action originally defined on hypergraphs by P. Duchet [13] and generalized to order ideals of an arbitrary finite poset by A. Brouwer and A. Schrijver [6].

Definition 1.2.5. Let P be a poset and $I \in J(P)$. $\mathrm{Row}(I)$ is the order ideal generated by the minimal elements of P not in I. In other words, if t is a minimal element of $P \setminus I$ and $s \leq t$, then $s \in \mathrm{Row}(I)$.

Rowmotion has recently generated significant interest as a prototypical action in the emerging subfield of dynamical algebraic combinatorics; see [44] for a detailed history and [3, 5, 9, 10, 11, 16, 17, 20, 21, 22, 24, 26, 30, 32, 33, 34, 41, 42, 43] for more recent developments.

In [6], Brouwer and Schrijver studied the order of rowmotion on a product of two chains poset, $[a] \times [b]$.

Theorem 1.2.6 ([6])**.** *$J([a] \times [b])$ under* Row *has order $a + b$.*

D. Fon-der-Flaass refined this further with a result on the length of any orbit of $J([a] \times [b])$ under rowmotion [18]. In [36], Stanley showed there exists an equivariant bijection between linear extensions of two disjoint chains $[a] + [b]$ (or equivalently, standard Young tableaux of disjoint skew shape) under promotion and $J([a] \times [b])$ under rowmotion.

Theorem 1.2.7 ([36])**.** *$J\left([a] \times [b]\right)$ under* Row *is in equivariant bijection with $\mathcal{L}\left([a] \oplus [b]\right)$ under* Pro.

Another instance where promotion on linear extensions and rowmotion are related is an equivariant bijection between linear extensions of the product of two chains poset $[2] \times [n]$ under promotion (or alternatively, rectangular, two-row standard Young tableaux under promotion) and order ideals of the type A_{n-1} positive root poset $\Phi^+(A_{n-1})$ under rowmotion. This is a restatement of the Type A case of a result of D. Armstrong, C. Stump, and H. Thomas in [1].

Theorem 1.2.8 ([44, Theorem 3.10]). *$J(\Phi^+(A_{n-1}))$ under* Row *is in equivariant bijection with $\mathcal{L}([2] \times [n])$ under* Pro*.*

J. Striker and N. Williams proved a general result [44] relating promotion and rowmotion which recovers Theorems 1.2.7 and 1.2.8 as special cases. They used the *toggle group* of P. Cameron and Fon-der-Flaass [8], which we describe below.

Definition 1.2.9. For any $p \in P$, the *toggle* $t_p : J(P) \to J(P)$ is defined as follows:

$$t_p(I) = \begin{cases} I \cup \{p\} & \text{if } p \notin I \text{ and } I \cup \{p\} \in J(P) \\ I \setminus \{p\} & \text{if } p \in I \text{ and } I \setminus \{p\} \in J(P) \\ I & \text{otherwise.} \end{cases}$$

The *toggle group* of P is the group generated by the t_p for all $p \in P$ with operation composition. Note that each t_p is an involution.

Remark 1.2.10. ([8, p. 546]) The toggles t_{p_1} and t_{p_2} commute whenever neither p_1 nor p_2 covers the other.

Cameron and Fon-der-Flaass showed a connection between rowmotion and toggling. Specifically, rowmotion can be performed by toggling each element of a poset from top to bottom, that is, in the reverse order of any linear extension. If the poset is ranked, this is equivalent to toggling top to bottom by ranks, or rows.

Theorem 1.2.11 ([8, Lemma 1]). *Let $f \in \mathcal{L}(P)$. Then $t_{f^{-1}(1)} t_{f^{-1}(2)} \cdots t_{f^{-1}(n)}$ acts as Row.*

The benefit of the toggle perspective is that we can study other actions that are closely related to rowmotion. In [44], Striker and Williams constructed a toggle group action that corresponds to linear extension promotion in the special cases of Theorems 1.2.7 and 1.2.8; they named this toggle group action *(order ideal-) promotion* because of this correspondence. Order ideal promotion first requires projecting to a two-dimensional lattice and defining *columns*. More specifically, order ideal promotion toggles poset elements from left to right by columns; see Example 1.2.14 for an example. As we will want to perform order ideal promotion in higher dimensions, we omit the formal definition of columns and instead define an n-dimensional lattice projection formally in Definition 1.2.21, as columns can be stated in terms of a two-dimensional lattice projection. Although we used Pro to denote promotion on a linear extension, we will also use Pro to denote this two-dimensional order ideal promotion; for the rest of this dissertation, we will only refer to the order ideal promotion Pro. In the following theorem, Striker and Williams showed that order ideal promotion and rowmotion are conjugate elements in the toggle group, and thus have the same orbit structure.

Theorem 1.2.12 ([44, Theorem 5.2])**.** *For any poset P with a two-dimensional lattice projection (in particular, any ranked poset), there is an equivariant bijection between $J(P)$ under* Pro *and $J(P)$ under* Row*.*

Striker and Williams found that in many cases, it was easier to prove the orbit sizes of Pro compared to Row. The reason for this in these cases is that the action of Pro on $J(P)$ is in equivariant bijection with rotation on another object. As a result, in order to study the orbits of Row, it is often useful to study Pro and apply Theorem 1.2.12. We will show such a rotation for a product of chains $[a] \times [b]$.

Definition 1.2.13. Define the *boundary path* of an order ideal I as a path of upsteps and downsteps that separates I from the rest of the poset. The *boundary path sequence*, $B(I)$, is a sequence of zeros and ones where zeros correspond to downsteps and ones correspond to upsteps in the boundary path.

Example 1.2.14. In Figure 1.6, we show an orbit of $J([3] \times [2])$ under Pro. The red path is the boundary path; the sequence of zeros and ones below each diagram is the boundary path sequence. We see that after applying Pro to move forward in the orbit, the boundary path sequence cyclically shifts to the left.

Striker and Williams showed that applying promotion to an order ideal of $[a] \times [b]$ corresponds to applying a leftward cyclic shift on the boundary path sequence of I.

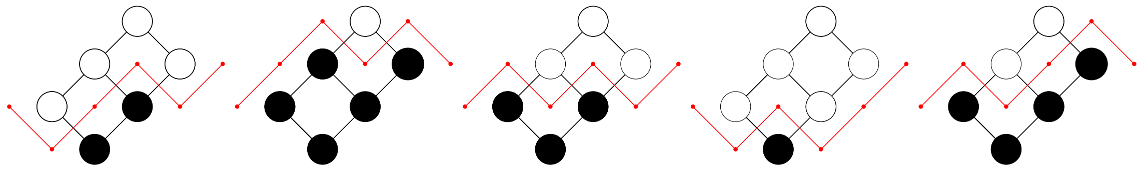


Figure 1.6. An orbit of $J([3] \times [2])$ under Pro with boundary path and boundary path sequence shown.

Theorem 1.2.15 ([44, Theorem 6.1])**.** *Let $I \in J([a] \times [b])$. The boundary path $B(\mathrm{Pro}(I))$ is a left cyclic shift of $B(I)$.*

This immediately gives the order of promotion on order ideals of $[a] \times [b]$.

Corollary 1.2.16 ([44])**.** *$J([a] \times [b])$ under* Pro *has order $a + b$.*

Additionally, combining this corollary and Theorem 1.2.12 gives an alternate proof of Theorem 1.2.6, that the order of rowmotion on $[a] \times [b]$ is $a + b$.

1.2.3. K-promotion on increasing tableaux and rowmotion on the product of three chains

In [27], O. Pechenik generalized promotion on standard Young tableaux to *K-promotion* on *increasing tableaux*, using the *K-jeu de taquin* of Thomas and A. Yong [45]. Increasing tableaux, a special subset of semistandard Young tableaux, first appeared in [7] in the context of K-theoretic Schubert calculus. We give the definitions of increasing tableaux and K-promotion below.

Definition 1.2.17. An *increasing tableau* of shape λ is a filling of boxes of partition shape λ with positive integers such that the entries strictly increase from left to right across rows and strictly increase from top to bottom along columns. We will use $\mathrm{Inc}^q(\lambda)$ to indicate the set of increasing tableaux of shape λ with entries at most q.

Figure 1.7 shows an increasing tableau in $\mathrm{Inc}^q(3, 3, 1)$ where q can be any integer greater than or equal to 6.

Definition 1.2.18 ([27])**.** Let $T \in \mathrm{Inc}^q(\lambda)$. Delete all labels 1 from T. Consider the set of boxes that are either empty or contain 2. We simultaneously delete each label 2 that is adjacent to an empty box and place a 2 in each empty box that is adjacent to a 2. Now consider the set of boxes that are either empty or contain 3, and repeat the above process. Continue until all empty boxes are

1	2	4
2	4	5
6		

Figure 1.7. An increasing tableau of shape $\lambda = (3, 3, 1)$.

located at outer corners of λ. Finally, label those boxes $q + 1$ and then subtract 1 from each entry. The result is the *K-promotion of T*, which we denote $K\text{-Pro}(T)$. Note that $K\text{-Pro}(T) \in \mathrm{Inc}^q(\lambda)$.

This is not the only way to describe K-promotion, however. In [11], K. Dilks, Pechenik and Striker showed that K-promotion can be performed using a sequence of local involutions analogous to those of Bender and Knuth for semistandard Young tableaux [2]. They called these involutions *K-Bender-Knuth involutions*, denoted by $K\text{-BK}_i$.

Proposition 1.2.19 ([11, Proposition 2.5])**.** *For $T \in \mathrm{Inc}^q(\lambda)$, $K\text{-Pro}(T) = K\text{-BK}_{q-1} \circ \cdots \circ K\text{-BK}_1$.*

With these K-Bender-Knuth involutions, we can also give an analogue of the evacuation action defined in Definition 1.2.4 for increasing tableaux.

Definition 1.2.20. Define *K-evacuation* on an increasing tableaux T as:

$$\mathcal{E}(T) = K\text{-BK}_1 \circ (K\text{-BK}_2 \circ K\text{-BK}_1) \circ \cdots \circ (K\text{-BK}_{q-3} \circ \cdots \circ K\text{-BK}_2 \circ K\text{-BK}_1) \circ$$

$$(K\text{-BK}_{q-2} \circ \cdots \circ K\text{-BK}_2 \circ K\text{-BK}_1) \circ (K\text{-BK}_{q-1} \circ \cdots \circ K\text{-BK}_2 \circ K\text{-BK}_1)(T).$$

In [11], Dilks, Pechenik, and Striker built on Proposition 1.2.19 to give a connection between increasing tableaux of rectangular shape with entries at most q and order ideals in a product of three chains poset. While the bijection between the two is straightforward, it is non-trivial that K-promotion on increasing tableaux is carried equivariantly to a toggle group action they called *hyperplane promotion* on order ideals in the product of three chains poset. We give the relevant definitions below.

Definition 1.2.21 ([11])**.** We say that an *n-dimensional lattice projection* of a ranked poset P is an order and rank preserving map $\pi : P \to \mathbb{Z}^n$, where the rank function on $\mathbb{Z}^n$ is the sum of the coordinates and $x \leq y$ in $\mathbb{Z}^n$ if and only if the componentwise difference $y - x$ is in $(\mathbb{Z}_{\geq 0})^n$.

Definition 1.2.22 ([11])**.** Let P be a poset with an n-dimensional lattice projection π, and let $v = (v_1, v_2, \ldots, v_n)$ where $v_j \in \{\pm 1\}$. Let $T^i_{\pi,v}$ be the product of toggles t_x for all elements x of P

that lie on the affine hyperplane $\langle \pi(x), v\rangle = i$. If there is no such x, then this is the empty product, considered to be the identity. Define *(hyperplane) promotion with respect to* π *and* v as the toggle product $\mathrm{Pro}_{\pi,v} = \dots T^{-2}_{\pi,v} T^{-1}_{\pi,v} T^{0}_{\pi,v} T^{1}_{\pi,v} T^{2}_{\pi,v} \cdots$

Note that for Chapters 2 and 3, we will almost exclusively let P be a product of chains poset and our lattice projection be a natural embedding into $\mathbb{N}^n$. However, in Chapter 4, we generalize one of our main results, using an arbitrary poset P with n-dimensional lattice projection. For ease of notation, whenever we use v we will mean $v = (v_1, v_2, \dots, v_n)$ where $v_j \in \{\pm 1\}$, where n will be inferred from context.

By Remark 1.2.10, toggles commute if there is no covering relation between their corresponding poset elements. So we note $\mathrm{Pro}_{\pi,v}$ is well-defined in the following way.

Remark 1.2.23 ([11])**.** Two elements of the poset that lie on the same affine hyperplane $\langle \pi(x), v\rangle = i$ cannot be part of a covering relation.

Now that we have established $\mathrm{Pro}_{\pi,v}$ and verified it is well-defined, we can relate it to rowmotion.

Proposition 1.2.24 ([11])**.** *For a finite ranked poset* P *with* n*-dimensional lattice projection* π*,* $\mathrm{Pro}_{\pi,(1,1,\dots,1)} = \mathrm{Row}$*. Additionally,* $\mathrm{Pro}_{\pi,(-1,1)}$ *is the two-dimensional promotion action* Pro*.*

This proposition and the next theorem show that any hyperplane promotion is conjugate to the more natural toggle group action of rowmotion.

Theorem 1.2.25 ([11, Theorem 3.25])**.** *Let* P *be a poset with an* n*-dimensional lattice projection* π*. Let* v *and* w *be vectors in* $\mathbb{Z}^n$ *with entries in* $\{\pm 1\}$*. Then there is an equivariant bijection between* $J(P)$ *under* $\mathrm{Pro}_{\pi,v}$ *and* $J(P)$ *under* $\mathrm{Pro}_{\pi,w}$*.*

For order ideals of a product of chains under Row, we also have a bijection to increasing tableaux under K-Pro.

Theorem 1.2.26 ([11, Theorem 4.4])**.** $J([a] \times [b] \times [c])$ *under* Row *is in equivariant bijection with* $\mathrm{Inc}^{a+b+c-1}(a \times b)$ *under* K*-*Pro*.*

This was a second, more general setting in which rowmotion was shown to have the same orbit structure as a previously studied promotion action. Along with orbit structure, a phenomenon, isolated by Propp and Roby, appears frequently among many posets and will be the subject of the next section.

1.3. The homomesy phenomenon

In this section, we define homomesy and state known results in two dimensions. We will generalize these results to higher dimensions in Chapter 2.

Definition 1.3.1. Given a finite set S, a bijective action $\tau : S \to S$, and a statistic $f : S \to \mathbb{K}$ where $\mathbb{K}$ is a field of characteristic zero, the triple (S, τ, f) exhibits *homomesy* if there exists $c \in \mathbb{K}$ such that for every τ-orbit $\mathcal{O}$

$$\frac{1}{\#\mathcal{O}} \sum_{x \in \mathcal{O}} f(x) = c$$

where $\#\mathcal{O}$ denotes the number of elements in $\mathcal{O}$. If such a c exists, we will say the triple is c-mesic.

A statistic can be any map from S to $\mathbb{K}$; however, it should have some combinatorial significance. For many of the results in this dissertation, our statistic will be the cardinality of an order ideal. Homomesy results have been observed in many well-known combinatorial objects. Propp and Roby proved the following results on a product of two chains.

Theorem 1.3.2 ([30])**.** *Let f be the cardinality statistic. Then $(J([a] \times [b]), \mathrm{Pro}, f)$ is c-mesic with $c = ab/2$.*

Theorem 1.3.3 ([30])**.** *Let f be the cardinality statistic. Then $(J([a] \times [b]), \mathrm{Row}, f)$ is c-mesic with $c = ab/2$.*

We will show proofs for both of these as one of our main results, Theorem 2.0.1, generalizes these theorems. The proof of Theorem 1.3.2 will follow Propp and Roby's proof in [30] with a few changes to notation.

Definition 1.3.4 ([30])**.** Let $P = [a] \times [b]$. Define the *file* $x_1 - x_2$ as all elements (x_1, x_2) with constant value $x_1 - x_2$. Furthermore, define the *height function* of I for file k as

$$h_I(k) = |k| + 2\#(\text{elements of } I \text{ in file } k).$$

We can relate the sum of the height functions to the size of $[a] \times [b]$ and the cardinality of I.

Lemma 1.3.5 ([30]). *Let $I \in J([a] \times [b])$. Then*

$$\sum_{k=-b}^{a} h_I(k) = \frac{a(a+1)}{2} + \frac{b(b+1)}{2} + 2f(I).$$

Proof. Summing $h_I(k)$ from the files $k = -b$ to $k = a$ means the $|k|$ term of $h_I(k)$ will sum from 1 to a and 1 to b, yielding the terms $\frac{a(a+1)}{2}$ and $\frac{b(b+1)}{2}$. Summing two times the number of elements of I in file k over all files k yields $2f(I)$. □

We now have the background to prove Theorem 1.3.2.

Proof of Theorem 1.3.2. Our strategy will be to show $J([a] \times [b])$ under Pro with statistic $h_I(k)$ is homomesic. From this, we will conclude $J([a] \times [b])$ under Pro with statistic $\sum_{k=-b}^{a} h_I(k)$ is homomesic, and by the previous proposition, $J([a] \times [b])$ under Pro with statistic $f(I)$ is as well. To show the homomesy result for $h_I(k)$, we rewrite $h_I(k)$ as a telescoping sum $h_I(k) = h_I(-b) + (h_I(-b+1) - h_I(-b)) + \cdots + (h_I(k) - h_I(k-1))$. As a result, if each term $h_I(k) - h_I(k-1)$ for $-b+1 \leq k \leq a$ exhibits homomesy, the sum $h_I(k)$ will as well.

To show the result for $h_I(k) - h_I(k-1)$, we will introduce a bijection between $h_I(k) - h_I(k-1)$ and the $(k+b)$th entry of the boundary path sequence of I, $B(I)$. Suppose the $(k+b)$th component of $B(I)$ is 1. This corresponds to an upstep between file $k-1$ and file k. If $k \leq 0$, then

$h_I(k) - h_I(k-1) = -1 + 2[\#(\text{elements of } I \text{ in file } k) - \#(\text{elements of } I \text{ in file } k-1)] = -1 + 2 = 1.$

If $k \geq 0$, then

$h_I(k) - h_I(k-1) = 1 + 2[\#(\text{elements of } I \text{ in file } k) - \#(\text{elements of } I \text{ in file } k-1)] = 1 + 0 = 1.$

In both cases, $h_I(k) - h_I(k-1) = 1$. Similarly, if the $(k+b)$th component of $B(I)$ is a 0, this corresponds to a downstep between file $k-1$ and file k, which results in $h_I(k) - h_I(k-1) = -1$. As a result, we have our desired bijection.

A boundary path sequence $B(I)$ for $I \in J([a] \times [b])$ must contain a ones and b zeros. By Theorem 1.2.15, $B(I)$ cyclically shifts to the left when Pro is applied to I. This implies that over an orbit of Pro, any component of $B(I)$ must average $\frac{a}{a+b}$ ones and $\frac{b}{a+b}$ zeros. By our previous bijection, this tells us $J([a] \times [b])$ under Pro with statistic $h_I(k) - h_I(k-1)$ is homomesic and hence, $f(I)$ is as well.

We have shown $(J([a] \times [b]), \mathrm{Pro}, f)$ is homomesic, but we still must show the orbit average is $ab/2$. Due to rotational symmetry, the order filters of $[a] \times [b]$ are in bijection with the order ideals of $[a] \times [b]$. More specifically, let $I \in J([a] \times [b])$ and let $H \in J([a] \times [b])$ be the order ideal isomorphic to $P \setminus I$. Therefore, $f(I) + f(H) = ab$. As a result, we can say the global average of f is $ab/2$, and hence c must also be $ab/2$. $\square$

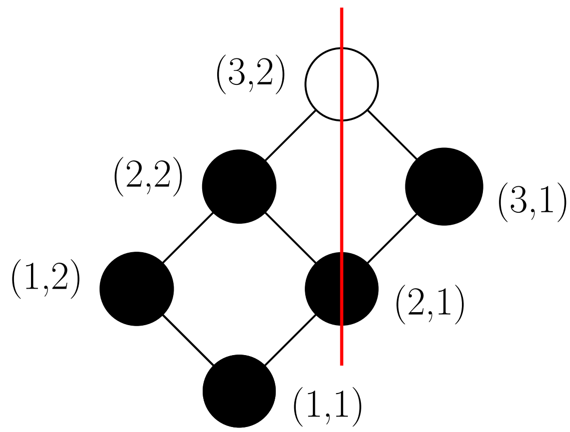


Figure 1.8. The poset $[3] \times [2]$ with order ideal I. The file 1 is denoted by the red line, with $h_I(1) = 3$.

Example 1.3.6. Figure 1.8 shows the poset $[3] \times [2]$ and an order ideal, denoted I. The red line going through the points $(3, 2)$ and $(2, 1)$ in the diagram shows the file 1, as 3-2 and 2-1 are both 1. Because $h_I(k) = |k| + 2\#(\text{elements of } I \text{ in file } k)$, we see $h_I(1) = 3$. Furthermore, if we write out each $h_I(k)$ from $k = -b$ to $k = a$, we obtain $2, 3, 4, 3, 4, 3$. Taking successive differences $h_I(k) - h_I(k-1)$ yields $1, 1, -1, 1, -1$. The bijection to a boundary path sequence merely changes negative ones to zeros, giving us $(1, 1, 0, 1, 0)$. We can see from Figure 1.6 that this is the boundary path sequence for I.

Propp and Roby also showed *refined* homomesy results in the product of two chains. In other words, they showed particular subcollections of elements also exhibited homomesy under rowmotion and promotion. We define the *indicator function* in order to state these results.

Definition 1.3.7. Let P be a poset, $I \in J(P)$, and $x \in P$. Denote the *indicator function* $1_x(I) : J(P) \to \{0, 1\}$ by

$$1_x(I) = \begin{cases} 1 & \text{if } x \in I \\ 0 & \text{if } x \notin I \end{cases}$$

One refined result of Propp and Roby involves *antipodal* elements in $[a] \times [b]$.

Definition 1.3.8. Let $P = [a] \times [b]$. If $x = (x_1, x_2)$ and $y = (a+1-x_1, b+1-x_2)$, then x and y are *antipodal* in P.

Theorem 1.3.9 ([30])**.** *Suppose x and y are antipodal elements in $[a] \times [b]$. Then $(J([a] \times [b]), \mathrm{Row}, 1_x + 1_y)$ and $(J([a] \times [b]), \mathrm{Pro}, 1_x + 1_y)$ are c-mesic with $c = 1$ if x and y are distinct and $c = 1/2$ if $x = y$.*

Theorem 1.3.10 ([30])**.** *Suppose k is a file in $[a] \times [b]$. Then $(J([a] \times [b]), \mathrm{Row}, \sum_{x \text{ in file } k} 1_x)$ and $(J([a] \times [b]), \mathrm{Pro}, \sum_{x \text{ in file } k} 1_x)$ are homomesic.*

In other words, sets of antipodal elements and sets of files of elements in $[a] \times [b]$ exhibit homomesy under both rowmotion and promotion.

Example 1.3.11. Figure 1.9 contains an orbit of $J([3] \times [2])$ under Pro. The red elements x and y are antipodal in $[3] \times [2]$. The average cardinality of these elements over this orbit is 1. Theorem 1.3.9 says that if we take any orbit of $J([3] \times [2])$ under Pro, we also obtain an average of 1.

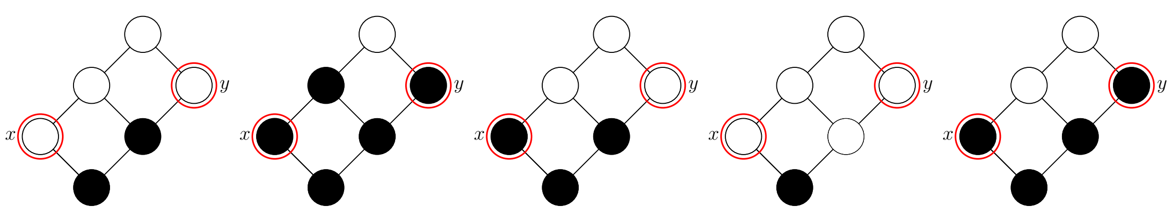


Figure 1.9. The average cardinality of x and y over this orbit is $\dfrac{0+2+1+0+2}{5} = 1$.

It is beneficial to study $J([a] \times [b])$ under Pro rather than Row, as $J([a] \times [b])$ under Pro is in bijection with boundary path sequences of length $a + b$ under a left cyclic shift. This fact makes the proof of Theorem 1.3.2 fairly straightfoward. Propp and Roby also have a direct proof of Theorem 1.3.3 in [30]; however, it is much more technical than in the promotion case. Einstein and Propp found a more elegant way to prove Theorem 1.3.3; we will expand on this in Chapter 2.

2. HOMOMESY ON $J([2] \times [a] \times [b])$ AND RECOMBINATION

In this chapter, we extend homomesy results discussed in Chapter 1 from two dimensions to three dimensions. We state our first main theorem and the primary motivation for Chapter 2, a higher dimensional analogue of Theorem 1.3.2 and 1.3.3.

Theorem 2.0.1. *Let f be the cardinality statistic. The triple $(J([2] \times [a] \times [b]), \mathrm{Pro}_v, f)$ is c-mesic with $c = ab$.*

We begin by introducing the idea of layers in Definition 2.1.2, as these are necessary for organizing the *recombination* proof technique of Einstein and Propp, along with our higher dimensional generalization. In Section 2.1, we summarize Einstein's and Propp's results. This includes Theorem 2.1.8, which shows how recombination connects Row and Pro for a two-dimensional product of chains. A key aspect of the proof of Theorem 2.0.1 is generalizing recombination to a higher dimensional product of chains. This appears in Section 2.2 and is our second main result of Chapter 2, Theorem 2.2.4. In Section 2.3, we complete the proof of Theorem 2.0.1 using recombination and a connection to increasing tableaux. Additionally, Corollaries 2.3.5 and 2.3.6 use symmetry to give two additional results similar to Theorem 2.0.1. In Section 2.4, we conclude with Propositions 2.4.1 and 2.4.2, showing that Theorem 2.0.1 does not generalize further. In the next chapter, we discuss a partial generalization. This chapter is based on work from [46].

2.1. An introduction to recombination

In [17] (with further details in [16]), Einstein and Propp found an elegant proof technique to prove Theorem 1.3.3; they called this technique *recombination.* The idea behind recombination is that we may start with an orbit from $J([a] \times [b])$ under Row and take sequential layers from order ideals to form a new orbit. We first introduce some useful notation.

Definition 2.1.1. Suppose $v = (v_1, v_2, \ldots, v_n) \in \mathbb{Z}^n$. Given $\gamma \in \{1, \ldots, n\}$, let $v^{\widehat{\gamma}} = (v_1, v_2, \ldots, v_{\gamma-1}, v_{\gamma+1}, \ldots, v_n)$.

We define our layers in the following way.

Definition 2.1.2. Define the jth γ-layer of $P = [a_1] \times \cdots \times [a_n]$ as

$$L_\gamma^j = \{(i_1, i_2, \ldots, i_n) \in P \mid i_\gamma = j\}$$

and the jth γ-layer of $I \in J(P)$ as

$$L_\gamma^j(I) = L_\gamma^j \cap I.$$

Additionally, given L_γ^j and $L_\gamma^j(I)$, define

$$(L_\gamma^j)^{\widehat{\gamma}} = \{(i_1, i_2, \ldots, i_n)^{\widehat{\gamma}} \mid (i_1, i_2, \ldots, i_n) \in L_\gamma^j\},$$
$$L_\gamma^j(I)^{\widehat{\gamma}} = \{(i_1, i_2, \ldots, i_n)^{\widehat{\gamma}} \mid (i_1, i_2, \ldots, i_n) \in L_\gamma^j(I)\}.$$

When taking layers, γ determines the component in which we are working. Additionally, j signifies which of the layers we are taking in that direction.

Einstein and Propp referred to each L_1^j as a negative fiber of P; we use the notation L_γ^j and $L_\gamma^j(I)$ as it more naturally describes our layers when we generalize recombination to higher dimensions in Section 2.2. Furthermore, we define $(L_\gamma^j)^{\widehat{\gamma}}$ and $L_\gamma^j(I)^{\widehat{\gamma}}$, which remove the jth coordinate, as it will be useful to view certain layers in the $(n-1)$-dimensional setting.

Using the idea of layers, Einstein and Propp defined the concept of recombination and proved the following proposition, which we restate in the above notation. See Figure 2.2 for an example.

Definition 2.1.3. Let $I \in J([a] \times [b])$. Define the *recombination* of I as $\Delta I = \cup_j L_1^j(\text{Row}^{j-1}(I))$.

In Theorem 2.2.4, we generalize the notion of recombination to higher dimensional products of chains. Here, we observe an important property of Row and Pro and how their toggles commute in the $[a] \times [b]$ case that will be helpful when generalizing to higher dimensions. In order to state this observation, we introduce an additional definition. This definition will also prove useful when discussing commuting toggles in n-dimensions.

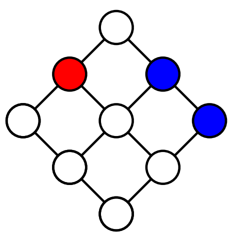
Definition 2.1.4. Let $P = [a_1] \times \cdots \times [a_n]$ and $\gamma \in \{1, 2, \ldots, n\}$. Define $T^j_{\text{Pro}_{v^{\widehat{\gamma}}}}$ as the toggle product of $\text{Pro}_{v^{\widehat{\gamma}}}$ on $(L_\gamma^j)^{\widehat{\gamma}}$.

The following result is discussed in [44], Theorem 5.4 and in [16], Section 8.

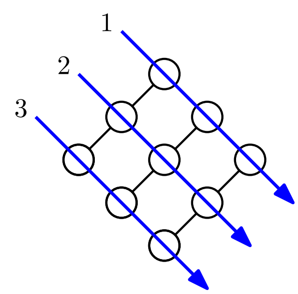
Proposition 2.1.5 ([16, 44])**.** *Let* $P = [a] \times [b]$. $\text{Row} = \text{Pro}_{(1,1)} = \prod_{j=1}^{a} T^j_{\text{Pro}_{(1,1)^{\widehat{1}}}}$ *and* $\text{Pro} = \prod_{j=1}^{a} T^{a+1-j}_{\text{Pro}_{(-1,1)^{\widehat{1}}}}$.

In other words, we can commute the toggles of Row so we toggle L_1^a, followed by L_1^{a-1}, and so on, toggling each layer from top to bottom. To see why, observe the following example.

Example 2.1.6. In Figure 2.1a, we can commute the toggle of the red element on the left with both toggles of the blue elements on the right, as the red element does not have a covering relation with either blue element. Therefore, when performing Row we can toggle both blue elements before the red element, and hence all of L_1^3 before the red element. Similar reasoning applies for each L_1^j, and as a result we can perform Row by toggling in the order denoted in Figure 2.1b, where Layer 1 is first, Layer 2 is second, and Layer 3 third. Additionally, the toggle order in each layer is denoted with an arrow. Note that for Pro, we would have a similar picture except we would toggle Layer 3 first, then Layer 2, then Layer 1.



(a) We can commute the toggle of either blue element with the red element, as there is no covering relation between them.



(b) We toggle Layer 1, then Layer 2, then Layer 3, with arrows denoting toggle order in each layer. This toggle order is equivalent to Row by commuting toggles.

Figure 2.1. We commute the toggles of Row as described in Example 2.1.6.

Our goal is to connect $\mathrm{Row}(I)$ and $\mathrm{Pro}(\Delta I)$. If we want to apply Pro to ΔI though, we must first verify that ΔI is an order ideal.

Lemma 2.1.7. *Let $I \in J([a] \times [b])$. Then ΔI is an order ideal of $[a] \times [b]$.*

Proof. Suppose $(i_1, i_2) \in \Delta I$. By Definition 2.1.3, $(i_1, i_2 - 1) \in \Delta I$ as $(i_1, i_2 - 1)$ is obtained from the same layer as (i_1, i_2). To show that ΔI is an order ideal, it suffices to show $(i_1 - 1, i_2) \in \Delta I$. If $i_1 = 1$ there is nothing to show. Because $(i_1, i_2) \in \Delta I$, we have $(i_1, i_2) \in L_1^{i_1}(\mathrm{Row}^{i_1 - 1}(I))$. By Proposition 2.1.5, $\mathrm{Row} = \prod_{j=1}^{a} T^j_{\mathrm{Pro}_{(1,1)\hat{1}}}$, which implies we can commute the toggle relations in Row so that $L_1^{i_1}$ is toggled before $L_1^{i_1 - 1}$. As a result, we must have $(i_1 - 1, i_2) \in L_1^{i_1 - 1}(\mathrm{Row}^{i_1 - 2}(I))$. Therefore, $(i_1 - 1, i_2) \in \Delta I$. $\square$

The idea behind recombination is the following: we take a single layer from each order ideal in a sequence of order ideals from a rowmotion orbit to form the layers of a new order ideal. Theorem 2.1.8 tells us that if we apply promotion to this new order ideal, the result is the same as if we move one step forward in the rowmotion orbit and apply recombination again. See Figure 2.2 for a specific example.

Theorem 2.1.8 ([16])**.** *Let* $I \in J([a] \times [b])$*. Then* $\mathrm{Pro}(\Delta I) = \cup_j L_1^j(\mathrm{Row}^j(I)) = \Delta(\mathrm{Row}(I))$.

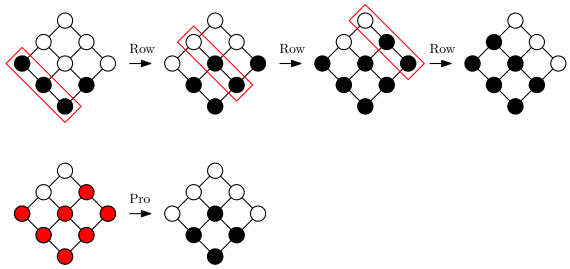
Proof. First, note that ΔI is an order ideal by Lemma 2.1.7. Also note that $\mathrm{Row} = \prod_{j=1}^{a} T^j_{\mathrm{Pro}_{(1,1)^{\hat{1}}}}$ and $\mathrm{Pro} = \prod_{j=1}^{a} T^{a+1-j}_{\mathrm{Pro}_{(-1,1)^{\hat{1}}}}$ by Proposition 2.1.5. We will show $\mathrm{Pro}(\Delta I)$=$\Delta(\mathrm{Row}(I))$ by showing $L_1^k(\mathrm{Pro}(\Delta I)) = L_1^k(\Delta(\mathrm{Row}(I)))$ for each $k \in \{1, 2, \ldots, a\}$. There are three cases.

Case $1 < k < a$**:** Let $J = \mathrm{Row}^{k-1}(I)$. We can commute the toggles of Row so that L_1^{k+1} of J is toggled before L_1^k of J, which is toggled before L_1^{k-1} of J. Thus, when applying the toggles of Row to L_1^k of J, the layer above is $L_1^{k+1}(\mathrm{Row}(J))$ whereas the layer below is $L_1^{k-1}(J)$. Additionally, we can also commute the toggles of Pro so L_1^{k-1} of ΔI is toggled before L_1^k of ΔI, which is toggled before L_1^{k+1} of ΔI. Therefore, when applying the toggles of Pro to L_1^k of ΔI, the layer below is $L_1^{k-1}(\mathrm{Pro}(\Delta I))$, whereas the layer above is $L_1^{k+1}(\Delta I)$. However, $L_1^{k-1}(\mathrm{Pro}(\Delta I)) = L_1^{k-1}(J)$, $L_1^k(\Delta I) = L_1^k(J)$, and $L_1^{k+1}(\Delta I) = L_1^{k+1}(\mathrm{Row}(J))$. Therefore, when applying Row to L_1^k of J and Pro to L_1^k of ΔI, both layers are the same and have the same layers above and below them. Because $(-1,1)^{\hat{1}} = (1,1)^{\hat{1}} = (1)$, we have $\mathrm{Pro}_{(-1,1)^{\hat{1}}} = \mathrm{Pro}_{(1,1)^{\hat{1}}}$ and so the result of toggling this layer is $L_1^k(\mathrm{Pro}(\Delta I))$, which is the same as $L_1^k(\mathrm{Row}(J)) = L_1^k(\mathrm{Row}^k(I)) = L_1^k(\Delta(\mathrm{Row}(I)))$.

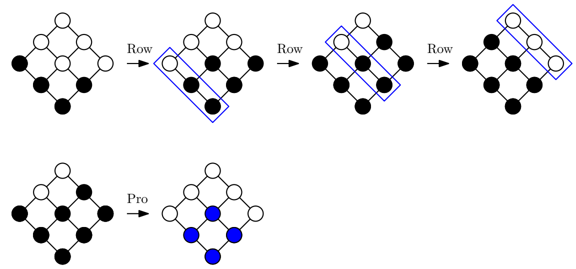
Case $k = 1$**:** As above, when applying Row to L_1^1 of I and Pro to L_1^1 of ΔI, both of these layers are the same, along with the layers above them. Because $k = 1$, there is not a layer below. As above, $\mathrm{Pro}_{(-1,1)^{\hat{1}}} = \mathrm{Pro}_{(1,1)^{\hat{1}}}$ and so we again obtain $L_1^1(\mathrm{Pro}(\Delta I)) = L_1^1(\Delta(\mathrm{Row}(I)))$.

Case $k = a$**:** Again, as above, when applying Row to L_1^a of J and Pro to L_1^a of ΔI, both of these layers are the same along with the layers below them. Because $k = a$ there is not a layer above. Again, $\mathrm{Pro}_{(-1,1)^{\hat{1}}} = \mathrm{Pro}_{(1,1)^{\hat{1}}}$ and so $L_1^a(\mathrm{Pro}(\Delta I)) = L_1^a(\Delta(\mathrm{Row}(I)))$. □

Example 2.1.9. To see an example of the proof technique for the $1 < k < a$ case, we will refer to Figures 2.2, 2.3, 2.4, and 2.5. We begin with the same orbit under Row as in Figure 2.2. Let I denote the first order ideal in this orbit; using recombination we form the order ideal ΔI. We



(a) From an orbit of Row, we use $L_1^1(I), L_1^2(\text{Row}(I))$, and $L_1^3(\text{Row}^2(I))$ to form a new order ideal, denoted here in red.



(b) From the same orbit of Row, we use $L_1^1(\text{Row}(I)), L_1^2(\text{Row}^2(I))$, and $L_1^3(\text{Row}^3(I))$ to form a new order ideal, denoted here in blue.

Figure 2.2. Performing Pro on the red order ideal results in the blue order ideal.

want to verify that by forming sequential recombination order ideals, we obtain an orbit under Pro. We will do so by showing that corresponding layers in the Row orbit and the recombination order ideal result in the same layer after performing Row and Pro, respectively. The boxed purple layers $L_1^2(I)$ in both orbits of Figure 2.3 correspond under recombination. We can commute the toggles of Row as we did in Figure 2.1b. We can also commute the toggles of Pro so we toggle Layer 3, then Layer 2, then Layer 1 in Figure 2.1b. This means when performing Row, we first toggle the layer indicated by the green arrow in Figure 2.4, left. Similarly, when performing Pro, we first toggle the layer indicated by the green arrow in Figure 2.4, right. Then, the next step of both Row and Pro is to toggle the boxed purple layer, as seen in Figure 2.5. We see that when we perform this step of Row and Pro, the boxed purple layer, the layer above, and the layer below are the same. Because we are toggling the same direction along the boxed purple layer, we are guaranteed the same result in both cases.

Propp and Roby gave a direct proof of Theorem 1.3.3 in [30]; however, using recombination, we can prove this result using Theorem 1.3.2. This is the proof technique used by Einstein and Propp in [16].

Proof of Theorem 1.3.3. Recombination gives a bijection between orbits of $J([a] \times [b])$ under Pro and $J([a] \times [b])$ under Row which preserves the cardinality of the order ideals. This result then follows immediately from Theorem 1.3.2. $\square$

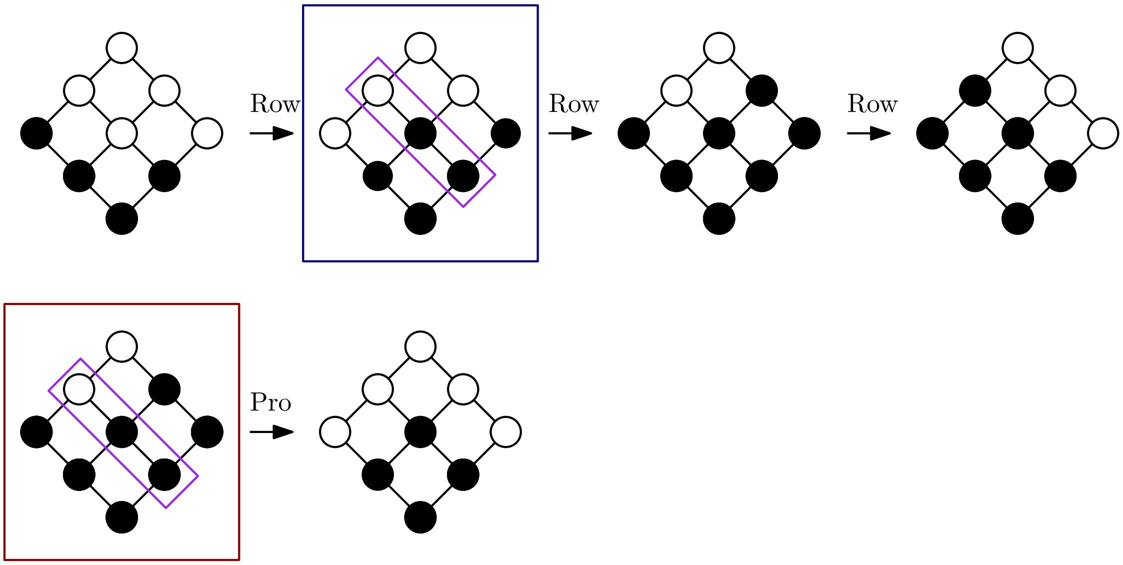


Figure 2.3. The boxed purple layers correspond under recombination. In Example 2.1.9, we demonstrate the idea of the proof using the order ideals in the large blue and red boxes.

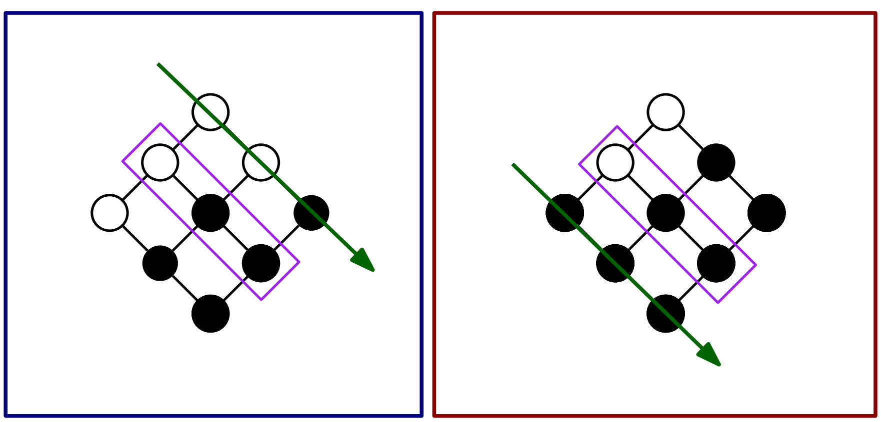


Figure 2.4. When performing Row on the left order ideal, L_1^3 is toggled first in the direction indicated. When performing Pro on the right order ideal, L_1^1 is toggled first in the direction indicated.

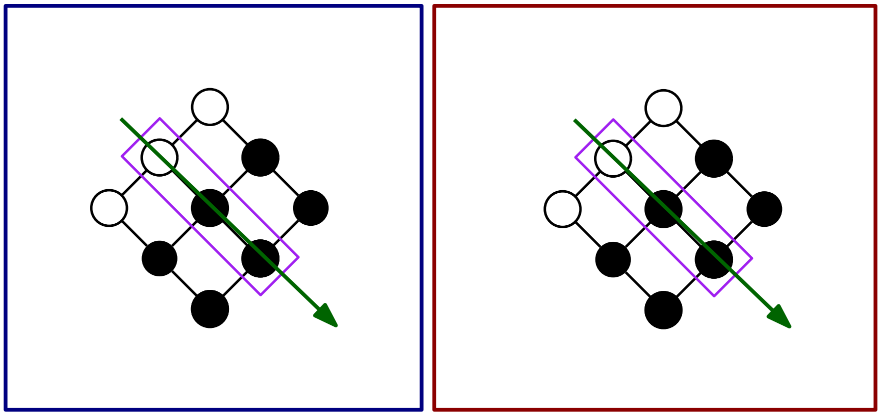


Figure 2.5. After performing the toggles from Figure 2.4, both order ideals now have $L_1^3(\mathrm{Row}^2(I))$ above the boxed purple layer and have $L_1^1(\mathrm{Row}(I))$ below the boxed purple layer. Therefore, when performing toggles on the purple layer, the three layers are the same.

2.2. Higher dimensional recombination

In order to prove Theorem 2.0.1, we will define the notion of recombination for a product of chains in full generality.

Definition 2.2.1. Let $P = [a_1] \times \cdots \times [a_n]$ and $I \in J(P)$. Define $\Delta_v^\gamma I = \cup_j L_\gamma^j(\mathrm{Pro}_v^{j-1}(I))$ where $\gamma \in \{1, \ldots, n\}$. We will call $\Delta_v^\gamma I$ the $(v, \gamma)-$*recombination* of I. When context is clear, we will suppress the (v, γ).

The idea behind recombination is the same as in the two-dimensional case: we take one layer from each order ideal in a sequence of order ideals from a promotion orbit to form the layers of a new order ideal. See Figure 2.6 for an example. In addition to generalizing recombination to n dimensions, we also generalize Proposition 2.1.5 to n dimensions. This is a toggle commutation result motivated by recombination, as we will make use of it when proving the recombination results that follow.

Lemma 2.2.2. *Let $P = [a_1] \times \cdots \times [a_n]$ and $\gamma \in \{1, 2, \ldots, n\}$. Then* $\mathrm{Pro}_v = \prod_{j=1}^{a_\gamma} T^{\alpha}_{\mathrm{Pro}_{v\widehat{\gamma}}}$ *where*

$$\alpha = \begin{cases} j & \text{if } v_\gamma = 1 \\ a_\gamma + 1 - j & \text{if } v_\gamma = -1. \end{cases}$$

Proof. Suppose $x := (x_1, \ldots, x_n), y := (y_1, \ldots, y_n) \in P$ with $x \in L_\gamma^j$ and $y \in L_\gamma^k$ for some j and k. We want to show that x and y are toggled in the same relatve order in Pro_v and $\prod_{j=1}^{a_\gamma} T^{\alpha}_{\mathrm{Pro}_{v\widehat{\gamma}}}$.

Case $j \neq k$**:** Without loss of generality, $j > k$. Furthermore, we can assume $x_\gamma = y_\gamma + 1$ and $x_i = y_i$ for $i \neq \gamma$. If this was not the case, x and y could not have a covering relation and we could commute the toggles.

If $v_\gamma = 1$, in $\prod_{j=1}^{a_\gamma} T^{\alpha}_{\mathrm{Pro}_{v^{\widehat{\gamma}}}}$, x is toggled before y by definition. Additionally,

$$\langle x, v\rangle = v_1x_1 + \cdots + v_\gamma x_\gamma + \cdots + v_nx_n > v_1y_1 + \cdots + v_\gamma y_\gamma + \cdots + v_ny_n = \langle y, v\rangle$$

and so x is toggled before y in Pro_v.

If $v_\gamma = -1$, in $\prod_{j=1}^{a_\gamma} T^{\alpha}_{\mathrm{Pro}_{v^{\widehat{\gamma}}}}$, y is toggled before x by definition. Additionally,

$$\langle x, v\rangle = v_1x_1 + \cdots + v_\gamma x_\gamma + \cdots + v_nx_n < v_1y_1 + \cdots + v_\gamma y_\gamma + \cdots + v_ny_n = \langle y, v\rangle$$

and so y is toggled before x in Pro_v.

Case $j = k$**:** In other words, $x_\gamma = y_\gamma$. Therefore,

$$\begin{aligned}\langle x, v\rangle > \langle y, v\rangle &\iff v_1x_1 + \cdots + v_\gamma x_\gamma + \cdots + v_nx_n > v_1y_1 + \cdots + v_\gamma y_\gamma + \cdots + v_ny_n \\ &\iff v_1x_1 + \cdots + v_{\gamma-1}x_{\gamma-1} + v_{\gamma+1}x_{\gamma+1} + \cdots + v_nx_n > \\ &\qquad v_1y_1 + \cdots + v_{\gamma-1}y_{\gamma-1} + v_{\gamma+1}y_{\gamma+1} + \cdots + v_ny_n \\ &\iff \langle x^{\widehat{\gamma}}, v^{\widehat{\gamma}}\rangle > \langle y^{\widehat{\gamma}}, v^{\widehat{\gamma}}\rangle\end{aligned}$$

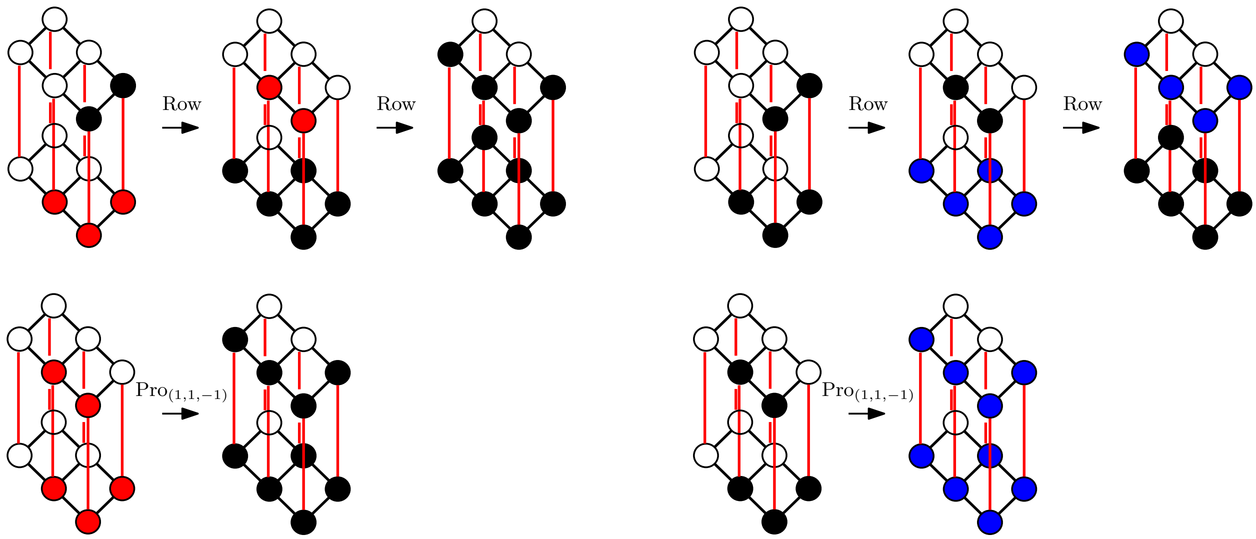
where $x^{\widehat{\gamma}}, y^{\widehat{\gamma}}$ are x and y with x_γ and y_γ deleted, respectively. Therefore, x can be toggled before y in Pro_v if and only if x can be toggled before y in $\prod_{j=1}^{a_\gamma} T^{\alpha}_{\mathrm{Pro}_{v^{\widehat{\gamma}}}}$. □

In other words, if we want to apply Pro_v, we can commute our toggles to toggle by layers of the form L^j_γ instead of using the toggle order given in Definition 1.2.22. More specifically, if $v_\gamma = 1$, we toggle in the order of $L^{a_\gamma}_\gamma, L^{a_\gamma-1}_\gamma, \ldots, L^1_\gamma$. If $v_\gamma = -1$, we toggle in the order of $L^1_\gamma, L^2_\gamma, \ldots, L^{a_\gamma}_\gamma$. This means that any promotion can be thought of as sequence of $n-1$ dimensional promotions on the layers of our product of chains poset.

Now that we have established n-dimensional recombination and toggle commutation, we determine conditions under which n-dimensional recombination results in an order ideal.

Lemma 2.2.3. *Let $I \in J([a_1] \times \cdots \times [a_n])$. Suppose we have v and γ such that $v_\gamma = 1$. Then $\Delta^\gamma_v I$ is an order ideal of P.*

Proof. Suppose $(i_1, \ldots, i_n) \in \Delta^\gamma_v I$. By Definition 2.2.1, $(i_1, \ldots, i_j - 1, \ldots, i_n) \in \Delta^\gamma_v I$ for $j \neq \gamma$ as these are obtained from the same layer as $(i_1, \ldots, i_n)$. To show that $\Delta^\gamma_v I$ is an order ideal, it suffices to show $(i_1, \ldots, i_\gamma - 1, \ldots, i_n) \in \Delta^\gamma_v I$ for $i_\gamma \geq 2$; if $i_\gamma = 1$ there is nothing to show. Because



(a) From an orbit of Row, we use $L_3^1(I)$ and $L_3^2(\mathrm{Row}(I))$ to form a new order ideal, denoted here in red.

(b) From the same orbit of Row, we use $L_3^1(\mathrm{Row}(I))$ and $L_3^2(\mathrm{Row}^2(I))$ to form a new order ideal, denoted here in blue.

Figure 2.6. Performing $\mathrm{Pro}_{(1,1,-1)}$ on the red order ideal results in the blue order ideal.

$(i_1, \ldots, i_n) \in \Delta_v^\gamma I$, we have $(i_1, \ldots, i_n) \in L_\gamma^{i_\gamma}(\mathrm{Pro}_v^{i_\gamma - 1}(I))$. By Lemma 2.2.2, $\mathrm{Pro}_v = \prod_{j=1}^{a_\gamma} T^j_{\mathrm{Pro}_{v^{\widehat{\gamma}}}}$, which implies we can commute the toggle relations in Pro_v so that $L_\gamma^{i_\gamma}$ is toggled before $L_\gamma^{i_\gamma - 1}$. As a result, we must have $(i_1, \ldots, i_\gamma - 1, \ldots, i_n) \in L_\gamma^{i_\gamma - 1}(\mathrm{Pro}_v^{i_\gamma - 2}(I))$. Therefore, $(i_1, \ldots, i_\gamma - 1, \ldots, i_n) \in \Delta_v^\gamma I$. $\square$

We now state our second main result, which shows how recombination relates different promotion actions. This result will allow us to prove Theorem 2.0.1.

Theorem 2.2.4. *Let $I \in J([a_1] \times \cdots \times [a_n])$. Suppose we have $v = (v_1, v_2, \ldots, v_n)$ where $v_j \in \{\pm 1\}$, $u = (u_1, u_2, \ldots, u_n)$ where $u_j \in \{\pm 1\}$, and γ such that $v_\gamma = 1$, $u_\gamma = -1$, and $v^{\widehat{\gamma}} = u^{\widehat{\gamma}}$. Then $\mathrm{Pro}_u(\Delta_v^\gamma I) = \Delta_v^\gamma(\mathrm{Pro}_v(I))$.*

Proof. First, note that $\Delta_v^\gamma I$ is an order ideal by Lemma 2.2.3. Also note that $\mathrm{Pro}_v = \prod_{j=1}^{a_\gamma} T^j_{\mathrm{Pro}_{v^{\widehat{\gamma}}}}$ and $\mathrm{Pro}_u = \prod_{j=1}^{a_\gamma} T^{a_\gamma + 1 - j}_{\mathrm{Pro}_{u^{\widehat{\gamma}}}}$ by Lemma 2.2.2. We will show $\mathrm{Pro}_u(\Delta_v^\gamma I)=\Delta_v^\gamma(\mathrm{Pro}_v(I))$ by showing $L_\gamma^k(\mathrm{Pro}_u(\Delta_v^\gamma I)) = L_\gamma^k(\Delta_v^\gamma(\mathrm{Pro}_v(I)))$ for each $k \in \{1, 2, \ldots, a_\gamma\}$. There are three cases.

Case $1 < k < a_\gamma$: Let $J = \mathrm{Pro}_v^{k-1}(I)$. We can commute the toggles of Pro_v so that L_γ^{k+1} of J is toggled before L_γ^k of J, which is toggled before L_γ^{k-1} of J. Thus, when applying the toggles of Pro_v to L_γ^k of J, the layer above is $L_\gamma^{k+1}(\mathrm{Pro}_v(J))$ whereas the layer below is $L_\gamma^{k-1}(J)$. Additionally, we can also commute the toggles of Pro_u so L_γ^{k-1} of $\Delta_v^\gamma I$ is toggled before L_γ^k of $\Delta_v^\gamma I$, which is toggled before L_γ^{k+1} of $\Delta_v^\gamma I$. Therefore, when applying the toggles of Pro_u to L_γ^k of $\Delta_v^\gamma I$, the layer below is $L_\gamma^{k-1}(\mathrm{Pro}_u(\Delta_v^\gamma I))$, whereas the layer above is $L_\gamma^{k+1}(\Delta_v^\gamma I)$. However, $L_\gamma^{k-1}(\mathrm{Pro}_u(\Delta_v^\gamma I)) = L_\gamma^{k-1}(J)$, $L_\gamma^k(\Delta_v^\gamma I) = L_\gamma^k(J)$, and $L_\gamma^{k+1}(\Delta_v^\gamma I) = L_\gamma^{k+1}(\mathrm{Pro}_v(J))$. Therefore, when applying Pro_v to L_γ^k of J and Pro_u to L_γ^k of $\Delta_v^\gamma I$, both layers are the same and have the same layers above and below them. Because $u^{\widehat{\gamma}} = v^{\widehat{\gamma}}$, we have $\mathrm{Pro}_{u^{\widehat{\gamma}}} = \mathrm{Pro}_{v^{\widehat{\gamma}}}$ and so the result of toggling this layer is $L_\gamma^k(\mathrm{Pro}_u(\Delta_v^\gamma I))$, which is the same as $L_\gamma^k(\mathrm{Pro}_v(J)) = L_\gamma^k(\mathrm{Pro}_v^k(I)) = L_\gamma^k(\Delta_v^\gamma(\mathrm{Pro}_v(I)))$.

Case $k = 1$: As above, when applying Pro_v to L_γ^1 of I and Pro_u to L_γ^1 of $\Delta_v^\gamma I$, both of these layers are the same, along with the layers above them. Because $k = 1$, there is not a layer below. As above, $\mathrm{Pro}_{u^{\widehat{\gamma}}} = \mathrm{Pro}_{v^{\widehat{\gamma}}}$ and so we again obtain $L_\gamma^1(\mathrm{Pro}_u(\Delta_v^\gamma I)) = L_\gamma^1(\Delta_v^\gamma(\mathrm{Pro}_v(I)))$.

Case $k = a_\gamma$: Again, as above, when applying Pro_v to $L_\gamma^{a_\gamma}$ of J and Pro_u to $L_\gamma^{a_\gamma}$ of $\Delta_v^\gamma I$, both of these layers are the same along with the layers below them. Because $k = a_\gamma$ there is not a layer above. Again, $\mathrm{Pro}_{u^{\widehat{\gamma}}} = \mathrm{Pro}_{v^{\widehat{\gamma}}}$ and so $L_\gamma^{a_\gamma}(\mathrm{Pro}_u(\Delta_v^\gamma I)) = L_\gamma^{a_\gamma}(\Delta_v^\gamma(\mathrm{Pro}_v(I)))$. □

Example 2.2.5. We give a three-dimensional example of the proof technique for the $k = 1$ case. Note that this is similar to the two-dimensional example from Example 2.1.9. We start with the partial orbits under Row and $\mathrm{Pro}_{(1,1,-1)}$ from Figure 2.6. As before, I denotes the first order ideal in the orbit of Row. In Figure 2.7, the purple layers L_3^1 in the blue and red boxes correspond under recombination. We want to verify that after applying Row to the order ideal in the top, blue box, the layer L_3^1 of the result is the same as the layer when we apply $\mathrm{Pro}_{(1,1,-1)}$ to the order ideal in the bottom, red box. When applying Row, we can commute the toggles so we first toggle the layer L_3^2 from top to bottom, then L_3^1 from top to bottom. On the other hand, when applying $\mathrm{Pro}_{(1,1,-1)}$, we can commute the toggles so we first toggle the layer L_3^1 from top to bottom, then L_3^2 from top to bottom. See Figure 2.8. As a result, when applying the toggles of Row to the purple layer L_3^1, the layer above L_3^2 has already been toggled and is therefore the layer $L_3^2(\mathrm{Row}(I))$. However, by construction, this is the layer $L_3^2(\Delta_{(1,1,1)}^3 I)$ from the recombination order ideal in the red box. Hence, in Figure 2.9, the layer L_3^2 above the purple layer L_3^1 is the same for both order ideals. Using similar reasoning, if there was a layer below the purple layer, these would also be the same. Because the purple layer L_3^1 is toggled in the same direction in both order ideals and the layer above is the same for both order ideals, the result of toggling this layer is the same for both order ideals.

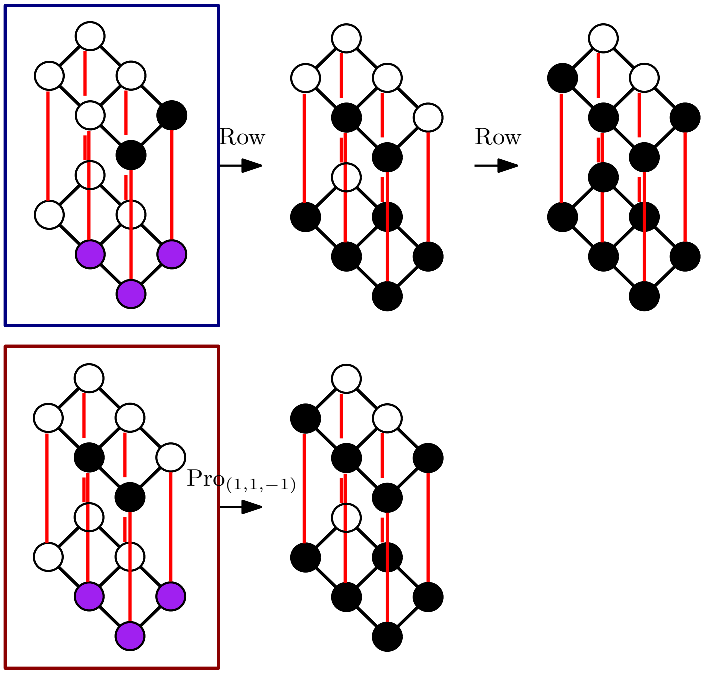


Figure 2.7. The purple layers L_3^1 in the blue and red boxes correspond under recombination.

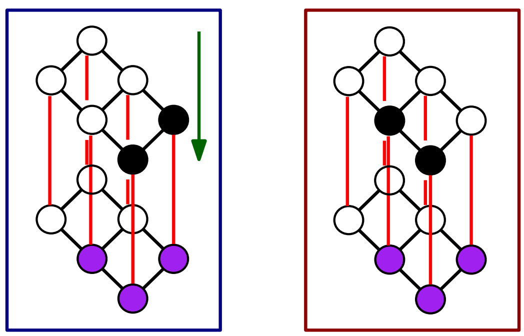


Figure 2.8. When performing Row on the left figure, L_3^2 is toggled first from top to bottom. When performing $\mathrm{Pro}_{(1,1,-1)}$ on the right figure, L_3^1 is the first layer toggled; in other words, there is no layer toggled before the purple layer.

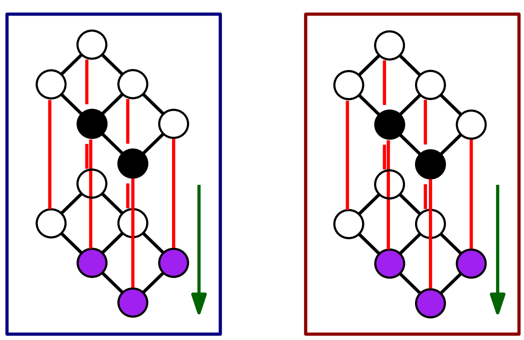


Figure 2.9. After performing the toggles from Figure 2.8, both order ideals now have $L_3^2(\mathrm{Row}(I))$ above the purple layer. Therefore, when performing toggles on the purple layer, the layer above is the same, so the result of toggling the purple layer from top to bottom is the same.

We have three immediate corollaries that will be useful in the proof of Theorem 2.0.1.

Corollary 2.2.6. $\mathrm{Pro}_{(1,1,-1)}(\Delta^3_{(1,1,1)}I) = \Delta^3_{(1,1,1)}(\mathrm{Pro}_{(1,1,1)}(I))$.

Proof. $v = (1,1,1)$, $u = (1,1,-1)$, and $\gamma = 3$ satisfy the assumptions of Theorem 2.2.4. □

Corollary 2.2.7. $\mathrm{Pro}_{(-1,1,-1)}(\Delta^1_{(1,1,-1)}I) = \Delta^1_{(1,1,-1)}(\mathrm{Pro}_{(1,1,-1)}(I))$.

Proof. $v = (1,1,-1)$, $u = (-1,1,-1)$, and $\gamma = 1$ satisfy the assumptions of Theorem 2.2.4. □

Corollary 2.2.8. $\mathrm{Pro}_{(1,-1,-1)}(\Delta^2_{(1,1,-1)}I) = \Delta^2_{(1,1,-1)}(\mathrm{Pro}_{(1,1,-1)}(I))$.

Proof. $v = (1,1,-1)$, $u = (1,-1,-1)$, and $\gamma = 2$ satisfy the assumptions of Theorem 2.2.4. □

Note that recombination gives us a bijection between orbits of order ideals under different promotion actions. Suppose v and u are as in Theorem 2.2.4. If we find the recombination of each order ideal in an orbit of Pro_v, we obtain a sequence of order ideals that form an orbit under Pro_u.

Remark 2.2.9. Let u, v be as in Theorem 2.2.4 and let $\mathcal{O}$ be an orbit of order ideals in $J([a_1] \times \cdots \times [a_n])$ under Pro_u. There is a unique orbit $\mathcal{O}'$ under Pro_v where the recombination of $\mathcal{O}'$ is $\mathcal{O}$. In other words, if we start with an orbit under Pro_u, we can invert recombination to get an orbit under Pro_v. For example, if we start with an orbit of $J([2] \times [a] \times [b])$ under $\mathrm{Pro}_{(-1,1,-1)}$, we can acquire an orbit of $J([2] \times [a] \times [b])$ under $\mathrm{Pro}_{(1,1,-1)}$.

This observation will be used to show $J([2] \times [a] \times [b])$ exhibits homomesy under $\mathrm{Pro}_{(-1,1,-1)}$ and $\mathrm{Pro}_{(1,-1,-1)}$.

2.3. Proving the main homomesy result

To prove Theorem 2.0.1, we relate the order ideals of our posets to increasing tableaux. To do so, we first need a map from $J([a] \times [b] \times [c])$ to increasing tableaux defined by Dilks, Pechenik, and Striker. Recall the definitions of increasing tableaux and K-Pro from Definition 1.2.17 and Definition 1.2.18, respectively.

Definition 2.3.1 ([11])**.** Define a map $\Psi : J([a] \times [b] \times [c]) \to \mathrm{Inc}^{a+b+c-1}(a \times b)$ in the following way. Let $I \in J([a] \times [b] \times [c])$. We can view I as a pile of cubes in an $a \times b \times c$ box; we then project onto the $a \times b$ face. More specifically, record in position (i, j) the number of boxes of I with coordinate (i, j, k) for some $0 \leq k \leq c - 1$. This results in a filling of a Young diagram of shape $a \times b$ with nonnegative entries that weakly decrease from left to right and top to bottom. By rotating the diagram 180°, our Young diagram is now weakly increasing in rows and columns. Now increase each label by one more than the distance to the upper left corner box. This results in an increasing tableau, which we denote $\Psi(I)$.

Along with defining Ψ, Dilks, Pechenik, and Striker also showed that Ψ intertwines $\mathrm{Pro}_{(1,1,-1)}$ and K-Pro.

Theorem 2.3.2 ([11])**.** *Ψ is an equivariant bijection between $J([a] \times [b] \times [c])$ under $\mathrm{Pro}_{(1,1,-1)}$ and $\mathrm{Inc}^{a+b+c-1}(a \times b)$ under K-Pro.*

Furthermore, we can relate the cardinality of I to the sum of the entries in $\Psi(I)$.

Lemma 2.3.3. *If $I \in J([2] \times [a] \times [b])$, the sum of the boxes in $\Psi(I)$ is equal to $f(I) + a(a+2)$ where f is the cardinality statistic.*

Proof. This follows from the definition of Ψ and the shape of $\Psi(I)$. $\square$

As a result of this lemma, if we can find an appropriate homomesy result on increasing tableaux, we can transfer the result to $J([2] \times [a] \times [b])$ under $\mathrm{Pro}_{(1,1,-1)}$ using Ψ, then to $J([2] \times [a] \times [b])$ under Row using Corollary 2.2.6. As it turns out, the appropriate homomesy result has already been discovered by J. Bloom, Pechenik, and D. Saracino.

Theorem 2.3.4 ([4])**.** *Let λ be a $2 \times n$ rectangle for any n, let $\mu \subseteq \lambda$ be a set of elements fixed under 180° rotation, and let σ_μ be the statistic of summing the entries in the boxes of μ. Then for any q, $(\mathrm{Inc}^q(\lambda)$, K-Pro, $\sigma_\mu)$ is homomesic.*

Note that the entire $2 \times n$ rectangle is fixed under 180° rotation. Moreover, for $I \in J([2] \times [a] \times [b])$, $\Psi(I)$ is an increasing tableau of shape $2 \times a$. With this theorem, we have sufficient machinery to prove Theorem 2.0.1.

Proof of Theorem 2.0.1. Each orbit of $J([2] \times [a] \times [b])$ under $\mathrm{Pro}_{(1,1,-1)}$ corresponds under Ψ to an orbit of $\mathrm{Inc}^{a+b+1}(\lambda)$ under K-Pro. Because Lemma 2.3.3 shows that the box sum of an increasing tableau differs by a constant with the cardinality of the corresponding order ideal, we can translate the increasing tableaux homomesy of Theorem 2.3.4 to the setting of $J([2] \times [a] \times [b])$. In other words, this shows that $J([2]\times[a]\times[b])$ under $\mathrm{Pro}_{(1,1,-1)}$ with cardinality statistic exhibits homomesy. Moreover, $\mathrm{Pro}_{(-1,-1,1)}$ reverses the direction that our hyperplanes sweep through our poset, which merely reverses our orbits of order ideals. As a result, we may conclude that $J([2]\times[a]\times[b])$ also exhibits homomesy under $\mathrm{Pro}_{(-1,-1,1)}$. To prove Theorem 2.0.1 for the remaining v, we begin with $v=(1,1,1)$, which is Row.

Let $\mathcal{O}_1, \mathcal{O}_2$ be orbits of $J([2]\times[a]\times[b])$ under Row. Additionally, let $R_1 = \{\Delta^3_{(1,1,1)} I : I \in \mathcal{O}_1\}$ and $R_2 = \{\Delta^3_{(1,1,1)} I : I \in \mathcal{O}_2\}$ be the corresponding recombination orbits. Because R_1 and R_2 are orbits under $\mathrm{Pro}_{(1,1,-1)}$, by Corollary 2.2.6 the average of the cardinality statistic over R_1 and R_2 must be equal. As a result, the average of the cardinality statistic over $\mathcal{O}_1$ and $\mathcal{O}_2$ must be equal. Hence, $J([2]\times[a]\times[b])$ is homomesic under Row. Again, because $\mathrm{Pro}_{(-1,-1,-1)}$ merely reverses the direction of hyperplane toggles, we conclude that $J([2]\times[a]\times[b])$ is homomesic under $\mathrm{Pro}_{(-1,-1,-1)}$.

We now turn our attention to $\mathrm{Pro}_{(-1,1,-1)}$ and $\mathrm{Pro}_{(1,-1,-1)}$. Using our recombination results in Corollaries 2.2.7 and 2.2.8, we can connect the cardinality of orbits of $J([2]\times[a]\times[b])$ under $\mathrm{Pro}_{(-1,1,-1)}$ and $\mathrm{Pro}_{(1,-1,-1)}$ to orbits of $J([2]\times[a]\times[b])$ under $\mathrm{Pro}_{(1,1,-1)}$. Therefore, because $J([2]\times[a]\times[b])$ under $\mathrm{Pro}_{(1,1,-1)}$ with cardinality statistic exhibits homomesy, we see $J([2]\times[a]\times[b])$ exhibits homomesy under both $\mathrm{Pro}_{(-1,1,-1)}$ and $\mathrm{Pro}_{(1,-1,-1)}$ as well. Furthermore, $\mathrm{Pro}_{(1,-1,1)}$ and $\mathrm{Pro}_{(-1,1,1)}$ reverse the orbits of $\mathrm{Pro}_{(-1,1,-1)}$ and $\mathrm{Pro}_{(1,-1,-1)}$, respectively, so $J([2]\times[a]\times[b])$ is homomesic under both $\mathrm{Pro}_{(1,-1,1)}$ and $\mathrm{Pro}_{(-1,1,1)}$.

We have shown the desired triples are homomesic, but we still must show the orbit average is ab. Due to rotational symmetry, the order filters of $J([2]\times[a]\times[b])$ are in bijection with the order ideals of $J([2]\times[a]\times[b])$. More specifically, let $I \in J([2]\times[a]\times[b])$. Let $H \in J([2]\times[a]\times[b])$ be the order ideal isomorphic to $P \setminus I$. Therefore, $f(I)+f(H)=2ab$. As a result, we can say the global average of f is ab, and hence c must also be ab. □

We obtain the following corollaries by symmetry.

Corollary 2.3.5. *Let f be the cardinality statistic. The triple $(J([a]\times[2]\times[b]), \mathrm{Pro}_v, f)$ is c-mesic with $c=ab$.*

Corollary 2.3.6. *Let f be the cardinality statistic. The triple $(J([a]\times[b]\times[2]), \mathrm{Pro}_v, f)$ is c-mesic with $c=ab$.*

Proof of Corollaries 2.3.5 and 2.3.6. Given an orbit $\mathcal{O}$ of $J([a]\times[2]\times[b])$ under Pro_v, we can use a cyclic rotation of coordinates and appropriate choice of v' to obtain an orbit $\mathcal{O}'$ of $J([2]\times[a]\times[b])$ under $\mathrm{Pro}_{v'}$ such that $\mathcal{O}$ and $\mathcal{O}'$ are in bijection. A similar argument applies to $J([a]\times[b]\times[2])$. $\square$

2.4. General products of chains

We conclude the section by determining that Theorem 2.0.1 does not generalize to an arbitrary product of three chains, a product of four chains, or a product of arbitrarily many two-element chains. Homomesy holds on order ideals of $[3]\times[3]\times[3]$ under Pro_v with cardinality statistic; however, this is not the case with order ideals of $[3]\times[3]\times[4]$.

Proposition 2.4.1. *Let f be the cardinality statistic. The triple $(J([3]\times[3]\times[3]),\mathrm{Pro}_v,f)$ is homomesic with $c=27/2$. However, the triple $(J([3]\times[3]\times[4]),\mathrm{Pro}_v,f)$ is not homomesic.*

Proof. A calculation using SageMath [38] shows that $J([3]\times[3]\times[3])$ under Row has 124 orbits, all with average cardinality 27/2. However, $J([3]\times[3]\times[4])$ under Row has 456 orbits with average cardinality 18, 2 orbits with average cardinality $161/9\approx 17.89$, and 2 orbits with average cardinality $163/9\approx 18.11$. Using recombination, we obtain the same result for any Pro_v. $\square$

We can further inquire about homomesy in higher dimensions. We find homomesy in the poset $[2]\times[2]\times[2]\times[2]$, but a negative result if any of the chains have size three. If we use only chains of size two, homomesy fails in dimension five.

Proposition 2.4.2. *Let f be the cardinality statistic. The triple $J([2]\times[2]\times[2]\times[2]),\mathrm{Pro}_v,f)$ is c-mesic with $c=8$. However, the triple $(J([2]\times[2]\times[2]\times[3]),\mathrm{Pro}_v,f)$ is not homomesic. Additionally, the triple $(J([2]\times[2]\times[2]\times[2]\times[2]),\mathrm{Pro}_v,f)$ is not homomesic.*

Proof. A calculation using SageMath [38] shows that $J([2]\times[2]\times[2]\times[2])$ under Row has 36 orbits, all with average cardinality 8. However, $J([2]\times[2]\times[2]\times[3])$ has 109 orbits with average cardinality 12, 6 orbits with average cardinality $82/7\approx 11.71$, and 6 orbits with average cardinality $86/7\approx 12.29$. Additionally, $J([2]\times[2]\times[2]\times[2]\times[2])$ has 771 orbits with average cardinality 16, 60 orbits with average cardinality $115/7\approx 16.43$, 60 orbits with average cardinality $109/7\approx 15.57$, 30 orbits with average cardinality $61/4=15.25$, 30 orbits with average cardinality $67/4=16.75$, 6 orbits with average cardinality 11, and 6 orbits with average cardinality 21. Using recombination, we once again obtain the same results for any Pro_v. $\square$

3. TABLEAUX AND REFINED RESULTS

In this chapter, we prove several related results and corollaries of the results in Chapter 2. Although Proposition 2.4.1 showed that cardinality does not exhibit homomesy with respect to promotion for order ideals of an arbitrary product of three chains, Corollary 3.2.4 gives a different statistic such that order ideals of a product of three chains under promotion do exhibit homomesy. Additionally, we use our main homomesy result, Theorem 2.0.1, to obtain a new homomesy result on increasing tableaux in Corollary 3.1.1. In Corollary 3.2.1, we use refined homomesy results on increasing tableaux to state more refined homomesy results on order ideals. Finally, we use results of Pechenik to obtain an antipodal homomesy result on $[2]\times[a]\times[b]$ in Theorem 3.3.4. The majority of this chapter is based on work from [46].

3.1. A corollary on increasing tableaux

For Theorem 2.0.1, we used the bijection Ψ^{-1} to translate a homomesy result on increasing tableaux to order ideals of a product of chains poset. Additionally, we used a cyclic rotation of the axes to obtain Corollary 2.3.6 on the product of chains $[a] \times [b] \times [2]$. From this corollary, we can translate back to increasing tableaux using Ψ to obtain an additional homomesy result on increasing tableaux. This is in the same spirit as the tri-fold symmetry used by Dilks, Pechenik, and Striker [11, Corollary 4.7].

Corollary 3.1.1. *Let λ be an $a \times b$ rectangle and let σ_λ be the statistic of summing the entries in the boxes of λ. Then $(\mathrm{Inc}^{a+b+1}(\lambda), K\text{-Pro}, \sigma_\lambda)$ is c-mesic with $c = \frac{ab(2+a+b)}{2}$.*

Proof. Each orbit of $\mathrm{Inc}^{a+b+1}(\lambda)$ under K-Pro corresponds to an orbit of $J([a] \times [b] \times [2])$ under $\mathrm{Pro}_{(1,1,-1)}$. For each $I \in J([a] \times [b] \times [2])$, $\sigma_\lambda(\Psi(I)) = f(I) + \frac{ab(a+b)}{2}$ where f is the cardinality statistic. Applying Corollary 2.3.6, the result follows. $\square$

Note that although this corollary is similar to the result of Bloom, Pechenik, and Saracino we stated as Theorem 2.3.4, the result is distinct as it applies to a larger class of shapes but is much more restrictive on the largest entry.

3.2. Refined column homomesy

In this section, we refine the homomesy result of Theorem 2.0.1 using the rotational symmetry condition of Theorem 2.3.4. Define columns $L_{1,2}^{j,k} = \{(i_1, i_2, i_3) \in [2] \times [a] \times [b] \mid i_1 = j, i_2 = k\}$. This notation is similar to the layer notation of Definition 2.1.2 with the exception that we fix two coordinates instead of one. To state this corollary, first recall Definition 1.3.8, the definition of antipodal elements in $[a] \times [b]$.

Corollary 3.2.1. *Let $L_{1,2}^{j_1,k_1}$ and $L_{1,2}^{j_2,k_2}$ be such that the coordinates (j_1, k_1) and (j_2, k_2) are antipodal in $[2] \times [a]$. If $f_L(I)$ denotes the cardinality of I on $L_{1,2}^{j_1,k_1}$ and $L_{1,2}^{j_2,k_2}$, then $(J([2] \times [a] \times [b]), \mathrm{Pro}_v, f_L)$ is c-mesic with $c = b$.*

Proof. The antipodal coordinates (j_1, k_1) and (j_2, k_2) are chosen so that the columns $L_{1,2}^{j_1,k_1}$ and $L_{1,2}^{j_2,k_2}$ correspond to a set of boxes in an increasing tableau fixed under 180° rotation. In other words, we can use the refined homomesy result on increasing tableaux from Theorem 2.3.4 and translate to $J([2] \times [a] \times [b])$ using the bijection Ψ^{-1}. As a result, we know $(J([2] \times [a] \times [b]), \mathrm{Pro}_v, f_L)$ is c-mesic. What remains to be shown is that $c = b$. Due to rotational symmetry, the order filters of $[2] \times [a] \times [b]$ are in bijection with the order ideals of $[2] \times [a] \times [b]$. More specifically, let $I \in J([2] \times [a] \times [b])$. Let $H \in J([2] \times [a] \times [b])$ be the order ideal isomorphic under rotation to the order filter $P \setminus I$. Therefore, $f_L(I) + f_L(H) = 2b$. As a result, we can say the global average of f_L is b, and hence c must also be b. This gives us that $(J([2] \times [a] \times [b]), \mathrm{Pro}_{(1,1,-1)}, f_L)$ is c-mesic with $c = b$; using recombination we obtain that $(J([2] \times [a] \times [b]), \mathrm{Pro}_v, f_L)$ is c-mesic with $c = b$. $\square$

Pechenik further generalized the results of [4] and the result stated in Theorem 2.3.4. From this, we obtain a more general analogue of Corollary 3.2.1. We summarize the relevant definition and theorem below.

Definition 3.2.2 ([28])**.** The *frame* of a partition λ is the set **Frame**(λ) of all boxes in the first or last row, or in the first or last column of λ.

Theorem 3.2.3 ([28])**.** *Let S be a subset of **Frame**$(m \times n)$ that is fixed under 180° rotation. Then $(\mathrm{Inc}^q(m \times n)$, K-Pro, $\sigma_S)$ is c-mesic with $c = \frac{(q+1)|S|}{2}$.*

The following is a new corollary of Theorem 3.2.3. It uses the bijection Ψ^{-1} and techniques similar to those of Corollary 3.2.1 to prove a more general analogue of Corollary 3.2.1 in the product of three chains.

Corollary 3.2.4. *Let $P = [a_1] \times [a_2] \times [a_3]$. Additionally, let $L_{1,2}^{j_1,k_1}$ and $L_{1,2}^{j_2,k_2}$ be such that the coordinates (j_1, k_1) and (j_2, k_2) are antipodal in $[a_1] \times [a_2]$ where each j_i is 1 or a_1 and each k_i is 1 or a_2. If $f_L(I)$ denotes the cardinality of I on $L_{1,2}^{j_1,k_1}$ and $L_{1,2}^{j_2,k_2}$, then $(J([a_1] \times [a_2] \times [a_3]), \mathrm{Pro}_v, f_L)$ is c-mesic with $c = a_3$.*

Proof. Similarly to the proof of Corollary 3.2.1, the antipodal coordinates (j_1, k_1) and (j_2, k_2) are chosen so that the columns $L_{1,2}^{j_1,k_1}$ and $L_{1,2}^{j_2,k_2}$ correspond to a set of boxes in an increasing tableau fixed under 180° rotation. Additionally, the columns correspond to boxes in the frame of the tableau. As a result, we know $(J([a_1] \times [a_2] \times [a_3]), \mathrm{Pro}_v, f_L)$ is c-mesic by translating the refined homomesy result on increasing tableaux from Theorem 3.2.3 to $J([a_1] \times [a_2] \times [a_3])$ using the bijection Ψ^{-1}. We must now show that $c = a_3$. Due to rotational symmetry, the order filters of P are in bijection with the order ideals of P. Let $I \in J(P)$ and let $H \in J(P)$ be the order ideal isomorphic under rotation to the order filter $P \setminus I$. Because the two columns $L_{1,2}^{j_1,k_1}$ and $L_{1,2}^{j_2,k_2}$ each contain a_3 elements, $f_L(I) + f_L(H) = 2a_3$. Therefore, the global average of f_L is a_3 and as a result, $c = a_3$. This gives the result for $v = (1, 1, -1)$; using recombination we obtain the result for all v. □

3.3. Refined antipodal homomesy

Corollary 3.2.1 is the most natural way to obtain a refined homomesy result from Theorem 2.3.4. However, there is a stronger homomesy result on *antipodal* elements in $[2] \times [a] \times [b]$. In other words, Theorem 1.3.9 generalizes to $[2] \times [a] \times [b]$. We define antipodal elements in $[a] \times [b] \times [c]$ in an analogous way to Definition 1.3.8, which specifies antipodal elements in $[a] \times [b]$.

Definition 3.3.1. Let x and y be elements in $[a] \times [b] \times [c]$. If $x = (x_1, x_2, x_3)$ and $y = (a + 1 - x_1, b + 1 - x_2, c + 1 - x_3)$, x and y are *antipodal* in $[a] \times [b] \times [c]$.

To study antipodal elements in $[2] \times [a] \times [b]$, we again use the bijection Ψ to increasing tableaux from Definition 2.3.1. We also use results of Pechenik on increasing tableaux.

Definition 3.3.2. Let $T \in \mathrm{Inc}^q(\lambda)$ and B a box in λ. Let $\mathrm{Dist}(T, B)$ denote the multiset of values B attains in an orbit of K-Pro. Additionally, let $\mathrm{arDist}(T, B)$ denote the alphabet reversal of $\mathrm{Dist}(T, B)$, that is, the multiset of values $q + 1 - b$ for every $b \in \mathrm{Dist}(T, B)$.

For an example of these definitions, see Example 3.3.5. We can now state the following result of Pechenik.

Lemma 3.3.3 ([29]). *Let $T \in \mathrm{Inc}^q(2 \times a)$. Let B and B^* be boxes in $2 \times a$ such that B^* is the box 180° rotated from B. Then* $\mathrm{Dist}(T, B) = \mathrm{arDist}(T, B^*)$.

Proof. Recall K-evacuation $\mathcal{E}(T)$ on an increasing tableaux T from Definition 1.2.20. In [4], Bloom, Pechenik, Saracino showed that $\mathrm{Dist}(T, B) = \mathrm{Dist}(\mathcal{E}(T), B)$. Additionally, in [27], Pechenik showed that if $T \in \mathrm{Inc}^q(2 \times a)$, then $\mathcal{E}$ performs a 180° rotation of T with alphabet reversal. As a result, we obtain $\mathrm{Dist}(T, B) = \mathrm{arDist}(T, B^*)$. $\square$

With Pechenik's result on rotationally symmetric boxes under K-Pro, we can now show the homomesy result on antipodal elements of $[2] \times [a] \times [b]$.

Theorem 3.3.4. *Suppose x and y are antipodal elements in $[2] \times [a] \times [b]$. Then $(J([2] \times [a] \times [b]), \mathrm{Pro}_v, 1_x + 1_y)$ is c-mesic with $c = 1$.*

Proof. Let $L_{1,2}^{x_1,x_2}, L_{1,2}^{y_1,y_2}$ be the two columns containing $x = (x_1, x_2, x_3)$ and $y = (y_1, y_2, y_3)$ and let B and B^* be the boxes in $2 \times a$ corresponding with $L_{1,2}^{x_1,x_2}$ and $L_{1,2}^{y_1,y_2}$, respectively, under Ψ. Because x and y are antipodal elements in $[2] \times [a] \times [b]$, B^* can be obtained by a 180° rotation from B. Let $I \in J([2] \times [a] \times [b])$ and $\mathcal{O}$ be the orbit under $\mathrm{Pro}_{(1,1,-1)}$ containing I. By Lemma 3.3.3, $\mathrm{Dist}(\Psi(I), B) = \mathrm{arDist}(\Psi(I), B^*)$.

We now relate $\mathrm{Dist}(\Psi(I), B)$ and $\mathrm{arDist}(\Psi(I), B^*)$ to elements in I. If $x \in I$ but $(x_1, x_2, x_3 + 1) \notin I$, the value of the box in $\psi(I)$ corresponding to $L_{1,2}^{x_1,x_2}$ will be $3 + a - x_1 - x_2 + x_3$. This is because x_3 counts the number of elements of I in $L_{1,2}^{x_1,x_2}$ and $3 + a - x_1 - x_2$ adjusts for rotation and the increase in values along diagonals. Let $\alpha = 3 + a - x_1 - x_2 + x_3$. Note that $x \in I$ if and only if the value of B in $\Psi(I)$ is greater than or equal to α. Because x and y are antipodal in $[2] \times [a] \times [b]$, $y = (3 - x_1, a + 1 - x_2, b + 1 - x_3)$. Using the same reasoning as above, $y \in I$ if and only if the value of B^* in $\Psi(I)$ is greater than or equal to $3 + a - (3 - x_1) - (a + 1 - x_2) + (b + 1 - x_3) = x_1 + x_2 - x_3 + b$ if and only if the corresponding value in $\mathrm{arDist}(\Psi(I), B^*)$ is less than or equal to

$q+1-(x_1+x_2-x_3+b) = \alpha - 1$. Therefore, because $\mathrm{Dist}(\Psi(I), B) = \mathrm{arDist}(\Psi(I), B^*)$, each value that appears in these multisets signifies exactly one of x or y appears in I. Thus, the sum of the numbers of times x and y appear in orbit $\mathcal{O}$ is $\#\mathcal{O}$. As a result, $(J([2]\times[a]\times[b]), \mathrm{Pro}_{(1,1,-1)}, 1_x+1_y)$ is c-mesic with $c = 1$. Using recombination, we obtain $(J([2] \times [a] \times [b]), \mathrm{Pro}_v, 1_x + 1_y)$ is c-mesic with $c = 1$ for any v. □

Example 3.3.5. In Figure 3.2, we have an orbit of $J([2] \times [2] \times [2])$ under $\mathrm{Pro}_{(1,1,-1)}$ and the corresponding orbit of $\mathrm{Inc}^5(2 \times 2)$ under K-Pro. Observe the boxes B and B^* as indicated by Figure 3.1. If I denotes any of the order ideals in the orbit, then $\mathrm{Dist}(\Psi(I), B) = \{1, 1, 1, 3, 2\}$, $\mathrm{Dist}(\Psi(I), B^*) = \{3, 5, 5, 5, 4\}$, and $\mathrm{arDist}(\Psi(I), B^*) = \{3, 1, 1, 1, 2\}$. Observe the two antipodal elements circled in red in Figure 3.2. The top element is in an order ideal when box B has value greater than or equal to 2. The bottom element is not in an order ideal when box B^* has value less than or equal to $5 + 1 - 2 = 4$. Because $\mathrm{Dist}(\Psi(I), B) = \mathrm{arDist}(\Psi(I), B^*)$, $\#\mathrm{Dist}(\Psi(I), B)$ gives the cardinality of the antipodal elements over the orbit, which will always be $\#\mathcal{O}$. This yields an average of 1. Theorem 3.3.4 says that if we take any orbit of $J([2] \times [2] \times [2])$ under $\mathrm{Pro}_{(1,1,-1)}$, we also obtain an average of 1.

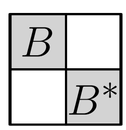


Figure 3.1. In Example 3.3.5, we focus on the shaded boxes B and B^* in $\mathrm{Inc}^5(2 \times 2)$.

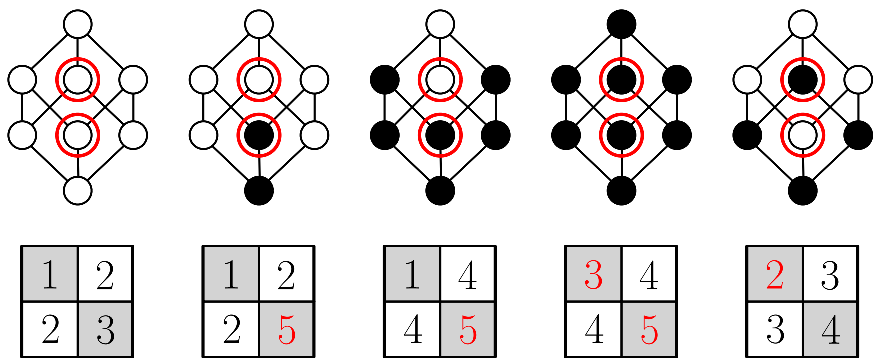


Figure 3.2. The multisets $\mathrm{Dist}(\Psi(I), B) = \{1, 1, 1, 3, 2\}$, $\mathrm{Dist}(\Psi(I), B^*) = \{3, 5, 5, 5, 4\}$ and $\mathrm{arDist}(\Psi(I), B^*) = \{3, 1, 1, 1, 2\}$ corresponding to the K-Pro orbit above. Here, a value is colored red if it corresponds to one of the circled elements being in an order ideal.

4. BEYOND THE PRODUCT OF CHAINS

In this chapter, we generalize the results of Chapter 2 beyond the setting of a product of chains to that of more general posets. We opted to state our recombination results in Chapter 2 for the product of chains rather than in full generality in order to emphasize the important aspects of the proofs without further complicating the notation. In Section 4.1, we generalize the definition of layers. We then generalize the work of Section 2.2 by defining recombination for any ranked poset in Definition 4.1.6, in addition to generalizing Theorem 2.2.4 with Theorem 4.1.9. We also state a bijection of Striker and Williams and present an n-dimensional analogue of it in Theorem 4.1.5. In Section 4.2, we utilize a previous homomesy result, Corollary 2.3.6, and our new recombination result to obtain Corollary 4.2.1, a new homomesy result on order ideals of a type B minuscule poset cross a two-element chain. We conclude the chapter with Example 4.2.2, illustrating recombination with an n-dimensional lattice projection. This chapter is based on work from [46].

4.1. Generalized recombination

Recall Definition 1.2.21 of an n-dimensional lattice projection. Promotion with respect to an n-dimensional lattice projection is defined in Definition 1.2.22. However, we need a notion of layers with respect to an n-dimensional lattice projection π. When generalizing the layers of Definition 2.1.2, we would like to define our layers on P, the poset on which we are performing our toggles. Because the notion of layers comes from $\mathbb{Z}^n$ and π is not necessarily injective, we will at times abuse notation in order to capture the same ideas as from Chapter 2.

Definition 4.1.1. Let P be a poset with n-dimensional lattice projection π. If $A \subseteq \mathbb{Z}^n$, let $\pi^{-1}(A)$ denote the preimage of A in P. Note since π is not necessarily injective, π^{-1} of a single element may include multiple poset elements.

Recall Definition 2.1.1 for the notation $v^{\widehat{\gamma}}$.

Definition 4.1.2. Let P be a poset with n-dimensional lattice projection π. Define the jth γ-layer of P as

$$L_\gamma^j = \{\pi^{-1}(i_1, i_2, \ldots, i_n) \mid i_\gamma = j \text{ and } (i_1, i_2, \ldots, i_n) \in \mathbb{Z}^n\}$$

and the jth γ-layer of $I \in J(P)$ as

$$L_\gamma^j(I) = L_\gamma^j \cap I.$$

Additionally, given L_γ^j and $L_\gamma^j(I)$, we abuse notation to define

$$(L_\gamma^j)^{\widehat{\gamma}} = \{\pi^{-1}((i_1, i_2, \ldots, i_n)^{\widehat{\gamma}}) \mid i_\gamma = j \text{ and } (i_1, i_2, \ldots, i_n) \in \mathbb{Z}^n\},$$
$$L_\gamma^j(I)^{\widehat{\gamma}} = (L_\gamma^j)^{\widehat{\gamma}} \cap I,$$

where $\pi^{-1}((i_1, i_2, \ldots, i_n)^{\widehat{\gamma}})$ denotes forming the poset given by the preimage of the $n-1$-dimensional poset obtained from deleting the coordinate γ and $(L_\gamma^j)^{\widehat{\gamma}} \cap I$ denotes using elements in the order ideal I to form an order ideal with the corresponding elements in $(L_\gamma^j)^{\widehat{\gamma}}$.

In order to prove results regarding recombination in Chapter 2, we relied heavily on the ability to commute the toggles of promotion. More specifically, we showed that any promotion could be thought of as sequence of $n-1$ dimensional promotions on the layers of our product of chains. Here we introduce notation for an analogous result.

Definition 4.1.3. Let P be a poset with n-dimensional lattice projection π and $\gamma \in \{1, 2, \ldots, n\}$. We define $T^j_{\mathrm{Pro}_{\pi, v^{\widehat{\gamma}}}}$ as the toggle product of $\mathrm{Pro}_{\pi, v^{\widehat{\gamma}}}$ on $(L_\gamma^j)^{\widehat{\gamma}}$.

This definition allows us to perform an $n-1$-dimensional promotion on a single layer of P. Before we give a general definition of recombination, we present Theorem 5.4 from [44] along with a higher dimensional analogue. Striker and Williams found a conjugating toggle element; in other words, the toggles necessary to state the explicit bijection from $J(P)$ under Row^{-1} to $J(P)$ under Pro. We state this result using our notation.

Theorem 4.1.4 ([44])**.** *Let P be a poset with two-dimensional lattice projection π, $v = (-1, 1)$ and $w = (-1, -1)$. There exists an equivariant bijection between $J(P)$ under $\mathrm{Pro}_{\pi,v} = \mathrm{Pro}$ and $\mathrm{Pro}_{\pi,w} = \mathrm{Row}^{-1}$ given by acting on an order ideal by $D = \prod_{i=1}^{b} \prod_{j=1}^{i} (T^{i+1-j}_{\mathrm{Pro}_{\pi,(1)}})^{-1}$ where L_2^b is the maximum non-empty layer in P.*

We generalize this theorem to n-dimensions by stating the toggle product needed to conjugate from one promotion to another.

Theorem 4.1.5. *Let P be a poset with n-dimensional lattice projection π, $v = (v_1, v_2, \ldots, v_n)$ where $v_j \in \{\pm 1\}$, $w = (w_1, w_2, \ldots, w_n)$ where $w_j \in \{\pm 1\}$ such that $v_\gamma = 1, w_\gamma = -1$, and $v^{\widehat{\gamma}} = w^{\widehat{\gamma}}$.*

There exists an equivariant bijection between $J(P)$ under $\mathrm{Pro}_{\pi,v}$ and $\mathrm{Pro}_{\pi,w}$ given by acting on an order ideal by $D_\gamma = \prod_{i=1}^{a_\gamma - 1}\prod_{j=1}^{i}(T^{i+1-j}_{\mathrm{Pro}_{\pi,v^{\widehat{\gamma}}}})^{-1}$ where $L_\gamma^{a_\gamma}$ is the maximum non-empty layer in P.

Proof. Without loss of generality, $v_\gamma = 1$ and $w_\gamma = -1$. As a result, $\mathrm{Pro}_{\pi,w} = \prod_{i=1}^{a_\gamma} T^{a_\gamma+1-i}_{\mathrm{Pro}_{\pi,w^{\widehat{\gamma}}}}$ and $\mathrm{Pro}_{\pi,v} = \prod_{i=1}^{a_\gamma} T^{i}_{\mathrm{Pro}_{\pi,v^{\widehat{\gamma}}}}$. Note that $w^{\widehat{\gamma}} = v^{\widehat{\gamma}}$. We will commute toggles to show $\mathrm{Pro}_{\pi,w}D_\gamma = D_\gamma \mathrm{Pro}_{\pi,v}$. When we expand, we obtain

$$\begin{aligned}\mathrm{Pro}_{\pi,w}D_\gamma =& T^{a_\gamma}_{\mathrm{Pro}_{\pi,w^{\widehat{\gamma}}}} T^{a_\gamma-1}_{\mathrm{Pro}_{\pi,w^{\widehat{\gamma}}}} \dots T^{1}_{\mathrm{Pro}_{\pi,w^{\widehat{\gamma}}}} (T^{1}_{\mathrm{Pro}_{\pi,w^{\widehat{\gamma}}}})^{-1}(T^{2}_{\mathrm{Pro}_{\pi,w^{\widehat{\gamma}}}})^{-1}(T^{1}_{\mathrm{Pro}_{\pi,w^{\widehat{\gamma}}}})^{-1}\dots(T^{a_\gamma-1}_{\mathrm{Pro}_{\pi,w^{\widehat{\gamma}}}})^{-1}\\ &(T^{a_\gamma-2}_{\mathrm{Pro}_{\pi,w^{\widehat{\gamma}}}})^{-1}\dots(T^{1}_{\mathrm{Pro}_{\pi,w^{\widehat{\gamma}}}})^{-1}\end{aligned}$$

and

$$\begin{aligned}D_\gamma\mathrm{Pro}_{\pi,v} =&(T^{1}_{\mathrm{Pro}_{\pi,w^{\widehat{\gamma}}}})^{-1}(T^{2}_{\mathrm{Pro}_{\pi,w^{\widehat{\gamma}}}})^{-1}(T^{1}_{\mathrm{Pro}_{\pi,w^{\widehat{\gamma}}}})^{-1}\dots(T^{a_\gamma-1}_{\mathrm{Pro}_{\pi,w^{\widehat{\gamma}}}})^{-1}(T^{a_\gamma-2}_{\mathrm{Pro}_{\pi,w^{\widehat{\gamma}}}})^{-1}\dots(T^{1}_{\mathrm{Pro}_{\pi,w^{\widehat{\gamma}}}})^{-1}\\ &T^{1}_{\mathrm{Pro}_{\pi,w^{\widehat{\gamma}}}}T^{2}_{\mathrm{Pro}_{\pi,w^{\widehat{\gamma}}}}\dots T^{a_\gamma}_{\mathrm{Pro}_{\pi,w^{\widehat{\gamma}}}}\\ =&(T^{1}_{\mathrm{Pro}_{\pi,w^{\widehat{\gamma}}}})^{-1}(T^{2}_{\mathrm{Pro}_{\pi,w^{\widehat{\gamma}}}})^{-1}(T^{1}_{\mathrm{Pro}_{\pi,w^{\widehat{\gamma}}}})^{-1}\dots(T^{1}_{\mathrm{Pro}_{\pi,w^{\widehat{\gamma}}}})^{-1}T^{a_\alpha}_{\mathrm{Pro}_{\pi,w^{\widehat{\gamma}}}}.\end{aligned}$$

However, we can commute $T^{k}_{\mathrm{Pro}_{\pi,w^{\widehat{\gamma}}}}$ and $T^{j}_{\mathrm{Pro}_{\pi,w^{\widehat{\gamma}}}}$ or $(T^{j}_{\mathrm{Pro}_{\pi,w^{\widehat{\gamma}}}})^{-1}$ if $|j-k|>1$ because the elements in these toggles could not share a covering relation. Therefore, we can commute toggles of $\mathrm{Pro}_{\pi,w}D_\gamma$ to obtain

$$\mathrm{Pro}_{\pi,w}D_\gamma =(T^{1}_{\mathrm{Pro}_{\pi,w^{\widehat{\gamma}}}})^{-1}(T^{2}_{\mathrm{Pro}_{\pi,w^{\widehat{\gamma}}}})^{-1}(T^{1}_{\mathrm{Pro}_{\pi,w^{\widehat{\gamma}}}})^{-1}\dots(T^{1}_{\mathrm{Pro}_{\pi,w^{\widehat{\gamma}}}})^{-1}T^{a_\alpha}_{\mathrm{Pro}_{\pi,w^{\widehat{\gamma}}}}.$$

Therefore, $\mathrm{Pro}_{\pi,w}D_\gamma = D_\gamma\mathrm{Pro}_{\pi,v}$ and so $\mathrm{Pro}_{\pi,v} = (D_\gamma)^{-1}\mathrm{Pro}_{\pi,w}D_\gamma$. □

We now present our generalized definition of recombination with respect to an n-dimensional lattice projection.

Definition 4.1.6. Let P be a poset with n-dimensional lattice projection π and $I \in J(P)$. Define $\Delta^{\gamma}_{\pi,v}I = \cup_j(L^j_\gamma(\mathrm{Pro}^{j-1}_{\pi,v}(I))$ where $\gamma \in \{1,\dots,n\}$. We will call $\Delta^{\gamma}_{\pi,v}I$ the $(\pi,v,\gamma)-$*recombination* of I. When context is clear, we will suppress the (π,v,γ).

The idea is the same as in Chapter 2; we take certain layers from an orbit of promotion to create a new order ideal. We can now state the analogue of Lemma 2.2.2, our result regarding toggling commutation, whose proof is similar to the proof of Lemma 2.2.2.

Lemma 4.1.7. *Let P be a ranked poset with lattice projection π and $\gamma \in \{1, 2, \ldots, n\}$. Then* $\mathrm{Pro}_{\pi,v} = \prod_{j=1}^{a_\gamma} T^{\alpha}_{\mathrm{Pro}_{\pi,v^{\widehat{\gamma}}}}$ *where*

$$\alpha = \begin{cases} j & \text{if } v_\gamma = 1 \\ a_\gamma + 1 - j & \text{if } v_\gamma = -1. \end{cases}$$

Proof. Suppose $x, y \in P$, which implies $\pi(x) := (x_1, \ldots, x_n), \pi(y) := (y_1, \ldots, y_n) \in \pi(P)$ with $x \in L^j_\gamma$ and $y \in L^k_\gamma$ for some j and k. We want to show that x and y are toggled in the same relative order in $\mathrm{Pro}_{\pi,v}$ and $\prod_{j=1}^{a_\gamma} T^{\alpha}_{\mathrm{Pro}_{\pi,v^{\widehat{\gamma}}}}$.

Case $j \neq k$: Without loss of generality, $j > k$. Furthermore, we can assume $x_\gamma = y_\gamma + 1$ and $x_i = y_i$ for $i \neq \gamma$. If this was not the case, x and y could not have a covering relation and we could commute the toggles of x and y.

If $v_\gamma = 1$, in $\prod_{j=1}^{a_\gamma} T^{\alpha}_{\mathrm{Pro}_{\pi,v^{\widehat{\gamma}}}}$, x is toggled before y by definition. Additionally,

$$\langle \pi(x), v \rangle = v_1 x_1 + \cdots + v_\gamma x_\gamma + \cdots + v_n x_n > v_1 y_1 + \cdots + v_\gamma y_\gamma + \cdots + v_n y_n = \langle \pi(y), v \rangle$$

and so x is toggled before y in $\mathrm{Pro}_{\pi,v}$.

If $v_\gamma = -1$, in $\prod_{j=1}^{a_\gamma} T^{\alpha}_{\mathrm{Pro}_{\pi,v^{\widehat{\gamma}}}}$, y is toggled before x by definition. Additionally,

$$\langle \pi(x), v \rangle = v_1 x_1 + \cdots + v_\gamma x_\gamma + \cdots + v_n x_n < v_1 y_1 + \cdots + v_\gamma y_\gamma + \cdots + v_n y_n = \langle \pi(y), v \rangle$$

and so y is toggled before x in $\mathrm{Pro}_{\pi,v}$.

Case $j = k$: In other words, $x_\gamma = y_\gamma$. Therefore,

$$\begin{aligned} \langle \pi(x), v \rangle > \langle \pi(y), v \rangle &\iff v_1 x_1 + \cdots + v_\gamma x_\gamma + \cdots + v_n x_n > v_1 y_1 + \cdots + v_\gamma y_\gamma + \cdots + v_n y_n \\ &\iff v_1 x_1 + \cdots + v_{\gamma-1} x_{\gamma-1} + v_{\gamma+1} x_{\gamma+1} + \cdots + v_n x_n > \\ &\qquad v_1 y_1 + \cdots + v_{\gamma-1} y_{\gamma-1} + v_{\gamma+1} y_{\gamma+1} + \cdots + v_n y_n \\ &\iff \langle \pi(x)^{\widehat{\gamma}}, v^{\widehat{\gamma}} \rangle > \langle \pi(y)^{\widehat{\gamma}}, v^{\widehat{\gamma}} \rangle \end{aligned}$$

where $\pi(x)^{\widehat{\gamma}}, \pi(y)^{\widehat{\gamma}}$ are $\pi(x)$ and $\pi(y)$ with x_γ and y_γ deleted, respectively. Therefore, x can be toggled before y in $\mathrm{Pro}_{\pi,v}$ if and only if x can be toggled before y in $\prod_{j=1}^{a_\gamma} T^{\alpha}_{\mathrm{Pro}_{\pi,v^{\widehat{\gamma}}}}$. $\square$

As in the product of chains setting, we have conditions which guarantee generalized recombination gives us an order ideal. The proof is similar to the proof of Lemma 2.2.3 with the inclusion of the lattice projection π.

Lemma 4.1.8. *Let $I \in J(P)$. Suppose we have v and γ such that $v_\gamma = 1$. Then $\Delta^{\gamma}_{\pi,v} I$ is an order ideal of P.*

Proof. Suppose $x \in \Delta^{\gamma}_{\pi,v} I$ where $\pi(x) = (i_1, \ldots, i_n)$. This means $x \in L^{i_\gamma}_{\gamma}(\mathrm{Pro}^{i_\gamma - 1}_{\pi,v}(I))$. Suppose $y \in P$ such that x covers y. To show $\Delta^{\gamma}_{\pi,v} I$ is an order ideal, it suffices to show $y \in \Delta^{\gamma}_{\pi,v} I$. Because π is rank-preserving, $\pi(y) = (i_1, \ldots, i_j - 1, \ldots, i_n)$ for some j. By Definition 4.1.6, $y \in L^{i_\gamma}_{\gamma}(\mathrm{Pro}^{i_\gamma - 1}_{\pi,v}(I))$ for $j \neq \gamma$. As a result, we must show $y \in L^{i_\gamma}_{\gamma}(\mathrm{Pro}^{i_\gamma - 1}_{\pi,v}(I))$ for $j = \gamma$ and $i_\gamma \geq 2$. If $i_\gamma = 1$ there is nothing to show. By Lemma 4.1.7, $\mathrm{Pro}_{\pi,v} = \prod_{j=1}^{a_\gamma} T^{a_\gamma + 1 - j}_{\mathrm{Pro}_{\pi,v^{\widehat{\gamma}}}}$, which implies we can commute the toggle relations in $\mathrm{Pro}_{\pi,v}$ so that $L^{i_\gamma}_{\gamma}$ is toggled before $L^{i_\gamma - 1}_{\gamma}$. As a result, we must have $y \in L^{i_\gamma - 1}_{\gamma}(\mathrm{Pro}^{i_\gamma - 2}_{\pi,v}(I))$. Therefore, $y \in \Delta^{\gamma}_{\pi,v} I$. □

We can now state our general recombination result. The proof is similar to Theorem 2.2.4 with the inclusion of the lattice projection π.

Theorem 4.1.9. *Let $I \in J(P)$. Suppose we have $v = (v_1, v_2, \ldots, v_n)$ where $v_j \in \{\pm 1\}$, $u = (u_1, u_2, \ldots, u_n)$ where $u_j \in \{\pm 1\}$, and γ such that $v_\gamma = 1$, $u_\gamma = -1$, and $v^{\widehat{\gamma}} = u^{\widehat{\gamma}}$. Then $\mathrm{Pro}_{\pi,u}(\Delta^{\gamma}_{\pi,v} I) = \Delta^{\gamma}_{\pi,v}(\mathrm{Pro}_{\pi,v}(I))$.*

Proof. First, note that $\Delta^{\gamma}_{\pi,v} I$ is an order ideal by Lemma 4.1.8. Also note that $\mathrm{Pro}_{\pi,v} = \prod_{j=1}^{a_\gamma} T^{j}_{\mathrm{Pro}_{\pi,v^{\widehat{\gamma}}}}$ and $\mathrm{Pro}_{\pi,u} = \prod_{j=1}^{a_\gamma} T^{a_\gamma + 1 - j}_{\mathrm{Pro}_{\pi,u^{\widehat{\gamma}}}}$ by Lemma 4.1.7. We will show $\mathrm{Pro}_{\pi,u}(\Delta^{\gamma}_{\pi,v} I) = \Delta^{\gamma}_{v}(\mathrm{Pro}_{\pi,v}(I)))$ by showing $L^{k}_{\gamma}(\mathrm{Pro}_{\pi,u}(\Delta^{\gamma}_{\pi,v} I)) = L^{k}_{\gamma}(\Delta^{\gamma}_{v}(\mathrm{Pro}_{\pi,v}(I)))$ for each $k \in \{1, 2, \ldots, a_\gamma\}$. There are three cases.

Case $1 < k < a_\gamma$**:** Let $J = \mathrm{Pro}^{k-1}_{\pi,v}(I)$. We can commute the toggles of $\mathrm{Pro}_{\pi,v}$ so that L^{k+1}_{γ} is toggled before L^{k}_{γ}, which is toggled before L^{k-1}_{γ}. Thus, when applying the toggles of $\mathrm{Pro}_{\pi,v}$ to L^{k}_{γ} of J, the layer above is $L^{k+1}_{\gamma}(\mathrm{Pro}_{\pi,v}(J))$ whereas the layer below is $L^{k-1}_{\gamma}(J)$. Additionally, we can also commute the toggles of $\mathrm{Pro}_{\pi,u}$ so L^{k-1}_{γ} is toggled before L^{k}_{γ}, which is toggled before L^{k+1}_{γ}. Therefore, when applying the toggles of $\mathrm{Pro}_{\pi,u}$ to L^{k}_{γ} of $\Delta^{\gamma}_{\pi,v} I$, the layer below is $L^{k-1}_{\gamma}(\mathrm{Pro}_{\pi,u}(\Delta^{\gamma}_{\pi,v} I))$, whereas the layer above is $L^{k+1}_{\gamma}(\Delta^{\gamma}_{\pi,v} I)$. However, $L^{k-1}_{\gamma}(\mathrm{Pro}_{\pi,u}(\Delta^{\gamma}_{\pi,v} I)) = L^{k-1}_{\gamma}(J)$, $L^{k}_{\gamma}(\Delta^{\gamma}_{\pi,v} I) = L^{k}_{\gamma}(J)$, and $L^{k+1}_{\gamma}(\Delta^{\gamma}_{\pi,v} I) = L^{k+1}_{\gamma}(\mathrm{Pro}_{\pi,v}(J))$. Therefore, when applying $\mathrm{Pro}_{\pi,v}$ to L^{k}_{γ} of J and $\mathrm{Pro}_{\pi,u}$ to L^{k}_{γ} of $\Delta^{\gamma}_{\pi,v} I$, both layers are the same and have the same layers above and below them. Because $u^{\widehat{\gamma}} = v^{\widehat{\gamma}}$, we have $\mathrm{Pro}_{\pi,u^{\widehat{\gamma}}} = \mathrm{Pro}_{\pi,v^{\widehat{\gamma}}}$ and so the result of toggling this layer is $L^{k}_{\gamma}(\mathrm{Pro}_{\pi,u}(\Delta^{\gamma}_{\pi,v} I))$, which is the same as $L^{k}_{\gamma}(\mathrm{Pro}_{\pi,v}(J)) = L^{k}_{\gamma}(\mathrm{Pro}^{k}_{\pi,v}(I)) = L^{k}_{\gamma}(\Delta^{\gamma}_{v}(\mathrm{Pro}_{\pi,v}(I)))$.

Case $k = 1$**:** As above, when applying $\mathrm{Pro}_{\pi,v}$ to L^{1}_{γ} of I and $\mathrm{Pro}_{\pi,u}$ to L^{1}_{γ} of $\Delta^{\gamma}_{\pi,v} I$, both of these layers are the same, along with the layers above them. Because $k = 1$, there is not a layer below. As above, $\mathrm{Pro}_{\pi,u^{\widehat{\gamma}}} = \mathrm{Pro}_{\pi,v^{\widehat{\gamma}}}$ and so we again have $L^{1}_{\gamma}(\mathrm{Pro}_{\pi,u}(\Delta^{\gamma}_{\pi,v} I)) = L^{1}_{\gamma}(\Delta^{\gamma}_{\pi,v}(\mathrm{Pro}_{\pi,v}(I)))$.

Case $k = a_\gamma$**:** Again, as above, when applying $\mathrm{Pro}_{\pi,v}$ to $L_\gamma^{a_\gamma}$ of J and $\mathrm{Pro}_{\pi,u}$ to $L_\gamma^{a_\gamma}$ of $\Delta_{\pi,v}^\gamma I$, both of these layers are the same along with the layers below them. Because $k = a_\gamma$ there is not a layer above. Again, $\mathrm{Pro}_{\pi,u^{\hat\gamma}} = \mathrm{Pro}_{\pi,v^{\hat\gamma}}$ and so $L_\gamma^{a_\gamma}(\mathrm{Pro}_{\pi,u}(\Delta_{\pi,v}^\gamma I)) = L_\gamma^{a_\gamma}(\Delta_v^\gamma(\mathrm{Pro}_{\pi,v}(I)))$. $\square$

4.2. Applications of generalized recombination

Recall the type B minuscule poset from Definition 1.1.9. Using this generalized recombination result and our homomesy result on $J([2]\times[a]\times[b])$, we can obtain an additional homomesy result on order ideals of the type B minuscule poset cross a two-element chain. Let $P_n = ([n]\times[n])/S_2$ denote a type B minuscule poset and let π be the natural embedding of $P_n \times [2]$ into $\mathbb{Z}^3$.

Corollary 4.2.1. *Let f be the cardinality statistic. The triple $(J(P_n \times [2]), \mathrm{Pro}_{\pi,v}, f)$ is c-mesic with $c = \frac{n^2+n}{2}$.*

Proof. Orbits of $J(P_n \times [2])$ under Row are in bijection with orbits of $J([n]\times[n]\times[2])$ under Row where the order ideals are symmetric about the plane $x - y = 0$. Let $\mathcal{O}$ be an orbit of $J(P_n \times [2])$ under Row and $\mathcal{O}'$ be the orbit of $J([n]\times[n]\times[2])$ in bijection with $\mathcal{O}$. We note $\#\mathcal{O} = \#\mathcal{O}'$. By Corollary 2.3.6, the cardinality of order ideals in $\mathcal{O}'$ is $(\#\mathcal{O}')n^2$. Alternatively, we can enumerate the cardinality of order ideals in $\mathcal{O}'$ by doubling the cardinality in $\mathcal{O}$ and removing what is double counted, namely, elements that appear on the plane $x - y = 0$. The cardinality of these elements is $(\#\mathcal{O}')n$ by Corollary 3.2.1. As a result, we have the following equality: $(\#\mathcal{O})n^2 = 2f(\mathcal{O}) - (\#\mathcal{O})n$ where $f(\mathcal{O})$ is the sum of the cardinalities of all order ideals in $\mathcal{O}$. Rearranging, we get $\frac{f(\mathcal{O})}{\#\mathcal{O}} = \frac{n^2+n}{2}$. Therefore, $(J(P_n \times [2]), \mathrm{Row}, f)$ is $\frac{n^2+n}{2}$-mesic. Additionally, because $\mathrm{Pro}_{\pi,(-1,-1,-1)}$ reverses the orbits of Row, $(J(P_n \times [2]), \mathrm{Pro}_{\pi,(-1,-1,-1)}, f)$ is $\frac{n^2+n}{2}$-mesic.

To obtain the result for the remaining v, we will use the recombination result of Theorem 4.1.9. From Theorem 4.1.9, we get $\mathrm{Pro}_{\pi,(1,1,-1)}(\Delta^3_{\pi,(1,1,1)} I) = \Delta^3_{\pi,(1,1,1)}(\mathrm{Pro}_{\pi,(1,1,1)}(I))$ and $\mathrm{Pro}_{\pi,(1,-1,1)}(\Delta^2_{\pi,(1,1,1)} I) = \Delta^2_{\pi,(1,1,1)}(\mathrm{Pro}_{\pi,(1,1,1)}(I))$ and
$\mathrm{Pro}_{\pi,(-1,1,1)}(\Delta^1_{\pi,(1,1,1)} I) = \Delta^1_{\pi,(1,1,1)}(\mathrm{Pro}_{\pi,(1,1,1)}(I))$. From this, we deduce $(J(P_n \times [2]), \mathrm{Pro}_{\pi,v}, f)$ is $\frac{n^2+n}{2}$-mesic for $v \in \{(1,1,-1),(1,-1,1),(-1,1,1)\}$. Finally, $\mathrm{Pro}_{\pi,(-1,-1,1)}, \mathrm{Pro}_{\pi,(-1,1,-1)}$, and $\mathrm{Pro}_{\pi,(1,-1,-1)}$ reverse the orbits of $\mathrm{Pro}_{\pi,(1,1,-1)}, \mathrm{Pro}_{\pi,(1,-1,1)}$, and $\mathrm{Pro}_{\pi,(-1,1,1)}$ respectively. As a result, $(J(P_n \times [2]), \mathrm{Pro}_{\pi,v}, f)$ is $\frac{n^2+n}{2}$-mesic for $v \in \{(-1,-1,1), (-1,1,-1), (1,-1,-1)\}$, completing the proof of the result. $\square$

Example 4.2.2. We now give an example of generalized recombination where we cannot use a simple embedding as our three-dimensional lattice projection. Let our poset be the tetrahedral poset on the left in Figure 4.1; for more on tetrahedral posets, see [40]. By Proposition 8.5 of [44], we see the significance of this poset is that its order ideals are in bijection with alternating sign matrices of size 4×4. We note that this poset cannot be embedded in $\mathbb{Z}^3$ since the element b is covered by four elements. We instead use the lattice projection π in Figure 4.1, projecting into $\mathbb{Z}^2$. We note that this lattice projection is not new, as it is used in Figure 18 in [44]. Figure 4.3 shows how we orient this in $\mathbb{Z}^2$.

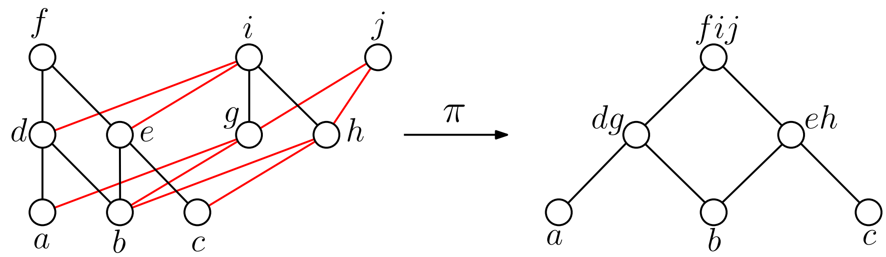


Figure 4.1. The poset on the left is a tetrahedral poset. For Example 4.2.2, we will use the lattice projection π to the subposet of $\mathbb{Z}^2$ on the right.

Figure 4.2 shows a partial orbit under rowmotion. We see from Figure 4.3 what our layers are: the first layer consists of a, the second layer consists of b, d, g, and the third layer consists of c, e, f, h, i, j.

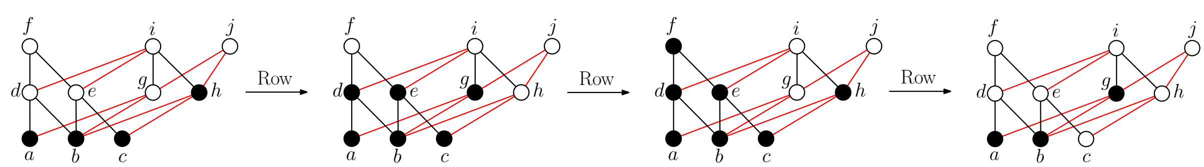


Figure 4.2. A partial orbit of order ideals under rowmotion. We use this example to demonstrate generalized recombination.

From the partial orbit, we take the first layer from the first order ideal, the second layer from the second order ideal, and the third layer from the third order ideal to form a new order ideal. These are indicated with red in Figures 4.4 and 4.5. We also take the first layer in the second order ideal, the second layer in the third order ideal, and the third layer from the fourth order ideal

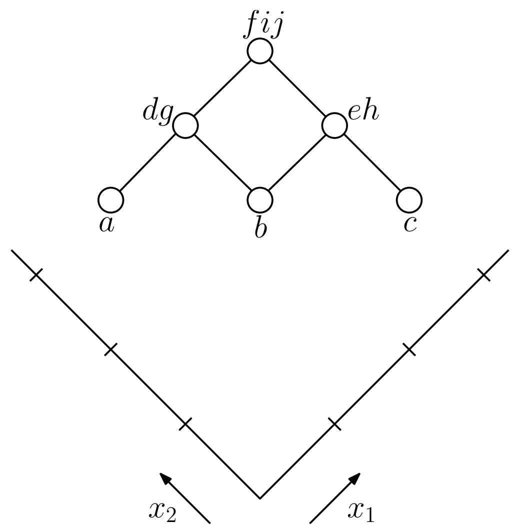


Figure 4.3. We orient this poset in $\mathbb{Z}^2$ in the following way. Our three layers are the diagonals with coordinates $x_1 = 1, 2$, and 3.

to form another new order ideal. These are indicated with blue in Figures 4.4 and 4.5. Generalized recombination tells us if we apply promotion to the red order ideal, we should obtain the blue order ideal, which we can see is the case.

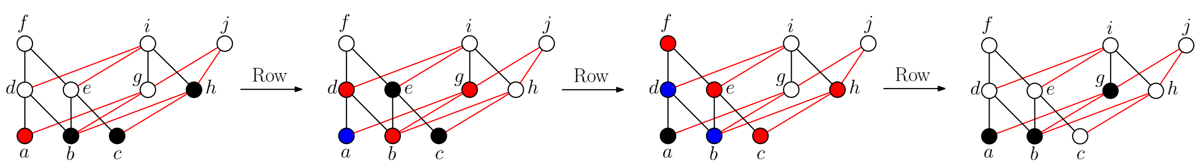


Figure 4.4. We use the red layers and blue layers from the partial orbit to form two new order ideals.

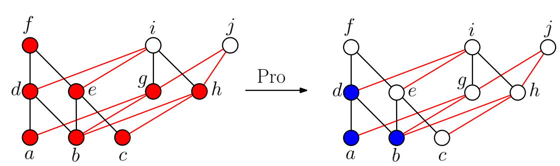


Figure 4.5. Applying promotion to the red order ideal gives us the blue order ideal.

5. INFINITE POSETS

In previous chapters, all posets have been finite. Moreover, toggle group actions on infinite posets have not been well studied. In particular, the promotion actions discussed in previous chapters have not been defined in the infinite case. In this chapter, our aim is to extend Definition 1.2.22 of promotion to infinite posets.

We begin by providing the framework to utilize toggles in the infinite setting with Definition 5.1.2. We also produce some preliminary results and examples to justify this definition is well chosen. In Proposition 5.1.8, we show our new definition, Definition 5.1.2, produces the same order ideal as Definition 1.2.22 of Dilks, Pechenik, and Striker if our poset is finite. In Theorem 5.1.9, we show that if we have an infinite poset with n-dimensional lattice projection, our new toggle definition of rowmotion matches the minimal generator definition of rowmotion. We also discuss the intuition of toggling in the infinite case in Remark 5.1.11, along with noting that promotion on infinite posets may not result in a bijective action in Remark 5.1.12. Because our action is no longer bijective and does not necessarily partition our order ideals into orbits, we instead investigate several interesting results from single applications of an action. More specifically, we connect order ideals of the poset $\mathbb{N}^2$ to monomial ideals. From this, we present two results in terms of minimal generators of monomial ideals in Theorems 5.2.9 and 5.2.10. We also introduce infinite boundary paths and in Lemma 5.2.7, generalize the left cyclic shift of the boundary path of a finite product of two chains under promotion to the infinite product of two chains under promotion. From this boundary path result, we give a homomesy result in the infinite product of two chains and conclude the chapter investigating a generalization of recombination for the infinite product of chains.

5.1. Defining rowmotion and promotion for infinite posets

In this section, we discuss what aspects of the intuition of rowmotion and promotion can be applied from the finite case to the infinite and in what situations this intuition fails. First, we observe the minimal generator definition of rowmotion from Definition 1.2.5 can still be applied, even if there are an infinite number of minimal elements of $P \setminus I$. If P is a poset and I an order ideal, we can always form $P \setminus I$. If this has minimal elements, these generate $\mathrm{Row}(I)$. If this does not have minimal elements, $\mathrm{Row}(I) = \varnothing$. As a result, when we refer to $\mathrm{Row}(I)$ in this chapter, we will be referring to this minimal generator definition. A good toggle definition of rowmotion should match Row, as for finite posets. With some posets, the intuition of rowmotion or the intuition of a toggle action may be less clear. We must determine which infinite posets to consider. We define promotion on an infinite poset with Definition 5.1.2 in the infinite case and justify why this definition is appropriate.

Remark 5.1.1. We use the notation $\mathbb{N} = \{0, 1, 2, \dots\}$ and $\mathbb{N}^+ = \{1, 2, \dots\}$.

Recall Definition 1.2.21 of an n-dimensional lattice projection. This definition is still valid if P is infinite and requires P to be ranked. However, in Example 5.1.7, we see we should project into $\mathbb{N}^n$ rather than $\mathbb{Z}^n$. In other words, our n-dimensional lattice projection will be an order and rank preserving map $\pi : P \to \mathbb{N}^n$. It is possible that future work may expand the class of infinite posets to consider, but for the remainder of this chapter, when we refer to actions on infinite posets, we will mean posets with an n-dimensional lattice projection $\pi : P \to \mathbb{N}^n$.

Recall Definition 1.2.22 of promotion on a finite poset P with n-dimensional lattice projection. By truncating at increasing ranks and using a union of finite promotions, we define promotion on an infinite poset P with an n-dimensional lattice projection into $\mathbb{N}^n$.

Definition 5.1.2. Let P be a poset with an n-dimensional lattice projection $\pi : P \to \mathbb{N}^n$ and let I be an order ideal of P. Let P_k be the subposet of P with elements of rank less than or equal to k and $I_k = I \cap P_k$. Define $\mathrm{Pro}_{\pi,v}(I) = \cup_{k \geq 1} \mathrm{Pro}_{\pi,v}(I_k)$.

A concern of this definition is that truncating at a particular rank might yield undesired elements that do not appear at larger and larger ranks; the use of a union would include these in the resulting order ideal $\mathrm{Pro}_{\pi,v}(I)$. In the following lemma, we show truncating at larger ranks gives nested order ideals, so the use of a union is appropriate.

Lemma 5.1.3. *Let P be a poset with an n-dimensional lattice projection $\pi : P \to \mathbb{N}^n$ and $I \in J(P)$. $\mathrm{Pro}_{\pi,v}(I_k) \subseteq \mathrm{Pro}_{\pi,v}(I_{k+1})$.*

Proof. By definition, $I_k \subseteq I_{k+1}$. We first look at elements of rank k, then induct on decreasing rank. Let $x \in \mathrm{Pro}_{\pi,v}(I_k)$ such that $\mathrm{rk}\, x = k$. If $x \in I_k$, because x has no covers in P_k, it would be toggled out when applying $\mathrm{Pro}_{\pi,v}$. This implies $x \notin I_k$ and x is toggled in when applying $\mathrm{Pro}_{\pi,v}$. Therefore, $x \notin I_{k+1}$. Because we are toggling I_{k+1} in the same direction and the covers of x in

P_{k+1} will not prevent x from being toggled in, $x \in \mathrm{Pro}_{\pi,v}(I_{k+1})$. Note that if we do have $x \in I_k$, then $x \in I_{k+1}$ might not be toggled out when applying $\mathrm{Pro}_{\pi,v}(I_{k+1})$, as a cover of x in P_{k+1} may be in the order ideal when t_x is applied. As a result, there will be the same or more elements with rank k in $\mathrm{Pro}_{\pi,v}(I_{k+1})$ as $\mathrm{Pro}_{\pi,v}(I_k)$.

Now let $x \in \mathrm{Pro}_{\pi,v}(I_k)$ such that $\mathrm{rk}\, x = r < k$ and if y is a cover of x and $y \in \mathrm{Pro}_{\pi,v}(I_{k+1})$, then $y \in \mathrm{Pro}_{\pi,v}(I_k)$. Also, note that if y is a cover of x and $y \in I_{k+1}$, then $y \in I_k$. Therefore, when t_x is applied as part of $\mathrm{Pro}_{\pi,v}(I_{k+1})$, x will have at least the same or more covers in the order ideal as when t_x is applied as part of $\mathrm{Pro}_{\pi,v}(I_k)$. As a result, $x \in \mathrm{Pro}_{\pi,v}(I_{k+1})$. Again, $\mathrm{Pro}_{\pi,v}(I_{k+1})$ may have more elements of rank r than $\mathrm{Pro}_{\pi,v}(I_k)$ as it may have more elements toggled in or not as many toggled out. As a result, if y covers an element of rank $r-1$ and $y \in \mathrm{Pro}_{\pi,v}(I_k)$, then $y \in \mathrm{Pro}_{\pi,v}(I_{k+1})$, which means we can induct to obtain the result for all ranks. Therefore, $\mathrm{Pro}_{\pi,v}(I_k) \subseteq \mathrm{Pro}_{\pi,v}(I_{k+1})$. □

We now provide several examples. Examples 5.1.4 and 5.1.5 demonstrate why Definition 5.1.2 is appropriate, as $\mathrm{Pro}_{(1,1)}$ is Row for these two examples. We will show this holds more generally in Theorem 5.1.9. On the other hand, Examples 5.1.6 and 5.1.7 show why we only consider posets with n-dimensional lattice projections $\pi : P \to \mathbb{N}^n$.

Example 5.1.4. Applying Row to the order ideal in Figure 5.1a results in the order ideal in Figure 5.1b. By truncating the poset and order ideal in Figure 5.1a at rank 2, we obtain the top left order ideal in Figure 5.2. Because this is a finite poset, we can apply the toggle definition of rowmotion to obtain the top right order ideal in Figure 5.2. We similarly obtain the middle left and bottom left order ideals in Figure 5.2 by truncating at rank 3 and rank 4, respectively. We see that when truncating at successive ranks, the order ideals obtained from applying rowmotion are nested and asymptotically grow to the desired order ideal.

Example 5.1.4 is an example in which the intuition of rowmotion is similar to the finite case and we can view rowmotion from the toggle perspective. We give another example on the infinite comb.

Example 5.1.5. Let P be the infinite comb poset in Figure 5.3 and I the order ideal on the left. Despite $P \setminus I$ having an infinite number of minimal generators, $\mathrm{Row}(I)$ produces the same order ideal as $\mathrm{Pro}_{(1,1)}(I)$ using Definition 5.1.2.

(a) The order ideal I in Example 5.1.4

(b) The order ideal generated by the minimal elements of $P \setminus I$

Figure 5.1. Using the minimal generator definition of Row, Definition 1.2.5, Row(I) is the order ideal on the right.

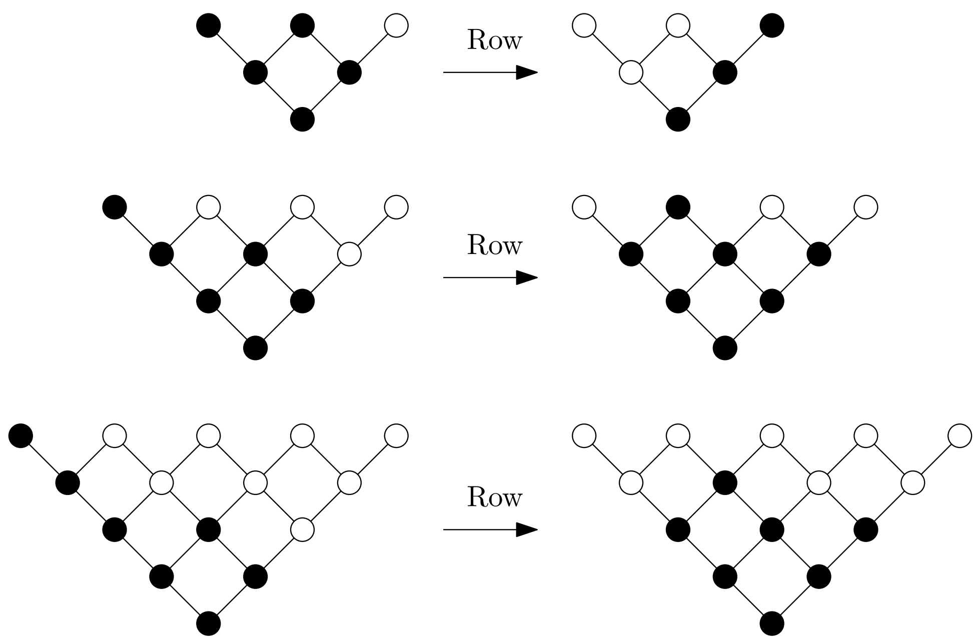


Figure 5.2. By truncating the order ideal in Figure 5.1a at larger and larger ranks before applying rowmotion, we see the results are nested order ideals that asymptotically grow to the desired order ideal.

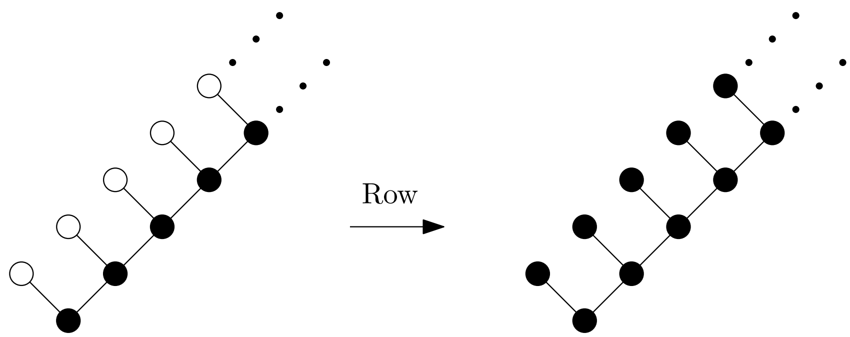


Figure 5.3. We can apply Row to the order ideal on the left even though $P \setminus I$ has an infinite number of minimal elements. For this example, we see $\mathrm{Row}(I) = \mathrm{Pro}_{(1,1)}(I)$.

We now give a poset where the intuition of the toggle perspective is less clear.

Example 5.1.6. Let P be the poset $\left\{\frac{1}{k} \mid k \in \mathbb{N}^+\right\} \cup \{0\}$ ordered by the standard less than or equal to $\leq$. With the order ideal $I = \{0\}$, $P \setminus I$ has no minimal elements, so $\mathrm{Row}(I) = \varnothing$.

This is an untuitive result because when P is finite, $\mathrm{Row}(I) = \varnothing$ if and only if $I = P$. Additionally, because Example 5.1.6 is not ranked, we cannot apply Definition 5.1.2. Using the work of [12], it could be possible to extend Definition 5.1.2 to nonranked posets. However, this example would still be difficult to work with, as it is not locally finite. In this chapter, we will not make this distinction, as we will only consider ranked posets, which are locally finite.

The next example shows why we only consider posets with n-dimensional lattice projections $\pi : P \to \mathbb{N}^n$.

Example 5.1.7. Let P be the poset $\mathbb{Z}$ ordered by the standard less than or equal to $\leq$. With the empty order ideal $I = \varnothing$, $P \setminus I$ has no minimal elements, so $\mathrm{Row}(I) = \varnothing$.

With this example, we might consider truncating in both directions. However, if we did this, we would lose the nesting property of Lemma 5.1.3, which is useful when proving results. The nesting property also justifies the use of the union in Definition 5.1.2. With the next result, we see that Definition 5.1.2 generalizes promotion from [11].

Proposition 5.1.8. *Suppose P is a finite poset with n-dimensional lattice projection $\pi : P \to \mathbb{N}^n$ and $I \in J(P)$. The definition of $\mathrm{Pro}_{\pi,v}(I)$ from Definition 5.1.2 coincides with the definition of $\mathrm{Pro}_{\pi,v}(I)$ in Definition 1.2.22.*

Proof. P is a finite poset with a rank function and $I \in J(P)$. Suppose P has rank r. Then, $P_r = P$, $I_r = I$, and $\bigcup_{k \geq 1} \mathrm{Pro}_{\pi,v}(I_k) = \cup_{k=1}^{k=r} \mathrm{Pro}_{\pi,v}(I_k)$. By Lemma 5.1.3, $\cup_{k=1}^{k=r} \mathrm{Pro}_{\pi,v}(I_k) = \mathrm{Pro}_{\pi,v}(I_r)$. Therefore, $\cup_{k \geq 1} \mathrm{Pro}_{\pi,v}(I_k) = \mathrm{Pro}_{\pi,v}(I_r) = \mathrm{Pro}_{\pi,v}(I)$, which is the desired result. $\square$

By the previous proposition, when applying promotion to an order ideal of either a finite or infinite poset, we can use the notation $\mathrm{Pro}_{\pi,v}(I)$ as this is unambigious. In Examples 5.1.4 and 5.1.5, we saw Row matched rowmotion obtained by Definition 5.1.2. We see that this is always the case.

Theorem 5.1.9. *Let P be an infinite poset with n-dimensional lattice projection $\pi : P \to \mathbb{N}^n$ and let $I \in J(P)$. Then* $\mathrm{Pro}_{\pi,(1,1,\ldots,1)}$ *acts as* Row.

Proof. Let $I \in J(P)$. By Proposition 1.2.24, $\mathrm{Pro}_{\pi,(1,1,\ldots,1)}$ acts as Row when our poset is finite. As a result, $\cup_{k \geq 1} \mathrm{Pro}_{\pi,(1,1,\ldots,1)}(I_k) = \cup_{k \geq 1} \mathrm{Row}(I_k)$. Therefore, to show the theorem, we show $\cup_{k \geq 1} \mathrm{Row}(I_k)$ produces $\mathrm{Row}(I)$, where $\mathrm{Row}(I)$ is given by the minimal generators of $P \setminus I$.

We first show $\cup_{k \geq 1} \mathrm{Row}(I_k) \subseteq \mathrm{Row}(I)$ by showing each $\mathrm{Row}(I_k) \subseteq \mathrm{Row}(I)$. Let $x \in \mathrm{Row}(I_k)$. The minimal generator definition of rowmotion implies there is a minimal element $s \in P_k \setminus I_k$ such that $x \leq s$. In order words, $s \notin I_k$ but every element s covers in P_k is in I_k. By the definition of I_k, this means $s \notin I$ and every element s covers in P is in I. Therefore, s is a minimal element of $P \setminus I$ and as a result $x \in \mathrm{Row}(I)$. Hence, $\mathrm{Row}(I_k) \subseteq \mathrm{Row}(I)$ and so $\cup_{k \geq 1} \mathrm{Row}(I_k) \subseteq \mathrm{Row}(I)$.

We now show $\mathrm{Row}(I) \subseteq \cup_{k \geq 1} \mathrm{Row}(I_k)$. Let $x \in \mathrm{Row}(I)$. The minimal generator definition of rowmotion implies there is a minimal element $s \in P \setminus I$ such that $x \leq s$. In other words, $s \notin I$ but every element s covers in P is in I. Suppose $\mathrm{rk}\, s = r$. Then $s \notin I_r$, but every element s covers in P_r is in I_r. Therefore, s is a minimal element of $P_r \setminus I_r$ and as a result, $x \in \mathrm{Row}(I_r) \subseteq \cup_{k \geq 1} \mathrm{Row}(I_k)$. We obtain $\mathrm{Row}(I) \subseteq \cup_{k \geq 1} \mathrm{Row}(I_k)$ and consequently the desired result, $\mathrm{Row}(I) = \cup_{k \geq 1} \mathrm{Row}(I_k) = \cup_{k \geq 1} \mathrm{Pro}_{\pi,(1,1,\ldots,1)}(I_k)$. $\square$

Remark 5.1.10. From the previous proof, we determined that if P is a ranked poset, $\mathrm{Row}(I) = \cup_{k \geq 1} \mathrm{Row}(I_k)$. This means that to perform $\mathrm{Row}(I)$, we can truncate at increasing ranks, perform finite rowmotion, and take the union of the results.

As a result, when referring to Row(I), we can now use the toggle definition of 5.1.2. We continue with a remark showing that Definition 5.1.2 matches the intuition of what we would expect from a toggle action on an infinite poset.

Remark 5.1.11. The definition of promotion on an infinite poset gives what one would intuitively expect. For example, first observe the any of the finite posets and order ideals on the left in Figure 5.2 under rowmotion. Because the first layer $L_1^0(I)$ is all of L_1^0, when toggling from top to bottom, we start toggling out elements of $L_1^0(I)$ and continue until an element has a cover in L_1^1. Now compare this to the infinite poset and order ideal in Figure 5.1a. Although we cannot toggle from top to bottom to apply rowmotion, because $L_1^0(I)$ is all of L_1^0, intuitively we would expect that we would start toggling elements out of $L_1^0(I)$ until an element has a cover in L_1^1. We see from Figure 5.1b that this is the case.

On the other hand, suppose we begin with the finite poset and empty order ideal in Figure 5.4. To match the notation of the finite case, we denote the action $\text{Pro}_{(-1,1)}$ as Pro. Pro toggles from left to right, toggling in elements of L_2^0 until the entire layer is in the order ideal. Compare this to the infinite poset and empty order ideal in Figure 5.5. Although we cannot toggle this poset from left to right, we would expect the same intuition to hold, that elements of L_2^0 would be toggled in until the entire layer is in the order ideal. The figure shows that after applying Definition 5.1.2, this is the case.

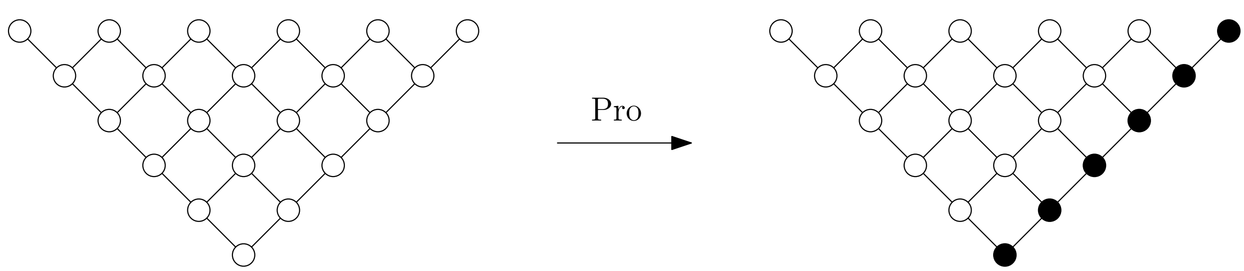


Figure 5.4. An empty order ideal of a finite poset. Performing Pro adds the layer L_2^0 to the order ideal.

With a finite poset, we saw that $\text{Pro}_{\pi,v}$ resulted in a bijective action. With an infinite poset, this is not necessarily the case.

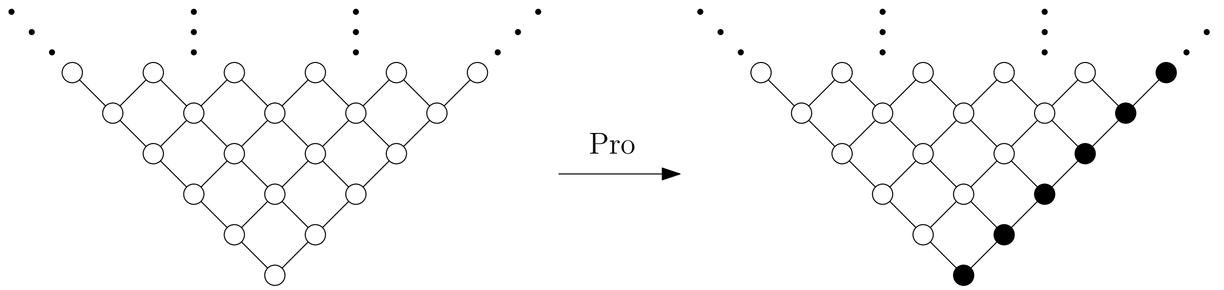


Figure 5.5. An empty order ideal in $\mathbb{N}^2$. Performing Pro adds the layer L_2^0 to the order ideal.

Remark 5.1.12. $\mathrm{Pro}_{\pi,v}$ does not necessarily result in a bijective action. Let $P = \mathbb{N}^2$, let I_1 be the empty order ideal, and I_2 be the order ideal with infinite layer L_1^0. However, $\mathrm{Pro}_{(-1,-1)}(I_1)$ and $\mathrm{Pro}_{(-1,-1)}(I_2)$ are both the full order ideal $\mathbb{N}^2$. See Figures 5.6 and 5.7 for this example. As a result, $\mathrm{Pro}_{(-1,-1)}$ is not invertible, so $\mathrm{Pro}_{\pi,v}$ is not necessarily bijective.

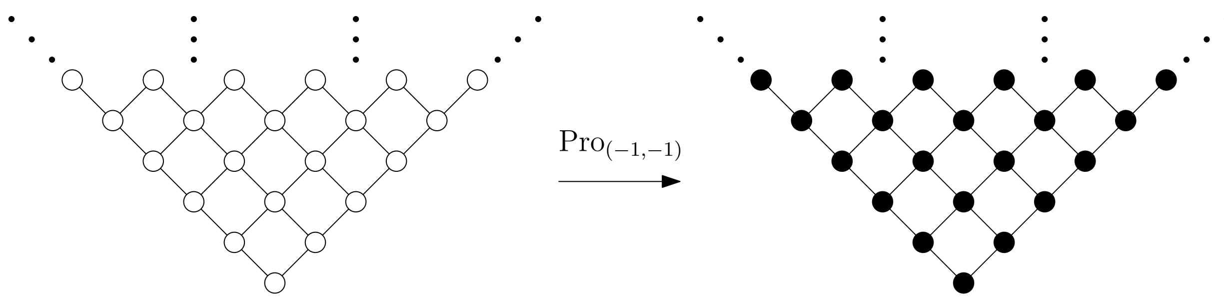


Figure 5.6. Applying $\mathrm{Pro}_{(-1,-1)}$ to the empty order ideal results in the full order ideal.

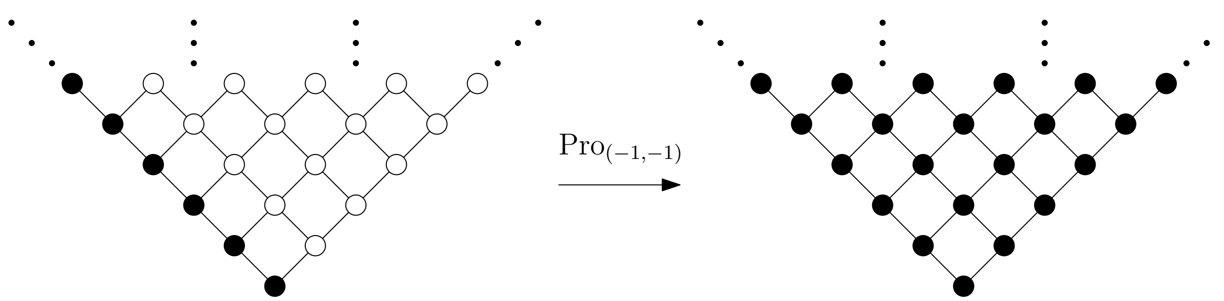


Figure 5.7. Applying $\mathrm{Pro}_{(-1,-1)}$ to the order ideal with infinite layer L_1^0 results in the full order ideal.

Promotion on an infinite poset was natural to define by truncating at increasing ranks. However, it is also natural to truncate using increasing finite products of chains $[k]^n$.

Remark 5.1.13. Here the notation $[k] = \{0, 1, \dots, k\}$ includes zero to match the inclusion of zero in our definition of $\mathbb{N}$. This differs from Definition 1.1.3 which did not include the element zero. In Section 5.2, our poset elements will represent exponents of *monomials*. We want these exponents to be non-negative integers, which is why we make this change. Despite this, any result on a product of chains poset from previous chapters can be rephrased in this new notation and retain its validity, as we have only shifted the labeling of the poset elements.

We now give Definition 5.1.14, a description of how to apply promotion on an infinite poset using truncated product of chains posets. In Proposition 5.1.15, we see that this new definition and Definition 5.1.2 are equivalent. Because we state this for a poset with n-dimensional lattice projection π, recall Definition 4.1.1 of π^{-1}.

Definition 5.1.14. Let P be a poset with an n-dimensional lattice projection $\pi : P \to \mathbb{N}^n$ and let I be an order ideal of P. Let $P'_k = \pi^{-1}(\pi(P) \cap [k]^n)$ be a subposet of P and $I'_k = I \cap P'_k$. Define $\mathrm{Pro}'_{\pi,v}(I) = \cup_{k\geq 1} \mathrm{Pro}_{\pi,v}(I'_k)$.

Proposition 5.1.15. *Suppose P is poset with n-dimensional lattice projection $\pi : P \to \mathbb{N}^n$. Then Definition 5.1.2 is equivalent to Definition 5.1.14. In other words, for $I \in J(P)$, $\mathrm{Pro}'_{\pi,v}(I)$ results in the same order ideal as $\mathrm{Pro}_{\pi,v}(I)$.*

Proof. First, note that using the same reasoning as in the proof of Lemma 5.1.3, $\mathrm{Pro}_{\pi,v}(I'_j) \subseteq \mathrm{Pro}_{\pi,v}(I'_{j+1})$. Also, using this same reasoning, for any j, $\mathrm{Pro}_{\pi,v}(I'_j) \subseteq \mathrm{Pro}_{\pi,v}(I_{2j}) \subseteq \cup_{k\geq 1} \mathrm{Pro}_{\pi,v}(I_k)$. As a result, $\mathrm{Pro}'_{\pi,v}(I) = \cup_{k\geq 1} \mathrm{Pro}_{\pi,v}(I'_k) \subseteq \cup_{k\geq 1} \mathrm{Pro}_{\pi,v}(I_k) = \mathrm{Pro}_{\pi,v}(I)$. On the other hand, for any j, $\mathrm{Pro}_{\pi,v}(I_j) \subseteq \mathrm{Pro}_{\pi,v}(I'_j) \subseteq \cup_{k\geq 1} \mathrm{Pro}_{\pi,v}(I'_k)$. Therefore, $\mathrm{Pro}_{\pi,v}(I) = \cup_{k\geq 1} \mathrm{Pro}_{\pi,v}(I_k) \subseteq \cup_{k\geq 1} \mathrm{Pro}_{\pi,v}(I'_k) = \mathrm{Pro}'_{\pi,v}(I)$. As we have subset inclusion in both directions, we obtain $\mathrm{Pro}'_{\pi,v}(I) = \mathrm{Pro}_{\pi,v}(I)$. $\square$

As a result of this proposition, if we wish to apply promotion to an infinite poset, we can truncate by increasing ranks or truncate by boxes increasing in size; both of these will give the same result. Therefore, for either case, we can use the notation $\mathrm{Pro}_{\pi,v}$ as this is unambiguous. We may find this helpful in specific instances, as we have a plethora of results for a finite product of chains.

5.2. Boundary paths and monomial ideals

In this section, we connect toggling actions on order ideals of $\mathbb{N}^n$ to *monomial ideals* of $\mathbb{K}[x_1, x_2, \ldots, x_n]$. We then define boundary paths on order ideals in $\mathbb{N}^2$ and show that Theorem 1.2.15, the shift of the finite boundary path on $[a] \times [b]$ under Pro, generalizes to $\mathbb{N}^2$. We conclude the section with Theorems 5.2.9 and 5.2.10, which investigate how the number of generators of a monomial ideal changes when applying Pro or Row to the corresponding order ideal $I \in J(\mathbb{N}^2)$. We begin by defining ideals, monomials, and monomial ideals.

Definition 5.2.1. An *ideal* I of a ring R is a subset of R such that I under addition is a subgroup of R under addition and $r \cdot x, x \cdot r \in I$ for all $x \in I, r \in R$.

With this explicitly stated, we can compare this definition of an ideal with Definition 1.1.10 of an order ideal. We will see how we can connect these two objects once we define monomial ideals.

Definition 5.2.2. Let $\mathbb{K}$ be a field and $\mathbb{K}[x_1, x_2, \ldots, x_n]$ be the polynomial ring over the variables $x_1, x_2, \ldots, x_n$. A *monomial* is a term of the form $\prod_{i=1}^{n} x_i^{\alpha_i} = x_1^{\alpha_1} \cdot x_2^{\alpha_2} \ldots x_n^{\alpha_n}$ where each α_i is a non-negative integer. A *monomial ideal* in $\mathbb{K}[x_1, x_2, \ldots, x_n]$ in an ideal generated by monomials.

Monomial ideals are well-studied by algebraists for a variety of reasons. They are well-behaved objects with nice properties. They are defined using polynomials, which are fundamental and natural algebraic objects. Additionally, they can be represented pictorially, giving further insight into their structure.

Definition 5.2.3. Let $P = \mathbb{N}^n$ and $I \subseteq P$. Define the monomial ideal $M(I) \subseteq \mathbb{K}[x_1, x_2, \ldots, x_n]$ such that $M(I) \cong P \setminus I$ where elements $(\alpha_1, \ldots, \alpha_n) \in P \setminus I$ correspond to monomials $x_1^{\alpha_1} \cdots x_n^{\alpha_n} \in M(I)$. We denote the minimal set of monomial generators of $M(I)$ as $G_{M(I)}$.

Because P is a product of chains, we use $\pi = id$ as the lattice projection for the remainder of this section. Also, although we give this definition for $\mathbb{N}^n$, our initial results focus on $\mathbb{N}^2$. Recall the definition of a boundary path and boundary path sequence from Definition 1.2.13. We use a similar definition for an infinite poset $\mathbb{N}^2$ with order ideal I.

Definition 5.2.4. Define the *boundary path* of an order ideal $I \subsetneq \mathbb{N}^2$ as a path of upsteps and downsteps that separates I from the rest of the poset. The *boundary path sequence* is $B(I) = (a_j)$

where $j \in \mathbb{Z}$ and each $a_j \in \{0,1\}$ where zeros correspond to downsteps and ones correspond to upsteps in the boundary path. We let a_0 correspond to the step immediately to the right of the line $y = x$. In this case, our boundary path sequence will have infinite length.

Our goal is to generalize the boundary path result in Theorem 1.2.15. However, the following remark shows us that the boundary path of an order ideal can aid us in studying the minimal generators of a corresponding monomial ideal.

Remark 5.2.5. Suppose $I \in J(\mathbb{N}^2)$ has boundary path $B(I)$. The $0,1$ subsequences in $B(I)$ are in bijection with the minimal generators of $\mathbb{N}^2 \setminus I$. This is because a $0,1$ subsequence gives us an element $p \in \mathbb{N}^2 \setminus I$, but guarantees that both elements covered by p in $\mathbb{N}^2$ are in I. As a result, $0,1$ subsequences in $B(I)$ are also in bijection with the minimal generators of $M(I)$, as $M(I) \cong \mathbb{N}^2 \setminus I$.

Example 5.2.6. Using the order ideal from Figure 5.1b, we show an example of a boundary path and boundary path sequence in Figure 5.8. The boundary path sequence of this order ideal is $(\ldots, a_{-6}, a_{-5}, a_{-4}, a_{-3}, a_{-2}, a_{-1}, a_0, a_1, a_2, a_3, a_4, a_5, \ldots) = (\ldots, 0, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 1, \ldots)$.

The result on the order of Pro, Corollary 1.2.16, does not generalize well from the finite case. However, the shift of the boundary path sequence under Pro of Theorem 1.2.15 does generalize to $\mathbb{N}^2$.

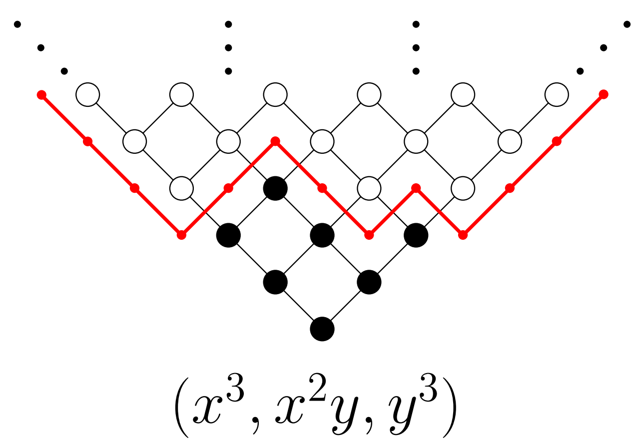
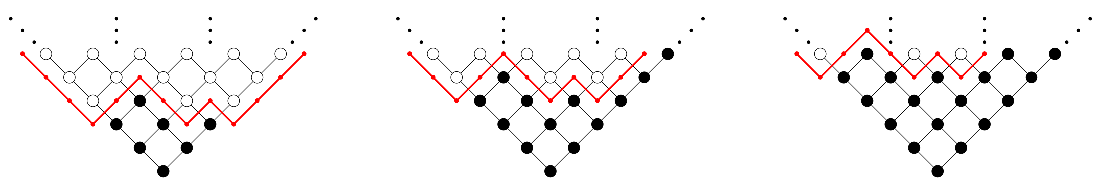


Figure 5.8. The boundary path is indicated in red. The corresponding boundary path sequence is $(\ldots, a_{-5}, a_{-4}, a_{-3}, a_{-2}, a_{-1}, a_0, a_1, a_2, a_3, a_4, a_5, \ldots) = (\ldots, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 1, \ldots)$.

Lemma 5.2.7. *Let $P = \mathbb{N}^2$ and $B(I)$ be the boundary path sequence of $I \in J(P)$ where $I \neq P$. Then $B(\mathrm{Pro}(I))$ is a left shift of $B(I)$.*

Proof. Since $B(I)$ is a boundary path sequence, $B(I) = (a_j)$ where $a_j \in \{0,1\}$. Denote $B(\text{Pro}(I)) = (b_j)$ where $b_j \in \{0,1\}$. Fix j; we must show $b_j = a_{j+1}$. We use Proposition 5.1.15 to work with the product of chains definition of Pro. When determining the behavior of the boundary path, we need to start with a sufficiently large finite poset. We find this by using the boundary paths a_j and a_{j+1}. Note that with a single boundary path step, the initial and terminal points of the boundary path step combined have at most 6 poset elements adjacent to them. With two consecutive boundary path steps, this number is at most 8. These will be the elements we use to ensure our box is of large enough size. Let k be greater than the maximum rank of the (at most) 8 poset elements adjacent to the boundary path points corresponding to a_j and a_{j+1}. Then $[k]^2$ is large enough such that the boundary path of $I'_k = I \cap [k]^2$ has a_j in position j and a_{j+1} in position $j+1$. By Theorem 1.2.15, $\text{Pro}(I'_k)$ has boundary path sequence a_{j+1} in position j. Because this hold for all sufficiently large k, it holds for $\text{Pro}(I)$; hence $b_j = a_{j+1}$. $\square$

Example 5.2.8. Let I denote the left order ideal in Figure 5.9. The middle order ideal is $\text{Pro}(I)$ and the right order ideal is $\text{Pro}^2(I)$. We observe the boundary path, denoted in red, is shifted by an application of Pro. As a result, the corresponding boundary path sequence is shifted to the left.



(x^3, x^2y, y^3) (x^3y, x^2y^2, y^4) (x^3y^2, x^2y^3, y^5)

$(\ldots, 0, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, \ldots)$ $(\ldots, 0, 0, 1, 1, 0, 0, 1, 0, 1, 1, 1, \ldots)$ $(\ldots, 0, 1, 1, 0, 0, 1, 0, 1, 1, 1, 1 \ldots)$

Figure 5.9. As described in Example 5.2.8, the boundary path sequence is shifted to the left when Pro is applied.

Because we can determine the generators of a monomial ideal from the boundary path sequence, we immediately obtain a result that when $P = \mathbb{N}^2$, the number of generators of the corresponding monomial ideal are invariant under Pro.

Theorem 5.2.9. *Let $P = \mathbb{N}^2$ and $I \in J(P)$. Then $|G_{M(\text{Pro}(I))}| = |G_{M(I)}|$.*

Proof. As mentioned in Remark 5.2.5, if $I \neq P$, a minimal generator of $M(I)$ corresponds to a subsequence $0, 1$ appearing in the boundary path sequence. Because $B(I)$ has a subsequence of infinitely many 0's to the left and Lemma 5.2.7 shows that promotion cyclically shifts the boundary path sequence to the left, the number of $0, 1$ subsequences in the boundary path sequence will not change. Hence, $|G_{M(\mathrm{Pro}(I))}| = |G_{M(I)}|$. Now suppose $I = P$. Then $\mathrm{Pro}(I) = I$, which implies $|G_{M(\mathrm{Pro}(I))}| = |G_{M(I)}| = 0$. Therefore, for any I, the theorem follows. $\square$

Additionally, using a result stated in [25], we see that under rowmotion, the number of generators of the corresponding monomial ideal increases by one.

Theorem 5.2.10. *Let $P = \mathbb{N}^2$ and $I \in J(P)$. Then $|G_{M(\mathrm{Row}(I))}| = |G_{M(I)}| + 1$.*

Proof. This is a consequence of Theorem 6.4.7 in [25], which says that if a monomial ideal in 2-dimensions has j generators, then it has $j-1$ corner elements. Suppose I is an order ideal such that $M(I)$ has j generators. In other words, $|G_{M(I)}| = j$. $\mathrm{Row}(I)$ is the order ideal generated by the minimal generators of $M(I)$, which means these are the corner points of $\mathrm{Row}(I)$. Therefore, $\mathrm{Row}(I)$ has j corner points and as a result, $|G_{M(\mathrm{Row}(I))}| = j + 1$, giving us the desired result. $\square$

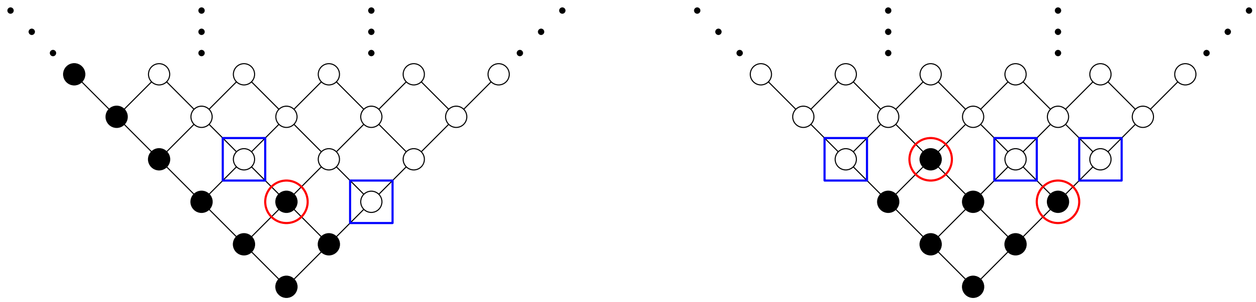
Example 5.2.11. For this example, we refer to Figure 5.10, which contains the same order ideal I and $\mathrm{Row}(I)$ as from Figure 5.1. Figure 5.10a shows I with the one corner element circled in red and the two generators of $M(I)$ boxed in blue. Figure 5.10b shows $\mathrm{Row}(I)$ with the two corner elements circled in red and the three generators of $M(\mathrm{Row}(I))$ boxed in blue. We see that the boxed monomial generators in $M(I)$ become the circled corner elements in $\mathrm{Row}(I)$. Because the number of monomial generators is one more than the number of corner elements for this case, $M(\mathrm{Row}(I))$ has exactly one more generator than $M(I)$.

5.3. Homomesy and recombination

In this section, Theorem 5.3.2 gives us a homomesy result on order ideals of the poset $\mathbb{N}^2$. Additionally, we generalize our recombination result from Theorem 2.2.4 to the infinite poset $\mathbb{N}^n$.

Without finite orbits, we cannot use Definition 1.3.1 to obtain homomesy results. However, in [31], Roby gives a more general definition of homomesy applicable to actions without finite orbits. We state this definition as follows.

Definition 5.3.1. Given a set S, an action $\tau : S \to S$, and a statistic $f : S \to \mathbb{K}$ where $\mathbb{K}$ is a field of characteristic zero, then (S, τ, f) exhibits *homomesy* if there exists $c \in \mathbb{K}$ such that



(a) This order ideal I has one corner element. $M(I)$ is generated by two elements.

(b) This order ideal is $\mathrm{Row}(I)$ and has two corner elements. $M(\mathrm{Row}(I))$ is generated by three elements.

Figure 5.10. We show I and $\mathrm{Row}(I)$ from Example 5.2.11 with the corner elements circled in red and the generators of the corresponding monomial ideal boxed in blue.

$$\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} f(\tau^i(x)) = c$$

is independent of the starting point $x \in S$. If such a c exists, we will say the triple is c-mesic.

We note that when τ is an invertible action with finite orbits, this reduces to Definition 1.3.1. Using this more general definition of homomesy, we obtain homomesy results from Lemma 5.2.7 and Theorem 5.2.10. Recall Definition 1.3.7 for the indicator function 1_x.

Theorem 5.3.2. *Let $P = \mathbb{N}^2$, $x \in P$. Then $(J(P), \mathrm{Row}, 1_x)$ and $(J(P), \mathrm{Pro}, 1_x)$ are both c-mesic with $c = 1$.*

Proof. Let $x \in P$ and $I \in J(P)$. By Theorem 5.2.10, the number of generators of the corresponding monomial ideal $M(I)$ increases by one after each application of Row. Therefore, there exists an N such that for all $i \geq N$, $x \in \mathrm{Row}^i(I)$. This implies $\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} 1_x(\mathrm{Row}^i(I)) = 1$. Thus, $(J(P), \mathrm{Row}, 1_x)$ is c-mesic with $c = 1$.

Similarly, by Lemma 5.2.7, the boundary path sequence shifts to the left after each application of Pro. Therefore, there exists an N such that for all $i \geq N$, $x \in \mathrm{Pro}^i(I)$. This implies $\lim_{N\to\infty} \frac{1}{N} \sum_{i=0}^{N-1} 1_x(\mathrm{Pro}^i(I)) = 1$. As a result, $(J(P), \mathrm{Pro}, 1_x)$ is c-mesic with $c = 1$. $\square$

We conclude this chapter by showing the recombination proof technique extends to the infinite setting. Using a similar approach as in Chapter 2, we let π be the natural embedding into $\mathbb{N}^n$. When $P = \mathbb{N}^n$, note that our definition for $\Delta_v^\gamma I$ from Definition 2.2.1 is still valid. We can see that with the same conditions as Lemma 5.3.3, performing recombination results in an order ideal.

Lemma 5.3.3. *Let $I \in J(\mathbb{N}^n)$. Suppose we have v and γ such that $v_\gamma = 1$. Then $\Delta_v^\gamma I$ is an order ideal of P.*

Proof. To show this, we use a similar strategy to the proof of Lemma 2.2.3. Pick $(i_1, \ldots, i_n) \in \Delta_v^\gamma I$. Because $(i_1, \ldots, i_n) \in \Delta_v^\gamma I$, we have $(i_1, \ldots, i_n) \in L_\gamma^{i_\gamma}(\mathrm{Pro}_v^{i_\gamma - 1}(I))$. Using a sufficiently large finite product of chains, we can show $(i_1, \ldots, i_\gamma - 1, \ldots, i_n) \in L_\gamma^{i_\gamma - 1}(\mathrm{Pro}_v^{i_\gamma - 2}(I))$ in a similar manner to Lemma 2.2.3. Therefore, $(i_1, \ldots, i_\gamma - 1, \ldots, i_n) \in \Delta_v^\gamma I$ and so $\Delta_v^\gamma I$ is an order ideal. $\square$

With the previous lemma, we can state our infinite recombination result. Recall Definition 2.1.1 for the notation $v^{\widehat{\gamma}}$.

Theorem 5.3.4. *Let $I \in J(\mathbb{N}^n)$. Suppose we have $v = (v_1, v_2, \ldots, v_n)$ where $v_j \in \{\pm 1\}$, $u = (u_1, u_2, \ldots, u_n)$ where $u_j \in \{\pm 1\}$, and γ such that $v_\gamma = 1$, $u_\gamma = -1$, and $v^{\widehat{\gamma}} = u^{\widehat{\gamma}}$. Then $\mathrm{Pro}_u(\Delta_v^\gamma I) = \Delta_v^\gamma(\mathrm{Pro}_v(I))$.*

Proof. As with Theorem 2.2.4, we would like to show for each layer we have $L_\gamma^k(\mathrm{Pro}_u(\Delta_v^\gamma I)) = L_\gamma^k(\Delta_v^\gamma(\mathrm{Pro}_v(I)))$. If we truncate to a finite product of chains $[\ell]^n$ where $\ell \geq k$, we have $L_\gamma^k((\Delta_v^\gamma I)'_\ell) = L_\gamma^k((\mathrm{Pro}_v^{k-1}(I))'_\ell)$ by definition. Using the same reasoning as in Theorem 2.2.4, our finite recombination result, $L_\gamma^k(\mathrm{Pro}_u((\Delta_v^\gamma I)'_\ell)) = L_\gamma^k(\mathrm{Pro}_v((\mathrm{Pro}_v^{k-1}(I))'_\ell))$. We use this to show subset inclusion of $L_\gamma^k(\mathrm{Pro}_u(\Delta_v^\gamma I)) = L_\gamma^k(\Delta_v^\gamma(\mathrm{Pro}_v(I)))$ in both directions.

Case $\subseteq$: Using $\mathrm{Pro}_v((\mathrm{Pro}_v^{k-1}(I))'_\ell) \subseteq \mathrm{Pro}_v(\mathrm{Pro}_v^{k-1}(I))$ and the statement above, we obtain $L_\gamma^k(\mathrm{Pro}_u((\Delta_v^\gamma I)'_\ell)) = L_\gamma^k(\mathrm{Pro}_v((\mathrm{Pro}_v^{k-1}(I))'_\ell)) \subseteq L_\gamma^k(\mathrm{Pro}_v^k(I))$. By definition, $L_\gamma^k(\mathrm{Pro}_v^k(I)) = L_\gamma^k(\Delta_v^\gamma(\mathrm{Pro}_v(I))$ and so $L_\gamma^k(\mathrm{Pro}_u((\Delta_v^\gamma I)'_\ell)) \subseteq L_\gamma^k(\Delta_v^\gamma(\mathrm{Pro}_v(I))$. As this is true for all truncated $(\Delta_v^\gamma I)'_\ell$, it holds for $\Delta_v^\gamma I$ as well. Therefore, $L_\gamma^k(\mathrm{Pro}_u(\Delta_v^\gamma I)) \subseteq L_\gamma^k(\Delta_v^\gamma(\mathrm{Pro}_v(I)))$.

Case $\supseteq$: Again, starting with $L_\gamma^k(\mathrm{Pro}_u((\Delta_v^\gamma I)'_\ell)) = L_\gamma^k(\mathrm{Pro}_v((\mathrm{Pro}_v^{k-1}(I))'_\ell))$, we can now use $(\Delta_v^\gamma I)'_\ell \subseteq \Delta_v^\gamma I$ to obtain $L_\gamma^k(\mathrm{Pro}_u(\Delta_v^\gamma I)) \supseteq L_\gamma^k(\mathrm{Pro}_u((\Delta_v^\gamma I)'_\ell)) = L_\gamma^k(\mathrm{Pro}_v((\mathrm{Pro}_v^{k-1}(I))'_\ell))$. Since this is true for all truncated $(\mathrm{Pro}_v^{k-1}(I))'_\ell$, it also holds for $\mathrm{Pro}_v^{k-1}(I)$. Therefore, $L_\gamma^k(\mathrm{Pro}_u(\Delta_v^\gamma I)) \supseteq L_\gamma^k(\mathrm{Pro}_v^k(I))$ and hence $L_\gamma^k(\mathrm{Pro}_u(\Delta_v^\gamma I)) \supseteq L_\gamma^k(\Delta_v^\gamma(\mathrm{Pro}_v(I)))$.

Because we showed subset inclusion in both directions, we obtain $L_\gamma^k(\mathrm{Pro}_u(\Delta_v^\gamma I)) = L_\gamma^k(\Delta_v^\gamma(\mathrm{Pro}_v(I)))$ for any layer, and hence, the desired result. $\square$

Note that although we have shown recombination for a product of chains, the same logic can be used for any infinite poset P with an n-dimensional lattice projection into $\mathbb{N}^n$.

6. FUTURE WORK

In this chapter, we present possible future avenues of research. In Section 6.1, we discuss generating all refined homomesies that involve only indicator functions in a product of chains. In Section 6.2, we theorize a homomesy result using an antichain cardinality statistics as opposed to our usual cardinality statistic. In Section 6.3, we further discuss minuscule posets cross a chain and possible homomesy results. In Section 6.4, we suggest possibilities using the work in Chapter 5 on infinite posets. This includes extending the recombination result, searching for additional homomesy results, and strengthening the connection between toggle dynamics and monomial ideals.

6.1. The subspace of homomesic statistics

Recall Theorems 1.3.9 and 1.3.10. These refined homomesy results of Propp and Roby showed the cardinality of antipodal elements and the cardinality of files in $J([a] \times [b])$ exhibit homomesy under Pro or Row. However, Propp and Roby were able to show a stronger result. For a poset P, consider the span of the set $S_P = \{1_x \mid x \in P\}$.

Theorem 6.1.1 ([30], with proof communicated by Einstein [14])**.** *Suppose $P = [a] \times [b]$. Then the set $\{1_x + 1_y \mid x, y \text{ are antipodal in } P\} \cup \left\{\sum_{x \in k} 1_x \mid k \text{ is a file in } P\right\}$ generates the subspace of homomesic statistics in span(S_P) for the case of* Row *acting on $J([a] \times [b])$.*

Theorem 6.1.2 ([30], with proof communicated by Einstein [14])**.** *Suppose $P = [a] \times [b]$. Then the set $\{1_x + 1_y \mid x, y \text{ are antipodal in } P\} \cup \left\{\sum_{x \in k} 1_x \mid k \text{ is a file in } P\right\}$ generates the subspace of homomesic statistics in span(S_P) for the case of* Pro *acting on $J([a] \times [b])$.*

In other words, on $J([a] \times [b])$ under Row or Pro, the only refined homomesic statistics that are linear combinations of indicator functions must be combinations of antipodal and file statistics.

In Theorem 3.3.4, we generalized the refined antipodal homomesy result to $J([2] \times [a] \times [b])$ under Pro_v for any v. Computations in SageMath [38] suggest that the subspace of homomesic statistics result should also generalize.

Conjecture 6.1.3. *Suppose $P = [2] \times [a] \times [b]$. The set $\{1_x + 1_y \mid x, y \text{ are antipodal in } P\}$ generates the subspace of homomesic statistics in span(S_P) for the case of Pro_v acting on $J([2] \times [a] \times [b])$ for any v.*

Note that no analogue of files are needed for this conjecture; it only requires antipodal statistics. This observation is obtained from our SageMath [38] computations. For several examples under Row, we found all homomesic statistics that are sums of indicator functions. For every example, the only statistics that appeared were antipodal. Additionally, we note that if the conjecture can be shown for any single Pro_v, we can obtain the result for all Pro_v using recombination.

6.2. Antichain cardinality

Recall our main homomesy result, Theorem 2.0.1, is a generalization of Theorems 1.3.2 and 1.3.3 of Propp and Roby. All of three of these theorems used the statistic of order ideal cardinality. However, Propp and Roby had an additional result using the statistic of antichain cardinality, or in other words, the cardinality of the generators of the order ideal.

Theorem 6.2.1 ([30])**.** *Let g be the antichain cardinality statistic. Then $(J([a] \times [b]), \mathrm{Row}, g)$ is c-mesic with $c = ab/(a + b)$.*

Using SageMath [38] to compute examples, we make the following conjecture.

Conjecture 6.2.2. *Let g be the antichain cardinality statistic. Then $(J([2] \times [a] \times [b]), \mathrm{Row}, g)$ and $(J([2] \times [a] \times [b]), \mathrm{Pro}_{(-1,-1,-1)}, g)$ are c-mesic with $c = 2ab/(a + b + 1)$.*

We note that Propp and Roby showed that Theorem 6.2.1 does not hold for Pro. Similarly, through computation, we note that homomesy does not hold for all other Pro_v.

Proposition 6.2.3. *Let g be the antichain cardinality statistic. The triple $(J([2]\times[3]\times[2]), \mathrm{Pro}_v, g)$ does not exhibit homomesy when $v \in \{(1, 1, -1), (-1, -1, 1), (1, -1, 1), (-1, 1, -1), (-1, 1, 1),$ $(1, -1, -1)\}$.*

Proof. A calculation using SageMath [38] shows that if $v \in \{(1, 1, -1), (-1, -1, 1), (-1, 1, 1),$ $(1, -1, -1)\}$ then $J([2] \times [3] \times [2])$ under Pro_v has 1 orbit with average antichain cardinality $7/6 \approx 1.17$, 2 orbits with average antichain cardinality $5/3 \approx 1.67$, 2 orbits with average antichain cardinality $11/6 \approx 1.83$, 2 orbits with average antichain cardinality $7/3 \approx 2.33$, 2 orbits with average antichain cardinality $5/2 = 2.5$, and 1 orbit with average antichain cardinality $8/3 \approx 2.67$. If $v \in \{(1, -1, 1), (-1, 1, -1)\}$ then $J([2] \times [3] \times [2])$ under Pro_v has 1 orbit with average antichain cardinality 1, 2 orbits with average antichain cardinality $5/3 \approx 1.67$, 2 orbits with average antichain cardinality 2, 1 orbit with average antichain cardinality $13/6 \approx 2.17$, 3 orbits with average antichain cardinality $7/3 \approx 2.33$, and 1 orbit with average antichain cardinality $7/2 = 3.5$. □

6.3. Minuscule posets

Our main homomesy result, Theorem 2.0.1, can be viewed as a result on order ideals of a poset obtained from taking a two-element chain cross a type A minuscule poset. Similarly, Corollary 4.2.1 is a homomesy result on a type B minuscule cross a two-element chain. However, there are additional minuscule posets to consider. More specifically, we consider the type D, type E_6, and type E_7 minuscule posets, which are sometimes referred to as the propeller, Cayley-Moufang, and Freudenthal posets, respectively. We show these in Figure 6.1.

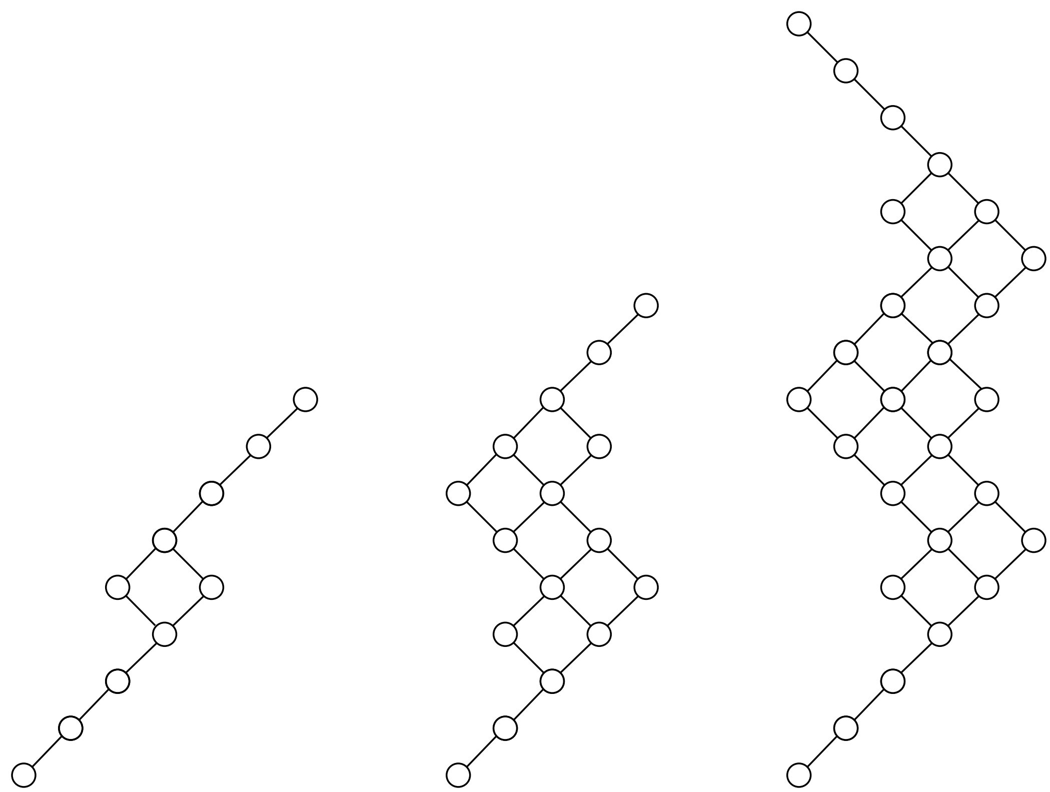


Figure 6.1. From left to right, we give examples a type D, the type E_6, and the type E_7 minuscule posets.

Based on personal communication with Pechenik and SageMath [38] computations, we make the following conjectures.

Conjecture 6.3.1. *Let f be the cardinality statistic, P be a type D minuscule poset, and $a \geq 2$ an integer. The triple $(J(P \times [a]), \mathrm{Pro}_v, f)$ exhibits homomesy.*

Conjecture 6.3.2. *Let f be the cardinality statistic, P be the type E_6 minuscule poset, and $a \geq 2$ an integer. The triple $(J(P \times [a]), \mathrm{Pro}_v, f)$ exhibits homomesy.*

Conjecture 6.3.3. *Let f be the cardinality statistic and P be the type E_7 minuscule poset. The triple $(J(P \times [2]), \mathrm{Pro}_v, f)$ exhibits homomesy.*

Again, note that if we can show these results for a single Pro_v, we obtain the results for all Pro_v using recombination. Also, if P is the type E_7 minuscule poset, a SageMath [38] computation shows $(J(P \times [3]), \mathrm{Pro}_v, f)$ does not exhibit homomesy.

Proposition 6.3.4. *Let f be the cardinality statistic and P be the type E_7 minuscule poset. $(J(P \times [3]), \mathrm{Pro}_v, f)$ does not exhibit homomesy for any v.*

Proof. A calculation using SageMath [38] shows that $J(P \times [3])$ under Row has 1214 orbits with average cardinality $81/2 = 40.5$, 1 orbit with average cardinality 40, and 1 orbit with average cardinality 41. Using recombination, we obtain the same result for any Pro_v. $\square$

6.4. Infinite posets

In Chapter 5, we introduced Definition 5.1.2 of promotion on an infinite poset P with n-dimensional lattice projection $\pi : P \to \mathbb{N}^n$. Because this is previously unstudied, there are many new directions of research we could take with infinite posets. We will mention several natural extensions to our results in Chapter 5.

Theorem 5.3.4 gave us a recombination result for order ideals of $\mathbb{N}^n$. We should be able generalize this from $\mathbb{N}^n$ to any poset P with n-dimensional lattice projection. Recall Definition 2.1.1 for the notation $v^{\widehat{\gamma}}$.

Conjecture 6.4.1. *Suppose P is a poset with n-dimensional lattice projection $\pi : P \to \mathbb{N}^n$ and let $I \in J(P)$. Suppose we have $v = (v_1, v_2, \ldots, v_n)$ where $v_j \in \{\pm 1\}$, $u = (u_1, u_2, \ldots, u_n)$ where $u_j \in \{\pm 1\}$, and γ such that $v_\gamma = 1$, $u_\gamma = -1$, and $v^{\widehat{\gamma}} = u^{\widehat{\gamma}}$. Then $\mathrm{Pro}_{\pi,u}(\Delta_v^\gamma I) = \Delta_v^\gamma(\mathrm{Pro}_{\pi,v}(I))$.*

Additionally, although we generalized recombination to the infinite case, we did not use it to prove any new results. If possible, we would like to find a use for recombination in the infinite case.

In Theorem 5.3.2, we obtained a homomesy result on $\mathbb{N}^2$. It would be natural to search for further homomesy results similar to this. More specifically, we should be able to generalize to a wider class of posets in higher dimensions.

In Theorems 5.2.9 and 5.2.10, we investigated how a single application of Pro or Row to $I \in J(\mathbb{N}^2)$ affects the number of generators of the corresponding monomial ideal $M(I)$ in $\mathbb{K}[x_1, x_2]$. As $\mathbb{K}[x_1, x_2]$ is a well-understood ring, it would be useful if we could extend our result to higher dimensions. Finally, to further connect our results to algebra, we would search for other algebraic properties that are predictable under single applications of promotion.

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