ASSESSING RELIABILITY OF HIGHLY RELIABLE PRODUCTS USING ACCELERATED DEGRADATION TEST DESIGN, MODELING, AND BAYESIAN INFERENCE

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Title

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The Supervisory Committee certifies that this ***disquisition*** complies with North Dakota State University's regulations and meets the accepted standards for the degree of

DOCTOR OF PHILOSOPHY

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ABSTRACT

The accelerated degradation test methods have proven to be a very effective approach to quickly evaluate the reliability of highly reliable products. However, the modeling of accelerated degradation test data to estimate reliability at normal operating condition is still a challenging task especially in the presence of multi-stress factors. In this study, a nonstationary gamma process is considered to model the degradation behavior assuming the strict monotonicity and non-negative nature of the product deterioration. It further assumes that both the gamma process parameters are stress dependent. A maximum likelihood method has been used for the model parameter estimation. The case study results indicate that traditional models that assume only shape parameter as stress dependent underestimate the product reliability significantly at normal operating conditions. This study further revealed that the scale parameter at a higher stress level is very close to the traditional constant assumption. However, at the normal operating condition, scale parameter value differs significantly with the traditional constant assumption value. This difference leads to the larger difference of reliability and lifetime estimates provided by the proposed approach. A Monte Carlo simulation with the Bayesian updating method has been incorporated to update the gamma parameters and reliability estimates when additional degradation data become available. A generalized reliability estimation framework for using the ADT data is also presented in this work.

Further, in this work, an optimal constant-stress accelerated degradation test plan is presented considering the gamma process. The optimization criteria are set by minimizing the asymptotic variance of the maximum likelihood estimator of the lifetime at operating condition under total experimental cost constraint. A heuristic based more specifically genetic algorithm approach has been implemented to solve the model. Additionally, a sensitivity analysis is performed which revealed that increasing budget causes longer test duration time with smaller sample size. Also, it reduces the asymptotic variance of the estimation which is very intuitive as more budget increase the possibility to generate more degradation information and helps to increase the estimation accuracy. The overall reliability assessment methodology and the test design has been demonstrated using the carbon-film resistor degradation data.

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DEDICATION

This disquisition dedicated to Iffat Toufique, Lamiha Shah, Monowara Begum, and Mafruha Akhter.

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LIST OF ABBREVIATIONS

AAF..Auxiliary Acceleration Factor

ADT ....Accelerated Degradation Test

ADDT ....Accelerated Destructive Degradation Testing

AF ....Acceleration Factor

AFT...Accelerated Failure Time

ALT...Accelerated Life Test

ASR...Asymptotic Sample Ratio

AT ....Accelerated Testing (or Test)

CE ....Cumulative Exposure

CSADT ....Constant-Stress Accelerated Degradation Test

CSALT..Constant-Stress Accelerated Life Test

FMEA ....Failure Modes and Effects Analysis

GP ....General Path

IG ....Inverse Gaussian

LED...Light Emitting Diode

LHD ....Latin Hypercube Design

LSE ....Least Square Estimate

MCMC ....Markov Chain Monte Carlo

MLE ....Maximum Likelihood Estimate

MSE ....Mean Squared Error

MTTF..Mean-Time-to-Failure

PALT...Partially Accelerated Life Test

PDD...Product Design and Development

PDF ....Probability Density Function

PH ....Proportional Hazards

PLD ....Penalized Local D-optimality

PO ....Proportional Odds

PSADT...Progressive Stress Accelerated Degradation Test

SSADT...Step-Stress Accelerated Degradation Test

SSALT ....Step-Stress Accelerated Life Test

UMVUE...Uniformly Minimum Variance Unbiased Estimator

USP ....Unique Selling Point

LIST OF SYMBOLS

α ....Gamma shape parameter

β ....Gamma scale parameter

c...Nonlinear constant

$\Gamma(.)$..Gamma function

i ....Number of observation

j ....Number of sample tested

k ....Level of stresses

y_{ijk} ....Degradation at i^{th} observation j^{th} sample and k^{th} stress level

Δy..Degradation increment

Δt ....Observation time interval

ω..Degradation threshold value

$f(.)$ ....Probability density function

$F(.)$ ....Cumulative density function

t_ω , ξ_ω...Lifetime at threshold point

$E(.)$ ....Expected value

P(.)...Probability function

P(. | .)...Conditional probability

L(.) ....Likelihood function

log L(.) ....Log-likelihood function

$\hat{x}$ ....Estimate of x

S_{1K}...Stress 1 at level k

S_{2K}...Stress 2 at level k

C_{op}...Operating costs

C_m ....Measurement costs

C_s ....Samples fixed costs

C_b ....Budget constraint

n_k ....Number of sample allocated at k^{th} stress level

m ....Measurement frequency

CHAPTER 1. INTRODUCTION

1.1. Overview

In this cutting-edge technological era, the manufacturers are increasingly under pressure to produce highly reliable products due to the immense pressure of global competition, increasing customer expectation, and maintain several regulatory compliances. This requires that manufacturers properly understand the failure behavior of the product and analyze product reliability for ensuring reliable design before it launches new products into the marketplace. Reliability is the probability that a product or a system will perform its intended function without failure for a specified period of time under specific operating conditions (O'Conor and Kleyner, 2012). The higher reliability of a product is not only a unique selling point (USP) of the product but it also reduces a huge amount of warranty related costs (Limon et al., 2016). For example, in the United States, manufacturers spend more than $25 billion per year towards warranty claims related issues (Mann et al., 2007).

It is, therefore, evident that predicting lifetime and reliability estimation during the product design and development (PDD) phases is a very critical issue. However, estimating reliability under normal operating conditions for sophisticated products is very time consuming and cumbersome effort due to the advancement of material technology and manufacturing processes. In most cases products are designed and built for working years without any failures and it is, therefore, difficult to get failure data or related information during the design phases. To overcome this difficulty, an accelerated test (AT) methods are being used to generate failure data, which have proven to be extremely useful not only to evaluate the reliability of highly reliable products but also to understand the failure behavior of the products.

In traditional AT methods, products are subjected to harsher than the normal operating condition to get the products fail and generate time-to-failure data, which is known as accelerated life test (ALT). Several acceleration and statistical methods are then utilized to model this time-to-failure information for estimating reliability under normal operating conditions. However, for highly reliable products, it is difficult even for ALT approach to make the product fail and generate time-to-failure data. To overcome these issues, an alternative approach known as an accelerated degradation test (ADT) has been employed to develop a better understanding of failure behavior and estimate the reliability of highly reliable products. The underlying assumption here is that the given product shows a measurable degradation characteristic before complete failure. The predetermined degradation threshold is used to obtain the time-to-failure data. Lu et al. (1996) argued that degradation approach provides more precise estimates of lifetime compared to the traditional failure time data analysis. Ling et al. (2015) also proposed to monitor the health and quality of a system and used observed accelerated degradation data to estimate several reliability metrics such as mean lifetime, reliability, and conditional reliability. The advantages of ADT is that it provides better reliability information than the traditional life test, requires fewer test samples, and observation of failures during the test is not required. Further, in ADT, the failure mechanism of the product can be visualized clearly, which provides better and clear insight into the degradation process that can be valuable information for the design improvement. Recently, several studies have used degradation data to assess the reliability parameters of highly reliable products such as aerospace electrical connector (Wenhua et al., 2011), LEDs (Liao and Elsayed, 2006), lithium-ion batteries (Tang et al., 2014b), and semiconductor ICs (Luo et al., 2014).

1.2. Dissertation Objectives

The ADT methods are used to expedite the failure mechanism of the product under higher stress levels. The degradation information from the test has been used to extrapolate the product-life under suitable physics, statistics, or combined model. However, modeling of accelerated degradation data to estimate reliability at normal operating condition is a challenging task. In the literature, the general degradation path and stochastic process approaches have been used to model the degradation behavior. The general degradation path model is a regression-type approach where the degradation of a product characteristic or functionality is considered as a linear or nonlinear function of time and applied stresses (Lu and Meeker, 1993). Recently, stochastic processes are becoming more popular for degradation modeling due to their capability of capturing the temporal variation in the degradation process and well established mathematical models (Limon et al., 2017a).

Among several stochastic processes, the gamma process is a suitable stochastic process to model the monotonic and strictly positive degradation behavior. However, the literature on gamma process to model ADT modeling is not plenty (Lawless and Crowder, 2004; Park and Padgett, 2005; Pan and Balakrishnan, 2011; Limon et al., 2017b). Most of the earlier studies on degradation modeling have considered single stress factor and linear degradation process under the assumption that only gamma shape parameter is dependent on stress factors (Lawless and Crowder, 2004; Park and Padgett, 2005). However, the increasing trend of using multiple stresses in ADT highlights the issues of interaction effects and nonlinearity affecting degradation process (Limon et al. 2017a). The recent studies have also revealed that consideration of only shape parameter as stress dependent may not be realistic (Bayel and Mettas, 2010; Balakrishnan and Ling, 2014; Limon et al., 2017b). Therefore, in this work, we aim to capture the interaction effects of multiple stress factors and construct a nonlinear degradation model using the stochastic gamma process. Further, to get a more realistic assessment, it is assumed that both gamma parameters (shape and scale) are stress dependent and are also affected by the interaction effect if it exists. Finally, the gamma parameters are estimated under normal operating conditions and approximated Birnbaum-Sanders failure model is used for assessing the reliability parameters.

To further improve the estimation accuracy of model parameter estimates, a Monte Carlo simulation study is conducted using initially estimated model parameters. We also propose a Bayesian framework to update model parameters when additional degradation data become available. Both the conjugate and non-conjugate Bayesian analysis have been proposed to update the gamma parameters and reliability estimates. The case example results indicate that the Bayesian update significantly reduces the estimation variability. A framework is presented to successfully model the ADT data for reliability parameter estimation considering the nonlinear degradation, multi-stress with interaction effect, and for both gamma parameters are dependent on multi-stresses. The proposed framework is also equally capable to model linear degradation behavior considering single stress effect. The case study example is discussed to demonstrate the applicability of the proposed approach and analyze the results.

Another important issue is that an ADT test is expensive and a poorly designed test will eventually yield wrong reliability estimation, which is a waste of both time and money. More importantly, inaccurate results could lead to poor decision making and increase warranty and liability issues for the manufacturer. Therefore, it is necessary to have an effective test plan that involved with optimal sample allocation at each stress level, incorporate the budget constraint, and simultaneously obtained a precise estimation. Realizing the importance of ADT planning, a constant-stress ADT design is proposed in this work considering the gamma process to capture the degradation behavior. The multiple stress loadings with possible interaction between stresses are also considered. To make the design more realistic, it is assumed that both the gamma parameters are stress dependent. To the best of our knowledge, there exists no research work that has attempted to consider both the gamma parameters as stress dependent on designing an ADT plan. Due to the multiple stresses, interaction effect, and assuming both gamma parameters are stress dependent, the Fisher information matrix as well as the asymptotic variance of the MTTF, become mathematically complex. To obtain the optimal solution analytically with the complex asymptotic variance, it is almost impossible to achieve an analytical solution of the objective function. Therefore, a heuristic search approach using the Genetic algorithm (GA) has been proposed to obtain the optimal solution. The sensitivity analysis of the budget constraint and model pre-estimates also shows that our proposed model is robust against these variables.

In summary, the primary research objectives of this work are: 1) to develop a realistic degradation modeling to estimate the reliability of a highly reliable product under normal operating condition using the ADT data and Bayesian approach, and 2) design an optimal ADT plan so that the reliability estimate become accurate under the budget constraints and constant stress loadings.

1.3. Dissertation Organization

Chapter two provides a detail of the accelerated test basics for example different accelerated loading conditions and life-stress models. Chapter three presented a literature review of accelerated tests, modeling approach of accelerated test especially for ADT test data and designing of AT plans. Chapter four provides the degradation modeling considering the gamma degradation process for accelerated degradation test data. The multi-stress acceleration with both gamma parameters are stress dependent model is described in this section. The possible interaction effects of stresses and the nonlinearity behavior also investigated. The Bayesian parameter updating methods and estimating remaining useful lifetime is discussed in chapter five. Chapter six consist of the optimal design of the constant-stress ADT considering the gamma process. Finally, chapter seven summarizes the whole work by describing the future research direction.

CHAPTER 2. FUNDAMENTALS OF AT METHODS

In an AT, products degradation as well as the failure mechanism are expedite to get failure information quickly. An effective test plan requires careful considerations of several issues, such as the type of AT methods, stress loadings, underlying product lifetime distribution, and the life-stress relationships. It is crucial that AT successfully imitates the field failure mechanisms in a laboratory environment. Therefore, to design effective AT methods, FMEA documents along with available failure/warranty data and engineering knowledge should be utilized to identify and select the relevant stress variables to reproduce failures during tests. In this chapter, the basic concepts of the AT methods and related general mathematical models are explained briefly under several sub-sections.

2.1. Accelerated Test (AT) Methods

AT methods can be broadly classified into two categories: accelerated life test (ALT) and accelerated degradation test (ADT). In ALT, the samples of a product are tested and the resulting failure times and censoring times are recorded. The data is then used to develop an ALT model for extrapolating the reliability of the product under the normal operating conditions. Usually, two censoring schemes are widely used in ALT: time censoring (type-I) where the number of actual failures is random upon the completion of the test (see Figure 2.1) and failure censoring (type-II) where the total test duration is random at the end of the test when a certain number of failures is observed. Compared to ALT, ADT methods are more suitable for highly reliable products that exhibit degradation before failure. In ADT, a performance or product characteristic is identified to measure the amount of degradation of the product. The degradation path is modeled and failure is defined when the degradation path reaches a pre-defined threshold. In a nondestructive ADT, several repetitive measurements can be taken during the experiment to continuously monitor the product's degradation behavior. In the case of destructive testing, only a single data point per unit can be observed leading to an accelerated destructive degradation test (ADDT) (Shi and Meeker, 2012). In the related literature, it is strongly argued the necessity of using the degradation tests for assessing the reliability of highly reliable products, if feasible (Lu and Meeker, 1993).

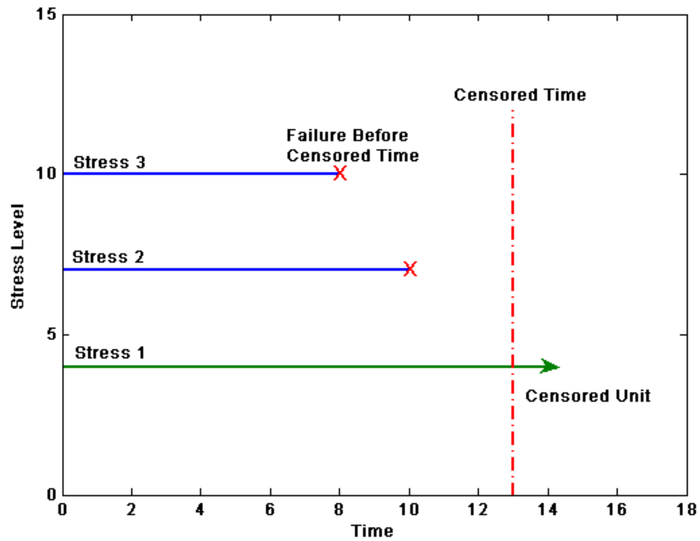


Figure 2.1: A graphical illustration of time-censored ALT

2.2. Acceleration Methods

An acceleration method is a way of increasing the usage rate or utilizing higher levels of stresses during AT. The method of usage rate acceleration is more applicable for products that are not in constant use, such as tires, light bulbs, and other similar products (Elsayed, 2012). The tests can be accelerated by simply increasing the operating hours per unit time (Yang, 2008). When it is not possible and/or efficient to make the product fail at a higher usage rate, accelerated stress test methods may be considered.

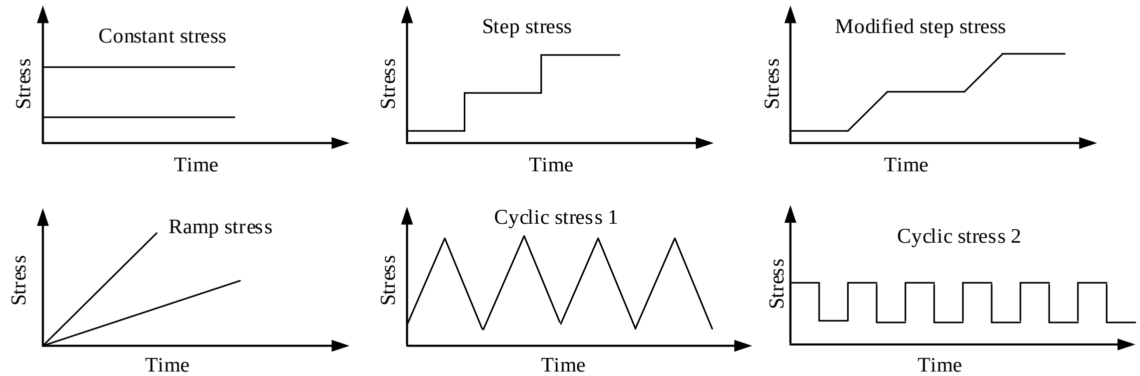


Figure 2.2: Different types of accelerated stress loadings

Regarding accelerated stress tests, four types of stress loading have been widely used: constant-stress, step-stress, progressive-stress, and cyclic-stress (Nelson, 2004). Figure 2.2 shows different types of stress loading usually applied in ATs. Among these stress loadings, constant-stress is the most commonly applied in AT (Nelson, 1981; Yang, 2008; and Zhu and Elsayed, 2013). Furthermore, the computational time, ease of stress application, and availability of existing theoretical models are the main factors that make the use of a constant-stress loading favorable. Nevertheless, the use of constant-stress loading sometimes may not result in failures during a specified test period and usually requires a lengthy test duration. To address this issue, step-stress loading yields failures relatively quicker than a constant-stress test (Miller and Nelson, 1983 and Xu and Fei, 2007). In a step-stress AT, the test specimen is subject to a constant-stress for a specific period of time, and then the stress is increased to the next higher level. This process continues until the specimen fails or is censored. The drawbacks of step-stress AT include the difficulty in parameter estimation and reliability extrapolation to the normal operating conditions, and the chances of introducing new failure modes. As another alternative, progressive-stress loading (a.k.a. ramp-stress loading) is also applied in AT. In a progressive-stress test, the units are subjected to a continually increasing stress over the test period (Nelson, 2004 and Yin and Sheng, 1987). For products that undergo cyclic-stresses in actual field operating conditions, cyclic-stress AT becomes the first choice, where test specimens are exposed to higher repetitive cyclic loadings, such as electrical sinusoidal voltage or fatigue stresses (Nelson, 2004).

2.3. Accelerating Stress Factors

The application of stress depends on the type of component under test and the stresses at the normal operating condition. For instance, vibration is often used to accelerate failures of mechanical components. In addition, humidity and random shock are other types of stresses applied to various mechanical components, such as bearing, shaft, and spring. For electronic components, temperature, humidity, vibration, electrical current, and voltage are common accelerating stress variables. In general, the most commonly applied stresses are temperature (Nelson, 1981; Munikoti and Dhar, 1998; and Wang and Chu, 2012), voltage (Munikoti and Dhar, 1998), current (Wang and Chu, 2012), humidity (Klinger, 1991), and UV radiation (Koo and Kim, 2005). The combination of these stress variables can also be applied according to actual operating conditions and the mechanisms behind a failure process (Vázquez et al., 2010).

Indeed, the number of accelerating stress variables to be used in AT is another important issue. The most common and preferable approach is a single-stress test. However, multiple-stress tests are receiving more attention in recent years (Zhu and Elsayed, 2013). The single-stress test method is simple to use and thus well documented and verified, whereas the multiple-stress test method has some issues, such as interaction effects, and the availability of appropriate models that relate life and stress. Despite these challenges, investigations on the multiple-stress ATs considering humidity and temperature (Klinger, 1991), and current and temperature (Vázquez et al., 2010) have been conducted.

2.4. Life-stress Relationship

The life-stress relationship plays an important role in data analysis and planning of an AT. For example, considering the lifetime of a product follows the normal or lognormal distribution, the simplest life-stress relationship that relates the product's mean or median life is a linear function (Kielpinski and Nelson, 1975 and Bai et al., 1989a) or log-linear function (Miller and Nelson, 1983 and Fard and Li, 2009) of applied stress as:

$$\mu(S) = \beta_0 + \beta_1 S \tag{2.1}$$

$$\ln[\mu(S)] = \beta_0 + \beta_1 S \tag{2.2}$$

respectively, where μ is the mean or median life, S represents the accelerating stress variable, and β_0 and β_1 are constants. The life-stress relationship for more than one stress variable considering an interaction effect can be expressed as (Park and Yum, 1997):

$$\ln[\mu(S_1, S_2)] = \beta_0 + \beta_1 S_1 + \beta_2 S_2 + \beta_3 S_1 S_2 \tag{2.3}$$

S_1 and S_2 are applied stresses. Unlike such purely statistical approaches, a life-stress relationship can also be obtained from various physics-based or empirical models. For example, the impact of temperature or thermal stress can be modeled by the Arrhenius law (Nelson and Kielpinski, 1976 and Gouno, 2007), and the inverse power law (Bai et al., 1992 and Bai et al., 1997) and exponential model (Park and Yum, 1997), which are appropriate for explaining the impact of non-thermal stresses on product lifetime. To capture a combined effect of thermal and non-thermal stresses, the generalized Eyring model is a popular choice (Park and Yum, 1996 and Tsai et al., 2014). The following transformations are often used to standardize stress variables in respective physical models:

For Arrhenius model: $S_i = \frac{1/S_0' - 1/S_i'}{1/S_0' - 1/S_M'}$ (2.4)

For power law model: $S_i = \frac{log(S_i') - log(S_0')}{log(S_M') - log(S_0')}$ (2.5)

For exponential model: $S_i = \frac{S_i' - S_0'}{S_M' - S_0'}$ (2.6)

where S_i', S_0', and S_M' represent the applied stresses of accelerated, normal, and maximum stress levels, respectively, whereas S_i *is* the transformed standardized stress ranging from zero to one.

2.5. Degradation Models

In ADT, degradation processes are usually modeled by either a general path (GP) model (Lu and Meeker, 1993) or a stochastic process model (Tang et al., 2014). Figure 2.3 illustrates a schematic of the degradation path.

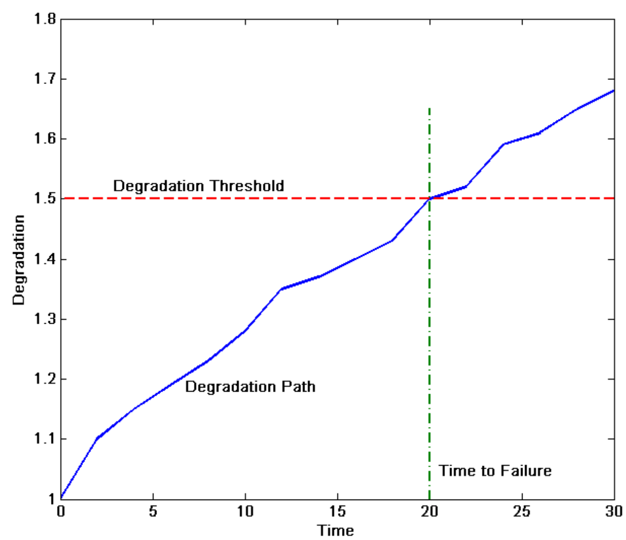


Figure 2.3: Schematic of a degradation path with the failure threshold

When using the GP model, the degradation path of a product is described by a linear or nonlinear function of time with the stress-dependent parameter(s). For example, a linear GP model states that the degradation measurement y_{ij} of unit i at time t_j can be described by:

$$y_{ij} = \eta_{ij}(t_{j,}\phi, \Theta_i) + \epsilon_{ij} \tag{2.7}$$

$$\eta_{ij}(t_{j,}\phi, \Theta_i) = \phi + \Theta_i t_j \tag{2.8}$$

where η_{ij} is the actual degradation path of unit *i* at time t_j, ϵ_{ij} is the measurement error having constant variance, ϕ is the fixed-effect parameter for all units, and Θ_i represents the random effect that varies from unit-to-unit and follows a certain distribution. In ADT, these parameters could be stress-dependent.

Sometimes appropriate physics-of-failure or empirical models, specific to a given product, are available to describe the degradation processes. For example, Wang and Chu (2012) used a nonlinear model for light emitting diode (LED) luminosity degradation and Tsai et al. (2013) proposed an exponential model to describe the degradation path of a polymer material. In a stochastic process model, the degradation measurements over time and the inherent variability of the degradation process are incorporated into a stochastic process. Recently, several popular stochastic processes have been used in modeling ADT data, such as the Wiener process (Tang et al., 2014a), Brownian motion process (Ge et al., 2010), gamma process (Tseng et al., 2009), and inverse Gaussian (IG) process (Ye et al., 2014). For example, using the Wiener process the degradation path $y(t, s)$ of a product is expressed in terms of time *t* and stress *s* as:

$$y(t, s) = \nu\eta(t, s) + \sigma B\big(\eta(t, s)\big) + \epsilon \tag{2.9}$$

where *ν* is the drift parameter, *σ* is the diffusion or volatility parameter, $B(\cdot)$ is the standard Brownian motion, $\eta(t, s)$ is a time and stress scale function, and ϵ is the measurement error with zero mean and a constant variance.

Among different stochastic processes, the gamma process has some attractive features, such as the monotonous increasing nature, that are suitable for modeling many degradation phenomena. Essentially, the degradation path $y(t,s)$ is described by the probability density function (pdf) of a gamma distribution as:

$$f_{Ga,\ (y(t,s))} = \frac{\beta^{\alpha(s)\eta(t)}}{\Gamma(\alpha(s)\eta(t))} y^{\alpha(s)\eta(t)-1} e^{(-y\beta)} \tag{2.10}$$

where α and β represent the shape and scale parameters, respectively, $\eta(\cdot)$ is a time scale function, and $\alpha(s)$ represents a shape parameter that is a function of stress s. The degradation increment and product lifetime both follow gamma distributions in the gamma process model.

The IG process has recently been used to model degradation phenomena. In particular, the probability density function of degradation path $y(t,s)$ is expressed as :

$$f_{IG,\ (y(t,s))} = \left(\frac{b}{2\pi y^3}\right)^{1/2} e^{\left[-\frac{b(y-a(s))^2}{2a^2(s)y}\right]} \tag{2.11}$$

where $a(s)$ is the stress-dependent mean and b is the shape parameter. The degradation rate and the lifetime both follow IG distributions in the IG process model. For more detailed discussions on modeling of ADT data refer to (Meeker et al., 1998a; Escobar and Meeker, 2006; and Ye and Xie, 2015).

2.6. Lifetime Distribution

Besides life-stress relationships, another important aspect that requires due consideration is the underlying lifetime distribution for a product. The popular accelerated failure time (AFT) models include the normal distribution, lognormal distribution (Nelson and Kielpinski, 1976), exponential distributions (Park and Yum, 1996), Weibull distribution (Meeker, 1984), and extreme value distribution (Nelson and Meeker, 1978). For the normal (or lognormal) distribution, it is often assumed that the distribution mean (or log-mean) depends on the stress level, and the variance is a constant (Nelson and Kielpinski, 1976). In earlier work considering the Weibull distribution, it was assumed that only the scale parameter is a stress-dependent while the shape parameter remains constant (Nelson and Meeker, 1978). The reason behind the constant shape parameter is that it is dependent on material properties and thus remains constant for all the stress levels (Klinger, 1992). However, recent studies question this assumption realizing that the variation in manufacturing processes, plants, and suppliers, and also a rapid improvement in the field of material science might cause stress-dependent shape parameter (Joyce et al., 1985 and Bayle and Mettas, 2010). Furthermore, Joyce et al. (1985) considered a log-linear relationship between the variance of lifetime and temperature stress for laser life analysis, and Seo et al. (2009) provided several examples where the shape parameter changes at higher stress levels. Subsequently, researchers considered that both the scale and shape parameters depend on stresses, which is more general and practical (Seo et al., 2009; Meeter and Meeker, 1994; and Hunt and Xu, 2012).

There are other probability distributions that have been considered in ALT data analysis, especially for periodic inspection-type ALT. These distributions include the Rayleigh distribution (Ahmad et al., 1994), exponentiated-Weibull distribution (Ahmad et al., 2006a), generalized-exponential distribution (Ahmad, 2010), and different kinds of Burr-type distributions (Ahmad and Islam, 1996; Ahmad et al., 2006b; and Ahmad et al., 2013). Ismail (2006) assumed that the lifetime follows the Gompertz distribution which is a relatively new distribution used in survival analysis. On the other hand, Elsayed and Zhang (2007) considered the proportional hazards (PH) model that does not assume any lifetime distribution.

In modeling ADT, a failure occurs when the product's degradation path crosses a pre-determined threshold for the first time (i.e., the first passage time). Let *D* be the failure threshold, the time to failure t_f under the linear GP model (Eqn. 2.8) is given by:

$$t_f = \left(\frac{D-\phi}{\Theta}\right) \tag{2.12}$$

where a probability distribution, such as lognormal, Weibull, and normal can be utilized to model the random-effect parameter (Wang and Chu, 2012 and Tsai et al., 2013).

CHAPTER 3. LITERATURE REVIEW

Accelerated tests have been getting much attention from the manufacturing and academic researcher during the late 1960s. This attention becomes more vibrant during the 1980s while there is a revolution in the quality management system, for example, customer satisfaction and warranty of the product is part of the quality assurance program. The methods and modeling of AT to predict lifetime and reliability have been investigated from different aspects. The existing literature can sharply categorize into two parts: AT for reliability matrices prediction and design of AT planning.

3.1. AT for Reliability Predictions

The initial works are heavily dominated in the area of ALT with constant stress loadings. Chernoff (1962) proposed accelerated life testing considering parametric distribution and censored data. The underlying lifetime is considered to follow the exponential distribution. Nelson (1975) demonstrated a lifetime estimation using ALT data with the least square method and inverse power law for complete type data. The two-parameter Weibull is assuming to represent the product lifetime in their work. However, the importance and superiority of the degradation data over failure-life data motivated researchers into the ADT to predict reliability measures. For instance, Nelson (1981) investigated the life-prediction of an insulator using its dielectric breakdown degradation due to temperature stress effect. The Arrhenius-lognormal model is chosen to model the ADT data and the maximum likelihood estimation (MLE) used for parameter estimates. Finally, insulator lifetime at normal operating condition is estimated using the model parameters. In another early work, Carey and Koenig (1991) also estimated the lifetime of an electrical component at normal use condition utilizing the degradation data measured at elevated temperature. Lu and Meeker (1993) first time proposed the regression-based general path (GP) model considering a two-stage method. In the first stage, model unknown parameters are estimated using the regression method and next, variances are estimated to obtain the overall estimates. Su et al. (1999) improved the estimation using the MLE instead of the least-square method and argued that MLE is statistically more efficient especially in case of small sample sizes. Shiau and Lin (1999) proposed a nonparametric regression method to assess the accelerated degradation path for reliability estimates. Meeker et al. (1998a) provide detail on the general path model for both linear and nonlinear degradation behavior.

3.1.1. Reliability assessment by stochastic processes

Because of several advantages mentioned in chapter 2, stochastics processes getting more attention in ADT data analysis. Whitmore and Schenkelberg (1997) considered the Wiener process to model the degradation behavior of ADT data. While modeling degradation behavior as Wiener process, several attempts have been made to capture the error in degradation measurement during the test (Whitmore 1995, Peng and Hsu 2012, Tang et al. 2014b) and also the variation within sample units known as random effects (Peng and Tseng, 2009; Si et al., 2013; Tang et al., 2014a). Liao and Elsayed (2006) consider the variation of the stress effect to model LED ADT data for reliability inferences. The author considered temperature and current as stress factors and their simulation results show that their method provides close to the actual lifetime prediction compared to the traditional method. The Geometric Brownian motion process is used by Park and Padgett (2005) for modeling ADT data. The authors argued that their model can approximate the failure time by Birnbaum–Saunders, and inverse Gaussian distributions efficiently.

To model monotonic degradation behavior, Gamma process is very suitable for reliability practitioners. For example, Bagdonavicius and Nikulin (2001) studied the effects of stress factors on product deterioration considering the gamma process. The authors also considered the intense stress effects (traumatic events) on the degradation process. Lawless and Crowder (2004) extended the gamma process model by capturing random effects or covariates and presented a tractable gamma process model. Realizing that lifetime estimation in gamma distribution is mathematically not tractable, Park and Padgett (2005) proposed the inverse Gaussian and Birnbaum-Saunders distributions as an efficient approximation of the gamma process model considering single stress factor. This work was further extended to consider two stress factors in degradation modeling (Park and Padgett, 2006). Recently, van Noortwijk (2009) presented a survey on the successful applications of the gamma process model in maintenance optimization. Pan and Balakrishnan (2011) introduced the reliability model for products subjected to the degradation of two performance characteristics by assuming the degradation of these two characteristics is governed by gamma processes. Very recently, the inverse Gaussian process has been getting attention to model the product deterioration (Wang and Xu, 2010; Ye and Chen, 2014; and Wang et al., 2016). Similar to the gamma process, the inverse Gaussian process also considers the nonnegative and monotonic changes in degradation.

3.2. Design of AT Planning

Selecting an appropriate AT method and proper test plan is essential for the effective use of available resources for ATs. There are several decision variables to be considered, such as the type of stress loading, number of stress levels, number of test units to be allocated to each stress level, and censoring schemes. Clearly, these decision variables must be determined under several constraints, such as the limited test time, budget, and availability of resources required for conducting the test. The optimal design of AT plan formulates and solves an optimization model considering these constraints. The rest of this section summarizes the state-of-the-art of optimal design of AT plans.

3.2.1. Designing ALT plan with constant stress

As the earliest effort in planning ALT, Chernoff (1962) developed an optimal ALT plan considering both complete and type-I censored data. It was assumed that the product's lifetime follows the exponential distribution and the failure rate is either a quadratic function or an exponential function of stress. Mann (1972) proposed a least-square curve fitting method to determine the stress levels and sample allocation for each stress level when the product's lifetime follows the Weibull distribution. Other optimal ALT plans were developed by assuming other lifetime distributions (Kielpinski and Nelson, 1975; Meeker and Nelson, 1975; and Nelson and Kielpinski, 1976). These optimal designs recommended two stress levels with a higher number of test units being allocated to the lower stress level (Kielpinski and Nelson, 1975). Because two-level ALT plans do not allow for validation of the life-stress relationship (Meeker, 1984), Nelson and Kielpinski (1976) proposed a robust compromise test plan with at least three stress levels. They also presented the compromise test plan theory that determines the third stress level as well as a sample allocation procedure. Meeker and Hahn (1985) presented 4:2:1 sample allocation ratio for the three-level compromise plan. Further, Yang (1994) presented a constant-stress ALT method with four stress levels and unequal censoring times. The result suggested a longer censoring time at the lower stress level and a shorter censoring time at a higher stress level. It also showed that the four-level ALT is more robust and shortens the test duration compared to the existing three-level tests.

Escobar and Meeker (1995) considered two stress variables in designing ALT with censoring. Park and Yum (1996) developed a test plan with two stress variables for the exponential lifetime distribution. The interaction effects of the two stresses were also considered using the generalized Eyring model. The experimental setup was arranged by a factorial design method, and the optimal test plan was obtained by minimizing the asymptotic variance of MLE of the mean life at the normal operating conditions. There are numerous applications on the use of multiple stresses in ALT (Zhu and Elsayed, 2013; Elsayed and Zhang, 2007; Elsayed and Zhang, 2009; Yang and Pan, 2013; and Balakrishnan and Ling, 2014).

In quality engineering, sampling is widely used as a tool for making an acceptance or rejection decision. The goals of designing a sampling plan are not only to reduce the cost of inspection but also reduce the risk of making the wrong decision. Similarly, in ALT it is important to reduce the time and cost of tests and assure that the tests are able to provide accurate reliability estimates. Bai et al. (1993a and 1995) developed ALT-based sampling plans that minimize the asymptotic variance of the test statistic for lot acceptability. The earlier research work on designing sampling plans assumed that the shape parameter of the Weibull distribution is constant at all stress levels, and later, cases with a non-constant shape parameter were considered. In particular, Seo et al. (2009) investigated both the time and failure censoring schemes. The optimal test criteria were chosen to satisfy both the producer’s and consumer’s risk requirements that minimize the asymptotic variance of the test statistic for lot acceptability.

A product may fail in more than one failure mode. The product fails whenever one of the competing failure modes occurs. For example, the piston-cylinder assembly of a hydraulic pump can fail due to wear-out or corrosion. Pascual (2007, 2008, and 2010) addressed the planning of ALT in the presence of independent competing risks under the Weibull (Pascual, 2007 and Pascual, 2008) or lognormal distribution (Pascual, 2010). The difference between the optimal test plans developed in (Pascual, 2007) and (Pascual, 2008) is whether or not the shape parameter of the Weibull distribution is known. A D-optimality criterion was considered to maximize the determinant of the Fisher information matrix of parameter estimates. Unlike most work on ALT, Liu and Tang (2010a) considered multiple independent risks for a repairable system. A power law life-stress relationship and Bayesian approach were used in test planning. The efficiency of the optimal test plan was compared with the compromise 4:2:1 test plan. It is worth pointing out that existing ALT plans considered independent competing risks. The most recent work by Zhang et al. (2014) considered the dependency of competing failures using the copula theory. They assumed a constant-stress loading and used the MLE for parameter estimation. It is obvious that the accuracy of statistical inference heavily relies on the choice of the copula model.

3.2.2. ALT design other than constant stress

To reduce the test duration further, step-stress ALT (SSALT) has attracted much attention in recent years. The drawback of SSALT is that the accuracy of reliability estimates from such tests is inversely proportional to the total testing time (Nelson, 2004). There are two ways to shift stress levels during SSALT. The popular way is to increase the stress to the next higher level after a certain period of time, and this process continues until all test units fail or at the end of the test. As an alternative, one can increase the stress to the next higher level upon a predetermined number or fraction of samples has failed under the current accelerated stress level. The popular cumulative exposure (CE) model is most commonly applied in analyzing SSALT data. According to this model, the remaining product life depends on the total stress exposure a product has experienced.

Among the early effort, Miller and Nelson (1983) considered the optimal design of simple SSALT for the exponential distribution with complete failure times. Bai et al. (1989a) extended the work by Miller and Nelson (1983) by introducing pre-determined censoring times. Bai and Kim (1993) developed an optimal simple SSALT considering the Weibull life distribution and type-I censoring. They also compared the simple SSALT plan to the three-level compromise test plan. They concluded that the simple SSALT plan suffers from the same drawback as two-level CSALT on the linear life-stress relationship and the use of two high stresses might introduce new failure modes.

To overcome the drawbacks of simple SSALT, Khamis and Higgins (1996) proposed a three-step SSALT plan considering a linear or quadratic life-stress relationship. Xu and Fei (2007) extended the earlier work by Escobar and Meeker (1995) into multi-stress SSALT plan. The extension of the model for an SSALT plan considering the CE model for the step-stress and the optimal test plan was obtained by minimizing the asymptotic variance of the MLE of a specific lifetime quantile. Ma and Meeker (208) considered a multi-step SSALT plan for both the Weibull and lognormal distributions and the CE model.

The progressive-stress loading can be applied to test units under a continuously increasing stress. The progressive-stress is also known as the ramp-stress loading when the stress linearly increases over time. Yin and Sheng (1987) first introduced progressive-stress ALT (PSALT) plans considering both the exponential and Weibull life distributions. Bai et al. (1992 and 1997) investigated PSALT with a simple ramp-stress loading. They assumed the Weibull lifetime distribution and the inverse power law as the life-stress relationship. The optimal test plan in (Bai et al., 1992) was obtained by minimizing the asymptotic variance of the MLE of a specific life quantile at the use stress level.

For a step-stress loading, a sudden change in stress level can cause shocks that might also introduce new failure modes of the product. To overcome this problem, Park and Yum (1998) introduced modified stress changing strategies. They proposed a finite rate of stress change instead of a sudden change in step-stress loading. It was claimed that the statistical efficiency of the model is not affected by the modified stress-loading except that the rate of stress change is extremely low. Liao and Elsayed (2010) developed an optimal ALT plan considering a ramp-stress loading and compared it with the equivalent CSALT plan. Zhu and Elsayed (2011) developed optimal step and ramp-stress ALT plans such that its estimation precision is equivalent to the CSALT plan. The advantage of these equivalent ALT plans is that they result in the same reliability estimation precision with fewer test resources. Hong et al. (2010) developed an optimal ALT method that determines both the ramp-rate and lower starting stress level simultaneously.

3.3. Designing ADT Plan

For highly reliable products that exhibit degradation before failure, ADT is more appropriate and effective than the ALT alternative. Compared to ALT, a few more decision variables need to be determined in ADT, such as measurement frequencies and the total number of measurements. Like ALT, constant stress, step stress, and progressive stress have been utilized in conducting ADT. The initial efforts in the design of ADT plans used an equal spacing of stress levels with equal sample allocation (Nelson, 1981). To improve the efficiency of ADT plan, Boulanger and Escobar (1994) developed an optimal ADT plan considering a constant-stress loading. The optimal test plan was obtained by minimizing the variance of the weighted least squares estimate (LSE) of the mean life at the use condition. Measurement times were determined under the D-optimality criterion, and several heuristic plans were also discussed. Another important issue in ADT planning is to determine the appropriate termination time of a test because it affects both the experimental cost and the estimation accuracy. Tseng and Yu

(1997) and Yu and Tseng (1998) developed several termination rules for a degradation test. They proposed an intuitive method that utilizes ALT and MLE for estimating the MTTF at normal operating conditions and determines degradation test termination time using the limiting property of estimator of MTTF. In addition to stress acceleration, the critical threshold value determines the times to failure in ADT. Yang and Yang (2002) suggested that tightening the critical threshold value can produce more failure data and reduces the asymptotic variance of reliability estimate. They also developed an approach to estimate the model parameters, which is more robust compared to the existing two-level ADT plans.

3.3.1. Destructive type ADT design

In planning ADT, the total budget has been considered as a major constraint (Wu and Chang, 2002; Yu, 2003; and Li and Kececioglu, 2004). For example, a nonlinear regression model (Wu and Chang, 2002) and nonlinear mixed integer programming methods (Yu, 2003) were used to obtain the optimal ADT plans. The test parameters were selected to minimize the mean squared error (MSE) of the estimated percentile of the product life at the operating conditions under the budget constraint. In another work, Li and Kececioglu (2006) studied ADT of LEDs and developed an analytical and simulation method to design the optimal test plan.

In an ADDT, only one degradation measurement can be collected from each test unit. Park and Yum (1997) and Shi et al. (2009) were the first few who attempted to design optimal ADDT plans. In these efforts, the optimal test plan was obtained by minimizing the variance of the MLE of a specific quantile of product lifetime. In particular, Park and Yum (1997) developed a non-linear constrained optimization model for the ADDT test plan. Similar to non-destructive tests, experimental costs were also taken into consideration in many ADDT plans (Tsai et al., 2013; Wang et al., 2009; and Yu and Peng, 2014). Wang et al. (2009) used the Monte Carlo simulation to design an optimal test plan. In addition, both linear and non-linear degradation models were considered in the design of optimal ADDT plans (Tsai et al., 2013 and Yu and Peng, 2014). The misspecification of lifetime distribution in ADDT method was also considered by Jeng et al. (2011).

3.3.2. Step and progressive-stress ADT design

The step and progressive-stress loadings were also considered in recently developed ADT methods. For instance, Park and Yum (2001) and Park et al. (2004) considered a step-stress accelerated degradation test (SSADT) plans with a constant degradation rate. The remaining useful life of a component under the step-stress loading was modeled by the popular CE model. The optimal test plans were determined by minimizing the asymptotic variance of the MLE of a specified quantile of the lifetime under the use condition. Haghighi (2014b) considered competing risks in designing an SSADT test plan. The author considered that the intensity functions of competing risks depend on the amount of degradation of the component. It is also assumed that the degradation process follows a known concave degradation path. Recently, Peng and Tseng (2010) investigated the progressive-stress ADT (PSADT) plan by assuming a non-linear degradation process. The exact relationship between the lifetime distributions of the PSADT and the CSADT was established to estimate the product lifetime under the normal operating conditions.

3.4. Bayesian Method

In many cases, information about the underlying degradation process and model parameters can be obtained from past studies. However, only little effort has been focused on Bayesian-based ADT plans. Recently, Liu and Tang (2010c) proposed a Bayesian approach to the optimal design of ADT plan considering the power law. The objective was to minimize the expected pre-posterior variance of model parameters under the use conditions. The results showed that the test plan is quite robust against model parameter uncertainty compared to a non-Bayesian approach. Shi and Meeker (2012) also investigated the Bayesian method and a non-linear degradation model to develop an ADDT plan. This work utilized the Bayesian method to incorporate the available historical information using a prior distribution. The large sample approximation was also used for analyzing the posterior distribution. Most recently, Li et al. (2017) proposed a Bayesian optimal ADT design considering the inverse Gaussian process to model the degradation behavior. The MCMC technique along with a surface fitting method was used to obtain the optimal design. It was claimed that the resulting Bayesian optimal design is more robust than the relative entropy and quadratic loss method.

3.5. Stochastic Processes

In recent years, stochastic process models are widely used in degradation modeling. Among those models, the Wiener process has been the most popular (Lim and Yum, 2011; Lim, 2012; Tang et al., 2004; and Hu et al., 2015). Tang et al. (2004) and Hu et al. (2015) considered the Wiener process in SSADT to capture unit-to-unit variation in a non-linear degradation process. They investigated both the constant and step-stress loadings in these tests. Both the budget constraint and estimation precision were taken into account in planning ADT (Lim, 2012 and Tang et al., 2004). Liao and Elsayed (2004) proposed an ADT method using the geometric Brownian motion to model the degradation rate that provides better statistical efficiency. In a similar work, Zhang et al. (2010) assumed the degradation path follows the drift Brownian motion in designing the ADT method. To further accelerate the test, step-stress loading was also considered in planning ADT (Zhang and Jiang, 2010 and Zhang et al., 2011). The optimal test plans were obtained using Monte Carlo simulation to minimize the MSE of reliability estimate at the normal operating conditions under a budget constraint. Liao and Tseng (2006) considered the Weiner process in designing an optimal SSADT plan. The asymptotic variance of the estimated percentile of the lifetime was minimized under a cost constraint. Moreover, Peng and Tseng [154] considered the Wiener degradation process and the linear drift rate to design an optimal PSADT plan.

Recently, gamma process (Tseng et al., 2009; Tsai et al., 2012; and Zhang et al., 2015) has been considered along with unit-to-unit variation (i.e., random effects) (Tsai et al., 2016) in modeling ADT and test plans. It has also been used in multi-stress ADT plans (Tsai et al., 2014 and Pan and Balakrishnan, 2010). For instance, Pan and Balakrishnan (2010) investigated both the Weiner and gamma degradation processes in multiple-step SSADT and used the Bayesian MCMC method to obtain efficient reliability estimates. The inverse Gaussian (IG) process has recently been taken into consideration to model degradation phenomena. Ye et al. (2014) investigated the IG degradation process in designing CSADT plans. The research considered the scenarios with and without random effects. The sensitivity analysis showed the robustness of the model to the presumed model parameters.

3.6. Others ADT Designs

Yu and Chiao (2002) proposed an ADT plan by considering a degradation process following the lognormal distribution. The optimal test plan was obtained such that the width of the confidence interval for the MTTF at the use condition was minimized. Similarly, Yu (2006) designed an optimal ADT plan where the degradation rate follows a reciprocal Weibull distribution. The MSE of the specified percentile of lifetime estimate at the normal operating conditions was minimized under a budget constraint. Tseng and Lee (2016) proposed a sample allocation method assuming the degradation model belongs to an exponential dispersion class.

They argued that their model is a generalized model that includes other stochastic models as its special cases. The optimal plan was obtained by minimizing the variance, and both two- and three- level ADTs were investigated.

CHAPTER 4. RELIABILITY ASSESSMENT BY ADT DATA

4.1. Degradation Modeling: Gamma Process

Abdel-Hameed (1975) first time considered the gamma process for modeling the deteriorating wear process. The gamma process represents the degradation behavior in a form of cumulative damage where the deterioration occurs gradually over the period of time. The increment of the degradation process is always considered to be monotonic and nonnegative. The schematic of a gamma deterioration process is illustrated in Figure 4.1. Assuming a random variable Y represents the deterioration then the gamma process that is a continuous-time stochastic process has the following mathematical properties (O'Connor and Kleyner, 2012)

a. $y(0) = 0$

b. $y(t)$ follow a gamma distribution with $Ga \sim (\alpha t, \beta)$

c. $y(t)$ has an independent increment in a time interval Δt ($\Delta t = t_i - t_{i-1}$)

d. The independent increment $\Delta y(t) = y_i - y_{i-1}$ also follows the gamma distribution $Ga \sim (\alpha \Delta t, \beta)$ with probability density function (PDF):

$$f_{\Delta y(t)} = \frac{\beta^{\alpha \Delta t}}{\Gamma(\alpha \Delta t)} \Delta y^{\alpha \Delta t - 1} e^{-(\beta \Delta y)} \tag{4.1}$$

where, $\alpha > 0$ and $\beta > 0$ represent the gamma shape and scale parameters, respectively, and $\Gamma(.)$ is a gamma function with $\Gamma(a) = \int_0^\infty x^{a-1} e^{-(x)} dx$. While using the gamma process for degradation modeling, it is also important to capture the relationship between deterioration and time. Several empirical studies on engineering applications show that expected deterioration ($E\big(y(t)\big) = \frac{\alpha}{\beta} t^c$) follows the power model and given as (van Noortwijk, 2009):

$$y(t) = bt^c \tag{4.2}$$

where t represents time and c is a nonlinearity parameter. To capture the nonlinearity in the degradation process, the equation (4.1) is modified by incorporating the nonlinearity parameter as given below:

$$f_{\Delta y(t)} = \frac{\beta^{\alpha(t_i^c - t_{i-1}^c)}}{\Gamma(\alpha(t_i^c - t_{i-1}^c))} \, \Delta y^{\alpha(t_i^c - t_{i-1}^c)-1} \, e^{-(\beta\Delta y)} \tag{4.3}$$

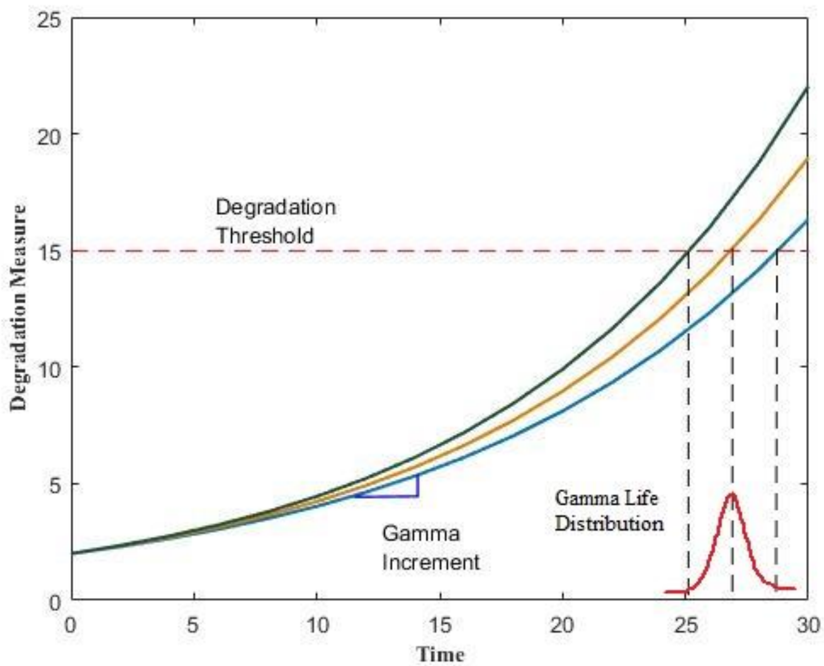


Figure 4.1: Schematic of a degradation process with gamma increment

4.2. Accelerated Degradation Modeling

In an accelerated test, to expedite the degradation process, product samples are subjected to higher stress levels than the normal operating conditions. The underlying assumption for designing an accelerated test is that it only expedites the identified failure process without introducing any new failure mechanism. There exist several physical or empirical life-stress models to express the effect of stresses on product lifetime. For example, the Arrhenius model provides a well-recognized relationship to capture the effect of temperature on the median life of the product. Equation (4.4) provides several well established life-stress models (Nelson, 2004):

$$
\begin{aligned}
t_m(s) &= a_1 e^{-\frac{a_2}{T}}; \quad Arrhenius\ model\ (s = T) \\
&= a_1 V^{a_2}; \quad Power\ law\ model\ (s = V) \\
&= a_1 e^{a_2 W}; \quad Exponential\ model\ (s = W)
\end{aligned}
\tag{4.4}
$$

Since the magnitude of stress measurement units may differ significantly in the multi-stress scenario, it is important to use standardized transform stresses to disregard the influence of stress measurement units. Chapter 2 provides an appropriate transformation formula for different life-stress models.

Suppose in an accelerated degradation test, y_{ijk} represents the i^{th} observation of the j^{th} sample under the k^{th} stress level at the time period t_{ijk} . If the degradation increment can be expressed by $\Delta y_{ijk} = y_{ijk} - y_{(i-1)jk}$, then, according to the properties of the gamma process and equation (4.3), the likelihood function of the degradation increment can be given as:

$$
L(\alpha_{jk}, \beta_{jk}) = \prod_{i=1}^{n} \frac{\beta_{jk}^{\alpha_{jk}\,(t_{ijk}^{c} - t_{(i-1)jk}^{c})}}{\Gamma(\alpha_{jk}\,(t_{ijk}^{c} - t_{(i-1)jk}^{c}))} \Delta y_{ijk}^{\alpha_{jk}\left(t_{ijk}^{c} - t_{(i-1)jk}^{c}\right) - 1} e^{(-\Delta y_{ijk}\beta_{jk})}
\tag{4.5}
$$

Here α_{jk} and β_{jk} represent the shape and scale parameter of j^{th} sample at k^{th} stress level, respectively. The estimates of shape and scale parameters for each sample can then be utilized to understand the random effects of the degradation.

Most of the earlier work on gamma degradation process had assumed that the only shape parameter (α) depends on stress factor while the scale parameter (β) remains constant (Park and Padget, 2005; Lawless and Crowder, 2004; Tseng et al., 2009). This assumption was based on the understanding that the activation energy for all samples is constant. The activation energy represents the minimum energy required to initiate the failure mechanism. The higher activation energy suggests the requirement of higher energy and higher stress levels to initiate failure. However, recent studies show that the activation energy can not only change over time due to improvement in product design, but also acquire different values for different failure mechanisms, production lots coming from the different manufacturing process, and different versions of the product (Bayel and Mettas, 2010). This implies that activation energy might vary from unit to unit and also could be dependent on stress levels. Additionally, the physical degradation rate is expressed by the ratio of shape and scale parameters by a gamma process. The variation of the degradation rate and increment process also can be expressed in terms of gamma parameters. Several recent studies have concluded that both the degradation rate and variability in degradation are stress dependent (Limon et al., 2017a; Rathod et al., 2011). It is, therefore, reasonable to assume that both the shape and scale parameters depend on stress factors. Moreover, several studies considered the non-constant scale parameter during the accelerated life test design considering different lifetime distribution (Meeter and Meeker, 1994; Balakrishnan and Ling, 2014). Further, in the presence of multiple stress conditions, the interaction between stresses could also be causing changes scale parameter. Thus, the effect of stress variables and their interactions on gamma parameters can be modeled using the general Eyring law as follows:

$$\alpha(s) = exp^{\left(a_0 + \sum_{i=1}^{r} a_i S_i + \sum_{i,j=1,i\neq j}^{q} a_{ij} S_i S_j\right)} \quad (4.6)$$

$$\beta(s) = exp^{\left(b_0 + \sum_{i=1}^{r} b_i S_i + \sum_{i,j=1,i\neq j}^{q} b_{ij} S_i S_j\right)} \quad (4.7)$$

where S_i represents the standardized transformed stress, *a* and *b* represent the corresponding stress coefficients that need to be estimated using the test data and r and q represent the number of stress factors and the number of interactions between stresses, respectively. Now considering the parameter-stress relationship in equations (4.6-4.7), the likelihood function of all degradation increments data is written as:

$$L\left(\hat{\theta}\right) = \prod_{i=1}^{n}\prod_{j=1}^{m}\prod_{k=1}^{p} \frac{\left[exp^{\left(b_0+\sum_{i=1}^{r} b_i S_i+\sum_{i,j=1,i\neq j}^{q} b_{ij} S_i S_j\right)}\right]^{exp^{\left(a_0+\sum_{i=1}^{r} a_i S_i+\sum_{i,j=1,i\neq j}^{q} a_{ij} S_i S_j\right)}\left(t_{ijk}^{c}-t_{(i-1)jk}^{c}\right)}}{\Gamma\left[exp^{\left(a_0+\sum_{i=1}^{r} a_i S_i+\sum_{i,j=1,i\neq j}^{q} a_{ij} S_i S_j\right)}\left(t_{ijk}^{c}-t_{(i-1)jk}^{c}\right)\right]} \Delta y_{ijk}^{\left[exp^{\left(a_0+\sum_{i=1}^{r} a_i S_i+\sum_{i,j=1,i\neq j}^{q} a_{ij} S_i S_j\right)}\left(t_{ijk}^{c}-t_{(i-1)jk}^{c}\right)-1\right]} exp^{\left[-\Delta y_{ijk} exp^{\left(b_0+\sum_{i=1}^{r} b_i S_i+\sum_{i,j=1,i\neq j}^{q} b_{ij} S_i S_j\right)}\right]} \quad (4.8)$$

where $\hat{\theta} = (\hat{a}_0, \hat{a}_i, \hat{a}_{ij}, \hat{b}_0, \hat{b}_i, \hat{b}_{ij}, \hat{c})$. Depending on the applied stresses and degradation behavior, the number of parameters in equation (4.8) will vary. Further, the nonlinear equation with multiple unknown parameters presents a greater challenge to estimate parameter values. The maximum likelihood method with advance optimization software R can be utilized to solve this complex equation. The built–in 'mle' function that uses the Nelder-Mead algorithm (optim) to optimize the equation can be used to estimate model parameters. Once the model parameter values $(\hat{a}_0, \hat{a}_i, \hat{a}_{ij}, \hat{b}_0, \hat{b}_i, \hat{b}_{ij}, \hat{c})$ are obtained, the gamma shape and scale parameters at any stress level can be estimated using equations (4.6-4.7).

4.3. Lifetime and Reliability Estimates

In a stochastic process, the failure is determined when the first passage of time reaches the threshold degradation value. Now assuming that a failure occurs while the degradation path reaches the threshold ω, then the time to failure, t_{ω}, is define the time when degradation path cross the threshold ω and the reliability function at time *t* will be,

$$R(t) = P(t < t_\omega) = 1 - \frac{\Gamma(\alpha t^c, \omega_\beta)}{\Gamma(\alpha t^c)} \quad (4.9)$$

where, $\omega_\beta = (\omega - y_0)\beta$ and y_0 is the initial degradation value. The cumulative distribution function (CDF) of t_ω is given as,

$$F(t) = \frac{\Gamma(\alpha t^c, \omega_\beta)}{\Gamma(\alpha t^c)} \quad (4.10)$$

Because of the gamma function, the evaluation of the CDF becomes mathematically intractable. To deal with this issue, Park and Padgett (2005) proposed an approximation of time-to-failure (t_ω) with Birnbaum-Saunders (BS) distribution and proposed the following CDF and PDF, respectively:

$$F_{BS}(t) \approx \phi\left[\frac{1}{a}\left(\sqrt{\frac{t^c}{b}} - \sqrt{\frac{b}{t^c}}\right)\right] \quad (4.11)$$

$$f_{BS}(t) = \frac{1}{2\sqrt{2ab}}\left[\sqrt{\frac{b}{t^c}} + \sqrt[3]{\frac{b}{t^c}}\right] e^{\left[-\frac{(b-t^c)^2}{2a^2 bt^c}\right]} \quad (4.12)$$

where $a = 1/\sqrt{(\omega_\beta)}$ and $b = \omega_\beta / \alpha$. Considering Birnbaum-Saunders approximation, the expected failure time can be estimated as:

$$t_\omega = \left(\frac{\omega_\beta}{\alpha} + \frac{1}{2\alpha}\right)^{\frac{1}{c}} \quad (4.13)$$

The equations (4.11) and (4.13) can be used to estimate the reliability and mean lifetime at the normal operating conditions, respectively.

The overall reliability assessment framework considering ATD data with stochastic gamma process and Bayesian inference is illustrated in Figure 4.2.

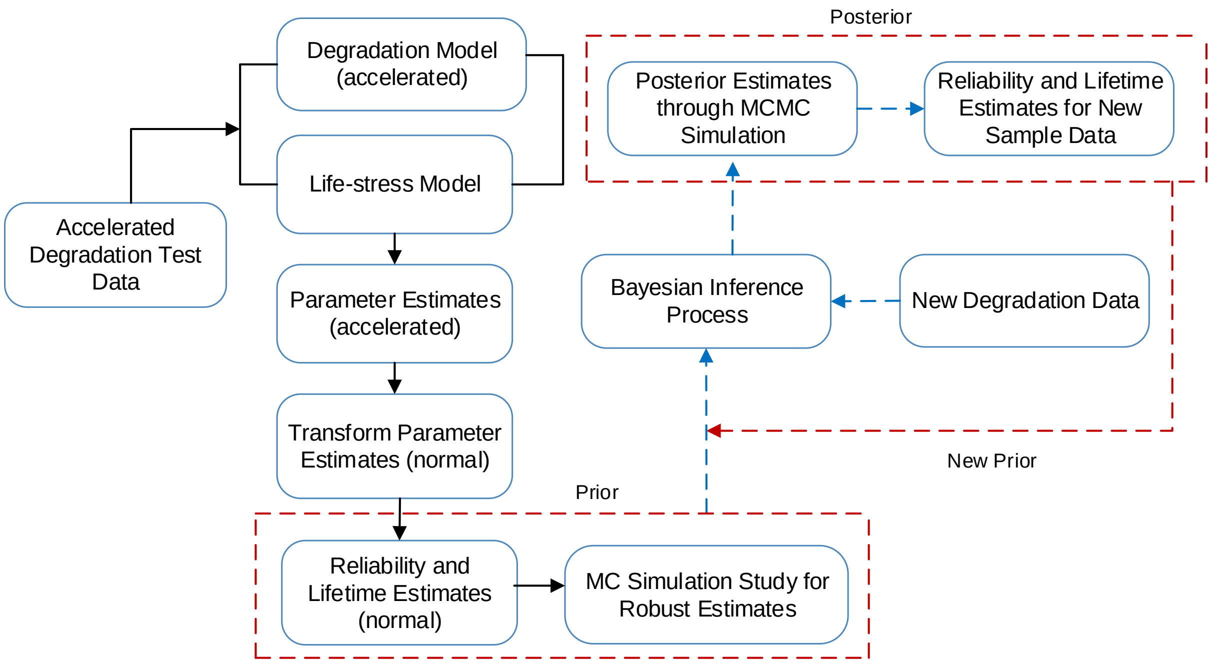


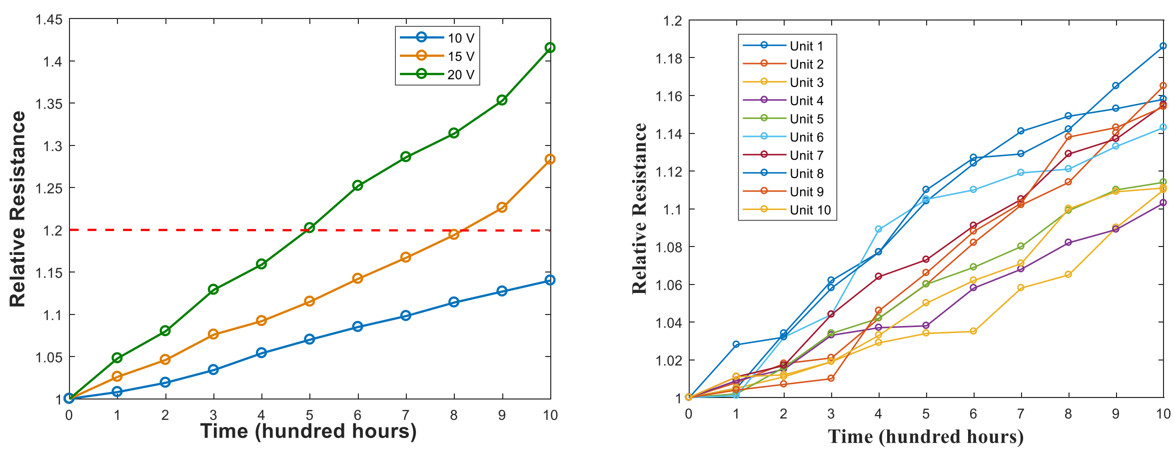
Figure 4.2: Reliability assessment framework for ADT data and Bayesian updates

4.5. Case Example of ADT Data Modeling

To demonstrate the proposed method, accelerated test data on carbon-film resistors is considered as a case study example and test data has been taken from the literature (Park and Padgett, 2006 and see Appendix A). There are several missing data points which have been simulated using the methods mentioned in Park and Padgett (206). A carbon-film resistor consists of carbon-film placed around the ceramic substrate is widely used in electronic circuits to restrict the electric current flow. The resistance of the carbon-film changes over time under applied temperature and voltage stresses and it is, therefore, considered as a degradation of the performance characteristic. For convenience, the relative resistance of carbon-film is taken as degradation measures with an initial relative resistance $y_0 = 1$ and a threshold value is determined at 20% increase of the relative resistance ($\omega = 0.20$). For accelerated degradation test, three levels of each stress factor are considered and ten samples are allocated at each treatment combination of two stress factors. The temperature levels are 350 K, 400 K, and 450 K and voltage levels include 10 V, 15 V, and 20 V. The normal operating conditions are set as a 323 K temperature and 5 V voltage. The degradation observations were taken at every hundred hour intervals.

4.5.1. Degradation data analysis

The effect of each stress factor and stress levels on the degradation process is illustrated in Figure 4.3. The data also shows the continuous deteriorating behavior of the relative resistance. The probability plot of degradation increment provides a good fit to gamma distribution compared to the normal distribution (see Figure 4.4). This justifies our assumption of considering gamma process to model the degradation path instead of using the Weiner process or Brownian motion process where degradation increment is assumed to follow the normal distribution. It is very critical to select the appropriate model in stochastic degradation analysis to ensure reduced model uncertainty and avoid wrong inferences.



a. Average degradation path at 350 K b. Sample degradation path at 350 K and 10 V

Figure 4.3: Effect of stresses on carbon-film resistors degradation path

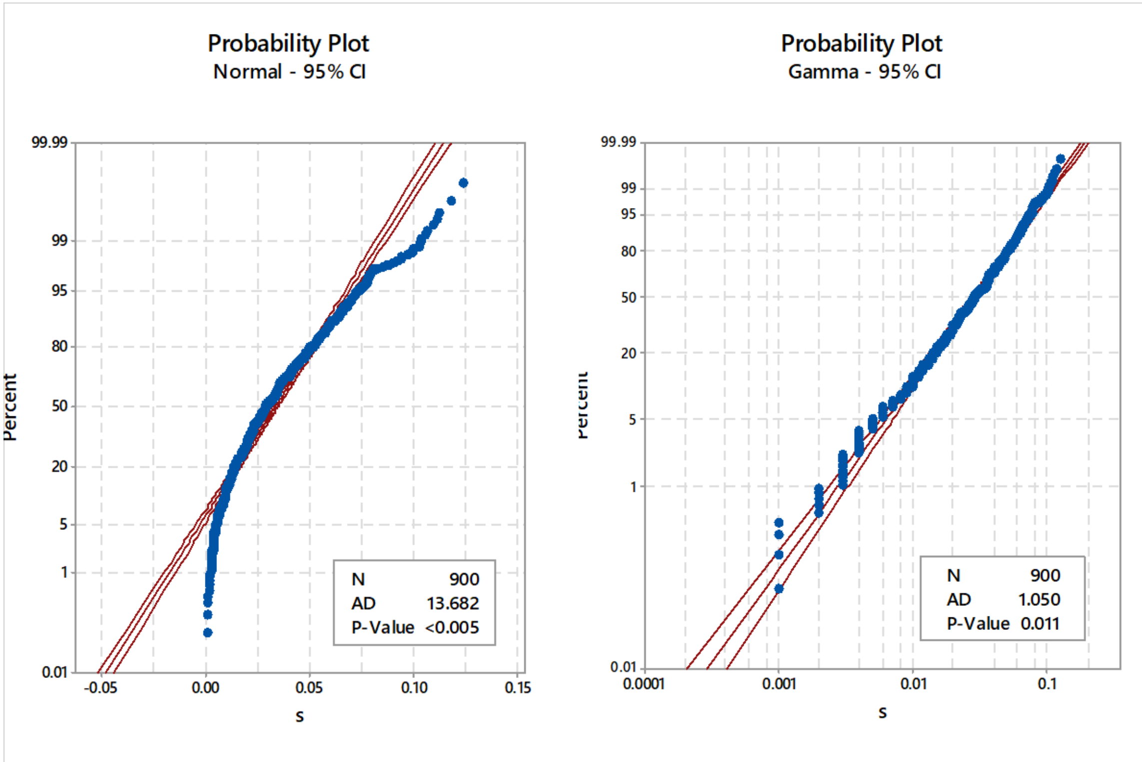


Figure 4.4: Comparative probability plot of degradation increment

To estimate model parameters, it is essential to first estimate overall model coefficient values of the likelihood function given in equation (4.8). However, due to a large number of unknown model coefficients in equation (4.8), it is extremely difficult to get the exact solution of the proposed likelihood function. Therefore, the nonlinear built-in optimization function *optim*, which is based on the Nelder-Mead algorithm, is used to maximize the log-likelihood function. Table 4.1 presents the estimated values of overall model coefficients considering all sample datasets.

Table 4.1: Estimated model coefficient parameters considering all datasets

Overall model parameter estimates								
$\hat{a}_0$	$\hat{a}_1$	$\hat{a}_2$	$\hat{a}_{12}$	$\hat{b}_0$	$\hat{b}_1$	$\hat{b}_2$	$\hat{b}_{12}$	$\hat{c}$
-0.4543	0.6672	1.5475	0.052	5.2107	-0.6791	-0.8338	1.0161	0.986

These model coefficient estimates are then utilized to estimate the stress-dependent gamma process parameters using Eqns. (4.6-4.7). Looking at the estimated model coefficient values, it seems stress factors acted differently on the shape and scale parameters. The estimated coefficient values suggest that both the temperature and voltage stresses have an increasing effect on the shape parameter and having a negative effect on scale parameter. Further, we also observe some significant interaction effect especially in the case of scale parameter, which is also illustrated in Figure 4.5.

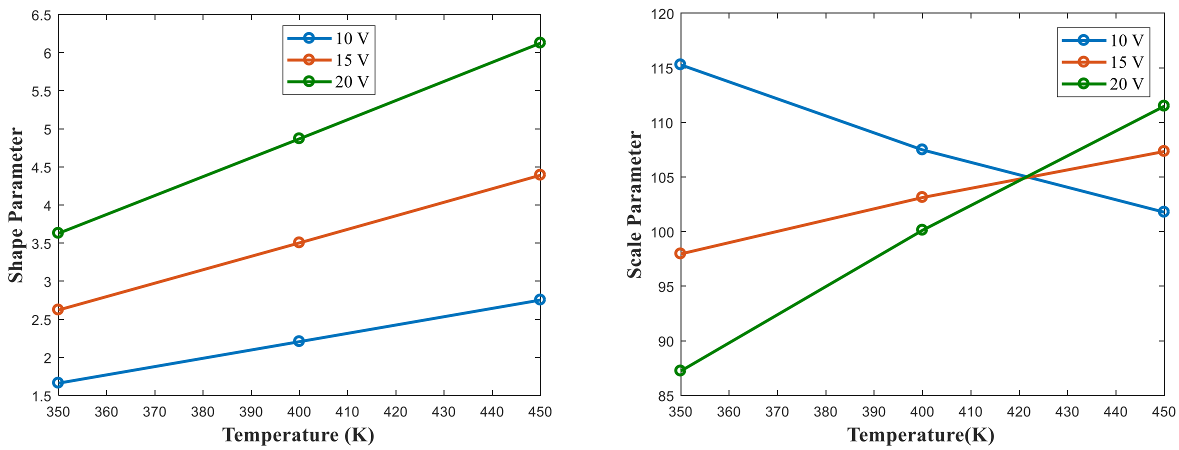


Figure 4.5: Effects of stresses on gamma parameter

This clearly supports our assumption that both the parameters are stress dependent though the effect of stress factors on the scale parameter is not straightforward. We believe this could be due to the possible interaction effect of stress factors. The presence of an interaction effect might be the reason behind the irregular pattern of gamma scale parameter values in Figure 4.5. It is, therefore, fair to infer that in the multi-stress environment, the gamma parameters, as well as other reliability measures, are impacted by stress levels and interaction effect of multiple stresses.

To validate our initial inference on significant interaction effect, the likelihood ratio (LR) test is performed to investigate the presence of the interaction between temperature and voltage stresses. The estimated LR test statistics is 8.69 that is greater than the critical chi-square value ($\chi^2_{1,0.05} = 3.841$) supporting the hypothesis of the presence of interaction. We, therefore, are fairly confident to conclude that there exists a statistically significant interaction between temperature and voltage stresses in this particular case example.

4.5.2. Reliability estimates using example data

Using the estimates of overall model coefficient parameters given in Table 4.1 and equations (4.6-4.7), the shape and scale parameters are estimated as 0.63 and 183.22, respectively. The expected lifetime of carbon-film resistor is estimated to be 62.47 hundred hours. Both the expected life and reliability estimates clearly demonstrate the highly reliable nature of the carbon-film products. The existing method in the literature (Park and Padgett, 2006) that considers only shape parameter is stress dependent and assumes linear model provides the expected lifetime estimate as 23.48 hundred hours, much lower estimate than the proposed approach. It seems the method proposed by Park and Padgett (2006) underestimates reliability parameters at operating conditions.

To further validate and improve the estimation accuracy, a Monte Carlo simulation study is conducted using the estimated model parameters (Liao and Elsayed, 2006). The flowchart of the MC simulation is provided in Figure 4.6. The simulation study provides an expected lifetime estimate as 61.45 hundred hours, which is close to the expected lifetime estimate given by the proposed approach.

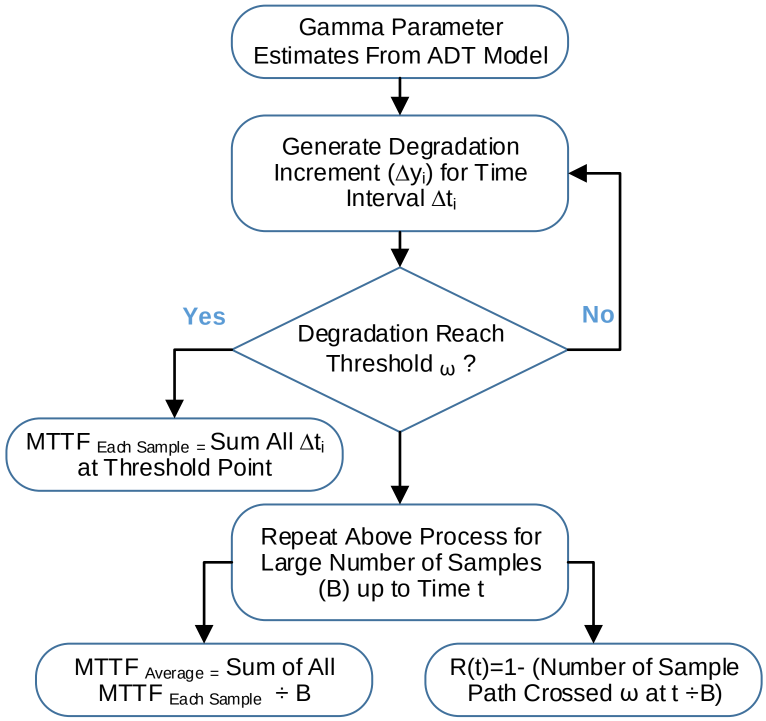


Figure 4.6: The MC simulation steps for robust estimates

Figure 4.7 shows the reliability plots obtained using the proposed approach, a simulation-based approach, and the existing method that considers the scale (β) parameter as a constant. The difference in reliability estimates provided by the proposed approach and existing method is mainly attributed to the assumptions in the proposed approach that both gamma parameters are stress dependent and the consideration of nonlinearity in the degradation process.

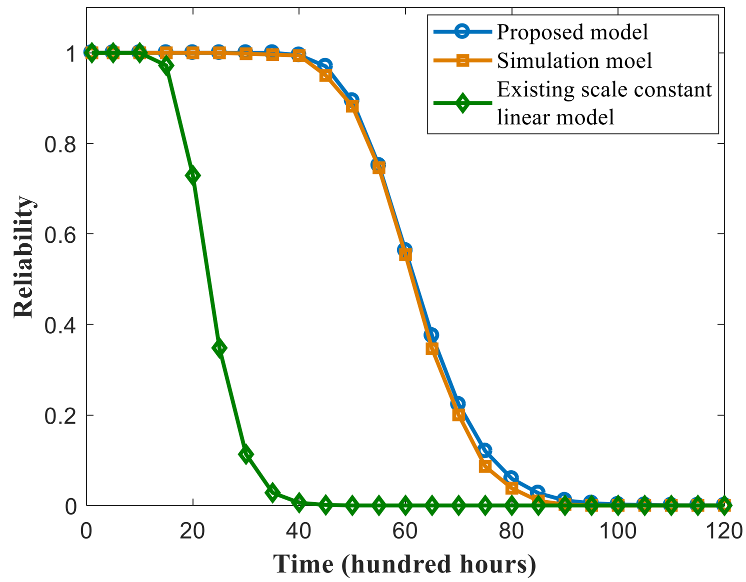
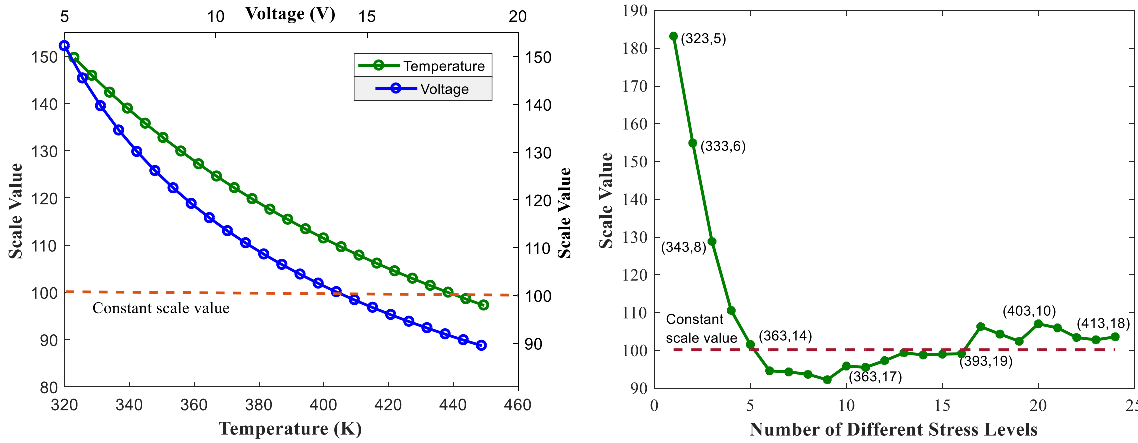


Figure 4.7: Reliability of carbon-film resistor at the normal operating condition

To further investigate this huge difference in reliability estimates, let us consider the expected degradation increment $E\big(y(t)\big) = \frac{\alpha}{\beta} t^{c}$. For a given time and nonlinearity parameter, it is clear from this relationship that the degradation behavior depends on gamma parameters only. It is also observed that the assumption of stress dependent gamma parameters leads to a higher estimated value of the scale parameter (β) at normal operating conditions as compared to the assumption of constant scale parameter. It seems this higher value of scale parameter represents slower degradation increment rate at normal operating conditions as compared to higher stress levels. To further understand this behavior, we plotted the changes in estimated values of scale parameters against different stress levels considering individual stress levels as well as combined levels as shown in Figure 4.8. We also plotted the constant scale parameter value to provide a visual comparison.



a. Individual stress effect

b. Combined stress effect

Figure 4.8: Change in scale parameter values with stresses levels

As shown in Figure 4.8, the stress-dependent scale parameter values, for both individual stress levels and combined stress effect, are higher at lower stress levels and then continuously decreasing as stress levels go up. After a certain level(s) of stress (we can call it steady stress level at which scale parameter value is same for both cases), the stress-dependent scale parameter value goes even below the constant scale parameter value obtained using existing method (Park and Padgett, 2006) or stays close to it. As mentioned earlier, the higher scale parameter value signifies the lower degradation rate meaning higher lifetime and reliability estimates. Since the stress levels at normal operating conditions are much lower than steady stress levels, the above rationale explains the reasoning for getting higher reliability estimate using the proposed approach as shown in Figure 4.7. The stress-dependent scale parameter value is lower than or close to constant scale parameter value beyond steady stress level, though the difference may not be very significant as shown in Figure 4.8b. This clearly explains that the methods based on the assumption of constant scale parameter will underestimate reliability parameters if operating conditions are below steady stress levels and sometimes overestimate reliability parameters above the steady stress levels though the difference may not be significant.

The same phenomenon is explained in Figure 4.9 where the difference is the lifetime estimate is very high at lower stress levels but not very significant at higher stress levels. This understanding justifies the consideration of stress-dependent gamma parameters that can provide more realistic reliability parameter estimates using ADT data. The overall pattern of a shift in lifetime estimates shown in these plots (see Figure 4.9) also indicates the effect of stress levels and their interaction.

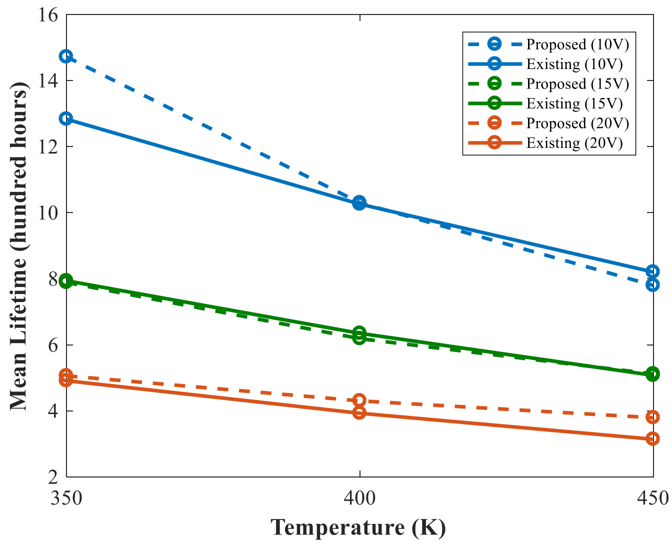


Figure 4.9: Illustration of lifetime comparison of the existing and proposed method

For the given data set, the nonlinearity parameter is not very strong mainly because the original data set is transformed into the relative measurement. However, to investigate the impact of nonlinearity on reliability parameter estimates, a sensitivity analysis was performed considering different values of nonlinearity parameter. Figure 4.10 shows the effect of nonlinearity on the degradation process and subsequently on product reliability. The analysis results show that higher nonlinearity in the degradation process means increasing degradation rate and a faster drop in reliability or lifetime. The results are intuitive because in a gamma process the expected degradation rate or increment is proportional to its time function ($E(y(t)) \propto t^c$) for given gamma process parameters.

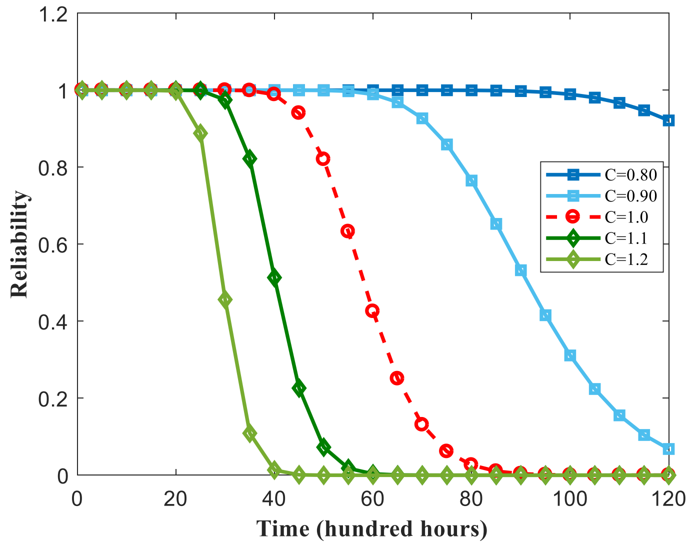


Figure 4.10: Effects of nonlinearity on reliability estimates considering remaining parameter constant

CHAPTER 5. BAYESIAN INFERENCE AND RUL ESTIMATION

Since initial model parameters are estimated based on the limited data available from ADT and hence will have higher uncertainty in the parameter estimates. Therefore, to further refine and update model parameter estimates if additional degradation data become available, we propose a Bayesian parameter updating approach. Especially, condition monitoring data has significant importance to predict remaining useful life estimation and decision making on maintenance planning. Seeing the practical importance, gamma parameters need to be updated when additional degradation data is available. In Bayesian updating, previously estimated model parameters are treated as random variables and uncertainty in these parameters is captured as the prior distribution. The prior distribution plays an important role that can also affect the overall Bayesian inference method. Therefore, a careful assumption and estimation are necessary to obtain the prior distribution. It is generally recommended to get the prior distribution using the past historical data or expert's opinion.

5.1. Conjugate Prior Distribution

In a conjugate prior distribution, both prior and posterior distributions belong to the same conjugate family of distributions (Martz and Waller, 1982). For analysis purpose, the scale parameter (β) is treated as a random variable and is assumed to follow the gamma distribution, β | α ~Ga (γ, η). The shape parameter (α) is assumed to be known. The newly observed degradation data are represented as y_i $(i = 1,2, \ldots, n)$ at the corresponding time t_i and the degradation increment at time increment $\Delta t_i (= t_i^c - t_{i-1}^c)$ is denoted as Δy_i. Now, suppose $P(\beta)$ and $L(\Delta y|\beta)$ represent the prior distribution and maximum likelihood function, respectively, the posterior distribution is given as (Martz and Waller, 1982):

$$P(\beta|\Delta y) = \frac{L(\Delta y|\beta)\, P(\beta)}{\int_0^\infty L(\Delta y|\beta)\, P(\beta)\, d\beta} \tag{5.1}$$

For *n* observed data and considering the kernel density function, the posterior relation can be written as:

$$P(\beta|\Delta y) \propto \prod_{i=1}^{n} L(\Delta y|\beta)\, P(\beta) \tag{5.2}$$

As both the data and scale parameter are considered to follow the gamma distribution, the maximum likelihood and prior distribution functions are given as:

$$L(\Delta y_i|\beta) = \prod_{i=1}^{n} \frac{\beta^{\alpha\Delta t_i}}{\Gamma(\alpha\Delta t_i)} \Delta y_i^{\alpha\Delta t_i - 1} e^{-(\beta\Delta y_i)} \tag{5.3}$$

$$P(\beta) = \frac{\eta^{\gamma}}{\Gamma(\gamma)} \beta^{\gamma-1} e^{-\eta\beta} \tag{5.4}$$

Considering equations (5.3) and (5.4), the posterior relation equation (5.2) can be presented as:

$$P(\beta|\Delta y) \propto \beta^{\alpha\sum_i^n \Delta t_i + \gamma - 1} e^{-\beta\left(\eta + \sum_i^n \Delta y\right)} \tag{5.5}$$

According to the property of the conjugate prior distribution, the posterior distribution is $\beta|\alpha \sim Ga(\gamma', \eta')$, where $\gamma' = \gamma + \alpha\sum_i^n \Delta t_i$ and $\eta' = \eta + \sum_i^n \Delta y_i$. Thus, the posterior estimate of β can be written as,

$$E(\beta|\Delta y_i) = \frac{\gamma + \alpha\sum_i^n \Delta t_i}{\eta + \sum_i^n \Delta y_i} \tag{5.6}$$

Equation (5.6) can now be used to update the posterior estimate of β whenever new degradation data become available.

5.2. Non-conjugate Prior Distribution

When both the shape and scale parameters of the gamma distribution are random variables, there exists no conjugate prior distribution. Suppose $P(\alpha, \beta)$ represent the joint prior distribution function and $L(\Delta y\,|\alpha, \beta)$ is the maximum likelihood function of the data, the posterior distribution of unknown parameters is given as (Martz and Waller, 1982):

$$P(\alpha,\beta|\Delta y) = \frac{L(\Delta y\ |\alpha,\beta)\ P(\alpha,\beta)}{\int_0^\infty \int_0^\infty L(\Delta y\ |\alpha,\beta)\ P(\alpha,\beta)\ d\alpha\ d\beta} \tag{5.7}$$

In statistically independent cases, the joint PDF of α and β can be expressed as, $P(\alpha,\beta) = P(\alpha)P(\beta)$. The best fitted distribution can be utilized to describe the randomness of α and β parameters. Considering the statistical independence, the posterior estimates of parameter α and β than can be given as:

$$E(\alpha|\Delta y) = \int_0^\infty \alpha\ P(\alpha|\Delta y)\ d\alpha \tag{5.8}$$

$$E(\beta|\Delta y) = \int_0^\infty \beta\ P(\beta|\Delta y)\ d\beta \tag{5.9}$$

where $P(\alpha|\Delta y)$ and $P(\beta|\Delta y)$ are known as the marginal distribution functions that are usually in unknown forms. To deal with an unknown form of the posterior distribution, a numerical computation method is required to estimate the posterior parameter values. The Markov Chain and Monte Carlo (MCMC) simulation with Gibbs sampler provides an efficient estimate of the posterior parameters (Gelman and Stren, 2004). The MCMC simulation is proposed in the following steps:

Step 1: Obtain the distribution of gamma parameters from ADT sample data

Step 2: Set prior distribution of parameters e.g. $\beta \sim Ga(\gamma',\eta')$ and new degradation increment Δy_i for time interval Δt_i

Step 3: Generate a large number (B) of sample observation using prior and newly available data from the proposed distribution until the equilibrium is reached

Step 4: Cut off the first T (say T =1000) number of initial observation to omit the noise effect

Step 5: Monitor the convergence of the posterior equilibrium, if not, generate more sample observation

Step 6: Obtain the mean value of parameters from the observed samples that will be the posterior estimates.

5.3. Remaining Useful Life Estimation

In the previous chapter, a reliability and lifetime estimation method has been developed using the accelerated degradation test data and gamma process for highly reliable products. The sudden failure of his highly valued critical components incurs huge costs including fatalities and environmental damages. Before any catastrophic failure, products usually deteriorates that reflects on the product performance or other condition monitoring parameters. These parameters as a measure of deterioration process can be captured by using modern sensor technology as a degradation data. A pre-defined threshold value of the product performance or these condition monitoring parameter values is set to obtain the mean-time-to-failure. Therefore, the inspections and condition-based maintenance (CBM) are critically important to the maintenance of highly reliable and critical components. The time left for failure occurs is predicted based on the current operating condition (Jardine et al., 2006). The time between the next failure and the current time is known as the remaining useful life (RUL). The RUL estimates provide critical information to predict and manage catastrophic failure, spare parts management, and maintenance strategies. Also, it is the key to the modern prognostics and health management (PHM) concept.

Prognostics and health management is broadly divided into two categories: physics-based and data-driven approach. In reality, getting the exact physics-of-failure model for product specific is difficult, sometimes impossible. On the other hand, the statistical data-driven approach is more accessible and therefore widely used. Considering the deterioration is a continuous process, it further divided the RUL prediction into two categories: general path model and the stochastic process model. In this work, we only focus on the stochastic process especially the gamma process model.

From the previous section, the lifetime according to the gamma process can be written as,

$$\xi_{\omega} = \left(\frac{\omega_{\beta}}{\alpha} + \frac{1}{2\alpha}\right)^{\frac{1}{c}} \tag{5.10}$$

Assume the field degradation data of a product is collected as y_1, y_2, …..,y_n. Then if $y_i < \omega_\beta$, the mean remaining useful life is given as:

$$\hat{\xi}_{y_i} = \left(\frac{(\omega - y_i)\beta}{\alpha} + \frac{1}{2\alpha}\right)^{\frac{1}{c}} \tag{5.11}$$

However, by updating the gamma parameter at each new degradation point using the Bayesian inference (previous sub-section), the Eqn. (5.10) can be also used to estimate the reaming useful life before its failures.

5.4. Bayesian Parameter Updates and RUL: Case Studies

To further reduce the uncertainty in parameter estimates obtained from ADT data, we update these initial parameter estimates once more data become available. The initial parameter estimates obtained from ADT data are 0.63 [0.28, 1.45] and 183.22 [73.30, 457.97], which are treated as prior information and provide a good fit to the gamma distribution. The prior gamma distribution parameters of α and β were obtained as $\hat{\alpha} \sim Ga\ (5.714, 9.2592)$ and $\hat{\beta} \sim Ga\ (11.19, 0.0645)$ (Appendix B). Table 5.1 shows degradation data obtained at normal operating conditions, which will be treated as additional information to obtain the posterior distribution of gamma parameters.

For conjugate prior, only the scale parameter is treated as a random variable to capture random effects with prior distribution parameters $\widehat{\beta} \sim Ga\ (11.19, 0.0645)$. The updated scale parameter with posterior estimates were given in Table 5.1.

Table 5.1: Posterior estimate of scale parameter using conjugate analysis

Time(x100)	15	25	35
y(t)	1.0523	1.0874	1.125
E (β|Δy)	173.702	172.785	169.759

For non-conjugate analysis, both shape and scale parameters are treated as random variables to capture random effects. As stated earlier, for both of these parameters gamma distribution is considered as a prior distribution with parameters estimates given as $\hat{\alpha} \sim Ga\ (5.714, 9.2592)$ and $\hat{\beta} \sim Ga\ (11.19, 0.0645)$. Because of the mathematical complexity, a numerical simulation method is used to obtain non-conjugate posterior distribution and subsequently estimates of the model parameters. The WinBUGS software, which is an excellent platform to carry out the MCMC simulation, is used to estimate posterior parameters (Ntzoufras, 2009). Table 5.2 provides the updated values of both the shape and scale parameters based on additional degradation information.

Table 5.2: Posterior estimate of shape and scale parameter using non-conjugate analysis

Time (x100)	15	25	35
y(t)	1.0523	1.0874	1.125
E(α|Δy)	0.616 [0.27, 0.96]	0.630 [0.33, 0.93]	0.651 [0.38, 0.92]
E(β|Δy)	170.90 [85.39, 256.41]	171.60 [95.16, 248.04]	170.40 [101.72, 239.08]

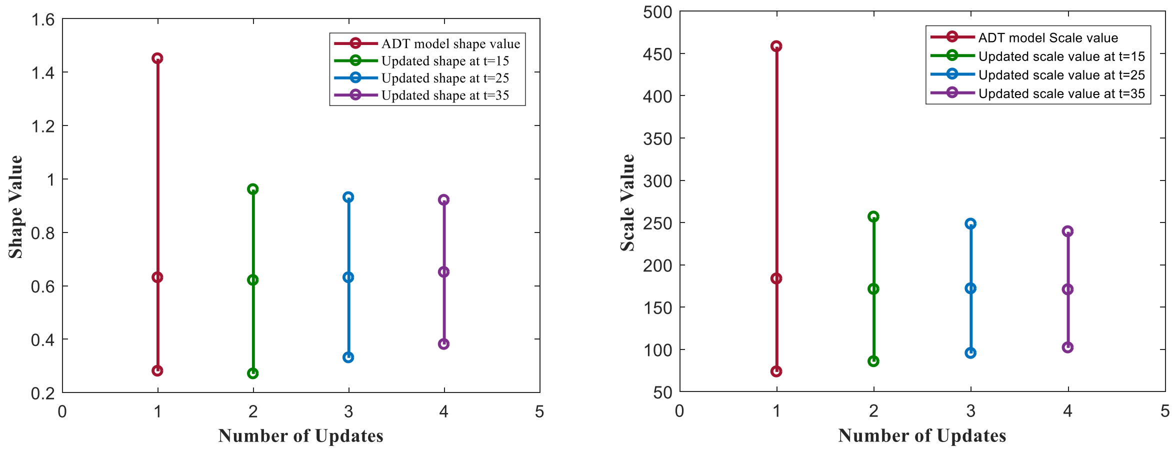
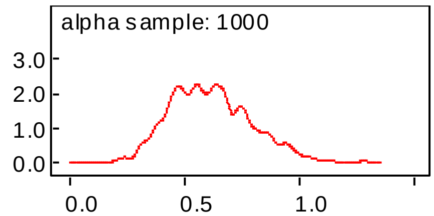
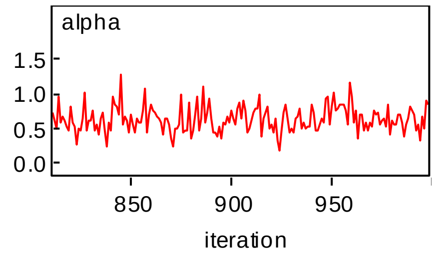


Figure 5.1: Uncertainty reduction of gamma parameter estimates with Bayesian updating

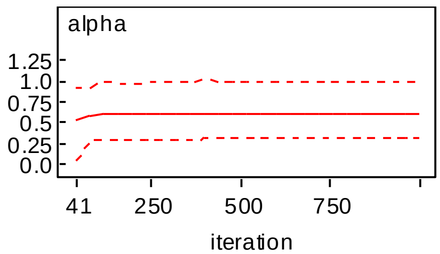
Table 5.2 and Figure 5.1 show reduction in the uncertainty of model parameters as a range of parameter estimates decreases when additional degradation information becomes available. It shows that the accuracy of model parameter estimates improves when more information or data on degradation become available. Reduction in the uncertainty of model parameter estimates leads to more precise reliability estimates and provides more confidence in reliability analysis efforts. Figure 5.2 shows the Bayesian updating process for the alpha parameter.

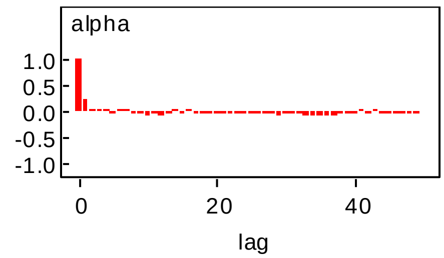




a. distribution of α

c. time series of α value





b. quantile value

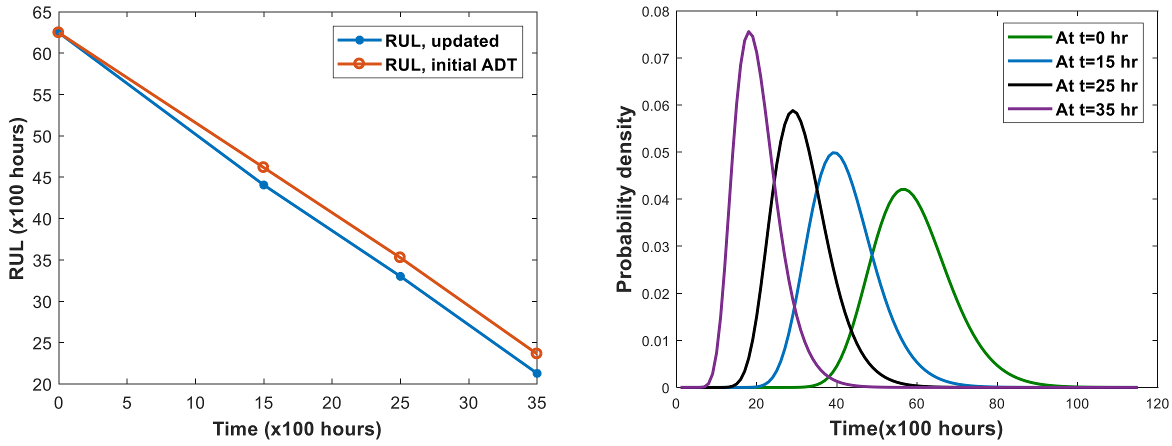
d. autocorrelation

Figure 5.2: Posterior α parameter updating for a new degradation data with MCMC simulation

Now in the presence of new degradation observation with an updated model parameter, it is obvious that the remaining useful life (RUL) of the product can be predicted. Table 5.3 shows the RUL estimation using both the updated parameter and initial ADT estimate. Figure 5.3 shows that the distribution of the RUL at different time observation.

Table 5.3: Estimates of RUL using initial and updated parameter

Time (x100)	0	15	25	35
y(t)	1	1.0523	1.0874	1.125
RUL, updated	62.47	44.06	33.01	21.28
RUL, initial	62.47	46.16	35.26	23.63



a. RUL vs time

b. RUL distribution

Figure 5.3: RUL and its distribution at different time observation

CHAPTER 6. ADT DESIGN AND PLANNING

To get the effective reliability estimation using the ADT method, it is necessary to have an effective test plan that involved with optimal sample allocation at each stress level, incorporate the budget constraint, and simultaneously obtained a precise estimation. A recent review by Limon et al. (2017a) provides the necessary and several other important issues on accelerated test designs.

Despite the fact that the gamma process is a suitable model for monotonic deterioration though ADT work considering the gamma process is not plenty. Most of the exiting work considered the single stress factor loadings, however, recent studies shows a necessity and inclination towards the multiple stress factor loading considering the actual product uses. Further, multiple stress factors arise the issue of interaction effect on the degradation process. Also, the above-mentioned works considered only the gamma shape parameter is stress dependent while the gamma scale parameter remains constant at different stress levels. The recent studies also show the drawback of this assumption. Therefore, to bridge the gap of existing literature, a constant-stress ADT design is proposed in this work. The multiple stress loadings with possible interaction between stresses are also considered in this work. To make the design more realistic, it is assumed that both the gamma parameters are stress dependent.

6.1. The ADT Design Optimization Model

6.1.1. Maximum-likelihood estimates

In this study, two stress factors, leveled by S_1 and S_2 are considered for a constant-stress ADT plan to accelerate the samples' degradation as well as the failure process. Suppose, y_{ijk} represents the i^{th} observation of the j^{th} sample under the k^{th} stress level at the time period t_{ijk} . It is assumed that the number of measurements for j^{th} sample unit at k^{th} stress level is same that is

$m_{jk} = m$ for all units. Also, measurement time t_{ijk} for jth sample at kth stress level are in equal interval that is $\Delta t_{ijk} = \Delta t$ for all units. The initial time $t_{0jk} = 0$ and the terminal observation time $t_{mjk} = m\, \Delta t$. In a gamma process, the degradation increment $\Delta y_{ijk} = y_{ijk} - y_{(i-1)jk}$ in time interval $\Delta t_{ijk=} t_{ijk} - t_{(i-1)jk}$ follow the gamma distribution with a PDF,

$$f_{\Delta y_{ijk}} = \frac{\beta_{(S_k)}^{\alpha(S_k)\Delta t}}{\Gamma(\alpha(S_k)\Delta t)} \Delta y_{ijk}^{(\alpha(S_k)\Delta t - 1)} e^{-(\Delta y_{ijk}\, \beta(S_k))} \tag{6.1}$$

For n number of test units $(n = \sum_{k=1}^{z} n_k)$ the log-likelihood function becomes,

$$\log L(\hat{\theta}) = \sum_{i=1}^{m} \sum_{j=1}^{n_k} \sum_{k=1}^{z} \left[A_k * \log(\beta_k) - \log \Gamma(A_k) + (A_k - 1) * \log(\Delta y_{ijk}) - \Delta y_{ijk} * \beta_k \right] \tag{6.2}$$

Here, $A_k = \alpha(s_k)\Delta t = e^{(\gamma_0 + \gamma_1 S_{1k} + \gamma_2 S_{2k} + \gamma_3 S_{1k} S_{2k})} \Delta t$ and $\beta(s_k) = e^{(\delta_0 + \delta_1 S_{1k} + \delta_2 S_{2k} + \delta_3 S_{1k} S_{2k})} = \beta_k$. The γ and δ represents the co-efficient of shape and scale parameters, respectively. Now, taking the expectations of the negative second partial derivatives of the Eqn. (6.2) with respect to the unknown parameters $\hat{\theta} = (\gamma_0, \gamma_1, \gamma_2, \gamma_3, \delta_0, \delta_1, \delta_2, \delta_3)$, the Fisher information matrix (**F**) can be derived as follows (see Appendix C).

$$\boldsymbol{F}(\hat{\theta}) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} \\ & & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} \\ & & & a_{44} & a_{45} & a_{46} & a_{47} & a_{48} \\ & & & & a_{55} & a_{56} & a_{57} & a_{58} \\ & & & & & a_{66} & a_{67} & a_{68} \\ & & & & & & a_{77} & a_{78} \\ & Symmetry & & & & & & a_{88} \end{bmatrix}$$

Then the asymptotic variance of the lifetime of the MLE at normal operating condition is obtained as,

$$Asvar(\hat{\xi}_{S_0}) = \boldsymbol{h}^T \boldsymbol{F}^{-1}(\hat{\theta}) \boldsymbol{h} \tag{6.3}$$

Here, $\boldsymbol{h}^T = \left[\frac{\partial \xi_{S_0}}{\partial \gamma_0}, \frac{\partial \xi_{S_0}}{\partial \gamma_1}, \frac{\partial \xi_{S_0}}{\partial \gamma_2}, \frac{\partial \xi_{S_0}}{\partial \gamma_3}, \frac{\partial \xi_{S_0}}{\partial \delta_0}, \frac{\partial \xi_{S_0}}{\partial \delta_1}, \frac{\partial \xi_{S_0}}{\partial \delta_2}, \frac{\partial \xi_{S_0}}{\partial \delta_3} \right]$.

6.1.2. The optimization model

At normal operating conditions, the lifetime can be rewritten as $\xi_{S_0} = \left(\frac{\omega_{\beta_0}}{\alpha_0} + \frac{1}{2\alpha_0}\right)$ and subsequently, $\boldsymbol{h^T} = [a, 0,0,0, b, 0,0,0]$ (See Appendix C). While the Fisher information matrix is derived, the inverse of the Fisher information matrix than can be expressed as follows,

$$\boldsymbol{F^{-1}}(\hat{\theta}) = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} & A_{17} & A_{18} \\ & A_{22} & A_{23} & A_{24} & A_{25} & A_{26} & A_{27} & A_{28} \\ & & A_{33} & A_{34} & A_{35} & A_{36} & A_{37} & A_{38} \\ & & & A_{44} & A_{45} & A_{46} & A_{47} & A_{48} \\ & & & & A_{55} & A_{56} & A_{57} & A_{58} \\ & & & & & A_{66} & A_{67} & A_{68} \\ & & & & & & A_{77} & A_{78} \\ & Symmetry & & & & & & A_{88} \end{bmatrix}$$

Now, using the expression of the $\boldsymbol{h^T}$, $\boldsymbol{F^{-1}}(\hat{\theta})$, and $\boldsymbol{h}$ matrices in the equation (6.3), the objective function of the model can be derived as $Asvar\left(\hat{\xi}_{S_0}(\hat{\boldsymbol{x}})\right) = a^2 A_{11} + 2abA_{51} + b^2 A_{88}$ (more detail in Appendix C). The objective function Asvar is a function of $\hat{\boldsymbol{x}} =$ $[\Delta t \; m \; n_1 \; n_2 \; n_3 \; n_4 \; S_{11} \; S_{12} \; S_{21} \; S_{22}]$, where $\hat{\boldsymbol{x}}$ represents the decision variable vector of the ADT plan.

The total accelerated degradation test cost can be divided into three categories: operating cost, degradation measurement cost, and sample units fixed cost. Considering C_{op} is the operating cost per unit, including the labor, the total experimental operating cost expression is $C_{op} \sum_{k=1}^{z} \Delta t_k \, m$. The degradation measurement cost of the test can be found by the expression $C_m \, m \, n$ where C_m represents the unit cost of measurement. Finally, if each sample unit costs C_s then the total fixed cost would be $C_S n$. Therefore, the total experimental cost can be written as,

$$TC = C_{op} \sum_{k=1}^{z} \Delta t_k \, m + C_m m \, n + C_S \, n \tag{6.4}$$

Now, the ADT design optimization model considering two stress levels is formulated as follows,

$$Minimize\ Asvar\left(\hat{\xi}_{S_0}(\hat{\boldsymbol{x}})\right) = a^2 A_{11} + 2abA_{51} + b^2 A_{88} \tag{6.5}$$

Subject to,

$$TC = C_{op} \sum_{k=1}^{Z} \Delta t_k\, m + C_m m\, n + C_S\, n \leq C_b$$

$$\sum_{k=1}^{z} n_k \leq n$$

$$0 \leq S_{1k} \leq 1$$
$$0 \leq S_{2k} \leq 1$$

$$n_k, m \in N(integer)$$

6.2. The Solution Approach

The objective function in Eqn. (6.5) depends on the unknown model parameters of $\hat{\theta} = (\gamma_0, \gamma_1, \gamma_2, \gamma_3, \delta_0, \delta_1, \delta_2, \delta_3)$. Before, solving the optimization model, it is required to get the pre-estimates of these parameter values. The historical test data and the MLE method can be used to obtain the initial pre-estimate values. The sensitivity analysis of the initial parameter estimates can provide an insight about the effect of these estimates in the overall ADT plan.

Further, the proposed model presented in Eqn. (6.5) is a complex nonlinear problem and it is very difficult or sometimes impossible to obtain an analytical solution in a reasonable time. To resolve this problem, a heuristic based search genetic algorithm (GA) method is proposed to obtain near-optimal solutions. The GA method is known as an evolutionary algorithm where the optimal solution is approached by a multiple search method. The decision variables are indicated as genetics and the objective function of the model is defined as the fitness function. As a first step, a population is randomly generated where each member represents a feasible solution also known as chromosomes. A set of chromosomes produced in an iteration of the algorithm is named as generation and the next generation is produced by the parents (eligible chromosomes of the previous generation) based on the fitness value. The GA search approach has the following steps (Schmitt, 2001) that also showed in Figure 6.1.

1. Randomly generate initial solutions of decision variables based on the population size
2. Generated solutions are evaluated by the fitness function
3. The selection and crossover of genes (decision variables)
4. Mutations of genes (partial)
5. Generated solution for the next generation and repeat from step 2
6. The evolution process continues until a convergent optimal solution is obtained

The GA requires several parameter settings to successfully obtain the optimal solution such as population size, crossover rate, and mutation rate. The larger population size helps to get a faster optimal solution. On the other hand, the complexity of the optimal search also increases with the population size. Similarly, a higher crossover rate expedites the optimal search method as well as increases the danger of breaking the solution at a higher fitness value. Further, there is always a chance to converge the solution into local optima rather than a global solution. The mutation rate resolves this problem. However, the excessively low mutation rates may cause failure to reach an optimal solution and a high mutation rate can break a better solution. Therefore, the moderate GA parameter settings are required to obtain the best solution in due time. The moderate GA parameter settings can be found by tuning and calibrating the parameters based on the desire fitness function value.

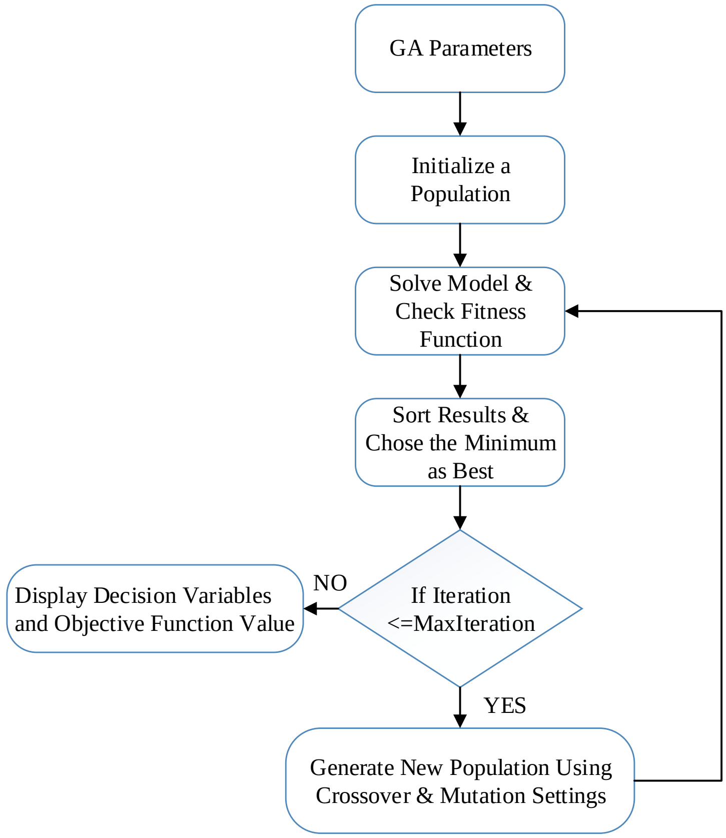


Figure 6.1: The algorithm flowchart of the GA search method

In this work, the GA is used to determine the optimal settings of the ADT design parameters. The decision variables ($\hat{\boldsymbol{x}}$) are considered the gene, and initial feasible solution of multiple sets of decision variables are generated based on the population size. The asymptotic variance (AsVar) is calculated using each of the chromosomes that results in the fitness function value. The smaller values of the AsVar have a better fitness as a solution. By iterating the above GA procedure, the optimal ADT design is achieved when the AsVar values of all feasible solutions converge to the same value.

6.3. Case Example and Sensitivity Analysis

The carbon film resistor used in electric circuits is taken as a case example in this study. The temperature and voltage cause the increase in resistance of the film-resistor. Therefore, the relative resistance is considered as a degradation characteristic and a 20% increase in the relative resistance is treated as the threshold value. The operating conditions of the film resistor are defined as 323 K and 5 V and the maximum stress level is set at 450 K and 20 V. Appendix A presented the relative resistance data of carbon film resistor at elevated stress level (Park and Padgett, 2006).

Since the increase in resistance is monotonic in nature, the assumption of the gamma process is justifiable. The initial pre-estimates of the unknown parameters are obtained using the MLE method to the existing test data set. Further, the cost coefficients are considered as C_{op}= \$3/hours, C_m = \$2/measurement, C_s=\$1/unit. The existing test data have been used to obtain the initial model coefficients as $(\hat{\gamma}_0, \hat{\gamma}_1, \hat{\gamma}_2, \hat{\gamma}_3)$ = (-5.0824, 0.6827, 1.5398, 0.0269) and $(\hat{\delta}_0, \hat{\delta}_1, \hat{\delta}_2, \hat{\delta}_3)$ = (5.2179, -0.6638, -0.8413, 0.9914). It is assumed that the total sample availability is 100. Two accelerated stress factors with two stress level will make a total of four accelerated stress test level combinations and four samples assigned at each combination. The optimization model is solved to obtain the optimal values of design variables $\widehat{\boldsymbol{x}} = [\Delta t\ m\ n_1\ n_2\ n_3\ n_4\ S_{11}\ S_{12}\ S_{21}\ S_{22}]$. As mentioned earlier that the objective function has no analytical solution and hence, a heuristic GA search is performed to obtain the optimal solution under the given constraint. The GA parameters are calibrated after several optimization runs and selected the parameter settings that provide the best objective function value. Table 6.1 provides several optimal ADT plans under different budget constraints.

Table 6.1: Optimal ADT plans under several budget conditions

C_b ($)	Δt	m	n_1	n_2	n_3	n_4	S_{11}	S_{12}	S_{21}	S_{22}	AsVar	Cost ($)
2000	34.693	3	19	27	23	20	0.809	0.120	1	0.221	6.942×10^{6}	1872
2500	37.315	4	20	28	16	11	1	0.476	1	0.416	5.506×10^{6}	2466
3000	35.053	5	13	25	20	9	1	0.286	0.890	0.336	3.669×10^{6}	2840
3500	42.063	5	28	10	19	24	0.830	0.087	0.858	0.321	2.784×10^{6}	3415

The optimal ADT plan for a budget constraint of $2000 has an objective function value of 6.942×10^{6} and decision variable values are $\hat{\boldsymbol{x}} = [34.693, 3, 19, 27, 23, 20, 0.809, 0.120, 1, 0.221]$. This optimal result suggests four accelerated test combinations of standardized temperature and voltage stress as (0.809, 1); (0.809, 0.221); (0.120, 1); and (0.120, 0.221). These standardized stress test combinations can easily be transformed into nominal stress values by using Eqns. (2.4-2.6). For example, the standardized stress combination (0.809, 1) will have normal stress combination as (418.56 K, 20 V). The total test time for each accelerated stress test would be 104.079 hours, and the number of samples allocated for each test combination levels are 19, 27, 23, and 20, respectively. The optimal test plan also suggests that during each accelerated test, three degradation measurements are taken (excluding the initial measurement) at a time interval of 34. 693 hours. Under this optimal test plan, the total costs of the experiment become $1872.

From the sensitivity analysis of the ADT model under different budget constraint (See Table 6.1), it is observed that the relaxing the budget constraint (increasing budget) resulted in longer test duration. The test duration increases in terms of the measurement frequency (m) or the observation interval time (Δt) or increases both the frequency and interval time. However, longer test durations result in reduced sample size requirements at each accelerated test combination, which is in line with the standard practice of having a smaller sample size for a longer test duration. The optimal stress level does not have any significant changes due to the budget variations. It is also observed that the AsVar decreases while the budget is increased. The higher budget increases the test duration as well as the degradation inspection that provides relatively more information about the degradation process, and eventually, more information helps reduce the variation.

Since the algorithm requires initial parameter values to start the search for an optimal solution, we used historical test data to get the MLE of the model parameters and treated them as initial parameter values also known as pre-estimates. To understand the impact of these initial pre-estimates on the optimal solution, a sensitivity analysis is performed by changing the initial parameter values. Different scenarios were created by changing the shape parameter coefficients ($\hat{\gamma}$) and scale parameter coefficients ($\hat{\delta}$) by $\pm 5\%$ of the original pre-estimates. Table 6.2 presents the optimal solution under a different scenarios of pre-estimates considering a budget constraint of $2000.

Table 6.2: Optimal ADT plans under different initial parameter settings

Scenario	Δt	m	n_1	n_2	n_3	n_4	S_{11}	S_{12}	S_{21}	S_{22}	AsVar	Cost ($)
+5% $\hat{\gamma}_i$	29.210	4	10	6	20	26	1	0.091	1	0.552	5.531×10^{6}	1960
-5% $\hat{\gamma}_i$	35.661	3	25	15	23	17	1	0.340	0.899	0.510	3.426×10^{6}	1844
+5% $\hat{\delta}_i$	31.310	3	24	14	26	15	0.789	0.210	1	0.572	1.363×10^{7}	1680
-5% $\hat{\delta}_i$	36.193	3	6	21	20	23	0.731	0.591	0.842	0.645	3.963×10^{6}	1793

The sensitivity analysis results (see Table 6.2) indicate that for all scenarios the total test duration varies from 93.93 to 116.84 hours in comparison to the original optimal test duration of

104.079 hours. Similarly, the measurement frequency interval variation is in the range of 29.210 to 36.193 hours with the earlier optimal frequency interval of 34.693 hours. The total number of measurements during the test remains almost constant as 3 with the exception of 4 measurements in one scenario only. It seems the variation in these variables (test duration, measurement frequency interval, and the total number of measurements) is not very significant as compared to initial optimal design variables obtained considering pre-estimates. Similarly, the accelerated stress level settings at different scenarios are very close to the original optimal plan except for the low voltage level (S_{22}), which does not seem to be very critical. The number of the sample varies from 62 to 81 whereas it is 89 in the original design. The AsVar value also changes (reduced in most cases) for the new setting conditions, which correlates to variation in total test duration. Increase in total test duration results in lowering AsVar that confirms with the earlier inference of decrease in uncertainty with longer test duration. It is also important to note that this slight increase in total test duration resulted in a reduction of sample requirements as expected. The total cost constraint varies from \$ 1793 to \$ 1960 indicating no significant difference from optimal cost (\$1872) obtained from original pre-estimates. Overall comparison of sensitivity analysis results with our optimal solution indicates that there is some variation in optimal values of decision variables but it does not seem to be very significant. However, authors would like to emphasize the importance of getting more accurate pre-estimates as sensitivity analysis results do show some variability.

CHAPTER 7. CONCLUSION AND SUMMARY

This dissertation presents a reliability assessment framework under normal operating conditions using the ADT data. The stochastic gamma process is considered to capture degradation behavior for multiple stress factors assuming that both the gamma parameters (shape and scale parameters) are stress dependent. The multiple stress factors with interaction effects have been investigated in the life-stress model to capture the effect of stress factors on gamma parameters and finally on reliability parameters. The case study results revealed that the assumption of only shape parameter is stress dependent leads to an underestimation of the reliability and lifetime parameters. To reduce the estimation bias, a Monte Carlo simulation study and Bayesian parameter updating method have been applied to reduce the uncertainty in reliability parameter estimation at the normal operating conditions. Both conjugate and non-conjugate cases were investigated with simulated data. The use of MCMC simulation technique with WinBUGS software provides an efficient Bayesian updating process. The updated parameters were further utilized to estimate the remaining useful lifetime for a particular component/product.

As the effectiveness of ADT models mostly depend on efficient test planning and therefore, an optimal ADT plan is developed considering the stochastic gamma process to model the degradation behavior. To represent the practical usage conditions, a multi-level and multiple stress ADT capturing possible interaction effects between stresses is incorporated in the model. Also, based on recent literature, both the gamma parameters are considered dependent on stress levels. The log-likelihood function, as well as the Fisher information matrix, has been developed for the model. The objective function is set as minimizing the asymptotic variance of the MLE of the lifetime under normal operating conditions. The total experimental cost also considered as a budget constraint in the optimization model. Due to the complexity of the objective function, a heuristic GA search algorithm has been used to obtain the optimal solution of the test plan. The sensitivity analysis is also performed under different budget constraint and for the different scenario of model pre-estimates. The results show the robustness of the model against this budget and pre-estimates.

Based on the research conducted in this dissertation, there exist numerous scope for further investigation and extensions in various area. In the following, several future scopes have been discussed briefly.

1. This work considered the gamma stochastic process with constant accelerated stress conditions. The methods can be investigated for other stochastic models, for example, Weiner and inverse Gaussian process. The different types of stress loadings, for instance, step or progressive type stress could also be implemented to further investigate the methods. For the proposed method, the effect of proper degradation model selection in terms of misspecification analysis will be a good research outcome.
2. This work considers two stress factors with their possible interaction effects. More than two stress factors would be more interesting as the interaction effect between stresses become more complicated. Also, more than one degradation indicator will make the computational complexity further especially to design the ADT. More efficient and advanced algorithms are needed to solve these complex optimization problems.
3. The existing proposed framework can be exhaustively extended for predicting the lifetime and RUL. This real-time prediction of the lifetime can be a very useful tool for prognostic health management of high valued assets. Also, lifetime prediction can be combined with maintenance optimization problem and extend it to several logistical

decision making such as spare parts inventory management, resource allocation, and warranty policy design.

4. The advancement in sensors and wireless technologies provides the scope of collecting multidimensional degradation data in real-time. More advanced data analytics tools, for example, machine learning and deep learning techniques can be implemented along with the Bayesian inference to constantly update the model parameters.

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APPENDIX A. CARBON FILM RESISTORS DEGRADATION DATA

Stress\Hours (1000)	0	1	2	3	4	5	6	7	8	9	10
	1.000	1.005	1.034	1.062	1.077	1.104	1.124	1.141	1.149	1.153	1.158
	1.000	1.008	1.018	1.021	1.042	1.06	1.082	1.102	1.114	1.14	1.165
	1.000	1.005	1.011	1.019	1.029	1.034	1.035	1.058	1.065	1.09	1.11
	1.000	1.009	1.015	1.033	1.037	1.038	1.058	1.068	1.082	1.089	1.103
	1.000	1.002	1.016	1.034	1.042	1.06	1.069	1.08	1.099	1.11	1.114
350 K/10 V	1.000	1.001	1.032	1.044	1.089	1.105	1.11	1.119	1.121	1.133	1.143
	1.000	1.011	1.017	1.044	1.064	1.073	1.091	1.105	1.129	1.137	1.155
	1.000	1.028	1.032	1.058	1.077	1.11	1.127	1.129	1.142	1.165	1.186
	1.000	1.004	1.007	1.010	1.046	1.066	1.088	1.103	1.138	1.143	1.154
	1.000	1.011	1.012	1.019	1.033	1.05	1.062	1.071	1.1	1.109	1.111
	1.000	1.032	1.052	1.059	1.075	1.09	1.119	1.124	1.133	1.15	1.16
	1.000	1.020	1.042	1.058	1.073	1.084	1.146	1.167	1.231	1.272	1.333
	1.000	1.017	1.055	1.068	1.084	1.111	1.128	1.18	1.214	1.225	1.27
	1.000	1.005	1.014	1.082	1.104	1.132	1.151	1.182	1.186	1.248	1.283
	1.000	1.042	1.049	1.062	1.067	1.086	1.134	1.145	1.185	1.205	1.265
350 K/15 V	1.000	1.012	1.049	1.103	1.111	1.12	1.134	1.162	1.214	1.226	1.278
	1.000	1.049	1.065	1.094	1.115	1.151	1.173	1.181	1.187	1.228	1.293
	1.000	1.007	1.011	1.014	1.039	1.057	1.064	1.083	1.087	1.153	1.157
	1.000	1.063	1.092	1.144	1.161	1.201	1.228	1.264	1.302	1.33	1.487
	1.000	1.014	1.033	1.071	1.094	1.121	1.141	1.179	1.201	1.227	1.299

Stress\Hours (1000)	0	1	2	3	4	5	6	7	8	9	10
	1.000	1.045	1.075	1.124	1.143	1.199	1.277	1.303	1.332	1.358	1.394
	1.000	1.078	1.107	1.214	1.25	1.295	1.351	1.386	1.402	1.449	1.479
	1.000	1.048	1.093	1.171	1.205	1.242	1.267	1.302	1.331	1.393	1.504
	1.000	1.049	1.074	1.132	1.169	1.204	1.225	1.247	1.262	1.273	1.377
	1.000	1.049	1.070	1.083	1.122	1.152	1.196	1.269	1.315	1.351	1.454
350 K/20 V	1.000	1.067	1.139	1.194	1.204	1.252	1.302	1.324	1.35	1.394	1.453
	1.000	1.044	1.086	1.109	1.162	1.22	1.261	1.275	1.304	1.374	1.437
	1.000	1.034	1.049	1.092	1.132	1.153	1.225	1.26	1.3	1.337	1.412
	1.000	1.036	1.071	1.091	1.095	1.143	1.204	1.257	1.272	1.287	1.322
	1.000	1.027	1.039	1.082	1.105	1.164	1.211	1.238	1.274	1.316	1.32
	1.000	1.032	1.038	1.061	1.084	1.112	1.178	1.191	1.197	1.22	1.256
	1.000	1.019	1.033	1.046	1.077	1.124	1.133	1.174	1.262	1.309	1.33
	1.000	1.017	1.045	1.058	1.060	1.07	1.09	1.11	1.113	1.141	1.155
	1.000	1.014	1.024	1.059	1.085	1.106	1.117	1.12	1.194	1.207	1.266
	1.000	1.025	1.039	1.045	1.058	1.079	1.115	1.118	1.151	1.159	1.174
400 K/10 V	1.000	1.013	1.023	1.057	1.081	1.097	1.142	1.145	1.167	1.171	1.184
	1.000	1.016	1.115	1.119	1.126	1.145	1.152	1.173	1.216	1.23	1.242
	1.000	1.041	1.044	1.078	1.088	1.152	1.16	1.179	1.189	1.195	1.229
	1.000	1.007	1.023	1.035	1.049	1.066	1.101	1.107	1.124	1.128	1.133
	1.000	1.010	1.024	1.043	1.078	1.101	1.113	1.124	1.146	1.161	1.197

Stress\Hours (1000)	0	1	2	3	4	5	6	7	8	9	10
	1.000	1.027	1.047	1.074	1.135	1.152	1.188	1.217	1.232	1.268	1.275
	1.000	1.047	1.101	1.127	1.176	1.2	1.224	1.251	1.268	1.279	1.299
	1.000	1.071	1.098	1.133	1.149	1.171	1.227	1.25	1.271	1.294	1.304
	1.000	1.072	1.104	1.135	1.169	1.191	1.213	1.223	1.244	1.269	1.276
	1.000	1.028	1.048	1.114	1.154	1.167	1.221	1.245	1.295	1.337	1.364
400 K/15 V	1.000	1.024	1.101	1.124	1.168	1.26	1.275	1.308	1.327	1.335	1.338
	1.000	1.077	1.089	1.101	1.110	1.145	1.159	1.176	1.187	1.236	1.25
	1.000	1.021	1.034	1.046	1.092	1.152	1.166	1.2	1.238	1.288	1.29
	1.000	1.018	1.057	1.097	1.120	1.155	1.178	1.206	1.235	1.29	1.328
	1.000	1.024	1.084	1.138	1.162	1.191	1.235	1.285	1.304	1.348	1.352
	1.000	1.061	1.095	1.131	1.161	1.196	1.267	1.275	1.28	1.294	1.338
	1.000	1.074	1.103	1.175	1.220	1.237	1.256	1.263	1.277	1.313	1.322
	1.000	1.048	1.080	1.123	1.241	1.263	1.272	1.327	1.334	1.396	1.412
	1.000	1.097	1.134	1.217	1.232	1.242	1.26	1.279	1.299	1.301	1.312
	1.000	1.053	1.094	1.160	1.224	1.226	1.239	1.3	1.332	1.377	1.381
400 K/20 V	1.000	1.028	1.076	1.148	1.204	1.281	1.295	1.31	1.323	1.358	1.37
	1.000	1.028	1.068	1.148	1.188	1.245	1.257	1.261	1.277	1.289	1.298
	1.000	1.043	1.095	1.148	1.202	1.225	1.246	1.257	1.283	1.305	1.311
	1.000	1.038	1.074	1.186	1.242	1.268	1.271	1.285	1.302	1.336	1.354
	1.000	1.054	1.087	1.102	1.123	1.188	1.275	1.299	1.345	1.354	1.384

Stress\Hours (1000)	0	1	2	3	4	5	6	7	8	9	10
	1.000	1.034	1.071	1.082	1.138	1.173	1.177	1.18	1.202	1.244	1.276
	1.000	1.013	1.055	1.073	1.096	1.132	1.158	1.174	1.194	1.214	1.229
	1.000	1.017	1.027	1.059	1.105	1.117	1.165	1.192	1.216	1.221	1.225
	1.000	1.016	1.041	1.048	1.065	1.096	1.12	1.153	1.162	1.198	1.222
	1.000	1.023	1.042	1.054	1.090	1.108	1.166	1.199	1.267	1.294	1.337
450 K/10 V	1.000	1.010	1.019	1.060	1.120	1.149	1.152	1.17	1.188	1.252	1.275
	1.000	1.006	1.018	1.038	1.054	1.078	1.088	1.104	1.17	1.185	1.203
	1.000	1.019	1.033	1.052	1.070	1.097	1.112	1.117	1.138	1.151	1.216
	1.000	1.028	1.063	1.073	1.117	1.129	1.155	1.184	1.244	1.258	1.32
	1.000	1.022	1.036	1.049	1.072	1.082	1.102	1.133	1.148	1.162	1.201
	1.000	1.040	1.072	1.133	1.186	1.211	1.219	1.223	1.272	1.295	1.314
	1.000	1.043	1.075	1.134	1.162	1.168	1.183	1.216	1.226	1.258	1.259
	1.000	1.031	1.092	1.136	1.153	1.179	1.195	1.222	1.233	1.258	1.277
	1.000	1.006	1.039	1.052	1.089	1.129	1.158	1.179	1.219	1.23	1.247
	1.000	1.041	1.079	1.121	1.141	1.169	1.202	1.26	1.289	1.308	1.32
450 K/15 V	1.000	1.063	1.087	1.140	1.193	1.248	1.286	1.291	1.31	1.318	1.35
	1.000	1.078	1.099	1.121	1.171	1.207	1.244	1.25	1.279	1.303	1.321
	1.000	1.041	1.068	1.109	1.189	1.241	1.272	1.297	1.319	1.323	1.358
	1.000	1.035	1.076	1.122	1.142	1.19	1.213	1.247	1.26	1.282	1.305
	1.000	1.015	1.037	1.066	1.090	1.125	1.156	1.183	1.19	1.202	1.216

Stress\Hours (1000)	0	1	2	3	4	5	6	7	8	9	10
	1.000	1.049	1.153	1.217	1.228	1.238	1.254	1.289	1.35	1.397	1.441
	1.000	1.103	1.126	1.226	1.253	1.289	1.29	1.31	1.336	1.344	1.394
	1.000	1.051	1.110	1.165	1.227	1.252	1.259	1.328	1.338	1.368	1.412
	1.000	1.059	1.138	1.247	1.253	1.274	1.29	1.348	1.373	1.388	1.392
	1.000	1.076	1.157	1.225	1.26	1.282	1.296	1.339	1.357	1.392	1.473
450 K/20 V	1.000	1.057	1.112	1.142	1.185	1.309	1.327	1.373	1.395	1.403	1.448
	1.000	1.031	1.110	1.180	1.248	1.291	1.306	1.322	1.37	1.391	1.392
	1.000	1.069	1.133	1.172	1.221	1.227	1.242	1.28	1.327	1.331	1.357
	1.000	1.068	1.146	1.204	1.252	1.261	1.269	1.29	1.33	1.338	1.388
	1.000	1.106	1.160	1.254	1.271	1.275	1.28	1.285	1.314	1.338	1.34

APPENDIX B. GAMMA PARAMETER ESTIMATES FOR EACH SAMPLE

Stress levels	Alpha (α)	Beta (β)	Constraint (c)	-2Log-like	AIC	Mean(α)	Mean (β)
	1.597	87.124	0.986	-65.086	-59.086		
	0.809	102.214	1.389	-68.353	-62.353		
	0.636	83.753	1.251	-70.955	-64.955		
	1.797	197.976	1.060	-74.265	-68.265		
	1.196	147.797	1.196	-74.004	-68.004		
350 K/10 V	0.793	72.981	1.141	-65.419	-59.419	1.290	114.539
	2.281	165.758	1.087	-71.015	-65.015		
	2.104	93.997	0.934	-62.316	-56.316		
	0.681	95.023	1.356	-66.342	-60.342		
	1.005	98.765	1.095	-70.857	-64.857		
	4.581	134.177	0.728	-69.814	-63.814		
	1.136	95.199	1.461	-55.398	-49.398		
	2.445	102.571	1.073	-59.100	-53.100		
	0.973	65.355	1.280	-54.033	-48.033		
	1.230	73.119	1.198	-54.884	-48.884		
350 K/15 V	1.585	75.904	1.127	-54.640	-48.640	2.148	89.840
	3.773	85.950	0.826	-54.650	-48.650		
	0.662	64.932	1.207	-63.555	-57.555		
	2.744	55.193	0.991	-44.692	-38.692		
	2.350	145.998	1.287	-61.632	-55.632		

Stress levels	Alpha (α)	Beta (β)	Constraint (c)	-2Log-like	AIC	Mean(α)	Mean (β)
	5.253	108.728	0.929	-55.154	-49.154		
	5.938	74.714	0.797	-50.058	-44.058		
	4.131	79.503	0.998	-48.548	-42.548		
	3.707	66.431	0.840	-50.180	-44.180		
	1.716	79.031	1.326	-49.906	-43.906		
350 K/20 V	5.091	65.902	0.798	-49.807	-43.807	3.830	84.197
	4.511	102.754	1.007	-52.097	-46.097		
	2.782	117.089	1.250	-54.922	-48.922		
	2.656	72.151	0.944	-52.101	-46.101		
	2.087	76.532	0.990	-54.421	-48.421		
	1.215	75.135	1.314	-53.288	-47.288		
	2.386	108.810	0.874	-65.908	-59.908		
	1.179	59.908	1.131	-53.848	-47.848		
	2.626	104.604	0.867	-64.247	-58.247		
400 K/10 V	2.036	92.282	0.933	-62.059	-56.059	1.856	85.815
	1.535	54.002	0.930	-54.892	-48.892		
	1.981	62.658	0.865	-56.503	-50.503		
	1.540	93.568	0.982	-68.481	-62.481		
	1.976	130.650	1.184	-66.811	-60.811		
	2.087	76.532	0.990	-54.421	-48.421		

Stress levels	Alpha (α)	Beta (β)	Constraint (c)	-2Log-like	AIC	Mean(α)	Mean (β)
	3.749	108.565	0.916	-58.603	-52.603		
	5.947	101.849	0.752	-61.361	-55.361		
	8.366	115.225	0.655	-60.885	-54.885		
	6.150	84.602	0.626	-62.315	-56.315		
	3.149	91.662	1.047	-54.693	-48.693		
400 K/15 V	2.598	47.780	0.794	-49.754	-43.754	4.381	89.358
	5.147	87.394	0.643	-58.549	-52.549		
	1.721	57.872	0.990	-52.223	-46.223		
	3.517	126.488	1.108	-61.699	-55.699		
	3.468	72.145	0.874	-51.672	-45.672		
	5.199	70.574	0.679	-53.309	-47.309		
	5.467	65.868	0.620	-56.131	-50.131		
	3.004	46.542	0.805	-46.303	-40.303		
	7.344	63.434	0.475	-58.074	-52.074		
	3.015	41.267	0.717	-47.533	-41.533		
400 K/20 V	3.562	67.026	0.845	-50.203	-44.203	4.439	64.482
	3.399	64.616	0.754	-53.425	-47.425		
	6.685	109.678	0.717	-58.247	-52.247		
	3.198	52.759	0.767	-49.508	-43.508		
	3.517	63.053	0.849	-49.803	-43.803		

Stress levels	Alpha (α)	Beta (β)	Constraint (c)	-2Log-like	AIC	Mean(α)	Mean (β)
	2.775	65.251	0.813	-53.899	-47.899		
	3.748	150.067	1.000	-65.971	-59.971		
	2.230	81.103	0.917	-58.095	-52.095		
	2.284	134.796	1.138	-64.141	-58.141		
	2.596	160.258	1.320	-58.180	-52.180		
450 K/10 V	1.178	63.373	1.173	-53.849	-47.849	2.279	108.209
	1.395	109.954	1.240	-63.814	-57.814		
	1.859	95.916	1.066	-60.814	-54.814		
	2.363	90.662	1.094	-54.199	-48.199		
	2.360	130.706	1.079	-66.450	-60.450		
	3.943	70.040	0.759	-53.576	-47.576		
	4.386	73.464	0.642	-57.490	-51.490		
	5.505	114.279	0.797	-61.164	-55.164		
	1.530	95.947	1.217	-60.315	-54.315		
450 K/15 V	4.882	91.535	0.826	-58.502	-52.502	4.520	90.748
	4.900	63.550	0.682	-52.852	-46.852		
	8.245	115.328	0.664	-58.033	-52.033		
	4.099	70.976	0.805	-51.935	-45.935		
	4.517	86.275	0.814	-59.144	-53.144		
	3.195	126.085	0.954	-64.454	-58.454		

Stress levels	Alpha (α)	Beta (β)	Constraint (c)	-2Log-like	AIC	Mean(α)	Mean (β)
	3.333	52.546	0.844	-46.102	-40.102		
	4.646	41.046	0.551	-48.715	-42.715		
	4.375	61.210	0.773	-49.117	-43.117		
	3.806	43.469	0.653	-48.288	-42.288		
	6.026	64.890	0.730	-48.921	-42.921		
450 K/20 V	4.182	55.439	0.782	-46.927	-40.927	4.686	54.067
	2.904	41.326	0.747	-47.315	-41.315		
	6.292	74.085	0.629	-52.783	-46.783		
	5.466	60.180	0.647	-50.660	-44.660		
	5.826	46.478	0.450	-55.438	-49.438		

APPENDIX C. FISHER MATRIX AND OPTIMIZATION MODEL FORMULATION

1st derivative of the log-likelihood function:

Here, $\psi_1(A_k)$ is a digamma function as defined by $\psi_1(A_k) = \frac{\delta \log \Gamma(A_k)}{\delta A_k} = \frac{\Gamma'(A_k)}{\Gamma(A_k)}$.

$$\frac{\partial \log L}{\partial \gamma_0} = \sum_{i=1}^{m} \sum_{j=1}^{n_k} \sum_{k=1}^{z} \left[A_k * \log(\beta_k) - A_k \psi_1(A_k) + A_k * \log\left(\Delta y_{ijk}\right)\right]$$

$$\frac{\partial \log L}{\partial \gamma_1} = \sum_{i=1}^{m} \sum_{j=1}^{n_k} \sum_{k=1}^{z} \left[A_k S_{1k} * \log(\beta_k) - A_k S_{1k} \psi_1(A_k) + A_k S_{1k} * \log\left(\Delta y_{ijk}\right)\right]$$

$$\frac{\partial \log L}{\partial \gamma_2} = \sum_{i=1}^{m} \sum_{j=1}^{n_k} \sum_{k=1}^{z} \left[A_k S_{2k} * \log(\beta_k) - A_k S_{2k} \psi_1(A_k) + A_k S_{2k} * \log\left(\Delta y_{ijk}\right)\right]$$

$$\frac{\partial \log L}{\partial \gamma_3} = \sum_{i=1}^{m} \sum_{j=1}^{n_k} \sum_{k=1}^{z} \left[A_k S_{1k} S_{2k} * \log(\beta_k) - A_k S_{1k} S_{2k} \psi_1(A_k) + A_k S_{1k} S_{2k} * \log\left(\Delta y_{ijk}\right)\right]$$

$$\frac{\partial \log L}{\partial \delta_0} = \sum_{i=1}^{m} \sum_{j=1}^{n_k} \sum_{k=1}^{z} \left[A_k - \Delta y_{ijk} \beta_k\right]$$

$$\frac{\partial \log L}{\partial \delta_1} = \sum_{i=1}^{m} \sum_{j=1}^{n_k} \sum_{k=1}^{z} \left[A_k S_{1k} - \Delta y_{ijk} \beta_k S_{1k}\right]$$

$$\frac{\partial \log L}{\partial \delta_2} = \sum_{i=1}^{m} \sum_{j=1}^{n_k} \sum_{k=1}^{z} \left[A_k S_{2k} - \Delta y_{ijk} \beta_k S_{2k}\right]$$

$$\frac{\partial \log L}{\partial \delta_3} = \sum_{i=1}^{m} \sum_{j=1}^{n_k} \sum_{k=1}^{z} \left[A_k S_{1k} S_{2k} - \Delta y_{ijk} \beta_k S_{1k} S_{2k}\right]$$

2nd derivative of log-likelihood function:

Here, $\psi_2(A_k)$ is a trigamma function as defined by $\psi_2(A_k) = \frac{\delta^2 \log \Gamma(A_k)}{\delta A_k^2} = \frac{\Gamma''(A_k)}{\Gamma(A_k)} = D_k$.

$$\frac{\partial^2 \log L}{\partial \gamma_0^2} = \sum_{i=1}^{m}\sum_{j=1}^{n_k}\sum_{k=1}^{z}\left[A_k * \log(\beta_k) - A_k\psi_1(A_k) - A_k^2 * \psi_2(A_k) + A_k * \log\left(\Delta y_{ijk}\right)\right]$$

$$\frac{\partial^2 \log L}{\partial \gamma_1^2} = \sum_{i=1}^{m}\sum_{j=1}^{n_k}\sum_{k=1}^{z}\left[A_k S_{1k}^2 * \log(\beta_k) - A_k S_{1k}^2\psi_1(A_k) - A_k^2 S_{1k}^2 * \psi_2(A_k) + A_k S_{1k}^2 * \log\left(\Delta y_{ijk}\right)\right]$$

$$\frac{\partial^2 \log L}{\partial \gamma_2^2} = \sum_{i=1}^{m}\sum_{j=1}^{n_k}\sum_{k=1}^{z}\left[A_k S_{2k}^2 * \log(\beta_k) - A_k S_{2k}^2\psi_1(A_k) - A_k^2 S_{2k}^2 * \psi_2(A_k) + A_k S_{2k}^2 * \log\left(\Delta y_{ijk}\right)\right]$$

$$\frac{\partial^2 \log L}{\partial \gamma_3^2} = \sum_{i=1}^{m}\sum_{j=1}^{n_k}\sum_{k=1}^{z}\left[A_k S_{1k}^2 S_{2k}^2 * \log(\beta_k) - A_k S_{1k}^2 S_{2k}^2\psi_1(A_k) - A_k^2 S_{1k}^2 S_{2k}^2 * \psi_2(A_k) + A_k S_{1k}^2 S_{2k}^2 * \log\left(\Delta y_{ijk}\right)\right]$$

$$\frac{\partial^2 \log L}{\partial \delta_0^2} = \sum_{i=1}^{m}\sum_{j=1}^{n_k}\sum_{k=1}^{z}\left[-\Delta y_{ijk}\beta_k\right]$$

$$\frac{\partial^2 \log L}{\partial \delta_1^2} = \sum_{i=1}^{m}\sum_{j=1}^{n_k}\sum_{k=1}^{z}\left[-\Delta y_{ijk}\beta_k S_{1k}^2\right]$$

$$\frac{\partial^2 \log L}{\partial \delta_2^2} = \sum_{i=1}^{m}\sum_{j=1}^{n_k}\sum_{k=1}^{z}\left[-\Delta y_{ijk}\beta_k S_{2k}^2\right]$$

$$\frac{\partial^2 \log L}{\partial \delta_3^2} = \sum_{i=1}^{m}\sum_{j=1}^{n_k}\sum_{k=1}^{z}\left[-\Delta y_{ijk}\beta_k S_{1k}^2 S_{2k}^2\right]$$

Other combination derivatives of the log-likelihood function:

$$\frac{\partial^2 \log L}{\partial \gamma_0 \delta_0} = \sum_{i=1}^{m} \sum_{j=1}^{n_k} \sum_{k=1}^{z} [A_k]$$

$$\frac{\partial^2 \log L}{\partial \gamma_0 \delta_1} = \sum_{i=1}^{m} \sum_{j=1}^{n_k} \sum_{k=1}^{z} [A_k S_{1k}]$$

$$\frac{\partial^2 \log L}{\partial \gamma_0 \delta_2} = \sum_{i=1}^{m} \sum_{j=1}^{n_k} \sum_{k=1}^{z} [A_k S_{2k}]$$

$$\frac{\partial^2 \log L}{\partial \gamma_0 \delta_3} = \sum_{i=1}^{m} \sum_{j=1}^{n_k} \sum_{k=1}^{z} [A_k S_{1k} S_{2k}]$$

$$\frac{\partial^2 \log L}{\partial \gamma_1 \delta_0} = \sum_{i=1}^{m} \sum_{j=1}^{n_k} \sum_{k=1}^{z} [A_k S_{1k}]$$

$$\frac{\partial^2 \log L}{\partial \gamma_1 \delta_1} = \sum_{i=1}^{m} \sum_{j=1}^{n_k} \sum_{k=1}^{z} [A_k S_{1k}^2]$$

$$\frac{\partial^2 \log L}{\partial \gamma_1 \delta_2} = \sum_{i=1}^{m} \sum_{j=1}^{n_k} \sum_{k=1}^{z} [A_k S_{1k} S_{2k}]$$

$$\frac{\partial^2 \log L}{\partial \gamma_1 \delta_3} = \sum_{i=1}^{m} \sum_{j=1}^{n_k} \sum_{k=1}^{z} [A_k S_{1k}^2 S_{2k}]$$

$$\frac{\partial^2 \log L}{\partial \gamma_2 \delta_0} = \sum_{i=1}^{m} \sum_{j=1}^{n_k} \sum_{k=1}^{z} [A_k S_{2k}]$$

$$\frac{\partial^2 \log L}{\partial \gamma_2 \delta_1} = \sum_{i=1}^{m} \sum_{j=1}^{n_k} \sum_{k=1}^{z} [A_k S_{1k} S_{2k}]$$

$$\frac{\partial^2 \log L}{\partial \gamma_2 \delta_2} = \sum_{i=1}^{m} \sum_{j=1}^{n_k} \sum_{k=1}^{z} [A_k S_{2k}^2]$$

$$\frac{\partial^2 \log L}{\partial \gamma_2 \delta_3} = \sum_{i=1}^{m} \sum_{j=1}^{n_k} \sum_{k=1}^{z} [A_k S_{1k} S_{2k}^2]$$

$$\frac{\partial^2 \log L}{\partial \gamma_3 \delta_0} = \sum_{i=1}^{m} \sum_{j=1}^{n_k} \sum_{k=1}^{z} [A_k S_{1k} S_{2k}]$$

$$\frac{\partial^2 \log L}{\partial \gamma_3 \delta_1} = \sum_{i=1}^{m} \sum_{j=1}^{n_k} \sum_{k=1}^{z} [A_k S_{1k}^2 S_{2k}]$$

$$\frac{\partial^2 \log L}{\partial \gamma_3 \delta_2} = \sum_{i=1}^{m} \sum_{j=1}^{n_k} \sum_{k=1}^{z} [A_k S_{1k} S_{2k}^2]$$

$$\frac{\partial^2 \log L}{\partial \gamma_3 \delta_3} = \sum_{i=1}^{m} \sum_{j=1}^{n_k} \sum_{k=1}^{z} [A_k S_{1k}^2 S_{2k}^2]$$

$$\frac{\partial^2 \log L}{\partial \gamma_0 \gamma_1} = \sum_{i=1}^{m} \sum_{j=1}^{n_k} \sum_{k=1}^{z} \left[A_k S_{1k} * \log(\beta_k) - A_k S_{1k} \psi_1(A_k) - A_k^2 S_{1k} * \psi_2(A_k) + A_k S_{1k} * \log\left(\Delta y_{ijk}\right)\right]$$

$$\frac{\partial^2 \log L}{\partial \gamma_0 \gamma_2} = \sum_{i=1}^{m} \sum_{j=1}^{n_k} \sum_{k=1}^{z} \left[A_k S_{2k} * \log(\beta_k) - A_k S_{2k} \psi_1(A_k) - A_k^2 S_{2k} * \psi_2(A_k) + A_k S_{2k} * \log\left(\Delta y_{ijk}\right)\right]$$

$$\frac{\partial^2 \log L}{\partial \gamma_0 \gamma_3} = \sum_{i=1}^{m} \sum_{j=1}^{n_k} \sum_{k=1}^{z} \Big[A_k S_{1k} S_{2k} * \log(\beta_k) - A_k S_{1k} S_{2k} \psi_1(A_k) - A_k^2 S_{1k} S_{2k} * \psi_2(A_k) + A_k S_{1k} S_{2k} * \log\big(\Delta y_{ijk}\big)\Big]$$

$$\frac{\partial^2 \log L}{\partial \gamma_1 \gamma_2} = \sum_{i=1}^{m} \sum_{j=1}^{n_k} \sum_{k=1}^{z} \Big[A_k S_{1k} S_{2k} * \log(\beta_k) - A_k S_{1k} S_{2k} \psi_1(A_k) - A_k^2 S_{1k} S_{2k} * \psi_2(A_k) + A_k S_{1k} S_{2k} * \log\big(\Delta y_{ijk}\big)\Big]$$

$$\frac{\partial^2 \log L}{\partial \gamma_1 \gamma_3} = \sum_{i=1}^{m} \sum_{j=1}^{n_k} \sum_{k=1}^{z} \Big[A_k S_{1k}^2 S_{2k} * \log(\beta_k) - A_k S_{1k}^2 S_{2k} \psi_1(A_k) - A_k^2 S_{1k}^2 S_{2k} * \psi_2(A_k) + A_k S_{1k}^2 S_{2k} * \log\big(\Delta y_{ijk}\big)\Big]$$

$$\frac{\partial^2 \log L}{\partial \gamma_2 \gamma_3} = \sum_{i=1}^{m} \sum_{j=1}^{n_k} \sum_{k=1}^{z} \Big[A_k S_{1k} S_{2k}^2 * \log(\beta_k) - A_k S_{1k} S_{2k}^2 \psi_1(A_k) - A_k^2 S_{1k} S_{2k}^2 * \psi_2(A_k) + A_k S_{1k} S_{2k}^2 * \log\big(\Delta y_{ijk}\big)\Big]$$

$$\frac{\partial^2 \log L}{\partial \delta_0 \delta_1} = \sum_{i=1}^{m} \sum_{j=1}^{n_k} \sum_{k=1}^{z} \Big[-\Delta y_{ijk} \beta_k S_{1k}\Big]$$

$$\frac{\partial^2 \log L}{\partial \delta_0 \delta_2} = \sum_{i=1}^{m} \sum_{j=1}^{n_k} \sum_{k=1}^{z} \Big[-\Delta y_{ijk} \beta_k S_{2k}\Big]$$

$$\frac{\partial^2 \log L}{\partial \delta_0 \delta_3} = \sum_{i=1}^{m} \sum_{j=1}^{n_k} \sum_{k=1}^{z} \Big[-\Delta y_{ijk} \beta_k S_{1k} S_{2k}\Big]$$

$$\frac{\partial^2 \log L}{\partial \delta_1 \delta_2} = \sum_{i=1}^{m} \sum_{j=1}^{n_k} \sum_{k=1}^{z} \Big[-\Delta y_{ijk} \beta_k S_{1k} S_{2k}\Big]$$

$$\frac{\partial^2 \log L}{\partial \delta_1 \delta_3} = \sum_{i=1}^{m} \sum_{j=1}^{n_k} \sum_{k=1}^{z} \Big[-\Delta y_{ijk} \beta_k S_{1k}^2 S_{2k}\Big]$$

$$\frac{\partial^2 \log L}{\partial \delta_2 \delta_3} = \sum_{i=1}^{m} \sum_{j=1}^{n_k} \sum_{k=1}^{z} \Big[-\Delta y_{ijk} \beta_k S_{1k} S_{2k}^2\Big]$$

The expected value of derivatives:

$$E(\log \Delta y_{ijk}) = \int_{0}^{\infty} \log \Delta y_{ijk} \frac{\beta_k^{A_k}}{\Gamma(A_k)} \Delta y_{ijk}^{(A_k-1)} e^{-\Delta y_{ijk}\beta_k} \, d\Delta y_{ijk}$$

$$= \frac{1}{\Gamma(A_k)} \left(\int_0^{\infty} \log x \;\; x^{A_k-1} e^{-x} \, dx - \int_0^{\infty} \log \beta_k \, x^{A_k-1} e^{-x} \, dx \right) \quad ; x = \Delta y_{ijk}\beta_k$$

$$= \frac{1}{\Gamma(A_k)} \left(\Gamma'(A_k) - \log \beta_k \Gamma(A_k) \right)$$

$$E(\log \Delta y_{ijk}) = \psi_1(A_k) - \log \beta_k$$

And $E\left(\Delta y_{ijk}\right) = A_k / \beta_k$

$$\mathrm{E}\left(-\frac{\partial^2 \log L}{\partial \gamma_0^2}\right) = m \sum_{k=1}^{z} n_k A_k^2 D_k$$

$$\mathrm{E}\left(-\frac{\partial^2 \log L}{\partial \gamma_1^2}\right) = m \sum_{k=1}^{z} n_k A_k^2 S_{1k}^2 D_k$$

$$\mathrm{E}\left(-\frac{\partial^2 \log L}{\partial \gamma_2^2}\right) = m \sum_{k=1}^{z} n_k A_k^2 S_{2k}^2 D_k$$

$$\mathrm{E}\left(-\frac{\partial^2 \log L}{\partial \gamma_3^2}\right) = m \sum_{k=1}^{z} n_k A_k^2 S_{1k}^2 S_{2k}^2 D_k$$

$$\mathrm{E}\left(-\frac{\partial^2 \log L}{\partial \delta_0^2}\right) = m \sum_{k=1}^{z} n_k A_k$$

$$\mathrm{E}\left(-\frac{\partial^2 \log L}{\partial \delta_1^2}\right) = m \sum_{k=1}^{z} n_k A_k S_{1k}^2$$

$$\mathrm{E}\left(-\frac{\partial^2 \log L}{\partial \delta_2^2}\right) = m \sum_{k=1}^{z} n_k A_k S_{2k}^2$$

$$\mathrm{E}\left(-\frac{\partial^2 \log L}{\partial \delta_3^2}\right) = m \sum_{k=1}^{z} n_k A_k S_{1k}^2 S_{2k}^2$$

$$\mathrm{E}\left(-\frac{\partial^2 \log L}{\partial \gamma_0 \gamma_1}\right) = m \sum_{k=1}^{z} n_k A_k^2 \, S_{1k} D_k$$

$$\mathrm{E}\left(-\frac{\partial^2 \log L}{\partial \gamma_0 \gamma_2}\right) = m \sum_{k=1}^{z} n_k A_k^2 \, S_{2k} D_k$$

$$\mathrm{E}\left(-\frac{\partial^2 \log L}{\partial \gamma_0 \gamma_3}\right) = m \sum_{k=1}^{z} n_k A_k^2 \, S_{1k} S_{2k} D_k$$

$$\mathrm{E}\left(-\frac{\partial^2 \log L}{\partial \gamma_0 \delta_0}\right) = -m \sum_{k=1}^{z} n_k A_k$$

$$\mathrm{E}\left(-\frac{\partial^2 \log L}{\partial \gamma_0 \delta_1}\right) = -m \sum_{k=1}^{z} n_k A_k \, S_{1k}$$

$$\mathrm{E}\left(-\frac{\partial^2 \log L}{\partial \gamma_0 \delta_2}\right) = -m \sum_{k=1}^{z} n_k A_k \, S_{2k}$$

$$\mathrm{E}\left(-\frac{\partial^2 \log L}{\partial \gamma_0 \delta_3}\right) = -m \sum_{k=1}^{z} n_k A_k \, S_{1k} S_{2k}$$

$$\mathrm{E}\left(-\frac{\partial^2 \log L}{\partial \gamma_1 \gamma_2}\right) = m \sum_{k=1}^{z} n_k A_k^2 \, S_{1k} S_{2k} D_k$$

$$\mathrm{E}\left(-\frac{\partial^2 \log L}{\partial \gamma_1 \gamma_3}\right) = m \sum_{k=1}^{z} n_k A_k^2 \, S_{1k}^2 S_{2k} D_k$$

$$\mathrm{E}\left(-\frac{\partial^2 \log L}{\partial \gamma_1 \delta_0}\right) = -m \sum_{k=1}^{z} n_k \, A_k S_{1k}$$

$$\mathrm{E}\left(-\frac{\partial^2 \log L}{\partial \gamma_1 \delta_1}\right) = -m \sum_{k=1}^{z} n_k \, A_k S_{1k}^2$$

$$\mathrm{E}\left(-\frac{\partial^2 \log L}{\partial \gamma_1 \delta_2}\right) = -m \sum_{k=1}^{z} n_k \, A_k S_{1k} S_{2k}$$

$$\mathrm{E}\left(-\frac{\partial^2 \log L}{\partial \gamma_1 \delta_3}\right) = -m \sum_{k=1}^{z} n_k \, A_k S_{1k}^2 S_{2k}$$

$$\mathrm{E}\left(-\frac{\partial^2 \log L}{\partial \gamma_2 \gamma_3}\right) = m \sum_{k=1}^{z} n_k \, A_k^2 S_{1k} S_{2k}^2 D_k$$

$$\mathrm{E}\left(-\frac{\partial^2 \log L}{\partial \gamma_2 \delta_0}\right) = -m \sum_{k=1}^{z} n_k \, A_k S_{2k}$$

$$\mathrm{E}\left(-\frac{\partial^2 \log L}{\partial \gamma_2 \delta_1}\right) = -m \sum_{k=1}^{z} n_k \, A_k S_{1k} S_{2k}$$

$$\mathrm{E}\left(-\frac{\partial^2 \log L}{\partial \gamma_2 \delta_2}\right) = -m \sum_{k=1}^{z} n_k \, A_k S_{2k}^2$$

$$\mathrm{E}\left(-\frac{\partial^2 \log L}{\partial \gamma_2 \delta_3}\right) = -m \sum_{k=1}^{z} n_k \, A_k S_{1k} S_{2k}^2$$

$$\mathrm{E}\left(-\frac{\partial^2 \log L}{\partial \gamma_3 \delta_0}\right) = -m \sum_{k=1}^{z} n_k\, A_k S_{1k} S_{2k}$$

$$\mathrm{E}\left(-\frac{\partial^2 \log L}{\partial \gamma_3 \delta_1}\right) = -m \sum_{k=1}^{z} n_k\, A_k S_{1k}^2 S_{2k}$$

$$\mathrm{E}\left(-\frac{\partial^2 \log L}{\partial \gamma_3 \delta_2}\right) = -m \sum_{k=1}^{z} n_k\, A_k S_{1k} S_{2k}^2$$

$$\mathrm{E}\left(-\frac{\partial^2 \log L}{\partial \gamma_3 \delta_3}\right) = -m \sum_{k=1}^{z} n_k\, A_k S_{1k}^2 S_{2k}^2$$

$$\mathrm{E}\left(-\frac{\partial^2 \log L}{\partial \delta_0 \delta_1}\right) = m \sum_{k=1}^{z} n_k\, A_k S_{1k}$$

$$\mathrm{E}\left(-\frac{\partial^2 \log L}{\partial \delta_0 \delta_2}\right) = m \sum_{k=1}^{z} n_k\, A_k S_{2k}$$

$$\mathrm{E}\left(-\frac{\partial^2 \log L}{\partial \delta_0 \delta_3}\right) = m \sum_{k=1}^{z} n_k\, A_k S_{1k} S_{2k}$$

$$\mathrm{E}\left(-\frac{\partial^2 \log L}{\partial \delta_1 \delta_2}\right) = m \sum_{k=1}^{z} n_k\, A_k S_{1k} S_{2k}$$

$$\mathrm{E}\left(-\frac{\partial^2 \log L}{\partial \delta_1 \delta_3}\right) = m \sum_{k=1}^{z} n_k\, A_k S_{1k}^2 S_{2k}$$

$$\mathrm{E}\left(-\frac{\partial^2 \log L}{\partial \delta_2 \delta_3}\right) = m \sum_{k=1}^{z} n_k\, A_k S_{1k} S_{2k}^2$$

Fisher information matrix: For two stress level

$$\begin{bmatrix} S_{11} & S_{12} & \rightarrow & k_1 \\ S_{11} & S_{22} & \rightarrow & k_2 \\ S_{12} & S_{21} & \rightarrow & k_3 \\ S_{12} & S_{22} & \rightarrow & k_4 \end{bmatrix}$$

$$a_{11} = a_{11} = m(n_1A_1^2D_1 + n_2A_2^2D_2 + n_3A_3^2D_3 + n_4A_4^2D_4)$$

$$a_{12} = a_{21} = m(n_1A_1^2S_{11}D_1 + n_2A_2^2S_{11}D_2 + n_3A_3^2S_{12}D_3 + n_4A_4^2S_{12}D_4)$$

$$a_{13} = a_{31} = m(n_1A_1^2S_{21}D_1 + n_2A_2^2S_{22}D_2 + n_3A_3^2S_{21}D_3 + n_4A_4^2S_{22}D_4)$$

$$a_{14} = a_{41} = m(n_1A_1^2S_{11}S_{21}D_1 + n_2A_2^2S_{11}S_{22}D_2 + n_3A_3^2S_{12}S_{21}D_3 + n_4A_4^2S_{12}S_{22}D_4)$$

$$a_{15} = a_{51} = -m(n_1A_1 + n_2A_2 + n_3A_3 + n_4A_4)$$

$$a_{16} = a_{61} = -m(n_1A_1S_{11} + n_2A_2S_{11} + n_3A_3S_{12} + n_4A_4S_{12})$$

$$a_{17} = a_{71} = -m(n_1A_1S_{21} + n_2A_2S_{22} + n_3A_3S_{21} + n_4A_4S_{22})$$

$$a_{18} = a_{81} = -m(n_1A_1S_{11}S_{21} + n_2A_2S_{11}S_{22} + n_3A_3S_{12}S_{21} + n_4A_4S_{12}S_{22})$$

$$a_{22} = a_{22} = m(n_1A_1^2S_{11}^2D_1 + n_2A_2^2S_{11}^2D_2 + n_3A_3^2S_{12}^2D_3 + n_4A_4^2S_{12}^2D_4)$$

$$a_{23} = a_{32} = m(n_1A_1^2S_{11}S_{21}D_1 + n_2A_2^2S_{11}S_{22}D_2 + n_3A_3^2S_{12}S_{21}D_3 + n_4A_4^2S_{12}S_{22}D_4)$$

$$a_{24} = a_{42} = m(n_1A_1^2S_{11}^2S_{21}D_1 + n_2A_2^2S_{11}^2S_{22}D_2 + n_3A_3^2S_{12}^2S_{21}D_3 + n_4A_4^2S_{12}^2S_{22}D_4)$$

$$a_{25} = a_{52} = -m(n_1A_1S_{11} + n_2A_2S_{11} + n_3A_3S_{12} + n_4A_4S_{12})$$

$$a_{26} = a_{62} = -m(n_1A_1S_{11}^2 + n_2A_2S_{11}^2 + n_3A_3S_{12}^2 + n_4A_4S_{12}^2)$$

$$a_{27} = a_{72} = -m(n_1A_1S_{11}S_{21} + n_2A_2S_{11}S_{22} + n_3A_3S_{12}S_{21} + n_4A_4S_{12}S_{22})$$

$$a_{28} = a_{82} = -m(n_1 A_1 S_{11}^2 S_{21} + n_2 A_2 S_{11}^2 S_{22} + n_3 A_3 S_{12}^2 S_{21} + n_4 A_4 S_{12}^2 S_{22})$$

$$a_{33} = a_{33} = m(n_1 A_1^2 S_{21}^2 D_1 + n_2 A_2^2 S_{22}^2 D_2 + n_3 A_3^2 S_{21}^2 D_3 + n_4 A_4^2 S_{22}^2 D_4)$$

$$a_{34} = a_{43} = m(n_1 A_1^2 S_{11} S_{21}^2 D_1 + n_2 A_2^2 S_{11} S_{22}^2 D_2 + n_3 A_3^2 S_{12} S_{21}^2 D_3 + n_4 A_4^2 S_{12} S_{22}^2 D_4)$$

$$a_{35} = a_{53} = -m(n_1 A_1 S_{21} + n_2 A_2 S_{22} + n_3 A_3 S_{21} + n_4 A_4 S_{22})$$

$$a_{36} = a_{63} = -m(n_1 A_1 S_{11} S_{21} + n_2 A_2 S_{11} S_{22} + n_3 A_3 S_{12} S_{21} + n_4 A_4 S_{12} S_{22})$$

$$a_{37} = a_{73} = -m(n_1 A_1 S_{21}^2 + n_2 A_2 S_{22}^2 + n_3 A_3 S_{21}^2 + n_4 A_4 S_{22}^2)$$

$$a_{38} = a_{83} = -m(n_1 A_1 S_{11} S_{21}^2 + n_2 A_2 S_{11} S_{22}^2 + n_3 A_3 S_{12} S_{21}^2 + n_4 A_4 S_{12} S_{22}^2)$$

$$a_{44} = a_{44} = m(n_1 A_1^2 S_{11}^2 S_{21}^2 D_1 + n_2 A_2^2 S_{11}^2 S_{22}^2 D_2 + n_3 A_3^2 S_{12}^2 S_{21}^2 D_3 + n_4 A_4^2 S_{12}^2 S_{22}^2 D_4)$$

$$a_{45} = a_{54} = -m(n_1 A_1 S_{11} S_{21} + n_2 A_2 S_{11} S_{22} + n_3 A_3 S_{12} S_{21} + n_4 A_4 S_{12} S_{22})$$

$$a_{46} = a_{64} = -m(n_1 A_1 S_{11}^2 S_{21} + n_2 A_2 S_{11}^2 S_{22} + n_3 A_3 S_{12}^2 S_{21} + n_4 A_4 S_{12}^2 S_{22})$$

$$a_{47} = a_{74} = -m(n_1 A_1 S_{11} S_{21}^2 + n_2 A_2 S_{12} S_{22}^2 + n_3 A_3 S_{12} S_{21}^2 + n_4 A_4 S_{12} S_{22}^2)$$

$$a_{48} = a_{84} = -m(n_1 A_1 S_{11}^2 S_{21}^2 + n_2 A_2 S_{11}^2 S_{22}^2 + n_3 A_3 S_{12}^2 S_{21}^2 + n_4 A_4 S_{12}^2 S_{22}^2)$$

$$a_{55} = a_{55} = m(n_1 A_1 + n_2 A_2 + n_3 A_3 + n_4 A_4)$$

$$a_{56} = a_{65} = m(n_1 A_1 S_{11} + n_2 A_2 S_{11} + n_3 A_3 S_{12} + n_4 A_4 S_{12})$$

$$a_{57} = a_{75} = m(n_1 A_1 S_{21} + n_2 A_2 S_{22} + n_3 A_3 S_{21} + n_4 A_4 S_{22})$$

$$a_{58} = a_{85} = m(n_1 A_1 S_{11} S_{21} + n_2 A_2 S_{11} S_{22} + n_3 A_3 S_{12} S_{21} + n_4 A_4 S_{12} S_{22})$$

$a_{66} = a_{66} = m(n_1 A_1 S_{11}^2 + n_2 A_2 S_{11}^2 + n_3 A_3 S_{12}^2 + n_4 A_4 S_{12}^2)$

$a_{67} = a_{76} = m(n_1 A_1 S_{11} S_{21} + n_2 A_2 S_{11} S_{22} + n_3 A_3 S_{12} S_{21} + n_4 A_4 S_{12} S_{22})$

$a_{68} = a_{86} = m(n_1 A_1 S_{11}^2 S_{21} + n_2 A_2 S_{11}^2 S_{22} + n_3 A_3 S_{12}^2 S_{21} + n_4 A_4 S_{12}^2 S_{22})$

$a_{77} = a_{77} = m(n_1 A_1 S_{21}^2 + n_2 A_2 S_{22}^2 + n_3 A_3 S_{21}^2 + n_4 A_4 S_{22}^2)$

$a_{78} = a_{87} = m(n_1 A_1 S_{11} S_{21}^2 + n_2 A_2 S_{11} S_{22}^2 + n_3 A_3 S_{12} S_{21}^2 + n_4 A_4 S_{12} S_{22}^2)$

$a_{88} = a_{88} = m(n_1 A_1 S_{11}^2 S_{21}^2 + n_2 A_2 S_{11}^2 S_{22}^2 + n_3 A_3 S_{12}^2 S_{21}^2 + n_4 A_4 S_{12}^2 S_{22}^2)$

The Fisher information matrix thus,

$$\boldsymbol{F}(\theta) = \begin{bmatrix} a_{11} & a_{12} & a_{13} & a_{14} & a_{15} & a_{16} & a_{17} & a_{18} \\ & a_{22} & a_{23} & a_{24} & a_{25} & a_{26} & a_{27} & a_{28} \\ & & a_{33} & a_{34} & a_{35} & a_{36} & a_{37} & a_{38} \\ & & & a_{44} & a_{45} & a_{46} & a_{47} & a_{48} \\ & & & & a_{55} & a_{56} & a_{57} & a_{58} \\ & & & & & a_{66} & a_{67} & a_{68} \\ & & & & & & a_{77} & a_{78} \\ & Symmetry & & & & & & a_{88} \end{bmatrix}$$

The lifetime,

$$\xi_S = \frac{\omega_\beta}{\alpha} + \frac{1}{2\alpha}$$

Here, $\omega_\beta = (\omega_1 - y_0)\beta$

At use condition, $\alpha(S_0) = e^{\gamma_0} = \alpha_0$

$\beta(S_0) = e^{\delta_0} = \beta_0$

$\omega_\beta = (\omega_1 - y_0)\beta_0 = \omega_{\beta_0}$

Therefore, $\xi_{S_0} = \left(\frac{\omega_{\beta_0}}{\alpha_0} + \frac{1}{2\alpha_0}\right)$

Now,

$$\frac{\partial \xi_{S_0}}{\partial \gamma_0} = -\left(\frac{\omega_{\beta_0}}{\alpha_0} + \frac{1}{2\alpha_0}\right)$$

$$\frac{\partial \xi_{S_0}}{\partial \gamma_1} = \frac{\partial \xi_{S_0}}{\partial \gamma_2} = \frac{\partial \xi_{S_0}}{\partial \gamma_3} = \frac{\partial \xi_{S_0}}{\partial \delta_1} = \frac{\partial \xi_{S_0}}{\partial \delta_2} = \frac{\partial \xi_{S_0}}{\partial \delta_3} = 0$$

$$\frac{\partial \xi_{S_0}}{\partial \delta_0} = \frac{\omega_{\beta_0}}{\alpha_0}$$

So, $\boldsymbol{h^T} = \left[\frac{\partial \xi_{S_0}}{\partial \gamma_0}, 0,0,0, \frac{\partial \xi_{S_0}}{\partial \delta_0}, 0,0,0\right] = \left[= -\left(\frac{\omega_{\beta_0}}{\alpha_0} + \frac{1}{2\alpha_0}\right), 0,0,0, \frac{\omega_{\beta_0}}{\alpha_0}, 0,0,0\right]$

The asymptotic variance of ξ_{S_0} can be written as follows,

$$Asvar\left(\hat{\xi}_{S_0}\right) = \boldsymbol{h^T F^{-1}}(\theta)\boldsymbol{h}$$

$$\boldsymbol{F^{-1}}(\theta) = \begin{bmatrix} A_{11} & A_{12} & A_{13} & A_{14} & A_{15} & A_{16} & A_{17} & A_{18} \\ & A_{22} & A_{23} & A_{24} & A_{25} & A_{26} & A_{27} & A_{28} \\ & & A_{33} & A_{34} & A_{35} & A_{36} & A_{37} & A_{38} \\ & & & A_{44} & A_{45} & A_{46} & A_{47} & A_{48} \\ & & & & A_{55} & A_{56} & A_{57} & A_{58} \\ & & & & & A_{66} & A_{67} & A_{68} \\ & & & & & & A_{77} & A_{78} \\ & Symmetry & & & & & & A_{88} \end{bmatrix}$$

Considering, $\boldsymbol{h^T} = \left[\frac{\partial \xi_{S_0}}{\partial \gamma_0}, 0,0,0, \frac{\partial \xi_{S_0}}{\partial \delta_0}, 0,0,0\right] = [a, 0,0,0, b, 0,0,0]$

Therefore,

$$Asvar\left(\hat{\xi}_{S_0}(\hat{\boldsymbol{x}})\right) = \boldsymbol{h^T F^{-1}}(\theta)\boldsymbol{h} = a^2A_{11} + abA_{51} + abA_{15} + b^2A_{88}$$

$$= a^2A_{11} + 2abA_{51} + b^2A_{88}$$