APPLICATIONS OF GROUPS OF DIVISIBILITY AND A GENERALIZATION OF KRULL DIMENSION

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ABSTRACT

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Groups of divisibility have played an important role in commutative algebra for many years. In 1932 Wolfgang Krull showed in [12] that every linearly ordered abelian group can be realized as the group of divisibility of a valuation domain. Since then it has also been proven that every lattice-ordered abelian group can be realized as the group of divisibility of a Bêzout domain. Knowing these two facts allows us to use groups of divisibility to find examples of rings with highly exotic properties. For instance, we use them here to find examples of rings which admit elements that factor uniquely as the product of uncountably many primes. In addition to allowing us to create examples, groups of divisibility can be used to characterize some of the most important rings most commonly encountered in factorization theory, including valuation domains, UFD's, GCD domains, and antimatter domains. We present some of these characterizations here in addition to using them to create many examples of our own, including examples of rings which admit chains of prime ideals in which there are uncountably many primes in the chain. Moreover, we use groups of divisibility to prove that every fragmented domain must have infinite Krull dimension.

As has already been stated, in using groups of divisibility we tended to run across rings that exhibited the strange property of having uncountable chains of prime ideals. As Krull dimension is a measure of "how high" one can continue to stack prime

ideals, it seemed natural to think of such rings as having a greater Krull dimension than rings in which every maximal ideal has finite height. Thus, we are motivated to create a means of being able to distinguish the Krull dimension of infinite dimensional rings. In so doing we have constructed a new definition which generalizes our present notion of Krull dimension and also allows us to make the kind of distinction we seek in the infinite dimensional case by associating to every ring a unique cardinal number which we define to be its Krull dimension. Here we find another use of groups of divisibility when we show that given any cardinal number σ, then we can find a ring whose Krull dimension is exactly σ. We also compute the dimension of infinite dimensional Noetherian rings as well as the dimension of $F[x_1, x_2, \ldots]$, where F is any countable field. Moreover, we prove that not only does every ring admit a unique Krull dimension, but in using our definition we also show that Krull dimension is preserved in all ring extensions that are INC and GU. In particular, Krull dimension is preserved in all integral extensions.

We close this dissertation by introducing a class of rings which we call *purgatory* domains. These are the rings which lie between the nicely behaved atomic domains and the nightmarish antimatter domains. In the last chapter, all integral domains are placed into one of the four following collections of isomorphism classes: fields, non-field atomic domains, non-field antimatter domains, and purgatory domains. It is shown that every isomorphism class of integral domains can be associated to a unique class of purgatory domains. Thus, if there is a "biggest" class of domains, it would have to be those that lie in purgatory.

ACKNOWLEDGEMENTS

Here I would like to name some of the folks who have played a special role in bringing me to my current station in life. Such a list is necessarily non-exhaustive; even embarrassingly so.

My father, Craig Trentham, taught me an invaluable mathematical skill. A person needs to be precise and articulate in one's communications. One's ideas need to be detailed, sharp, and focused. If a father has done his job right (and mine certainly has!), then any attempt to offer thanks must be met with frustration and embarrassment. We cannot keep a straight face and proclaim to our nearest star, "By the way, thanks for the light and the heat". Work hard, rely on yourself, and be precise in your thoughts and words. Necessary skills for any mathematician! I thank my dad for these lessons he taught me; these and the billions of others that a son will pick up watching his father.

I am terribly proud of the NDSU Department of Mathematics. The rate at which the department is graduating Ph.D's and the quality of these Ph.D's points to a math department that is on a consistent and accelerating rise. I have enjoyed the relaxed, friendly environment of the department and have benefitted enormously from its high mathematical standards. The department has offered me the opportunity to learn the art of teaching mathematics to undergraduates as well as affording me numerous occasions to give talks in seminars and colloquia regarding various aspects of my research. I am enormously grateful to the department for making travel funds available to me, thereby making it possible to give talks at a number of conferences on commutative algebra around the country. Special thanks must be given to Ben Duncan and Sean Sather-Wagstaff. Ben gave me invaluable advice on numerous occasions regarding particulars with various of my talks, applying for jobs, and advice on surviving my first years as a faculty member. Sean has also served as a mentor to me and oftentimes made sure I had all my ducks in a row. Sean made numerous helpful suggestions regarding my thesis and I think that the notation is cleaner and more consistent because of them. I appreciate your "A.R.", Sean! Both Sean and Ben were instrumental in helping me find employment upon graduating. Many thanks, gentlemen!

DEDICATION

I have dreamed of this particular moment for quite some time. There are many people who have played an important role in shaping me into a mathematician - more than I can list. But there are three that stand out about all others. Children of the Great Smoky Mountains all!

I am named after my grandfather, William Carl Headrick - "Pop" to me. It is Pop who first taught me who my best teacher is and ought to be. Having grown up during the Great Depression, he and his family did not have the money to send him to college. Yet, he is one of the most learned men I have ever known (and I've known a few!). Pop was an artist. He was an artist in the way he played his guitar, grew his garden, lived his life. Many people in my life would describe me as fiercely independent, but my entire life I have always been haunted by the prospect of doing *anything* that could ever let this man down or disappoint him in any way. Pop would take immense pride in the work I have done here - as proud as when he would come watch me pitch from the mound, make (occasional) good grades, read a book he had suggested, or see me walking through his door. There is no higher praise than praise from Pop.

My mother, Mary Ann Trentham, is Pop's daughter. She is a musician, writer, and an educator. More than anyone else, Mother taught me the importance of appreciating and creating beautiful things. I think she knew I was going to be something along the lines of a mathematician early on in my childhood. She even tried telling me as much on a number of occasions. Most likely I would roll my eyes. On many occasions I have told her that I think that I inherited much of my mathematical ability from her. I think she finds this somewhat laughable. However, mathematical research requires all the imagination of the artist and it can easily be argued that the practice of conducting mathematical research is an art. My mother is one that tends

to look at the big picture in life. If I have inherited my mathematical tendencies from her, then it is no surprise that I have gravitated toward algebra. As an algebraist, it is not my primary concern that my work might have some application in the "real world" in the right here and right now. It is the beauty of the topic that holds sway over me and motivates me. I do it because I no longer know how to not do it. I think my mom, the pianist, would understand perfectly.

Jim Coykendall - my mentor, guru, and sage during my stay at NDSU. I would also add "academic advisor" and "close friend" to that list. Jim and I crossed paths many years ago. His first words to me were a not-so-veiled threat upon my continuing existence. I *think* he was joking. No doubt I have afforded him many occasions since that time to regret not having followed through. I tend to bless my friends and loved-ones with a mighty need for patience. While my mom might have known that I was going into the mathematical sciences when I was a youngster, it was Jim who took my hand and led me down the rabbit-hole. Many were the times when I would approach Jim with a clever proof of an oh-so-sexy result only to have him effortlessly swat it down with a quick counterexample. But it has been in this way that I have learned that a good example carries all the weight of a good theorem and requires just as much imagination. Thank you for everything, sir! Rock on, my friend!

All three of you have shown me the power of a well-placed example. I thank you all for the example you have been to me.

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CHAPTER 1. BASIC CONCEPTS

Before proceeding, we address some ground rules and notational matters which will be used throughout the entirety of this dissertation. The term "ring" denotes a commutative ring with identity. Also, if I is an ideal of a ring R, then we often denote this by writing $I \leq R$. It will always be assumed that an ideal of a ring never contains a unit, i.e., all ideals are proper ideals. We denote the set of all prime ideals in a ring R by $Spec(R)$ and the set of all maximal ideals in R is denoted by $Max(R)$. The set of units of a ring R is denoted by $U(R)$. Given a ring R, then R^* denotes the nonzero elements of the ring. If S is a set, then $|S|$ denotes the cardinality of that set. If a set S "misses" another set T, then we mean $S \cap T = \emptyset$. In any multiplicatively closed set S, we always have that $0 \notin S$.

1.1. Atoms and Primes

Our aim in this section is to establish some basic facts regarding irreducible and prime elements in a domain as well as some of the influences these types of elements may have on the rings which contain them.

Definition 1.1.1. Let D be a domain and $p \in D^* - U(D)$. We say p is an **atom** or **irreducible** if, whenever $p = xy$ for some $x, y \in D$, then either x or y is a unit of D. Further, the set of all atoms in D is denoted by **Irr(D)**.

The reader will recognize that the notions of irreducible and atom coincide. We use this somewhat unconventional language in light of the following definition.

Definition 1.1.2. Let D be a domain and let $x \in D$. If $x = \prod_{i=1}^{n} p_i$ with each $p_i \in Irr(D)$, then we will call this is an **atomic factorization** of x. If every nonzero nonunit in D admits an atomic factorization, then we will call D **atomic**.

Example 1.1.3. $\mathbb{Z}$ is an atomic domain because every nonzero nonunit can be expressed as a product of primes, all of which are easily verified to be atoms.

Here is a useful characterization of atoms.

Theorem 1.1.4. *[10] Let D be a domain with $p \in D$. Then $p \in Irr(D)$ if and only if pD is maximal with respect to being a proper principal ideal.*

Proof. ($\Rightarrow$) We regard p as an atom of D and assume $pD \subseteq xD$ for some nonunit $x \in D$. Then $x|p$ and we write $p = xy$. This forces $y \in U(D)$ and so $pD = xD$.

($\Leftarrow$) Now assume pD is maximal with respect to being principal and let $x \in D - U(D)$ such that $p = xy$. Then $pD \subseteq xD$. The maximality of pD forces $pD = xD$, making p and x associates, i.e., $y \in U(D)$. $\square$

We have introduced the idea of atomicity and now move to the stronger notion of primality.

Definition 1.1.5. Let D be a domain. We say $p \in D^* - U(D)$ is **prime** if, whenever $p|ab$, then $p|a$ or $p|b$. In the event that $x = \prod_{i=1}^{n} p_i$ where each p_i is prime, then we will say that this is a **prime factorization** of x.

We can use prime ideals to characterize prime elements.

Theorem 1.1.6. *[10] Let $p \in D^*$. Then p is prime if and only if pD is a nonzero prime ideal of D.*

Proof. ($\Rightarrow$) We assume p is prime and say $ab \in pD$. Then $ab = pr$ for some $r \in D$, i.e., $p|ab$. Without loss of generality, assume $p|a$. Then $a \in pD$.

($\Leftarrow$) We now assume pD is a prime ideal and say $p|ab$. Then $ab \in pD$. Without loss of generality, assume $a \in pD$ so that $a = pr$ for some $r \in D$. Then $p|a$. $\square$

There are those who would like to say that if D is a domain, then the zero element of D is prime. There is good reason for this because they regard primality as an ideal-theoretic notion. We choose to require that primes be nonzero. As a consequence, a routine argument shows that every prime in a domain is an atom. But atoms need not be prime. For example, in $D = \mathbb{Z}[\sqrt{-3}]$ we have that $2 \in Irr(D)$. But note $2(2) = (1+\sqrt{-3})(1-\sqrt{-3})$ are both atomic factorizations of the same element and 2 divides neither factor on the right side of the equation. Atomicity is a nice property for a ring to have, but the example we just gave shows that atomic factorizations may fail to be unique. This never happens with prime factorizations. Dr. Jim Coykendall shares the following result.

Theorem 1.1.7. *Suppose D is a domain and $x = p_1 p_2 \cdots p_n$ is a prime factorization in D. Assume $x = a_1 \cdots a_m$ is another factorization of x with each $a_i \in D^* - U(D)$. Then $m \leq n$. Further, $m = n$ if and only if each $a_i = u_i p_{\sigma(i)}$ for some permutation σ of the set $\{1, 2, \ldots, n\}$.*

Proof. We begin by proving the first statement. We proceed by induction on n. Suppose $n = 1$. Without loss of generality, assume $p_1 | a_1$ and write $a_1 = r_1 p_1$ for some $r_1 \in D$. Then we write $a_1 a_2 \cdots a_1 = p_1 = r_1 p_1 a_2 a_3 \cdots a_n$. Canceling we get $1 = r_1 a_2 a_3 \cdots a_n$. Thus, $r_1 \in U(D)$. Further, since each $a_i \notin U(D)$, then we must have $n = 1$.

Assume now there exists some $n \in \mathbb{N}$ such that the statement holds for all $t \leq n$. Suppose then $a_1 a_2 \cdots a_n = p_1 p_2 \cdots p_n p_{n+1}$. Since each $a_i \in D^* - U(D)$, then there is no loss in generality in assuming $p_{n+1} | a_1$. We write $r_2 p_{n+1}$. Then $r_2 p_{n+1} a_2 a_3 \cdots a_m = p_1 p_2 \cdots p_{n+1}$. Canceling, we get $(r_1 a_2) a_3 a_4 \cdots a_m = p_1 p_2 \cdots p_n$. Note that $r_2 a_2 \in D^* - U(D)$. By our induction hypothesis we have that $m - 1 \leq n$ and so $m \leq n + 1$. Thus, the first statement holds for all $n \in \mathbb{N}$.

We now prove the second statement.

($\Rightarrow$) We proceed by induction on n. If $n = 1$, then the statement is obvious. Suppose the statement is true for some $n \in \mathbb{N}$. Assume then $a_1a_2\cdots a_{n+1} = p_1p_2\cdots p_{n+1}$. Without loss of generality, assume $p_{n+1}|a_{n+1}$ and write $r_np_{n+1} = a_{n+1}$ for some $r_n \in D$. Then $a_1a_2\cdots a_{n+1} = p_1p_2\cdots p_{n+1} = a_1a_2\cdots a_nr_np_{n+1}$. Upon cancelations we get $p_1p_2\cdots p_n = x_1x_2\cdots x_n$, where $x_i = a_i$ when $1 \leq i \leq n-1$ and $x_n = a_nr_n$. By the induction hypothesis, each $x_i = u_ip_{\sigma(i)}$ for some permutation σ of $\{1,2,\ldots,n\}$. Thus, each $a_i = u_ip_{\sigma(i)}$ for some permutation σ of $\{1,2,\ldots,n\}$ and $a_nr_n = up_j$ for some j. Since p_j is prime, then either $p_j|a_n$ or $p_j|r_n$. If $p_j|r_n$, then $r_n \in D^* - U(D)$. Letting $x_{n+1} = r_n$, it would follow that $x_1x_2\cdots x_{n+1} = p_1p_2\cdots p_n$ with each $x_i \in D^8 - U(D)$, contradicting the first statement of the theorem. Thus, $p_j|a_n$. To finish, we need to show that $r_n \in U(D)$. Since each a_i is an associate of a prime for all $i \in \{1,2,\ldots,n\}$, then without loss of generality we may write $a_i = b_ip_i$ when $1 \leq i \leq n$. We also assume $a_{n+1} = r_np_{n+1}$. Then $a_1a_2\ldots a_{n+1} = (b_1b_2\cdots b_{n-1}b_nr_n)p_1p_2\cdots p_{n+1} = p_1p_2\cdots p_{n+1}$. Hence, $b_1b_2\cdots b_nr_n = 1$ and so $r_n \in U(D)$.

($\Leftarrow$) Suppose now each $a_i = u_ip_{\sigma(i)}$ for some permutation σ for all $i \in \{1,2,\ldots,n\}$. We already know $m \leq n$. If $m < n$, then upon canceling we would have that $p_j \in U(D)$ for some prime p_j, a contradiction. So $m = n$. □

1.2. Polynomial and Power Series Rings

We now present some basic facts regarding polynomial and power series extensions. The proofs of these theorems can be found in any standard algebra text such as [10].

Theorem 1.2.1. *Let R be a ring and $I \leq R$. Then $R[x_i]_\Lambda/I[x_i]_\Lambda \cong (R/I)[x_i]_\Lambda$.*

Proof. Let $\pi : R[x_i]_\Lambda \to (R/I)[x_i]_\Lambda$ be the natural projection. Then we have that $R[x_i]_\Lambda/Ker(\pi) \cong (R/I)[x_i]_\Lambda$. Hence, we need only to show $Ker(\pi) = I[x_i]_\Lambda$. Clearly, $I[x_i]_\Lambda \subseteq Ker(\pi)$. Let σ be the multiplicative semigroup generated by $\{x_i|i \in \Lambda\}$.

Suppose then $f \in Ker(\pi)$ and write $f = r_0 + r_1t_1 + \ldots + r_nt_n$, where each $r_i \in R$, each $t_i \in \sigma$ and $t_i \neq t_j$ for all $i \neq j$. Let $\widetilde{r_i} = r_i + I$. Note that $\pi(\sigma)$ is linearly independent in R/I and $\widetilde{r_0} \notin Span(\{(t_1+I, \ldots, t_n+I\}$. Now $\pi(f) = \widetilde{r_0} + \widetilde{r_1}t_1 + \ldots + \widetilde{r_n}t_n = 0 \Rightarrow \widetilde{r_i} = 0$ for each $\widetilde{r_i}$. But this simply means every $r_i \in I$ and therefore $f \in I[x_i]_\Lambda$. $\square$

Theorem 1.2.2. *[10] Let R be a ring. Then $U(R[[x]]) = \{f \in R[[x]] | f(0) \in U(R)\}$.*

Proof. We begin with the case $f = 1 + a_1x + a_2x^2 + \ldots \in R[[x]]$. Recall that $(1 + a_1x + a_2x^2 + \ldots)(1 + b_1x + b_2x^2 + \ldots) = 1 + c_1x + c_2x^2 + \ldots$, where each $c_n = b_n + b_{n-1}a_1 + b_{n-2}a_2 + \ldots + b_1a_{n-1} + a_n$. Our strategy is to show that each b_n can be constructed in such a way as to guarantee that every $c_n = 0$. If we need $b_1 + a_1 = 0$, then we can solve and say $b_1 = -a_1$. For every $i \in \mathbb{N}$, let A_i be the ideal given by $A_i = (a_1, \ldots, a_i)R$. Now if we require $0 = b_2 + b_1a_1 + a_2$, then solving we get $b_2 \in A_2$ by using the fact that $b_1 \in A_1$. Continuing inductively in this fashion, we can construct each b_n so that each $b_n \in A_n$, i.e., we can build each b_n as to force $c_n = 0$ for all n.

Now we tackle the general case and let $f = u + a_1x + a_2x^2 + \ldots \in R[[x]]$ with $u \in U(R)$. Then $(u^{-1}f)(0) = 1 \Rightarrow u^{-1}f \in U(R[[x]]) \Rightarrow f \in U(R[[x]])$. $\square$

Suppose D is a domain contained in a field F and let $R = D + xK[[x]]$. Then the proof of Theorem 1.2.2 can be easily modified to establish Theorem 1.2.3. We skip its proof. But as we will have use for it, we put in the record.

Theorem 1.2.3. *Let D be a domain and F a field containing D. Let $R = D + xF[[x]]$. Then $U(R) = \{f \in R | f(0) \in U(D)\}$.*

Definition 1.2.4. Let R be a ring and $0 \neq f = a_0 + a_1x + a_2x^2 + \ldots \in R[[x]]$. We define the **order** of f, denoted $ord(f)$, to be $ord(f) = min\{n \in \mathbb{N}_0 | a_n \neq 0\}$.

The idea of the order of an element in a power series ring is analogous to the notion of degree in polynomial rings. For example, if D is a domain and one wishes to show that $D[x]$ is also a domain, then all one needs to do is employ a "degree argument". To show that $D[[x]]$ is a domain, we can use an "order argument" instead. Furthermore, if D is a domain, then using the convention $deg(0) = -\infty$ it is easily shown that $deg(fg) = deg(f) + deg(g)$ for every $f, g \in D[x]$. Similarly, defining $ord(0) = -\infty$, it is easy to show $ord(fg) = ord(f) + ord(g)$ for all $f, g \in D[[x]]$. The requirement that D should be a domain is critical in both scenarios.

1.3. The Axiom of Choice

The Axiom of Choice is usually presented as a set-theoretical statement and therefore it comes as no surprise that it plays a fundamental role in an extremely wide range of mathematical disciplines such as set theory, topology, real analysis, linear algebra, commutative algebra, and functional analysis, to name a few. Historically, many mathematicians were uneasy about its use, even going so far as to reject results which depended upon it. The reason for this is understandable once one begins to appreciate its power.

Various texts which address the Axiom of Choice may present it in different, but equivalent, ways. The definitions and claims made in this section can be found in almost any text that deals with the fundamentals of set theory, such as [8]. Although not a universal practice, most authors opt to define the Axiom of Choice in the following way.

Axiom 1.3.1 (The Axiom of Choice)**.** Every cartesian product of nonempty sets indexed by a nonempty set is a nonempty set.

At first, this statement seems innocent and perhaps, on initial inspection, even obvious. Its power resides in how it can be used to prove the existence of things which are virtually impossible to find. For example, given any indexing set Λ, it is easy to find an element of $\prod_\Lambda \mathbb{R}$. But note that here we are no longer dealing with an *arbitrary* product of sets. Perhaps the reader might try to find an element in an arbitrary product of arbitrarily nonempty sets! Those who oppose the Axiom of Choice (or any of its equivalents statements) will usually have the same understandable response: "Show me"! The Axiom of Choice often goes by the name *Zorn's Lemma*, which is logically equivalent to the Axiom of Choice.

Theorem 1.3.2. *(Zorn's Lemma) Suppose S is a nonempty partially ordered set. If every chain of elements of S admits an upper bound which also in S, then S admits a maximal element.*

Theorem 1.3.3 is of fundamental importance in commutative algebra.

Theorem 1.3.3. *[10] Every commutative ring with 1 admits a maximal ideal.*

Proof. Let R be a commutative ring with 1. Let S be the collection of all proper ideals of R and note that $S \neq \emptyset$ because $0 \in S$. Partially order S by (set theoretic) inclusion and let $(C_i)_\Lambda$ be a chain of ideals in S. Let $J = \bigcup_\Lambda C_i$. Because each C_i is a proper ideal, then each C_i contains no units. Also, note that because J is a union of a chain of ideals, then J is an ideal. Thus, J contains no units and J is a proper ideal of R, i.e., $J \in S$. By Zorn, S admits a maximal element. But the elements of S are the proper ideals of R. So R admits a maximal ideal. □

Note in the proof just presented that we never actually *found* a way of constructing a maximal ideal in any given ring because of our reliance on Zorn's Lemma. Yet, we are assured of the existence of such an object! We may sometimes make statements such as "Let P be a prime ideal of R". Note how the Axiom of Choice lurks in this statement because one might wonder how we can be sure R actually *has* a prime ideal at all. The answer lies in the fact that we assume the Axiom of Choice and therefore we believe every ring admits a maximal ideal, which is prime.

We now consider a third statement which is equivalent to the Axiom of Choice. Before presenting its statement, let us recall that a nonempty partially ordered set S is said to be **well-ordered** if every nonempty subset of S admits a minimal element *in that subset.* For example, $\mathbb{N}$ in its usual order is well-ordered whereas $\mathbb{Z}$ and $\mathbb{R}$ in their usual orders are not. To see that $\mathbb{Z}$ is not well-ordered (in its usual order), simply note that $\mathbb{Z}$ itself admits no minimal element. In the case of $\mathbb{R}$, we observe that while the interval $(0, 1)$ admits an infimum, this infimum is not a member of the interval. Of course, we could have also said that $\mathbb{R}$, like $\mathbb{Z}$, admits no minimal element. However, in the case of $\mathbb{Z}$, $\mathbb{Q}$, or any other infinitely countable set S, we can order the elements of S in such a way as to transform S into a well-ordered set. Indeed, because S is infinitely countable, then there exists a bijection $\mathbb{N} \leftrightarrow S$ and so we can say $S = \{s_1, s_2, \dots\}$. Now we can partially order S by saying $s_i < s_j \Leftrightarrow i < j$. Note that this is actually a well-ordering of S. We now know that every countable set can be well-ordered because we can *find* such an ordering. It therefore seems natural to ask whether or not we can well-order $\mathbb{R}$. The Well-Ordering Principle provides an answer to this query.

Theorem 1.3.4. *(Well-Ordering Principle) Every nonempty set can be well-ordered.*

That the Well-Ordering Principle, Zorn's Lemma, and the Axiom of Choice are equivalent is astounding. After all, at first glance the Axiom of Choice seems as obviously true as the Well-Ordering Principle may appear to be clearly false. To illustrate our meaning, we observe again that $\mathbb{R}$ is not well-ordered in its usual ordering. However, if the well-ordering principle holds, then there exists another way to order the elements of $\mathbb{R}$ so as to make it well-ordered. To this day, a well-ordering of $\mathbb{R}$ has never been discovered. Further, just as we can well-order any infinitely countable set S by using a bijection $S \leftrightarrow \mathbb{N}$, we could find a well-ordering of $\mathbb{R}$ by well-ordering any set which has the same cardinality of $\mathbb{R}$. We now consider another equivalent way the Axiom of Choice can manifest itself both in mathematics at large as well as in our own proceedings.

Theorem 1.3.5. *Given sets S and T, then either there exists a injection $S \to T$ or an injection $T \to S$. Equivalently, either $|S| \leq |T|$ or $|T| \leq |S|$. Furthermore, if $|S| \leq |T| \leq |S|$, then $|S| = |T|$.*

Like the Axiom of Choice, Theorem 1.3.5 asserts the existence of something which is not constructed. One of the most mind-boggling results which follows from the Axiom of Choice is the famous Banach-Tarski Paradox, which basically says that given any two compact spaces of $A, B \subsetneq \mathbb{R}^3$, then there exists a way to break up these spaces into finitely many parts and reassemble these parts so as to transform A into B and vice versa. In "real world" terms, one might then be able to break apart a grain of sand into finitely many pieces and put these pieces back together again to create an something the size of Jupiter. Given this seemingly untenable situation it is understandable that one might become at least a little leery of the Axiom of Choice. On the other hand, a rejection of the Axiom of Choice yields results which, to most mathematicians, are no less horrifying. For example, in the absence of the Axiom of Choice, one is able to construct uncountable sets which admit no infinitely countable subsets. We choose to assume the validity of the Axiom of Choice.

1.4. Localization

In this section we establish some of the fundamentals of localizations without which much of our discussion would come to a quick stand-still. In the interests of not completely reinventing the wheel, we will take the liberty of assuming the reader understands our meaning when we refer to *multiplicatively closed* sets and a *localization* of a ring. However, to create the illusion of completeness and self-containment we nevertheless provide some fundamental results regarding multiplicatively closed sets and localizations. The proof of this result is a typical application of Zorn's Lemma and can be found in [10].

Theorem 1.4.1. *[10] Let R be a ring and let $S \subsetneq R^*$ be multiplicatively closed Given any ideal $I \leq R$ such that $I \cap S = \emptyset$, then there exists an ideal $P \leq R$ that contains I and is maximal with respect to $P \cap S = \emptyset$. Moreover, $P \in Spec(R)$.*

Proof. Let S be the set of all ideals of R that contain I. Certainly $S \neq \emptyset$ because $I \in S$. Letting $(C_i)_\Lambda$ be a chain in S, then it is easily verified that $\bigcap_\Lambda C_i \in S$. By Zorn, we let $P \in S$ be maximal. We now wish to show $P \in Spec(R)$. To this end, assume $a, b \in R$ with $ab \in R$. Assume $a, b \notin P$. Then $(P, a) \cap S \neq \emptyset \neq (P, b)$. Let $s_1 \in (P, a) \cap S$ and $s_2 \in (P, b) \cap S$. Write $s_1 = p_1 + r_1 a$ and $s_2 = p_2 + r_2 b$ with $p_1, p_2 \in P$ and $r_1, r_2 \in R$. Then $s_1 s_2 = p_1 p_2 + p_1 r_2 b + p_2 r_1 a + r_1 r_2 ab \in P$, i.e., $S \cap P \neq \emptyset$, a contradiction. □

Theorem 1.4.1 is a powerful result. One can think of it as saying that every ideal that misses S can be expanded to a prime ideal that is maximal with respect to missing S. Whenever we expand an ideal in this way, we are implicitly invoking Theorem 1.4.1.

Definition 1.4.2. Let R be a ring. We define a set $S \subseteq R^*$ to be **saturated** if $xy \in S \Rightarrow x, y \in S$. Further, given a set S we define its **saturation** to be the set $\{s \in R | st \in S \text{ for some } t \in R\}$.

The saturation of a set is saturated and the saturation of a multiplicatively closed set is multiplicatively closed Should a set S be saturated and multiplicatively closed in a ring R, then we will refer to S as being a **multiplicative system** of R. Further, there are those who assume multiplicatively closed sets to be saturated from the start, or at the very least contain the identity of the ring. Unless stated otherwise, we will always assume that localizations occur at multiplicative systems. The following theorem justifies this action and was shared by Dr. Jim Coykendall.

Theorem 1.4.3. *Let R be a ring and assume $S \subseteq D^*$ is multiplicatively closed. Let $\overline{S}$ be the saturation of S. Then $R_S \cong R_{\overline{S}}$*

Proof. Define $\phi : R_S \to R_{\overline{S}}$ by $\phi(\frac{r}{s}) = \frac{r}{s}$. We first need to show that our map is well-defined. Suppose $\frac{r_1}{s_1} = \frac{r_2}{s_2}$. Then there exists some $s_3 \in S$ such that $(r_1s_2 - r_2s_1)s_3 = 0$. That ϕ is well-defined now follows from the fact that $S \subseteq \overline{S}$.

It is not difficult to show that ϕ is a ring homomorphism and skip the verification. We now show ϕ is monic. Suppose then $\frac{r}{s} = \frac{0}{1}$ in $R_{\overline{S}}$, where $s \in S$. Then $(r(1) - s(0))t_1 = 0$ for some $t_1 \in \overline{S}$. Let $t_2 \in \overline{S}$ such that $t_1t_2 \in S$. Then $rt_1t_2 = 0$ and so $(r(1) - s(0))t_1t_2s' = 0$ for some $s' \in S$. Since $t_1t_2, s \in S$, then $\frac{r}{s} = \frac{0}{s'}$ in R_S. Thus, ϕ is monic.

We now show ϕ is epic. Let $\frac{r}{t_1} \in R_{\overline{S}}$ for some $r \in R$ and $t_1 \in \overline{S}$. Let $t_2 \in \overline{S}$ such that $t_1t_2 \in S$. Then $\frac{r}{t_1}\frac{rt_2}{t_1t_2} = \phi(\frac{rt_2}{t_1t_2})$, making ϕ epic. □

Although it is not always necessary, Theorem 1.4.3 allows us to assume that localizations always occur at multiplicative systems. We now continue with a useful characterization of multiplicative systems. We will expand upon this characterization in Theorem 3.2.4.

Theorem 1.4.4. *[11] Let R be a ring and $S \subseteq R^*$. Then S is a multiplicative system if and only if S is the complement of a union of prime ideals of R.*

Proof. ($\Rightarrow$) We assume S is a multiplicative system. Since $0 \notin S$, then by Theorem 1.4.1 we can expand (0) to a prime ideal that is maximal with respect to missing S. Now let $\sigma = \{P \in Spec(R) | P \cap S = \emptyset\}$. We show $S = R - \bigcup_\sigma P$. Since $S \cap P = \emptyset$ for all $P \in \sigma$, then $S \subseteq R - \bigcup_\sigma P$. For the reverse containment, we assume

$s \in R - \bigcup_\sigma P$. Assume $s \notin S$. Using Theorem 1.4.1 we can expand sR to a prime ideal P that misses S. Then $P \in \sigma \Rightarrow s \in \bigcup_\sigma P$, a contradiction.

($\Leftarrow$) Suppose now S is a complement of a union of primes of R and write $S = R - \bigcup_{i\in\Lambda} P_i$. Let $x, y \in S$. If $xy \notin S$, then $xy \in \bigcup_{i\in\Lambda} P_i$. Hence, $xy \in P_i$ for some $i \in \Lambda$. Without loss of generality, say $x \in P_i$. Then $x \in \bigcup_{i\in\Lambda} P_i$, a contradiction. So S is multiplicatively closed

We now show S is saturated and assume $xy \in S$ for some $x, y \in R$. If $x \notin S$, then $x \in \bigcup_{i\in\Lambda} P_i$. But then $xy \in \bigcup_{i\in\Lambda} P_i$, a contradiction. Having shown $x \in S$, a similar argument verifies $y \in S$. So S is saturated and we are done. □

Remark 1.4.5. If R is a ring, $P \in Spec(R)$, and $S = R - P$, then it is conventional to write R_P instead of R_S. We will adhere to this practice. Further, some authors use the notation $S^{-1}R$ instead of R_S.

Our next result shows us what all the ideals in a localization "look like". Apart from being of some interest in its own right, we will put it to use in the proof of Theorem 1.4.8 when we characterize prime ideals in a localization.

Theorem 1.4.6. *Suppose D is a domain and $S \subseteq D^*$ is multiplicatively closed Then every ideal of D_S is of the form I_S, where I is an ideal of D.*

Proof. Suppose $Q \leq D_S$ and let $I = Q \cap D$. It is easily verified that $I \leq D$ and that $I_S \leq D_S$. Since $I \subseteq Q$, then we already have that $ID_S \subseteq Q$. Suppose $\frac{x}{s} \in Q$ for some $x \in D$ and $s \in S$. Then $(\frac{x}{s})s = x \in Q \cap D = I$. Hence, $\frac{x}{s} \in I_S$ and we are done. □

The reader should be careful not to misread what Theorem 1.4.6 is saying. In particular, this theorem is *not* asserting that distinct ideals in D correspond to distinct ideals in D_S. For example, if we let $S = \mathbb{Z} - 2\mathbb{Z}$, then $2\mathbb{Z}_S = 6\mathbb{Z}_S$ and $3\mathbb{Z}_S = 5\mathbb{Z}_S$. Thus, even distinct *prime* ideals may collapse into the same ideal in a localization. Gladly, this is not the case for the set of primes that miss S. Theorem 1.4.8 characterizes primes in a localization. First we consider a lemma.

Lemma 1.4.7. *Suppose R is a ring, $P \in Spec(R)$, and $S \subseteq R$ is a multiplicative system such that $P \cap S = \emptyset$. If $\frac{x}{s_1} \in PR_S$ for some $x \in R$ and $s_1 \in S$, then $x \in P$.*

Proof. From Theorem 1.4.6 we know that $\frac{x}{s_1} = \frac{p}{s_2}$ for some $p \in P$ and $s_2 \in S$. Hence, $(xs_2 - ps_2)s_3 = 0$ for some $s_3 \in S$. Thus, $xs_2s_3 \in P$ and $s_2s_3 \notin P$. So $x \in P$. $\square$

Theorem 1.4.8. *[10] Let D be a domain and $S \subseteq D$ a multiplicative system. Then there exists a 1-1 correspondence between the prime ideals of D that miss S and the prime ideals of D_S given by $P \leftrightarrow P_S$.*

Proof. Assume first that $P \leq D$ with $P \cap D = \emptyset$. From Theorem 1.4.6 we know that $P_S \leq D_S$. We now show that $P \in Spec(D)$ if and only if $P_S \in Spec(D_S)$. For the forward implication, assume $(\frac{x}{s_1})(\frac{y}{s_2}) \in P_S$. From Lemma 1.4.7 we know that $xy \in P$. Without loss of generality, we assume $x \in P$. Then $\frac{x}{s_1} \in P_S$ and so $P_S \in Spec(D_S)$. Conversely, assume $x, y \in D$ with $xy \in D$. Then $xy \in PD_S$. Without loss of generality, assume $x \in P_S$. Then Lemma 1.4.7 gives us that $x \in P$. So $P \in Spec(D)$.

The argument in the previous paragraph shows that the function $P \to P_S$ is well-defined and epic. To show that the assignment is monic, assume P and Q are distinct primes in D that miss S. Suppose $x \in P - Q$. If $x \in Q_S$, then Lemma 1.4.7 tells us that $x \in Q$, a contradiction. Thus, the assignment is monic. $\square$

Remark 1.4.9. Although it was not explicitly pointed out, it is clear that the correspondence in Theorem 1.4.8 is order-preserving.

Recall that every nonzero prime in a domain D is an atom. As it happens, primes behave very nicely in localizations. Dr. Jim Coykendall shares the following result.

Theorem 1.4.10. *Let D be a domain, $S \subseteq D^*$ a multiplicative system, and assume $p \in D^*$ is prime. Then either $p \in U(D_S)$ or p remains prime in D_S.*

Proof. Assume $p \notin U(D_S)$ and $p|\frac{x}{s_1}\frac{y}{s_2}$ for some $x, y \in D$ and $s_1, s_2 \in S$. Then $\frac{x}{s_1}\frac{y}{s_2} = p\frac{r}{s_3}$ for some $r \in D$ and $s_3 \in S$. Rewriting, we have $xys_3 = rps_1s_2$ in D. If $p|s_1s_2$, then using the saturation of S we would have that $p \in S$. But this would indicate $p \in U(D_S)$, a contradiction. Hence, $p|xy$ in D. Without loss of generality, say $p|x$ in D. Then it is easy to see that $p|\frac{x}{s_1} \in D_S$. □

Generally speaking, atoms are not as nicely behaved in localizations as primes. The following example demonstrates how an irreducible may admit a nontrivial factorization in a localization.

Example 1.4.11. Let $D = \mathbb{Z}[\sqrt{-5}]$. Using a norm argument it can be shown that $2 \in Irr(D)$. Let $P \in Spec(D)$ such that $2 \in P$. Then we also have that $2 \in PD_P$. In D we have that $2(3) = (1+\sqrt{-5})(1-\sqrt{-5})$. Since $PD_P \in Spec(D_P)$, then $1+\sqrt{-5} \in PD_P$ or $1-\sqrt{-5} \in PD_P$. Assume now that $1+\sqrt{-5} \in PD_P$. Then $1+\sqrt{-5}+1-\sqrt{-5} = 2 \in PD_P \Rightarrow 1-\sqrt{-5} \in PD_P$. Similarly, $1-\sqrt{-5} \in PD_P \Rightarrow 1+\sqrt{-5} \in PD_P$. Thus, $1+\sqrt{-5}, 1-\sqrt{-5} \in PD_P$. Now note $3 \notin PD_P$ because otherwise $3-2 = 1 \in PD_P$. Hence, $3 \in U(D_P)$ and so $2 = \frac{1}{3}(1+\sqrt{-5})(1-\sqrt{-5})$, i.e., $2 \notin Irr(D_P)$.

In later sections we will see that not only can localizations produce new units, but in addition we can localize certain rings and produce new atoms. In addition, we will be exploring various factorization properties that are preserved under localizations.

1.5. Quotient Rings

In this section we only wish to characterize the prime ideals in a quotient ring.

Theorem 1.5.1. *[10] Let R be a ring and $I \leq R$. Then every ideal of R/I is of form Q/I, where Q is an ideal of R which contains I.*

Proof. The desired result follows from the fact that if $f : D \to T$ is a ring-epimorphism, then $f^{-1}(H)$ is an ideal for every ideal H of D. $\square$

We now record the characterization we seek.

Theorem 1.5.2. *[10] There exists a 1-1 correspondence between prime ideals of R containing I containing I and the prime ideals of R/I given by $P \leftrightarrow P/I$.*

Proof. Assume P be an ideal that contains I. We first show that $P \in Spec(R)$ if and only if $P/I \in Spec(R/I)$. For the forward implication we assume $P \in Spec(R)$. Then $(R/I)/(P/I) \cong R/P$. Since P is prime, then R/P is a domain. So P/I is prime in R/I. For the converse, assume $x, y \in R$ with $xy \in P$. Then $xy + I \in P/I$. Without loss of generality, assume $x + I \in P/I$. Then $x \in P$, as desired.

Assume now that Q and P are distinct primes of R both of which contain I. Say $x \in P - Q$. Then $x \notin Q/I$, i.e., P/I and Q/I are distinct primes in R/I. $\square$

As with the correspondence of primes in localizations, the correspondence in Theorem 1.5.2 is order-preserving.

CHAPTER 2. FUNDAMENTALS OF FACTORIZATION

2.1. Factorization Properties

This section will introduce us to some of the rings of factorization that we will be encountering most often in these writings. In particular, we will look at some very useful characterizations of these rings that will greatly facilitate many of the arguments in the sequels to this section.

There is no place better to begin than with the class of rings which exhibit the best of all possible worlds from a factorization point of view - the class of unique factorization domains (UFD's). We will assume the reader is familiar with the traditional definition of what it means to be a UFD so we can proceed to more interesting topics. First let us consider an immensely useful characterization of these gems.

Theorem 2.1.1. *[11] Let D be an integral domain. The following are equivalent:*

a) D is a UFD

b) Every nonzero nonunit of D is a product of primes.

c) Every nonzero prime ideal of D contains a nonzero prime element.

Proof. *a)* $\Rightarrow$ *b)* It is enough to show that every irreducible in D is prime. Suppose $p \in Irr(D)$ and $p|ab$ for some $a, b \in D$. Then $rp = ab$ for some $r \in D$. We may assume $a, b \notin U(D)$. Because D is a UFD, then we may say $a = p_1 p_2 \cdots p_n$ and $b = p_{n+1} p_{n+2} \cdots p_{n+m}$ with each $p_i \in Irr(D)$. Using the unique factorization of D, we then know that p is an associate of some p_i.

b) $\Rightarrow$ *a)* This follows immediately from Theorem 1.1.7.

b) $\Rightarrow$ *c*) If $P \in Spec(D)$ is nonzero, then let $x \in P - 0$. Write $x = p_1 \cdots p_n$ where each p_i is prime in D. Because P is prime in D, then $p_i \in P$ for some i.

c) $\Rightarrow$ *b*) Let S be the set of all nonzero elements in D which are products of primes and note that this makes S multiplicatively closed By Theorem 1.4.1 we may let $P \in Spec(D)$ be maximal with respect to missing S. Suppose there exists some $x \in P - 0$. Then under our assumptions, x is a product of primes. But because P is prime in D, this means P must contain a prime, a contradiction since $P \cap S = \emptyset$. Thus, $P = 0$ and so every nonzero nonunit is a product of primes.

□

Here is a famous result which can be found in any text on basic abstract algebra. However, most texts do not use the proof we employ. For example we invite the reader to compare our proof with that given in [10].

Corollary 2.1.2. *[10] Every PID is a UFD.*

Proof. If $P \in Spec(D)$ is nonzero, then it is generated by a prime element. Using part *c*) of Theorem 2.1.1 we rest our case. □

We can use UFD's to formulate a nice characterization of PID's. Dr. Jim Coykendall shares the following result.

Theorem 2.1.3. *Let D be a non-field domain. The following are equivalent:*

a) D is a PID.

b) D is a UFD and every nonzero prime ideal is maximal.

c) Every prime ideal of D is principal.

Proof. *a*) $\Rightarrow$ *b*) We already know every PID is a UFD. Suppose now $P_1 \subseteq P_2$ are nonzero prime ideals in D. As D is a PID, we let $P_1 = (p_1)$ and $P_2 = (p_2)$. Then

$(p_1) \subseteq (p_2) \Rightarrow p_2|p_1$. But nonzero prime elements are irreducible. Thus, p_1 and p_2 must be associates and so $(p_1) = (p_2)$.

b) $\Rightarrow$ *c)* Suppose $P \in Spec(D)$ is nonzero. Then from Theorem 2.1.1 we know that P contains a nonzero prime element p. Note $(p) \subseteq P$ and $(p) \in Spec(D)$. Thus, (p) is maximal and so $(p) = P$.

c) $\Rightarrow$ *a)* We are given that every prime ideal is principal. Assume D contains an ideal I that is not principal. Zorn's Lemma can be used to show that we may expand I to an ideal M that is maximal with respect to not being principal. We claim $M \in Spec(D)$, thereby deriving the necessary contradiction.

To show M is prime in D, let $a, b \in D - M$ with $ab \in M$. Then $M \subsetneq (M, a)$. From the maximality of M, then we know $(M, a) = xD$ for some $x \in D$. Now consider the ideal $J = \{r \in D | rx \in M\}$ and observe $M \subseteq J$. It is important to note that $b \in J$. Indeed, because $(M, a) = (x)$, then $x = m + ra$ for some $m \in M$ and $r \in D$. Hence, $bx = bm + rab \in M$. Thus, $M \subsetneq (M, b) \subseteq J$. Thus, J is principal. Now because $(M, a) = xD$, then $Jx = J(M, a) = J(M + (a)) = JM + Ja \subseteq M$. Also, if $m \in M$, then $m \in (M, a) = (x)$. So $m = r_m x$ for some $r_m \in D$ and we have shown $M \subseteq Jx$. But being a product of principal ideals, Jx is itself principal. Therefore, M is principal, a contradiction. □

Theorem 2.1.4. *[10] Every localization of a UFD is a UFD and every localization of a PID is a PID.*

Proof. Recall from Theorem 1.4.8 that every prime ideal of a localization R_S is of form P_S, where $P \in Spec(R)$ and $P \cap S = \emptyset$. Now let D be a UFD and consider the localization D_S. Let $0 \neq P_S \in Spec(D_S)$ and let $0 \neq p \in P$ be a prime element. Theorem 1.4.10 assures us p remains prime in P_S. Theorem 2.1.1 now tells us D_S is a UFD.

Now let D be a PID and let $P_i nSpec(D_S)$. Say $PD = pD$ for some prime element $p \in D$. It is now easy to show $PD_S = pD_S$. Now Theorem 2.1.3 forces D_S to be a PID. $\square$

Recall that a **Noetherian** ring is one which satisfies the ascending chain condition on ideals. It is not difficult to prove that this is equivalent to saying that every ideal of the ring is finitely generated and so we feel comfortable omitting the verification. Remarkably, this characterization is equivalent to saying that every *prime* ideal of the ring is finitely generated. We have already seen an instance in Theorem 2.1.3 in which it was sufficient to examine the behavior of the prime ideals to determine certain properties of a ring. The proof of this fact regarding Noetherian rings and their prime ideals is readily available in any number of texts, but for the convenience of the reader it is recorded here, as well.

Theorem 2.1.5. *[11] Let R be a ring. The following are equivalent:*

a) R is Noetherian.

b) Every prime ideal of R is finitely generated.

Proof. The implication $a) \Rightarrow b)$ is clear and so we will prove only the other implication. We begin by letting S be the set of all ideals of R that are not finitely generated and we would like to show $S = \emptyset$. Assuming otherwise we partially order S by inclusion and Zornify to find a maximal $M \in S$. If $M \in Spec(R)$, then we will have our desired contradiction.

Suppose $a, b \in R$ with $ab \in M$. Assume, to the contrary, that $a, b \notin M$. Then (M, a) must be finitely generated because of the maximality of M. Thus, we may say $(M, a) = (m_1 + r_1 a, \ldots, m_n + r_n a)$, where each $m_i \in M$ and every $r_i \in R$. Now let $J = \{r \in R | ra \in M\}$. It is easily verified that $J \leq R$ and $(M, b) \subseteq J$. Thus, J is forced to be finitely generated. Now clearly $(m_1, \ldots, m_n, Ja) \subseteq M$. Let $m \in M$.

Then certainly $m \in (M, a)$ and so $m = \sum_{i=1}^{n} x_i(m_i + r_i a)$ for some $x_1, \ldots, x_n \in R$. This gives us that $m - \sum_{i=1}^{n} x_i m_i = a \sum_{i=1}^{n} x_i r_i$ and so we deduce $\sum_{i=1}^{n} x_i r_i \in J$. It follows that $m \in (m_1, \ldots, m_n, Ja)$. We have shown that $M = (m_1, \ldots, m_n, Ja)$, which a finitely generated ideal. However, this contradicts the fact that M is *not* finitely generated and so we are finished. □

Theorem 2.1.6. *[10] Every localization of a Noetherian domain D is Noetherian.*

Proof. From Theorem 2.1.5 it is enough to show that every prime ideal of the localization D_S is finitely generated. Let $0 \neq PD_S \in Spec(D_S)$, where $P \in Spec(D)$. Using the fact that D is Noetherian we may say $P = (p_1, \ldots, p_n)D$. It is now easy to verify that $P_S = (p_1, \ldots, p_n)D_S$ and so we are done. □

Many of our favorite factorization properties survive in polynomial extensions. We now record the famous Hilbert Basis Theorem. The highly nontrivial proof is omitted.

Theorem 2.1.7. *[11](Hilbert Basis Theorem) Let R be a Noetherian ring. Then $R[x]$ is Noetherian.*

Remark 2.1.8. We should also point out that an analogous result to the Hilbert Basis Theorem holds for power series rings. That is, if R is Noetherian, then $R[[x]]$ is also Noetherian. A proof of this result is quite different from that of the Hilbert Basis Theorem and can be found in [10].

An important class of rings which generalize Noetherian domains are the domains which satisfy the *ascending chain condition on principal ideals*, that is, every chain of principal ideals in the ring stabilizes. For the sake of brevity, we will refer to such a domain as being an **ACCP-domain**, or simply as being *ACCP*. Clearly, every Noetherian domain is ACCP and it is not difficult to verify that every UFD is ACCP.

The class of ACCP domains is properly larger than the class of Noetherian domains. Before presenting an example which demonstrates this, we remind that reader of the fact that if D is UFD, then so is $D[x_i]_{i\in\Lambda}$ for any indexing set Λ.

Example 2.1.9. Let F be a field and $D = F[x_1, x_2, \ldots]$. Clearly, D is non-Noetherian because the ideal $(x_i)_{i=1}^{\infty}$ is not finitely generated. However, D is a UFD and therefore is an ACCP-domain.

Our next result shows us how the ACCP property is preserved in polynomial extensions.

Theorem 2.1.10. *If D be an ACCP-domain, then the same is true of $D[x]$.*

Proof. Let $f_1 \in D[x]$. If $deg(f_1) = 0$, then any chain of principal ideals ascending from $f_1D[x]$ must be finite because D is ACCP. Further, if $f_1 \in Irr(D[x])$, then by Theorem 1.1.4 we know that $f_1D[x]$ is maximal with respect to being principal. Thus, we assume $deg(f_1) > 0$ and $f_1 \notin Irr(D[x])$. Let $f_2 \in D[x]$ such that $deg(f_2) < deg(f_1)$ and $f_2|f_1$. Then $f_1D[x] \subsetneq f_2D[x]$. If $f_3 \in D[x]$ with $deg(f_3) < deg(f_2)$ and $f_3|f_2$, then $f_1D[x] \subsetneq f_2D[x] \subsetneq f_3D[x]$. We can continue in this way a maximum of $deg(f_1)$-times. Without loss of generality, assume $f_1D[x] \subsetneq f_2D[x] \subsetneq \ldots f_nD[x]$ is a chain of principal ideals with $deg(f_i) > deg(f_{i+1})$. If $f_nD[x] \subsetneq rD[x]$, then it must be true that $r \in D$. Since D is ACCP, then any chain of principal ideals ascending from $rD[x]$ must be finite. Lastly, observe that any chain ascending from $f_1D[x]$ that starts off as $f_1D[x] \subsetneq aD[x]$ for some $a \in D$ is finite because D is ACCP. Thus, $D[x]$ is ACCP. $\square$

Remark 2.1.11. In the proof just given we utilized a degree argument to show that $D[x]$ must be ACCP. This proof can be easily adapted to verify that $D[[x]]$ is ACCP by using an order argument instead. This argument would also require the use of Theorem 1.2.2.

Here is a demonstration of how the ACCP property can assert itself in linear algebra. We assume a bit of familiarity on the part of the reader with some basics of tensor products, exact sequences, and flat modules. The following example comes to us via Dr. Sean Sather-Wagstaff.

Example 2.1.12. It is well known that tensor products commute with direct sums. We offer here a means of constructing examples which shows this is not generally the case with direct products. To begin, let D be an ACCP-domain with quotient field $K \neq D$. Because D is atomic, then we can choose some $p \in Irr(D)$. Consider the product $M = \prod_{n \in \mathbb{N}} D/p^n D$. It is easily verified that $K \otimes_D D/p^n D = 0$, whence $\prod_{n \in \mathbb{N}}(K \otimes_D D/p^n D) = 0$. Now consider $\phi : D \to \prod_{n \in \mathbb{N}} D/p^n D$ given by $\phi(d) = (d + p^n D)_{n \in \mathbb{N}}$. A routine exercise shows that ϕ is a well-defined D-map. Assume now $(\alpha + p^n D)_{n \in \mathbb{N}} = (p^n D)_{n \in \mathbb{N}}$. Then $\alpha \in p^n D$ for all $n \in \mathbb{N}$. Let us say $\alpha = r_n p^n$ so that $\alpha p^{-n} = r_n$. Observe that $r_1 D \subseteq r_2 D \subseteq \ldots$ is an ascending chain of principal ideals. As D is ACCP, then there must exist an $n \in \mathbb{N}$ such that $r_n D = r_m D$ for all $m > n$. Hence, $\alpha p^{-n} = u \alpha p^{-n-1}$ for some $u \in U(D)$. This gives us $p\alpha = u\alpha$. Should we have $\alpha \neq 0$, then $p \in U(D)$, a contradiction. Thus, $\alpha = 0$ and therefore ϕ is monic.

We recall that localization preserves exactness and $K = D_{(0)}$, making K a flat D-module. Hence, the exactness of

$$0 \longrightarrow D \xrightarrow{\phi} M$$

gives us exactness of the sequence

$$0 \longrightarrow K \otimes_D D \xrightarrow{\phi \otimes M} K \otimes_D M\,.$$

Since $K \otimes_D D \cong K$, then $Im(\phi \otimes M) \neq 0$. From this we get

$K \otimes_D M \neq 0$. Hence, products do not necessarily commute with tensors.

The class of ACCP-domains captures an enormous range of the rings most often encountered in factorization. However, there is an even larger class of domains which generalizes ACCP-domains. We recall from Chapter 1 that the class of *atomic* domains are those integral domains in which every nonzero nonunit can be expressed as a (finite) product of atoms.

Theorem 2.1.13. *[2] Every ACCP-domain is atomic.*

Proof. Suppose $x \in D^* - U(D)$. Because D is ACCP, then every chain of principal ideals ascending from xD must stabilize. Put another way, xD is contained in a principal ideal of $pD \subsetneq D$ and pD is maximal with respect to being principal. But this just means $p \in Irr(D)$. Now let $p_1 \in Irr(D)$ such that $p_1|x$. Then $xD \subseteq \frac{x}{p_1}D$. If there is some $p_2 \in Irr(D)$ where $p_2|xp_1$, than we have a chain $xD \subsetneq \frac{x}{p_1}D \subsetneq \frac{x}{p_1p_2}D$. Because D is ACCP, then this chain must stabilize. Hence, there must exist some $\pi \in Irr(D)$ such that $\frac{x}{p_1p_2\cdots\, p_n}D = \pi D$ where each $p_i \in Irr(D)$. This tells us that $\frac{x}{p_1p_2\cdots\, p_n} = u\pi$ for some $u \in U(D)$. Now solve for x to conclude x admits an atomic factorization. $\square$

Remark 2.1.14. In a remarkable example by Roitman in [14] it was shown that atomicity need not be preserved in polynomial extensions. The proof of Theorem 2.1.10 gives us a clue as to what kind of pathology we might encounter. It would still be the case in an atomic domain D that in factoring in $D[x]$ we could decrease our degree only finitely many times,i.e., every nonzero in $D[x]$ is a product of *indecomposable* elements. Hence, there might exist elements in $D[x]$ in which we can unendingly factor out elements from D. This means that in D, although every element admits an atomic factorization, there must exist at least one nonzero nonunit $x \in D$ that admits an atomic factorization as well as a chain of principal ideals ascending from xD which never stabilizes.

Example 2.1.15. Let $D = \mathbb{Z}[x, y, \frac{2}{x}, \frac{2}{y}, \frac{2}{y^2}, \ldots]$. Note that $2 = x\frac{2}{x}$ is an atomic factorization. On the other hand, $2D \subsetneq (\frac{2}{y})D \subsetneq (\frac{2}{y^2})D \subsetneq \ldots$ is a chain of principal ideals that never stabilizes. Now we do not claim that D is atomic. In fact, we know that this ring is not atomic because $\frac{2}{y}$ does not admit an atomic factorization. The point here is that although an element x in a ring may admit an atomic factorization, this does not mean that we cannot find an infinite chain of principal ideals ascending from xD.

We now know that not only may atomic domains fail to be ACCP and that the class of ACCP-domains resides properly within the class of atomic domains, but we also have an intuitive grasp as to *why* any atomic domain which is not ACCP must fail to be ACCP. Understandably, this may strike one as a radical circumstance. In Chapter 6 we will present evidence that suggests that perhaps pathological ring structure is the norm. Should this not be enough to turn the stomach (or make one salivate...depending on one's taste), we observe that the property of being an integral domain in the proof of Theorem 2.1.13 was crucial. Were it not for this fact, we could not use the kind of degree argument which was employed. We now turn our attention to a class of domains which will dominate much of all that follows.

A **valuation domain** is any domain V such that given any $x, y \in V$, then either $x|y$ or $y|x$. As contrived as this definition may at first appear, valuation domains are ubiquitous in the literature and are extremely useful in a wide variety of contexts. For example, they prove to be the building blocks of the integral closure of a domain in the sense that the integral closure of any domain is the intersection of its valuation overrings. Further, their seemingly simple structure make them indispensable when looking for strange behavior or counterexamples. They will prove to be immensely important in later discussions. Among their many talents, we find in Theorem 2.1.24 that they can be used to provide us with a very nice characterization of the class of rings known as Prüfer domains. Perhaps without knowing it, many who have taken a course in calculus have probably encountered them when dealing with power series over a field. Indeed, $F[[x]]$ is a Noetherian valuation domain for any field F. Before providing a (non-exhaustive) characterization of valuation domain's, we remind the reader that a **Bêzout** domain D is one in which every finitely generated ideal is principal. Equivalently, every pair of elements $x, y \in D$ in the domain admits a greatest common divisor which can be expressed as a D-linear combination of x and y. Also, we recall that an **overring** T of a domain D with quotient field K is a ring such that $D \subseteq T \subseteq K$.

Theorem 2.1.16. *[7] Let V be a domain with quotient field K. The following are equivalent:*

a) V is a valuation domain.

b) Given any $k \in K^*$, then either $k \in V$ or $k^{-1} \in V$.

c) The ideals of V are linearly ordered by inclusion.

d) The principal ideals of V are linearly ordered by inclusion.

e) D is Bêzout and quasi-local.

f) Every overring of V is a valuation domain.

Proof. *a)* $\Rightarrow$ *b)* We assume V is a valuation domain and let $k = ab^{-1} \in K$ where $a, b \in D^*$. Now either $a|b$ or $b|a$. If $b|a$, then $\frac{a}{b} \in D$. On the other hand, if $a|b$, $\frac{1}{k} = \frac{b}{a} \in V$.

b) $\Rightarrow$ *c)* Let I_1, I_2 be ideals in V with $I_1 \nsubseteq I_2$. Let $\alpha \in I_1 - I_2$ and let $0 \neq \beta \in I_2$. Now let $k = \frac{\alpha}{\beta}$. Either $k \in V$ or $\frac{1}{k} \in$. If $k \in V$, then $\beta|\alpha$. But then $\alpha = k\beta \in I_2$, a contradiction. So $\frac{1}{k} \in V$. Hence $\alpha|\beta$ and so $\beta \in I_1$. Thus, $I_2 \subseteq I_1$.

$c) \Rightarrow d)$ Clear.

$d) \Rightarrow a)$ Clear.

$a) \Rightarrow e)$ If V is a valuation domain with maximal ideals M_1 and M_2, then we may assume $M_1 \subseteq M_2$ from part $c)$. But the maximality of M_1 then implies $M_1 = M_2$. So V is quasi-local. To demonstrate that V is Bêzout, let $I = (x_1, \dots, x_n)$ be a finitely generated ideal of V. From $d)$ we may assume $x_1V \subseteq \dots x_nV$. Now certainly $x_nV \subseteq I$. However, every generator of I is contained in x_nV. So it must therefore be true that $I = x_nV$.

$e) \Rightarrow a)$ We now assume V is Bêzout and quasi-local. Let $x, y \in V$. We wish to show $y|x$ or $x|y$. Since V is Bêzout, then the ideal $(x, y)V$ is principal. Say $(x, y)V = zV$. Then we can find $r_1, r_2 \in V$ such that $r_1x + r_2y = z$. Hence, $r_1(\frac{x}{z}) + r_2(\frac{y}{z}) = 1$. Because V is quasi-local, this implies one of $\frac{x}{z}$ or $\frac{y}{z}$ is a unit. Assume $\frac{x}{z} \in U(V)$ so that $x = uz$ for some $u \in U(V)$. Since x and z are associates and $z|y$, then we must have that $x|y$.

$a) \Rightarrow f)$ Let T be an overring of V and let $k \in K$. From $b)$ we know that $k \in V$ or $\frac{1}{k} \in V$. As $V \subseteq T$, then either $k \in T$ or $\frac{1}{k} \in T$, making T a valuation domain.

$f) \Rightarrow a)$ Clear. $\square$

Our next two theorems provide us with many examples of valuation domain's. We will see many more examples later.

Example 2.1.17. Let F be a field. We show that $F[[x]]$ is a valuation domain. By Theorem 1.2.2, every nonzero nonunit of $F[[x]]$ is of form ux^n for some $n \in \mathbb{N}$. Choosing $u_1x^n, u_2x^m \in F[[x]]$, we may assume $n \leq m$. Then $u_1x^n | u_2x^m$. Theorem 2.1.16 gives us that $F[[x]]$ is a valuation domain.

Knowing that every valuation domain is Bêzout, it is easy to see that the converse need not hold. One need look no further than to $\mathbb{Z}$ as evidence of this.

Having made mention of greatest common divisors, we now motivate our discussion further with an example.

Example 2.1.18. Let F be a field and consider the polynomial extension $D = F[x, y]$. Note that the finitely generated ideal (x, y) is not principal even though $[x, y] = 1$ because x and y are non-associate primes. Thus, D is not a Bêzout domain.

Bêzout domains are part of a larger family of domains called GCD domains. A **GCD domain** is a domain in which every pair of elements in the ring admits a greatest common divisor. We have already seen examples of them when we looked at UFD's and Bêzout domains. Henceforth, given a domain D and $x, y \in D$, we will adopt the less cumbersome notation $[x, y]$ to denote "the" great common divisor of x and y. From a set-theoretic point of view, there may be many greatest common divisors of x and y. However, they are all associates and so there is little risk of confusion in our notation.

Generally speaking, $[x, y]$ need not exist in a domain. For example, in $\mathbb{Q}[x^2, x^3]$, it can be shown that $[x^5, x^6]$ does not exist. In turn, Theorem 2.1.20 shows us that every GCD domain is an AP domain. An **AP domain** is a domain in which all atoms are prime. Upon verification of a few facts regarding factorizations in GCD domain, verifying that every GCD domain is an AP domain becomes a simple task.

Lemma 2.1.19. *[11] Let D be a GCD domain with $x, y, z \in D^*$.*

a) $[xy, xz] = x[y, z]$.

b) If $d = [x, y]$, *then* $1 = [\frac{x}{d}, \frac{y}{d}]$.

c) If $[x, y] = [x, c] = 1$, *then* $[x, yc] = 1$.

d) If $[x, y] = 1$ *and* $x|yz$, *then* $x|z$.

Proof. *a)* Let $d = [xy, xz]$. Then $d|xy$ and $d|xz$. But we also have that $x|xy$ and $x|xz$. Thus, $x|d$. Write $d = rx$ for some $r \in D$. Write $dt = xy$ for some $t \in D$ so that $rxt = xy$. Then $r|y$. Similarly, $r|z$. Assume now $g \in D$ such that $g|y$ and $g|z$. Then $xg|xy, xz \Rightarrow xg|d \Rightarrow ag|rx \Rightarrow g|r$. Thus, $r = [y, z]$, and so $[xy, xz] = d = xr = x[y, z]$, as desired.

b) We are assuming $d = [x, y]$. In particular, $d|x$ and $d|y$ and so $\frac{x}{d}, \frac{y}{d} \in D$. Let $k = [\frac{x}{d}, \frac{y}{d}]$. From part *a)* we then get $d = [x, y] = [(\frac{x}{d})d, (\frac{y}{d})d] = d[\frac{x}{d}, \frac{b}{y}] = dk$. We now see that d and dk are associates. Hence, $k \in U(D)$, i.e., $1 = [\frac{x}{d}, \frac{y}{d}]$.

c) We are given that $[x, y] = [x, z] = 1$ and wish to deduce $[x, yz] = 1$. Assume $d|x$ and $d|yz$. Then certainly $d|xy$. Hence, $d|xy, yz$. Therefore, $d|[xy, yz] = y[x, z] = y$. We are saying now that $d|y$ and $d|x$. But it was assumed that $[x, y] = 1$. Thus, $d|1$.

d) If $[x, y] = 1$, then from *a)* we know that $z = [xz, yz]$. Since $x|yz$, then we can write $rx = yz$ for some $r \in D$. Now we have $z = [xz, yz] = [xz, rx] = x[z, r]$. So $x|z$. □

Here is a useful result that puts Lemma 2.1.19 to work.

Theorem 2.1.20. *Every GCD domain is an AP domain.*

Proof. Suppose D is a GCD domain and $p \in Irr(D)$. Suppose now $p|xy$ and write $rp = xy$. Assume $p \nmid x$. If $d = [x, p]$, then $d|p$. Since $p \in Irr(D)$, then either $d \in U(D)$ or $d = up$ for some $u \in U(D)$. But if d and p are associates, then $p|d$ and so $p|x$, a contradiction. Thus, $1 = [x, y]$. Lemma 2.1.19 tells us $p|y$ and so p is prime. □

We will be needing our next result in a later proof and it is of some interest in its own right. It relies on a beautiful result which states that every invertible ideal in a quasi-local domain is principal, a proof of which can be found in [11]. Theorem 2.1.22 can be found in [11] as an (unproven) exercise. First, we need the following definition.

Definition 2.1.21. Let I be an ideal in a domain D with quotient field K. We define the **inverse** of I, denoted I^{-1}, by $I^{-1} = \{k \in K | kI \subseteq D\}$. If $II^{-1} = D$, then we say I is **invertible**.

Theorem 2.1.22. *Every invertible ideal in a GCD domain is principal.*

Proof. Let D be a GCD domain with an invertible ideal I. Let M be a maximal ideal containing I. As I is invertible in D, then it is easily seen that ID_M is invertible, also. As D_M is quasi-local, then ID_M is principal and we write $ID_M = xD_M$ for some $x \in I$. Now certainly $xD \subseteq I$. Letting $a \in I$, then $a \in xD_M$. Hence, $a = x\frac{d}{s}$ for some $d \in D$ and $s \in D - M$. Thus, $as = xd$. Since $s \notin M$, then $[s, x] = 1$. As D is a GCD domain, it follows that $x|a$. Thus, $a \in xD \Rightarrow I \subseteq xD$ and we are done. $\square$

Having now surveyed some of the rings which lie at the heart of factorization, let us tinker with a few of them by seeing how they react to having atomicity thrust upon them.

Theorem 2.1.23. *Let D be an atomic domain.*

a) If D is a valuation domain, then D is Euclidean (and hence, a PID).

b) If D is Bêzout, then D is a PID.

c) If D is an AP domain, then D is a UFD. In particular, every atomic GCD domain is a UFD.

Proof. *a)* We assume D is an atomic valuation domain (not a field) and wish to show D is Euclidean. Note that if $p_1, p_2 \in Irr(D)$, then these atoms must be associates of each other. Indeed, from Theorem 2.1.16 we may assume $p_1 | p_2$. But because

$p_2 \in Irr(D)$, then p_2D is maximal with respect to being principal and so we must have $p_2|p_1$. Now since D is atomic and there is only atom in D (up to associates), then every nonzero nonunit $x \in D$ is of form $x = u_x p_1^n$ for some $u_x \in U(D)$ and $n \in \mathbb{N}$. Now use the function $\varphi : D^* \to \mathbb{N}_0$ given by $u_x p_1^n$ to show that D is a Euclidean domain.

b) Let $P \in Spec(D)$ be nonzero. Since D is atomic and $P \neq 0$, then P must contain atoms. Let $p_1, p_2 \in P \cap Irr(D)$. Let $z = [p_1, p_2]$. If z is not an associate of p_1, then $z \in U(D)$. But because D is Bêzout, then $z \in (p_1, p_2)D \subseteq P$. Hence, P contains a unit, a contradiction. A similar argument shows that z is an associate of p_2. It now follows that p_1 and p_2 must be associates. Because D is atomic, then every nonzero nonunit of P is a product of atoms. But P contains only one atom (up to associates). Thus, P is principal. By part *c*) of Theorem 2.1.3 D must be a PID.

c) That an atomic AP domain is a UFD is simply a restatement of Theorem 2.1.1. □

We can find PID's that are not Euclidean and we can find UFD's that are not PID's. We can also find AP domains that are not GCD domains. However, Theorem 2.1.23 shows us that in the realm of atomic domains, the notions of GCD domain and AP domain collapse down to equivalent notions.

Theorem 2.1.24 ushers in another important class of domains. Some highly nontrivial background is needed to establish this result and we will skip its proof.

Theorem 2.1.24. *[11] Let D be a domain. The following are equivalent:*

a) Every nonzero finitely generated ideal of D is invertible.

b) D_P *is a valuation domain for all* $P \in Spec(D)$.

c) D_M *is a valuation domain for all* $M \in Max(D)$.

Any domain satisfying the equivalent conditions of Theorem 2.1.24 is called a **Prüfer** domain. Theorem 2.1.25 shows us that Bêzout domains are characterized as lying at the crossroads of GCD domains and Prüfer domains.

Theorem 2.1.25. *Let D be a domain. The following are equivalent:*

a) D is a Bêzout domain.

b) D is both GCD and Prüfer.

Proof. ($\Rightarrow$) We already know every Bêzout domain is GCD. Further, every nonzero finitely generated ideal of a Bêzout domain is principal and therefore invertible. So D is Prüfer.

($\Leftarrow$) Suppose I is a nonzero finitely generated ideal of D. As D is Prüfer, we know I is invertible. But every invertible ideal of a GCD domain is principal by Theorem 2.1.22. All nonzero principal ideals in a domain are invertible. So D is Bêzout. □

In Theorem 2.1.23 it was seen that every atomic Bêzout domain is a PID. We now offer an alternative proof of this result.

Proof. As every Bêzout domain is a GCD domain and an atomic GCD domain is a UFD, then every atomic Bêzout domain is automatically a UFD. Let $0 \neq P \in Spec(D)$ and consider the localization D_P. Recall from Theorem 2.1.16 that every overring of a valuation domain is again a valuation domain. In addition, a localization of a UFD is again a UFD and therefore atomic. Thus, D_P is an atomic valuation domain. That D_P is a PID now follows from Theorem 2.1.23. Theorem 1.4.8 then gives that P a ht-1 prime in D, i.e., there is no prime ideal Q such that $0 \subsetneq Q \subsetneq P$. Every nonzero prime ideal is therefore maximal and from part $b)$ of Theorem 2.1.3 we know that D is a PID. □

Theorem 2.1.27 requires the following lemma which is shared by Dr. Jim Coykendall. The proof is omitted.

Lemma 2.1.26. *Any prime ideal that properly contains an invertible prime ideal cannot be invertible.*

Theorem 2.1.27. *[7] Suppose D is a Prüfer domain. Then D is a PID if and only if D is a UFD.*

Proof. We already know that every PID is a UFD. So we assume D is a Prüfer UFD. Suppose M is a maximal ideal of D. By Theorem 2.1.24, D_M is a valuation domain. But from Theorem 2.1.1 we also know that D_M is a UFD. Thus, D_M is an atomic valuation domain, i.e., D_M is a PID. Thus, every nonzero prime ideal of D is maximal. By Theorem 2.1.3, D is a PID. $\square$

Foreshadowing the idea of integral closure, we remark that AP domains need not be GCD domains. An efficient means of verifying this is by finding an AP domain which does not enjoy the property of being integrally closed which, as we will see, is a feature shared by all GCD domains as we will see in Theorem 2.2.10.

2.2. Integrality

The notion of integrality is an indispensable tool in commutative algebra. One of the means by which this idea most often presents itself is in its wonderful ability to serve as a testing ground. For example, a very quick way of seeing that $D = \mathbb{Z}[\sqrt{-3}]$ is not a UFD is by noting that it is not integrally closed and, as we will presently see, all UFD's *are* integrally closed. Let us begin with the necessary definition.

Definition 2.2.1. Let $R \subseteq T$ be an extension of rings. We say that $t \in T$ is **integral** over R provided $f(t) = 0$ for some monic $f(x) \in T[x]$. If every element of T is integral over R then we say $R \subseteq T$ is an **integral extension**.The set of elements in T that are integral over R is called the **integral closure of R in T**, denoted $\overline{R_T}$.

If R is a domain and T is the quotient field of R, then we refer to the set of all elements of T that are integral over R as the **integral closure of R**, denoted $\overline{R}$. If $R = \overline{R}$, then we say R is **integrally closed**.

Obviously $R \subseteq R$ is an integral extension. Less trivially, it can be seen that in the extension $\mathbb{Z} \subseteq \mathbb{Z}[\sqrt{-3}]$ we have that $\sqrt{-3}$ is integral over $\mathbb{Z}$. On the other hand, in the ring extension $\mathbb{Z} \subseteq \mathbb{Q}$, the element $\frac{1}{2}$ is *not* integral over $\mathbb{Z}$. To see this, suppose $\frac{1}{2}$ is integral over $\mathbb{Z}$. Then we may say $(\frac{1}{2^n}) + r_{n-1}(\frac{1}{2^{n-1}}) + \ldots + r_1(\frac{1}{2}) + r_0 = 0$, where each $r_j \in \mathbb{Z}$ and $r_0 \neq 0$. Multiplying both sides of the equation by 2^n we then see that $1 + r_{n-1}2 + r_{n-2}2^2 + \ldots + r_1 2^{n-1} + 2^n r_0 = 0$. But from this we could deduce $2 \in U(\mathbb{Z})$, a contradiction. Soon we will find a far more efficient way of seeing this same result.

Here is a wonderful characterization of integrality. In short order we will see a result that uses the following characterization and tells us that integral extensions are rings. A proof can be found in [11] and we omit the proof.

Theorem 2.2.2. *[11] Let $R \subseteq T$ be a ring extension and let $u \in T$. The following are equivalent:*

a) u is integral over R

b) $R[u]$ is a finitely generated R-module

c) There exists an R-submodule A of T such that $uA \subseteq A$ and the annihilator of A in T is 0

Here is a technical result which will ease some of our arguments later on.

Lemma 2.2.3. *Let $R \subseteq T \subseteq L$ be ring extensions in which T is a finitely generated R-module and L is a finitely generated T-module. Then L is a finitely generated R-module.*

Proof. Say $T = (t_1, \ldots, t_n)$ as an R-module and $L = (\ell_1, \ldots \ell_m)$ as a T-module.We claim that $L = (t_j \ell_k)$ as an R-module. Certainly $(t_j \ell_k) \subseteq L$. So let $a \in L$ and write $a = \sum_{i=1}^{m} x_i \ell_i$ with each $x_i \in T$. As each $x_i \in T$, then we may write $x_i = \sum_{j=1}^{n} r_{j,i} t_j$ with each $r_{j,i} \in R$. Then $a = \sum_{i=1}^{m} \sum_{j=1}^{m} r_{j,i} t_j \ell_i \in (t_j \ell_k)$, as desired. □

The next theorem is as lovely as it is powerful and our uses for it will be legion. It assures us that integral closures are rings.

Theorem 2.2.4. *[11] Let $R \subseteq T$ be a ring extension. Then $\overline{R_T}$ is a ring.*

Proof. Assume $u, v \in \overline{R_T}$ and set $A = R[u, v]$. We regard A as an R-submodule of T. By Theorem 2.2.2 we know that $R[u]$ is a finitely generated R-module and A is a finitely generated $R[u]$-module. From Theorem 2.2.3 it follows that A is a finitely generated R-module. As $1 \in A$, then the annihilator of A in T is 0. Clearly, $(u+v)A, uvA \subseteq A$. By Theorem 2.2.2 we now have that $u+v, uv \in \overline{R_T}$, as desired. □

We now present the transitivity of integral extensions.

Theorem 2.2.5. *[11] Let $R \subseteq T \subseteq L$ be rings in which $R \subseteq T$ and $T \subseteq L$ are both integral extensions. Then $R \subseteq L$ is an integral extension.*

Proof. Let $\ell \in L$ and write $\ell^n + t_{n-1}\ell^{n-1} + \ldots + t_1 \ell_1 + t_0 = 0$ with each $t_i \in T$. Now let $A = R[t_0, \ldots, t_{i-1}, \ell]$. Then certainly $\ell A \subseteq A$ and the annihilator of A in T is 0 because $1 \in A$. By Theorem 2.2.2 it suffices to show A is a finitely generated R-module. Because ℓ is integral over A, then A is a finitely generated $R[t_0, \ldots, t_{i-1}]$-module. As every t_j is integral over R, then each t_j is all the more so integral over $R[t_0, \ldots, t_{i-1}]$. Thus, $R[t_0, \ldots, t_i]$ is a finitely generated $R[t_0, \ldots, t_{i-1}]$-module. Now by inductive application of Theorem 2.2.2, we conclude that A is a finitely generated R-module. $\square$

Theorem 2.2.6. *[7] Let $R \subseteq T$ be rings. Then $\overline{R_T}$ is integrally closed in T. In particular, the integral closure of a domain is integrally closed.*

Proof. We begin by letting K be the quotient field of T. If $t \in K$ is integral over $\overline{R_T}$, then we can use Theorem 2.2.4 to see that $\overline{R_T} \subseteq \overline{R_T}[t]$ is an integral extension. But $R \subseteq \overline{R_T}$ is also an integral extension. From Theorem 2.2.5 we then know $R \subseteq \overline{R_T}[t]$ is also an integral extension and so t is integral over R, i.e., $t \in \overline{R_T}$. $\square$

Not only is the integral closure of a domain D integrally closed, but it is the smallest integrally closed overring of D.

Theorem 2.2.7. *Let D be a domain and assume T is an integrally closed overring of D. Then $\overline{D} \subseteq T$.*

Proof. If $\alpha \in \overline{D}$, then α is integral over D. As $D \subseteq T$, then α is also integral over T. Being integrally closed, we get $\alpha \in T$. $\square$

Now we show that integral closure commutes with localizations.

Theorem 2.2.8. *[7] Suppose R is a ring and $S \subseteq R$ is multiplicatively closed Then $(\overline{R})_S = \overline{R_S}$. In particular, every localization of any integrally closed domain is integrally closed.*

Proof. Suppose first that $\frac{x}{s} \in (\overline{R})_S$ for some $r \in \overline{R}$ and $s \in S$. We write $x^n + r_{n-1}x^{n-1} + r_{n-2}x^{n-2} + \ldots + r_1x + r_0 = 0$. Then $s^{-n}(x^n + r_{n-1}x^{n-1} + r_{n-2}x^{n-2} + \ldots + r_1x + r_0) = (\frac{x}{s})^n + \frac{r_{n-1}}{s}(\frac{x}{s})^{n-1} + \frac{r_{n-2}}{s^2}(\frac{x}{s})^{n-2} + \ldots + \frac{r_1}{s^{n-1}}(\frac{x}{s}) + \frac{r_0}{s^n} = 0$. Thus, $\frac{x}{s}$ is integral over R_S, i.e., $\frac{x}{s} \in \overline{R_S}$. For the reverse containment, assume $d \in \overline{R_S}$ and write $d^n + \frac{r_{n-1}}{s_{n-1}}d^{n-1} + \frac{r_{n-2}}{s_{n-2}}d^{n-2} + \ldots + \frac{r_1}{s_1}d + \frac{r_0}{s_0} = 0$. Let $s = s_1s_2\cdots s_{n-1}$. Then

$(sd)^n + \frac{r_{n-1}s}{s_{n-1}}(sd)^{n-1} + \frac{r_{n-2}s^2}{s_{n-2}}(sd)^{n-2} + \ldots + \frac{r_1 s^{n-1}}{s_1}(sd) + \frac{r_0 s^n}{s_0} = 0$. Note that $\frac{r_i s}{s_i} \in R$ since each $s_i | s$. Thus, $sd \in \overline{R}$ and so $d \in (\overline{R})_S$, finishing the proof. □

Here is another way of constructing new integrally closed domains from old ones. This result can be found in [11] and we omit the straightforward proof.

Theorem 2.2.9. *Let $\{D_i | i \in \Lambda\}$ be a family of integrally closed domains, all of which are contained in some larger domain.*

a) $\bigcap_{i\in\Lambda} D_i$ is integrally closed.

b) If $\{D_i | i \in \Lambda\}$ is a chain, then $\bigcup_{i\in\Lambda} D_i$ is integrally closed.

Our next theorem shows that many of our favorite domains are integrally closed.

Theorem 2.2.10. *[11] Every GCD domain is integrally closed.*

Proof. Let D be a GCD domain with quotient field K and let $\frac{t}{s} \in K$ be integral over D, where $t, s \in D$. Since D is a GCD domain, then it may be assumed that $[t, s] = 1$. Now we write $(\frac{t}{s})^n + r_{n-1}(\frac{t}{s})^{n-1} + \ldots + r_1(\frac{t}{s}) + r_0 = 0$. Multiplying by s^n we get $t^n + r_{n-1}st^{n-1} + \ldots + r_1 s^{n-1}t + r_0 s^n = 0$. Solving for t^n in this equation, it is then evident that $s | t^n$, i.e., $s | t$. But $[s, t] = 1$ and so $s \in U(D)$. Therefore, $ts^{-1} \in D$ and we are done. □

As every UFD is a GCD domain, we obtain the following corollary.

Corollary 2.2.11. *Every UFD is integrally closed.*

Example 2.2.12. Let F be a field and consider the ring $D = F[x^2, x^3]$. Note that D is not integrally closed because $x \in \overline{D} - D$. By Theorem 2.2.4, $F[x] \subseteq \overline{D}$. Because $F[x]$ is a PID, then $F[x]$ must be integrally closed. Hence, Theorem 2.2.7 guarantees $\overline{D} \subseteq F[x]$. Therefore, $\overline{D} = F[x]$.

Example 2.2.13. Let $D = \mathbb{Z} + x\mathbb{R}[[x]]$ and note D is not integrally closed (note $\sqrt{2} \notin D$, for example). However, $\mathbb{R}[[x]]$ is integrally closed and so by Theorem 2.2.7 $\overline{D} \subseteq \mathbb{R}[[x]]$. Suppose now $f(x) \in \overline{D}$ and write $f^n + r_{n-1}f^{n-1} + \ldots + r_1 f + r_0 = 0$, where every $r_i \in D$. Observe that $f^n(0) + r_{n-1}(0)f^{n-1}(0) + \ldots + r_1(0)f(0) + r_0(0) = 0$, i.e., $f(0) \in \overline{\mathbb{Z}_{\mathbb{R}}}$. But as $f \in \mathbb{R}[[x]]$, it follows that $f \in \overline{\mathbb{Z}_{\mathbb{R}}} + x\mathbb{R}[[x]]$. We have shown $\overline{D} \subseteq \overline{\mathbb{Z}_{\mathbb{R}}} + x\mathbb{R}[[x]]$. Assume now we have some $f \in \overline{\mathbb{Z}_{\mathbb{R}}} + x\mathbb{R}[[x]]$ and write $f = z_0 + \sum_{i=1}^{\infty} r_i x^i$, where $z_0 \in \overline{\mathbb{Z}_{\mathbb{R}}}$ and each $r_i \in \mathbb{R}$. Note $\overline{\mathbb{Z}_{\mathbb{R}}} \subseteq \overline{D}$ and $\sum_{i=1}^{\infty} r_i x^i \in \overline{D}$, i.e., $z_0, \sum_{i=1}^{\infty} r_i x^i \in \overline{D}$. By Theorem 2.2.4 we conclude $f \in \overline{D}$. Thus, $\overline{D} = \overline{\mathbb{Z}_{\mathbb{R}}} + x\mathbb{R}[[x]]$.

As an added observation, we can use Theorem 1.2.3 to show that $f \in Irr(D)$ if and only if $f(0) \in Irr(\mathbb{Z})$. Since $\mathbb{Z}$ is an AP domain, it then follows that D is also an AP domain. But as D is not integrally closed, we can see that it is not a GCD domain. Recalling Theorem 2.1.20 it follows that the class of AP domains is properly larger than the class of GCD domains.

Example 2.2.14. Let $F \subsetneq K$ be fields and $R = F + xK[[x]]$. A similar argument as in the previous example can be used to show that R is integrally closed if and only if F is algebraically closed in K. Thus, $\overline{\mathbb{Q}_{\mathbb{R}}} + x\mathbb{R}[[x]]$ is integrally closed, where $\overline{\mathbb{Q}_{\mathbb{R}}}$ is the algebraic closure of $\mathbb{Q}$ in $\mathbb{R}$.

The next big result we would like to establish is to show that the GCD property is preserved in polynomial extensions. This will be a useful aid when we investigate the question of whether or not the property of being integrally closed survives polynomial in extensions. In addition, we will use it to provide an unconventional proof of the fact that a polynomial extension of a UFD is again a UFD. We remind the reader that if D is a GCD domain, then a **primitive** polynomial in $D[x]$ is one in which the greatest common divisor of the coefficients is 1. We proceed with a few lemmas which will do all the heavy lifting for the bigger result we are targeting.

Lemma 2.2.15. *Assume D is a GCD domain and let S be the set of all primitive polynomials in $D[x]$. Suppose $r|py$ for some $r, y \in D$ and $p \in S$. Then $r|y$.*

Proof. Let us begin by saying $p = p_0+p_1x+\ldots p_nx^n$ and $yp = r(f_0+f_1x+\ldots+f_nx^n)$, where each $f_i \in D$. Then $r|yp_i$ for each i. Let $s_o, s_1, \ldots, s_n \in D$ such that $yp_i = rs_i$. Since $p \in S$, then $1 = [p_0, \ldots, p_n]$. From Theorem 2.1.19 we have $y = [yp_0, \ldots, yp_n] = [rs_0, \ldots, rs_n] = r[s_0, \ldots, s_n]$. Thus, $r|y$. □

For the next lemma we recall that an **LCM-domain** is one in which every pair of elements admits a least common multiple. In Theorem 3.5.14 it is shown that D is a GCD domain if and only if D is an LCM-domain. We borrow this fact now.

Lemma 2.2.16. *Let D be a GCD domain with quotient field K and let S be as in Lemma* 2.2.15. *For any $p_1, p_2 \in S$, $[p_1, p_2]$ exists.*

Proof. Since $K[x]$ is a PID, then we may let $[p_1, p_2] = p_3$ for some $p_3 \in S$. That is, $[p_1, p_2] = p_3$ in $K[x]$. Suppose now $p_4 \in S$ such that $p_4|p_1, p_2$ in $K[x]$. Then $\alpha p_4 = p_3$ for some $\alpha \in K[x]$. Using the fact that D is a GCD domain we write $\alpha = \sum_{i=0}^{n} \frac{r_i}{s_i}x^i$ where each $[r_i, s_i] = 1$. As every GCD domain is an LCM-domain, we can let $s = lcm(s_0, \ldots, s_n)$. Hence, $s\alpha = p_5 \in S$ and we have $sp_3 = (s\alpha)p_4 = p_5p_4$. Since S is multiplicatively closed, then by Lemma 2.2.15 we have $s \in U(R)$ and so each $s_i \in U(R)$. Thus, $\alpha \in D[x]$.

We have shown that if $p, p' \in S$ such that $p = bp'$ for some $b \in K[x]$, then $b \in D[x]$. Thus, $p_3|p_1, p_2$ in $D[x]$ and if $p_4|p_1, p_2$, then $p_4|p_3$ in $K[x]$, i.e., $p_4|p_3$ in $D[x]$. Thus, $[p_1, p_2] = p_3$ in $D[x]$, as desired. □

Lemma 2.2.17. *Let D be a GCD domain and let S be as in Lemma* 2.2.15. *Let $p_1, p_2, p_3 \in S$ such that $p_3 = [p_1, p_2]$. If $r_3 = [r_1, r_2]$ for some $r_1, r_2, r_3 \in D$, then $r_3p_3 = [r_1p_1, r_2p_2]$.*

Proof. Clearly $r_3p_3|r_1p_1, r_2p_2$. Suppose $h|r_1p_1, r_2p_2$ and write $h = r_4p_4$ with $r_4 \in D$ and $p_4 \in S$. Since $r_4|r_1p_1$ and $p_1 \in S$, then by Lemma 2.2.16 $r_4|r_1$. Similarly, $r_4|r_2$. So $r_4|r_3$. Now because $r_4p_4|r_1p_1$, then we may write $(r_5p_5)(r_4p_4) = r_1p_1$ for some $r_5 \in D$ and $p_5 \in S$. Since S is multiplicatively closed, then $r_1|r_4r_5$. Thus, $p_1 = (\frac{r_4r_5}{r_1})(p_5p_4)$ and so $p_4|p_1$. Similarly, $p_4|p_2$, and so $p_4|p_3$. We have now demonstrated $r_4|r_3$ and $p_4|p_3$, i.e., $r_4p_4|r_3p_3$, as desired. □

Theorem 2.2.18 can be found as an exercise in [11].

Theorem 2.2.18. *[11] Let S be the same as in Lemma* 2.2.15. *If D is a GCD domain, then so is $D[x]$.*

Proof. Let $f, g \in D[x]$ with $f = r_1p_1$ and $g = r_2p_2$ for some $r_1, r_2 \in D$ and $p_1, p_2 \in S$. We use Lemma 2.2.16 and let $p_3 = [p_1, p_2]$. Allowing $r_3 = [r_1, r_2]$, then from Lemma 2.2.17 we have $r_3p_3 = [r_1p_1, r_2p_2] = [f, g]$, as desired. □

Here are some applications of Theorem 2.2.18. Theorem 2.2.19 is a well-known result but the proof is original.

Theorem 2.2.19. *If D is a UFD, then so is $D[x]$.*

Proof. As every UFD is a GCD domain, then we already know $D[x]$ must be a GCD domain. By Theorem 2.1.1 it suffices to show that every prime ideal of $D[x]$ contains a prime element. First we choose some nonzero $P \in Spec(D[x])$. If it happens that $P \cap D \neq 0$, then we are done. Thus, we should assume $P \cap D = 0$. As P is prime, this then forces P to contain a primitive element. Let $f \in P$ be primitive and of minimal (positive) degree. Note that the primitive elements in $D[x]$ constitute a saturated set. Hence, $f \in Irr(D[x])$. Since $D[x]$ is a GCD domain and every GCD domain is an AP domain, then f must be prime and so we are done. □

Theorem 2.2.20 makes use of the fact that every integrally closed domain can be realized as the intersection of all its valuation overrings. A proof of this beautiful and highly nontrivial fact can be found in [11].

Theorem 2.2.20. *[7] Let D be a domain with quotient field K The following are equivalent:*

a) D is integrally closed

b) $D[x]$ is integrally closed

c) D_P is integrally closed for all $P \in Spec(D)$

Proof. *a)* $\Rightarrow$ *b)* Using the fact that $D = \bigcap_{V \in S} V$, we make the substitution $D[x] = (\bigcap_{V \in S} V)[x]$. It is easily argued that $(\bigcap_{V \in S} V)[x] = \bigcap_{V \in S}(V[x])$. Since every valuation domain is a GCD domain, then each $V[x]$ is a GCD domain. Using Theorem 2.2.10 we deduce each $V[x]$ must be integrally closed. Now implement Theorem 2.2.9 to reach the desired result.

b) $\Rightarrow$ *a)* Clear.

That *a)* $\Rightarrow$ *c)* follows from Theorem 2.2.8. The converse follows from Theorem 2.2.9 and the fact $D = \bigcap_{M \in Max(D)} D_M$. □

Theorem 2.2.21. *If D is integrally closed, then so is $D[\{x^q\}]_{q \in \mathbb{Q}_+}$*

Proof. For each $n \in \mathbb{N}$, define $R_n = D[x^{\frac{1}{n!}}]$ and note $R_n \subsetneq R_{n+1}$. Further, $D[x^q]_{q \in \mathbb{Q}_+} = \bigcup_{n=1}^{\infty} R_n$. For a fixed $n \in \mathbb{N}$ we observe that the map $D[x] \to D[x^{\frac{1}{n}}]$ given by $f(x) \to f(x^{\frac{1}{n!}})$ is a ring isomorphism. Thus, since $D[x]$ is integrally closed, then so is each R_n. Now we can use Theorem 2.2.9 to close the argument. □

Here is an alternative (and more efficient!) proof of Theorem 2.2.21

Proof. We denote $R = D[x^q]_{q\in\mathbb{Q}_+}$ and let $x^r \in [x^q]_{q\in\mathbb{Q}_+}$. Write $r = \frac{a}{b}$ for some $a, b \in \mathbb{N}$. Then x^r is a root of $t^b - x^a \in D[x, t]$. Now we can use Theorem 2.2.4 to deduce $D[x] \subseteq D[x^q]_{q\in\mathbb{Q}_+}$ is an integral extension. Theorem 2.2.6 then guarantees that $D[x^q]_{q\in\mathbb{Q}_+}$ is integrally closed. □

Now we turn our attention to some ideas that are intimately connected with integral extensions. We will see that these ideas carry tremendous weight in the next section when we begin discussing Krull dimension.

Definition 2.2.22. Let $R \subseteq T$ be a ring extension. We say that the extension is **Going Up** (GU) if, whenever we have a chain of prime ideals $P_1 \subsetneq P_2$ in R and a prime ideal Q_1 in T that contracts to P_1, then there exists a prime ideal Q_2 in T such that $Q_1 \subsetneq Q_2$ and Q_2 contracts to P_2. In addition, we say that the ring extension is **Lying Over** (LO) if every prime ideal of R can be realized as a contraction of some prime ideal in T.

The next two theorems provide us with a useful characterization of GU extension and a relationship between LO and GU.

Theorem 2.2.23. *[11] Let $R \subseteq T$ be rings. The following are equivalent:*

a) $R \subseteq T$ is GU

b) If $P \in Spec(R)$, $S = R - P$, and $Q \in Spec(T)$ is maximal with respect to missing S, then $Q \cap R = P$

Proof. *a)* $\Rightarrow$ *b)* Suppose $R \subseteq T$ is GU, $P \in Spec(R)$, $S = R - P$, and $Q \in Spec(T)$ is maximal with respect to missing S. Since Q is prime in T, then $Q \cap R \in Spec(R)$. Further, $Q \cap S = \emptyset$ guarantees $Q \cap R \subseteq P$. As $R \subseteq T$ is GU, then we know there exists some $Q' \in Spec(T)$ such that $Q \subset Q'$ and $Q' \cap R = P$. It is necessary that $Q' \cap S = \emptyset$. Now use the maximality of Q to conclude $Q = Q'$ and therefore $Q \cap R = P$.

$b) \Rightarrow a)$ Assuming the hypothesis of $b)$, let us assume $P \subseteq P'$ are prime ideals in R and Zornify to find some $Q \in Spec(T)$ such that $Q \cap R = P$. Now let $S = R - P'$ and observe $Q \cap S = \emptyset$. By Theorem 1.4.1 we can expand Q to some $Q' \in Spec(T)$ that is maximal with respect to $Q' \cap S = \emptyset$. Then $Q \subseteq Q'$ and by the hypotheses of $b)$ we conclude $Q' \cap R = P'$, i.e., $R \subseteq T$ is GU. $\square$

Theorem 2.2.24. *[11] Every ring extension that is GU is LO.*

Proof. Let $P \in Spec(R)$ and $S = R - P$. Since S is multiplicatively closed in R, then S is also multiplicatively closed in T. Using Zorn we can find some $Q \in Spec(T)$ that is maximal with respect to missing S. Now use the GU property to conclude $Q \cap R = P$ to reach the conclusion we seek. $\square$

Definition 2.2.25. Let $R \subseteq T$ be a ring extension. We say the extension is **Incomparable** (INC) provided that whenever distinct prime ideals in T contract to the same prime in R, then said primes in T are not comparable, i.e., neither is a subset of the other.

Theorem 2.2.26. *[11] Let $R \subseteq T$ be a ring extension. The following are equivalent:*

a) $R \subseteq T$ is INC

b) If P and Q are prime ideals in R and T, respectively, with $Q \cap R = P$, then Q is maximal with respect to missing $S = R - P$

Proof. $a) \Rightarrow b)$ We assume $R \subseteq T$ is INC and let $P \in Spec(R)$ and $Q \in Spec(T)$ such that $Q \cap R = P$. We also name $S = R - P$. Suppose $Q \subseteq Q'$ are prime ideals in T with $Q' \cap S = \emptyset$. Then $Q' \cap R \subseteq P = Q \cap R \subseteq Q' \cap R$. Thus, $Q' \cap R = P$. Since the extension is INC, then we must have $Q = Q'$.

$b) \Rightarrow a)$ Suppose $Q_1 \subseteq Q_2$ are primes lying over the same prime in R. Since Q_1 is maximal with respect to missing S, then we must have $Q_1 = Q_2$, as desired. $\square$

Upon reflection of the previous two theorems, the reader might notice that in ring extensions that are INC and GU, the way in which prime ideals line up in one ring says something about how the prime ideals should be lined up in the other. Our next theorem packs a lot of punch.

Theorem 2.2.27. *[11] Integral extensions are INC and GU.*

Proof. First we establish INC. Let $Q_1, Q_2 \in Spec(T)$ be distinct with $Q_1 \cap R = Q_2 \cap R$. Assume $Q_1 \subsetneq Q_2$ and let $q \in Q_2 - Q_1$. As $R \subseteq T$ is an integral extension, then we may say $q^n + r_{n-1}q^{n-1} + \ldots + r_1 q + r_0 = 0$ where each $r_i \in R$ and $r_0 \neq 0$. It follows from this that $q(q^{n-1} + r_{n-1}q^{n-2} + \ldots + r_2 q + r_1) = -r_0 \in Q_2 \cap R = Q_1 \cap R$. Thus, $q^{n-1} + r_{n-1}q^{n-2} + \ldots + r_2 q + r_1 \in Q_1 \subseteq Q_2$ and we have that $r_1 \in Q_1 \cap R$. Continuing inductively we conclude $r_0, \ldots, r_{n-1} \in Q_1$. But then $q^n \in Q_1 \Rightarrow q \in Q_1$, a contradiction. Thus, $Q_1 \not\subsetneq Q_2$ and we have that the extension is INC.

To show that $R \subseteq T$ is GU, assume $P_1 \subsetneq P_2$ are prime ideals in R and let $Q_1 \in Spec(T)$ such that $Q_1 \cap R = P_1$. Let $S = R - P_2$. By Theorem 2.2.26 we know Q_1 is maximal with respect to missing $R - P_1$. Since $P_1 \subsetneq P_2$, we can expand Q_1 to some $Q_2 \in Spec(T)$ that is maximal with respect to missing S. Note $Q_1 \subsetneq Q_2$ because of the maximality of Q_1. As $Q_2 \cap (R - P_2) = \emptyset$, then $Q_2 \cap R \subseteq P_2$. Assume $Q_2 \cap R \subsetneq P_2$ and let $p \in P_2 - (Q_2 \cap R)$. Then $(Q_2, p) \cap S \neq \emptyset$. Say $s \in (Q_2, p) \cap S$ and write $s = q + tp$ for some $q \in Q_2$ and $t \in T$. Since $t \in T$, then $t^m + y_{m-1}t^{m-1} + \ldots + y_1 t + y_0 = 0$, where each $y_j \in R$. Multiplying both sides of this equation by p^m we then get $(tp)^m + y_{m-1}p(tp)^{m-1} + \ldots + y_1 p^{m-1}(tp) + y_0 p^m = 0$. Now $tp = s - q$ and so $(s-q)^m + y_{m-1}p(s-q)^{m-1} + \ldots + y_1 p^{m-1}(s-q) + y_0 p^m = 0$. Hence, $(s)^m = -(y_{m-1}p(s)^{m-1} + \ldots + y_1 p^{m-1}(s) + y_0 p^m) \in Q_2 \cap R \subseteq P_2$, a contradiction. Therefore, $P_2 \subseteq Q_2 \cap R$ and so $P_2 = Q_2 \cap R$, as desired. ☐

Here is an interesting application of some of the ideas we have thus far presented.

Theorem 2.2.28. *Suppose $F \subseteq K$ is an extension of fields and set $D = F + xK[[x]]$. Then D is Noetherian if and only if $[K : F]$ is finite.*

Proof. ($\Rightarrow$) We assume D is Noetherian and wish to show $[K : F]$ is finite. Assume, to the contrary, that $[K : F]$ is infinite. Then we may choose a set $\{k_1, k_2, \ldots\} \subseteq K$ that is linearly independent over F. Observe now that if $n > m$, then $k_n x \notin (k_1 x, \ldots, k_m x)$. To verify this, suppose $k_n x = f_1 k_1 x + \ldots + f_m k_m x$ for some $f_1, \ldots, f_m \in D$. Because $ord(k_n x) = 1$, then we must have that $ord(f_i) = 0$ for all f_i. But then $k_n \in Span(k_1, \ldots, k_m)$, contradicting linear independence. We deduce now that $k_{n+1} x \notin (k_1 x, \ldots, k_n x)$ for all $n \in \mathbb{N}$. But then $(k_1 x) \subsetneq (k_1 x, k_2 x) \subsetneq \ldots$ is an ascending chain of ideals in D, a contradiction because D was assumed to be Noetherian.

($\Leftarrow$) Now we suppose $[K : F]$ is finite. As such, $F \subseteq K$ is an algebraic extension. Hence, $K \subsetneq \overline{D} \Rightarrow K[[x]] \subseteq \overline{D}$. But because $K[[x]]$ is integrally closed, then $\overline{D} \subseteq K[[x]]$. Now knowing that $\overline{D} = K[[x]]$, it follows from Theorem 2.2.27 that $xK[[x]]$ is the unique nonzero prime ideal of D. Because $[K : F]$ is finite, then we let $\{k_1, \ldots, k_n\}$ be a basis for K over F. Let $kx \in xK[[x]]$. Then $k = f_1 k_1 + \ldots + f_n k_n$ for some $f_1, \ldots, f_n \in F$. So $k_1 x = f_1 k_1 x + \ldots + f_n k_n x$. It is now obvious that $xK[[x]] = (x, k_1 x, k_2 x, \ldots, k_n x)$. Having shown every prime ideal of D is finitely generated, Theorem 2.1.5 tells us that D must be Noetherian. $\square$

Example 2.2.29. Let $\overline{\mathbb{Q}}$ be the algebraic closure of $\mathbb{Q}$ and note that $\mathbb{Q} \subsetneq \overline{\mathbb{Q}}$ is an infinite field extension. Then $D = \mathbb{Q} + x\overline{\mathbb{Q}}[[x]]$ must be non-Noetherian by Theorem 2.2.28. Observe further that $\overline{\mathbb{Q}}[[x]] = \overline{D}$. Thus, the integral closure of a non-Noetherian domain can be Noetherian.

From our previous theorem it is evident that $\mathbb{R} + x\mathbb{C}[[x]]$ is Noetherian. Thanks to Nagata, it is well known that the integral closure of a Noetherian domain need not be Noetherian [13]. In turn, the integral closure of a non-Noetherian domain may be Noetherian.

Recalling that every Noetherian domain is atomic, we can use Theorem 2.2.28 to produce examples of Noetherian domains which contain no primes. Using the notation from Theorem 2.2.28, we see that if $F \subsetneq K$ and K is finite, then D will admit only finitely many atoms, none of which are prime. If F is finite and K is the algebraic closure of F, then D will admit infinitely many atoms, none of which are prime.

2.3. Krull Dimension

We now begin to look at Krull dimension. In a certain sense, Krull dimension gives us a way of measuring the size of a ring. In Chapter 5 we will be presenting a generalization of Krull dimension and much of the material presented here serves a good pedagogical role. To get our feet wet and to motivate some of the ideas in later chapters, we opt for the traditional approach.

Definition 2.3.1. A chain of prime ideals of form $P_0 \subsetneq P_1 \subsetneq \ldots \subsetneq P_n$ in a ring R is said to have length n. The **Krull dimension** of R, denoted $dim(R)$, is defined to be the supremum taken over all such possible chains. If this supremum is finite, then R is said to **finite dimensional**. Otherwise, R is said to be **infinite dimensional**.

It is obvious that a domain D is 0-dimensional if and only if D is a field. Moreover, if D is non-field domain, then we could rephrase Theorem 2.1.3 by saying D *is a PID if and only if D is both a UFD and 1-dimensional.* In particular, $dim(\mathbb{Z}) = 1$. Also, $dim(\mathbb{Z}[x]) \geq 2$ because $0 \subsetneq (x) \subsetneq (2, x)$ is a chain of prime ideals in $\mathbb{Z}[x]$ of length 2. To see an example of an infinite dimensional domain, let F be a field and consider the polynomial ring $F[x_1, x_2, \ldots]$. This ring is infinite dimensional because $(x_1) \subsetneq (x_1, x_2) \subsetneq \ldots$ is an infinite chain of prime ideals. However, rings may be infinite dimensional without having any infinite chains of primes. Indeed, Nagata showed this with his construction of an infinite dimensional Noetherian ring. This is the reason for using *supremum* in the definition. Further, one should not be misled by the definition into thinking that all chains of primes are even *countable.*

Observe that as a consequence of the prime correspondence in Theorem 1.4.8 and Theorem 1.5.2, it is easily shown that $dim(R_S) \leq dim(R)$ and $dim(R/I) \leq dim(R)$. Many authors define the notions of the *height* and *depth* of a prime in terms of dimension. It is said that given $P \in Spec(R)$, then the **depth** of P, denoted $d(P)$, is given by $d(P) = dim(R/P)$. Further, the **height** of P, denoted by $ht(P)$, is defined as $ht(P) = dim(R_P)$. Because of prime correspondence, note that $ht(P)$ is the supremum taken over all lengths of chains of primes that contain P and $d(P)$ is the supremum taken over all lengths of chains of primes contained within P. Using prime correspondence it is not difficult to prove the following useful theorem.

Theorem 2.3.2. *If R is ring, then $dim(R) = sup\{ht(M)|M \in Max(R)\}$. Equivalently, $dim(R) = sup\{dim(R_M)|M \in Max(R)\}$.*

We now examine GU and INC as they relate to Krull dimension.

Theorem 2.3.3. *[11] Suppose the ring extension $R \subseteq T$ is GU. Then $dim(R) \leq dim(T)$. In particular, if R is infinite dimensional, then so is T.*

Proof. Suppose $P_0 \subsetneq P_1 \subsetneq \ldots \subsetneq P_n$ is a chains of primes in R. By GU, we are guaranteed the existence of a chain of primes $Q_0 \subsetneq Q_1 \ldots \subsetneq Q_n$ where each Q_i lies over P_i. Thus, the existence of a chain of primes of length n in R implies the existence of a chain of primes of length n in T, i.e., $dim(R) \leq dim(T)$. $\square$

Theorem 2.3.4. *[11] Suppose the ring extension $R \subseteq T$ is INC. Then $dim(R) \geq dim(T)$.*

Proof. Suppose $Q_0 \subsetneq Q_1 \subsetneq \ldots \subsetneq Q_n$ is a chain of primes in T and let $P_i \in Spec(R)$ be given by $P_i = Q_i \cap R$. Then we have a chain of primes $P_0 \subseteq P_1 \subseteq \ldots \subseteq P_n$ in

R. By INC, Q_i and Q_{i+1} cannot lie over the same prime in R. Thus, each $P_i \subsetneq P_{i+1}$. This means that given a chain of length n in T, we must also have a chain of length n in R. So $dim(R) \geq dim(T)$. $\square$

The following theorem is one of the most important in the annals of dimension theory. One might summarize it by saying "Krull dimension is preserved in integral extensions". One can find extremely similar statements as that in Theorem 2.3.5 in any number of books on commutative algebra such as [4], [7], or [11] . However, the difference between our statement and those found elsewhere can be found in the last sentence of the theorem. This difference is one of the primary motivations for the work we will see in Chapter 4 and is the reason we provide no citation.

Theorem 2.3.5. *Suppose $R \subseteq T$ is an integral extension. If $dim(R)$ is finite, then $dim(R) = dim(T)$. If R is infinite dimensional, then so is T.*

Proof. Integral extensions are GU and INC. BY GU, if R is infinite dimensional, then T is, as well. So assume R is finite dimensional. By GU and INC, we have $dim(R) \leq dim(T) \leq dim(R)$ and we are done. $\square$

We now briefly consider dimension behavior in polynomial extensions. First, observe that if R is a ring and $P \in Spec(R)$, then $P[x]$ and (P, x) are both primes in $R[x]$ that lie over P. In other words, the extension $R \subsetneq R[x]$ is LO. Thus, if R is infinite dimensional, then so is $R[x]$. Also, if R is finite dimensional, then $dim(R) < dim(R[x])$. That this is true in the case R is a domain is obvious. For the more general case, suppose $dim(R) = n$ and $P_0 \subsetneq P_1 \subsetneq \ldots \subsetneq P_n$ is a chain of primes in R. Then we have a chain of primes of length $n+1$ in $R[x]$ given by $P_0[x] \subsetneq P_1[x] \subsetneq \ldots \subsetneq P_n[x] \subsetneq (P_n, x)$. We conclude that in the finite dimensional case, we always have $dim(R[x]) \geq dim(R) + 1$. Of course, if R is a field, then R is a field and $R[x]$ is a PID. So it possible that equality is attained. It is also possible $dim(R[x]) > dim(R) + 1$. The next theorem helps us construct bounds on the dimension of a polynomial extension.

Theorem 2.3.6. *[7] Suppose $Q_1 \subsetneq Q_2 \subsetneq Q_3$ is a chain of primes in $R[x]$. Then $Q_1 \cap R \subsetneq Q_3 \cap R$.*

Put another way, the previous theorem states that no more than two primes of a polynomial ring can lie over the same prime in the base ring. It follows that given a chain of primes in a ring R of length n, then any chain of primes in $R[x]$ whose elements lie over the chain in designated in R can have length no greater than $2n+1$. Thus if $dim(R) = n$, then $n + 1 \leq dim(R[x]) \leq 2n + 1$. Further, it has been shown that given any $n \in \mathbb{N}$ and any $m \in \mathbb{N}$ with $n+1 \leq m \leq 2n+1$, then there exists a ring R such that $dim(R) = n$ and $dim(R[x]) = m$. The dimension behavior of polynomial extensions of Noetherian rings are extremely nicely behaved. The following theorem is established in [7]. We point out that this nice behavior is not limited to Noetherian rings.

Theorem 2.3.7. *If R is Noetherian, then $dim(R[x]) = dim(R) + 1$. It follows from the Hilbert Basis Theorem that $dim(R[x_1, \dots x_n]) = dim(R) + n$.*

It should also be noted that the dimension behavior of power series extensions of Noetherian rings are as nicely behaved as their polynomial extensions. In [4] it is shown that if R is a Noetherian ring and $P \in Spec(R)$ with a basis of n elements, then $ht(P) = n$. As every prime is finitely generated, it follows that every prime ideal has finite height. In particular, every maximal ideal has finite height.

Remark 2.3.8. If D is an infinite dimensional UFD, then D must admit a ht-1 prime which is necessarily generated by a prime element. It is not immediately clear that D must admit a ht-2 prime. We again caution the reader against the temptation into thinking that primes always line up nicely as might be suggested in the definition of

Krull dimension. Put another way, given a chain of primes ascending from a given prime ideal, there is no guarantee that this chain is well-ordered. We will soon see that such examples are not so difficult to generate. However, none of these examples involve UFD's. Thus, we think it would be interesting to find an example of a UFD which does not admit a ht-2 prime. We further note that it would be curious to find an example of a 2-dimensional UFD that is non-Noetherian. Should it be demonstrated that every 2-dimensional UFD is Noetherian, then we might conjecture that every finite dimensional UFD is Noetherian.

2.4. Almost Integrality

As the name suggests, the idea of almost integrality is a generalization of integrality. It will be seen that almost integrality is a non-Noetherian notion. That is, integrality and almost integrality collapse into the same meaning under the Noetherian condition, as is shown in Theorem 2.4.5. In addition, almost integrality is a good tool by which we can quickly construct GCD domains that fail to be UFD's. Let us proceed with the definitions we need.

Definition 2.4.1. Let D be a domain with quotient field K. We say that an element $k \in K$ is **almost integral** over D if there exists some $r \in D^*$ such that $k^n r \in D$ for all $n \in \mathbb{N}$. Moveover, the collection of elements of K that are almost integral over D is called the **complete integral closure** of D, denoted $C(D)$. If $D = C(D)$, then we say that D is **completely integrally closed**.

Example 2.4.2. Let D be a domain with quotient field K and let $R = D + xK[x]$. Then every element of K is almost integral over R.

Note in the previous example that if $D \neq K$, then we can easily see that almost integral elements need not be integral. As the name suggest, every integral element is almost integral.

Theorem 2.4.3. *[7] Let D be a domain. Then $\overline{D} \subseteq C(D)$.*

Proof. Suppose $t \in \overline{D}$ and let $m \in \mathbb{N}$. Write $t^n + r_{n-1}t^{n-1} + \ldots + r_1 t + r_0 = 0$ where each $r_i \in R$. By Theorem 2.2.2 we know that $t^m \in R[t^1, \ldots, t^{n-1}]$. Since $t \in K$, then we may write $t = \frac{a}{b}$ for some $a, b \in R$. Thus, $tb^n, t^2b^n, \ldots, t^{n-1}b^n \in R$. Since $t^m \in R[t^1, \ldots, t^{n-1}]$, then we may write $t^m = s_0 + s_1 t + s_2 t^2 + \ldots + s_{n-1}t^{n-1}$ and so $t^m b^n = s_0 b^n + s_1(tb^n) + s_2(t^2b^n) + \ldots + s_{n-1}(t^{n-1}b^n) \in R$ and we are done. □

One clear consequence of Theorem 2.4.3 is that any domain that is completely integrally closed is automatically integrally closed. Happily, the nicest rings in factorization are completely integrally closed.

Theorem 2.4.4. *[11] Every UFD is completely integrally closed.*

Proof. Suppose D is a UFD with quotient field K and assume $t \in K$ is almost integral over D. Choose $r \in D^*$ such that $t^m r \in D$ for all $m \in \mathbb{N}$. We write $t = \frac{a}{b}$ for some $a, b \in D$. Further, since every UFD is a GCD domain, then it can be assumed that $[a, b] = 1$. Now if $r \in U(D)$, then we are done. So we assume $r \notin U(D)$. By Theorem 2.1.1, we let $r = p_1 \cdots p_n$ be a prime factorization. It now suffices to show that $b \in U(D)$. Letting $t^m r = d_m$, then we see that $a^m r = d_m b^m$. Since $[a, b] = 1$, then from Theorem 2.1.19 we know that $b^m | r$ for all m. If $b \notin U(D)$, then by Theorem 2.1.1 there exists some prime $\pi \in D$ that divides b. Hence, $\pi^{n+1} | p_1 \cdots p_n$, contradicting Theorem 1.1.7. □

We have seen that every GCD domain is integrally closed. Thus, a quick way to find a GCD domain that fails to be a UFD is by finding a GCD domain that is not completely integrally closed. We will see in Theorem 3.3.4 that a valuation domain is completely integrally closed if and only if its Krull dimension is less than two. Thus, any two dimensional valuation domain will do the trick.

In the introduction we stated that Noetherian domains cannot distinguish between the ideas of integrality and almost integrality. We establish this fact now.

Theorem 2.4.5. *[7] Let D be a Noetherian domain. Then $\overline{D} = C(D)$.*

Proof. Let D be a Noetherian domain with quotient field K. We already know from Theorem 2.4.3 that integral implies almost integral in any domain. Hence, we need only demonstrate the converse. Let $k \in K$ be almost integral over D and choose a nonzero $r \in D$ such that $k^n r \in D$ for all n. Say $k^n r = x_n$. Then we always have the ideal containment $(x_1, \ldots, x_m) \subseteq (x_1, \ldots, x_{m+1})$. Because D is Noetherian, then $x_{n+1} \in (x_1, \ldots, x_n)$. Thus, $k^{n+1}r = a_1(rk) + a_2(rk^2) + \ldots + a_n(rk^n)$ for some $a_1, \ldots, a_n \in D$. Hence, $k^n = a_1 + a_2 k + \ldots + a_n k^{n-1}$, as desired. □

Some of the properties of integrality carry over into the world of almost integrality. We saw in Theorem 2.2.4 that the integral closure of a domain is a ring. The same is true of complete integral closures.

Theorem 2.4.6. *The complete integral closure of a domain is a ring.*

Proof. Let K be the quotient field of a domain D and suppose $k_1, k_2 \in K$ are almost integral over D. Choose nonzero $r_1, r_1 \in D$ such that $k_1^n r_1, k_2^n r_2 \in D$ for all n. Then $(k_1 k_2)^n (r_1 r_2) \in D$, i.e., $k_1 k_2$ is almost integral over D. Lastly, $(k_1 + k_2)^n (r_1 r_2) = (\sum_{i=0}^{n} z_i k_1^{n-i} k_2^i)(r_1 r_2)$, where each $z_i \in \mathbb{Z}$. Since $k_1^{n-i} r_1, k_2^i r_2 \in D$, then it follows that $k_1 + k_2$ is almost integral over D and we are done. □

Complete integral closures also have an answer to Theorem 2.2.7.

Theorem 2.4.7. *Let D be a domain and suppose T is a completely integrally closed overring of D. Then $C(D) \subseteq T$.*

Proof. Letting K be the quotient field of D, assume $k \in K$ is almost integral over D. Then there exists a nonzero $r \in D$ such that $k^n r \in D$ for all n. As $D \subseteq T$, then

$r, k^n r \in T$. Hence, k is almost integral over T, as well. As T is completely integrally closed, then we have $k \in T$ and we are done. $\square$

Theorem 2.4.8. *Suppose D is a domain with quotient field K. Then $R = D + xK[x]$ and $T = D + xK[[x]]$ are integrally closed if and only if D is integrally closed. Further, R and T are completely integrally closed if and only if $D = K$.*

Proof. We begin with the first statement of the theorem and show that R is integrally closed if and only if D is integrally closed.

($\Rightarrow$) We assume R is integrally closed and let $k \in \overline{D}$. Then $k \in \overline{R}$. As R is integrally closed, then $k \in R$. Thus, $k \in D$ and so $D = \overline{D}$.

($\Leftarrow$) Let us assume D is an integrally closed domain with quotient field $K \neq D$. Because $K[x]$ is an integrally closed overring of R, then from Theorem 2.2.7 we know that $\overline{R} \subset K[x]$. Assume now $0 \neq f \in \overline{R}$ and write $f = f_0 + k_1 x + k_2 x^2 + \ldots k_n x^n$, where $f_0 \in D$ and each $k_i \in K$. Because $f \in \overline{R}$, then it is easily verified that $f_0 \in \overline{D}$. But because D is integrally closed, then we must have that $f_0 \in D$. Thus, R is integrally closed.

We now show that R is completely integrally closed if and only if $D = K$.

($\Rightarrow$) We assume R is completely integrally closed and let $k \in C(D)$. Then $k \in R \Rightarrow k \in D$. So $K = D$.

($\Leftarrow$) If $K = D$, then $R = K[x]$, which is a PID. As every PID is a UFD, we quote Theorem 2.4.4 to finish.

To verify the claims of the theorem with respect to T, simply mimic the arguments above. $\square$

Remark 2.4.9. It has been pointed out that Krull dimension is preserved in integral extensions. It therefore seems natural to ask whether or not Krull dimension is preserved in the complete integral closure of a ring. In light of Theorem 2.4.8, the answer is clearly no. Indeed, since $K[x]$ is a PID, then $K[x]$ is completely integrally closed. By Theorem 2.4.7 it follows $C(R) \subseteq K[x]$. Moreover, since every element of $K \subseteq C(R)$, then Theorem 2.4.6 implies that $K[x] \subseteq C(R)$. Hence, $K[x] = C(R)$. As $K[x]$ is a PID, then we see that the complete integral closure of R is one-dimensional. However, if P is a nonzero prime ideal in D, then $0 \subsetneq xK[x] \subsetneq (P, xK[x])$ is a chain of primes in R of length 2, i.e., $dim(R) > 1$. As an added observation, we note that if R is integrally closed but not completely integrally closed, then R must fail to be Noetherian. Of course, we might have deduced that R is non-Noetherian by simply observing that the ideal $xK[x]$ is not finitely generated.

Here is another application of some these ideas.

Example 2.4.10. Let $\iota = \sqrt{-1}$ and $D = \mathbb{Z}[2\iota]$. We wish to show $C(D) = \mathbb{Z}[\iota]$. First, since ι is integral over D and $\overline{D}$ is a ring, then $\mathbb{Z}[\iota] \subseteq \overline{D}$. Also, $\mathbb{Z}[\iota]$ is integrally closed. By Theorem 2.2.7 it follows that $\overline{D} \subseteq \mathbb{Z}[\iota]$ and so $\overline{D} = \mathbb{Z}[\iota]$. By the Hilbert Basis Theorem, $\mathbb{Z}[x]$ is Noetherian. We can therefore realize $\mathbb{Z}[\iota]$ as a homomorphic image of a Noetherian ring, making $\mathbb{Z}[\iota]$ Noetherian. By Theorem 2.4.5 it follows that $\mathbb{Z}[\iota]$ is completely integrally closed. Now Theorem 2.4.7 assures us that $C(D) \subseteq \mathbb{Z}[\iota]$. Note further that $2\iota, -2, -2\iota, 2 \in D$. Thus, $2\iota^n \in D \Rightarrow \iota \in C(D)$. Theorem 2.4.6 guarantees that $C(D)$ is a ring. Thus, because $\mathbb{Z} \subseteq C(D)$ and $\iota \in C(D)$, then $\mathbb{Z}[\iota] \subseteq C(D)$, as desired.

We saw in Theorem 2.2.20 that a polynomial extension of an integrally closed domain is integrally closed. Conspicuously absent is the determination of whether or not integral closure is preserved in power series extensions. Later we will show that it is quite easy to find examples showing that this need not be the case. Now because the complete integral closure of a domain contains its integral closure, it is clear that the property of being completely integrally closed is stronger than that of being merely integrally closed. Our next result is illustrative of this fact.

Theorem 2.4.11. *[7] Let D be a domain. The following are equivalent:*

a) D is completely integrally closed

b) $D[x]$ is completely integrally closed

c) $D[[x]]$ completely integrally closed

Proof. *a)* $\Rightarrow$ *b)* Let T be the complete integral closure of $D[x]$ and note that Theorem 2.4.7 assures us $T \subseteq K[x]$ since $K[x]$ is completely integrally closed. Now let us say $f = k_0+k_1x+\ldots k_nx^n \in T$ and let $0 \neq r \in D[x]$ such that $f^n r \in D[x]$. If $x|r$, then $r = x^t(r_0 + r_1x + \ldots r_nx^n)$, where $r_0 \neq 0$. Hence, $f_0^n r = (f_0^n x^t)(r_0 + r_1x + \ldots r_nx^n) \in D[x]$. This means $k_0^n x^t r_0 \in D[x] \Rightarrow k_0 r_0 \in D$. Thus, there is no loss in generality in assuming $x \nmid r$. We know $k_0 \in C(D) \Rightarrow k_0 \in D$. Because $D \subseteq T$ and T is a ring, then $f - k_0 = k_1x + k_2x^2 + \ldots k_nx^n \in T$. Using similar logic, we argue that $k_1 \in D$. Continuing inductively we get $k_0, k_1, \ldots k_n \in D$ and so $f \in D[x]$.

a) $\Rightarrow$ *c)* We let T be the complete integral closure of $D[[x]]$. To obtain what we need it will suffice to show that $T \subseteq K[[x]]$ because upon doing so we can argue inductively as in the previous paragraph. Now since $D[[x]] \subseteq K[[x]]$, then we must have that $T \subseteq K((x))$, the quotient field of $K[[x]]$. Because $D[[x]] \subseteq K[[x]]$, then any element in T must also be almost integral over $K[[x]]$. Being a UFD, $K[[x]]$ is completely integrally closed. Thus, $T \subseteq K[[x]]$ and we are done.

The implications *b)* $\Rightarrow$ *a)* and *c)* $\Rightarrow$ *a)* are clear. □

It is well know that if D is a UFD, then $D[[x]]$ need not be a UFD. However, Theorem 2.4.4 and Theorem 2.4.11 assure us that a power series extension of a UFD is completely integrally closed. All the more so we know that any power series extension of a UFD must also be integrally closed.

As an added observation, let us imagine that R is an integrally closed Noetherian domain. Then $R[[x]]$ is also Noetherian and completely integrally closed. We will see later how power series extensions of GCD domains need not be integrally closed. Therefore, any such GCD domain must not only fail to be Noetherian, but they cannot even be atomic.

2.5. Branched Primes

In Chapter 3 we are going to find some important uses for the notion of branched prime ideal. Before giving the definition we need, let us recall that an ideal Q in a ring R is said to be **primary** if, given $a, b \in R$ and $ab \in Q$, then either $a \in Q$ or $b^n \in Q$ for some $n \in \mathbb{N}$. Moreover, it will be recalled that if Q is primary in a ring R, then $\sqrt{Q} \in Spec(R)$. In this event, if we designate $P = \sqrt{Q}$, then we say that Q is **P-primary**.

Definition 2.5.1. Let R be a ring and $P \in Spec(R)$. We call P **branched** if there exists a P-primary ideal other than P itself.

It is obvious from this definition that the zero-ideal of any domain is never branched. However, if a ring has zero-divisors, a minimal prime may or may not be branched. For example, in $\mathbb{Z}/4\mathbb{Z}$, $(2 + 4\mathbb{Z})$ is an unbranched prime. However, in $\mathbb{Z}/8\mathbb{Z}$, $(2 + 8\mathbb{Z})$ is branched. Here is another way to find branched primes. Given a ring R and $P \in Spec(R)$, it is not difficult to show that $P = \sqrt{P^n}$ for any $n \in \mathbb{N}$. Hence, so long as $P^n \neq P$, then P would be a branched prime of R. In particular, any nonzero prime ideal in a PID would have to be branched. We will prove that every nonzero prime ideal in a Noetherian domain is branched. Further, if V is any finite-dimensional valuation domain, then every nonzero prime ideal is branched. In due course, we are going to present a result which characterizes branched primes in valuation domain's and this characterization will allow us to characterize what it means to be a branched prime in any Prüfer domain. This characterization will also allow us to easily produce examples of rings with unbranched primes. But before we do any of this and because the notion of a branched prime is defined in terms of primary ideals, we are going to run through some facts regarding primary ideals. We begin with a useful characterization of primary ideals.

Theorem 2.5.2. *[10] Let Q and P be ideals in a ring R with $P \in Spec(R)$. Then Q is P-primary if and only if*

a) $Q \subsetneq P \subseteq \sqrt{Q}$; and

b) if $ab \in Q$ and $a \notin Q$, then $b \in P$

Proof. ($\Rightarrow$) We assume Q is P-primary. Then certainly $Q \subseteq P = \sqrt{Q}$. So suppose we have $ab \in Q$ with $a \notin Q$. Because Q is primary, it follows that $b^n \in Q$ for some $n \in \mathbb{N}$. Hence, $b^n \in P \Rightarrow b \in P$ because $P \in Spec(R)$.

($\Leftarrow$) Assuming the conditions of the converse, we wish to show that Q is P-primary, i.e., we wish to show that Q is primary and $\sqrt{Q} = P$. Suppose first that $ab \in Q$ with $a \notin Q$. Then $a \in P$. But because $P \subseteq \sqrt{Q}$, then we know $b \in \sqrt{Q}$. Hence, $b^n \in Q$ for some $n \in \mathbb{N}$, making Q a primary ideal. To show that $P = \sqrt{Q}$, we need only to prove $\sqrt{Q} \subseteq P$. So choose $x \in \sqrt{Q}$. Then $x^n \in Q$ for some $n \in \mathbb{N}$. But then $x^n \in P \Rightarrow x \in P$ and we are done. □

Here we present another useful tool. It states that localization commutes with radicals.

Theorem 2.5.3. *[10] Suppose R is a ring, $S \subsetneq R$ is multiplicatively closed, and $I \leq R$. Then $\sqrt{S^{-1}I} = S^{-1}\sqrt{I}$.*

Proof. Suppose $\frac{a}{s} \in S^{-1}\sqrt{I}$ with $s \in S$ and $a \in \sqrt{I}$. Then $a^n \in I$ for some n so that $(\frac{a}{s})^n \in S^{-1}I$. Hence, $\frac{a}{s} \in \sqrt{S^{-1}I}$. For the reverse containment, we assume $\frac{a}{s} \in \sqrt{S^{-1}I}$. Then $\frac{a^n}{s^n} = \frac{i}{\sigma}$ for some $i \in I$ and $\sigma \in S$. Hence, $a^n = \frac{j}{\sigma}$ for some $j \in I$. It now follows that $a^n s' \in I$ for some $s' \in S$ and so $a^n(s')^n \in I$, as well. Write $as' = k$

and note $k \in \sqrt{I}$. We now have $as's's = ks's \Rightarrow (as's - ks)s' = 0 \Rightarrow \frac{a}{s} = \frac{k}{ss'} \in S^{-1}\sqrt{I}$ and we are done. $\square$

We are going to be making implicit use of the previous theorem on a number of occasions, particularly when we are localizing at a prime ideal. For example, given a ring R, $P \in Spec(R)$, and an ideal $I \leq R$, we can say $\sqrt{IR_P} = \sqrt{I}R_P$. One should be careful here with notation. In the event that $IR_P \neq R_P$, we cannot conclude that $I \subseteq P$. This does mean, however, that there is some ideal $J \leq R$ such that $J \subseteq P$ and $JR_P = IR_P$. For a more concrete illustration of this, let F be a field and consider the ring $R = F[x, y]$. Let $P = xR$ and $I = (x^2, y)R$. Then $I \nsubseteq P$ but $IR_P = x^2R_P$.

Theorem 2.5.4. *[10] Suppose $f : R \to T$ is a ring epimorphism, $J \leq T$, and $I = f^{-1}(J)$. Then the ideal I is primary in R if and only if J is primary in T. Further, if J is P-primary, then I is $f^{-1}(P) - primary$.*

Proof. We begin by proving the first statement of the theorem. Assuming I is primary, we suppose $ab \in J$ with $a \notin J$. Let $x, y \in R$ such that $f(x) = a$ and $f(y) = b$. Then $xy \in I$ with $x \notin I$. Hence, $y^n \in I$ for some $n \in \mathbb{N}$, which implies $b^n \in J$. Conversely, we assume J is primary and let $x, y \in R$ with $xy \in I$ and $x \notin I$. Then $f(xy) \in J$ and $f(x) \notin J$. So $f(y^n) \in J \Rightarrow y^n \in I$.

To establish the second statement of the theorem, we assume J is P-primary. In particular, J is primary and so I is primary in R. Now $\sqrt{I} = f^{-1}(\sqrt{J}) = f^{-1}(P)$ and so we are done. $\square$

Theorem 2.5.5. *Suppose H is a primary ideal in a ring R, $P \in Spec(R)$, and $H \subseteq P$. If HR_P is $PR_P - primary$, then H is P-primary.*

Proof. Because H is primary, then we already know $\sqrt{H} \in Spec(R)$ and so $\sqrt{H}R_P = \sqrt{HR_P} \in Spec(R_P)$. But $\sqrt{HR_P} = PR_P$. By correspondence, $\sqrt{H} = P$. $\square$

Our next three results are correspondence theorems. The first two are regarding primary ideals in quotient rings and localizations. The third one characterizes branched primes in localizations.

Theorem 2.5.6. *Let D be a domain and $P \in Spec(D)$. Then there exists a 1-1 correspondence between the set S_1 of primary ideals contained in P and the set S_2 of primary ideals of the localization D_P given by $Q \leftrightarrow QD_P$.*

Proof. First, recall that every ideal of D_P is of form ID_P, where $I \leq D$. Secondly, assume $Q \in S_1$ so that $QD_P \leq D_P$. Now choose $\frac{ab}{s_1 s_2} \in QD_P$ for some $a, b \in D$ and $s_1, s_2 \in D - P$. Then $\frac{ab}{s_1 s_2} = \frac{q}{s_3}$ for some $q \in Q$ and $s_3 \in D - P$. Now $s_3 ab = s_1 s_2 q \in Q$. Wishing to show QD_P is primary, we assume $\frac{a}{s_1} \notin QD_P$ so that $a \notin Q$. Because $a \notin Q$, then $s_3 b \in \sqrt{Q} \subseteq P$. But $s_3 \notin P$ and so we must have $b \in \sqrt{Q}$. Hence, $b^n \in Q$ for some $n \in \mathbb{N}$, whence $(\frac{b}{s_2})^n \in QD_P$.

The argument in the preceding paragraph shows that if $Q \in S_1$, then $QD_P \in S_2$. With this fact in hand, it is obvious that $\phi : S_1 \to S_2$ given by $\phi(Q) = QD_P$ is well-defined. To show that ϕ is monic, we assume $Q_1 D_P = Q_2 D_P$ for some $Q_1, Q_2 \in S_1$. Letting $q \in Q_1$, then we have $q \in Q_2 D_P$ so that $q = \frac{q'}{s}$ for some $q' \in Q_2$ and $s \in D_P$. Since $\sqrt{Q_2} \subseteq P$ and $s \notin P$, then by Theorem 2.5.2 it follows that $q \in Q_2$. Similarly, $Q_2 \subseteq Q_1$ and so $Q_1 = Q_2$, i.e., ϕ is monic.

We now show that ϕ is epic and choose $QD_P \in S_2$, where $Q_2 \leq D$. Let $I = QD_P \cap D$. As I is an ideal of D and $I \subseteq P$, we wish to show $I \in S_1$. Assume $a, b \in D$ with $ab \in I$ and $a \notin I$. Then $a \notin QD_P$, which forces $b^n \in QD_P \cap D$. So $I \in S_1$ and $I \subseteq Q$. Hence, $ID_P \subseteq QD_P$. Now should we have $q \in Q$, then $q \in QD_P \cap D = I$. So $Q \subseteq I$, whence $QD_P \subseteq ID_P$. Therefore, $QD_P = \phi(I)$ and we are done. □

Theorem 2.5.7. *Let R be a ring and $I \leq R$. Then there is a 1-1 correspondence between the primary ideals of R that properly contain I and the primary ideals of R/I*

given by $Q \leftrightarrow Q/I$.

Proof. Assume Q is an ideal containing I. We first show that Q is primary in R if and only if Q/I is primary in R/I. This will show that the map $Q \to Q/I$ is well-defined and epic. Assuming Q is primary, we assume $ab + I \in Q/I$ and $a + I \notin Q/I$. Then $ab \in Q$ and $a \notin Q$. Since Q is primary, then $b^n \in Q$ for some n. Hence, $b^n + I \in Q/I$, making Q/I primary in R/I. Conversely, assume Q/I is primary in R/I and let $\pi : R \to R/I$ be the natural projection. By Theorem 2.5.4 it follows that $Q = \pi^{-1}(Q/I)$ is primary.

To show that the map is monic, simply observe that if $Q_1/I = Q_2/I$, then $Q_1 = \pi^{-1}(Q_1/I) = \pi^{-1}(Q_2/I) = Q_2$. □

Theorem 2.5.8. *Let R be a ring and $P \in Spec(R)$. Then there exists a 1-1 correspondence between the set S_1 of branched primes contained in P and the set S_2 of branched primes in R_P given by $Q \leftrightarrow QR_P$.*

Proof. As a result of prime correspondence in localizations, it will be enough to show that $Q \in S_1$ if and only if $QR_P \in S_2$. Suppose first $Q \in S_1$. Then there exists a Q-primary ideal J distinct from Q. By correspondence, JR_P is primary and $JR_P \subsetneq QR_P$. Further, $\sqrt{JR_P} = \sqrt{J}R_P = QR_P$ and so $QR_P \in S_2$.

Conversely, assume $QR_P \in S_2$ for some prime ideal $Q \subseteq P$. Let JR_P be a QR_P-primary ideal distinct from QR_P. We may make the added assumption that $J \subseteq Q$. Because $JR_P \subsetneq QR_P$, then $J \subsetneq Q$. Since JR_P is QR_P-primary, then by Theorem 2.5.5 we know J is Q-primary. Hence, $Q \in S_1$. □

Theorem 2.5.9. *Let D be a quasi-local domain with nonzero maximal ideal M. Then every ideal of D is M-primary if and only if dim(D)=1.*

Proof. If every nonzero ideal of D is M-primary, then there can be only one nonzero prime ideal because the radical of any ideal is the intersection of the primes containing it. For the converse, choose a nonzero $I \leq D$. Because $dim(D) = 1$ and D is quasi-local, then $\sqrt{I} = M$. Hence, we wish only to show that I is primary. Suppose $ab \in I$ with $a \notin I$. By Theorem 2.5.2 it suffices to show $b \in M$. If $b \notin M$, then we must have $b \in U(D)$. But then $ab \in I \Rightarrow a \in I$, a contradiction. □

Definition 2.5.10. Let R be a ring and $Q, P \in Spec(R)$. We say Q and P are **adjacent** primes if there is no prime lying properly between them. If Q and P are adjacent primes, we may also call them **adjacent neighbors**.

Theorem 2.5.11. *Suppose R is a ring and $P \in Spec(R)$. If there exists some $Q \in Spec(R)$ such that $Q \subsetneq P$ is a chain of adjacent primes, then P is branched.*

Proof. If $Q \subsetneq P$ are adjacent primes, then R_P/QR_P is 1-dimensional and quasi-local. By Theorem 2.5.9 it follows that PR_P/QR_P is branched. Thus, we may let IR_P/QR_P be a PR_P/QR_P-primary distinct from PR_P/QR_P for some ideal $I \subsetneq P$. From Theorem 2.5.7 we have that IR_P is PR_P-primary. Now use Theorem 2.5.5 to conclude that I is P-primary. Since $I \subsetneq P$, we then know that P is branched. □

The hypotheses in Theorem 2.5.11 may seem a little strange at first, but we will soon find examples of rings that admit primes having no immediate predecessors. Also, we will soon find examples of rings which admit chains of primes which are not only infinite but are uncountable. Moreover, if R is any Noetherian ring with $dim(R) > 0$, then every prime admits an adjacent neighbor. It seems natural to wonder whether or not there exists a ring in which all of the prime ideals admit no immediate predecessors. Our next result answers this question.

Theorem 2.5.12. *Let R be a ring such that $dim(R) > 0$. Then R admits a chain of adjacent primes.*

Proof. Let $x \in R^* - U(D)$ and let $M \in Max(R)$ such that $x \in M$. If MR_M is finite dimensional, then by prime correspondence we are done. So we assume MR_M

is infinite dimensional. Now let $S = \{s, s^2, s^3, \ldots\}$ and note that S is multiplicatively closed in R_M. By Theorem 1.4.1 we let $PR_M \in Spec(R_M)$ such that PR_M is maximal with respect to missing S. By correspondence, it then follows that $PR \subsetneq (P, s)R \subseteq MR$. By maximality of P and prime correspondence, there does not exist a $Q \in Spec(R)$ such that $P \subsetneq Q \subseteq (P, s)$. Now let σ be the set of prime ideals contained in M and that contain S. We partially order σ by saying $P_1 \leq P_2$ whenever $P_2 \subseteq P_1$. Letting $(C_i)_\Lambda$ be a chain in σ, note that $\bigcap_\Lambda C_i \in \sigma$. By Zorn, we may let μ be maximal in σ. Then $P \subsetneq (P, s) \subseteq \mu$. Suppose now $T \in Spec(R)$ such that $P \subseteq T \subseteq \mu$. If $s \notin T$, then by maximality of P we would have that $P = T$. If $s \in T$, then by maximality of μ we would have that $T = \mu$. Hence, $P \subsetneq \mu$ is a chain of adjacent primes. ☐

Theorem 2.5.13 is a sharpening of Theorem 2.5.12 in that we can find adjacent primes "between" any two distinct and comparable primes.

Theorem 2.5.13. *Let R be a ring and let $Q \subsetneq P$ be a chain of primes in R. Then there exist adjacent primes $Q' \subsetneq P'$ such that $Q \subseteq Q' \subsetneq P' \subseteq P$.*

Proof. First, note that $D = (R/Q)_{(P/Q)}$ is a ring. Now every prime ideal in D is of form $(T/Q)_{P/Q}$, where $Q \subseteq T \subseteq Q$ is a chain of primes in R. Now use Theorem 2.5.12 to find a chain of adjacent primes in D given by $(Q'/Q)_{(P/Q)} \subsetneq (P'/Q)_{(P/Q)}$. By correspondence, we then have that $Q \subseteq Q' \subsetneq P' \subseteq P$, as desired. ☐

An immediate consequence of Theorem 2.5.11 and Theorem 2.5.12 is that every ring that is not zero-dimensional admits a branched prime. In particular, given any chain $Q \subsetneq P$ of primes in a ring R, then there exists a branched prime T such that $Q \subsetneq T \subseteq P$. It is also clear from Theorem 2.5.13 and Theorem 2.5.11 that if R admits an infinite chain of primes, then R must also admit an infinite chain of branched primes.

Our next result gives us a means of producing a wealth of examples of branched prime ideals. In particular, it becomes evident that every nonzero prime ideal in a finite dimensional domain is branched.

Theorem 2.5.14. *Suppose R is a finite dimensional ring and $P \in Spec(R)$ is non-minimal, i.e., $ht(P) > 0$. Then P is branched.*

Proof. Since R is finite dimensional, then R_P is both finite dimensional and quasi-local. Now since $dim(R_P) < \infty$, then we can let $Q \subsetneq P$ be adjacent primes in R. By Theorem 2.5.9, every nonzero ideal of R_P/QR_P must be PR_P/QR_P-primary. Let IR_P/QR_P be a nonzero non-maximal ideal with $Q \subsetneq I \subsetneq P$. Then IR_P/QR_P is PR_P/QR_P-primary. By correspondence, IR_P is PR_P-primary. Again, by correspondence we know that $IR_P = MR_P$ for some primary ideal $M \leq R$ such that $M \subseteq P$. As $IR_P \subsetneq PR_P$, then we must have $M \subsetneq P$. We now know M is a P-primary ideal distinct from P, i.e., P is branched. $\square$

The previous theorem shows how easily we can produce examples of branched primes. We have only to turn to finite dimensional rings. But it opens the door to an even greater wealth of examples than what might at first be evident.

Theorem 2.5.15. *Suppose R is a ring such that $0 < ht(M) < \infty$ for all $M \in Max(R)$. Then every non-minimal prime ideal of R is branched.*

Proof. Suppose $P \in Spec(R)$ is non-minimal. As P is contained in some maximal ideal of finite height, then we know R_P must be finite dimensional. Since PR_P is non-minimal and R_P is finite dimensional, then from Theorem 2.5.14 it follows that PR_P is branched. By correspondence, P must be branched in R. $\square$

We now know that every non-minimal prime ideal in any finite dimensional ring is branched. Further, because every maximal ideal in a Noetherian ring has finite height, then every non-minimal prime in a Noetherian ring must be branched. Because Noetherian rings can be infinite dimensional, then we can easily produce examples of infinite dimensional domains in which every nonzero prime ideal is branched. It also follows from these observations that any ring which admits a non-minimal unbranched prime must be infinite dimensional and non-Noetherian. Additionally, we point out that it is possible to produce examples of infinite dimensional valuation domain's where every nonzero prime ideal is branched. Further, we can produce examples of infinite dimensional valuation domain's which admit prime ideals P_n and Q_n such that $ht(P) = n = depth(Q)$ for all $n \in \mathbb{N}$. It is not so difficult to produce such examples with the aid of groups of divisibility, which will be examined in Chapter 3. We put off producing these example until then.

We now characterize branched primes in a valuation domain.

Theorem 2.5.16. *[7] Let V be a valuation domain and $0 \neq P \in Spec(V)$. The following are equivalent:*

a) P is branched;

b) There exists an ideal $A \leq V$ such that $A \subsetneq P$ and $\sqrt{A} = P$;

c) P is the radical of a principal ideal;

d) P is not the union of the set of primes of V properly contained in P;

e) There exists a prime ideal $M \subsetneq P$ such that M and P are adjacent.

Proof. $a) \Rightarrow b)$ Clear.

$b) \Rightarrow c)$ Let $x \in P - A$. Then $A \subsetneq xV \subseteq P \Rightarrow P = \sqrt{A} \subseteq \sqrt{xV} \subseteq P$, i.e., $P = \sqrt{xV}$.

$c) \Rightarrow d)$ Suppose $P = xV$ for some $x \in P$ and let $S = \{Q \in Spec(V) | Q \subsetneq P\}$. Then $x \notin \bigcup_{Q \in S} Q$ and $\bigcup_{Q \in S} Q \in Spec(V)$. So $\bigcup_{Q \in S} Q \subsetneq xV \subseteq P$.

$d) \Rightarrow e)$ This is obvious because the prime ideals of a valuation domain are linearly ordered by inclusion.

$e) \Rightarrow a)$ Suppose $M \in Spec(V)$ such that $M \subsetneq P$ and M is adjacent to P. Then $dim(V_P/MV_P) = 1$. Let QV_P/MV_P be a nonzero nonmaximal ideal and let $\pi : V_P \to V_P/MV_P$ be the natural projection. By Theorem 2.5.14 we know QV_P/MV_P is PV_P/MV_P-primary. Letting $L = \pi^{-1}(QV_P/MV_P)$, we have $MV_P \subsetneq L \subsetneq PV_P$. Now we can use Theorem 2.5.4 to deduce that L is PV_P-primary. Theorem 2.5.6 assures us $L = HV_P$ for some primary ideal $H \subseteq P$. As $L \subsetneq PV_P$, then $H \subsetneq P$. Now quoting Theorem 2.5.5 we see that H is P-primary, from which it follows from the definition that P is branched. $\square$

In Chapter 3 we are going to be exploring groups of divisibility. Among other things, we can use these groups to easily produce examples of valuation domain's with interesting properties. So for now let us make the assumption that we can produce an example of an infinite dimensional valuation domain with the property that every nonmaximal prime ideal has finite height. Then the maximal ideal M of this ring must be unbranched and Theorem 2.5.16 makes the verification of this quite simple. Indeed, letting $P \subsetneq M$ be a chain of primes in this valuation domain, we know that $ht(P) < \infty$. Thus, there must exist some $Q \in Spec(V)$ such that $P \subsetneq Q \subsetneq M$. This means there is no prime adjacent to M, making M unbranched. On the other hand, using groups of divisibility we can construct examples of valuation domain's in which the only unbranched prime ideal is the zero-ideal. One way to do this would be to construct our valuation domain in such a way that the zero ideal is the only prime that does not have finite depth. Moreover, we can use these groups to construct examples of valuation domain's such that for any given $n \in \mathbb{N}$, then there exists a valuation domain with precisely n unbranched primes. Pushing a little further, we can produce examples of valuation domain's whose branched primes form an infinitely countable set while the set unbranched primes is uncountable.

Now, having characterized the branched primes of a valuation domain, we can use Theorem 2.5.16 to characterize the branched primes of Prüfer domains. This should not be too surprising given the intimate connections between valuation domain's and Prüfer domains. It is interesting to compare and contrast the following theorem to Theorem 2.5.16 given that we have widened our scope from valuation domains to the class of Prüfer domains. The proof of this theorem highlights the strong ties between Prüfer domains and valuation domain's.

Theorem 2.5.17. *[7] Suppose D is a Prüfer domain and $O \neq P \in Spec(D)$. The following are equivalent:*

a) P is branched;

b) There exists an ideal $A \subsetneq P$ such that $\sqrt{A} = P$;

c) P is a minimal prime of a principal ideal;

d) P is a minimal prime of a finitely generated ideal;

e) P is not the union of the primes properly contained in P;

f) There exists a prime ideal $M \subsetneq P$ such that M and P are adjacent.

Proof. $a) \Rightarrow b)$ Clear.

$b) \Rightarrow c)$ Choose $a \in P - A$. Then $AD_P \subsetneq aD_P \subseteq PD_P$. Suppose $Q \in Spec(D)$ such that $a \in QD \subsetneq PD$. Then $aD_P \subseteq QD_P \subseteq PD_P$. But $\sqrt{A} = P \Rightarrow \sqrt{AD_P} = PD_P$. So $AD_P \subseteq QD_P$, whence $QD_P = PD_P$. By correspondence, this forces $Q = P$.

$c) \Rightarrow d)$ Clear.

$d) \Rightarrow e)$ We now assume P is a minimal prime of a finitely generated ideal J. Since J is finitely generated in D, then JD_P is finitely generated in D_P. But D_P is a valuation domain, whence JD_P is principal. Hence, PD_P is the radical of a principal ideal of D_P. Using Theorem 2.5.16 we can see that PD_P is not the union of the primes properly contained in PD_P. By correspondence, it follows that P is not the union of the primes which it properly contains.

$e) \Rightarrow f)$ This is clear upon observing that since D is Prüfer, then the primes of D contained in P must be linearly ordered by inclusion.

$f) \Rightarrow a)$ Suppose we have a prime ideal $M \subsetneq P$ such that M and P are adjacent. Then PV_P must be branched. Thus, there exists a PD_P-primary ideal distinct from PD_P. By correspondence and Theorem 2.5.5, there exists a P-primary ideal distinct from P, i.e., P is branched.

$\square$

CHAPTER 3. GROUPS OF DIVISIBILITY

3.1. Partially Ordered Abelian Groups

In this section we lay some groundwork which will allow us to streamline some of our arguments later on. We begin with the definition of partially ordered abelian groups.

Definition 3.1.1. Suppose $(G, +)$ is an abelian group and $(G, \leq)$ is a partially ordered set. We say G is a **partially ordered abelian group** (POAG) if the relation $\leq$ is compatible with the group operation of G, i.e., given any $a, b, c \in G$ with $a \leq b$, then $a + c \leq b + c$. If G is a POAG and $(G, \leq)$ is linearly ordered, then we say G is a **linearly ordered abelian group** (LOAG).

Examples of such groups are easy to come by. Of course $\mathbb{R}$ or any of its subgroups can be viewed as a LOAG. In fact, it is not difficult to show that any subgroup of a POAG is again a POAG with the inherited order. Letting $G = \mathbb{Z} \oplus \mathbb{Z}$, we define $(a, b) \leq (x, y)$ whenever $a \leq x$ and $b \leq y$. Then G is a POAG but not a LOAG because we can find non-comparable elements, e.g., $(1, 0)$ and $(0, 1)$. In [7] it is shown that every torsion-free abelian group admits a linear ordering. It is also easy to see that every abelian group admits at least one partial ordering, namely, the trivial ordering, i.e., $a \leq b$ if and only if $a = b$. In some cases, the trivial ordering is the only ordering that an abelian group admits.

Theorem 3.1.2. *Suppose G is a POAG. If G is a torsion group, then G admits only the trivial ordering. In particular, finite abelian groups can be partially ordered only via the trivial ordering.*

Proof. Suppose $0 < a$ in G. Then by compatibility of the group operation we must have that $na > 0$ for all $n \in \mathbb{N}$, a contradiction since G is a torsion group. Thus,

$0 \leq a \Rightarrow 0 = a$. Assume now that $a \leq b$ for some $a, b \in G$. Then $0 \leq b - a \Rightarrow b - a = 0 \Rightarrow a = b$. $\square$

For the sake of simplicity we will assume that if G is an abelian group and $H \leq G$, then $\pi : G \to G/H$ always denotes the canonical projection. Further, if G is a POAG, then we will say $G_+ = \{x \in G | x \geq 0\}$. Lastly, if G is an abelian group, then a **positive subset** of G is a subset $P \subseteq G$ such that:

a) $0 \in P$

b) $P \cap (-P) = \{0\}$

c) $P + P \subseteq P$.

Theorem 3.1.3. *[7] Let G be an abelian group and P a positive subset of G. The relation $\leq$ given by $a \leq b$ if and only if $b - a \in P$ defines a partial ordering on G that is compatible with the group operation on G.*

Proof. Since $0 \in P$, then clearly $a \leq a$ for all $a \in G$. Suppose $a \leq b$ and $b \leq a$. Then $a - b, b - a \in P \Rightarrow b - a \in P \cap (-P) = \{0\} \Rightarrow a = b$. Assume now $a \leq b \leq c$ so that $b - a, c - b \in P$. Then $c - a = (c - b) + (b - a) \subseteq P + P \subseteq P$. So $\leq$ is a partial ordering on G. To demonstrate compatibility of the relation with operation on G we take$a, b, c \in G$ with $a \leq b$. We have $b - a \in P \Rightarrow (b + c) - (a + c) \in P \Rightarrow a + c \leq b + c$, as desired.

$\square$

Definition 3.1.4. Let S be a partially ordered set and $H \subseteq S$. We say that H is a **convex** subgroup of S if, given any $h_1, h_2 \in H$ and $s \in S$ with $h_1 \leq s \leq h_2$, then $s \in H$.

Theorem 3.1.5. *[7] Let G be a POAG, $H \leq G$, and $\pi : G \to G/H$. Then $\pi(G_+)$ is a positive subset of G/H if and only if H is convex in G.*

Proof. ($\Rightarrow$) Suppose $\pi(G_+)$ is a positive subset of G/H. Let $h_1, h_3 \in H$ and $h_3 \in G$ such that $h_1 \leq h_2 \leq h_3$. Then $h_2 - h_1 \geq 0$ and $h_2 - h_3 \leq 0$. So $h_2 - h_1 + H \in \pi(G_+)$ and $h_2 - h_3 + H \in -\pi(G_+)$. But $h_2 - h_1 + H = h_2 + H = h_2 - h_3 + H$ since $h_1, h_3 \in H$. So $h_2 + H \in \pi(G_+) \cap (-\pi(G_+)) = \{0 + H\}$ since $\pi(G_+)$ is a positive subset of G/H. Hence, $h_2 \in H$, as desired.

($\Leftarrow$) Suppose H is convex in G. We wish to show that $\pi(G_+)$ is a positive subset of G/H. It is clear that $0 + H \in \pi(G_+)$ and $\pi(G_+) + \pi(G_+) \subseteq \pi(G_+)$. Assume $g + H \in \pi(G_+) \cap (-\pi(G_+))$. Then $p + H = g + H = n + H$ for some $p \in G_+$ and $n \in -G_+$. We now have $g - p, g - n \in H$. As $p \geq 0$ and $n \leq 0$, then $g - p \leq g \leq g - n$. Using the convexity of H we may conclude $g \in H$. $\square$

The upshot of the previous two theorems is that we can regard the factor group G/H as a POAG whenever G is a POAG and $H \leq G$ is convex. For the sake of clarity, let us describe the partial ordering on G/H precisely. We say $a + H \leq b + H$ whenever $b - a + H \in \pi(G_+)$. For the remainder of the paper, we will require that for any convex subgroup H of a POAG G, the factor group G/H is to be regarded as a POAG and the partial ordering will always be assumed to be that which is induced by $\pi(G_+)$. Before moving further, we need the following terminology.

Definition 3.1.6. Let G_1 and G_2 be POAG's and let $\phi : G_1 \to G_2$ be a group map. We say that ϕ is **order-preserving** if whenever $a, b \in G_1$ with $a \leq b$, then $\phi(a) \leq \phi(b)$. If ϕ is both order-preserving and monic (epic), then we say ϕ is an **order monomorphism (epimorphism)**. If ϕ is both order-preserving and an isomorphism, then we say G_1 and G_2 are **order isomorphic** and we denote this by writing $G_1 \cong_o G_2$.

We remark that if G_1 and G_2 are POAG's and $\phi : G_1 \to G_2$ is an order-preserving group map, then for the sake of brevity we will say that ϕ is an **order homomorphism** or an **order map**. It is easily verified that a composition of order maps is again an order map. Moreover, if $\phi : G_1 \to G_2$ is an order map, then it may happen that $\phi(a) = 0$ even when $a \in G_+$. This does not contradict the definition because $a \geq 0$ and $\phi(a) \geq 0$. Also, while convexity is preserved under order isomorphisms, convexity need not be preserved under order monomorphisms. To see this, simply look at the inclusion map $\mathbb{Z} \to \mathbb{Q}$

Example 3.1.7. Let $G_1 = \mathbb{Z} \oplus \mathbb{Z}$ in the lexicographic order and let $G_2 = \mathbb{Z} \oplus \mathbb{Z}$ in the product order. Then as groups we have that $G_1 \cong G_2$. However, $G_1 \ncong_o G_2$ because G_1 is linearly ordered while G_2 is not.

Theorem 3.1.8. *Suppose G and L are POAG's and $\phi : G \to L$ is a group map. The following are equivalent.*

a) ϕ is order-preserving

b) Whenever $a \in G_+$, then $\phi(a) \geq 0$.

Proof. That a) implies b) is obvious. Conversely, assume that whenever $a \in G_+$, then $\phi(a) \geq 0$. Suppose $a \leq b$ in G. Then $b - a \in G_+ \Rightarrow \phi(b) - \phi(a) = \phi(b - a) \geq 0$, i.e., $\phi(b) \geq \phi(a)$. □

Theorem 3.1.9. *[7] Let G be a POAG and $H \leq G$ such that $H_+ = H \cap G_+ \neq 0$. The following are equivalent:*

a) H is convex in G

b) H_+ is a convex subset of G_+.

Proof. That $a) \Rightarrow b)$ is clear. Assuming the hypotheses of b), we let $h_1, h_2 \in H$ and $g \in G$ such that $h_1 \leq g \leq h_2$. Then $0 \leq g - h_1 \leq h_2 - h_1$. Since $h_2 - h_1 \in H_+$, then $g - h_1 \in H_+$. In particular, $g - h_1, h_1 \in H$ and so $g \in H$. □

Our next theorem allows us to determine under what circumstances the natural projection $G \to G/H$ is order preserving.

Theorem 3.1.10. *[7] Suppose H is a convex subgroup of a G, a POAG. Then the projection $\pi : G \to G/H$ is order-preserving. Further, given $a, b \in G$, then $a + H \leq b + H$ if and only if $a \leq b + h$ for some $h \in H$.*

Proof. Suppose $a, b \in G$ with $a \leq b$. Then $b - a \in G_+ \Rightarrow b - a + H \in \pi(G_+)$. This implies $a + H \leq b + H$. Thus, π is order-preserving. We now prove the second statement of the theorem. We let $a, b \in G$ and wish to show $a + H \leq b + H$ if and only if $a \leq b + h$ for some $h \in H$

($\Rightarrow$) Suppose first that $a+H \leq b+H$ so that $b-a+H \geq H$. Then $b-a \in \pi(G_+)$ and we can say $b - a + H = p + H$ for some $p \in G_+$. Now $b - a - p = h$ for some $h \in H$. We then conclude $b \geq b - p = a + h$, as desired.

($\Leftarrow$) Assume $a \leq b+h$ for some $h \in H$. Then $0 \leq b-a+h$, i.e., $b-a+h \in G_+$. Hence, $b-a+h+H \in \pi(G_+)$. Thus, $b-a+h+H \geq H$. But $b-a+h+H = b-a+H$ since $h \in H$. Hence, $b - a + H \in \pi(G_+)$ and so $a + H \leq b + H$. $\square$

Theorem 3.1.11. *If G is a LOAG and $H \leq G$ is convex, then G/H is a LOAG.*

Proof. Suppose $a + H, b + H \in G/H$. Since G is a LOAG, then we may assume $a \leq b$ in G. As $\pi : G \to G/H$ is order-preserving, then $a + H = \pi(a) \leq \pi(b) = b + H$. $\square$

In dealing with branched primes and Krull dimension, we have already seen the power of correspondence theorems. Recall that in any group G with normal subgroup H, then there is a 1-1 correspondence between the subgroups of G which contain H and subgroups of G/H given by $Q \leftrightarrow Q/H$. Our next result is a correspondence theorem which characterizes convexity in partially ordered factor groups.

Theorem 3.1.12. *Suppose G is a POAG and $H \leq G$ is convex. Then there is a 1-1 correspondence between the convex subgroups of G which contain H and the convex subgroups of G/H given by $Q \leftrightarrow Q/H$.*

Proof. Assume $H \leq Q \leq G$. From the remarks preceding the theorem, it suffices to show that Q/H is convex in G/H if and only if Q is convex in G. For the forward implication, we assume $0 \leq g \leq q$ for some $g \in G, q \in Q$. By Theorem 3.1.10 it follows that $H \leq g + H \leq q + H$. Now use the convexity of Q/H to deduce that $g + H \in Q/H$, i.e., $g \in Q$. That Q is convex now follows from Theorem 3.1.9.

For the converse, we assume Q is convex in G and $H \leq g + H \leq q + H$ for some $q \in Q$. From Theorem 3.1.10 we know that $0 \leq g + h_1$ and $g \leq q + h_2$ for some $h_1, h_2 \in H$. Thus, $0 \leq g + h_1 \leq q + h_2 + h_1 \Rightarrow -h_1 \leq g \leq q + h_2$. Since Q is convex and contains H, then $g \in Q$. Thus, $g + H \in Q/H$. The desired result now follows from Theorem 3.1.9. □

If our next theorem were given a name, it would have to be "The First Isomorphism Theorem for partially ordered abelian groups".

Theorem 3.1.13. *Let G and L be POAG's and $\phi : G \rightarrow L$ an order epimorphism. Then $Ker(\phi)$ is convex in G and $L \cong_o G/Ker(\phi)$.*

Proof. Suppose $a \leq b \leq c$ for some $a, c \in Ker(\phi)$ and some $b \in G$. Then $0 = \phi(a) \leq \phi(b) \leq \phi(c) = 0$. Thus, $b \in Ker(\phi)$, as desired.

For the second part of the result, recall that the First Isomorphism Theorem for groups assures us that the group map $g + Ker(\phi) \rightarrow \phi(g)$ describes a group isomorphism and so we need only show that this assignment is order-preserving. Assume $a + Ker(\phi) \leq b + Ker(\phi)$ in $G/Ker(\phi)$. By Theorem 3.1.10 it follows that $a \leq b + h$ for some $h \in Ker(\phi)$. Hence, $\phi(a) \leq \phi(b + h) = \phi(b) + \phi(h) = \phi(b)$ and we are done. □

For the next result we recall that a LOAG is **Archimedean** if, given any $a, b \in G$ with $0 < a < b$, then there exists an $n \in \mathbb{N}$ such that $na > b$. We will always assume Archimedean groups are nontrivial. It has been shown that every Archimedean LOAG is order isomorphic to an additive subgroup of $\mathbb{R}$. We will make implicit use of this fact frequently. Lastly, if there exists an order monomorphism of POAG's $\phi : H \to G$, then we will say that G contains a copy of H.

Theorem 3.1.14. *Let G be a LOAG. Then G is Archimedean if and only if G does not contain a copy of the lexicographic sum $\mathbb{Z} \oplus \mathbb{Z}$.*

Proof. ($\Rightarrow$) This result follows from the fact that a subgroup of an Archimedean group is Archimedean and $\mathbb{Z} \oplus \mathbb{Z}$ is clearly not Archimedean.

($\Leftarrow$) We assume G is not Archimedean and that G contains $\mathbb{Z} \oplus \mathbb{Z}$. As G is not Archimedean, then there exist $x, y \in G_+$ such that $x > ny$ for all $n \in \mathbb{N}$. Now consider the map $\mathbb{Z} \oplus \mathbb{Z} \to G$ given by $(a, b) \to ax + by$. That this map is a well-defined group map is easily verified. Suppose now $(a, b) \geq (0, 0)$. If $a = 0$, then $b \geq 0$ and we have $ax + by = by \geq 0$. If $a > 0$, then $ax > 0$. Now as $x > ny$ for all $n \in \mathbb{N}$ and because $y > 0$, then $x + my > 0$ for all $m \in \mathbb{Z}$. Hence, $ax + by \geq x + by > 0$. From Theorem 3.1.8 it follows that our map is order-preserving and so we have only left to show that the map is monic. So assume $ax + by = 0$. If $a > 0$, then $ax + by \geq x + by > 0$. We must therefore have that $a \leq 0$. If $a < 0$, then $0 = -ax - by > 0$, a contradiction. Now we have that $a = 0$ and so $ax + by = by = 0$. Since $y > 0$, we conclude $b = 0$ and so we are done. $\square$

Theorem 3.1.15. *Suppose $\varphi : G_1 \to G_2$ is an order isomorphism of POAG's and let $H_1 \leq G_1$ be convex. Then $H_2 = \varphi(H_1)$ is convex in G_2 and $G_1/H_1 \cong_o G_2/H_2$.*

Proof. That H_2 is convex is simply a consequence of the fact that φ is an order isomorphism. Letting $\pi : G_2 \to G_2/H_2$ be the natural projection, then we know from Theorem 3.1.10 that π is an order map. Hence, $\pi \circ \varphi : G_1 \to G_2/H_2$ is an order epimorphism. Suppose now that $h \in H_1$. Then $\varphi(h) \in H_2 \Rightarrow (\pi \circ \varphi(h)) = 0$. This establishes the inclusion $H \subseteq Ker(\pi \circ \varphi)$. For the reverse containment, let $g \in Ker(\pi \circ \varphi)$ so that $\varphi(g) \in Ker(\pi)$. But $Ker(\pi) = H_2$. Since φ is monic and $H_2 = \varphi(H_1)$, then $g \in H_1$. Knowing that $H_1 = Ker(\pi \circ \varphi)$ we quote Theorem 3.1.13 to conclude $G_1/H_1 = G_1/Ker(\pi \circ \varphi) \cong_o G_2/H_2$, as desired. □

We now consider suprema and infima in POAG's. Recall that given a partially ordered set S and a subset $A \subseteq S$, we say an element $s \in S$ is the **supremum** (**infimum**) of A, denoted $s = sup(A)$ ($s = inf(A)$), provided that $s \geq a$ ($s \leq a$) for all $a \in A$ and if we have some $d \in S$ such that $d \geq a$ *($d \leq a$)* for all $a \in A$, then $s \leq d$ *($s \geq d$)*. It should be remembered that $A \subseteq S$ need not admit a supremum or infimum and this is still true in the case of POAG's. Further, should $sup(A)$ exist, it need not be true that $sup(A) \in A$. Of course, the same is true for $inf(A)$.

Theorem 3.1.16. *[7] We let G be a POAG and $a, b \in G$. Then $sup(a,b)$ exists if and only if $inf(a,b)$ exists. Further, if $sup(a,b)$ exists, then $sup(a,b) + inf(a,b) = a + b$.*

Proof. In proving the equivalent statements in the first half of the theorem, we will also establish the validity of the statement in the second half.

($\Rightarrow$) We assume $d = sup(a,b)$ exists and let $c = a + b - d$. We then have $b \leq sup(a,b) \Rightarrow b - d \leq 0$. Hence, $a + b - d \leq a$. Similarly, $a + b - d \leq b$. So now $c \leq a, b$. Suppose we find $t \leq a, b$ for some $t \in G$. Then $a - t \geq 0 \Rightarrow b + a - t \geq b$. Similarly, $b + a - t \geq a$. Now $b + a - t \geq a, b \Rightarrow b + a - t \geq d = a + b - c$. So $-t \geq -c$, i.e., $t \leq c$. Thus, $c = inf(a,b)$.

($\Leftarrow$) Making the obvious adjustments, simply mimic the argument just made.

□

Sometimes we might have a POAG in which $sup(a,b)$ exists for all $a, b \in G$. For example, this is clearly true in the case that G is a LOAG. This observation motivates the following definition.

Definition 3.1.17. Let G be a POAG. If $sup(a, b)$ exists for all $a, b \in G$, then we say G is **lattice ordered**.

Examples of lattice ordered groups include any LOAG or any free abelian group in the product order. There are many others and we will take a look at some of them in later sections.

3.2. Groups of Divisibility

In this section we introduce groups of divisibility. We begin by recalling that given a domain D, then $U(D)$ forms a group using the multiplication of D. Further, if D is contained in a field F, then $U(D)$ is a subgroup of F^* and so we can form the factor group $G = F^*/U(D)$. This factor group is called the **group of divisibility of F with respect to D**. We observe that a typical element of G is denoted by $kU(D)$, where $k \in F^*$. Further, given $xU(D), yU(D) \in G$, we write $xU(D) + yU(D) = xyU(D)$. The additive notation on the left side of the previous equation is not only used because we are talking about an abelian group, but in addition there will be occasions where this notation makes an argument less cluttered. When we speak of a group of divisibility, the field F and domain D will always be understood. Thus, we will oftentimes use the less cumbersome notation $xU + yU$. The case in which F is the quotient field of D will be of particular interest to us and in this case we will refer to the factor group $F^*/U(D)$ as being the **group of divisibility of D**. It is customary in the literature to denote the group of divisibility of a domain D by $G(D)$.

The next theorem is of a fundamental nature and its importance here cannot be overemphasized. Its statement should start to give the reader a hint as to why we were interested in taking a brief tour through the theory of POAG's in the previous section. Our usage of this result will be somewhat ubiquitous and usually made without mention. Its proof, although instructive, is routine and will be omitted.

Theorem 3.2.1. *[7] Let D be a domain contained in a field F and let G be the group of divisibility of F with respect to D. Then G is a POAG via the relation $xU \leq yU$ if and only if $\frac{y}{x} \in D$.*

Given a group of divisibility $F^*/U(D)$, our primary interest is going to be in the case in which F is the quotient field of D. However, the more general definition does allow us the opportunity to create some interesting examples. We start out by considering the ring $D = \mathbb{Z}[2\iota]$, where $\iota = \sqrt{-1}$. Notice that $\iota \notin D$. However, because $\iota^2 = -1$, then we see that ιU is a torsion element of $G(D)$. But this could never happen when D is integrally closed. We recall for our next theorem that given a domain D with quotient field K, then we say D is **root closed** whenever, given any $k \in K$ and $n \in \mathbb{N}$ such that $k^n \in D$, then $k \in D$.

Theorem 3.2.2. *If a domain D is root closed, then $G(D)$ is torsion-free.*

Proof. Suppose $k^n U = U$ for some $k \in K$. Then $k^n = u$ for some $u \in U(D)$. Since D is root-closed, then $k \in D$. Since $k^n \in U(D)$, it then follows that $k \in U(D)$. So $kU = U$, making $G(D)$ torsion-free. □

Having seen that groups of divisibility may admit torsion elements, we inquire as to whether or not a group of divisibility may be a torsion group. Using our more general definition we see that the answer is yes. For example, let F be a finite field and let $\overline{F}$ be its algebraic closure. Given any $f \in \overline{F}$, then there exists some $n \in \mathbb{N}$ such that $f^n = 1$. Thus, if G is the group of divisibility of $\overline{F}$ with respect to F, then we see that G is an infinite torsion POAG. We also take note of the fact that if a group of divisibility is torsion, then by Theorem 3.1.2 it admits only the trivial ordering. Hence, the group of divisibility of a domain D is torsion if and only if D is a field. The overwhelming majority of groups of divisibility which will be of interest to us will not be torsion.

Theorem 3.2.3. *Let D be a domain with $G(D) = G$. Let $H \leq G$ and consider the set $S = \{s \in D | sU \in H_+\}$. Then S is a multiplicative system in D if and only if H is convex in G.*

Proof. ($\Rightarrow$) We assume S is saturated and multiplicatively closed in D and wish to show H is convex in G. From Theorem 3.1.9 it suffices to show that H_+ is convex in G_+. First, note that $H_+ = \{sU | s \in S\}$. So if $hU \in H_+$, then $h \in S$ by definition and we get $hU \in \{sU | s \in S\}$. On the other hand, should we have $\sigma U \in \{sU | s \in S\}$, then $\sigma U = sU$ for some $s \in S$. But if $s \in S$, then $sU \in H_+$ by definition.

With the previous observation in hand, let $h_1, h_2 \in H_+$ and $g \in G$ with $h_1 \leq g \leq h_2$. Say $g = xU$ and $h_2 = sU$ for some $s \in S$. Since $U \leq h_1 \leq xU \leq sU$, then $x \in D$ and $x|s$. Since S is saturated, we then have that $x \in S$ and so $g \in H_+$, as desired.

($\Leftarrow$) We now assume H is convex in G. By Theorem 3.1.9 we know that H_+ is convex in G_+. To verify that S is multiplicative closed we first choose $x, y \in S$ so that $xU, yU \in H_+$. Then $xyU = xU + yU \in H_+$ and so $xyU \in H_+ \Rightarrow xy \in S$. Now to verify that S is saturated we pick $x, y \in D$ such that $xy \in S$. This implies $xyU \in H_+$. Note $U \leq xU \leq xyU$. As $U \in H_+$, we utilize the convexity of H to conclude $xU \in H_+$, i.e., $x \in S$. Similarly, $y \in S$ and we are done. $\square$

Our next result establishes a nice relationship between multiplicative systems in a domain D and convex subgroups of $G(D)$. We will then be in a position to determine the effect of localization on the group of divisibility in Theorem 3.2.5. Recall from Theorem 1.4.4 where it was shown that S is a multiplicative system if and only if S is the complement of a union of prime ideals. Theorem 3.2.4 expands upon this characterization.

Theorem 3.2.4. *Let D be a domain and let $S \subseteq D^*$ be multiplicatively closed. Let H be the subgroup of $G(D)$ generated by $\sigma = \{sU | s \in S\}$. The following are equivalent:*

a) S is a multiplicative system in D

b) S is the complement of the union of some family of prime ideals of D

c) H is convex in $G(D)$

Proof. We have already demonstrated $a) \Leftrightarrow b)$ in Theorem 1.4.4.

$a) \Rightarrow c)$ We assume S is a multiplicative system and wish to show H is convex in G. By Theorem 3.1.9 it suffices to show H_+ is convex in G_+. Let $xU, yU \in H_+$ and $kU \in G$ such that $xU \leq kU \leq yU$. As $xU \in H_+$, then $k \in D^*$. Now $yU \in H_+$ and so we may write $yU = (\sum_{i=1}^{n} s_iU) - (\sum_{i=n+1}^{n+m} s_iU)$ with each $s_j \in S$. Note $yU \leq \sum_{i=1}^{n+m} s_iU$. Thus, $kU \leq \sum_{i=1}^{n+m} s_i$. We have that $k \in D^*$, $k | \prod_{i=1}^{n+m} s_i$, and $\prod_{i=1}^{n+m} s_i \in S$. Using the saturation of S we conclude $k \in S$. Put another way, $kU \in H_+$, as desired.

$c) \Rightarrow a)$ Now we assume H is convex in G and let $x, y \in D^*$ such that $xy \in S$. Then $U \leq xU \leq xyU \in H$. Utilizing the convexity of H we deduce $xU \in H$. Now H is generated by σ and so we may write $xU = (\sum_{i=1}^{n} s_iU) - (\sum_{i=n+1}^{n+m} s_iU)$ where each $s_j \in S$. Therefore, $xU + (\sum_{i=n+1}^{n+m} s_iU) = \sum_{i=1}^{n} s_iU$. So $uxs_{n+1} \cdots s_{n+m} = s_1 \cdots s_n$ for some $u \in U(D)$. As S is multiplicatively closed it now follows that the product $uxs_{n+1} \cdots s_{n+m} \in S$.

We wish to show $x \in S$. Assume, to the contrary, $x \notin S$. Then there exists some $P \in Spec(D)$ such that $x \in P$ and $P \cap S = \emptyset$. Now $x \in P \Rightarrow uxs_{n+1} \cdots s_{n+m} \in P$. But then $P \cap S \neq \emptyset$, a contradiction.

□

Theorem 3.2.5. *Suppose $S \subseteq D$ is a multiplicative system and let H be the subgroup of $G = G(D)$ generated by $\{sU | s \in S\}$. Then $G(D_S) \cong_o G/H$.*

Proof. Define $\phi : G \to G(D_S)$ by $\phi[kU(D)] = kU(D_S)$. It is easily verified that ϕ is an order epimorphism. Suppose now $\phi[kU(D)] = U(D_S)$. Then $k = \frac{a}{s}$ for some $a \in D^*$ and $s \in S$. If $a \notin S$, then $a \in P$ for some $P \in Spec(D)$ such that $P \cap S = \emptyset$. But then $PD_S \in Spec(D_S) \Rightarrow k \notin U(D_S)$, a contradiction. Hence, $U(D_S) = \{s_1 s_2^{-1} U(D_S) | s_1, s_2 \in S\}$ and so $Ker(\phi) \subseteq H$. On the other hand, every element of H is of form $s_1 s_2^{-1} U(D_S)$ and so $H \subseteq Ker(\phi)$. Note that from Theorem 3.2.4 we know that H is convex in G. As $H = Ker(\phi)$, the desired result now follows from Theorem 3.1.13. $\square$

Definition 3.2.6. Recall that given a domain D and $x \in D$, then we say x **fragments** in D if there exists a nonzero nonunit $y \in D$ such that $y^n | x$ for all $n \in \mathbb{N}$. If every nonzero nonunit in D fragments, then we say D is **fragmented**.

We will be considering fragmented domains a little more in Chapter 4. In Example 2.1.15 we saw that an element can both fragment and admit an atomic factorization. We now proceed with a theorem that shows how the group of divisibility can be utilized to characterize the existence of fragmentation. It also demonstrates the way in which fragmentation both prevents a domain from being Noetherian and influences the Krull dimension of a ring.

When we say a domain "allows fragmentation", we mean that there is an element in the domain that fragments.

Theorem 3.2.7. *Let D be a domain with quotient field K and let H be the lexicographic sum $\mathbb{Z} \oplus \mathbb{Z}$. Then D admits fragmentation if and only if $G(D)$ contains an order isomorphic copy of H. Further, if D admits fragmentation, then $dim(D) > 1$ and D is non-Noetherian.*

Proof. We first establish the characterization of the existence of fragmentation and postpone the proof of the latter half of the statement of the theorem to the end. We let $G = G(D)$ throughout.

($\Rightarrow$) Suppose $x, y \in D^* - U(D)$ with $x \in \bigcap_{n=1}^{\infty} y^n D$ and consider $\phi : H \to G$ defined by $\phi(a, b) = x^a U + y^b U$. We omit the routine verification that ϕ is a well-defined group map and we will content ourselves to demonstrate that ϕ is monic and order-preserving.

First we show that ϕ is monic. To this end, assume $x^a y^b U = x^a U + y^b U = U$ and note that since $x \in \bigcap_{n=1}^{\infty} y^n D$, then $xy^{-n} \in D$ for all $n \in \mathbb{N}$. Observe that $xy^{-n} \in D^* - U(D)$. Indeed, otherwise we would have $x = uy^n$ for some $u \in U(D)$ and some $n \in \mathbb{N}$. But $x \in y^{n+1}D$ and so could deduce $uy^n = x = dy^{n+1} \Rightarrow y \in U(D)$, a contradiction. Now as $xy^{-n} \in D^* - U(D)$, then $xy^{-n}U > U$ for all $n \in \mathbb{N}$. Because $y \in D^* - U(D)$ we have that $xy^n U > U$ for all $n \in \mathbb{N}$ and so we have demonstrated $xy^m U > U$ for all $m \in \mathbb{Z}$. We would like to show ϕ is monic and it suffices to show $a = b = 0$. Assume, to the contrary, that $a < 0$. Then $x^{-a}U > U \Rightarrow x^{-a}y^{-b}U > U \Rightarrow x^a y^b U < U$, a contradiction. Now we know $a \geq 0$. But if $a > 0$, then $x^a y^b U > xy^b U > U$, a contradiction. We conclude $a = 0$. Now that $a = 0$ we have $x^a y^b U = y^b U = U$. It is easily established that we must now have $b = 0$ and so ϕ is monic.

Now we show ϕ is order preserving. If $(a, b) \geq (0, 0)$, then either $a > 0$ or $a = 0$ and $b \geq 0$. If $a > 0$, then $x^a y^b U > U$, as was shown in the previous paragraph. So assume $a = 0$. If $b = 0$, then $y^b \in U(D)$. If $b > 0$, then $y^b \in D^* - U(D) \Rightarrow y^b U > U$. Now use Theorem 3.1.8 to see that ϕ is order preserving.

($\Leftarrow$) Now we posit that G contains a copy of H. Equivalently, there exists an order monomorphism $\phi : H \to G$. Let $x, y \in D$ such that $\phi(e_1) = xU$ and $\phi(e_2) = yU$, where $e_1 = (1, 0)$ and $e_2 = (0, 1)$. Given any $n \in \mathbb{N}$ we have that

$e_1 > ne_2 \Rightarrow xU > y^nU$. This implies $y^n|x$ in D for all $n \in \mathbb{N}$. Put another way, $x \in \bigcap_{n=1}^{\infty} y^nD$. Since ϕ is order preserving, then $x, y \in D^* - U(D)$ and so we are done.

We now move on to prove $dim(D) > 1$ and that D is non-Noetherian whenever D admits fragmentation. Let us show that D is non-Noetherian by demonstrating how it fails to satisfy the ascending chain condition. Choose some elements $x, y \in D^* - U(D)$ such that $x \in \bigcap_{n=1}^{\infty} y^nD$ and then simply observe that we have proper ideal containments $\frac{x}{y^n}D \subsetneq \frac{x}{y^{n+1}}D$ for all $n \in \mathbb{N}$.

To show $dim(D) > 1$, we maintain the assumptions of the previous paragraph and consider the set $S = \{y^n | n \in \mathbb{N}\}$. It is obvious that S is multiplicatively closed in D. If $a \in xD \cap S$, then there exists some $r \in D^*$ such that $rx = y^n$ for some n. Rephrasing, $\frac{rx}{y^n} = 1$. But recall $xy^{-n}U > U$ in G. Thus, $U = rxy^{-n}U \geq xy^{-n}U > U$, a contradiction. We may now say $xD \cap S = \emptyset$. As $xD \cap S = \emptyset$, we expand $0 \subsetneq xD$ to some $P_1 \in Spec(D)$ such that $P_1 \cap S = \emptyset$. Then $P_1 \subsetneq (P_1, y)$. Expanding (P_1, y) to some $P_2 \in Spec(D)$, we conclude $0 \subsetneq P_1 \subsetneq P_2$. Therefore, $dim(D) > 1$. □

3.3. Applications to Valuation Domains

The group of divisibility has an enormous range of applications to the class of valuation domain's and in this section we are going to explore some of those which will prove to be most useful to us. We begin by showing how groups of divisibility characterize valuation domain's. Thus, Theorem 3.3.1 is an expansion of Theorem 2.1.16.

Theorem 3.3.1. *[7] Let D be a domain with $G = G(D)$. Then D is a valuation domain if and only if G is linearly ordered.*

Proof. ($\Rightarrow$) We assume D is a valuation domain with quotient field K. From Theorem 2.1.16 we know that given any $k \in K^*$, either $k \in D$ or $\frac{1}{k} \in D$. In terms of G, this means either $kU \geq U$ or $k^{-1}U \geq U$. Let $k_1U, k_2U \in G$. Assume both $k_1U, k_2U \geq U$. Then $k_1, k_2 \in V$. By Theorem 2.1.16 we may assume $k_1|k_2$ so that $k_1U \leq k_2U$. If both $k_1U, k_2U \leq U$, then we may assume $U \leq k_1^{-1}U \leq k_2^{-1}U$ and so $k_2U \leq k_1U$. The only other case left is $k_1U \leq U \leq k_2U$. In any event, all elements of G are comparable and so G is linearly ordered.

($\Leftarrow$)We assume now G is linearly ordered. Let $a, b \in D^*$. Without loss of generality, say $U \leq aU \leq bU$. Then $a|b$ in D and so D is a valuation domain by Theorem 2.1.16. $\square$

In Theorem 3.3.1 we saw that we can associate to any valuation domain a LOAG. The converse of this result is also true, i.e, if G is a LOAG, then we can realize G as the group of divisibility of a valuation domain. The argument is instructive in the way it utilizes semigroup rings and we present the construction in Theorem 3.3.2. Wolfgang Krull first proved this result in [12].

Theorem 3.3.2. *Let G be a LOAG. Then there exists a valuation domain V such that $G \cong_o G(V)$.*

Proof. We let F be a field and $D = F[x^g]_{g \in G_+}$. Letting M be the canonical maximal ideal of D we look at the localization $V = D_M$. If $f \in V^* - U(V)$, then $f = ux^g$ for some $u \in U(V)$ and $g \in G_+$. Therefore, it is easily verified that $\phi : V \to G$ defined by $\phi(x^gU) = g$ is an order isomorphism. That V is a valuation domain follows from Theorem 3.3.1. $\square$

We now characterize Noetherian valuation domain's.

Theorem 3.3.3. *Let V be a non-field valuation domain. The following are equivalent:*

a) V is a PID.

b) V is Noetherian.

c) V is atomic.

d) $G(V) \cong_o \mathbb{Z}$.

Proof. We already know every PID is Noetherian and every Noetherian domain is atomic. Moreover, we showed in Theorem 2.1.23 that an atomic valuation domain is a PID. To finish the proof, we show that $a) \Rightarrow c)$ and $d) \Rightarrow c)$.

$a) \Rightarrow d)$ Because V is a valuation domain, then $G(V)$ is linearly ordered. Moreover, since V is a PID, then D admits no fragmentation and $dim(V) = 1$. By Theorem 3.2.7 and Theorem 3.1.14 it follows that $G(V)$ must be Archimedean. But V is atomic and so $G(V)$ must admit a minimal positive element. Thus, $G(V)$ is order isomorphic to a nonzero subgroup of $\mathbb{R}$ with a minimal positive element. This can only mean $G(V) \cong_o \mathbb{Z}$.

$d) \Rightarrow c)$ Suppose $G(V) \cong_o \mathbb{Z}$ and let $a \in V^* - U(V)$. Let $\pi U \in G(V)$ be minimal and positive. Then $aU = \pi^n U$. Hence, $a = u\pi^n$ for some $u \in U(V)$ and therefore V is atomic. □

Our next result provides us with a powerful characterization of one-dimensional valuation domains and can used to easily generate useful counterexamples. Of particular note, it will be recalled from Theorem 2.2.20 and Theorem 2.2.8 that the property of being integrally closed is preserved in all polynomial extensions and localizations of a domain. Conspicuously absent was the determination of whether or not integral closure survives in power series extensions. Theorem 3.3.4 shows us that a power series extension of a valuation domain is, in fact, *rarely* integrally closed.

Theorem 3.3.4. *Let V be a valuation domain with quotient field K and $V \subsetneq K$. The following are equivalent:*

a) $dim(V) = 1$.

b) V is completely integrally closed.

c) V admits no fragmentation.

d) $G(V)$ is Archimedean.

Proof. $a) \Rightarrow b)$ We assume $dim(V) = 1$ and let $k \in K$ be almost integral over V. Then there exists some $x \in V^*$ such that $k^n x \in V$ for all $n \in \mathbb{N}$. We may assume $x \in V^* - U(V)$ because otherwise $k \in V$. We would like to show $k \in V$. Assume, to the contrary, that $k \notin V$. Then $k = \frac{1}{a}$ for some $a \in V^* - U(V)$. Since $\frac{x}{a^n} \in V$, then $x \in \bigcap_{n=1}^{\infty} a^n V$ and so $\bigcap_{n=1}^{\infty} a^n V \neq 0$. Let $S = \{a^n | n \in \mathbb{N}_0\}$ and note that S is multiplicatively closed. By Theorem 1.4.1 there exists a $P \in Spec(V)$ such that $P \cap S = \emptyset$. We know $0 \subsetneq P$ because $x \in P$. But since $dim(V) = 1$, then P must be maximal in V. But then $a \in U(V)$, a contradiction.

$b) \Rightarrow c)$ Suppose V is completely integrally closed and let $a, b \in V^*$ such that $a \in \bigcap_{n=1}^{\infty} b^n V$. We wish to show $b \in U(V)$. As $a \in \bigcap_{n=1}^{\infty} b^n V$, then $\frac{a}{b^n} \in V$ for all $n \in \mathbb{N}$, i.e., $\frac{1}{b}$ is almost integral over V. Having V be completely integrally closed then implies $\frac{1}{b} \in V$, as desired.

$c) \Rightarrow d)$ Assuming V admits no fragmentation, it then follows from Theorem 3.2.7 that $G(V)$ does not contain a copy of the lexicographic sum $\mathbb{Z} \oplus \mathbb{Z}$. Using Theorem 3.1.14 we see that $G(V)$ must therefore be Archimedean.

$d) \Rightarrow a)$ Our assumption here is that $G(V)$ is Archimedean and we wish to establish that $dim(V) = 1$. Since $V \subsetneq K$, then $dim(V) \geq 1$. If $dim(V) > 1$, then we can choose a chain $0 \subsetneq P_1 \subsetneq P_2$ of prime ideals of V. Let $0 \neq p_1 \in P_1$ and $p_2 \in P_2 - P_1$. Taking advantage of the primality of P_1 we may say $p_2^n \in P_2 - P_1$. But V is a valuation domain and so we have that $p_2^n | p_1$ for all $n \in \mathbb{N}$. Hence, $U(V) < p_2^n U(V) < p_1 U(V)$, contradicting the fact that $G(V)$ is Archimedean. $\square$

Recall now that because integral extensions enjoy the lying over property, then the integral closure of a non-field domain must also fail to be a field. The complete integral closure of a domain does not enjoy this property, as we are about to exemplify. Before doing so, we prove a theorem which will facilitate the process.

Theorem 3.3.5. *[11] Suppose V is a valuation domain with quotient field K and let T be an overring of V. Then $T = V_P$ for some $P \in Spec(V)$.*

Proof. First note that if $k \in K$, then by Theorem 2.1.16 it follows that $k \in V$ or $\frac{1}{k} \in V$. As $V \subseteq T$, then the same is true of T. Hence, Theorem 2.1.16 assures us that T is a valuation domain. As such, T is quasi-local. Let μ be the maximal ideal of T and let $P = \mu \cap V$. Then $P \in Spec(V)$. We claim $T = V_P$. Let $\frac{v}{s} \in V_P$ with $v \in V$ and $s \in V - P$. Since $s \notin P$, then $s \notin \mu$. Hence, $s \in U(T)$ and so $\frac{v}{s} \in T$. For the reverse containment, let $t \in T$. Then either $t \in V$ or $\frac{1}{t} \in V$. If $t \in V$, then $t \in V_P$. So assume $t \notin V$ so that $\frac{1}{t} \in V$. Then $\frac{1}{t} \in V_P$ and write $t = \frac{v}{s}$ for some $v \in V$ and $s \in V - P$. Observe that $t \in U(T)$ and $t = \frac{s}{v}$. It now suffices to show $v \notin P$. If $v \in P$, then $v \in \mu \cap V$. But then μ contains a unit, a contradiction. □

Theorem 3.3.5 can be generalized to the case of Prüfer domains. Indeed, if D is a Prüfer domain, then it turns out that every valuation overring is of the form D_P for some $P \in Spec(D)$. The proof of this result is quite different than that just presented and can be found in [11]. Theorem 3.3.5 also has an interesting generalization to the case of Bêzout domains as we will see in Theorem 4.1.4.

In [7] we see that the complete integral closure of a domain need not be completely integrally closed. However, such a domain must always fail to be a valuation domain.

Theorem 3.3.6. *Let V be a valuation domain with quotient field K. Then $C(V)$ is completely integrally closed.*

Proof. By Theorem 3.3.5, $C(V) = V_P$ for some $P \in Spec(V)$. Let $k \in K$ be almost integral over $C(V)$. Then $k^n(\frac{r}{s}) \in V_P$ for some $r \in V$ and $s \in V - P$. By Theorem 2.1.16 we know that either $r|s$ or $s|r$ in V. If $r|s$, then $\frac{r}{s} \in U(V_P)$ and so we would have $k \in V_P$. So assume $s|r$ in V. Then $\frac{r}{s} \in V$. But then k is almost integral over V, i.e., $k \in C(V)$. In either case, $k \in C(V)$ and we are done. □

Theorem 3.3.7. *Let V be a valuation domain with complete integral closure T and quotient field K. Then $T \subsetneq K$ if and only if V admits a ht-1 prime.*

Proof. ($\Rightarrow$) Suppose $T \subsetneq K$. As an overring of V we know from Theorem 3.3.5 that $T = V_P$ for some $P \in Spec(V)$. We know from Theorem 3.3.6 that T is completely integrally closed. As T is a valuation domain and $T \subsetneq K$, it follows from Theorem 3.3.4 that $dim(T) = 1$. Now Theorem 1.4.8 assures that P is a ht-1 prime.

($\Leftarrow$) Assume $P \in Spec(V)$ is a ht-1 prime. Then $dim(V_P) = 1$ and so we may use Theorem 3.3.4 to deduce V_P is completely integrally closed. From Theorem 2.4.7 it follows that $T \subseteq V_P$. Hence, $T \subseteq V_P \subsetneq K$. □

It is clear from Theorem 3.3.7 that if V is a valuation domain whose complete integral closure is its quotient field, then the Krull dimension of V is either infinite or zero. Here is an example showing how the complete integral closure of a non-field domain might actually turn out to be its quotient field.

Example 3.3.8. Consider the lexicographic sum $G = \ldots \oplus \mathbb{Z} \oplus \mathbb{Z} \oplus \mathbb{Z}$. Because G is linearly ordered, then Theorem 3.3.2 allows us to let V be a valuation domain such that $G(V) \cong_o G$. We claim that $C(V) = K$, the quotient field of V. To this end, it suffices to show that V does not admit a ht-1 prime ideal. Let $\mathfrak{B} = \{e_i | i \in -\mathbb{N}\}$ be the standard basis for G and let $\phi : G \to G(V)$ be an order isomorphism. For any $\alpha \in V^* - U(V)$ we let $\phi(\alpha) = \sum_{r=-1}^{-n} r_i e_i$ with $r_{-n} \neq 0$. Since ϕ is epic, then there exists some $\beta \in V$ such that $\phi(\beta U) = e_{-n-1}$. Note now $e_{-n-1} > m\phi(\alpha U)$ for all $m \in \mathbb{N}$. From this we deduce $\beta U > \alpha^m U$ for all m. This means $\beta \in \bigcap_{i=1}^{\infty} \alpha^n V$. Note that $\bigcap_{i=1}^{\infty} \alpha^n V = \sqrt{\alpha V} \in Spec(V)$. Hence, $\sqrt{\alpha V} \neq 0$. It follows that no element in V can be contained in a ht-1 prime ideal and so we are done.

Here is another correspondence theorem. It will help us build valuation domain's of any given Krull dimension.

Theorem 3.3.9. *[7] Let V be a valuation domain. For any $P \in Spec(V)$, define H_P to be the subgroup of $G(V)$ generated by $\{sU | s \in V - P\}$.*

a) Each H_P is convex in $G(V)$.

b) If $P \subsetneq Q$ are primes in V, then $H_Q \subsetneq H_P$.

c) There exists a 1-1 correspondence between the prime ideals of V and the convex subgroups of $G(V)$ given by $P \leftrightarrow H_P$.

Proof. *a*) Note that $V - P$ is saturated and multiplicatively closed. That each H_P is convex now follows from Theorem 3.1.12.

b) We assume $P \subsetneq Q$ are primes in V so that $V - Q \subsetneq V - P$. Hence, $H_Q \subseteq H_P$. To establish the proper containment, simply note that if $xU \in H_Q$ for some $x \in V$, then $x \in V - Q$. Hence, if $x \in V - P$ and $x \notin V - Q$, then $xU \in H_P$ and $xU \notin H_Q$.

c) That the map $P \rightarrow H_P$ is well-defined follows immediately from *b*). Further, because V is a valuation domain, then the prime ideals of V are linearly ordered by inclusion. Thus, *b*) now assures us that the map is also monic. To show that the map is epic, start with a convex subgroup $H \leq G(V)$. Now let $S = \{s \in V | sU \in H\}$. Since H is convex, then Theorem 3.2.4 tells us that S is saturated and multiplicatively closed. As such, S is a complement of a union of primes. But the primes of V are linearly ordered by inclusion and so this union is some prime ideal P. Hence, $H = H_P$ and we are done. □

With regards to the previous theorem, we may refer to H_P as the convex **subgroup associated with** P or that P **is the prime ideal of** V **associated with** H_P. The following theorem gives us another interesting property of these convex subgroups.

Theorem 3.3.10. *Let V be a valuation domain, $P \in Spec(V)$, and H_P the convex subgroup of $G(V)$ associated with P. Then $H_P \cong_o G(V/P)$.*

Proof. First, if K is the quotient field of V, then $K^* = V^* \cup \{\frac{1}{a} | a \in V^*\}$. Thus, $G(V) = \{aU | a \in V^*\} \cup \{a^{-1}U | a \in V^*\}$. Consider now the map $H_P \to G(V/P)$ given by $aU \to \tilde{a}$ and $a^{-1}U \to (\tilde{a})^{-1}U$, where $a \in V^*$, $\tilde{a}U = (a+P)U(V/P)$, and $(\tilde{a})^{-1}U = (\frac{1+P}{a+P})U(V/P)$. It is easily verified that this map is an order isomorphism. □

We remark that if V is a valuation domain with $P \in Spec(V)$, then we are now able to compute the groups of divisibility of both V_P and V/P. Indeed, Theorem 3.2.5 states that $G(V_P) \cong_o G(V)/H_P$ and Theorem 3.3.9 tells us that $G(V/P) \cong_o H_P$.

Here is another interesting application of Theorem 3.3.9. It shows us how to use groups of divisibility to construct valuation domain's of any given dimension. We will use this idea again when we generalize Krull dimension in Chapter 5. Note the implicit use of Theorem 3.3.2 in the hypothesis of the theorem.

Theorem 3.3.11. *Let V be a valuation domain such that $G(V)$ is order isomorphic to the lexicographic sum $\bigoplus_{i=1}^{n} A_i$, where each A_i is nonzero and Archimedean. Then V is n-dimensional.*

Proof. Because each A_i is nonzero and Archimedean, then it is clear that the direct sum has exactly n+1 convex subgroups (including the zero subgroup). By Theorem 3.3.9, there are exactly n+1 prime ideals. As the ideals of V are linearly ordered, then $dim(V) = n$. □

As has already been pointed out, any valuation domain which admits a nonzero unbranched prime is infinite dimensional. We would now like to know whether or not this characterizes infinite dimensional valuation domains. The following theorem settles this matter for us.

Theorem 3.3.12. *Let S be a linearly ordered set with the property that given any nonempty subset $\sigma \subseteq S$, then there exists some $x \in \sigma$ such that $x \geq y$ for all $y \in \sigma$. Now let V be a valuation domain such that $G(V)$ is order isomorphic to the lexicographic sum $G = \bigoplus_{\Lambda} A_i$, where each A_i is a Archimedean group. Then every nonzero prime ideal of V is branched.*

Proof. Suppose H is nonzero proper convex subgroup of G. For each $i \in \Lambda$, let $\iota_i : A_i \to G$ be the natural injection. Now let $\sigma = \{i \in S | Im(\iota_i) \subseteq H\}$. Since $H \neq G$ then $\sigma \neq S$, i.e., $S - \sigma \neq \emptyset$. Hence, we may choose a maximal $j \in S - \sigma$. Let Q be the subgroup of G generated by $H \cup Im(\iota_j)$. Since $Im(\iota_j) \subsetneq H$, then $H \subsetneq Q$. If $K \leq G$ such that $H \subsetneq H \subseteq Q$, then $Im(\iota_j) \subseteq K$. Thus, $H \cup Im(\iota_j) \subseteq K \Rightarrow Q \subseteq K \Rightarrow K = Q$. We have now shown that H and Q are adjacent in G. Moreover, because $Q = Im(\iota_j) + H$, then clearly Q is convex in G.

Now recall from Theorem 3.3.9 that there exists a 1-1 correspondence between the set of prime ideals of V and the set of convex subgroups of G(V) given by $P \leftrightarrow H_P$, where $P_1 \subsetneq P_2 \Rightarrow H_{P_2} \subsetneq H_{P_1}$. Thus, if every nonzero convex subgroup $H \leq G$ admits an adjacent convex subgroup $Q \subsetneq H$, then every nonzero prime P admits an adjacent prime $\mathfrak{P} \subsetneq P$ and we are done. ☐

Example 3.3.13. Let V be a valuation domain such that $G(V) \cong_o \bigoplus_{i=-1}^{-\infty} \mathbb{Z}$. Applying Theorem 3.3.12 we see that every nonzero prime ideal of V is branched. Moreover, since $\bigoplus_{i=-1}^{-\infty} \mathbb{Z}$ is linearly ordered and clearly contains infinitely many convex subgroups, then by Theorem 3.3.9 we may conclude that V is infinite dimensional.

Note that our indexing set in the previous example need not be countable. Indeed, since any set S can be well-ordered, then we can consider S in the reverse ordering. That is given a well-ordered set $(S, \leq_1)$, then we can introduce a new ordering $\leq_2$ by saying $a \leq_2 b$ if and only if $b \leq_1 a$. Thus, $(S, \leq_2)$ is a linearly ordered set satisfying the conditions needed in Theorem 3.3.12. Note also that this is a means of creating rings with chains of uncountably many primes. Observations such as this is what motivate much of our discussion in Chapter 5.

3.4. More Applications to Valuation Domains

Our discussion in this section is motivated by the idea of trying to characterize a certain class of valuation domain's via groups of divisibility. In particular, we are interested in those valuation domain's called *discrete* valuation domains. Some remarks regarding our verbiage are in order here. Many algebraists use this terminology when referring to Noetherian valuation domain's. Nowadays, though, there are some who use this nomenclature when referring to any valuation domain whose group of divisibility is order isomorphic to the lexicographic sum $\bigoplus_{i=1}^{n} \mathbb{Z}$. In particular, such a valuation domain would necessarily be finite dimensional. The definition that we are going to use is the same as that found in [7]. It will be shown that domains satisfying this definition can be infinite dimensional.

Definition 3.4.1. Let V be a valuation domain. We call V **discrete** if every primary ideal is a power of its radical.

As every Noetherian valuation domain is a PID, then it is clear that every Noetherian valuation domain is discrete because every nonzero ideal is a power of the maximal ideal. However, one of our goals here is to show that discrete valuation domain's can also be infinite dimensional. Theorem 3.4.3 is a characterization of discrete valuation domain's using branched prime ideals. We will continue to build on this characterization. However, we will need the following lemma which can be found in [7].

Lemma 3.4.2. *[7] Suppose V is a valuation domain with a nonzero prime ideal P. If $P \neq P^2$, then $\{P^n\}_{n=1}^{\infty}$ is the set of all P-primary ideals.*

Theorem 3.4.3. *[7] Suppose V is a valuation domain. Then V is discrete if and only if every branched prime ideal of V is not idempotent.*

Proof. ($\Rightarrow$) We assume V is discrete, i.e., every primary ideal is a power of its radical. Now let $P \in Spec(V)$ be branched. By definition, there exists a P-primary ideal distinct from P. Suppose Q is P-primary with $Q \subsetneq P$. Write $Q = P^n$. Should $P = P^2$, then we would have $P = P^2 = P^3 = \ldots P^n = Q$, contradiction.

($\Leftarrow$) Assume Q is P-primary. We wish to show that Q is a power of P. If $Q = P$, then we are done. So assume $Q \neq P$. Then there exists a P-primary ideal distinct from P, i.e., P is branched. Applying Lemma 3.4.2 it follows that $Q \in \{P^n\}_{n=1}^{\infty}$ and so the desired result is achieved. □

Before proceeding we pause to make an observation that will be useful in the upcoming proof. We start by letting V be a valuation domain and $P \in Spec(V)$. It follows from Theorem 2.5.6 that if $Q_1 \subsetneq Q_2 \subseteq P$ are primes in V, then we are guaranteed that Q_1 and Q_2 are adjacent in V if and only if Q_1V_P and Q_2V_P are adjacent in V_P. Using prime correspondence, Theorem 2.5.16 gives us that Q_2 is branched in V if and only if Q_2V_P is branched in V_P. Also, because of the correspondence of prime ideals in V/P and the ideals of V containing P, it readily follows that if Q is a prime in V that *properly* contains P, then Q is branched in V if and only if Q/P is branched in V/P. Moreover, if $Q/P \in Spec(V/P)$ is branched, then it must be true that $P \subsetneq Q$ because the zero ideal is never branched in a domain.

Theorem 3.4.4. *Let V be a valuation domain and $P \in Spec(V)$. If V is discrete, then V_P and V/P are also discrete.*

Proof. We begin by showing V_P must be discrete. Suppose $Q \subseteq P$ are primes in V. Assume $QV_P \in Spec(V_P)$ is branched and $QV_P = Q^2V_P$. Let $q \in Q$. Then $q \in Q^2V_P \Rightarrow q = \frac{q_1q_2}{s}$ for some $q_1, q_2 \in Q$ and $s \in V - Q$. Because $s \in V - Q$, then $\frac{q_2}{s} \in V$. Note $(\frac{q_1}{s})s \in Q$. Since $s \notin Q$, then $\frac{q_2}{s} \in Q$. So $\frac{q_1q_2}{s} \in Q^2V$. This implies $QV \subseteq Q^2V$. But this is impossible because QV must be branched and, thus, must fail to be idempotent by Theorem 3.4.3.

We now show V/P is discrete and begin by assuming Q/P is a branched prime in V/P. Then Q must be a branched prime in V. By Theorem 3.4.3 we know that $Q^2 \neq Q$ and so we must have that $Q^2/P \neq Q/P$. As Q/P is not idempotent, we know that V/P must be discrete. $\square$

We offer our next result as consolation to our friends in number theory. We will find good use for it, as well. It should come as no surprise. We do introduce some notation here, though. When we call a ring a **DVR**, we will mean that we are speaking of a discrete valuation domain.

Theorem 3.4.5. *Suppose V is a DVR with nonzero maximal ideal M. Then V is Noetherian if and only if dim(V)=1.*

Proof. ($\Rightarrow$) From Theorem 2.1.16 we know that every Noetherian valuation domain is a PID. Now use the fact that every nonfield PID is one dimensional.

($\Leftarrow$) Suppose dim(V)=1. Since V is discrete, then Lemma 3.4.2 tells us that $\{M^n\}_{n=1}^{\infty}$ is the set of all M-primary ideals. Choose some $m \in M - M^2$. By Theorem 2.5.9, mV is M-primary. Also, note $M^2 \subsetneq mV$. Thus, $mV = M$. We have shown that every prime ideal of V is finitely generated and so we are done. $\square$

We can now consider our first expansion of the characterization of DVR's given in Theorem 3.4.7. First we need the following lemma.

Lemma 3.4.6. *Let V be a valuation domain with maximal ideal M and $P \in Spec(V)$. Then M is principal in V if and only if M/P is principal in V/P.*

Proof. Suppose $M/P = (m + P)$ for some $m \in M$ and let $x \in M$. If $m \nmid x$, then $mV \subsetneq xV \Rightarrow (m + P) \subsetneq (x + P) \subseteq M$, a contradiction. So $M/P = (m + P)$. The converse is trivial. □

Theorem 3.4.7. *Let V be a valuation domain. Then V is discrete if and only if PV_P is principal for any branched $P \in Spec(V)$.*

Proof. ($\Rightarrow$) Suppose V is a DVR and $P \in Spec(V)$ is branched. Let $Q \subsetneq P$ be adjacent primes in V. By Theorem 3.4.4, V_P/QV_P is a DVR. Note further that $dim(V_P/QV_P) = 1$. Using Theorem 3.4.5 we deduce that V_P/QV_P is Noetherian and so PV_P/QV_P is principal. Lemma 3.4.6 now assures us that PV_P is principal.

($\Leftarrow$) We assume PV_P is principal for branched $P \in Spec(V)$. Then letting P be a branched prime of V, let us say $PV_P = xV_P$ for some $x \in PV$. Now $P^2V_P = x^2V_P \subsetneq xV_P$. If $P^2V = PV$, then $P^2V_P = PV_P$. Hence, $P^2V \subsetneq PV$ and so V is a DVR by Theorem 3.4.3. □

We would like to characterize DVR's in terms of their groups of divisibility. Our next result makes it clear that we are moving in this direction.

Theorem 3.4.8. *Let V be a valuation domain with prime ideals $Q \subsetneq P$. Let $H_P \subsetneq H_Q$ be the associated convex subgroups of $G = G(V)$. Then $GV_P/QV_P \cong_o H_Q/H_P$.*

Proof. For the sake of simplicity, let us say $V' = V_P/QV_P$ and $(a + QV_P)U(V') = \tilde{a}U(V')$. We consider the order map $\phi : H_Q \to G(V')$ induced by $\phi(sU(V)) = \tilde{s}U(V')$. Clearly ϕ is epic and so $H_Q/Ker(\phi) \cong_o G(V')$. The only thing left to show is $Ker(\phi) = H_P$. First, if $\phi(sU(V) = \tilde{s}U(V') = U(V')$, then $\tilde{s} \in U(V')$, i.e., $s + QV_P \in U(V')$. This means $s \in V - P$ and so $sU(V) \in H_P$, establishing $Ker(\phi) \subseteq H_P$. Conversely, if $s \in V - P$, then $\tilde{s} \in U(V')$. So $H_P \subseteq Ker(\phi)$ and we are done. $\square$

Definition 3.4.9. A LOAG is **discrete** if, given adjacent convex subgroups $H_1 \subsetneq H_2$, then $H_2/H_1 \cong_o \mathbb{Z}$

Our next theorem characterizes DVR's in terms of groups of divisibility. It will help us build interesting examples. This result can be found as an exercise in [7].

Theorem 3.4.10. *Let V be a valuation domain with G=$G(V)$. Then V is discrete if and only if G is discrete.*

Proof. ($\Rightarrow$) Suppose V is discrete and let $H_1 \subsetneq H_2$ be adjacent convex subgroups of $G = G(V)$. We wish to show $H_2/H_1 \cong_o \mathbb{Z}$. Let P_1 and P_2 be the adjacent primes in V corresponding to H_1 and H_2 and note $P_2 \subsetneq P_1$. We already know from Theorem 3.3.10 that $G(V/P_2) \cong_o H_2$. Now consider the localization $R = (V/P_2)_{P_1/P_2}$. Note that Theorem 3.4.4 can be used to verify that R is a DVR. Since $P_2 \subsetneq P_1$ are adjacent, then $dim(R) = 1$. Now Theorem 3.4.5 tells us that R is Noetherian. From Theorem 3.3.3 it follows that $\mathbb{Z} \cong_o G(R)$. We now wish to show that $G(R) \cong_o H_2/H_1$. To this end, observe that because R is a localization of V/P_2, then we have the natural projection $\pi : G(V/P_2) \to G(R)$ given by $\pi(kU(V/P_2)) = kU(R)$. Since $G(V/P_2) \cong_o H_2$, then $G(R) \cong_o H_2/Ker(\pi)$. It is not difficult now to verify that $Ker(\pi) = H_1$ and so we are done.

($\Leftarrow$) Assuming G is discrete, we wish to show V is discrete. Let $P \in Spec(V)$ be branched. Then there exists adjacent primes $Q \subsetneq P$ in V. Now let $H_P \subsetneq H_Q$ be the associated adjacent convex subgroups of G. Because G is discrete, we have $\mathbb{Z} \cong_o H_Q/H_P \cong_o G(V_P/QV_P)$. Thus, V_P/QV_P is a one-dimensional valuation domain with principal maximal ideal. Quoting Lemma 3.4.6 we deduce that PV_P is principal in V_P. Theorem 3.4.7 now assures us that V is a DVR. □

Having characterized DVR's in terms of groups of divisibility, we can turn our attention to producing examples of infinite-dimensional DVR's. The following lemma will prove useful in this endeavor.

Lemma 3.4.11. *Let G be a free LOAG in the lexicographic order with nonzero convex subgroup H. Say $G = \bigoplus_\Lambda \mathbb{Z}$ and let $\mathcal{B}$ be the standard basis for G. Then H is free with basis $\mathcal{B} \cap H$.*

Proof. Let $0 \neq r_1e_1 + \ldots + r_ne_n = h \in H$, where each $r_i \in \mathbb{Z}$, each $e_i \in \mathcal{B}$, and $e_1 > e_2 > \ldots > e_n$. Since $-h \in H$, then we may assume $r_1 > 0$. Note $0 < e_1 < 2h \in H$. The convexity of H allows us to conclude $e_1 \in H$. Hence, $r_2e_2 + \ldots + r_ne_n \in H$. Continuing inductively, we deduce $e_1, \ldots, e_n \in H$. Thus, $\mathcal{B} \cap H$ is a basis for H and we are done. □

Theorem 3.4.12. *Suppose V is a valuation domain and consider the lexicographic sum $G = \bigoplus_\Lambda \mathbb{Z}$. If $G(V) \cong_o G$, then V is a DVR.*

Proof. By Theorem 3.4.10 it suffices to show that G is discrete. Let $H \subsetneq Q$ be adjacent convex subgroups of G. Letting $\mathcal{B}$ be the standard basis for G then we know from Lemma 3.4.11 H and Q are free LOAG's with bases $\mathcal{B} \cap H$ and $\mathcal{B} \cap Q$, respectively. As H and Q are adjacent, then we know there is a unique $e \in \mathcal{B} \cap (Q - H)$. Thus, Q/H is a rank-1 free abelian group, i.e., $Q/H \cong_o \mathbb{Z}$. □

Theorem 3.4.12 furnishes us a means of constructing a plethora of DVR's because given any linearly ordered set Λ we can find a valuation domain whose group of divisibility is $\bigoplus_\Lambda \mathbb{Z}$. In particular, if Λ is an infinite set, then our valuation domain must also be infinite dimensional. We ask ourselves whether or not every DVR has a group of divisibility which can be realized as a free abelian group. The answer is no. To see this, we again use the fact that given *any* LOAG (call it G), then we can construct a valuation domain whose group of divisibility is order isomorphic to G. Thus, all we have to do is construct G in such a way as to guarantee that G is discrete but cannot be realized as a direct sum. It seems natural, therefore, to turn our gaze upon direct products.

Theorem 3.4.13. *Suppose Λ is a linearly ordered set and $\{G_i\}_\Lambda$ is a family of nontrivial POAG's. Let $G = \prod_\Lambda G_i$. Given $(a_i), (b_i) \in G$, say $(a_i) < (b_i)$ if there exists some $k \in \Lambda$ such that $a_k < b_k$ and if $j < k$, then $a_j = b_j$. Then $(G, \leq)$ is a POAG.*

Proof. □

In Theorem 3.4.13 we call G the **lexicographic product** of the family $\{G_i\}_\Lambda$. As one might expect, lexicographic products behave very differently than lexicographic sums. Like sums, lexicographic products may admit elements that are not comparable, i.e., these products need no be linearly ordered. In the case of direct sums, this can happen if and only if at least one of the summands is not a LOAG. Unlike sums, a lexicographic product may admit non-comparable elements even when each G_i is a LOAG. Our next example illustrates this and provides motivation for Theorem 3.4.16, where we show precisely when a lexicographic product of POAG's is a LOAG.

Example 3.4.14. Consider the lexicographic product $G = \prod_{\mathbb{Q}} \mathbb{Z}$. Let $S = \mathbb{Q} \cap (\pi, \infty)$. Now choose $(a_i), (b_i) \in G$ such that $a_i = b_i$ for all $i \in \mathbb{Q} - S$. For every $i \in S$, let $a_i = 1$ and $b_i = -1$. Since $a_i \geq b_i$ for all $i \in \mathbb{Q}$, then $(a_i) < (b_i)$ is impossible. Also, it is clear that $(a_i) \neq (b_i)$. Now should we have $(b_i) < (a_i)$, then there would be some $j \in \mathbb{Q}$ such that $b_j < a_j$ and $b_k = a_k$ for all $k < j$. This then forces $j \in S$.

But because $\pi < j \in \mathbb{Q}$, then there exists some $k \in \mathbb{Q}$ such that $\pi < k < j$, in which case $a_k = b_k$ (since $k < j$ and $a_k \neq b_k$ (since $k \in S$), a contradiction. Hence, G is not a LOAG.

Example 3.4.15. Consider the lexicographic product $G = \prod_{i=-1}^{-\infty} \mathbb{Z}$. Then the elements $(\ldots, -1, 0, -1, 0)$ and $(\ldots, 0, -1, 0, -1)$ are not comparable. Thus, G is not a LOAG.

Observe in both of the previous examples that we are indexing families of LOAG's with linearly ordered sets that are not well-ordered. This ushers us in the direction of the following characterization.

Theorem 3.4.16. *Let Λ be a linearly ordered set and consider the lexicographic product $G = \prod_\Lambda L_i$, where each L_i is a nontrivial LOAG. Then G is a LOAG if and only if Λ is well-ordered.*

Proof. ($\Rightarrow$) We assume G is a LOAG and let $\emptyset \subsetneq S \subseteq \Lambda$. Given any L_i, choose some positive $a_i \in L_i$. Now let $(x_i), (y_i) \in G$ such that $x_i = y_i$ for all $i \notin S$. If $i \in S$, let us say $x_i = a_i$ and $y_i = -a_i$. Since S is nonempty, then $(x_i) \neq (y_i)$. But G is a LOAG and so we must have $(y_i) < (x_i)$. Hence, there is some $j \in \Lambda$ such that $y_j < x_j$ and $y_k = x_k$ for all $k < j$. Hence, j is minimal in S.

($\Leftarrow$) We choose $(a_i), (b_i) \in G$ with $(a_i) \neq (b_i)$. Let $S = \{\lambda \in \Lambda | a_\lambda \neq b_\lambda\}$. As Λ is well-ordered, then S admits a minimal element c. Without loss of generality, say $a_c < b_c$. Then $(a_i) < (b_i)$ and we are done. $\square$

We remark here that some authors require that lexicographic products of LOAG's be indexed by well-ordered sets when defining them. Theorem 3.4.16 explains why someone might do this; they want the product to be linearly ordered.

Theorem 3.4.16 gives us an opportunity to create yet more examples of DVR's by using direct products instead of direct sums as we did in Theorem 3.4.12.

Theorem 3.4.17. *Suppose Λ is a well-ordered set and V is a valuation domain such that $G(V) \cong_o G = \prod_\Lambda \mathbb{Z}$. Then V is a DVR.*

Proof. For every $\lambda \in \Lambda$, let $\pi_\lambda : G \to \mathbb{Z}$ be the natural projection, i.e., $\pi_\lambda((a_i)) = a_\lambda$. If $B = \{e_i | i \in \Lambda\}$ is the standard basis for $\bigoplus_\Lambda \mathbb{Z}$, then $B \subseteq G$. Now as Λ is well-ordered, then there exists a minimal $\alpha \in \Lambda$ such that $e_\alpha \in Q$. Note that because H is convex and $H \subsetneq Q$, then $e_\alpha \notin H$. Hence, $H \subseteq Ker(\pi_\alpha)$. But H and Q are adjacent and so we must have $H = Ker(\pi_\alpha)$. Thus, $Q/H = Q/Ker(\pi_\alpha) \cong_o \mathbb{Z}$. Having shown that G is discrete, it follows from Theorem 3.4.10 that V is a DVR. □

Observe that Theorem 3.4.17 allows us to create examples of rings that exhibit an interesting pathology. For example, if we index by $\mathbb{N}$, then there are elements in the valuation domain which are products of infinitely many primes. We will see more of this type of behavior in the next section.

3.5. Applications in a Wider Setting

Much of the inspiration for the work done in this section comes from the considerations of the previous section. In addition to providing some useful characterizations of domains via groups of divisibility, we push further and attempt to construct groups of divisibility with properties that reflect interesting ring-theoretic properties. We begin with some observations regarding POAG's.

Theorem 3.5.1. *Consider a set $\{G_i\}_\Lambda$ of POAG's. Then $\prod_\Lambda G_i$ is a POAG via the relation $(a_i) \leq (b_i)$ if and only if each $a_i \leq b_i$.*

The proof is easy enough to be omitted. The partially ordering in the previous theorem is called a **product** ordering. Unless otherwise stated, it will always be assumed that $\prod_\Lambda G_i$ takes on the product order. We should point out that every subgroup of POAG is itself a POAG via the inherited product order. It is easily seen that a subgroup of a LOAG is also a LOAG. However, if G is a POAG and H is a subgroup with the inherited order, it may happen that the inherited order of H and the partial ordering of G are not the same. For example, even though a subgroup of a lattice-ordered abelian group is partially ordered via the inherited order, the subgroup may fail to be lattice ordered. We show how this can happen in Example 3.5.21. Like products, we will always regard a direct sum of POAG's in the product order unless otherwise stated.

One of the motifs of this chapter has been an examination of the interplay between ring theory and the theory of POAG's. The motivation for the following definitions should be no mystery.

Definition 3.5.2. Suppose G is a POAG. If $x \in G_+$ is minimal, we call x an **atom** or **irreducible** G. We denote the set of all atoms of G by Irr(G). If $x, y, z \in G_+$ such that whenever $x = y + z$, then $x \leq a$ or $x \leq z$, then we say x is **prime** in G. We denote the set of all primes of G by $Pr(G)$. Additionally, if every $g \in G_+$ can be expressed as a (finite) sum of atoms, then we say G is **atomic**.

Theorem 3.5.3. *Let D be a domain and $G = G(D)$.*

a) $x \in Irr(D)$ if and only if $xU \in Irr(G)$

b) x is a prime element in D if and only if $xU \in Pr(G)$

Proof. $a)$ Assuming $p \in Irr(D)$, we let $U \leq xU, yU$ for some $x, y \in D$ with $pU = xU + yU$. Then $xyU = pU > U$ so that either $xU > U$ or $yU > U$. Assume $xU > U$. Then $x \notin U(D)$ and $p = u(xy)$ for some $u \in U(D)$. Since $p \in Irr(D)$, then $uy \in U(D)$ and so $y \in U(D)$, i.e., $yU = U$. Conversely, we assume $pU \in Irr(G)$ and let $p = ab$ for some $a, b \in D$. Then $pU = aU + bU$. Since $p \in Irr(D)$, then we may say $bU = U$, i.e., $b \in U(D)$.

b) Assuming p is a prime element in D, say $pU \leq aU + bU$ for some $a, b \in D$. Then $pU \leq abU \Rightarrow p|ab$. Thus, $p|a$ or $p|b$, i.e., $pU \leq aU$ or $pU \leq bU$. Conversely, assume $pU \in Pr(G)$ and say $p|ab$ for some $a, b \in D$. Then $pU \leq aU + bU$ and so, without loss of generality, we may say $pU \leq aU$. Then $p|a$ and we are done. □

Theorem 3.5.4. *Let G be a POAG. Then $Pr(G) \subseteq Irr(G)$.*

Proof. Suppose $x \in Pr(G)$ and say $x = a + b$ for some $a, b \geq 0$. We may assume $x \leq a$ so that $0 = (a - x) + b$. If $b > 0$, then because $a - x \geq 0$, then it would follow that $(a - x) + b > 0$, a contradiction. Hence, $b = 0$. □

Theorem 3.5.5. *Assume G is a POAG and $0 \leq a, b \in G$. If $p \in Irr(G)$ and $a + b < p$, then $a = b = 0$.*

Proof. As $0 \leq a+b < p$, then $p = (a+b)+(p-a-b)$. Thus, $a+b = 0$ or $p-a-b = 0$. But $p - a - b > 0$. So $0 = a + b$. Since $a \geq 0$, then $0 \leq b \leq a + b = 0 \Rightarrow b = 0$. Similarly, $a = 0$. □

Theorem 3.5.6. *Let $G_1 = \bigoplus_\Lambda \mathbb{Z}$ and $G_2 = \prod_\Lambda \mathbb{Z}$ be partially ordered in the product order. Let $\lambda : G_1 \to G_2$ be the natural inclusion and assume that $\mathcal{B} = \{e_i | i \in \Lambda\}$ is the natural basis for G_1.*

a) $B = Pr(G_1) = Irr(G_1)$

b) $\lambda(B) = Pr(G_2) = Irr(G_2)$

Proof. We prove only part a) as the proof of part b) is virtually identical (with only some notational differences). To establish the equalities, we show $\mathcal{B} \subseteq Pr(G_1) \subseteq Irr(G_2) \subseteq \mathcal{B}$. If $e_i < (x_j)$, then each $x_j \geq 0$ and $x_i \geq 1$. Take $(a_j), (b_j) \in G_1$ with $e_i \leq (a_j), (b_j)$. Then $e_i \leq (a_i + b_i) \Rightarrow 0 \leq a_i + b_i$ for all i and $a_i + b_i \geq 1$. Hence, $a_i \geq 1$ or $b_i \geq 1$. Without loss of generality, assume $a_i \geq 1$. Then $e_i \leq (a_i)$.

In Theorem 3.5.4 it was already shown that $Pr(G) \subseteq Irr(G)$ for any POAG. So we need only show now that $Irr(G_1) \subseteq \mathcal{B}$. Suppose then $(x_j) \in Irr(G_1)$. Then $x_j \geq 0$ for all j and $(x_j) > 0$. Say $x_i > 0$. Then we have $0 < e_i \leq (x_j)$. If $e_i < (x_j)$, then $0 + e_i < (x_j)$. By Theorem 3.5.5 it would follow that $e_i = 0$, a contradiction. Therefore, $e_i = (x_j)$ and we are done. $\square$

Theorem 3.5.7. *Suppose G is a POAG such that $G = \langle G_+ \rangle$. Then every $a \in G_+$ is a (finite) sum of primes if and only if G is (order isomorphic to) a free abelian group in the product order.*

Proof. We prove only the forward implication as the reverse implication is obvious. Let $Pr(G) = \{p_i | i \in \Lambda\}$ and consider the map $\phi : G \to \bigoplus_\Lambda \mathbb{Z}$ induced by $\phi(p_i) = e_i$, where $\mathcal{B} = \{e_i | \Lambda\}$ is the standard basis. Suppose $g \in G_+$. Then $g = \sum_{j=1}^{n} s_j e_{i_j}$ with each $e_{i_j} \in \mathcal{B}$ and each $s_j \in \mathbb{N}$. Then $\phi(g) > 0$. Quoting Theorem 3.1.8 we deduce that ϕ is order-preserving. It also readily follows now that ϕ is monic. Further, since each $e_i \in \mathcal{B}$ admits a pre-image, then ϕ is epic and so we are done.

$\square$

We now characterize UFD's via groups of divisibility. Dr. Jim Coykendall shares this result.

Theorem 3.5.8. *A non-field domain D is a UFD if and only if $G(D)$ is (order isomorphic to) a free abelian group in the product order.*

Proof. ($\Rightarrow$) If D is a UFD, then every nonzero nonunit of D is a (finite) product of primes. Further, given any domain with group of divisibility G, then $G = \langle G \rangle$. Hence, every element in $G(D)_+$ is a (finite) sum of primes of $G(D)$. Now quote Theorem 3.5.7.

($\Leftarrow$) Suppose $G(D) \cong_o \bigoplus_\Lambda \mathbb{Z}$ in the product order. Then every $aU \in G(D)_+$ is a sum of primes of G(D). Hence, every nonzero nonunit in D is a product of primes, i.e., D is a UFD. $\square$

Note that the group of divisibility of a UFD is telling us a little more than we might at first realize. In particular, the cardinality of the indexing set tells us precisely how many atoms a particular UFD admits (up to associates).

Example 3.5.9. If D is a domain and $G(D) \cong_o \bigoplus_{i=1}^{n} \mathbb{Z}$, then D is a UFD with only finitely many atoms.

Example 3.5.10. Since $Irr(\mathbb{Z})$ is countably infinite and $\mathbb{Z}$ is a UFD, then $G(\mathbb{Z}) \cong_o \bigoplus_{i=1}^{\infty} \mathbb{Z}$.

Example 3.5.11. Let R be a countable UFD. Then $D = R[x_1, x_2, \ldots]$ is a countable UFD. Thus, $G(D) \cong_o G(R) \cong_o G(\mathbb{Z})$.

It is evident from the previous example that groups of divisibility cannot tell the difference between PID's and UFD's in the general case. Continuing in this vein, note that it would not be difficult to characterize domains that are completely integrally closed with groups of divisibility. This is because the property of almost integrality is strictly a multiplicative property, which is exactly the kind of property groups of divisibility can pick up on. Thus, because the notion of integrality is both an additive and multiplicative notion, we cannot hope to characterize integrally closed domains via groups of divisibility.

Having characterized UFD's in terms of groups of divisibility, we would like to do the same for GCD domains. First, we make some remarks about least common multiples. Given a domain D and some $x, y \in D$, then $z \in D$ is their *least common multiple* provided both $x, y | z$ and given any $w \in D$ such that $x, y | w$, then $z | w$. Recall that an LCM-domain is a domain in which every pair of elements admits a least common multiple.

Remark 3.5.12. Given a domain D and $x, y \in D$, then $lcm(x, y)$ need not exist even when $[x, y]$ exists. For example, letting F be a field and $D = F[[x^2, x^3]]$, then $[x^2, x^3] = 1$. However, $lcm(x^2, x^3)$ does not exist because $x^2, x^3 | x^5, x^6$ but $x^5 \nmid x^6$.

Here is a theorem which will prove useful when we use the group of divisibility to characterize GCD domains. It is also a nice illustration of the interplay that is possible between theory of POAG's and factorization. The proof is very straightforward and will be skipped.

Theorem 3.5.13. *[7] Let D be a domain, $x, y, z \in D^*$, and $xU, yU, zU \in G(D)$.*

a) If $zU = inf(xU, yU)$, then $z = [x, y]$.

b) If $inf(xU, yU)$ exists and $z = [x, y]$, then $zU = inf(xU, yU)$.

c) $zU = sup(xU, yU)$ if and only if $z = lcm(x, y)$

The mention of LCM-domains is a rare occurrence in the literature. Our next theorem demonstrates why this is not an accident.

Theorem 3.5.14. *[7] Let D be a domain with group of divisibility G and quotient field K. The following are equivalent:*

a) D is a GCD domain

b) D is an LCM-domain

c) G is lattice ordered

Proof. a) $\Rightarrow$ b) Let $x, y \in D^*$ and $z = [x, y]$. Observe that we already have that $zU \leq xU, yU$. From Theorem 3.1.16 and Theorem 3.5.13, it suffices to show that $zU = inf(xU, yU)$. To this end, we let $a, b \in D^*$ such that $[a, b] = 1$ and $ab^{-1}U \leq xu, yU$. Then $aU \leq xbU, ybU$. Since $aU \leq xbU$, then $a | xb$. But as $[a, b] = 1$, we must have that $a|x$. Similarly, we may deduce $a|y$. Since $z = [x, y]$, then $a|z$. Since $aU \leq zU$, then all the more so we have that $aU \leq bzU$. Hence, $ab^{-1}U \leq zU$ and we are done.

$b) \Rightarrow a)$ Now let us posit that D is an LCM-domain. We pick some $a, b \in D^*$ and let $c = lcm(a, b)$. From Theorem 3.1.16 we know that $inf(aU, bU)$ exists. Further, from Theorem 3.1.16 it follows that $cU + inf(aU, bU) = aU + bU$. Let $k \in K$ such that $inf(aU, bU) = kU$ so that $cU + kU = aU + bU$. As $c|ab$, then $cU \leq aU + bU$. Equivalently, $aU + bU - cU \geq U$. We have therefore demonstrated that $kU \geq U$, i.e., $k \in D$. From part $a)$ of Theorem 3.5.13 we obtain the desired result.

$c) \Rightarrow b)$We assume G is lattice ordered and let $x, y \in D^*$. Then $sup(xU, yU)$ exists. That D is an LCM-domain follows from Theorem 3.1.16.

$a), b) \Rightarrow c)$ We begin by letting $a, b, c, d \in D^*$ such that $[a, b] = 1 = [c, d]$. We would like to show G is lattice ordered, i.e., demonstrate the existence of some $xy^{-1}U \in G$ such that $xy^{-1}U = sup(ab^{-1}U, cd^{-1}U)$. Having already shown that GCD domain if and only if LCM-domain, we let $x = lcm(a, c)$ and $y = [b, d]$. First, observe that $ab^{-1}U \leq xy^{-1}U$ because $a|x$ and $y|b$. So $U \leq xby{-}1a^{-1}U$. Similarly we have that $U \leq xcy^{-1}d^{-1}U$. Suppose now $ab^{-1}U, cd^{-1}U \leq mn{-}1U$ for some relatively prime $m, n \in D^*$. Then $U \leq mbn^{-1}a^{-1}U$. Since $[a, b] = 1$, then $a|m$. Similarly, $n|b$. We can again mimic this argument to that $c|m$ and $n|d$. Restating, we have $a, c|m$ and $n|b, d$. Having assumed $x = lcm(a, c)$ and $y = [b, d]$ we conclude $x|m$ and $n|y$. Put another way, $xU \leq mU$ and $nU \leq yU$. Thus, $mU - xU + yU - nU \geq U$ and so $xy^{-1}U \leq mn^{-1}U$. We have shown that $xy^{-1}U = sup(ab^{-1}U, cd^{-1}U)$, as desired.

□

Here is one we simply cannot resist. This result can be found in [7] but the proof is original.

Theorem 3.5.15. *Every localization of a GCD domain is a GCD domain.*

Proof. Let D be a GCD domain and $S \subseteq D^*$ a multiplicative system. We wish to show D_S is a GCD domain. Let H be the subgroup of $G(D)$ generated by the set $\{sU|s \in S\}$. We then know from Theorem 3.2.5 $G(D_S) \cong_o G/H$. Now we let $aU + H, bU + H \in G/H$. As D is a GCD domain, then we know G is lattice ordered. Let $cU = sup(aU, bU)$. Using the fact that $U \in H$ we have from Theorem 3.1.10 that $aU + H, bU + H \leq cU + H$. Let $aU + H, bU + H \leq yU + H$ for some $yU + H \in G/H$. By Theorem 3.1.10 we are guaranteed the existence of some $h_1U, h_2U \in H$ such that $aU \leq yU + h_1U$ and $bU \leq yU + h_2U$. Since H is generated by $\{sU|s \in S\}$, then we may say $h_1U = s_1s_2^{-1}U$ and $h_2U = s_3s_4^{-1}U$ for some $s_1, s_2, s_3, s_4 \in S$. Observe now that $h_1U, h_2U \leq s_1s_3U \in H$ and so $aU, bU \leq yU + s_1s_3U$. As $cU = sup(aU, bU)$ we have that $cU \leq yU + s_1s_3U$. Using Theorem 3.1.10 again we conclude $cU + H \leq yu + H$ and so $cU + H = sup(aU + H, bU + H)$.

We have now shown that G/H is lattice ordered. From Theorem 3.2.5 we know that $G(D_S) \cong_o G/H$. Now quote Theorem 3.5.14 to reach the desired conclusion. □

Recall in Theorem 3.3.2 it was shown that given every linearly ordered abelian group can be realized as the group of divisibility of a valuation domain. Kaplansky and Jaffard went even further and proved the following amazing result.

Theorem 3.5.16. *Every lattice ordered abelian is order isomorphic to the group of divisibility of a GCD domain.*

Actually, a little more is true than what is stated in the previous theorem. Not only can every lattice ordered abelian group be realized as the group of divisibility of a GCD domain, but it has been shown that every such POAG can be realized as the group of divisibility of a Bêzout domain. This astounding result is known as the Jaffard-Ohm-Kaplansky Theorem. We use it implicitly in the first sentence of the proof of the following theorem.

Theorem 3.5.17. *Suppose G is a lattice ordered abelian group and let $H \leq G$ be convex. Then G/H is lattice ordered.*

Proof. Suppose G is a lattice ordered abelian group and let D be a GCD domain such that $G(D) \cong_o G$. Say $\varphi : G(D) \to G$ is an order isomorphism. Let $S = \{s \in D | sU \in \varphi^{-1}(H_+)\}$. Since H is convex, then H_+ is convex in G_+ by Theorem 3.1.9. Now Theorem 3.2.4 can be used to deduce that S is saturated and multiplicatively closed. Now we have that $G(D_S) \cong_o G(D)/\varphi^{-1}(H) \cong_o G/H$. But Theorem 3.5.15 tells us that D_S is a GCD domain. Now we use Theorem 3.5.14 to conclude that $G(D_S)$ is lattice ordered and we are done. □

Our next result can be used to produce rings with exotic factorization behavior. The proof is a straightforward application of the appropriate definitions.

Theorem 3.5.18. *Let $\{G_i\}_\Lambda$ be a set of POAG's.*

a) $\bigoplus_\Lambda G_i$ is atomic if and only if each G_i is atomic.

b) $\bigoplus_\Lambda G_i$ and $\prod_\Lambda G_i$ are lattice-ordered if and only if each G_i is lattice-ordered.

c) If each G_i is atomic, then $\prod_\Lambda G_i$ is atomic if and only if Λ is finite.

In Theorem 3.5.8 it was shown that a domain is a UFD if its group of divisibility is free in the product order. Theorem 3.5.18 allows us to streamline this argument a bit. Indeed, if the group of divisibility G of a domain D is free in the product order, then by Theorem 3.5.18 we already know that G is lattice ordered. Moreover, as a free abelian group can be realized as a direct sum of the integers, then Theorem 3.5.18 also assures us that G is atomic. Now use Theorem 3.5.14 to deduce that D is an atomic GCD domain. As GCD implies AP, then D is both atomic and AP, i.e., D is a UFD.

Here is a construction that yields some very strange, if not troubling, factorization properties.

Example 3.5.19. Let $G = \prod_{\mathbb{R}} \mathbb{Z}$ in the product order. Since $\mathbb{Z}$ is lattice ordered, then Theorem 3.5.18 assures us G is lattice ordered. As such, Theorem 3.5.16 allows us to let D be a GCD domain such that $G(D) \cong_o G$. Note that, like UFD's, every nonzero nonunit can be expressed as a product of primes. However, the products in D are very different than in a UFD. Remember that in the case of UFD's all products are finite. But in D, not only do we have elements that are products of infinitely many primes, there are elements that factor uniquely as a product of uncountably many primes.

Theorem 3.5.18 can be used to demonstrate that subgroups of lattice-ordered abelian groups need not be lattice-ordered. Our argument is non-constructive and will use the following lemma.

Lemma 3.5.20. *Let D be a domain and say $Max(D) = \{M_i | i \in \Lambda\}$. Then there exists an order monomorphism $G(D) \to \prod_{\Lambda} G(D_{M_i})$ given by $kU(D) \to kU(D_{M_i})$.*

Proof. If $kU(D) \geq U(D)$, then $k \in D$. Consequently, $k \in D_{M_i}$ for all i. It follows from Theorem 3.1.8 that the map is order-preserving. Moreover, if $kU(D_{M_i}) = U(D_{M_i})$, then $k \in U(D_{M_i})$ for all i. But then $k \in \bigcap_{\Lambda} D_{M_i} = D$. Also, because $k \in U(D_{M_i})$ for all i, then $k \notin M_i$ for all i. Thus, $k \in U(D)$. The kernel of the map is therefore trivial and so we are done. □

We now demonstrate that a subgroup of a lattice-ordered abelian group need not be lattice ordered. Note the implicit use of the Jaffard-Ohm-Kaplansky Theorem.

Example 3.5.21. Let D be a Prüfer domain (non-field). Say $Max(D) = \{M_i | i \in \Lambda\}$. Because D is Prüfer, then D_M is a valuation domain for all $M \in Max(D)$.

Consequently, $G(D_{M_i})$ is a LOAG for all $i \in \Lambda$. Now let $G = \prod_{i\in\Lambda} G(D_{M_i})$. Because each $G(D_{M_i})$ is a LOAG and therefore lattice-ordered, then by Theorem 3.5.18 we know that G is lattice-ordered. Using Lemma 3.5.20 we have an order monomorphism $G(D) \to G$. Hence, we can think of $G(D)$ as a subgroup of a lattice-ordered abelian group. Now if every subgroup of a lattice-ordered abelian group was lattice-ordered, then it would follow that $G(D)$ must be lattice ordered. From Theorem 3.5.14 it would then follow that D is a GCD domain. We could then conclude that every Prüfer domain is also a GCD domain. But if this were so, then by Theorem 2.1.25 it would follow that every Prüfer domain must also be a Bêzout domain, a contradiction because $D = \mathbb{Z}[\sqrt{-5}]$ is Prüfer but not Bêzout. That D is Prüfer follows from the fact that D is Dedekind. If D were Bêzout, then because D is Noetherian it would follow that D is both Bêzout and atomic, i.e., D is a PID. But note that $D = \mathbb{Z}[\sqrt{-5}]$ is not even a UFD.

The Jaffard-Ohm-Kaplansky Theorem asserts that every lattice ordered abelian group can be realized as the group of divisibility of a Bêzout domain. Theorem 3.5.22 provides us with a means of producing examples of lattice ordered abelian groups which can *only* be realized as the group of divisibility of a Bêzout domain.

Theorem 3.5.22. *Suppose $L_1, L_2, \ldots, L_n$ are nonzero LOAG's, D is a domain, and $G = \bigoplus_{i=1}^{n} L_i$ in the product order. Assume that there exists an order isomorphism $G(D) \to G$. Then $|Max(D)| = n$. Further, for each $M \in Max(D)$ we have that $G(D_M) \cong_o L_i$ for some L_i. Lastly, D must be a Bêzout domain.*

Proof. We first observe that by Theorem 3.5.18 G is lattice-ordered and so D is a GCD domain by Theorem 3.5.14. Now let $H = \{(0, a_2, a_3, \ldots, a_n | a_i \in L_i\}$ and note that H is a convex subgroup of G. Hence, there exists a multiplicative system $S \subseteq D^*$ such that $G(D_S) \cong_o G/H$. Noting that the map $L_1 \to G/H$ given by $\ell \to (\ell, 0, 0, \ldots, 0) + H$ is an order isomorphism, then $G(D_S) \cong_o L_1$. As L_1 is a

LOAG, then D_S must be a valuation domain and therefore quasi-local. It follows that $D_S = D_{P_1}$ for some $P_1 \in Spec(D)$. Similarly, we can construct prime ideals $P_1, P_2, \ldots, P_n$ such that $G(D_{P_i}) \cong_o L_i$.

We now show that $P_1 \in Max(D)$. Let $M \in Max(D)$ such that $P_1 \subseteq M$. Let $0 \neq m \in M$ and note that if $mU \to (\ell_1, \ldots, \ell_n)$ with $\ell_1 > 0$, then $m \notin U(D_{P_1})$. As D is a GCD domain, it follows that $m \in P_1$ and we conclude that $P_1 = M$. Assume then, to the contrary, that $mU \to (0, \ell_2, \ldots, \ell_n)$. Because G has the product order, then each $\ell_i \geq 0$. Since $m \in P_1$, then we must have $\ell_i > 0$ for at least one ℓ_i. Without loss of generality, assume $\ell_2 > 0$. Then $MD_{P_2} \in Spec(D_{P_2}) \Rightarrow MD_{P_2} \subseteq P_2D_{P_2} \Rightarrow M \subseteq P_2$. Since $M \in Max(D)$, then $P_2 = M$. But then $P_1 \subseteq P_2$, a contradiction since $P_1D_{P_2} = D_{P_2}$. Hence, $P_1 \in Max(D)$. Similarly, each $P_i \in Max(D)$.

We have now shown that there exist $M_1, \ldots, M_n \in Max(D)$ such that $G(D_{M_i}) \cong_o L_i$. Suppose now $M \in Max(D)$ and choose some $0 \neq m \in M$. Say $mU \to (x_1, \ldots, x_n) > 0$. Then some $x_i > 0$ and so $x \in M_i$. It follows that $M \subseteq \bigcup_{i=1}^{n} M_i$ and so $M \subseteq M_i$ for some M_i. Thus, $M = M_i$ and so the first two assertions of the theorem are now established.

We now establish the last assertion of the theorem. We have shown $|Max(D)| = n$ and $G(D_M) \cong_o L_i$ for one of our LOAG's L_i. As each $G(D_M)$ is a LOAG, then D_M is a valuation domain. Therefore, D is Prüfer domain. Now use the characterization in Theorem 2.1.25 to conclude that D is a Bêzout domain. $\square$

Remark 3.5.23. In Theorem 3.5.22, note that if each L_i is Archimedean, then D_M is a 1-dimensional valuation domain for each maximal ideal M. By prime correspondence for localizations, it follows that every maximal ideal is a ht-1 prime and so $dim(D) = 1$. Also, notice that if G is a direct sum of more than 2 or more LOAG's in the product order, then D is never a valuation domain. Hence, this theorem shows that we can use groups of divisibility to construct Bêzout domains of any given Krull dimension and these domains are never valuation domain's.

We would not find it very surprising if this result actually characterized semi-quasi-local Prüfer domains. The proof might rest on the following conjecture.

Conjecture 3.5.24. Suppose D is a semi-quasi-local Prüfer domain, $M \in Max(D)$, and $S = \{\mu \in Max(D) | \mu \neq M\}$. If $0 \neq x \in M$, then there exists some $y \in \bigcup_S \mu$ such that $xU(D_M) = yU(D_M)$.

Proving Conjecture 3.5.24 would allow us to give the following characterization of semi-quasi-local Prüfer domains.

Theorem 3.5.25. *Suppose D is a semi-quasi-local domain with $Max(D) = \{M_1, \dots, M_n\}$. The following are equivalent:*

a) D is a Bêzout domain

b) D is a Prüfer domain

c) $G(D) \cong_o \bigoplus_{i=1}^n G(D_{M_i})$ (product order)

Proof. We already know from Theorem 2.1.25 that every Bêzout domain is Prüfer. Further, the implication $c) \Rightarrow a)$ was established in Theorem 3.5.22. Thus, we need establish the implication $b) \Rightarrow c)$. Assuming D is a Prüfer domain, it follows that D_M is a valuation domain for any $M \in Max(D)$ and so each $G(D_M)$ is a LOAG. Say $L_i = G(D_{M_i})$. From Lemma 3.5.20 we know the map $G(D) \to \bigoplus_{i=1}^n L_i$ given by $kU \to (kU_i)$ is an order monomorphism. Letting $x = (a_1U_1, \dots, a_nU_n) \in \bigoplus_{i=1}^n L_i$, we have that $x = \sum_{i=1}^n \lambda_i(a_iU_i)$, where $\lambda_i : L_i \to \bigoplus_{i=1}^n L_i$ is the natural inclusion. Invoking Conjecture 3.5.24, we know there exists a $b_i \in M_i - \bigcup_{j \neq i} M_j$ such that $b_iU_i = a_iU_i$. Observe now that if $i \neq j$, then $b_iU_j = U_j$. Hence, $b_1b_2 \cdots b_nU_1 = b_1U_1 = a_1U_1$. Similarly, $b_1 \cdots b_nU_j = a_jU_j$. Letting $x = b_1b_2 \cdots b_n$, we then have that

$xU \to (xU_1, xU_2, \ldots, xU_n) = (a_1U_1, \ldots, a_nU_n)$. Thus, the map $G(D) \to \bigoplus_{i=1}^n L_i$ is epic and we are done. $\square$

Note that the requirement that D be semi-quasi-local is crucial. For example, if F is a countable field, then $D = F[x, y]$ is a countable UFD with infinitely many primes and so $G(D) \cong_o \bigoplus_{i=1}^\infty \mathbb{Z}$. But D is not Prüfer because D is a 2-dimensional UFD.

We end this chapter with a corollary to Theorem 3.5.22 which characterizes semi-quasi-local PID's.

Corollary 3.5.26. *Suppose D is a non-field domain. The following are equivalent:*

a) D is a semi-quasi-local PID

b) $G(D) \cong_o \bigoplus_{i=1}^n \mathbb{Z}$ for some $n \in \mathbb{N}$

Proof. *a)* $\Rightarrow$ *b)* Since D is a semi-quasi-local PID, then D is one-dimensional and there are only finitely many maximal ideals. Say $|Max(D)| = n$. Then D contains exactly n primes. As such, $G(D) \cong_o \bigoplus_{i=1}^n \mathbb{Z}$.

b) $\Rightarrow$ *a)* We assume $G(D) \cong_o \bigoplus_{i=1}^n \mathbb{Z}$ for some $n \in \mathbb{N}$. Then D is a UFD by Theorem 3.5.8. From Theorem 3.5.22 we know that D is a Bêzout domain. As every Bêzout domain is Prüfer, it follows from Theorem 2.1.27 that D is a PID. $\square$

CHAPTER 4. A GENERALIZATION OF KRULL DIMENSION

4.1. Antimatter Domains

There is another class of rings which has not been mentioned up to now and which will be very important to us. These rings feel pathological from a factorization point of view because their elements cannot be broken down into atoms. We say that a domain D is an **antimatter domain** (AMD) if D admits no atoms. To begin with the trivial case we point out that every field is an AMD. We call an AMD R a *nontrivial* AMD if R is not a field. Here are some less boring examples.

Example 4.1.1. Let F be a field and consider the ring $R = F[x^q]_{q\in\mathbb{Q}_+}$. Then $M = (x^q)_{q\in\mathbb{Q}_+}$ is a maximal ideal. The localization R_M is a 1-dimensional antimatter valuation domain. To see that R_M is a valuation domain note that given any nonzero nonunit in R_M is of the form ux^q for some $u \in U(R_M)$ and some $x^q \in M$. Thus, if $u_1x^{q_1}, u_2x^{q_2} \in R_M$ with $q_1 \leq q_2$, then $u_1x^{q_1} | u_2x^{q_2}$, making R_M a valuation domain. Alternatively, it is not very hard to verify that $G(R_M) \cong_o \mathbb{Q}$. Quoting Theorem 3.3.1 it follows that R_M is a valuation domain. It is also evident from these observations that R_M is an AMD because we can always find a rational between 0 and any other positive rational. Lastly, $dim(R_M) = 1$ because M is a ht-1 prime in R.

The term "antimatter" domain was coined by Coykendall in [6]. However, in [7] it is evident that they had already been noticed. Gilmer observed that if D is an integrally closed domain with an algebraically closed quotient field K, then given any nonzero nonunit $x \in R$, then there exists some nonzero nonunit $y \in R$ and some $n_x \in \mathbb{N}$ such that $x = y^{n_x}$. The reason for this is because given $t^n - x^n \in R[t]$, then as K is algebraically closed we know there is some $y \in K$ satisfying this polynomial.

But now because D is integrally closed, then $y \in D$. It is obvious that D is an AMD. Moreover, we may also deduce from this fact that any integrally closed domain admitting an atom cannot have a quotient field which is algebraically closed. Thus, for example, the quotient field of an integrally closed Noetherian domain or the quotient field of a non-field UFD must fail to be algebraically closed. In particular, $\mathbb{C}$ cannot be realized as the quotient field of a non-field UFD. We can go even further and say that $\mathbb{C}$ is not even an algebraic extension of a quotient field of a non-field UFD. The proof of this follows from the fact that any algebraic extension of a field F is contained in the algebraic closure of F and that an algebraic extension of an algebraic extension is an algebraic extension. Thus, not only is $\mathbb{C}$ not the integral closure of a non-field UFD, but neither is $\mathbb{R}$.

Recall now that a fragmented domain is any domain such that given any nonzero nonunit $x \in D$, then there exists another nonzero nonunit $y \in D$ such that $y^n|x$ for all $n \in \mathbb{N}$. Clearly, such a ring must be an AMD. Moreover, in [5] it was shown that every non-field fragmented domain must be infinite dimensional. We will revisit this result in Theorem 4.3.23 where we use groups of divisibility to offer an alternative proof of this result. We will also utilize groups of divisibility as a means of producing examples of AMD's. In [6], we find the following example showing how we can embed any domain D into an AMD T such that $dim(D) = dim(T)$.

Example 4.1.2. Let D be a domain with quotient field K and let L be an algebraically closed field containing K. Now let $\overline{D}$ be the integral closure of D in L. As $D \subseteq \overline{D}$ is an integral extension, then we know $dim(D) = dim(\overline{D})$. Now show $\overline{D}$ is an AMD and that K is the quotient field of $\overline{D}$. Now suppose $a \in \overline{D}$ and say $a^n + r_{n-1}a^{n-1} + \ldots + r_1 a + r_0 = 0$ with each $r_i \in D$. Since $a \in L$, then the polynomial $x^2 - a = 0$ has a root in L. Say $t \in L$ with $t^2 = a$. Then we have that $t^{2n} + r_{n-1}t^{2(n-1)} + \ldots + r_1 t^2 + r_0 = 0$, i.e., $t \in \overline{D}$. It follows that $a \notin Irr(\overline{D})$ and so $\overline{D}$ admits not atoms.

Our next example shows that many UFD's admit nontrivial antimatter overrings.

Example 4.1.3. Let F be a field and let $R = F[x_1, x_2, \ldots]$. Then R is a UFD. Now consider the overring of R given by $L = F[x_1, x_2, \ldots; \frac{x_i}{x_{i+1}^n}]_{n \in \mathbb{N}}$. Now let M be the canonical maximal ideal of L and consider the localization L_M. This localization is an example of a fragmented domain. To see this, first observe that if $f \in L_M$, then f also lies in the quotient field of $F[x_1, \ldots, x_n]$ for some n. Hence, every power of x_{n+1} divides f and so we see not only that L_M is an AMD but, in fact, L_M is a fragmented domain.

In light of the previous example, we pose some questions. Can a finite dimensional UFD admit an antimatter overring? Does every infinite dimensional UFD admit an antimatter overring? In Corollary 4.1.5 it becomes evident that PID's never admit nontrivial antimatter overrings. Dr. Jim Coykendall shares the following result.

Theorem 4.1.4. *Every overring of a Bêzout domain is a localization.*

Proof. Suppose D is a Bêzout domain and T is an overring of D. Let $S = \{s \in D | \frac{1}{s} \in T\}$. Clearly S is multiplicatively closed because T is a ring. Hence, $D_S \subseteq T$. Assume now $\frac{d}{s} \in T$ for some $d, s \in D$. Since D is a GCD domain, then we may assume $[d, s] = 1$. As D is Bêzout, then we may write $1 = r_1 d + r_2 s$ for some $r_1, r_2 \in D$. Thus, $\frac{1}{s} = \frac{r_1 d + r_2 s}{s} = r_1 \frac{d}{s} + r_2 \in T$. Hence, $s \in S$ and so $T \subseteq D_S$. We have therefore shown that $T = D_S$, as desired. □

Corollary 4.1.5. *Every overring of a PID is a PID.*

Proof. If D is a PID, then D is Bêzout. Thus, every overring of D is a localization by Theorem 4.1.4. Now use Theorem 2.1.4 for the desired result. □

Our next example shows another way to construct AMD's. It is a different type of construction than seen in the Example 4.1.2 because of the way new variables are introduced to "kill" atomicity.

Example 4.1.6. Let D_0 be a non-field domain containing an atom. Now partition the atoms of D_0 by associates, i.e., form equivalence classes among the atoms via $\pi_1 \sim \pi_2$ whenever π_1 and π_2 are associates. Now choose a representative from each equivalence class to create a new set Λ. Now for each $p_i \in \Lambda$ introduce a variable x_i. We can now build the ring extension $D_0 \subsetneq D_1 = D[x_i; \frac{p_i}{x_i}]_{i\in\Lambda}$. Note that if $\pi \in Irr(D_0)$, then $\pi \notin Irr(D_1)$. Similarly, we can build a domain D_2 in which every atom of D_1 ceases to be an atom in D_2. Continuing inductively, we have an ascending chain of domain $(D_i)_{i=0}^{\infty}$ in which the atoms of D_n cease to be atoms in D_{n+1}. Now let $L = \bigcup_{i=0}^{\infty} D_i$. It is now not very difficult to see that T is an AMD.

We now characterize AMD's via groups of divisibility. This result is fundamental and can be used to produce many more examples of AMD's. Its proof follows immediately from Theorem 3.5.3 and so we will skip it.

Theorem 4.1.7. *Let D be a domain. Then D is an AMD if and only if $G(D)$ has no atoms.*

We could rephrase Theorem 4.1.7 by saying that D is an AMD if and only if its $G(D)$ has no elements which are minimal with respect to being positive. Let us put this result to work.

Example 4.1.8. Suppose V is a valuation domain and $G(V) \cong_o \mathbb{Z} \oplus \mathbb{Q}$. Then V is a 2-dimensional AMD. On the other hand, $G(V) \cong_o \bigoplus_{i=1}^{n} A_i$, where each A_i is a nonzero Archimedean group and A_n is dense in $\mathbb{R}$, then V is an n-dimensional AMD.

Example 4.1.9. Suppose D is a domain and $G(D) \cong_o \mathbb{Z} \oplus \mathbb{Q}$ in the product order. By Theorem 3.5.22 D is a Bêzout domain with exactly 2 maximal ideals. Moreover, Theorem 4.1.4 assures us that every overring of a Bêzout domain is a localization. But $G(D)$ has only two nontrivial convex subgroups. Thus, D admits only 2 non-field overrings. One of the overrings of D is a Noetherian valuation domain while the other is a 1-dimensional valuation AMD.

It is evident that we can use groups of divisibility to create AMD's that are infinite dimensional or of any given finite dimension. Here is another way we can use AMD's to create "bigger" AMD's. Theorem 1.2.3 makes the proof of the following result an easy task.

Theorem 4.1.10. *Suppose D is an AMD with quotient field K. Then $D + xK[[x]]$ is an AMD if and only if $D \neq K$.*

We recall now that an **underring** of a domain D is a subring of D which has the same quotient field as D. For example, $D[x^2, x^3]$ is an underring of $D[x]$ while D itself is not. Additionally, if D is a domain with quotient field K, then $D + xK[[x]]$ is an underring of $K[[x]]$. We see, therefore, from the previous theorem that PID's may admit antimatter underrings. This should be contrasted with Theorem 4.1.5 in which it was shown that the only antimatter overring of a PID is its quotient field. It follows from the same theorem that a nontrivial AMD cannot admit a PID underring. We have already seen examples of fields which admit antimatter subrings. For example, the integral closure of $\mathbb{Z}$ in $\mathbb{C}$ is a 1-dimensional AMD. These observations suggest an interesting line of questioning. Given a field F, under what circumstances do F, $F[x]$, or $F[[x]]$ admit antimatter subrings? Some fields do not admit nontrivial antimatter subrings.

Example 4.1.11. Suppose D is an antimatter subring of $\mathbb{Q}$. Then because D has an identity, it is necessary that $\mathbb{Z} \subsetneq D$. As $\mathbb{Z}$ is a PID, it follows from Theorem 4.1.5 that D is a localization of $\mathbb{Z}$. But a localization of a PID is a PID. Thus, D is an atomic AMD. This can only mean D is a field and so we must have that $D = \mathbb{Q}$.

We observe that there are countable fields which admit nontrivial antimatter subrings. For example, we might consider the integral closure of $\mathbb{Z}$ in the algebraic closure of $\mathbb{Q}$. Having seen that a field might or might not admit an antimatter subring, we might extend our these ideas to a more general setting. We will not pursue these queries very far at all as our motivation is merely to open a line of questioning. The following theorem might serve as a starting point.

Theorem 4.1.12. *Let D be a domain. Then D admits a non-field antimatter subring if and only if $D[x_i]_{i\in\Lambda}$ admits a non-field antimatter subring for any given indexing set Λ.*

Proof. ($\Rightarrow$) Clear.

($\Leftarrow$)Suppose R is a non-field antimatter subring of $D[x_i]_{i\in\Lambda}$ and we consider the subring $R' = \{f \in R | deg(f) = 0\}$ of R. If $R' = R$, then we are done. So we should require $R' \neq R$. It now suffices to show R' is not a field. Assume, to the contrary, that R' is a field. Since $R' \neq R$, then there must exist some $g \in R$ such that g is of minimal positive degree. As R is an AMD, then there must exist some $k, k \in R^* - U(R)$ such that $g = hk$. Note that $deg(g) = deg(h) + deg(k)$. By the minimality of $deg(g)$, there is no loss of generality in saying $deg(h) = 0$. Hence, $h \in R'$. But since R' is a field, we may conclude $h \in U(R') = U(R)$, a contradiction. $\square$

Corollary 4.1.13. *$\mathbb{Q}[x_i]_{i\in\Lambda}$ contains no nontrivial antimatter subrings.*

Theorem 4.1.14. *Assume F is a field of characteristic 0 with the property that given any nonzero $x \in F$, there exists some integer $n \geq 2$ and some $y_x \in F$ such that $y_x^n = x$. Then F admits a non-field antimatter subring.*

Proof. Since $char(F) = 0$, then the subring $\mathbb{Z}_F$ generated by 1_F is isomorphic to $\mathbb{Z}$. Letting $\overline{\mathbb{Z}_F}$ be the integral closure of $\mathbb{Z}_F$ in F, then $\overline{\mathbb{Z}_F}$ is a non-field AMD. $\square$

We remark that $\mathbb{R}$ or any algebraically closed field of characteristic 0 would satisfy the hypotheses of Theorem 4.1.14. Moreover, fields of nonzero characteristic may or may not admit antimatter subrings. For example, given any prime p, then the algebraic closure K of F_p admits no nontrivial antimatter subrings because K can be realized as a union of a chain of fields. Hence, given any $a \in K$, then $a^n = 1$ for some $n \in \mathbb{N}$. On the other hand, the integral closure of $F_p[x]$ in the algebraic closure of its quotient field is a 1-dimensional AMD. We close this section with the following conjecture.

Conjecture 4.1.15. Given any field F, then $F[[x]]$ admits a nontrivial antimatter subring.

4.2. The Integral Closure of an Antimatter Domain

In the previous section we saw we can use groups of divisibility to construct examples Bêzout AMD's. Recall that every Bêzout domain is a GCD domain and all GCD domains are integrally closed. Moreover, we showed that the integral closure T of a domain D in the algebraic closure of its quotient field is an AMD. Again, such a domain would be integrally closed because $D \subseteq T$ is an integral extension. The "hands-on" examples of AMD's which have been presented so far are all integrally closed. One might wonder if *all* AMD's are integrally closed. The answer is no. Consider the following construction.

Example 4.2.1. Let F be a field and consider the ring

$$R = F[x_1, x_2, \dots ; \tfrac{y^2}{x_1}, \tfrac{y^2}{x_1^2} \dots ; \tfrac{y^3}{x_1}, \tfrac{y^3}{x_1^2} \dots ; \tfrac{x_1}{x_2}, \tfrac{x_1}{x_2^2} \dots ; \tfrac{x_2}{x_3}, \tfrac{x_2}{x_3^2} \dots ; \dots].$$

Letting M be the canonical maximal ideal of R, we see that the localization R_M is an AMD. To verify this, let α be any nonzero nonunit in R_M. Then there exists some $n \in \mathbb{N}$ such that $\alpha \in F(y^2, y^3, x^1, x^2, \dots x^n)$. Thus, $\alpha R_M \subsetneq x_{n+1} R_M$. Note that $y \in \overline{R_M}$ but $y \notin R_M$. So R_M is not integrally closed.

We also point that from the previous example that $y \notin Irr(\overline{R_M})$ because we have $\frac{y}{x_1}, x_1 \in \overline{R_M}$, whence $y = \frac{y}{x_1}(x_1)$. Indeed, it is not immediately clear whether or not $\overline{R_M}$ admits any atoms at all.

Example 4.2.2. Let V be a valuation domain such that $G = G(V) \cong_o \mathbb{Z} \bigoplus \mathbb{Q}$. As G_+ admits no minimal elements, then V is an AMD. But note that since V is 2-dimensional, then Theorem 3.3.4 tells us this ring is not completely integrally closed. We use Theorem 3.3.7 to deduce that the complete integral closure of V is V_P, where P is the ht-1 prime of V. Since V_P is 1-dimensional, then V_P is completely integrally closed. Further $G(V_P) \cong_o \mathbb{Z}$, making V_P a Noetherian valuation domain. Since all Noetherian valuation domain's are PID's, we see not only that the complete integral closure of an AMD may admit an atom but even more so it can even be Noetherian. This is also a good illustration that localization can be used to produce atoms where there none.

The following example is due to Dr. Jim Coykendall. It shows that the integral closure of an atomic domain can be an AMD.

Example 4.2.3. Let $F \subseteq L$ be an algebraic extension of fields and consider the ring $R = F + xL[x^q]_{q \in \mathbb{Q}}$. Let $M = xL[x^q]_{q \in \mathbb{Q}^+}$. Then R_M is atomic. Indeed, if $\alpha \in R_M^* - U(R_M)$, then $\alpha = ux^q$ for some $u \in U(R_M)$ and $q \in \mathbb{Q} \cap [1, \infty)$. Note that if $j \in \mathbb{Q}_+$, then $x^j \in Irr(R_M)$ if and only if $1 \leq j < 2$. Thus, if $\alpha = ux^q$ and $q = n + h$ for some $n \in \mathbb{N}$ and $h \in \mathbb{Q} \cap [1, 2)$, then $\alpha = (ux^n)x^h$ is an atomic factorization of α. So R_M atomic.

As an added note, we point out that if $D = L[x^q]_{q \in \mathbb{Q}}$ and if $\mathfrak{M}$ is the canonical maximal ideal of D, then $\overline{R_M} = D_{\mathfrak{M}}$. Also, we see that $G(D_{\mathfrak{M}}) \cong_o \mathbb{Q}$, making $D_{\mathfrak{M}}$ an antimatter valuation domain. Thus, the integral closure of an atomic domain may be an AMD.

Let us take assessment of the results so far in this section. First, we have seen that AMD's may or not be integrally closed. Moreover, the complete integral closure of an AMD may be a 1-dimensional PID. Also, the integral closure of an atomic domain may be an AMD. Therefore, we begin to wonder if it is possible for the integral closure of an AMD to admit an atom. We consider another example.

Example 4.2.4. Consider the ring given by

$$R = F[x_1, x_2 \ldots ; \tfrac{y^2}{x_1}, \tfrac{y^2}{x_1 x_2} \ldots ; \tfrac{y^3}{x_1}, \tfrac{y^3}{x_1 x_2} \ldots ; \tfrac{x_1}{x_2}, \tfrac{x_1}{x_2 x_3} \ldots ; \tfrac{x_2}{x_3}, \tfrac{x_2}{x_3 x_4} \ldots ; \ldots].$$

Let M be the canonical maximal ideal of R and look at the localization R_M. Using a similar argument as was employed in Example 4.2.1 it can be verified that R_M is an AMD. It is also clear that R_M is not integrally closed since $y \in \overline{R_M}$ but $y \notin R_M$. Note that $y \notin Irr(\overline{R_M})$. To see this, note that $\frac{y^2}{x_1 x_2}, \frac{x_1}{x_2} \in \overline{R_M}$. From Theorem 2.2.4 it follows that $\frac{y^2}{x_1 x_2}\frac{x_1}{x_2} = \frac{y^2}{x_2^2} \in \overline{R_M}$. Thus, $\frac{y}{x_2} \in \overline{R_M}$ and so $y = \frac{y}{x_2}(x_2)$ is a nontrivial factorization in $\overline{R_M}$.

We pose the following conjectures.

Conjecture 4.2.5. a) There exists an AMD whose integral closure admits an atom.

b) If D is an AMD and $\overline{D}$ admits an atom, then D is infinite dimensional.

Working under the assumption that R being an AMD does not force $\overline{R}$ to be an AMD, we push further and inquire as to the possibility of the integral closure of an AMD being atomic or even Noetherian. To motivate, we consider the following example.

Example 4.2.6. Suppose we have a field F and consider the ring $D = F[[x^2, x^3]]$. Clearly, D is not integrally closed and $F[[x]] \subseteq \overline{D}$. But $\overline{D}$ is the smallest integrally closed overring of D and so we must have $\overline{D} = F[[x]]$.

Observe that $\overline{D}$ is a valuation domain while D is not. Also, $x^2, x^3 \in Irr(D)$. If one is searching for AMD's whose integral closures might admit an atom, then we might first target a known integrally closed domain which admits an atom and try to realize it as the integral closure of one of its antimatter underrings. The following theorem some guidance in showing where we ought *not* use this strategy.

Theorem 4.2.7. *Suppose D is an AMD and $D \subseteq V$ is an integral extension. If V is a valuation domain, then V is an AMD.*

Proof. We assume V is a valuation domain with maximal ideal M. Recall that a valuation domain admits an atom if and only if it maximal ideal is principal (and nonzero). So we assume to the contrary that $M = mV$ for some $m \in V$. Letting $S = \{P \in Spec(V) | P \subsetneq M\}$ we see that $M - \bigcup_{P \in S} P \neq \emptyset$ since $m \in M - \bigcup_{P \in S} P$. Let $\mu = M \cap D$. Since V is a valuation domain and integral extensions are INC and GU, then the prime ideals of D must be linearly ordered by inclusion. Hence, μ is the unique maximal ideal of D. Further, since $M - \bigcup_{P \in S} P \neq \emptyset$, then we must also have that $\mu - \bigcup_{P \in S}(P \cap D) \neq \emptyset$. Now given any $a \in \mu$, we have that $a \in M$ all the more so. Also, given any $b \in M - \bigcup_{P \in S} P$, we have $bU(V) = m^t U(V)$ for some $t \in \mathbb{N}$, where t depends on b. As $\mu - \bigcup_{P \in S}(P \cap D) \neq \emptyset$, we may let $\alpha \in \mu - \bigcup_{P \in S}(P \cap D)$ such that $\alpha U(V) = m^n U(V)$ for some minimal $n \in \mathbb{N}$. As D is an AMD, then $\alpha D \subsetneq \beta D$ for some $\beta \in D^* - U(D)$. Hence, $\beta \in \mu - \bigcup_{P \in S}(P \cap D)$ and so $\beta U(V) = m^k U(V)$ for some $k \in \mathbb{N}$. Since $\beta | \alpha$, then $k \leq n$. By minimality of n we also have that $n \leq k$. Hence, α and β are associates in V. Now $\alpha D \subsetneq \beta D$ and so $\alpha = \beta \varphi$ for some nonunit $\varphi \in D$. But as α and β are associates in V, we must have that $\varphi \in U(V)$. We now have that $\varphi \in U(V) \cap D = U(D)$, a contradiction since $\varphi \notin U(D)$. $\square$

If we are searching for integrally closed domains which can be realized as the integral closure of an AMD, then in addition to showing us where *not* to look, the following theorem gives us an idea as to how truly exotic such a domain would have to be. Recalling that all Noetherian domains are ACCP, it also brings an abrupt end to any question regarding the possibility of an AMD having a Noetherian integral closure.

Theorem 4.2.8. *If D is an AMD, then $\overline{D}$ is not an ACCP domain.*

Proof. Suppose $x_1 \in D^* - U(D)$. Since D is an AMD, then there exist some $r_2, x_2 \in D^* - U(D)$ such that $x_1 = r_2 x_2$. Using this same idea we can construct a sequence $(x_i)_{i=1}^{\infty}$ of nonzero nonunits in D such that $x_i = r_{i+1}x_{i+1}$ with each $r_j \in D^* - U(D)$. Since any nonunit in a domain remains a nonunit in its integral closure, then $x_1\overline{D} \subsetneq x_2\overline{D} \subsetneq \ldots$ is an ascending chain of principal ideals in $\overline{D}$, achieving the desired result. □

Atomic domains that fail to be ACCP have been shown to exist but are a rarity in the literature. Thus, finding examples of AMD's with atomic integral closures might well be worth our while.

Now if we are targeting an integrally closed atomic domain D first and trying to realize such a domain as the integral closure of an antimatter subring, it would be nice to know whether or not D even *admits* antimatter subrings. Hence, our discussion on the existence of antimatter subrings in the previous section becomes all the more pertinent.

4.3. A Generalization of Krull Dimension

In the previous section, most of our discussion was centered around the notion of integral extensions. One of the most fundamental results of dimension theory is that Krull dimension is preserved in integral extensions. More generally, Krull dimension is preserved in extensions that are GU and INC. We have seen numerous instances where knowing the dimension of a ring provided us with some highly nontrivial information about the ring. For example, if V is a non-field valuation domain, then V is completely integrally closed if and only if $dim(V) = 1$. Also, we have seen that a non-field UFD is a PID if and only if it is 1-dimensional. By way of an application, we can see that given any field F, then $F[x^2, x^3]$ is 1-dimensional because its integral closure is a PID. From an intuitive point of view, we can think of Krull dimension as a certain measure of the size of a ring because of the way in which are able to "stack up" prime ideals. In this section, we generalize the idea of Krull dimension in an effort to speak with more specificity with regards to infinite dimensional rings. In addition to being intimately connected with the idea of integrality, we will see that many of the ideas in this section are motivated by our work on groups of divisibility.

We begin by recalling the definition of Krull dimension given in Chapter 2. A chain of prime ideals $P_0 \subsetneq P_1 \subsetneq \cdots \subsetneq P_n$ in a ring R is said to have length n. The Krull dimension of R is then defined to be the supremum of the lengths of all such chains of primes in R. If this supremum is finite, then R is said to be finite dimensional and the dimension of R is equal to this (finite) supremum. Otherwise, R is said to be infinite dimensional. We pose the following question: If two rings have infinite Krull dimension, then should they be regarded as having the *same* Krull dimension? It would seem as though rings having infinite Krull dimension have historically been lumped together and treated as though not having *finite* Krull dimension were equivalent to having the *same* Krull dimension. We think the answer to the question posed should be a resounding no. It is the primary purpose of the next two sections to argue in favor of this shift in outlook. As the new definition will be indistinguishable from the one of tradition in the finite dimensional case, our focus will be almost exclusively on infinite dimensional rings. We will be using the notation $|S|$ will denote the cardinality of the set S. As the Axiom of Choice is enthusiastically embraced here, we recall that the collection of cardinal numbers is linearly ordered. In addition, given cardinal numbers α and β with $\alpha \leq \beta \leq \alpha$, then $\alpha = \beta$. We now consider some examples to motivate our discussion. The following example is due to Dr. Jim Coykendall. We use the symbol $\aleph_1$ to denote $|\mathbb{R}|$. (back here)

Example 4.3.1. We show that $dim(R[x_1, x_2, ...]) \geq \aleph_1$. Letting $P \in Spec(R)$, we have the isomorphism $R[x_1, x_2, ...]/P[x_1, x_2, ...] \cong (R/P)[x_1, x_2, ...]$. Hence, we have the inequality $dim(R[x_1, x_2, ...]) \geq dim((R/P)[x_1, x_2, ...])$. It may therefore be assumed, without loss of generality, that R is a domain. Now let $\phi : \mathbb{Q} \to \mathbb{N}$ be a bijection so that $R[x_1, x_2, ...] = R[x_{\phi(q)}]_{q \in \mathbb{Q}}$. Now given any $q \in \mathbb{Q}$, define $S_q = \{x_{\phi(t)} | t \leq q\}$. Note that R being a domain guarantees that each $(S_q) \in Spec(R)$. Now for every $r \in \mathbb{R}$, let $P_r = \bigcup_{q \leq r}(S_q)$. If $r_1, r_2 \in \mathbb{R}$ with $r_1 < r_2$, then $r_1 < q < r_2$ for some $q \in \mathbb{Q}$. Thus, $P_{r_1} \subsetneq P_q \subsetneq P_{r_2}$ and so we have a chain$(P_r)_{r \in \mathbb{R}}$ of nonzero pairwise distinct prime ideals in $R[x_1, x_2, ...]$, thereby achieving our aim.

Example 4.3.2. Suppose V_1 and V_2 are infinite dimensional valuation domains such that $Spec(V_1)$ is countable while $Spec(V_2)$ is uncountable. Then $|Spec(V_1| < |Spec(V_2|$. It therefore seems reasonable to conclude $dim(V_1) < dim(V_2)$.

It should be pointed out that under the traditional definition of Krull dimension, the dimensions of the rings in Example 4.3.1 and Example 4.3.2 cannot be distinguished because neither ring has finite dimension. We are therefore compelled to offer a definition by which we are able to make such a distinction in the infinite dimensional case but which also agrees with the traditional definition in the finite dimensional case.

Definition 4.3.3. Let S be the set of all chains of prime ideals in R. For each $C \in S$ choose some $x_C \in C$. We define the **Krull dimension** of R, denoted $dim(R)$, by $dim(R) = sup\{|C - x_C|\}_{C \in S}$. If this supremum is finite, then we will say R is finite dimensional. Otherwise, we will refer to R as having infinite Krull dimension.

Naturally, if we are going to use this definition our first order of business should be to make sure that it is well-defined. In [1] it is shown that if X is any *set* of cardinal numbers, then there exists a cardinal number σ such that $\sigma = sup(X)$. Thus, not only is our new definition well-defined, but we get the added bonus of knowing that under our new definition every ring *admits* a uniquely determined Krull dimension. To ease the tenor of our discussion, we will assume for the remainder of the paper that the Krull dimension of R and the dimension of R are synonymous notions.

An additional noteworthy aspect of using this definition lies in the fact that instead of thinking of the dimension a ring R in terms of what it is not (finite), we may identify the dimension of R with a cardinal number by which we may know what the dimension of R actually *is*. Our new definition also allows us to make the appropriate definitions for the height and depth of a prime ideal. The theorem which will follow our next definition should feel familiar and will ease some of our calculations.

Definition 4.3.4. Let $P \in Spec(R)$. The height and depth of P, denoted $ht(P)$ and $d(P)$,respectively, are given by the following equations:

a) $ht(P) = dim(R_P)$

b) $d(P) = dim(R/P)$

We observe that since the definition of Krull dimension is well-defined, then so also are the definitions just presented.

Theorem 4.3.5. *Let R be a ring. Then $dim(R) = sup\{ht(M)|M \in Max(R)\} = sup\{d(P)|P \in Spec(R)\}$.*

Proof. We first show that $dim(R) = sup\{ht(M)|M \in Max(R)\}$. Given $P \in Spec(R)$ and $\sigma \subseteq R$ multiplicatively closed we know that every prime in the localization R_σ is of form PR_σ where $P \cap \sigma = \emptyset$. So if $M \in Max(R)$, then it is obvious that $dim(R_M) \leq dim(R)$, i.e., $sup\{ht(M)|M \in Max(R)\} \leq dim(R)\}$. For the reverse inequality we let S be the set of all chains of prime ideals in R and choose some $C \in S$. Let $x_C \in C$ and $M \in Max(R)$ such that $P \subseteq M$ for all $P \in C$. Then $|C - x)|_C \leq ht(M) \Rightarrow dim(R) \leq sup\{ht(M)|M \in Max(R)\}$, as desired.

We now show $dim(R) = sup\{d(P)|P \in Spec(R)\}$. Assuming $P \in Spec(R)$ we know that every prime ideal in R/P is of form Q/P where $Q \in Spec(R)$ such that $P \subseteq Q$. Thus, $dim(R/P) \leq dim(R) \Rightarrow sup\{d(P)|P \in Spec(R)\} \leq dim(R)$. Letting S be the set of all primes in R we choose some $C \in S$ and $x_C \in C$. Then $\mathfrak{P} = \bigcap_{P \in C} P \in Spec(R)$ and so $|C - x_C| \leq D(\mathfrak{P})$. Hence, $sup\{|C - x_C|\}_{C \in S} \leq sup\{d(P)|P \in Spec(R)\}$, i.e., $dim(R) \leq sup\{d(P)|P \in Spec(R)\}$. So we are done. □

Theorem 4.3.6. *Suppose R is a ring and $dim(R) > \sigma$ for some cardinal number σ. Then there exists a chain of primes $(P_i)_{i \in \Lambda}$ in R such that $|\Lambda| > \sigma$.*

Proof. By definition, $dim(R) = sup\{|C - x_C|\}_{C \in S}$, where S is the set of all chains of primes in R. If $|C - x_C| \leq \sigma$ for all chains of primes in S, then $sup\{|C - x_C|\}_{C \in S} \leq C$. □

An immediate consequence of Theorem 4.3.6 is that if $dim(R) > \aleph_0$, then R must admit an infinite chain of primes. We now compute the Krull dimension of all infinite dimensional Noetherian rings.

Example 4.3.7. If R is Noetherian and $M \in Max(R)$, then $ht(M) < \infty$. So $dim(R) = sup\{ht(M)|M \in Max(R)\} \leq \aleph_0$. If R happens to be infinite dimensional, then we have $dim(R) \leq \aleph_0 \leq dim(R)$, i.e., $dim(R) = \aleph_0$. Thus, if R is Noetherian, the following are equivalent:

a) R is infinite dimensional

b) $dim(R) = dim(R[x]) = dim(R[[x]]) = \aleph_0$.

Hence, if R is a ring and $dim(R) > \aleph_0$, then R is non-Noetherian. Later we will recall this example when we broaden our scope to consider the dimension of polynomial and power series extensions of any ring.

Having computed the Krull dimension of all infinite dimensional Noetherian, we would like to use our generalization to compute the dimension of some other rings. The next two results will help us do this.

Theorem 4.3.8. *Let $\{A_i | i \in \sigma\}$ be a set of (nonzero) Archimedean groups and let V be a valuation domain whose group of divisibility is order isomorphic to the lexicographic sum $\bigoplus_\sigma A_i$ via the order map $\varphi : \bigoplus_\sigma A_i \to G(V)$. For each A_i, choose some $\alpha_i \in A_i$ with each $\alpha_i > 0$. For each α_i, choose $a_i \in V$ such that $a_i U = \varphi(\lambda_i(\alpha_i))$, where $\lambda_i : A_i \to \bigoplus_\sigma A_i$ is the natural injection.*

a) Then $P \in Spec(V)$ is branched if and only if $P = \sqrt{a_i V}$ for some a_i.

b) There exists a bijection between σ and the set of all branched primes given by $i \to P_i$, where $P_i \in Spec(V)$ such that $P_i = \sqrt{a_i V}$.

Proof. *a)* Let $P \in Spec(V)$. From Theorem 2.5.16 we know that if P is the radical of a principal ideal, then P is branched. Hence, we have only to prove the forward implication. So we assume P is branched. We again use Theorem 2.5.16 and write $P = \sqrt{xV}$ for some nonzero $x \in P$. Say $\varphi^{-1}(xU) = \sum_{j=1}^{t} \lambda_{i_j}(y_{i_j})$ where each $y_{i_j} \in A_{i_j}$, $y_{i_1} > 0$, and $i_1 < i_2 < \ldots i_t$. Since each A_i is Archimedean, then there exists an $n \in \mathbb{N}$ such that $y_{i_1} < n\alpha_{i_1}$. Hence, $\varphi^{-1}(xU) < \lambda_{i_1}(n\alpha_{i_1}) \Rightarrow xU < \varphi(\lambda_{i_1}(n\alpha_{i_1})) = a_{i_1}^n U$. Thus, $x | a_{i_1}^n \Rightarrow a_{i_1} \in P \Rightarrow \sqrt{a_{i_1} V} \subseteq P = \sqrt{xV}$.

To finish the proof, we show that $\sqrt{xV} \subseteq \sqrt{a_{i_1} V}$. Suppose then $z \in \sqrt{xV}$ so that $z^m \in xV$ for some $m \in \mathbb{N}$. Then $xU \leq z^m U \Rightarrow \sum_{j=1}^{t} \lambda_{i_j}(y_{i_j}) \leq \varphi^{-1}(z^m U)$. Since each A_i is Archimedean, then there exists an $n \in \mathbb{N}$ such that $ny_{i_1} > \alpha_{i_1}$. Hence,

$\lambda_{i_1}(\alpha_{i_1}) < n\sum_{j=1}^{t}\lambda_{i_j}(y_{i_j}) \le n\varphi^{-1}(z^m U)$. We now have that $a_{i_1}U = \varphi(\lambda_{i_1}(\alpha_{i_1})) \le z^{mn}U \Rightarrow a_{i_1}|z^{mn} \Rightarrow z^{mn} \in a_{i_1}V \Rightarrow z \in \sqrt{a_{i_1}V}$. Hence, $\sqrt{xV} \subseteq \sqrt{a_{i_1}V}$, as desired.

b) We begin by first showing the map is well-defined. If $i = j$, then $\alpha_i = \alpha_j \Rightarrow \lambda_i(\alpha_i) = \lambda(\alpha_j) \Rightarrow \varphi(\lambda_i(\alpha_i)) = \varphi(\lambda_j(\alpha_j)) \Rightarrow a_iU = a_jU \Rightarrow a_iV = a_jV \Rightarrow \sqrt{a_iV} = \sqrt{a_jV} \Rightarrow P_i = P_j$. Knowing now the map is well-defined, it is evident from part a) that the map is onto. So we show $P_i = P_j \Rightarrow i = j$. Assume, to the contrary, that $i < j$. We have $P_i = \sqrt{a_iV}$ and $P_j = \sqrt{a_jV}$. Since $i < j$, then $n\lambda_j(\alpha_j) < \lambda_i(\alpha_i)$ for all $n \in \mathbb{N}$. Hence, $a_j^nU < a_iU \Rightarrow a_j \notin \sqrt{a_iV}$, i.e., $a_j \notin P_i$, a contradiction since $P_i = \sqrt{a_jV}$. Thus, $j \le i$. Similarly, if $j < i$, then $a_i \notin P_j$, a contradiction. So we also have that $i \le j$ and we are done. $\square$

Remark 4.3.9. It is worth pointing out that the map in part *b*) of Theorem 4.3.8 is order-preserving in the sense that if $i < j$, then $P_i \subsetneq P_j$. We will be implicitly using this fact in some arguments to come.

In our next result we show how to construct a ring of any given Krull dimension. Before doing so, recall that every set can be well-ordered. Say $(S, \le_1)$ is a well-ordered set. We now introduce a new relation $\le_2$ on S and say that $a \le_2 b$ whenever $b \le_1 a$. We will refer to this new ordering as the **reverse ordering** of $(S, \le_1)$. Thus, if we regard a set S as being well-ordered, then in the inverse well-ordering S has the property that given any subset $\sigma \subseteq S$, then σ admits a maximal element. Moreover, because every set can be well-ordered, then it is evident that every set also admits an inverse well-ordering, as well.

Theorem 4.3.10. *Given any cardinal number σ, then there exists a ring R such that $dim(R) = \sigma$.*

Proof. Let S be a set such that $|S| = \sigma$. We may regard S as being inversely well-ordered. Now let V be a valuation domain such that $G(V) \cong_o \bigoplus_S \mathbb{Z}$. Theorem 3.3.12

assures us that every nonzero prime of V is branched. Further, Theorem 4.3.8 tells us that there exists a bijection between S and the set of branched primes. Thus, $dim(V) = |S| = \sigma$ and we are done. $\square$

In Theorem 4.3.10 we could have indexed the free abelian group with a well-ordered set to achieve the same result. However, using an inversely well-ordered set makes the argument more streamlined.

Theorem 4.3.12 makes use of the following result, which can be found in [7].

Theorem 4.3.11. *If D is a domain and $P \in Spec(R)$, then D admits a valuation overring V with maximal ideal M such that $M \cap D = P$.*

Theorem 4.3.12. *Let D be a domain and let S be any linearly ordered set. Let $H = \bigoplus_S \mathbb{Z}$ be ordered lexicographically. If $G(D)$ contains an order isomorphic copy of H, then $dim(D) \geq |S|$.*

Proof. Let $\mathfrak{B} = \{e_i | i \in S\}$ be the standard basis for H and let $\phi : H \to G(D)$ be an order monomorphism. For each $e_i \in \mathfrak{B}$, let $\varepsilon_i \in D$ such that $\phi(e_i) = \varepsilon_i U(D)$. Note that $e_j < e_i \Rightarrow \varepsilon_i \in \bigcap_{n=1}^{\infty}(\varepsilon_j^n)$. In particular, if $i_1, i_2 \in S$ with $i_1 < i_2$, then $\varepsilon_{i_1} D \subsetneq \varepsilon_{i_2} D$. Thus, we have a chain of principal ideals $(\varepsilon_i D)_{i \in S}$. Now consider the ideal $I = \bigcup_{i \in S}(\varepsilon_i)$. Since $e_i > 0$ for all $i \in S$, then $\varepsilon_i U(D) > U(D)$ for all $i \in S$, i.e., $I \neq D$. Let P be a prime ideal of D containing I. Using Theorem 4.3.11 we can find a valuation overring V with maximal ideal M such that $M \cap D = P$. Now note that if $\varepsilon_i D \subsetneq \varepsilon_j D$, then $\varepsilon_i V \subsetneq \varepsilon_j V$. Indeed, if $\varepsilon_i D \subsetneq \varepsilon_j D$, then $\varepsilon_i D \subseteq \bigcap_{n=1}^{\infty} \varepsilon_j^n D \Rightarrow \varepsilon_i V \subseteq \bigcap_{n=1}^{\infty} \varepsilon_j^n V$. Now if $\varepsilon_i V = \varepsilon_j V$, then $\varepsilon_i V \subseteq \varepsilon_j^2 V \subseteq \varepsilon_j V \subseteq \varepsilon_i V \Rightarrow \varepsilon_j V = \varepsilon_j^2 V \Rightarrow \varepsilon_j = v\varepsilon_j^2$ for some $v \in V$. But then $1 = v\varepsilon_j \Rightarrow 1 \in M$, a contradiction.

Given $i \in S$ define $\beta_i = \{e_j \in \mathfrak{B} | i < j\}$. Let G_i be the subgroup of H spanned by β_i. Now consider the set $\sigma_i = \{r \in D | rU(D) \in \phi(G_i)\}$. Suppose $r_1, r_2 \in \sigma_i$. Then $r_1 U(D), r_2 U(D) \in \phi(G_i) \Rightarrow r_1 r_2 U(D) = r_1 U(D) + r_2 U(D) \in \phi(G_i) \Rightarrow r_1 r_2 \in \sigma_i$.

We have shown that σ_i is multiplicatively closed for all $i \in S$. Further, since each σ_i is multiplicatively closed in D, then each σ_i is multiplicatively closed in V, as well. Recall that the complement of any multiplicatively closed set is a union of primes. As the ideals of a valuation domain are linearly ordered by inclusion, we conclude $V - \sigma_i \in Spec(V)$. Say $V - \sigma_i = P_i$. Given $i \in S$, then $\varepsilon_i \notin \sigma_i \Rightarrow \varepsilon_i \in P_i$. In addition, if $i_1, i_2 \in S$ with $i_1 < i_2$, then $\varepsilon_{i_2} \in \sigma_{i_1}$. This implies $\varepsilon_{i_2} \in P_{i_2} - P_{i_1}$. So $i_1 < i_2 \Rightarrow P_{i_1} \subsetneq P_{i_2}$. We have produced a chain $(P_i)_{i \in S}$ of prime ideals in V with the property that $\varepsilon_i \in P_i$ for all $i \in S$ and if $i_1, i_2 \in S$ with $i_1 < i_2$, then $P_{i_1} \subsetneq P_{i_2}$. Now for every $i \in S$, define $\mathfrak{P}_i = P_i \cap D$. If $i_1, i_2 \in S$ with $i_1 < i_2$, then $\varepsilon_{i_2} \in \mathfrak{P}_{i_2} - \mathfrak{P}_{i_1}$. We have therefore produced a chain $(\mathfrak{P}_i)_{i \in S}$ of prime ideals in D with the property that if $i_1, i_2 \in S$ with $i_1 < i_2$, then $\mathfrak{P}_{i_1} \subsetneq \mathfrak{P}_{i_2}$. Since $\varepsilon_i \in \mathfrak{P}_i$ for all $i \in S$, then $(\mathfrak{P}_i)_{i \in S}$ is a chain of nonzero pairwise distinct prime ideals in D and so we are done. □

Example 4.3.13. In Example 3.3.13 we saw that if V is a valuation domain with $G(V) \cong_o \bigoplus_{i=-1}^{-\infty} \mathbb{Z}$, then every nonzero prime ideal is branched. From part b) of Theorem 4.3.8 it then follows that $|Spec(V)| = \aleph_0$. We then conclude that $dim(V) = \aleph_0$.

Example 4.3.14. Recall in Example 4.3.1 that the ring $R[x_1, x_2, \ldots]$ admits a chain of primes that can be indexed by $\mathbb{R}$. Of particular interest in Example 4.3.1 are the cases when R is either Noetherian or countable. In the case that R is a finite dimensional Noetherian domain we have $dim(R[x_1, x_2, ..., x_n]) = dim(R) + n$, which is finite. Hence, one might have expected that since $dim(R[x_1, x_2, ..., x_n]) \leq \aleph_0$, then $dim(R[x_1, x_2, ...])$ might be $\aleph_0$, which we see now to never be the case.

Should R be countable we are able to determine the exact value of $dim(R)$. As R is countable, then so is $R[x_1, x_2, ...]$. It follows that $|\mathcal{P}(R))| = \aleph_1$, where $\mathcal{P}(R)$ is the power set of R. Hence, $\aleph_1 \leq dim(R) \leq |\mathcal{P}(R)| = \aleph_1$. We conclude that $dim(R) = \aleph_1$ whenever R is countable. We find it delightfully surprising to discover that a countable ring may admit an uncountable chain of pairwise distinct primes. However, should we take the union of any such chain, then we would have a prime ideal with only countably many elements.

The following example is inspired by Example 4.3.1.

Example 4.3.15. Let V be a valuation domain such that $G(V) \cong_o \bigoplus_{\mathbb{Q}} \mathbb{Z}$. From Theorem 4.3.12 it follows that $dim(V) \geq \aleph_0$. Moreover, Theorem 4.3.8 assures us that the set S of branched primes of V is countably infinite. Now for every $r \in \mathbb{R}$, set $Q_r = \bigcup_{i \leq r} P_i$, where each $P_i \in S$. Using similar methods as in Example 4.3.1 we can construct a chain of primes indexed by $\mathbb{R}$. Thus, $dim(V) \geq \aleph_1$. It is also interesting to note that although every prime ideal of V can be expressed as a chain of branched primes, the branched primes are exceedingly rare in this ring.

Example 4.3.16. Let V be a valuation domain such that $G(V) \cong_o \bigoplus_{\mathbb{R}} \mathbb{Z}$. From Theorem 4.3.12 we already know that $dim(V) \geq \aleph_1$. In particular, Theorem 4.3.8 assures us that the set if $S = \{P_i | i \in \mathbb{R}\}$ is the set of branched primes, then $|S| = \aleph_1$. Now for each $i \in \mathbb{R}$, define $Q_i = \bigcup_{i<r} P_i$, where each $P_i \in S$. If $Q_i \in S$, then $Q_i = P_k$ for some $P_k \in S$. But if $P_k = \bigcup_{i<r} P_i$, then it follows readily from part a) of Theorem 4.3.8 that $P_k = P_j$ for some $j < i$. But then $k = j < i$ and we can choose $t \in \mathbb{R}$ such that $j < t < i$. It follows that $\bigcup_{i<r} P_i = P_k \subsetneq P_t \subseteq \bigcup_{i<r} P_i$, a contradiction. Thus, each Q_i is unbranched.

We now show that $Q_i \subsetneq P_i$ are adjacent primes. We know from Theorem 2.5.13 that there exist adjacent primes $Q' \subsetneq P'$ with $Q_i \subseteq Q' \subsetneq P' \subseteq P_i$. Thus, P' is branched and so $P' = P_t$ for some $t \leq i$. If $t < i$, then $P_t \subsetneq Q_i$, a contradiction. Thus, $P' = P_i$. We deduce now that there are no branched primes lying properly between Q_i and P_i. It follows then that Q_i and P_i are adjacent.

In the previous paragraph we showed that given any nonzero nonmaximal unbranched prime Q, then there exists a branched prime P such that $Q \subsetneq P$ are adjacent. Letting W be the set of all nonmaximal nonzero unbranched primes of V, then $|W| = |S|$. Since V is infinite dimensional, we can ignore the zero ideal and the maximal ideal in computing the dimension of V. Thus, $dim(V) = |S| + |W| = \aleph_1 + \aleph_1 = \aleph_1$.

We point out that although the rings in the previous two examples are both infinite dimensional, the ring in Example 4.3.16 has greater Krull dimension than the ring in Example 4.3.15. Theorem 4.3.18 shows us what is really at work in some of the examples we have just seen. First we need a definition.

Definition 4.3.17. Let $(P_i)_\Lambda$ be a chain of primes in a ring R. We say this chain is **dense** if, whenever $P_i \subsetneq$ are primes in the chain, then there exists a P_j in the chain such that $P_i \subsetneq P_j \subsetneq P_k$.

Theorem 4.3.18. *Let R be a ring. Then the following are equivalent:*

a) R admits a dense chain of primes

b) R admits a chain of primes indexed by $\mathbb{Q} \cap [0, 1]$

c) R admits a chain of primes indexed by the closed interval $[0, 1]$

Proof. Employing an argument similar to the one in Example 2.10, it is clear that $b) \Rightarrow c)$. The implication $c) \Rightarrow a)$ is obvious. To see $a) \Rightarrow b)$, we start with a dense chain of primes in R given by $(P_i)_{i\in\Lambda}$. Then given primes $P_i \subsetneq P_j$ in the chain, there exists some P_k in the chain such that $P_i \subsetneq P_k \subsetneq P_j$. Reindexing if necessary, we can find for each $n \in \mathbb{N}$ a subchain C_n of form $P_0 \subsetneq P_{1/2^n} \subsetneq P_{2/2^n} \subsetneq \dots P_{(2^n-1)/2^n} \subsetneq P_1$. Now define $C = \bigcup_{n=0}^{\infty} C_n$. For each $\alpha \in \mathbb{Q} \cap [0, 1]$, define $\mathfrak{P}_\alpha = \bigcup_{k\leq\alpha} P_k$. Suppose $q_1, q_2 \in \mathbb{Q} \cap [0, 1]$ with $q_1 < q_2$. Then there exists $m, n \in \mathbb{N}$ such that $q_1 < (2^n - m)/2^n < q_2$. This gives us $\mathfrak{P}_{q_1} = \bigcup_{k\leq q_1} P_k \subsetneq P_{(2^n-m)/2^n} \subsetneq \bigcup_{k\leq q_2} P_k = \mathfrak{P}_{q_2}$, i.e., $q_1 < q_2 \Rightarrow \mathfrak{P}_{q_1} \subsetneq \mathfrak{P}_{q_2}$. Now we have a chain $(\mathfrak{P}_q)_{q\in\mathbb{Q}\cap[0,1]}$, as desired. $\square$

Obviously, the equivalent conditions of Theorem 4.3.18 imply $dim(R) \geq \aleph_1$. The next two examples demonstrate that the converse need not hold, i.e., if $dim(R) \geq \aleph_1$, then R need not admit a dense chain of primes.

Example 4.3.19. Let S be an infinite set that is inversely well-ordered and let V be a valuation domain such that $G(V) \cong_o \bigoplus_S \mathbb{Z}$. We have seen in Theorem 4.3.10 that $dim(V) = |S|$. Assume now $(P_i)_\Lambda$ is an infinite chain of primes in V and choose any P_i in this chain. If Λ is finite, then clearly the chain would not be dense. So we assume Λ is infinite and pick $a, b, c \in \Lambda$ with $a < b < c$. Since the map in part b) of Theorem 4.3.8 is order-preserving, then we have three primes $P_a \subsetneq P_b \subsetneq P_c$ from the chain. Let $\sigma = \{i \in \Lambda | i < b\}$. Since $\sigma \subseteq S$ and S is inversely well-ordered, then there exists some $t \in \sigma$ such that $t \geq i$ for all $i \in \sigma$. Hence, t is an immediate predecessor of b in Λ. It now follows that $P_t \subsetneq P_b$ are adjacent primes in the chain, i.e., the chain is not dense.

The following example achieves the same goal as the previous but is constructive and does not rely on the Axiom of Choice.

Example 4.3.20. Let $\mathcal{C}$ denote the standard Cantor set and assume V is a valuation domain such that $G(V)$ is order isomorphic to the lexicographic sum $\bigoplus_{i \in \mathcal{C}} \mathbb{Z}$. Then there exists a chain $(Q_i)_{i \in \mathcal{C}}$ of nonzero pairwise-distinct primes in V. Given any $P \in Spec(V)$ we denote the subgroups of $G(V)$ which are generated by the sets $\{jU(V) | j \in P - 0\}$ and $\{kU(V) | k \in V - P\}$ as J_P and K_P, respectively. Then $G(V) = J_P + K_P$. Further, if $P \subsetneq P'$ are primes in V, then it is evident that $J_P \subsetneq J_{P'}$. Let $\psi : G(V) \to \bigoplus_{i \in \mathcal{C}} \mathbb{Z}$ be an order isomorphism and let $\mathfrak{B}$ be the standard basis for $\bigoplus_{i \in \mathcal{C}} \mathbb{Z}$. Now let $\mathfrak{B}_P = \mathfrak{B} \cap \psi(J_P)$ for each $P \in Spec(V)$. Then $P \subsetneq P' \Rightarrow \mathfrak{B}_P \subsetneq \mathfrak{B}_{P'}$. Also, given any $P \in Spec(V)$, we let $r(P) = sup\{i \in \mathcal{C} | e_i \in \mathfrak{B}\}$ and note that each $r(P) \in \mathcal{C}$ since $\mathcal{C}$ is closed. Given primes $P_1 \subsetneq P_2 \subsetneq P_3$ in V, we have $\mathfrak{B}_{P_1} \subsetneq \mathfrak{B}_{P_2} \subsetneq \mathfrak{B}_{P_3}$. Let $e_{i_2} \in \mathfrak{B}_{P_2} - \mathfrak{B}_{P_1}$ and $e_{i_3} \in \mathfrak{B}_{P_3} - \mathfrak{B}_{P_2}$. We then get $e_{i_2} > e_{i_3} \Rightarrow i_2 < i_3$, from which it follows that $r(P_1) \leq i_2 < i_3 \leq r(P_3)$.

Assume now that V admits a dense chain of primes denoted by $(P_i)_{i\in\Lambda}$. We then have a subsequence $(r(P_i))_{i\in\Lambda}$ of $\mathcal{C}$. Suppose $r(P_a) < r(P_e)$ for some $P_a, P_e \in (P_i)_{i\in\Lambda}$. Then $P_a \subsetneq P_e$. Using the fact that our chain of primes is dense we can generate a subchain $P_a \subsetneq P_b \subsetneq P_c \subsetneq P_d \subsetneq P_e$ of $(P_i)_{i\in\Lambda}$. Now we have $r(P_a) < r(P_c) < r(P_e)$. We have therefore constructed a dense subset of $\mathcal{C}$, a contradiction since $\mathcal{C}$ contains no dense subsets.

It follows very readily from Definition 4.3.3 that if σ is a cardinal number such that $dim(R) > \sigma$, then R must admit a chain of primes $(P_i)_{i\in\Lambda}$, where $|\Lambda| > \sigma$. If V is the valuation domain in Example 4.3.20, then $dim(V) \leq \aleph_1$. But Theorem 4.3.12 implies $\aleph_1 \leq dim(V)$. Hence, $dim(V) = \aleph_1$. It is also evident that if one were to attempt to construct a ring R where $dim(R) = \aleph_1$ and given any chain of primes $(P_i)_{i\in\Lambda}$ in R we have $|\Lambda| < \aleph_1$ (in the spirit of Nagata), then the Continuum Hypothesis would have to be abandoned. No such attempt will be made here.

It was proven in [5] that all non-field fragmented domains have infinite Krull dimension. As an added application of Theorem 4.3.12 we revisit this result in Theorem 4.3.23 to provide it with a more economical proof, which will make good use of the following lemma.

Lemma 4.3.21. *Let D be a fragmented domain and assume $(x_i)_{i=1}^{\infty}$ is a sequence in $D^* - U(D)$ such that $x_j \in \bigcap_{m=1}^{\infty}(x_{j+1}^m)$. Then given any $a_2, \ldots, a_n \in \mathbb{Z}$ we have $\frac{x_1}{x_2^{a_2}\cdots x_n^{a_n}} \in D^* - U(D)$.*

Proof. Should $x_1 = ux_2^k$ for some $u \in U(D)$, then because $x_1 \in (x_2^{k+1})$ we would have $(x_2^k) = (x_1) \subseteq (x_2^{k+1}) \subseteq (x_2^k)$, whence $x_2^k = rx_2^{k+1}$ for some $r \in D$. But then $rx_2 = 1$, a contradiction since $x_2 \notin U(D)$.

Let us first consider the case when $a_2, \ldots, a_n \in \mathbb{N}$. Since D is fragmented, then from our observation in the previous paragraph we may write $x_1 = r_2 x_2^{a_2+1}, x_2 = r_3 x_3^{a_3+1}, \ldots x_{n-2} = r_{n-1} x_{n-1}^{1+a_{n-1}}, x_{n-1} = r_n x_n^{1+a_n}$ for some $r_2, \ldots, r_n \in D^* - U(D)$. Hence, $x_1 = (r_2 x_2^{a_2}) x_2$. But $x_2 = r_3 x_3^{a_3+1}$ and so $(r_2 x_2^{a_2}) x_2 = (r_2 x_2^{a_2})(r_2 x_2^{a_2}) x_3$. Continuing in this way we conclude $x_1 = (r_2 x_2^{a_2})(r_3 x_3^{a_3}) \cdots (r_n x_n^{a_n})$. We conclude $\frac{x_1}{x_2^{a_2} \cdots x_n^{a_n}} = r_2 \cdots r_n$. Since $r_2, \ldots, r_n \in D^* - U(D)$, then so is their product. To finish the proof, we assume $a_2, \ldots, a_n \in \mathbb{Z}$. We have already shown that $\frac{x_1}{x_2^{|a_2|} \cdots x_n^{|a_n|}} \in D^* - U(D)$. In G_D this means $x_1 x_2^{-|a_2|} \cdots x_m^{-|a_m|} U(D) > U(D)$. But $x_1 x_2^{a_2} \cdots x_m^{a_m} U(D) \geq x_1 x_2^{-|a_2|} \cdots x_m^{-|a_m|} U(D)$ and so we are done. $\square$

Theorem 4.3.22. *Let D be a domain and assume $H = \bigoplus_{i=1}^{\infty} \mathbb{Z}$ is ordered lexicographically. The following conditions are equivalent:*

a) D is fragmented.

b) Given any $x \in D^* - U(D)$, there exists a subgroup $H_x \leq G(D)$ such that $xU(D) \in G(D)$ and $H_x \cong_o H$.

Proof. *a)* $\Rightarrow$ *b)* We assume D is fragmented and let $x = x_1 \in D^* - U(D)$. Since D is fragmented, we can find a sequence $(x_i)_{i=1}^{\infty}$ in $D^* - U(D)$ such that every $x_i \in \bigcap_{n=1}^{\infty} x_{i+1}^n D$. We let $\{e_i | i \in \mathbb{N}\}$ be the standard basis for H and consider the group map $\phi : H \to G(D)$ determined by $\phi(e_i) = x_i U(D)$. Let us assume $\sum_{j=1}^{m} r_j e_{i_j} > 0$ with $e_{i_1} > \ldots > e_{i_m}$ and $r_1 \neq 0$. As H enjoys the lexicographic ordering, then we deduce $r_1 > 0$. We have $\phi(\sum_{j=1}^{m} r_j e_{i_j}) = x_{i_1}^{r_1} \cdots x_{i_m}^{r_m} U(D)$. Notice that since $e_{i_1} > \ldots > e_{i_m}$, then $i_1 < \ldots < i_m$. Hence, $x_{i_j} \in \bigcap_{n=1}^{\infty} x_{i_{j+1}}^n D$. Now $x_{i_1}^{r_1} \cdots x_{i_m}^{r_m} \in D^* - U(D)$, i.e., $x_{i_1}^{r_1} \cdots x_{i_m}^{r_m} U(D) > U(D)$. By Theorem 3.1.8 ϕ an order-preserving map. Note that it also follows from Theorem 3.1.8 that ϕ is monic since H is linearly ordered. Now just define $H_x = Im(\phi)$ and we are done.

b) $\Rightarrow$ *a)* : We let $x \in D^* - U(D)$. By our hypothesis there exists a subgroup $H_x \leq G_D$ such that $xU(D) \in H_x$ and $H_x \cong_o H$. Say $\psi : H_x \to H$ is an order isomorphism so that $\psi(xU(D)) = \sum_{i=1}^{m} r_i e_i$ for some $r_1, \ldots, r_m \in \mathbb{Z}$. Note that $n(e_{m+1}) < \psi(xU(D))$ for all $n \in \mathbb{N}$. If $\psi(yU(D)) = e_{m+1}$, $y \notin U(D)$ and $y^n U(D) < xU(D)$ for all $n \in \mathbb{N}$, i.e., $x \in \bigcap_{n=1}^{\infty} y^n D$. So D is fragmented. $\square$

Theorem 4.3.22 not only makes the next theorem painless, but it also demonstrates the strong relationship between fragmentation in a domain and lexicographically ordered subgroups in its group of divisibility. The following result was originally published in [5].

Theorem 4.3.23. *If D is a fragmented domain, then $dim(D) \geq \aleph_0$.*

Proof. We assume D is fragmented. By Theorem 4.3.22 we know that given any $x \in D^* - U(D)$, there exists a subgroup $H_x \leq G_D$ such that $xU(D) \in G_D$ and $H_x \cong_o H$. The desired result now follows immediately from Theorem 4.3.12. $\square$

4.4. Dimension Behavior in Some Classical Ring Extensions

In this section we first turn our attention to ring extensions satisfying GU and INC. If $R \subseteq T$ is such an extension, then it is asserted in the literature that $dim(R) = dim(T)$. In the finite dimensional case, the veracity of this claim is indisputable. Should R and T be infinite dimensional, however, then it would seem that the tradition heretofore has been to regard all infinite dimensional rings as having the *same* Krull dimension. But having now seen examples that illustrate this need not be the case, we would like to know whether or not Krull dimension is preserved ring extensions that are GU and INC, e.g., integral extensions.

Lemma 4.4.1. *Suppose $R \subseteq T$ is a GU ring extension and let $P \in Spec(R)$. By LO, let $Q \in Spec(T)$ such that Q lies over P. The $R_P \subseteq T_Q$ is GU.*

Proof. Suppose $P_1 R_P \subsetneq P_2 R_P$ is a chain of primes in R_P. By Theorem 1.4.8 $P_1 \subsetneq P_2 \subseteq P$ is a chain of primes in R. By GU, we have a chain of primes $Q_1 \subsetneq Q_2 \subseteq Q$

in T where each Q_i lies over P_i. Thus, we use Theorem 1.4.8 again to build a chain of primes $Q_1T_Q \subsetneq Q_2T_Q \subseteq$ in T_Q. We now wish to show Q_1T_Q lies over P_1R_P. To this end, let $\frac{r}{s} \in Q_1T_Q \cap R_P$ for some $r \in R$ and $s \in R - P$. Since $s \notin P$, $s \notin Q$. It is also easily verified that $r \in Q_1$. Thus, $r \in Q_1 \cap R = P_1$ and so $\frac{r}{s} \in P_1R_P$. For the reverse containment, since $P_1 \subseteq Q_1$ and $R_P \subseteq T_Q$, then $P_1R_P \subseteq Q_1T_Q$. Thus, $P_1R_P = Q_1T_Q \cap R_P$. A similar argument shows that $P_2R_P = Q_2T_Q \cap R_P$ and so we are done. $\square$

Lemma 4.4.2. *Suppose $R \subseteq T$ is a ring extension satisfying GU and let $P_1 \subsetneq P_2$ be prime ideals in R. By GU, we let $Q_1 \subsetneq Q_2$ be primes in T that lie over P_1 and P_2, respectively. Assume now there is a $P_3 \in Spec(R)$ such that $P_1 \subsetneq P_3 \subsetneq P_2$. Then there exists a $Q_3 \in Spec(T)$ such that $Q_1 \subsetneq Q_3 \subsetneq Q_2$ and Q_3 lies over P_3.*

Proof. From Theorem 4.4.1 we know that $R_{P_1} \subseteq T_{Q_1}$ is a GU extension. Also, since $P_1 \subsetneq P_3 \subsetneq P_2$ in R, then by correspondence we have $P_1R_{P_2} \subsetneq P_3R_{P_2} \subsetneq P_2R_{P_2}$. Now use GU to find a chain of primes $Q_1T_{Q_2} \subsetneq Q_3T_{Q_2} \subsetneq Q_2T_{Q_2}$ such that each $Q_iT_{Q_i}$ lies over $P_iR_{P_i}$.

We now wish to show each Q_i lies over P_i. To this end, assume $r \in Q_1 \cap R$ so that $r \in Q_1T_{Q_2} \cap R_{P_2} = P_1R_{P_2}$. Then $r = \frac{p}{s}$ for some $p \in P_1$ and $s \in R - P_2$. Thus, $(rs - p)s' = 0$ for some $s' \in R - P_2$ and so $rss' \in P_1$. But $ss' \notin P_1$ and so have $r \in P_1$. We have now shown $Q_1 \cap R \subseteq P_1$. A similar argument establishes the reverse containment. Hence, Q_1 lies over P_1. A similar argument shows that each Q_i lies over P_i and we are done. $\square$

In our next lemma we will refer to a chain of primes in a ring R *between* some prime ideals P_1 and P_2 of R. By this we mean that we have a chain of primes $(B_i)_\sigma$ with the properties that $P_1, P_2 \in \{B_i\}_\sigma$, $\bigcap_{i\in\sigma} B_i = P_1$, and $\bigcup_{i\in\sigma} B_i = P_2$.

Lemma 4.4.3. *Assume the ring extension $R \subseteq T$ is GU and let $P_1 \subsetneq P_2$ be prime ideals in R. Let $(B_i)_\Lambda$ be a chain of primes between P_1 and P_2. Then there exists a chain $(C_i)_\Lambda$ of primes in T such that $C_i \cap R = B_i$ for all $i \in \Lambda$.*

Proof. Since GU implies LO, we may let $Q_1 \subsetneq Q_2$ be primes in T that lie over P_1 and P_2, respectively. Let S be the set of all chains of primes in T between Q_1 and Q_2 such that if $(M_i)_I \in S$, then $M_i \cap R = B_k$ for some $B_k \in \{B_i\}_\Lambda$. Note that $S \neq \emptyset$ since the chain $Q_1 \subsetneq Q_2$ is a member of S. Now partially order S by set theoretic inclusion. Let $(L_i)_\varphi$ be a chain (of chains) in S and say $L_i = (M_{i,j})_{j\in\phi_i}$. We can linearly order $F = \bigcup_\varphi L_i = \bigcup_{i\in\varphi}\bigcup_{j\in\phi_i} M_{i,j}$ by set inclusion to conclude that $F \in S$. By Zorn, we may choose some maximal $M \in S$.

To finish the proof, we now show that M satisfies the claim of Lemma 4.4.3. Since $M \in S$, then every $Q \in M$ lies over some prime in $\{B_i\}_\Lambda$. Assume now, to the contrary, that there is some $B_\ell \in \{B_i\}_\Lambda$ such that $B_\ell \neq M_i \cap R$ for all $M_i \in M$. As every $C \in S$ contains primes that lie over P_1 and P_2, then $P_1 \subsetneq B_\ell \subsetneq P_2$. Let $D = \{\mathfrak{P} \in M | \mathfrak{P} \cap R \subsetneq B_\ell\}$ and $N = \{\mathfrak{Q} \in M | B_\ell \subsetneq \mathfrak{P} \cap R\}$. Observe that D and N are linearly ordered by inclusion so that $\bigcup_D \mathfrak{P}, \bigcap_N \mathfrak{Q} \in Spec(T)$. Now we will say that $\prod_1 = \bigcup_{\mathfrak{P}\in D}(\mathfrak{P} \cap R)$ and $\prod_2 = \bigcap_{\mathfrak{Q}\in N}(\mathfrak{Q} \cap R)$. Then $\prod_1 \subseteq B_\ell \subseteq \prod_2$ is a chain of prime ideals in R.

We proceed with a couple of observations. First, $\prod_1 \subsetneq B_\ell$. Indeed, if we attain equality, then $\bigcup_D \mathfrak{P}$ lies over B_ℓ and upon linearly ordering $M \cup \{\bigcup_D \mathfrak{P}\}$ we have contradicted the maximality of M. Lastly, a similar argument assures us that $B_\ell \subsetneq \prod_2$.

We are now in a position to utilize Lemma 4.4.2. We have a chain of primes in R given by $\prod_1 \subsetneq B_\ell \subsetneq \prod_2$ and we have primes $\bigcup_D \mathfrak{P}, \bigcap_N \mathfrak{Q} \in Spec(T)$ that lie over $\prod_1$ and $\prod_2$, respectively. By Lemma 4.4.2 there exists a prime $Q \in Spec(T)$ such that $\bigcup_D \mathfrak{P} \subsetneq Q \subsetneq \bigcap_N \mathfrak{Q}$ and $Q \cap R = B_\ell$. Upon linearly ordering $M \cup \{Q\}$ by inclusion, we have contradicted the maximality of M. $\square$

Let us consider for a moment what the previous lemma is really telling us. Should we have a ring extension $R \subseteq T$ satisfying GU, then given any chain of primes $(P_i)_\Lambda$ in R, we can find a corresponding chain of primes $(Q_i)_{i\in\Lambda}$ in T. Thus, whenever $R \subseteq T$ is GU, then $dim(R) \leq dim(T)$ regardless of the cardinal numbers associated with the respective dimensions of R and T. It should come as little surprise when we demonstrate in our next theorem that equality can be attained whenever the extension is also INC.

Theorem 4.4.4. *Let $R \subseteq T$ be a ring extension satisfying GU and INC. Then $dim(R) = dim(T)$.*

Proof. Since $R \subseteq T$ is GU, then $dim(R) \leq dim(T)$. Now let $M \subsetneq T$ be a maximal ideal. By Zorn we may let $(B_i)_\psi$ be a saturated chain of primes in T between 0 and M. Then $(B_i \cap R)_\psi$ is a chain of primes in R between 0 and $M \cap R$. As $R \subseteq T$ is GU, then $M \cap R$ must be maximal in R. Since $R \subseteq T$ is INC, then given primes $B_j \subsetneq B_k$ in T we have $B_j \cap R \subsetneq B_k \cap R$. We conclude that $ht(M) \leq ht(M \cap R)$. It now follows from Theorem 4.3.5 that $dim(T) \leq dim(R)$ and we are done. $\square$

We will now address the question as to how the dimensions of R, $R[x]$, and $R[[x]]$ compare when R is infinite dimensional. We have two primary objectives here. Our first aim is to establish that $dim(R) = dim(R[x])$ when R is infinite dimensional. The second objective is then to use the this result to suggest a line of inquiry into the dimension behavior of $R[[x]]$. Theorem 4.4.5 is a celebrated result in dimension theory and is proven in [7].

Theorem 4.4.5. *If $P_1 \subsetneq P_2 \subsetneq P_3$ are prime ideals in $R[x]$, then $P_1 \cap R \subsetneq P_3 \cap R$.*

Theorem 4.4.5 assures us that no more than 2 primes of a polynomial ring $R[x]$ may lie over the same prime in R. Gilmer then uses this fact to prove that if R is finite dimensional with $dim(R) = n$, then $n+1 \leq dim(R[x]) \leq 2n+1$. Theorem 4.4.5 also gives us the following characterization of infinite dimensional rings.

Theorem 4.4.6. *Let R be a ring. Then R is infinite dimensional if and only if* $dim(R) = dim(R[x])$.

Proof. It has already been shown that if R is finite dimensional, then $dim(R) < dim(R[x])$. Hence, if $dim(R) = dim(R[x])$, then R must be infinite dimensional. For the converse, we observe that since at most two primes in $R[x]$ may lie over any given prime in R, then we have that $dim(R[x]) \leq 2dim(R) + 1$. But since R is infinite dimensional, then $dim(R) = 2dim(R) + 1$. Thus, $dim(R) \leq dim(R[x]) \leq dim(R)$ and we are done. □

The nice dimension behavior of polynomial extensions do not carry over to power series extensions. In fact, there are examples of zero-dimensional rings whose power series extensions are infinite dimensional. We believe this kind of pathology extends to the infinite dimensional case. Therefore, we end this chapter with the following conjecture.

Conjecture 4.4.7. Given any cardinal number σ, there exists a ring R_σ such that $\sigma = dim(R_\sigma) < dim(R_\sigma[[x]])$.

CHAPTER 5. PURGATORY DOMAINS AND A CORRESPONDENCE PROBLEM

5.1. Motivation

In this chapter we pull back and take in a panoramic view. One of the ultimate objectives of the work being done in any branch of algebra is the classification of the algebraic objects in that particular field of study. In meandering through the literature of commutative algebra one finds that certain domains seem to enjoy a certain ubiquity. We are bombarded with discussion of Noetherian domains, fields, UFD's, HFD's, Dedekind domains, Krull domains, valuation domains, Prüfer domains, and on and on. That we should naturally gravitate toward these domains comes as no surprise as they are used to achieve results of spectacular depth and beauty. But how commonplace are they in the grand scheme of things? One of the primary sources of inspiration for this line of thinking comes from experience in classroom teaching. Often is the occasion when it becomes convenient to have a number chosen at random by a student. The vast majority of the time the number given by the student would be an integer and, virtually invariably, the number would be rational at the very least. No doubt, the psychological reasons for this are numerous. Now let us imagine that we could take the set of real numbers, throw them into a bag and shuffle them. Then if we were to pick a number at random, the probability that the number would be rational (or even algebraic over any countable subfield of $\mathbb{R}$!) is zero. This is because the Lebesgue measure of any countable set of reals is zero. Of course, what this is really saying is that in a certain sense rational numbers are an extremely rare occurrence in $\mathbb{R}$, despite the fact that we encounter them on a routine basis. Surely one reason the calculus student will usually give an integer when asked to choose a number is because these are the types of numbers that are most often encountered in the classroom. After all, it is no less convenient to say "pi" than it is to say "eight". Perhaps there has been a similar tendency in commutative algebra to regard the rings with which we are so familiar and toward which we have concentrated so much of our time and energy as being as commonplace as the student of calculus believes the integers to be. Certainly, these rings are not difficult to find. But perhaps this is only because they are useful in solving those particular types of problems which attract our fancies. But what if these rings are comparable to the integers sitting within the reals? What if they are actually extremely rare occurrences? There will be those who will say they have no use for such questions. This is understandable. Then again, there are those outside of the mathematical sciences who have little use for understanding that π is transcendental over $\mathbb{Q}$, much less *why*.

From the point of view of factorization, atomic domains afford a certain degree of luxury because every nonzero nonunit can be broken down into parts that cannot be broken down any further. On the other side of the ledger we have the AMD's, where given any nonzero nonunit, there is no end as to how far you can keep pulling things apart in nontrivial ways. If we liken atomic domains to a calm sunny day at the lake, then AMD's must be the stuff of our most ghastly nightmares. But there is a third option.

Definition 5.1.1. A **purgatory** domain is any domain which is neither atomic nor antimatter.

When we refer to a domain as being *in purgatory*, we simply mean that the said domain is a purgatory domain. Once we know where to look, we find that purgatory domains are not so difficult to find. Later, we present some ideas that give credence to the idea that not only are they not so difficult in tracking down, but perhaps they are far more common than the classes of atomic and antimatter domains combined! Indeed, we will see that to every integral domain D, there exists a purgatory domain which we can associate to D in a unique way (up to isomorphism). We consider this highly significant given the amount of attention lavished upon atomic domains versus that bestowed to domains in purgatory. But first, perhaps we should get our feet wet by looking at some easy ways to generate purgatory domains.

Example 5.1.2. Let D be a non-field AMD. Then $D[x]$ lies in purgatory. Certainly, $D[x]$ is non atomic because D is an AMD. However, $x \in Irr(D[x])$.

Example 5.1.3. As Roitman exemplified in [14], atomicity is not necessarily preserved in polynomial extensions of domains. Thus, it is possible for D to be atomic and $D[x]$ to be in purgatory.

Example 5.1.4. If V is a non-field valuation domain with a principal maximal ideal, then V admits an atom. If, in addition, $dim(V) > 1$, then V must fail to be atomic and therefore we would find V residing in purgatory. In particular, if we consider the lexicographic sum $\bigoplus_{i=1}^{n} \mathbb{Z}$, then for any positive integer $n > 2$ we can find a valuation domain, V_n, such that $G(V_n) \cong_o \bigoplus_{i=1}^{n} \mathbb{Z}$. Such a valuation domain admits an atom but would fail to be atomic by Theorem 2.1.16. To find an example of a 1-dimensional purgatory domain, we consider the $G = \mathbb{Z} \oplus \mathbb{Q}$ in the product order. As G is a sum of lattice ordered abelian groups, then G itself is lattice ordered by Theorem 3.5.18. The Jaffard-Ohm-Kaplansky Theorem states that we can find a domain D such that $G(D) \cong_o G$. Clearly now $\mathbb{Z} \oplus \mathbb{Q}$ admits an element that is minimal with respect to being positive. Hence, D admits an atom. Also, Theorem 3.5.22 assures us that D is a 1-dimensional Bêzout domain. However, if D were atomic, then D would a PID by Theorem 2.1.23. But Theorem 3.5.8 tells us that D is not even a UFD.

5.2. A Correspondence Problem

We are motivated by the following questions. To each atomic domain, can associate a unique purgatory or AMD? To each AMD, can we associate a unique atomic or purgatory domain? To each purgatory domain, can we associate a unique atomic or AMD? It is tempting to first turn our attention to polynomial or power series extensions of rings. After all, we already seen that if D is either atomic or an AMD, then it is possible for $D[x]$ and $D[[x]]$ to be in purgatory. In fact, if D is any non-field AMD, then $D[x]$ and $D[[x]]$ are always in purgatory. Therefore, it seems natural to assume that perhaps this function is monic. That is, if $D[x] \cong R[x]$, then perhaps $D \cong R$. Rings that satisfy this property are said to be **1-stable**. The fact that such rings are even given a name gives us good reason to doubt our hypothesis. Indeed, Hochster showed in [9] shows that our hypothesis does not hold. That is to say, there are examples of (Noetherian) rings R and S such that $R[x] \cong S[x]$ but $R \ncong S$! Rings that are 1-stable are of interest in their own right, but it is not our purpose here to study such rings. However, Theorem 5.2.2 is a basic characterization of 1-stable rings and we will find some use for it. It also highlights some of the pathologies that can occur for rings that are not 1-stable. In the proof of Theorem 5.2.2 we will find use of the following lemma. We omit the proof.

Lemma 5.2.1. *Suppose $f : R \to S$ is a ring isomorphism and $I \leq R$. Then $R/I \cong S/f(I)$.*

Suppose R and D are isomorphic rings. The we denote the set of all isomorphisms $R \to S$ by *Iso(R,D)*. We use this notation in the next two theorems.

Theorem 5.2.2. *Let R and S be rings such that $R[x] \cong S[x]$. The following are equivalent:*

a) $R \cong S$

b) $\exists \varphi \in Iso(R[x], S[x])$ *such that* $\varphi(x) = x$

c) $\exists \varphi \in Iso(R[x], S[x])$ *such that* $\varphi(x) \in xS[x]$

d) $\exists \varphi \in Iso(R[x], S[x])$ *such that* $\varphi(xR[x]) \subseteq xS[x]$

e) $\exists \varphi \in Iso(R[x], S[x])$ *such that* $\varphi(xR[x]) = xS[x]$

f) $\exists \varphi \in Iso(R[x], S[x])$ *such that* $\varphi(R[x] - xR[x]) = S[x] - xS[x]$

Proof. The implications $a) \Rightarrow b)$, $b) \Rightarrow c)$, $c) \Rightarrow d)$, and $e) \Leftrightarrow f)$ are clear. That $d) \Rightarrow e)$ follows from the fact that $xR[x]$ and $xS[x]$ are both ht-1 primes. Thus, upon verification of the implication $e) \Rightarrow a)$ we will be done. But this is not difficult. We have $R \cong R[x]/xR[x]$. Let $\varphi : R[x] \to S[x]$ be a ring isomorphism. From Lemma 5.2.1 we then get $R[x]/xR[x] \cong S[x]/xS[x]$. But $S[x]/xS[x] \cong S$. Therefore, $R \cong S$. $\square$

We can obtain a similar result to Theorem 5.2.2 by replacing $R[x]$ and $S[x]$ with $R[[x]]$ and $S[[x]]$, respectively. For the sake of completeness, we state this result. However, we will skip its proof because it simply mimics the proof of the previous theorem.

Theorem 5.2.3. *Let R and S be rings such that $R[[x]] \cong S[[x]]$. The following are equivalent:*

a) $R \cong S$

b) $\exists \varphi \in Iso(R[[x]], S[[x]])$ *such that* $\varphi(x) = x$

c) $\exists \varphi \in Iso(R[[x]], S[[x]])$ *such that* $\varphi(x) \in xS[[x]]$

d) $\exists \varphi \in Iso(R[[x]], S[[x]])$ *such that* $\varphi(xR[[x]]) \subseteq xS[[x]]$

e) $\exists \varphi \in Iso(R[[x]], S[[x]])$ *such that* $\varphi(xR[[x]]) = xS[[x]]$

f) $\exists \varphi \in Iso(R[[x]], S[[x]])$ *such that* $\varphi(R[[x]] - xR[[x]]) = S[[x]] - xS[[x]]$

Some rings are 1-stable for polynomial and power series extensions. It should come as no surprise that fields are example of such rings and, of course, $\mathbb{Z}$ is, also.

Theorem 5.2.4. *Let F and L be fields with $F[x] \cong L[x]$. Then $F \cong L$.*

Proof. We begin by allowing $\varphi : F[x] \to L[x]$ be a ring isomorphism. Then $L^* = U(L[x]) = U(\varphi(F[x])) = \varphi(U(F[x])) = F^*$. Also, $\varphi(0) = 0$. So $\varphi(F) = L$. $\square$

Theorem 5.2.5. *Let F and L be fields with $F[[x]] \cong L[[x]]$. Then $F \cong L$.*

Proof. Recall that for any field K, the ring $K[[x]]$ is a Noetherian valuation domain and so $xK[[x]]$ is its unique nonzero prime ideal. Thus, if $\varphi : F[[x]] \to L[[x]]$ is a ring isomorphism, then we must have $\varphi(xF[[x]]) = xL[[x]]$. That $F \cong L$ is now a direct consequence of Theorem 5.2.3. $\square$

We begin now to show that we may associate to every atomic or AMD a unique purgatory domain. Indeed, we will show even more than this in Theorem 5.2.12 where we show that we can associate a unique purgatory domain to every integral domain.

Theorem 5.2.6. *Let D be a domain with quotient field K and let $R = D + xK[[x]]$. Then $xK[[x]]$ is the unique $ht-1$ prime ideal of R.*

Proof. First, suppose $0 \neq P \subseteq xK[[x]]$ is a chain of primes in R and let $\beta = k_n x^n + k_{n+1}x^{n+1} + \ldots \in P$ with $k_n \neq 0$. Then $(\frac{1}{k_n})x\beta = x_{n+1}u$ for some $u \in U(R)$. Hence, $x^{n+1}u \in P \Rightarrow x_{n+1} \in P \Rightarrow x \in P$. Now let $0 \neq \pi \in xK[[x]]$ and note that $x|\beta^2$. Hence, $\beta^2 \in P$ and so $\beta \in P$. Thus, $P = xK[[x]]$, i.e., $xK[[x]]$ is a ht-1 prime.

We now show that $xK[[x]]$ is the only ht-1 prime in R. To this end, assume now P is a ht-1 prime and let $0 \neq f(x) \in P$. If $f(0) \neq 0$, then $f(x) = f(0)u$ for some $u \in U(R)$. We would then have that $f(0) \in P$. But if this was the case, then we would have that $xK[[x]] \subsetneq f(0)R \subseteq P$, contradicting the assumption that P is a ht-1 prime. Thus, $f(0) = 0$ and so $P \subseteq xK[[x]]$. Since $xK[[x]]$ is also a ht-1 prime, then $P = xK[[x]]$ and we are done. $\square$

Theorem 5.2.7. *Suppose D_1 and D_2 are non-field domains with quotient fields K_1 and K_2, respectively. Let $R_i = D_i + xK_i[[x]]$. Then $R_1 \cong R_2 \Leftrightarrow D_1 \cong D_2$.*

Proof. ($\Rightarrow$) First note that we know that $xK_i[[x]]$ is the unique ht-1 prime in R_i from Theorem 5.2.6. If $\varphi : R_1 \to R_2$ is a ring isomorphism, then $\varphi(xK_1[[x]]) = xK_2[[x]]$ because of the necessary correspondence of ht-1 primes. From Theorem 5.2.1 we know that $R_1/xK_1[[x]] \cong R_2/Im(\varphi) = R_2/xK_2[[x]]$. Thus, $D_1 \cong R_1/xK_1[[x]] \cong R_2/Im(\varphi) = R_2/xK_2[[x]] \cong D_2$, as we wished to show.

($\Leftarrow$)We let $\varphi : D_1 \to D_2$ be a ring isomorphism. Then we can extend φ to a ring isomorphism $\overline{\varphi} : K_1 \to K_2$ by $\overline{\varphi}(\frac{x}{y}) = \frac{\varphi(x)}{\varphi(y)}$. This then gives a ring isomorphism $\Phi : K_1[[x]] \to K_2[[x]]$ given by $\Phi(\sum_{i=0}^{\infty} k_i x^i) = \sum_{i=0}^{\infty} \overline{\varphi}(k_i)x^i$. Hence, $\tilde{\Phi} : R_1 \to R_2$ defined by $\tilde{\Phi}(d_0 + \sum_{i=1}^{\infty} k_i x^i) = \varphi(d_0) + \sum_{i=1}^{\infty} \overline{\varphi}(k_i)x^i$ is a ring isomorphism. $\square$

Let us pause for a moment and consider some of the things Theorem 5.2.7 allows us to do. First, if D is a non-field AMD with quotient field K, then the same can be said of $D + xK[[x]]$. Moreover, Theorem 5.2.7 allows us to associate to any non-field AMD D another such AMD and this association is unique up to isomorphism. Theorem 5.2.7 further allows us to associate to every non-field atomic domain a unique (up to isomorphism) purgatory domain. Theorem 5.2.11 is similar to Theorem 5.2.7 but will serve our future purposes a little better. First we need some help.

Theorem 5.2.8. *Let R and D be domains with complete integral closures $C(R)$ and $C(D)$, respectively. Assume $\varphi : R \to D$ is a ring isomorphism and let $\iota_1 : R \to C(R)$ and $\iota_2 : D \to C(D)$ be the natural inclusions. Then there exists a ring isomorphism $\overline{\varphi} : C(R) \to C(D)$ making the following diagram commute.*

$$\begin{tikzcd} R \arrow[r, "\varphi"] \arrow[d, "\iota_1"] & D \arrow[d, "\iota_2"] \\ C(R) \arrow[r, "\overline{\varphi}"] & C(D) \end{tikzcd}$$

Proof. For every $t \in C(R)$, choose some $a_t, b_t \in R$ such that $t = \frac{a_t}{b_t}$. Define $\overline{\varphi}(\frac{a_t}{b_t}) = \frac{\varphi(a_t)}{\varphi(b_t)}$. First, if $\frac{a_t}{b_t} = \frac{\alpha_t}{\beta_t}$, then $a_t\beta_t = \alpha_t b_t \Rightarrow \varphi(a_t)\varphi(\beta_t) = \varphi(\alpha_t)\varphi(b_t) \Rightarrow \frac{\varphi(a_t)}{\varphi(b_t)} = \frac{\varphi(\alpha_t)}{\varphi(\beta_t)}$, assuring $\overline{\varphi}$ is well-defined. That $\overline{\varphi}$ is a ring map now follows easily from the fact that

φ is a ring map. Now letting $h = \varphi^{-1}$, it is easily shown that $\overline{\varphi} \circ \overline{h} = 1_{C(R)}$ and $\overline{h} \circ \overline{\varphi} = 1_{C(D)}$. Therefore, $\overline{\varphi}$ is a ring isomorphism. The only thing left to demonstrate is the commutativity of the diagram, which is easy enough to omit. $\square$

If $R \subsetneq T$ is any ring extension, then the **conductor** of R in T is defined by $I = \{r \in R | rT \subseteq R\}$. It is easily shown that I is an ideal of both R and T. Theorem 5.2.9 tells us that I is characterized as being the largest ideal common to both R and T.

Theorem 5.2.9. *Let $R \subsetneq T$ be a ring extension and let I be an ideal of both R and T. Then I is the conductor of R in T if and only if I contains every ideal common to both R and T.*

Proof. Suppose first I is the conductor ideal of R in T and let K be any other ideal common to both rings. Letting $k \in K$, we wish to show that $k \in I$. So let $t \in T$. Then $kt \in K$. But since K is an ideal of R, then $kt \in R$. So $k \in I$. Conversely, assume I contains every ideal common to both R and T. Letting $t \in I$ and $t \in T$, we wish to show $rt \in R$. But this follows immediately from the fact that $rt \in I \subseteq R$. $\square$

Lemma 5.2.10. *Let D be a domain with quotient field $K \neq D$ and let $R = D + xK[x]$. Then the conductor of $D + xK[x]$ in $K[x]$ is $xK[x]$.*

Proof. Let J be conductor of $D + xK[x]$ in $K[x]$. Then certainly $xK[x] \subseteq J$. Let $j_0 + j_1x + j_2x^2 + \ldots j_nx^n \in J$ and note $j_1x + j_2x^2 + \ldots j_nx^n \in xK[x]$. Because $xK[x] \subseteq J$, then $j_1x + j_2x^2 + \ldots j_nx^n \in J$. We deduce $j_0 \in J \cap K$. As J is an ideal of $K[x]$ and K is a field, then we must have $j_0 = 0$ and so $J \subseteq xK[x]$. $\square$

Theorem 5.2.11. *Let D_1 and D_2 be domains with quotient fields K_1 and K_2. Let $R_i = D_i + xK_i[x]$. Then $R_1 \cong R_2 \Leftrightarrow D_1 \cong D_2$.*

Proof. ($\Rightarrow$)We begin by letting $\varphi : R_1 \to R_2$ be a ring isomorphism. As in the proof of Theorem 5.2.2, it suffices to show $\varphi(xK_1[x]) = xK_2[x]$. Observe now that $K_1[x]$ and $K_2[x]$ are the complete integral closures of R_1 and R_2, respectively. Let $\iota_i : R_i \to K_i[x]$ be the natural inclusion. From Theorem 5.2.8 we can extend φ to a ring isomorphism $\overline{\varphi} : K_1[x] \to K_2[x]$ giving us the following commutative diagram of ring maps.

$$\begin{tikzcd} R_1 \arrow[r, "\varphi"] \arrow[d, "\iota_1"] & R_2 \arrow[d, "\iota_2"] \\ K_1[x] \arrow[r, "\overline{\varphi}"] & K_2[x] \end{tikzcd}$$

Using Lemma 5.2.10 we know that $xK[x]$ is the conductor of R_1 in $K_1[x]$. The commutativity of the diagram makes it evident that $\varphi(xK_1[x] = \overline{\varphi}(xK_1[x])$. Because φ and $\overline{\varphi}$ are isomorphisms, then $\varphi(xK[x])$ is an ideal in both R_2 and $K_2[x]$. From Lemma 5.2.10 it follows that $\varphi(xK_1[x]) \subseteq xK_2[x]$. But each $xK_i[x]$ are ht-1 prime ideals. Thus, $\varphi(xK_1[x]) = xK_2[x]$ and we are done.

($\Leftarrow$) Mimic the converse argument in the proof of Theorem 5.2.7.

$\square$

To see what all this hubbub does for us, we need to introduce some notation. First we will let $\mathbb{D}$, $\mathbb{A}$, $\mathbb{F}$, $\mathbb{M}$, and $\mathbb{P}$ denote the collections of all isomorphism classes of integral domains, non-field atomic domains, fields, non-field AMD's, and purgatory domains, respectively. Note that $\mathbb{D} = \mathbb{A} \cup \mathbb{F} \cup \mathbb{M} \cup \mathbb{P}$ and the "factors" on the right side of this equation are pairwise disjoint. For any domain D, we are going to now let $\overline{\overline{D}}$ denote the isomorphism class of D. For example, if $\overline{\overline{R}} = \overline{\overline{D}} \in \mathbb{A}$, then we mean R and D are isomorphic non-field atomic domains.

Theorem 5.2.12. *Define $\phi : \mathbb{D} \to \mathbb{P}$ by*

$$\phi(\overline{\overline{D}}) = \begin{cases} \overline{\overline{D + xK_D[x]}} & \textit{if } \overline{\overline{D}} \in \mathbb{A} \cup \mathbb{M} \cup \mathbb{P} \\ \overline{\overline{D[x] + yK_{D[x]}[[y]]}} & \textit{if } \overline{\overline{D}} \in \mathbb{F} \end{cases}$$

Then ϕ is a well-defined injection.

Proof. We first show that ϕ is well-defined. Assume $D_1 \cong D_2$. If $\overline{\overline{D_1}} \in \mathbb{A} \cup \mathbb{M} \cup \mathbb{P}$, then $D_1 + xK_{D_2}[x] \cong D_2 + xK_{D_2}[x]$ by Theorem 5.2.11. Suppose now $\overline{\overline{D}} \in \mathbb{F}$. Then $D_1[x] \cong D_2[x]$ and so $D_1[x] + yK_{D_1[x]}[[y]] \cong D_2[x] + yK_{D_2[x]}[[y]]$ by Theorem 5.2.7. In any case, $\phi(\overline{\overline{D_1}}) = \phi(\overline{\overline{D_2}})$, making ϕ well-defined.

We now show ϕ is monic. Assuming $\phi(\overline{\overline{D_1}}) = \phi(\overline{\overline{D_2}})$, we wish to show $\overline{\overline{D_1}} = \overline{\overline{D_2}}$. If $\overline{\overline{D_1}} \in \mathbb{A} \cup \mathbb{M} \cup \mathbb{P}$ and $D_1 + xK_{D_2}[x] \cong D_2 + xK_{D_2}[x]$, then it follows from Theorem 5.2.11 that $D_1 \cong D_2$. If $\overline{\overline{D_1}} \in \mathbb{F}$ and $D_1[x] + yK_{D_1[x]}[[y]] \cong D_2[x] + yK_{D_2[x]}[[y]]$, then from Theorem 5.2.7 we have that $D_1[x] \cong D_2[x]$. Now use Theorem 5.2.5 to conclude $D_1 \cong D_2$. In any case, we have now shown that $\phi(\overline{\overline{D_1}}) = \phi(\overline{\overline{D_2}}) \Rightarrow D_1 \cong D_2 \Rightarrow \overline{\overline{D_1}} = \overline{\overline{D_2}}$, as desired. □

Theorem 5.2.12 does not settle the correspondence problems we are interested in and we will pursue this matter no further here. However, what Theorem 5.2.12 does assure us of is that *if* there a"largest" isomorphism class of domains, then it must be $\mathbb{P}$. We further caution the reader against the temptation into thinking of these isomorphism classes as sets. They are not. Thus, for example, should we discover a 1-1 map $\mathbb{P} \to \mathbb{M}$, it does not follow that $\mathbb{P}$ and $\mathbb{M}$ are the same size, i.e., it does not follow that there exists a bijection $\mathbb{P} \leftrightarrow \mathbb{M}$ by virtue of the fact that these classes do not constitute sets.

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