SQUARE PEG THINKING, ROUND HOLE PROBLEMS: AN INVESTIGATION OF
STUDENT THINKING ABOUT AND MATHEMATICAL PREPARATION FOR VECTOR
CONCEPTS IN CARTESIAN AND NON-CARTESIAN COORDINATES USED IN
UPPER-DIVISION PHYSICS

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ABSTRACT

As part of the broader effort to explore the math-physics interface in the upper-division our research team seeks to develop research-based curriculum to aid students in translating middle-division mathematics course content to middle- and upper-division physics courses. Toward that goal, an investigation into student thinking regarding vector concepts, such as basis unit vectors, position vectors, and velocity vectors in both Cartesian and non-Cartesian coordinates was performed. Through analysis of students' written responses to in-class assessments, we identified several themes of student thinking that were emergent. Think-aloud interview protocols targeting those themes were developed. Seven subjects were interviewed and their responses were analyzed through a Resources and Framing theoretical framework. Analyzing interview responses allowed us to name specific resources that were activated and categorize those resources into thematic clusters or groups. A case-study provided the opportunity to map how the coordination of a single student’s resources can activate together and how the non-activation of a key resource can cause a dramatic shift in a student’s thinking. The interview data also revealed a propensity for “pattern-matching”: writing the algebraic expressions of position vectors in spherical coordinates in a form morphologically similar to that of Cartesian coordinates. The data showed that many resources that are productive in Cartesian coordinates are inappropriately applied to non-Cartesian coordinates, although rarely in the same way across students. The consistency with which such resource activations across coordinate systems occurred led to an investigation of what Calculus I-III students are taught and what they learn about these vector concepts in various coordinate systems by the end of their multivariable calculus courses. Content that used Cartesian coordinates dominated textbook material, both quantitatively by proportion, and qualitatively through the nature of the presentation. Students were surveyed at the end of Calculus III courses using questions with notation consistent with both physics expectations and that used in those calculus texts, revealing an emerging understanding of vector concepts both in Cartesian and non-Cartesian coordinate systems. These findings will be further elaborated upon in this dissertation. Commentary on how these findings can inform instructional material development will also be presented.
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DEDICATION

This dissertation is dedicated to future physics majors. May your experience toward a physics degree improve as a result of this work.
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1. INTRODUCTION

1.1. Motivation & Purpose

Mathematics is often described as a language [6,7,8,9]. Mathematics is also a critical aspect of communication within physics [10,11,12,13,14,15,16,17]. Yet the disciplines of math and physics often use mathematical language in different ways. Redish & Kuo [18] characterize these differences in terms of linguistic analogies in two different ways: at times the mathematics in the disciplines of math and physics can behave as two dialects of the same language, or in other cases as two distinct languages. This characterization illustrates that mathematicians and physicists think about and interpret math in different ways. Redish & Kuo cite “Corrine's Shibboleth” [19] as an example. This example presents a situation where the temperature of a metal plate placed above an outlet pipe that emits cool air is modeled in Cartesian coordinates with the function $T(x, y) = k(x^2 + y^2)$ where $k$ is a constant. The shibboleth then asks what should be written if the function were to be written as $T(r, \theta) = ?$, where the question mark is to be replaced with the response. Redish & Kuo [18] discuss how physicists tend to answer that question with $T(r, \theta) = kr^2$ while mathematicians tend to answer with $T(r, \theta) = k(r^2 + \theta)$. They argue that different responses are because physicists often apply physical meaning to symbols, which in the case of the mathematicians’ answer would lead to adding two quantities with different units ($r$ being in units of length and $\theta$ being in radians). The mathematicians object to the physicists’ answer by focusing on the symbols of the expression and treating them as dummy variables, and then applying algebraic rules to complete the problem. Through this approach, the physicists’ answer does not properly account for all the variables. The shibboleth successfully illustrates how mathematicians and physicists can look at the same algebraic expression and interpret it in two different ways. Going beyond this shibboleth example, Redish & Kuo [18] also state:

“To succeed in physics, students need not just to be fluent with mathematical processing in the context of physics, but also with the mathematical modeling of physical systems, blending physical meaning with mathematical structures, and interpreting and evaluating results.”
Differences in mathematical language across the disciplines of physics and mathematics create the need for learners and practitioners of physics to translate the math taught in math classes to how math is used in physics. Some research has shown that this translation is challenging for physics students [10,11,20,21]. Research has also shown that mathematical skill is positively correlated with success in physics courses [12,13,14,15,16,17]. Some universities have attempted to ameliorate the difficulty of translating across the math-physics interface by implementing Mathematical Methods courses, or “Math Methods courses” for short, for physics students. Students typically take these courses after the introductory physics and Calculus sequences (i.e. University Physics I & II and Calculus I-II), and before upper-division physics courses (i.e. Classical Mechanics, Electromagnetic Theory, Quantum Mechanics, Statistical Mechanics, Thermodynamics). Currently, there is no published scientific literature on the effectiveness of the Math Methods courses that do exist, and not every undergraduate physics program has such a course. Additionally, there are several studies within Physics Education Research (PER) that show that upper-division physics students struggle with the mathematical concepts used in upper-division physics courses. A review of that literature was performed by Caballero, Wilcox, Doughty, and Pollock [22] with individual studies relevant to the research in this dissertation discussed further in the literature review below (see Secs. 1.3.3-4). These documented mathematical struggles suggest that the current state of students’ mathematical preparation for upper-division physics course content is insufficient, motivating this project of developing research based curricula for Math Methods courses.

The curriculum from Math Methods courses frequently includes instruction on coordinate systems, vector calculus, complex numbers, differential equations, and linear algebra, just to name a few [4,23]. The work presented in this dissertation describes an investigation of student thinking and textbooks as part of a larger effort to develop research-based instructional materials that will eventually be used as part of a complete curriculum for such a Math Methods course. This effort will henceforth be referred to as the “Math Methods project”. The first research step toward developing such materials is to explore student thinking about the concepts presented in a Math Methods courses. The mathematical concepts this dissertation will discuss are basis vectors or unit vectors, position vectors, and velocity vectors in both Cartesian and polar, spherical and cylindrical coordinates. Polar, spherical, and cylindrical coordinates are hereafter referred to holistically as non-Cartesian coordinates.
Classical Mechanics and Electromagnetic Theory frequently use various coordinate systems to describe the regions of space in which physical phenomena of interest occur [1, 24, 25, 26, 27]. Cartesian coordinates, also known as rectangular coordinates, are common and useful for a wide range of phenomena. However, there are also numerous physical situations that involve a high degree of spatial symmetry. Such situations can be cumbersome to model with Cartesian coordinates and it can be difficult or impossible to do some mathematical operations in Cartesian coordinates. For example, rotating reference frames, for which a merry-go-round is a simple example, are conveniently modeled by polar coordinates. Polar coordinates can use the circular symmetry of rotational motion to simplify mathematical expressions and some mathematical operations. In polar coordinates, a circle of radius $R$ centered at the origin can be modeled with the simple expression $r = R$. By contrast, that same circle in Cartesian coordinates would be modeled with the more complex expression $R^2 = x^2 + y^2$. Extending into three dimensions, spherical coordinates are useful for the case of modeling the electric field surrounding a point charge and cylindrical coordinates are useful for modeling the magnetic field around a long, straight wire carrying an electrical current [1]. In another upper-division content example, spherical coordinates are useful in determining the allowed energies of a hydrogen atom [28]. In general, as coursework increases in complexity, so too do the associated mathematical models. Upper-division physics curricula addresses increasingly complex physical phenomena using different coordinate systems.

Within the coordinate system models just described it is also frequently necessary to describe positions, velocities, and other physical quantities. Often doing so requires an algebraic vector expression such as a position vector, a velocity vector, a force vector, an electric field vector, or a magnetic field vector. Within such expressions, unit vectors are used to communicate the directional aspect of vector quantities. Many upper-division physics textbooks make use of unit-vector notation, meaning mathematical notation that uses unit vectors within an algebraic expression, in multiple coordinate systems to communicate the physical concepts the books contain [1, 24, 25, 26, 27, 28]. For instance, the inside front cover of Griffiths’s *Classical Electrodynamics*, a standard upper-division Electromagnetic Theory text, contains a table of vector formulas and their derivatives in Cartesian, spherical, and cylindrical coordinates, all in unit-vector notation. Section 1.4 of the same textbook also gives explicit instructions on unit vectors and how to use them to define positions in Cartesian, spherical, and cylindrical coordinates. Various Classical Mechanics textbooks also dedicate sections
of their first chapters to explaining unit vector notation in Cartesian coordinates and at least polar coordinates [24, 25, 26, 27]. Griffiths’s Introduction to Quantum Mechanics textbook also provides explicit instruction on spherical coordinates and uses associated unit vector notation as part of derivations for some three-dimensional quantum mechanical examples [28]. Collectively, our research team interprets the presence of this content in so many textbooks to mean that the ability to communicate quantitatively through the use of unit vector notation in various coordinate systems is essential in understanding the physics concepts presented in the texts, and by extension, upper-division physics courses.

The work presented in this dissertation will have two related foci: 1) student thinking about vector concepts in Cartesian and non-Cartesian coordinates as they are used in upper-division physics courses and 2) what students are taught about and what they have learned about these vectors in Cartesian and non-Cartesian coordinates in their middle-division mathematics courses, namely their multivariable calculus course. Unit vector notation comes with its own set of symbols and syntax, both of which change with the different coordinate systems (see Sec. 1.2). Thus, these two research foci are analogous to determining how well students speak the “language” that involves unit vectors in various coordinate systems in the upper-division physics level and how they are taught to speak that language in mathematics. This analogy might imply we are seeking student fluency in these areas – in this context “fluency” means the ease of which and how effectively students use this aspect of mathematical language. However, fluency is not directly measurable; one reason being there are multiple definitions and types of fluency [29]. What is measurable — through careful observation and analysis — are what ideas and techniques students use when answering questions about unit vectors and applications of unit vectors in different coordinate systems. Thus, within the two research foci just defined, the research questions this dissertation work seeks to answer are:

1) What mathematical ideas do students use to answer questions about unit vectors and applications of unit vectors in Cartesian and non-Cartesian coordinates?

2) What ideas about unit vectors and unit vector applications in Cartesian and non-Cartesian coordinates are students taught and, presumably, expected to learn during their calculus courses?
3) What ideas about unit vectors and unit vector applications in Cartesian and non-Cartesian coordinates do students actually have at the end of their calculus courses?

As a research team, we determined that we could best explore students’ thinking about unit vector notation by asking them direct questions about unit vectors and then by asking them to apply unit vector concepts to other problems that reveal their thinking about unit vectors and other vector concepts in a physical context. Two of those other vector concepts are position and velocity vectors. These two vector types were chosen because they rely on relatively simple physical contexts. In other words, the physical concepts of position and velocity were chosen because they should be understood well enough by upper-division students that it wouldn’t confound their ability to think about and engage with non-Cartesian unit vector and position vector concepts. Indeed, position and velocity are physical concepts presented early in introductory physics texts [30,31,32] and are also core to K12 level science curricula [33,34].

In three of the four studies contained in this dissertation, a theoretical framework of resources and framing [18,35,36,37,38] was used to model student thinking and guided the identification and description of student “resource” and “resource clusters”. More will be said about this theoretical framework and the decision to use it in this research in (see Sec. 1.4.).

The findings of this work will be used to inform the development of instructional materials. This work also contributes to a wider call within PER to increase the number of investigations on student thinking in upper-division physics courses [39]. It also exists as a contribution to a growing research base on student use and understanding of mathematics in upper-division physics courses [22].

Due to the very specific nature of the mathematical language studied in this dissertation work, a substantial section of the introduction will be dedicated to explaining the relevant math. That math discussion will be followed by a review of literature that has general applicability to this dissertation research as a whole. To close the Introduction, there will be a discussion of the theoretical bases underlying this work.

1.2. Mathematics Discussion

This section will define the mathematical terms used throughout this dissertation. There are two subsections to this section. The first subsection will define the relevant coordinate systems.
The second subsection will further define the vector terms used and how those vectors behave within the various coordinate systems.

1.2.1. Coordinate Systems

A coordinate system can be thought of as a geometric model superimposed on either a real or abstract space. Such models provide a way to communicate quantitatively about the geometry of that space and locations or events that occur within that space. The coordinate systems discussed in this dissertation are all used as models for real spaces in either two or three dimensions. The coordinates themselves then provide information about where along each dimension a given point is found. For example, Cartesian coordinates use the symbols \(x\), \(y\), and in three-dimensional cases \(z\), to represent the coordinates. The values of the coordinates communicate the location of a point relative to an origin using the notational form \((x,y)\). In other words, the point \((3,5)\) would be found 3 units in the \(x\) direction from the origin, and 5 units in the \(y\) direction.

A space can be defined using a set of linearly-independent vectors. This linear independence means that the directions of these vectors are completely independent of one another. For real spaces, that means that all of these vectors are orthogonal, meaning perpendicular, to each other. The set of vectors that defines the directions of a space are called the “basis vectors.” It is common for these basis vectors to be defined as having unit length, meaning a length of one, and point in the direction they are defining. A “unit vector” is any vector of unit length. In this dissertation, “unit vector” will be used synonymously with “basis unit vector”. The orthogonality of these vectors is important for real spaces because it is possible for there to be situations where information in only one of two or three dimensions changes. For example, consider a person standing on a spot marked “X” on the ground and throwing a ball directly upward. If the ball is only under the influence of gravity and thrown directly upward, its vertical position relative to that X will change in time, but its horizontal position will not. Having orthogonal basis vectors provides a mathematical means of communicating such events.

Cartesian coordinates in three dimensions, describing a point with coordinates \((x,y,z)\), will have basis unit vectors \(\hat{x}\), \(\hat{y}\), and \(\hat{z}\). The circumflex above each symbol looks like a hat. Therefore, \(\hat{x}\) is pronounced “\(x\) hat.” The circumflexes also reflect the fact that each of those vectors are unit vectors. The unit vectors \(\hat{x}\), \(\hat{y}\), and \(\hat{z}\) point in the directions of increasing \(x\), \(y\), and \(z\), respectively. Thus, a Cartesian coordinate system is defined by the basis of unit vectors \(\hat{x}\), \(\hat{y}\), and \(\hat{z}\), and then...
the coordinates provide information about where a point is along the direction parallel to their respective unit vectors.

Non-Cartesian coordinates, i.e. plane polar coordinates (which we will call polar coordinates), polar spherical coordinates (spherical coordinates), and polar cylindrical coordinates (cylindrical coordinates), leverage the geometry of situations where there are high degrees of circular, spherical, or cylindrical symmetry to make mathematical modeling and calculations more tractable, and in some cases to make such calculations possible. Upper-Division physics courses make extensive use of polar, spherical, and cylindrical symmetries to model physical phenomena [1,24,25,26,27,28]. Such situations include rotational motion and rotating reference frames [24,25,26,27] and electromagnetic fields around variously shaped charge distributions [1]. As such, being able to communicate mathematically about such phenomena in the language of non-Cartesian coordinates is an essential skill for upper-division physics students, as well as professional physicists.

Polar coordinates are an effective model of circular symmetry. They involve a central point called the pole. Any other point in a polar-coordinate plane then has coordinates \((r, \theta)\) where \(r\) is the distance from pole to point and \(\theta\) is the angle between a reference line and the imaginary line connecting the pole and the point (Figure 1.1). The unit vectors that define a polar coordinate plane, \(\hat{r}\) and \(\hat{\theta}\), point one unit in the increasing \(r\) and \(\theta\) directions, respectively.

Spherical coordinates are one way of extending polar coordinates into three dimensions. In spherical coordinates, another reference line passes through the pole perpendicular to the polar plane. A third coordinate, usually \(\phi\) with associated unit vector \(\hat{\phi}\) is then used as a measure of the angle between the line that connects that pole and the point and this new reference line. Thus, spherical coordinates define a point using the coordinates \((r, \theta, \phi)\).

Cylindrical coordinates are another way of extending polar coordinates into three dimensions. Cylindrical coordinates simply take the polar coordinate plane and add a \(z\)-axis to it perpendicular to the circular plane. Thus, a point in a cylindrical coordinate system has coordinates \((r, \theta, z)\) with unit vectors \(\hat{r}\), \(\hat{\theta}\), \(\hat{z}\) each pointing one unit in the directions in which their respective coordinates increase. It should be noted that some texts use different symbols for cylindrical coordinates. For example, in *Introduction to Electrodynamics* by David J. Griffiths, \(s\), \(\phi\), and \(z\) are used in place of \(r\), \(\theta\), and \(z\), respectively.
1.2.2. Unit Vectors & Position Vectors

Given the often-nuanced descriptions of the mathematics used in this dissertation, it’s essential to have a clear definition of unit and position vectors in non-Cartesian coordinate systems. First, the position vector is defined because it contains many of the mathematical concepts studied in this research. Consider a particle moving in space. At time $t_0 = 0$, the particle is at point A, and at some time $t > 0$ the particle is at some other point B. A displacement vector, commonly denoted $\vec{d}$, is then defined as the directed quantity from point A to point B (Figure 1.2). A position vector, commonly denoted $\vec{r}$, is defined as a displacement vector where point A is coincident with the origin or pole of some coordinate system (Figure 1.3). Thus, the position vector is a vector with its tail at the origin and tip at the coordinates of the point of interest. In Cartesian coordinates, a position vector is typically written in the algebraic form $\vec{r} = a\hat{x} + b\hat{y} + c\hat{z}$ (or equivalently $\vec{r} = a\hat{i} + b\hat{j} + c\hat{k}$) where $a$, $b$, and $c$ are the coordinates of the point of interest. These $a$, $b$, and $c$ coefficients also have a magnitude equal to the absolute value of their respective coordinates. In a single term, the magnitude and the unit vector together form the component vector. For example, the term “$a\hat{x}$” is
the $x$ component vector, often said simply as the $x$ component, of the position vector $\vec{r}$. It should also be noted than any algebraic expression written with unit vectors, such as $\vec{r} = a\hat{x} + b\hat{y} + c\hat{z}$ above, is said to be written in unit vector notation.

Position vectors in polar, spherical, and cylindrical coordinates are somewhat distinct from position vectors in Cartesian coordinates. Spherical unit vectors are illustrated in Figure 1.4 [1]. Cylindrical coordinates will be discussed further below. In polar coordinates, the algebraic expression for the position vector $\vec{r}$ terminating at any point $P$ would be $\vec{r} = R\hat{r}$ where $R$ is the magnitude of the distance from the pole to point $P$ and $\hat{r}$ is a vector of length one that points radially outward from the pole in the direction of increasing radius. Thus, the symbol “$\hat{r}$” does not have a static meaning like $\hat{x}$, $\hat{y}$, and $\hat{z}$. The Cartesian unit vectors point in directions that are location-independent. Regardless of where any two points are in Cartesian plane, $\hat{x}$ and $\hat{y}$ will point in the same directions
for both points. Unit vectors in polar coordinates point in directions that are location-dependent. At any two non-coincident points in a polar coordinate plane, the \( \hat{r} \) unit vectors will point away from the pole. Pointing away from the pole will result in the \( \hat{r} \) unit vectors pointing in different direction. Thus, \( R\hat{r} \) is sufficient to describe a polar position vector because both the magnitude, \( R \), and the direction, \( \hat{r} \), are given. Despite these differences between the location-independent and location-dependent natures of Cartesian and non-Cartesian unit vectors, there is a conversion of Cartesian unit vectors to polar with the transformations

\[
\hat{r} = \cos \theta \hat{x} + \sin \theta \hat{y} \quad (1.1)
\]

\[
\hat{\theta} = -\sin \theta \hat{x} + \cos \theta \hat{y} \quad (1.2)
\]

where it can be seen the angular dependence accounts for the location-dependent directionality of \( \hat{r} \). For spherical coordinates these transformations (with the physics convention of \( \theta \) as the polar angle and \( \phi \) as the azimuthal) are

\[
\hat{r} = \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z} \quad (1.3)
\]

\[
\hat{\theta} = \cos \theta \cos \phi \hat{x} + \cos \theta \sin \phi \hat{y} - \sin \theta \hat{z} \quad (1.4)
\]

\[
\hat{\phi} = -\sin \phi \hat{x} + \cos \theta \hat{y} \quad (1.5)
\]

These transformations often represent some of students’ first exposure to polar and spherical coordinates and their corresponding unit vectors [40,41,42] (also see Chapter 4).

The location-independent behavior of the directions of unit vectors in non-Cartesian coordinates leads to situations in which vector expressions at two or more non-coincident points can be symbolically identical but have different mathematical and/or physical meanings. Two points in a polar plane that are both a distance \( R \) from the pole will both be modeled by the position vector expression \( \vec{r} = R\hat{r} \) (See Figure 1.5). The two expressions, while having identical symbols,
Figure 1.5. In polar coordinates $\vec{r}_A$ and $\vec{r}_B$ have identical-looking algebraic expressions on the right-hand sides of their equations: $\vec{r}_A = 2\hat{r}$, $\vec{r}_B = 2\hat{r}$. Despite this symbolic similarity, their respective $\hat{r}$ unit vectors point in different directions, making $\vec{r}_A \neq \vec{r}_B$.

have two different meanings because the $\hat{r}$ unit vectors point in different directions. The symbolic non-uniqueness of the algebraic expressions for polar position vectors does not mean that $\hat{r}$ rotates about the origin so it can match its associated position vector, it means there is an infinite number of $\hat{r}$ unit vectors that all use the same symbol. Such algebraic ambiguity can lead to some confusion. It’s tempting to say that in polar coordinates $\vec{r} = R\hat{r}$ defines a circle since there is no way of knowing which $\hat{r}$ is being used without any other information. However, such thinking would imply that a vector and a surface are equal, which is mathematically incorrect. It would be reasonable and correct to say the set of all position vectors, $\{R\hat{r}\}$, defines a circle, however, the reality is, upon seeing just the algebraic expression $\vec{r} = R\hat{r}$, there is no way to determine which specific vector is being referenced without additional information. Thus, the correct mathematical interpretation is that $\vec{r} = R\hat{r}$ without additional information refers to a single, unique but unknown polar or spherical position vector.

Some of the work to be presented in this dissertation also involves cylindrical coordinates. Cylindrical coordinates are a hybrid of polar coordinates and Cartesian coordinates. They are polar in the circular cross section and behave like Cartesian coordinates along the length of the cylinder. Thus, the proper notation for a position vector in cylindrical coordinates is $\vec{r} = r\hat{r} + z\hat{z}$. Some
physics textbooks – such as Griffiths’s EM [1] – write this as $\vec{r} = s \hat{s} + z \hat{z}$. See Figure 1.6 for a visual representation of cylindrical coordinates.

Both the location-dependent behavior of unit vectors in Cartesian coordinates and the location-independent behavior of non-Cartesian unit vectors have their uses. In an analogy, the English language also has both location-dependent and location-independent sets of directional vocabulary. Verbiage such as ‘north’, ‘south’, ‘east’, and ‘west’ communicates directional information that does not depend on location or physical orientation. Giving someone an instruction to travel north has the same meaning regardless of a particular person’s location on the globe – unless they are already at the north pole. Verbiage such as ‘left’, ‘right’, or ‘up’ and ‘down’ communicates directional information that does change based on the locations and/or orientations of those communicating such directions. On the Earth’s surface, “up” has a different absolute direction if one is in the United States or in China. In navigation, both north-south-east-west language and right-left language have their uses depending on the context. For example, in cities with very grid-like road systems, north-south-east-west navigation instructions can be very effective. North-south-east-west-based language can also be helpful when multiple individuals are at different locations and need to move to a common location. However, in mountainous regions or areas with numerous bodies of water where roads must twist and wind, telling someone to turn east at a given intersection might lead to considerable confusion. In such cases, an instruction to turn ‘right’ or ‘left’ is more helpful. Cartesian coordinates behave much like north-south-east-west directions. Regardless of where in a given Cartesian coordinate system points exist, the $\hat{x}$ direction will be the same. Polar, spherical, and the circular part of cylindrical coordinates are then analogous to giving right-left directional in-
formation. In a manner similar to how two people facing each other will point in opposite directions when asked to point to their right, people at two points in a polar coordinate system will point in different directions when asked to point in their “\( \hat{r} \)” direction. Importantly, it is following the same “rule” that leads to people in both situations to point in different directions. In the case of two people facing each other and pointing to their right, “point to your right” is the same instruction for both. It’s the consequence of following the rule that leads to them pointing in different directions. Likewise, for two people standing at different points in a polar plane, the instruction “point in your \( \hat{r} \) direction” is the same for both, but the consequence of following that instruction leads to them to point in different directions. Given that physics uses mathematical models to represent physical phenomena that occur in spaces with different geometries, it is necessary for students to be able to communicate using both these location-dependent and location-independent models as they are both useful for different contexts. Such ability includes fluency with unit-vector notation in multiple coordinate systems.

1.3. Literature Review

There has been a relatively small amount of published research about student thinking in non-Cartesian coordinate systems in either PER or Research in Mathematics Education (RUME). The following subsections will outline many of key aspects of that extant literature. More research exists on student thinking and use of vectors in Cartesian coordinates and in various physical contexts. Most such work has occurred at the introductory physics level. There is also a growing research base about student thinking and use of math in upper-division physics courses. This review will provide a survey of these introductory and upper-division research bases followed by more detailed descriptions of the few papers directly related to research in this dissertation. Other papers that apply more directly to the research in the following individual chapters will be discussed in the introductions and literature reviews contained within those chapters.

1.3.1. Student Thinking About Vectors in Introductory Physics

Some research has been done on student thinking about vectors in introductory physics. This work has studied how students think about vectors and vector operations in Cartesian coordinates without physical context. Knight [43] and Nguyen & Meltzer [44] studied initial vector understanding among introductory physics students through development of vector concept tests. Both studies emphasized the need to spend more instructional time developing student understanding of vector
concepts. Other studies have more specifically focused on student understanding of vector operations. Heckler & Scaife [45] and Buncher [46,47] studied student thinking about vector addition, finding students perform better with unit-vector-based representations of vectors than with arrow representations. Student thinking and struggles with vector subtraction has been researched by Wang & Sayre [48] and Flores, Kanim, and Kautz [49]. The latter found that students struggle to add and subtract vectors graphically after traditional instruction. Barniol & Zavala investigated student difficulties with vector scaling [50]. Van Deventer & Wittman [51], Van Deventer [52], and additional work by Barniol & Zavala [53] investigated student thinking about dot and cross products. A synthesis of this work reveals students generally do not have a firm understanding of vectors or vector concepts both before and after introductory physics courses. Van Deventer [52] also studied student performance on vector tasks with and without physical context with isomorphic questions finding that while students perform similarly poorly on such isomorphic questions before instruction, they perform slightly better on math only versions of questions after instruction.

1.3.2. Student Use of Vectors in Introductory Physics

A number of investigations look beyond the purely mathematical understanding of vectors by taking a more specific look at how students use and think about vectors in various contexts within introductory physics. Some of these investigations also overlap with those studying students’ purely mathematical understanding of vectors. A portion of Barniol & Zavala’s work [53] compared student understanding of vector concepts on a purely mathematical test with isomorphic physics questions on a second concept test targeting the basic concepts of force, velocity, and work. They found no statistical difference between overall performance on each test. However, performance on individual items favored context-free and physical context dependent questions a roughly equal number of times. Flores, Kanim and Kautz [49] found that students’ vector addition and subtraction difficulties also left them unable to interpret the results of such operations. These interpretive difficulties meant students were unable to answer qualitative questions about physical phenomena involving vectors, i.e. force, velocity, acceleration, etc. Shaffer & McDermott [54] studied student understanding of vector concepts in kinematics and developed instructional materials to address those difficulties. They identified a number of student difficulties with vector concepts at the introductory level, and – of particular note to the current research – presented evidence that suggests those difficulties persist as students move on to upper-division and graduate level courses.
1.3.3. Student Understanding and Use of Mathematics in Upper-Division Physics

A growing effort to study student understanding and use of mathematics in upper-division physics has been underway for slightly more than a decade. Caballero, Wilcox, Doughty, and Pollock [22] provide a review of this effort. The two main approaches to this work have been macroscopic – seeking to transform courses by identifying and addressing student difficulties – and microscopic – seeking to describe students’ in-the-moment thinking through identification and mapping of fine-grain ideas used while problem solving. However, the review finds much of the work it presents disconnected from the other works within the review. Prior to the Math Methods Project, there had been no effort to synthesize the results described by Caballero et al. and other original studies into a single Math Methods course curriculum. There are also relatively few papers related to student understanding and use vectors and/or coordinate systems in the upper-division. However, tangentially related to the research in this dissertation is a portion of the work by Pepper, Chasteen, Pollock and Perkins [55], which contains observations made of student thinking of vector derivatives in the context of electromagnetism. They observe that students sometimes attend to one of the magnitude or directional natures of vectors at a time, but not both. These observations were primarily in the context of curls and divergences of vectors in Cartesian coordinates.

1.3.4. Student Understanding and use of Unit and Position Vectors and non-Cartesian Coordinate Systems

A portion of the previously mentioned 2012 work of Barniol & Zavala [50] looks specifically at student difficulties with unit vectors in the Cartesian coordinate system. Students were asked to draw the unit vector for a vector of length $2\sqrt{2}$ with tail at the origin and pointing at a $45^\circ$ angle between the $x$ and $y$ axes. They found students can know a unit vector is of length 1 but still not draw it to proper length. Difficulties with students reasoning the correct length and direction of the unit vector from the basis unit vectors $\hat{i}$ and $\hat{j}$ were also found. While Barniol & Zavala’s work is about unit vectors, the questions were not directly about basis unit vectors and were limited to the Cartesian coordinate system. However, Barniol & Zavala’s findings do point out that student thinking about unit vectors is problematic.

Two studies by Montiel, Vidakovic and various colleagues focused on student thinking while transitioning from use of Cartesian coordinates to polar coordinates. The first, by Montiel, Vi-
dakovic and Kabaël [40], interviewed second semester calculus students individually and after instruction to probe their understanding of functions in polar coordinates. They found a student tendency to apply the vertical line test – a test that works in Cartesian coordinates but not polar coordinates – to polar coordinates. Montiel and colleagues situate this research in the context that the definition of function is frequently taught in precalculus only at the moment knowledge of the definition is needed, and that prior to such instruction students have been instructed to convert back and forth between the Cartesian and polar coordinate systems. A key takeaway from this research is that student understanding of a particular concept – functions in this example – are often tied to the coordinate system in which that concept is learned; which can influence the ability to make sense of other coordinate systems. The second study, by Montiel, Wilhemi, Vidakovic and Elstak [42], used an onto-semiotic approach to explain the fundamental concepts of the different coordinate systems, particularly the polar coordinate system. They further explain the method and sequence in which students learn about the polar coordinate system, which is Cartesian-centric and presented as the polar coordinate system being a conversion from Cartesian coordinates. Through analyzing student responses to written questions and small-group interviews of multivariate calculus students conducted soon after their final exam, Montiel et al. again observed the practice of students inappropriately applying the vertical line test to polar coordinates. Additionally, they found more general evidence that mathematical meaning understood in the context of one coordinate system can be lost in the transition to other coordinate systems due to differences in representation.

Expanding on the work of Montiel and colleagues, Paoletti, Moore, Gammaro and Musgrave [41] performed a teaching experiment on pre-service secondary teachers who had completed Calculus II and at least two other math courses. The intent of the study was to gain insight into student thinking of the polar coordinate system during instruction, as opposed to post-instruction as Montiel and colleagues had done. Whole-class discussions were videotaped (n = 21) and 4 students were video recorded as they worked in pairs on a written teaching instrument developed by Moore. Their results show that even though the students had completed Calculus II and sometimes beyond, constructing knowledge of the polar coordinate system was a novel experience. They report student misunderstanding starts with incomplete understanding of radian measure. They also observed the problematic function understandings described by Montiel and colleagues, as well as confusion regarding the respective conventions of the origin and the pole and ordered pairs – a Cartesian
ordered pair \((x, y)\) behaves as (input, output) when their is functional dependence between the two, while a polar ordered pair \((r, \theta)\) behaves as (output, input). In sum, Paoletti et al. claim that students have an emerging understanding of the polar coordinate system and when uncertain of how to solve a problem in the polar coordinate system, will return to their more solid understanding of the Cartesian coordinate system. In turn, students often misapply Cartesian conventions to polar coordinates. Furthermore, they claim that their work, in conjunction to the work of Montiel and colleagues, shows that meaning of mathematical concepts such as function and origin often become tied to the coordinate system predominantly used.

Sayre & Wittman [56] obtained similar results to Montiel et al. and Paoletti et al. in a physical context using a combination of theories to investigate the connections between student thinking in physics and mathematics in the context of junior level Classical Mechanics. They analyzed data from video recordings of homework help sessions and small group interviews built around a problem asking students to set up equations of motion for a swinging pendulum. They concluded that student understanding of Cartesian coordinates is quite solid while student understanding of polar coordinates is much more plastic – a cognitive psychology term meaning more flexible and under formation. This conclusion stemmed from results showing students often start a problem presuming the use of Cartesian coordinates while not stopping to consider if that coordinate system is the best choice. Students also persist in the use of Cartesian coordinates when it’s not the most appropriate coordinate system, derive polar coordinate models starting from Cartesian models, and make additional problem-solving mistakes while using the polar coordinate system.

Hinrichs [57] developed a simple concept test to probe student thinking regarding unit and position vectors in non-Cartesian coordinate systems. The test was given to a total of 46 electromagnetic theory students at the undergraduate and graduate level at multiple universities and university types (small private and public liberal arts schools in the Midwest and a large public research university in the southwest). Different textbooks were also used. Hinrichs found that while both undergraduate and graduate physics students seemed to understand spherical coordinates generally, they have difficulty expressing position vectors in terms of the spherical basis unit vectors in 3D space. Specifically, there was difficulty translating the ordered triple defining the coordinates of a point in space – form \((x, y, z)\) in Cartesian and \((r, \theta, \phi)\) in spherical – to the algebraic expression of the associated position vector. A common response was “pattern-matching” spherical position
vectors to Cartesian notation. In Cartesian coordinates the relationship between the ordered triple representing the coordinates of a point and the associated position vector expression is, in the notation Hinrichs uses, 
\[(x, y, z) = x\mathbf{x} + y\mathbf{y} + z\mathbf{z} \]. Many students wrote spherical position vectors as 
\[(r, \theta, \phi) = r\mathbf{r} + \theta\mathbf{\theta} + \phi\mathbf{\phi} \], which is incorrect yet was the answer for 46% of respondents. Hinrichs also reported responses that appeared as \[5\mathbf{r}, \frac{\pi}{2}\mathbf{\theta}, 0\mathbf{\phi} \] or \[(5\mathbf{r}, \frac{\pi}{2}\mathbf{\theta}, 0\mathbf{\phi}) \] which he states are possibly blends of the left and right hand sides of the equation \[(x, y, z) = x\mathbf{x} + y\mathbf{y} + z\mathbf{z} \]; i.e. that students may be thinking \[(x, y, z) = x\mathbf{x} + y\mathbf{y} + z\mathbf{z} = (x\mathbf{x}, y\mathbf{y}, z\mathbf{z}) \]. Hinrichs points out these responses are possibly another form of pattern-matching but that is uncertain without more explanation from the students as to their reasoning. If all three of these responses are considered pattern-matching in one form or another, they account for 66% of his respondents. Another 24% of respondents did not include unit vectors in their answers at all, with the remainder giving some type of un-categorizable response. Hinrichs makes four overarching claims: 1) very few students can write a correct position vector from the basis unit vectors in spherical coordinates, 2) the most frequent mistakes made were of the pattern-matching variety, 3) the second most common mistake was to not use unit vectors at all, and 4) the results did not depend on level of student (graduate vs. undergraduate), university, or course text.

Vega, Christensen, Farlow, Passante and Loverude [3] studied student thinking about polar unit vectors. The research team initially sought to identify student difficulties with non-Cartesian unit vectors. However, after analyzing written data they observed such a framework was limited in its ability to describe the data and was unable to account for the correct ideas found within. Thus, they transitioned to a framework more consistent with a resources and framing theoretical framework (see Sec. 1.4.). Data consisted of both responses to written questions and video recorded interviews of students after instruction in a Math Methods course and before instruction in an Electromagnetic Theory course. These students were asked to indicate the direction of polar unit vectors in a plane. Many responses were given showing the directions to be out of the page or drawing the unit vectors as curved instead of linear. The interviews revealed a tendency toward using the motion of an object – using words such as ‘traveling’ and ‘moving’ – to reason about the direction of the basis unit vectors. Furthermore, the interviews revealed there is relatively little understanding of unit vectors generally. The interviews also included examples of situations where students can appear to have the necessary productive resources to answer a question yet activate
other resources unproductively which cause them to answer incorrectly and, in some cases, rethink previously stated productive resources.

Emphasizing the point made in the Motivation & Purpose section (Sec. 1.1), the research findings summarized above suggest that the mathematical preparation that most upper-division physics students receive prior to taking upper-division physics courses is insufficiently helping them translate across the math-physics interface. This synthesis of findings provides motivation for the current project.

1.4. Theoretical Bases

1.4.1. Theoretical Frameworks

A theoretical framework is the structure that holds the underlying theories and assumptions of research project together [58,59,60]. In essence, the theoretical framework serves as the blueprint that guides and informs the data analysis and the research in general [61]. By explicitly stating the which theoretical framework or describing the theoretical framework that was used to inform the research, the author strengthens their presented argument by connecting the reader to relevant theories and forming the intellectual basis for the warrants connecting questions, data, and claims [59]. In PER there are two prevalent theoretical frameworks for analyzing student content thinking: a student difficulties framework [62,63,64,65] and the framework of resources and framing [18,36,37,38]. In upper-division PER, work guided by an identifying student difficulties framework is more common than work guided by a resources framework [39].

The framework of identifying student difficulties emerged as the guiding framework of the Physics Education Group (PEG) at the University of Washington and has been widely adopted as a guide for transforming undergraduate physics curricula [63]. Given that course transformation is a stated goal of the PEG, they champion a two-step approach that first identifies common student errors in thinking – difficulties [64] – and then elicits and addresses those difficulties with targeted instructional materials [65]. The instructional materials rely on a cognitive conflict model by creating situations where students are made aware of the inconsistency between their own thinking and observations of observable phenomena. Then, they are forced to reconcile the conflict [63]. This two-step process has resulted in reformed curricula that has led to substantial learning gains and/or improved conceptual understanding of physics concepts. A notable example of such curricula is Tutorials in Introductory Physics [66]. The difficulties framework also takes a quantitative approach
to researching student thinking. It affords the ability to predict what approximate proportions of a group of students will exhibit various relevant difficulties and allows instructors to prepare for such group compositions. As such, the difficulties framework takes a lab-bench-like approach to researching student thinking and designing curriculum [62]. Making this quantitative nature possible is that within a given context a difficulty is persistent and consistently applied. An example of this is the thinking that electric current is “used up” in an electrical circuit. This difficulty is observed across demographics and in a wide variety of electrical circuit related questions [64].

A resources and framing framework emerged from a knowledge-in-pieces [67,68] approach and is rooted in results from broader education research and both behavioral and cognitive psychology [18,35,36,37,38,69]. A resources and framing framework posits that knowledge can be modeled as fine-grain ideas that exist in an individual’s brain – broadly defined as resources – that are gained through experience. These resources can be activated or not activated depending on a given stimulus [36]. Resources that are closely connected to other resources can be modeled as forming a cluster. Resource clusters can form networks which can become hierarchical cognitive structures known as schemas. Those clusters and schemas which activate together eventually become strongly tied together such that they always activate together. This model is thus explicitly built on well-established neurological models describing the interconnection of neurons [70]. As of yet, the resources framework does not provide a means for quantitative models or quantitative predictions [38]. Instructional material development informed by a resources framework is less common in the published literature than that of identifying student difficulties, but what work does exist relies on encouraging students to analyze and be critical of their underlying epistemologies rather than the cognitive conflict model at the core of difficulties-based materials [37,71]. In PER, there is a practice of identifying the resources students activate while thinking about a problem and then describing how those resources connect or don’t connect to each other. These interconnected resources can be represented in a diagram known as a resource graph [72,73].

In this dissertation, both the resources and difficulties frameworks will be used to varying extents. Both frameworks offer unique affordances of value to the research foci and questions described in the Background & Purpose section (Sec. 1.1). In the research presented herein, when the research questions are focused on what students are thinking, the resources framework is primarily used. Similarly, when research questions are about proportions of students who exhibit
a certain way of thinking, methods more consistent with a difficulties framework are used. The previous paragraph outlines some advantages for exploring and describing thinking afforded by the resources framework. The difficulties framework can be useful for instructors and curriculum developers insofar as providing a means to gain insight in student group compositions. For example, an instructor of a course might know there are three common student difficulties for a given concept and that 20% of their students are likely to exhibit difficulty A, 15% are likely to exhibit difficulty B, and another 10% exhibit difficulty C. This knowledge allows the instructor to make evidence-based instructional decisions for developing learning objectives, lesson planning, and overall course design. Such group compositions can also inform researchers seeking to develop evidence-based curricula.

Chapters 2 and 3 of this dissertation primarily focus on individual students’ thinking and thus are based on the resources theoretical framework. Chapter 5 of this dissertation presents work that uses a hybrid approach of these frameworks in its research design. The research questions for that work required the use of both the resources framework and some of the methods more consistent with the difficulties framework. More will be said about this synthesis of frameworks in Chapter 5.

1.4.2. The Cognitive Psychology Theory of Pattern Recognition

The combination of activation of fine-grain ideas in the resources framework and “pattern-matching” as described by Hinrichs [57] overlaps with the cognitive psychology theory of pattern recognition. Broadly speaking, pattern recognition is the cognitive process of responding to environmental stimuli and using those stimuli to build coherence in the understanding of the local environment. Such coherence includes understanding the way in which that environment operates, how objects and individuals behave within and respond to that environment, and allowing predictions to be made about future events both in that environment and in unfamiliar environments [74,75,76,77,78,79]. In the late 1950’s and early 1960’s two theories about pattern recognition emerged: template-matching [75,80,81] and feature-analysis [82,83,84,85]. Template-matching involves the brain responding to a stimulus by matching the input information to a previously existing template – or prototypical form – stored in memory. Feature-analysis involves the brain recognizing the salient features of the input stimulus. Some features can overlap, but the combination of recognized salient features in the brain allows one to recognize the stimulus. Of these two theories, feature-analysis is the accepted theory in current cognitive psychology literature [74]. Template-matching was considered problematic shortly after its introduction as it would require memory to
store a nearly infinite number of templates. Therefore, template-matching was considered not feasible [74], although it does have some uses in artificial intelligence and computer programming [81]. Feature-analysis requires the existence of some number of feature-detectors. When certain regions of the eye respond to a visual sensory input, associated neurons are activated in the brain. The set of neurons that activate for a given input are considered the feature-detectors for that input. Multiple feature-detectors can be simultaneously activated as the sensory information can activate several receptors simultaneously. The combination of activated feature-detectors forms a “neural code” for the incoming stimulus [74]. While no explicit connection has been found in our literature search, the behavior of feature-detectors and the theoretical descriptions of the resources and framing theoretical framework [18,35,36,37,38,69] appear to be quite analogous, and both are patterned after neurological models of the brain [70]. Chapter 3 presents work that further explores how pattern recognition overlaps with our theoretical lens of resources and framing.

1.5. Dissertation Summary

The following chapters will articulate four studies within the two foci described in the Motivation & Purpose section (Sec. 1.1). Chapters 2 and 3 report on findings for the first focus: student thinking about vector concepts in Cartesian and non-Cartesian coordinates and unit vectors used in upper-division physics courses. The data for the research in these chapters are student responses given during one-on-one think aloud interviews. Chapter 2 is a case study about one high-achieving upper-division physics student’s thinking about vector concepts primarily in polar and spherical coordinates. The case study was published in Physical Review: Physics Education Research in September 2019. Chapter 3 is an analysis of responses across the interview sample that sheds light on how a resources framework can account for Hinrichs’s description of pattern-matching. This chapter is to be submitted to Physical Review: Physics Education Research. The findings from the work to be described in chapters 2 and 3 also revealed that a more thorough investigation of what students are taught in their math courses about vector concepts in various coordinate systems needed to be performed. Chapters 4 and 5 therefore report on findings from the second focus: what students are expected to learn about and what they have learned about vectors in Cartesian and non-Cartesian coordinates in their middle-division mathematics courses. Chapter 4 is an analysis of calculus textbooks that seeks to determine the nature of instruction given to students about vector concepts in various coordinate systems. This chapter is to be submitted to
Focus 1: Exploring physics students’ thinking

- Physics student interviews
  - Chapter 2: Case Study
  - Chapter 3: “Pattern-Matching”

Focus 2: Exploring students’ mathematical preparation before upper-division physics

- Results prompted
  - Investigate math community expectations
  - Math students’ content understanding

Chapter 4: Calculus Textbook Analysis
Chapter 5: Calculus III Student Surveys

Figure 1.7. A flowchart of how the two research foci described in Sec. 1.1 and the four research chapters presented in this dissertation fit together.

either Physical Review: Physics Education Research or The American Journal of Physics. Chapter 5 reports on an analysis of student responses to a survey on understanding of Cartesian unit and position vectors from students at the end of a Calculus III course. This chapter will be submitted to either Physical Review: Physics Education Research or American Journal of Physics. Figure 1.7 is a visual overview of how the two research foci and these four individual research chapters fit together. The subsections that follow are more detailed descriptions of the content of each chapter.

1.5.1. Summary of Chapter 2

Chapter 2 is a research paper entitled “Mapping activation of resources among upper division physics students in non-Cartesian coordinate systems: A case study.” The authors and their affiliations, in the published order, are: Brian Farlow, North Dakota State University; Marlene Vega, California State University - Fullerton; Michael E. Loverude, California State University - Fullerton; Warren M. Christensen, North Dakota State University. This paper was published in Physical Review: Physics Education Research in September 2019 [2]. After interviewing seven subjects about vector and coordinate system concepts, we had a data set of wide-ranging ideas and seven unique patterns of connected ideas. Therefore, we decided to drill down on a single interview subject and see if we could determine some of the underlying ideas that subject was describing while answering math questions and name those ideas as resources using a resources framework. Doing so allowed us to develop a methodology for identifying and naming resources and seeing how those
resources activate – or don’t activate – together. We were able to name several resources and group them according to the context of the question the subject was answering. Further, we were able to track how the activation of some ideas led to the activation of other ideas. We were also able to provide an example of how the non-activation of a key resource led the subject to attempt to define vector concepts in non-Cartesian coordinates in terms of the behavior of those same concepts in Cartesian coordinates. The introduction of this case study also includes a streamlined version of the Mathematics Discussion presented above in Sec. 1.2. This mathematics discussion was included in this paper because it was the first of the series of papers we are planning to publish, making it necessary to clarify what the mathematical terms and symbols mean.

1.5.2. Summary of Chapter 3

This chapter is a research paper entitled “Using a Resources Framework to account for ‘pattern-matching’ responses in the context of non-Cartesian coordinates and unit vectors” that will be submitted to Physical Review: Physics Education Research. The planned published author list, author affiliations, and order are: Brian Farlow, North Dakota State University; Alden Bradley, Humboldt State University; Marlene Vega, California State University - Fullerton; Michael E. Loverude, California State University - Fullerton; Warren M. Christensen, North Dakota State University. All seven of our interview subjects across two interview protocols gave responses to questions asking them to write position vectors in non-Cartesian coordinates. The results appear to be consistent with what Hinrichs described as pattern-matching [57]. However, our study utilizes a resources framework to further identify and refine resources and resource clusters that builds on our previous work [2,3]. The RUME work of Montiel et al. [40,42], Paoletti et al. [41], and the theory of Pattern Recognition [74,75,76,77,78,79] were connected to this work, as well. It was found that subjects apply resources for Cartesian unit vectors and coordinates to problems requiring thinking about non-Cartesian coordinates and unit vectors. The framework of resources provides an explanation for why pattern-matched responses, as described by Hinrichs, are common. This chapter also further outlines some emerging understanding of vector concepts in non-Cartesian coordinates and provides initial insight into curricular development to leverage the identified resources.

1.5.3. Summary of Chapter 4

Chapter 4 is a research paper entitled “Multivariable Calculus Textbook Representation of Non-Cartesian Coordinates: A Misalignment Between Multivariable Calculus Textbook Content
and Upper-Division Physics Application” that will be submitted to the American Journal of Physics or Physical Review: Physics Education Research. The planned authors’ list, authors’ affiliations, in the order of planned publication, are: Chaelee Dalton, Pomona College; Brian Farlow, North Dakota State University; Warren M. Christensen, North Dakota State University. This chapter also marks the transition of this dissertation from discussing work in Focus 1 to work in Focus 2. The work is a report on a detailed analysis of seven popular Calculus textbooks. The bolded definitions, worked example problems, expository text, and end-of-chapter exercises of these books were coded for the coordinate system – i.e., Cartesian, polar, spherical, cylindrical, or a combination of these – in which their content was presented. Proportions of content in each coordinate system are reported. A qualitative analysis discussing the nature of the presentation of concepts is also reported. It was found that the overwhelming majority of the content is presented in Cartesian coordinates. Furthermore, six of the seven textbooks did not mention non-Cartesian unit vectors at all and the one that did, mentioned only polar unit vectors a single time. Qualitatively, when non-Cartesian coordinates were expected to be used, explicit instructions were given to do so. Such explicit instructions did not exist for Cartesian coordinates. Such presentation established Cartesian coordinates as the assumed or default coordinate system, giving students very little opportunity to determine which coordinate system provided the most useful model for a given situation. These results provide some evidence for why the physics interview subjects from chapters 2 and 3 transferred many resources productive in Cartesian coordinates to non-Cartesian contexts; the bulk of their mathematical preparation is in Cartesian coordinates. Thus, with the backing from Montiel and colleagues [40,42] and Paoletti et al. [41] that understanding of mathematical concepts is tied to the coordinate system in which those concepts are learned, it follows that students activate Cartesian resources when faced with non-Cartesian problems as was seen in Chapters 2 and 3.

1.5.4. Summary of Chapter 5

Chapter 5 is a paper entitled “Using the Identification of Resources to Explore Mathematics Students’ Thinking About Unit Vectors and Unit Vector Notation at the End of Their Multivariable Calculus Courses” intended for submission to Physical Review: Physics Education Research or to the American Journal of Physics. The planned authors’ list, authors’ affiliations, in the order of planned publication, are: Brian Farlow, North Dakota State University; Jordan Brainard, North Dakota State University, Warren M. Christensen, North Dakota State University. In chapter 5, we
investigate the ideas that students have after exposure to the curriculum we studied in chapter 4, at the end of their Calculus III courses. In other words, chapters 4 and 5 together answer the related questions 1) ‘What are math students expected to learn about vector concepts in various coordinate systems?’ and 2) ‘what do students understand about these vector concepts?’, respectively. The research in chapter 5 used written responses to surveys given to Calculus III students during the last week of their Calculus III courses. The study used a methodological approach that arose from a synthesis of the student difficulties and resources and framing theoretical frameworks; which will be discussed further in chapter 5. Through analysis of the written data, ideas that were activated productively and unproductively were identified and coded. Generally speaking, there were low activation rates of productive ideas and a wide range of ideas that were not productive for the given contexts. Additionally, the hybrid methodology provided the opportunity to see how different combinations of ideas could further illuminate student thinking. The idea combinations discussed led to identification of both a learning goal and learning objective we hypothesize will be helpful in developing curricular materials.
2. MAPPING ACTIVATION OF RESOURCES AMONG UPPER DIVISION PHYSICS STUDENTS IN NON-CARTESIAN COORDINATE SYSTEMS: A CASE STUDY*

2.1. Introduction

2.1.1. Background and Purpose

Using non-Cartesian coordinate systems continues to be difficult for undergraduate physics students [40,41,42] even in upper-division physics courses (e.g., Classical Mechanics, Electromagnetism, Thermodynamics, Quantum Mechanics, Statistical Mechanics) where application of these mathematical ideas plays a more significant role [1,24,26,27]. As part of a larger initiative into exploring the math-physics interface in upper-division physics courses [22,39], our current collaboration has initiated the development of a research-based curriculum for a mathematical methods course for undergraduate physics majors. This paper discusses a portion of that effort by focusing on the identification of cognitive resources students activate in the context of basis unit vectors, position vectors, and velocity vectors in plane polar and spherical polar coordinate systems.

The polar coordinate system, both planar and spherical, is commonly used in a number of upper-division physics courses, including mechanics, electrodynamics, and quantum mechanics [1,24,26,27,28]. Despite this focus, research has shown that students struggle using these coordinate systems. Sayre & Wittmann [56] found that students did not always recognize when applying polar coordinates would be advantageous. They also found that while Cartesian coordinate understanding is typically quite solid, polar coordinate understanding is less robust and still in the formation process. Paoletti et al. [41] found that students, spend the greatest proportion of their math training using Cartesian coordinates, and apply Cartesian coordinate system conventions to polar coordinates. Paoletti et al.’s findings are consistent with findings by Montiel and various colleagues [40,42] who claim mathematical meaning is often tied to the coordinate system in which a concept is learned. As an example, Montiel et al. describe students attempting to apply the vertical line test to determine if an expression is a function in the polar coordinate system. The vertical


B. Farlow had primary responsibility for drafting and revising all versions of this chapter.
line test uses a vertical line to sweep across a coordinate plane. If the curve represented on the
given plane intersects this vertical line at one and only one point for every \( x \)-value, the curve
represents a function. The vertical line test works for Cartesian – the coordinate system in which
functions and the vertical line test are initially taught – but frequently fails in polar coordinates.
Hinrichs [57] observed similar results while studying spherical unit vectors in the context of position
vectors noting students “pattern-matched” position vector notation in spherical coordinates to match
that of Cartesian coordinates. This pattern-matching resulted in the inclusion of unnecessary unit
vectors, which frequently appeared as students writing spherical position vectors in the incorrect
(but Cartesian-imitating) form \( \vec{r} = r \hat{r} + \theta \hat{\theta} + \phi \hat{\phi} \).

Our initial work on students’ thinking on unit vectors in non-Cartesian coordinate systems
categorized student thinking into “clusters” of similar but not identical ways that students thought
about unit vectors [3]. Specifically, we found that students conflated \textit{unit vectors at a point} with
\textit{position vectors to a point}. Additionally, there was a tendency for students to conflate ideas about
the \textit{direction of a unit vector} having to do with either 
\textit{the location of an object at a point} or with the
\textit{direction of motion of an object at a point}. Vega et al. [3] further describe a “\textit{Unit Vector Cluster}”
that has essential resources to fit a canonical definition of a unit vector: a unit vector is a vector;
has a length of one unit; points in the direction of increasing coordinate; and is dimensionless. The
naming of these resources is done for completeness and from an expert perspective, and not all of
these resources were identified by the students in the interview sample.

The research presented in this paper used think-aloud interviews to gain additional insight
into student thinking about a wider range of vector concepts in non-Cartesian – and specifically
planar polar, spherical, and, at times, cylindrical coordinate systems. There are many other non-
Cartesian coordinate systems, however, hereafter, when we refer to non-Cartesian coordinates, unit
vectors, and coordinate systems we imply only these three. Two interview protocols were developed
that included questions about unit vectors, position vectors, and velocity vectors in polar and
spherical coordinates and polar and spherical unit vectors. The second protocol included cylindrical
coordinates and unit vectors, as well. Four interviews were conducted using the first protocol. Two
of the four interview subjects were senior undergraduate physics majors, one was a first-year physics
graduate student, and one was a third-year physics graduate student. Using the second protocol,
three subjects were interviewed; one subject a junior undergraduate physics major, one a second-
year physics graduate student, and the third a physics faculty member. Across the two protocols, a total of three of the seven subjects responded to unit vector questions in a manner consistent with the resource clusters described in our initial work [3] that were not productive for these particular questions. The remaining four subjects answered the initial unit vector questions in our protocol successfully. Later in their interviews, these four subjects struggled when attempting to apply their productive thinking about non-Cartesian unit vectors in the context of position and velocity vectors.

The responses from the interviewees were nuanced and, in subtle ways, were different across individuals. Responses demonstrated activation of ideas that were likely previously useful for students, and some of these ideas had commonalities across interview subjects, making them suitable for analysis using a resources framework (Sec 2.1.3). All seven subjects, regardless of their activating productive or non-productive unit vector resources initially, responded in a similar fashion to questions on position vectors using non-Cartesian coordinates and unit vectors. For example, all subjects included \( \hat{\phi} \) and/or \( \hat{\theta} \) terms in algebraic expressions for polar and spherical position vectors where only \( \hat{r} \) terms are needed. Given the consistency with which all the subjects included such unnecessary terms, identifying resources that students activated when answering questions on position and velocity vectors was determined to be a useful framing for better understanding students’ vector thinking. The work presented here is a case study of a single subject from our interview pool that will be used to present and discuss some of the subject’s thinking and identify that thinking as cognitive resources. Presentation of this case study allows us to build on our prior work for unit vectors, as well as explore the extent to which previously identified resources are appropriate for the context of position and velocity vectors. This essential step builds toward the analysis of our remaining interview sample, wherein we may be able to identify resource clusters for positions vectors and velocity vectors.

2.1.2. Mathematics Discussion

Non-Cartesian coordinates, e.g. plane polar coordinates (which we will call polar coordinates), polar spherical coordinates (spherical coordinates), and polar cylindrical coordinates, leverage the geometry of a given situation to make mathematical modeling and calculations more tractable, and in some cases to make such calculations possible. Such situations include rotational motion and rotating reference frames [24,26,27] and electromagnetic fields around variously shaped charge distributions [1]. As such, being able to communicate mathematically about such phenomena
Figure 2.1. A displacement vector \( \vec{d} \) is the directed quantity from some initial point A to some other point B.

in the language of non-Cartesian coordinates and unit vectors is an essential skill for upper-division physics students, as well as professional physicists.

Given the often-nuanced descriptions of the mathematics used in this paper, it’s essential to have a clear definition of unit and position vectors in non-Cartesian coordinate systems to establish an operational definition. First, the position vector is defined because it contains many of the mathematical concepts studied in this research. Consider a particle moving in space. At time \( t_0 = 0 \), the particle is at point A, and at some time \( t > 0 \) the particle is at some other point B. A displacement vector, commonly denoted \( \vec{d} \), is then defined as the directed quantity from point A to point B (Figure 2.1).

A position vector, commonly denoted \( \vec{r} \), is defined as a displacement vector where point A is coincident with the origin or pole of some coordinate system (Figure 2.2). Thus, the position vector is a vector with its tail at the origin and tip at the coordinates of the point of interest. In Cartesian coordinates, a position vector is typically written in the algebraic form \( \vec{r} = a\hat{x} + b\hat{y} + c\hat{z} \) or \( \vec{r} = a\hat{\imath} + b\hat{\jmath} + c\hat{k} \) where \( a \), \( b \), and \( c \) are the magnitudes of the \( x \), \( y \), and \( z \) components, respectively, and the vectors with circumflexes above them connote unit vectors. A unit vector is a dimensionless vector of length one that points in the direction of the increasing coordinate it connotes.

Position vectors in polar, spherical, and cylindrical coordinates operate differently from position vectors in Cartesian coordinates. Spherical unit vectors are illustrated in Figure 2.3 [1].

In the polar coordinate system, the algebraic expression for the position vector \( \vec{r} \) terminating at some point \( P \) a distance \( R \) from the pole would be \( \vec{r} = R\hat{\rho} \) where \( \hat{\rho} \) is a vector of length one that points radially outward from the pole in the direction of increasing radius. Thus, the symbol \( \hat{\rho} \) does not have a static meaning like \( \hat{x} \), \( \hat{y} \), and \( \hat{z} \) and algebraic expressions of polar position vectors are
Figure 2.2. A position vector $\vec{r}$ is a displacement vector with tail at the origin and terminal point at the point of interest.

Figure 2.3. An illustration of the spherical basis unit vectors from Griffiths’ EM textbook [1]. This diagram uses the physics convention of $\phi$ as the azimuthal angle and $\theta$ as the polar angle (the mathematics convention reverses these symbols).
not symbolically unique (Figure 2.4). This dynamic behavior of polar unit vectors results from the bases of non-Cartesian coordinate systems being location-dependent. Furthermore, the expression $R\hat{r}$ is sufficient to describe a polar position vector because both its magnitude, $R$, and its direction, $\hat{r}$, are given. Despite this difference in the directional natures of Cartesian and non-Cartesian unit vectors, there is a conversion of Cartesian unit vectors to polar with the transformations:

$$\hat{r} = \cos \theta \hat{x} + \sin \theta \hat{y}$$ (2.1)

$$\hat{\theta} = -\sin \theta \hat{x} + \cos \theta \hat{y}$$ (2.2)

where it can be seen the angular dependence accounts for the non-static directionality of $\hat{r}$. For spherical coordinates these transformations (with the physics convention of $\theta$ as the polar angle and $\phi$ as the azimuthal) are:

$$\hat{r} = \sin \theta \cos \phi \hat{x} + \sin \theta \sin \phi \hat{y} + \cos \theta \hat{z}$$ (2.3)

$$\hat{\theta} = \cos \theta \cos \phi \hat{x} + \cos \theta \sin \phi \hat{y} - \sin \theta \hat{z}$$ (2.4)

$$\hat{\phi} = -\sin \phi \hat{x} + \cos \theta \hat{y}$$ (2.5)

These transformations often represent some of students’ first exposure to polar and spherical coordinates and their corresponding unit vectors [40,41,42].

The non-uniqueness of the algebraic expressions for polar position vectors does not mean that $\hat{r}$ rotates about the origin so it can match its associated position vector, it means there is an infinite number of $\hat{r}$ unit vectors for any given polar coordinate system. Such algebraic ambiguity can lead to some confusion. It’s tempting to say that when using polar basis-vector notation in polar coordinates that $\vec{r} = R\hat{r}$ defines a circle since there is no way of knowing which $\hat{r}$ is being used without any other information. However, such thinking would imply that a vector and a surface are equal, which is mathematically incorrect. It would be reasonable and correct to say the set of all position vectors, $\{R\hat{r}\}$, defines a circle, however, the reality is, upon seeing just the
Figure 2.4. In polar coordinates \( \vec{r}_A \) and \( \vec{r}_B \) have identical-looking algebraic expressions on the right-hand sides of their equations: \( \vec{r}_A = 2\hat{r}, \vec{r}_B = 2\hat{r} \). But, because their respective \( \hat{r} \) unit vectors point in different directions, \( \vec{r}_A \neq \vec{r}_B \).

algebraic expression \( R\hat{r} \), there is no way to determine which specific vector is being referenced without additional information. Thus, for the purposes of this paper, we claim that \( R\hat{r} \) without additional information refers to a single, unique but unknown polar or spherical position vector.

2.1.3. Theoretical Framework

The initial data of student written responses were analyzed using the theoretical framework of Identifying Student Difficulties \[62,63,64\]. Within this framework student difficulties and successes are identified through written or interview responses and are utilized to develop curricular materials that use a variety of strategies, including cognitive conflict, to guide students to realize inconsistencies within their own thinking and lead them to correct understanding. The analysis of initial written data uncovered a number of student difficulties, however only a handful of student responses fell into common categories and among those responses there were a variety of ideas. As a result, the analysis did not disentangle specific and consistent difficulties with sufficient clarity to be of use as the foundation for a targeted curriculum. Some students would report correct answers but with unclear or incomplete reasoning. Some incorrect answers appeared superficially similar but presented slight differences that made it unclear what the underlying difficulty might be \[3\]. Very often, it simply wasn’t clear what the student was trying to do, and a considerable
amount of interpretation and inference on the part of the researcher would have been required to
categorize a response into any category. Such varied responses are inconsistent with the description
of persistent, firmly held, and consistently applied incorrect ideas, or difficulties, associated with
a student difficulties framework. Despite these variations and the limited sample size, a handful
of thematic patterns of thinking were observable and useful for designing additional questions for
further exploration guided by a different theoretical framework.

The theoretical framework of resources and framing was selected to analyze students’ in-
the-moment thinking [36,37,38]. Within this framework, a cognitive resource refers to a content
specific, fine-grain idea that students unconsciously use, or sometimes don’t use, to address a ques-
tion or problem posed to them. These resources may be closely associated or connected to other
resources within a frame or schema that individual students have assembled and are activated when
presented with some stimulus. Stated more generally and according to Hammer (through personal
communications), a resource is a nugget of information or a nugget of an idea that a student has
found useful at some time in the past. The resources framework affords the opportunity to dissect
student reasoning into pieces that can be identified, understood, labeled, and described; increasing
opportunities for the development of targeted curricular materials. It’s worth noting that the pre-
cise naming of individual resources doesn’t need to be “correct” in order to be productive to the
researcher [37]. Meaning, if the chosen nomenclature for individual resources was somehow incorrect
or non-intuitive, the claims of the research are unaffected because the utility of the framework is in
its ability to assign working labels to individual resources which allows the use – or lack of use – of
those resources to be tracked and analyzed.

2.2. Methodology

2.2.1. Interview Protocol

Using the written data described in the previous study [3], a think-aloud interview protocol
was developed to further probe student thinking and reasoning regarding non-Cartesian unit, po-
position, and velocity vectors. The interview sample was described in the introduction. This paper
focuses on results from the first set of interviews. Interview subjects completed the interview tasks
in front of a whiteboard. Images for the interview tasks were projected onto the whiteboard so that
interview subjects could draw and write answers directly on top of projected images.
Our interview protocol first presents interview subjects with a two-dimensional spiral with two points, A and B (Figure 2.5) [3]. Subjects are asked to draw the directions of the polar unit vectors \( \hat{r} \) (radial) and \( \hat{\theta} \) (azimuthal) originating at those two points, and to explain their reasoning as to why they drew them in their illustrated directions. This question will henceforth be referred to as Spiral Question A (SQA). A correct student response is shown in Figure 2.6, where the \( \hat{r} \) vectors are drawn pointing radially outward from the pole and the \( \hat{\theta} \) unit vectors are drawn (approximately) tangentially to the circles of radii equal to the distances the points are from the pole. This drawing was done by this paper’s case study subject.

Part B of the Spiral Question (SQB) instructs subjects to assume the magnitude of the radial distance of the particle from the origin is \( r = R_0 - b\theta \) and the angle \( \theta = \omega t \), where \( R_0 \), \( b \), and \( \omega \) are constants, and then asks them to identify the units of each constant and then to use those constants to write a polar position vector \( \vec{r} \) as a function of time. The correct position vector is \( \vec{r} = (R_0 - b\theta)\hat{r} \). Subjects are then asked to write a velocity vector \( \vec{v} \) describing the situation. As with SQA, subjects are asked to explain their thinking verbally. The velocity vector is the first time derivative of the position vector and because \( \hat{r} \) is non-static and changes with time, a product rule operation is required in the differentiation, resulting in \( \vec{v} = -b\omega \hat{r} + (R_0 - b\omega t)\omega \hat{\theta} \). This differentiation
is shown in Section 2.6. While it is possible that an interview subject could recall that \( \frac{d\hat{r}}{dt} = \omega \hat{\theta} \) where \( \omega \) is angular speed, the researchers assumed it is more likely that such a result would be obtained from transforming to the Cartesian components of \( \hat{r} \), differentiating those components, then re-transform the resulting expression into polar coordinates. Such a task is not trivial. This spiral question was thus not an attempt to probe the ability to do basis transformations, but to probe the understanding of the consequences of non-Cartesian bases being location-dependent, i.e. the basis unit vectors being directionally non-static. An interview subject attempting to differentiate or stating that \( \hat{r} \) needs to be differentiated would be sufficient evidence of such understanding. Thus, for the purposes of our analysis, any indication from the interviewees that the time derivative of \( \hat{r} \) was non-zero would have been evidence of productive thinking that non-Cartesian unit vectors have nontrivial time derivatives.

The next interview question is based on a question developed by Hinrichs [57] and presents students with a three-dimensional Cartesian coordinate system with four points, labeled A, B, C, and D, respectively, with one point each along the positive \( x \)-, \( y \)-, and \( z \)-axes and one along the
negative $y$-axis (Figure 2.7). For each point, subjects are asked to write a spherical polar position vector in terms of $\hat{r}$, $\hat{\theta}$, and $\hat{\phi}$ that defines that point and to explain their reasoning as they do so. Henceforth this question will be referred to as the 3D Question (3DQ). The correct position vectors are $\vec{r}_A = 4\hat{r}$, $\vec{r}_B = 3\hat{r}$, $\vec{r}_C = 3\hat{r}$, and $\vec{r}_D = 2\hat{r}$.

The interview protocol includes additional questions. However, this paper focuses on only the questions described above. The interviews took 50-70 minutes for students to complete.

2.2.2. Case Study

The variance seen in student responses to the written questions described in the Theoretical Framework section (Sec. 2.1.3) were also present in the interview results. Each interview subject responded with a unique combination of productive and non-productive ideas. There was not a consistent underlying thinking present that the authors could identify as a “student difficulty”. For example, while the four interview subjects would often produce similar answers consistent with “pattern-matching” as defined by Hinrichs [57], identical pattern-matched answers across interviews were quite rare. Through repeated viewings of the recorded interview videos, it was observed that there were some fine-grain ideas that seemed to be activated in most, if not all, of the students' thinking. We determined that the clearest way forward was to present a careful analysis of a single
interview. Doing so allows for the meticulous task of identifying and justifying specific resources for this student and developing a model of how those resources seem to be connected and organized. Once completed, we will further elaborate on these resources and their possible organization and structure in subsequent papers with the remaining interview sample.

The subject of the case study will be known by the pseudonym Mark. Mark is an undergraduate physics major who was enrolled in both classical and quantum mechanics and had completed upper-division electromagnetism, modern physics, and all math credits necessary for a physics degree (Calculus I, II, III, Differential Equations, Linear Algebra). We selected Mark's interview for detailed analysis because his description of the mathematics was similar to those of the other interviewees and those responses seen in previously analyzed written data. Such similarities include both mathematically correct and incorrect responses, several of which demonstrate what Hinrichs describes as pattern-matching [57]. Mark demonstrates a firm understanding of the polar unit vectors on SQA, however his thinking about these unit vectors is challenged as he progresses through the interview. Through the lens of our previous work, Vega et al. [3], Mark productively activated many of the resources in the Unit Vectors Cluster described and did not show evidence of activation of resources from the other clusters such as Motion Cluster, the Coordinate Cluster, or the location-dependent Cluster. Herein, we expand upon our previous analysis, and analyze Mark’s activation of unit vector resources and how he used them in the context of position and velocity vectors. Mark was also chosen for this analysis due to the clear articulation of his thought processes during his interviews. He provides clear descriptions of his thinking which enable us to confidently identify resources he was activating. Overall, Mark is one of the more successful students from our interview sample in terms of answering questions correctly.

In this paper, the results of the interview and the discussion/analysis of those results will be presented in separate sections. This presentation not only mirrors the process in which the research team collected and analyzed the data, it also allows for a more holistic analysis. A goal of analyzing this case study was to develop a vocabulary and framework for analyzing the other interviews. Thus, it is not only important to identify individual, fine-grain ideas and name them as resources, it is also informative to identify and map how groups of such resources activate, or don’t activate, together. Presenting a small portion of Mark’s responses as evidence for the activation of an individual resource together could inadvertently shift the focus of the analysis more toward the
naming individual resources rather than the simultaneous goals of naming individual resources and the mapping of the activation of groups of resources that are thematically connected in some way.

2.3. Results

2.3.1. Unit Vector Reasoning

Initially on SQA, Mark gave accurate and complete definitions for ̂r and ̂θ, drawing ̂r vectors in the same direction as the corresponding r vectors (Figure 2.6) and stating the following two justifications respectively (M# are Mark’s statements):

M1 “[̂r points] away from the origin”
M2 “[̂θ] is in the direction of increasing θ where θ is like your azimuthal.”

Mark later stated that regarding point A the direction of the ̂r vector is

M3 “…from the origin pointing toward A.”

Mark clearly stated and illustrated the correct definitions of the polar unit vectors, with no evidence of any confusion.

2.3.2. Position Vector Reasoning - Initial Conflict and Resolution

When asked to write the general polar position vector on SQB Mark responded with \( \vec{r} = (R_0 - b\theta)\hat{r} + \theta\hat{\theta} \). The interviewer asked Mark why he included both \( \hat{r} \) and \( \hat{\theta} \) terms. Mark paused for a moment, and stated:

M4 “so the way I previously described my description to \( \hat{r} \) you wouldn’t necessarily need \( \hat{\theta} \) – ’cause like what I said for \( \hat{r} \) earlier was that it was measured from the origin to whatever point you’re talking about, but now I’m believing that that was not right.”

Then after a long pause, the interviewer encouraged him to verbalize his thoughts, and the following exchange occurred (I# are interviewer statements or questions):

M5 “I’m thinking that \( \hat{r} \) thing is right, ’cause otherwise what other direction would \( \hat{r} \) be in?”
I1 “So your answer way back in part A?”
M6 “So I’m thinking that’s still right. I’m thinking this extra θ term is not supposed to be here.”
I2 “OK. And why are you now thinking that?”
M7 “Well, because given the definition that I gave of \( \hat{r} \), \( \vec{r} = (R_0 - b\theta)\hat{r} \) tells you everything you need to know about your position vector.”

Mark further stated that \( \hat{\theta} \) would be needed when describing a change in position:

M8 “Oh right, yeah. It’ll only be relevant in the next question when you talk about the change in position.”
Mark used correct reasoning based on his understanding of the unit vectors to inform his thinking about this polar position vector. Such understanding appeared to include the realization that polar unit vectors are not directionally static. To confirm this, the interviewer asked what the Cartesian unit vectors $\hat{x}$ and $\hat{y}$ would look like at points A and B. Mark drew them pointing in the correct directions at both points.

Finally, the interviewer asked Mark to draw the position vector for point A on the spiral. Mark drew a vector with its tail at the origin and arrowhead terminating at point A, which is consistent with the definition of a position vector discussed in Section 2.1.2, and as can be seen in part of Figure 2.8. In his Figure 2.8 sketch, Mark initially labeled the position vector as “$\vec{r}$” and subsequently added a subscript “i” to the label while answering SQB.

2.3.3. Velocity Vector Reasoning - Attempting to Resolve Two Competing Ideas

When asked to write the velocity vector for SQB, Mark’s initial response was to write $\vec{v} = \Delta \vec{r}$. He then drew the velocity vector as the difference between initial and final position vectors, which can be seen in Figure 2.8. He further wrote $\vec{v} = \frac{\Delta \vec{r}}{\Delta t}$, and, with some leading questions by the interviewer, realized he needed to take the first time derivative of his previous position vector to get the velocity vector. At this point, Mark stated that $\dot{\theta}$ was needed in the velocity vector and should come out in the result of differentiating the position vector, $\vec{r}$. However, when he attempted this differentiation he missed the necessary product rule operation (see Sec. 2.6) and wrote $\dot{\vec{r}} = -b\omega \vec{r}$. Mark said nothing about the time derivative of $\hat{r}$, which is the only unit vector in his correct position vector expression, being non-zero and thereby requiring the use of the product rule.

While contemplating these ideas, Mark questioned his initial and correct definitions of the polar unit vectors:

**M9** “So what I originally came up with [pointing to his correct position vector], this isn’t right. This needs to have some $\theta$ dependence of some sort because, um, it should depend on $\dot{r}$ - it should have dependence on $r$ direction because in this case $|\vec{r}|$ is getting shorter, it spirals inwards. But it should also have $\theta$ - it should also have $\dot{\theta}$ dependence...”

**M10** “I wonder if I defined my definition wrong, or something, with $\hat{r}$.”

**I3** “Well, you’ve already questioned yourself on that. Why are you questioning your questioning?”

**M11** “I feel like this [pointing to velocity vector] should definitely not just be one term. It should have $\theta$ in here.”

**I4** “So we didn’t - you did not have it in your position vector - the $\dot{\theta}$ dependence - now you want to say that it shows up again in velocity?”
Figure 2.8. Mark’s initial approach to $\vec{v}$ was to show that $\vec{v}$ is the change in position vectors. He was able to follow this line of thinking to eventually reach taking the first time derivative of the position vector.

M12 “If it doesn’t show up [in $\vec{r}$] it definitely won’t show up [in $\vec{v}$], but I’m, I’m feeling like it should.”
I5 “OK. Just tell me why you think it should.”
M13 “Because, um, I mean when you think about $\hat{\theta}$, $\hat{\theta}$ we said is like this direction [pointing in direction of increasing $\theta$] sorta...um, and in this case the higher that - or the larger values of $b$ and $\omega$ take on, the faster conceivably we’ll be changing the $\theta$. So I mean, these [pointing to $b$ and $\omega$] should have some bearing on what $\theta$ is going to be doing. I’m just totally blanking on...[long pause]”
I6 “So if I could summarize, you do think that you need to include a $\theta$ term somehow in your velocity vector but you are just not sure how to do that?”
M14 “Yes... why am I struggling with this? I feel like it should be really easy.”

2.3.4. Changing Unit Vector Definitions

After saying he thought it should be easy to get $\hat{\theta}$ out of differentiating $\vec{r}$ and struggling to do so, Mark questioned his original definitions. He stated:

M15 “So, I defined $\hat{r}$ incorrectly... OK, so $\hat{r}$ should not, it does not have any actual bearing on where the point is, the position vector is what matters there.”

Mark then developed a new strategy so he could include a $\hat{\theta}$ term in the position vector. He determined how $\hat{r}$ would behave if it were static, that is, always pointing in the same direction like a Cartesian unit vector. He explained this static nature by making $\hat{r}$ an explicit transformation of $\hat{x}$ and $\hat{y}$ and a “scaling factor” (as he called it) to result in unit length. He wrote $\hat{r} = \sqrt{\hat{x} + \hat{y}}$ as a representation of his thinking. His illustration accompanying his explanation (Figure 2.9) clearly
Figure 2.9. Mark’s illustration used during his attempt to explain $\hat{r}$ as a transformation of $\hat{x}$ and $\hat{y}$, and therefore directionally static.

shows $\hat{x}$ along the $x$-axis and $\hat{y}$ along the $y$-axis resulting in a $\hat{r}$ that makes a $45^\circ$ angle between $\hat{x}$ and $\hat{y}$. His verbal explanation explicitly demonstrates a return to a Cartesian way of thinking:

M16 “...now that I’m thinking about this so, because like what I was talking about earlier [his initially correct definition of $\hat{r}$], it was like totally nonsense because think about it like parallel to Cartesian coordinates; like $\hat{x}$ is just here [as he drew $\hat{x}$] and $\hat{y}$ is just here [as he drew $\hat{y}$]. Like it doesn’t matter what point I’m looking at, right? So looking at A, $\hat{r}$ should always be right there [as he drew $\hat{r}$ at a $45^\circ$ angle between $\hat{x}$ and $\hat{y}$].”

Mark flipped from his initial, non-static definition of $\hat{r}$ to a static definition of $\hat{r}$. After talking through all of this, he remained uncertain as to which of his two definitions was better or which one was correct. He described this uncertainty as a “moral dilemma.” When asked to elaborate on this moral dilemma, he explained:

M17 “I think when I was defining $\hat{r}$ before [his original, correct definition] I was defining $\hat{r}$ to be the position vector $\vec{r}$ over the magnitude of $\vec{r}$, which is not the same thing that you’re talking about in polar coordinates, I don’t believe... I’m not thinking that’s right because I’m thinking $|\vec{r}|$ should just be something that is a strict transformation of $\hat{x}$ and $\hat{y}$ so it would be pointing like $\hat{r}$ at a $45^\circ$ angle between $\hat{x}$ and $\hat{y}$.”

On the surface, this explanation of his moral dilemma appeared to be a complete change of thinking from his correct definitions to a more Cartesian-like definition.
2.3.5. Questioning the Efficacy of New Unit Vector Definitions

Before Mark gave any answers to the 3DQ, the interviewer asked him which of his two stated definitions of \( \mathbf{r} \) he would use to answer the question. After spending a few moments in thought, he replied:

M18 “I think it would be easier using my first one.”

As he prepared to write his answers to the position vectors on the board, he further stated:

M19 “OK, so, now I’m thinking I’m gonna go contradict everything I just said [his second definition].”

2.3.6. Returning to and Attempting to Apply Initially Productive Unit Vector Definitions

Mark’s initial responses to three of the four points on the 3DQ are examples of Hinrichs’s description of pattern-matching. Mark’s four position vectors were

\[
\mathbf{r}_A = 4 \hat{r} + \frac{\pi}{2} \hat{\phi}, \quad \mathbf{r}_B = 3 \hat{r}, \quad \mathbf{r}_C = 3 \hat{r} + \frac{\pi}{2} \hat{\theta}, \quad \mathbf{r}_D = 2 \hat{r} + \frac{3\pi}{2} \hat{\phi}.
\]

(Of note, Mark made the decision to use the physics convention of \( \phi \) as the azimuthal angle and \( \theta \) as the polar angle) As he answered \( \mathbf{r}_A \), he counted out the number of intervals point A was from the origin and said that the radial term should be \( 4\hat{r} \) and then stated:

M20 “So then \( \mathbf{r}_A \) would be \( 4\hat{r} \) plus - it would be \( \pi \) in the \( \hat{\theta} \) direction - 'cause this [while pointing to +x-axis] would be like our zero \( \phi \). \phi \) would be equal to zero along here [still pointing to +x-axis].”

When the interviewer asked him if \( \phi \) was the angle between the \( x \)- and \( y \)-axes he said it was, and drew \( \phi \) as the angle between them. After he wrote all the position vectors, the interviewer asked him why there was \( \phi \) and/or \( \theta \) dependence for points A, C, and D but not for B. He replied:

M21 “Well, I mean there is a dependence per se, right? I mean you could write all these out and just say that [missing \( \theta \) term] is you know plus 0 times \( \theta \) but uh, for B, just the way it, and I think you’re, basically when I think you’re using this you’re free to define \( \phi \) equals 0 to be anywhere, but I’ve sorta defined it to be like, ya know, you’re measuring it with respect to like, [points to +x-axis] sort of, \( \phi \) yeah, \( \phi \) would be like you measure it with respect to [+x] so [+x-axis] is just 0 because it lies on that \( \phi \) equals 0 line,”

indicating that he saw a need for explicit \( \phi \) and/or \( \theta \) dependence but only wrote such terms when he believed those dependencies were nonzero.

2.4. Discussion

2.4.1. Identifying Mark’s Resources in the Context of our Previous Resource Clusters

As noted in the results section, some of Mark’s responses to position and velocity vector questions can be described using resources that are consistent with our previous description of
the Unit Vector Cluster [3]. Additional responses share similarities with the previously identified location-dependent Cluster; however, the ideas described in that cluster don’t map well to Mark’s responses, particularly in the context of position and velocity vector questions. Mark’s initial (and later) idea, that non-Cartesian unit vectors change direction with location and his use of the spiral-shaped path to inform his construction of a velocity vector are examples of location informing his thinking about unit vectors. However, in the context of position and velocity vectors, Mark appeared to activate resources from both of these previously identified clusters and combine those activated resources in patterns dependent on whether he was answering a position vector question or a velocity vector question. For clarity in the presentation of Mark’s thinking, we are first naming individual resources Mark activates and then grouping those activated resources by the context of the question he was answering, specifically, the “Position Vector Resources group” and the “Velocity Vector Resources group”. Grouping Mark’s resources based on the context of the question, allows for a particular coherent mapping of Mark’s thinking which we find useful. We acknowledge that this is only one of many possible groupings, and that alternative groupings of resources could illuminate other essential features of Mark’s thinking.

In this paper, we have chosen to use the term “group” in part to reflect that we are also sorting resources into thematic categories. But, we also use “group” as opposed to “Cluster” to reflect that this approach to thematic categorization is different in nature than that used in our previous work [3]. In the Vega et al. paper, the clusters were themes of thinking that emerged from responses from multiple interview subjects and exclusively within the context of unit vectors. In this present paper, these new resource groups align with the context of questions about additional vector types and emerge from the responses of a single subject. Additionally, this present work seeks to name individual resources, whereas Vega et al.’s work sought to name groups of ideas in thematic clusters. We use similar naming structures, ie “[Theme Title] Cluster” in Vega et al. and “[Theme Title] group” in this paper to reflect that the goal of understanding and describing student thinking about vector and coordinate-system concepts remains the same across individual studies within the broader project. Finally, it is not our intention to add the term “group” to the vocabulary of Resource Theory, only to use the term as a tool of categorization in the context of this research.

To more clearly communicate the flow of Mark’s thinking and introduce our new resource nomenclature, we developed two toy models. The first consists of resource graphs [73] organized by
the context of each question – one each for unit vectors, position vectors, and velocity vectors (Figure 2.10). The second (Figure 2.11) is a combination of resource graphs and a box-and-arrow model that presents a chronological map of Mark’s thinking and an expanded *Velocity Vector Resources* group to illustrate how his activated resources are connected within it. Within each group, a solid-color bubble indicates a resource that is coordinate system invariant, i.e. productive in all coordinate systems and not specific to any individual coordinate system. A cross-hatched background in a bubble indicates a resource that is productive in the context in which Mark is working but not necessarily productive in all coordinate systems. The solid white background bubble for a resource in the *Velocity Vector Resources* group is explained in detail in Section 2.4.3.

### 2.4.2. Mark’s Unit Vector Resources

At the beginning of the interview, Mark clearly and correctly illustrates and defines both Polar and Cartesian unit vectors. While answering SQA, he draws all four requested unit vector arrows in the correct directions and provides clear and correct reasoning for why he drew them in those directions, as seen in statements M1-3 and his drawing in Figure 2.6. Based on previous work, this is not a trivial task [3]. The combination of statements M1 and M2, Figure 2.6, and Mark’s illustration of the Cartesian unit vectors in their correct directions suggests that Mark activates productive resources for unit vectors pointing in the direction of increasing coordinate, having unit length, and being orthogonal within their respective coordinate systems. "*Increasing coordinate direction*" is a more refined version of our previously identified location-dependent *Cluster* and is a statement of the general behavior of unit vectors in all coordinate systems. Previous work identified a non-productive activation of this resource for Cartesian unit vectors, however Mark activates this in a productive fashion and the model is more clear when using the description of *Increasing coordinate direction*. Working together these resources form at least a portion of a resource group we have called “*Unit Vector Resources* group” (Figure 2.10). This *Unit Vector Resources* group contains three of the four resources identified previously from Vega et al. [3], and, in this case, is identified by evidence from a single student, as opposed to our previous work were we identified them from an expert perspective. The only missing resource from our previous expert-like *Unit Vector Cluster* is one where unit vectors are identified as dimensionless, which is not directly probed by this question. It should be noted that Mark did not explicitly state that the basis unit vectors are orthogonal in his responses to SQA, SQB, and 3DQ. However, Mark draws the unit vectors perpendicular to each
Figure 2.10. Resource groups identified from Mark’s statements and drawings, organized by the content of the question. A solid-color background indicates a resource that is coordinate system invariant. A cross-hatched bubble background indicates a resource that is productive in the coordinate system Mark is currently working. In the Velocity Vector Resources group, the $\frac{d\hat{r}}{dt} \neq 0$ resource bubble has a solid white background because it is an expert-identified resource that Mark does not activate.
Figure 2.11. A chronological box-and-arrow plot that illustrates the connections between Mark’s resources and resource groups. He starts with productive unit vector resources then activates a nonproductive position vector resource. He then compares that non-productive resource to his productive unit vector resources, namely the Increasing coordinate direction resource, and then seems to continue using that productive resource. But, Mark’s productive thinking was then challenged when he moved to a velocity vector question. The chronological plot is then expanded into a “zoomed-in” representation of the Velocity Vector Resources group. The individual resources Mark activated in that group are then linked by how he appeared to activate them chronologically during the interview. There are two paths of thinking in this expanded group view that lead him to the same two resources which he is ultimately unable to reconcile with one another. This unsuccessful reconciliation could possibly be resolved with the non-activated $\frac{d\hat{r}}{dt} \neq 0$ resource. After unsuccessfully reconciling those two resources, Mark activates a new resource we call $\hat{r}$ depends on $\hat{x}$ and $\hat{y}$ that we sort into the Unit Vector Resources group because he is questioning his unit vector definitions when he activates this new resource. However, Mark does not recall the actual unit vector transformations, which leads him to challenge his previously stated productive unit vectors resources and develop a model in which polar unit vectors behave more like Cartesian ones.
other whenever he draws them together. Furthermore, in a later interview question not analyzed in-depth for this paper, Mark stated the following:

M22 “...as long as \( \hat{\theta} \) and \( \hat{\phi} - \hat{\theta} \) is perpendicular to \( \hat{r} \) and \( \hat{\phi} \) is perpendicular to both of them it should be ok.”

This statement appears to confirm the thinking behind his consistent orthogonal illustrations. The fourth resource in our new Unit Vector Resources group seen in Figure 2.10 is discussed in Section 2.4.4. We did not find evidence of Mark’s activation of resources associated with the previously identified Coordinate Cluster, or Motion Cluster [3]. Despite Mark’s productive activation of the three expert-identified resources, Mark struggled to activate them successfully when faced with problems using position and velocity vectors that triggered additional resources.

2.4.3. Mark’s Position Vector Resources

On several occasions, Mark talked about and drew pictures of position vectors when answering questions. The ideas he presented seemed distinct from his thinking about unit vectors and therefore we categorized them separately. Resources associated with Mark’s thinking about position vectors are summarized here and are represented in Position Vector Resources group of Figure 2.10.

Mark consistently constructed position vector expressions and drew position vector sketches that had a vector beginning at the origin/pole of the coordinate system and terminating at the point of interest. Examples of such can be seen in his drawings for Figure 2.8 as well as his statement M20. Thus, we have included a resource in the Position Vector Resources group graph of Figure 2.10 called “From origin to point”. This resource is invariant across coordinate systems, and as such, has a solid background in Figure 2.10. Similar to identified resources in the Unit Vector Resources group, this is a resource that aligns with an expert definition of a position vector (see Mathematics Discussion).

Mark’s first attempt at writing a polar position vector on SQB was \( \vec{r} = (R_0 - b\theta)\hat{r} + \theta\hat{\theta} \). Later in the interview, Mark’s written response to 3DQ and his statements, M20 and M21, demonstrate his propensity to write algebraic expressions for polar/spherical position vectors that include azimuthal and/or polar terms. It is challenging to identify a single fine-grain resource from Mark’s data exclusively, however, drawing from Hinrichs’s work, we identify “\( \vec{r} \) has form \( \vec{r} = a\hat{a} + b\hat{b} \)” as a resource to denote Mark’s activation of an idea that position vector expressions have the general form \( \vec{r} = a\hat{a} + b\hat{b} \), with a \( c\hat{c} \) added in three dimensional cases. As identified, this \( \vec{r} = a\hat{a} + b\hat{b} \) resource is
both productive in Cartesian coordinates – i.e. mapping a point’s coordinates to a vector expression – and consistent with what Hinrichs calls “pattern-matching” [57]. Mathematics education research literature indicates this type of response is not surprising given the primacy of Cartesian coordinates in the education system. Students spend the vast majority of their time learning about and applying the Cartesian coordinate system [40,41,42]. Note the cross-hatched background of this resource in Figure 2.10 denoting its lack of universal utility across coordinate systems. A deeper look into the thinking of the study’s remaining interview subjects may reveal that there is more going on in the *Position Vector Resources* group than Mark’s data alone demonstrates. We aim to further explore these “pattern-matched” responses [57] and expand upon the *Position Vector Resources* group in a future paper.

After his initial response to SQB, Mark recalled his previous description of \( \hat{r} \) and through his statements M4-M7, ultimately decided that “\( \hat{r} \) tells you everything you need to know,” and dropped the extra \( \hat{\theta} \) term. Based on the direct cues in his statements, we attribute this reasoning to Mark activating his *Unit Vector Resources* group, and activating a new resource, that the position vector in polar coordinates only has an \( \hat{r} \) term which we identify as: \( \vec{r} \text{ has only } \hat{r} \). The activation of these resources is represented in the timeline graphic at the top of Figure 2.11, as the red boxes (*Unit Vector Resources*) and green circle (*\( \vec{r} \text{ has form } \vec{r} = a\hat{a} + b\hat{b} \) from the *Position Vector Resources* group).

### 2.4.4. Mark’s Velocity Vector Resources

A combination of Mark’s verbal and written responses, when considering the velocity vector question in SQB, present a nuanced look into Mark’s thinking about velocity vectors. Because of the context Mark is considering when they are activated, we define these resources to be within a grouping of Velocity Vector resources. The *Velocity Vector Resources* group (Figure 2.10) contains five resources activated by Mark (solid or cross-hatched blue background) and one resource (white background) that is not activated by Mark. In Figure 2.11, Mark’s *Velocity Vector Resources* group is connected via arrows to highlight the order in which Mark activated them. This expanded look into Mark’s *Velocity Vector Resources* group is embedded in the timeline at the top of Figure 2.11, which identifies the order in which Mark activates specific resources, and/or groups of resources. All three resource groupings were included into Figure 2.10 for completeness; however, all information
for the Velocity Vector Resources group in Figure 2.10 is presented in Figure 2.11 with additional chronological information. As such, we refer only to Figure 2.11 henceforth, unless otherwise noted.

When Mark first defined the velocity vector, \( \vec{v} \), in his response to SQB, he stated that \( \vec{v} = \Delta \vec{r} \). He later expanded that to \( \vec{v} = \frac{\Delta \vec{r}}{\Delta t} \) and drew what can be seen in Figure 2.8. Synthesizing these very similar ideas it is clear Mark has a nugget of knowledge that \( \vec{v} \) and \( \Delta \vec{r} \) are closely related. Therefore, we identify a resource “\( \vec{v} \Leftrightarrow \Delta \vec{r} \)” in the Velocity Vector Resources group (Figure 2.11). Note the placement of this resource on the far left of the cluster signifying it being activated first chronologically.

While contemplating the problem, Mark sketched the geometric representation shown in Figure 2.8 without any prompt. We identify this new representation as a distinct resource. It’s possible that this is an activation of the From origin to point resource within the Position Vector Resources group, however Mark appears to be using a position vector resource within a different context, to reason about what the “change” in position vectors would look like. Therefore, the sketch is included as one of Mark’s resources in a different grouping in Figure 2.10 and as one of the two paths (colored black to denote an unprompted activation) emanating from \( \vec{v} \Leftrightarrow \Delta \vec{r} \) in Figure 2.11.

The interview protocol was written with the expectation that interview subjects would attempt to determine the velocity vector by differentiating the position vector with respect to time. When Mark appeared stuck and had not yet performed this differentiation, the interviewer asked him gently leading questions in an attempt to elicit this idea. Mark came up with this relationship almost immediately, and we identify it as the “\( \vec{v} = \dot{\vec{r}} \)” resource in Figure 2.11, connected by an orange arrow to indicate a prompted activation. It is possible that relating the velocity vector to some kind of change in the position vector and specifically equating the velocity vector to the time rate of change of the position vector are similar, and perhaps even the same resource. For Mark, however, he does not make an explicit connection between the two resources as being the same or a very similar idea. Therefore, for the purposes of this analysis, we consider them as distinct resources.

Before answering SQB, in statement M8, Mark notes that \( \dot{\theta} \) would be a necessary term to “talk about the change in position”. Mark activates this idea again when responding to SQB in statement M9: “This needs to have some \( \theta \) dependence of some sort...” We identify this resource as \( \vec{v} \rightarrow \dot{\theta} \) and connect it to the resource of the drawing of two position vectors. He also reasons
that the velocity vector will have an \( \hat{r} \) component because it’s “spiraling inward” (from statement M9).

Mark performs the differentiation of his final (and correct) answer for the position vector \( \vec{r} = (R_0 - b\theta)\hat{r} \) and gets an expression with only an \( \hat{r} \) term. From each of these resources independently and his correct observation that both the radial and angular positions change as an object moves along the spiral (statement M9), he correctly reasoned \( \vec{v} \) needed both \( \hat{r} \) and \( \hat{\theta} \) terms; hence the resources \( \vec{v} \xrightarrow{\text{needs}} \hat{r} \) and \( \vec{v} \xrightarrow{\text{needs}} \hat{\theta} \) inside the Velocity Vector Resources group and the connections to each from the spiral sketch and \( \vec{v} = \dot{\vec{r}} \) in Figure 2.11. This thinking is markedly different from our previously reported Motion Cluster, which articulated student ideas associated with the unit vector \( \hat{r} \) pointing in the direction of motion or tangential to the path as the definition of \( \hat{r} \) [3]. Mark isn’t asserting that \( \hat{r} \) points in the direction of motion, but rather that the velocity vector should have components of both \( \hat{r} \) and \( \hat{\theta} \). His dialogue in statements M10-M14 demonstrate more of his thinking around these ideas, but his only conclusion at that time is that “So what I originally came up with [pointing to his correct position vector], this isn’t right."

2.4.5. Attempting to Resolve Productive Resources Leads to a Shift Toward Cartesian Thinking

Mark’s activated resources are not sufficient to resolve his issue of the velocity vector needing both \( \hat{r} \) and \( \hat{\theta} \) terms, and he believes one of his previous answers is problematic and requires revision. Differentiating his expression for \( \vec{r} \) should result in both \( \hat{r} \) and \( \hat{\theta} \) terms. As mentioned in the Mathematics Section, a challenging aspect of this differentiation is the recognition that the time derivative of \( \hat{r} \) is non-zero. As discussed at the end of the Methodology Section, if Mark had mentioned that \( \hat{r} \) was dependent on time, or that \( \hat{r} \) itself needed to be differentiated, or attempted to differentiate it while differentiating \( \vec{r} \), there would be evidence to suggest that he activated productive resources regarding the behavior of \( \hat{r} \). However, Mark first came to the realization that the velocity vector needed an azimuthal term through physical reasoning – using the motion of the particle – not from the differentiation of the position vector.

While Mark has several productive resources that are helpful in deriving \( \vec{v} \), Mark appears to be missing or simply not activating one that would connect his productive resources: \( \frac{d\hat{r}}{dt} \neq 0 \) (shown with a white bubble background in Figure 2.10 & Figure 2.11). Remarkably, this non-activated resource combined with his not reconciling the resources \( \vec{v} \xrightarrow{\text{needs}} \hat{r} \) and \( \vec{v} \xrightarrow{\text{needs}} \hat{\theta} \) led him to
question and attempt to redefine the productive resources from the Unit Vector Resources group he had already activated. His thinking is illustrated in Figure 2.11 with the arrows from $\vec{v} \xrightarrow{\text{needs}} \hat{r}$ and $\vec{v} \xrightarrow{\text{needs}} \hat{\theta}$ bypassing the non-activated $\frac{df}{dt} \neq 0$ resource and pointing to activation of a new resource that we attribute to the Unit Vector Resources group.

Mark’s new analysis resulted in his statement that $\hat{r}$ is an “explicit transformation of $\hat{x}$ and $\hat{y}$”, and we name this new resource as “$\hat{r}$ depends on $\hat{x}$ and $\hat{y}$”. This resource has the potential to be productive, since $\hat{r}$ is an explicit transformation of $\hat{x}$ and $\hat{y}$ (see Eqn. 2.1 in Sec. 2.1.2). However, Mark wrote down the relationship as $\hat{r} = \sqrt{\hat{x}^2 + \hat{y}^2}$. This relationship is possibly consistent with some vector operations in using the Pythagorean theorem while calculating the magnitude of a vector in Cartesian coordinates, i.e. $|\vec{r}| = \sqrt{x^2 + y^2}$. Further evidence of this Pythagorean- and vector-operation-like thinking can be seen in Figure 2.9, where, along with his reasoning in M16, $\hat{x}$ and $\hat{y}$ are drawn as components of $\hat{r}$. This drawing appears to be consistent with the position vector, in Cartesian coordinates, being the hypotenuse of the right triangle formed by the vector and its $x$ and $y$ components arranged tip-to-tale. Within our analysis, this $\hat{r} = \sqrt{\hat{x}^2 + \hat{y}^2}$ relationship straddles the Unit Vector Resources group – by providing a definition for – and the Position Vector Resources group – by using reasoning that is consistent with position vector and Pythagorean Theorem rules/conventions. During initial analysis of Mark’s thinking, we had identified this as a resource; however, our thinking has evolved. We now think it is unlikely that this relationship is a resource, i.e., a “nugget of useful information”, that Mark has found useful in the past. Rather, we suggest that it can be viewed as a “conceptual blend” [86,87] of two separate resources. Bing and Redish [87] specifically used conceptual blending to make sense of what appeared to be newly generated student ideas at the math-physics interface. This framework could provide tremendous insight into future analysis of our data.

During Mark’s attempt to transform from Cartesian unit vectors to polar unit vectors, he attempted to make $\hat{r}$ static, likening its behavior to that of Cartesian unit vectors:

(A portion of M16) “...now that I’m thinking about this so, because like what I was talking about earlier [his initially correct definition of $\hat{r}$], it was like totally nonsense because think about it like parallel to Cartesian coordinates [emphasis added]”

From an expert perspective, it seems that Mark attempted to develop a justification for why $\hat{r}$ is a static quantity. While taking the time derivative of the position vector, his math is
consistent with treating it as a static quantity, and then he articulated that \( \hat{r} \) depends on \( \hat{x} \) and \( \hat{y} \). Thus, it seems that he attempted to develop a model where \( \hat{r} \) is static because \( \hat{x} \) and \( \hat{y} \) are static, using Cartesian-coordinate behavior as a guide (Statement M16). This new thinking demonstrates a distinct shift away from his Increasing coordinate direction resource.

It’s possible that a more complete recollection of how the polar unit vectors can be derived from the Cartesian unit vectors would have helped Mark see that \( \hat{r} \) needs to be differentiated. However, the interview did not prompt Mark to do so, nor did it explicitly probe Mark’s ability to convert from Cartesian to polar coordinates. Mark made no mention that \( \hat{r} \) would need to be differentiated despite his prior productive activation of the Increasing Coordinate Direction resource and illustrations of a directionally non-static \( \hat{r} \) (Figure 2.6). While having access to the correct mathematical transformations could have helped him, there was a connection that was not made between his productively stated behavior of \( \hat{r} \) and the mathematical consequences of that behavior. Further probing thinking concerning the conversion of Cartesian to spherical unit vectors is an area of future research interest. It also seems likely that the non-activated \( \frac{d\hat{r}}{dt} \neq 0 \) resource could serve as a point for curricular emphasis, at least for Mark.

Despite these new ideas, when Mark continued on to 3DQ, he returned to his first definition of \( \hat{r} \) as pointing in the Increasing coordinate direction and abandoned his newly activated \( \hat{r} \) depends on \( \hat{x} \) and \( \hat{y} \) resource. Statements M18-19 show he recognized his initial definitions of polar unit vectors are more convenient for answering the 3DQ. However, Mark wrote the position vectors to 3DQ in the same form as he did initially in SQB, with additional terms of \( \hat{\theta} \) and/or \( \hat{\phi} \). We claim Mark is activating the same resource as he had previously, \( \vec{r} \) has form \( a\hat{x} + b\hat{y} \) with an additional \( c\hat{c} \) term for the third dimension. Unlike his response to SQB, Mark did not compare his responses with his Increasing coordinate direction resource, even after probing questions by the interviewer. In Statement M21, Mark used the measures of the angles \( \phi \) and \( \theta \) of the angular coordinates from \( (r, \phi, \theta) \) as the “magnitudes” of the \( \hat{\phi} \) and \( \hat{\theta} \) terms. As was previously mentioned in Section 2.4.3, this \( \vec{r} = a\hat{a} + b\hat{b} + c\hat{c} \) form is likely based on the Cartesian notation for position vectors, and is a result consistent with findings from math education research literature [40,41,42] and Hinrichs [57].

There is also evidence Mark expected the algebraic notation of a position vector to define a unique position in space and that he didn’t trust that correct polar/spherical notation did so. While constructing his spherical position vectors in 3DQ he continually and consistently gestured to
show how his algebraic expression would lead a reader to the exact point his expression attempted to define. He explicitly gestured and stated that azimuthal angles are measured from the $x$-axis and polar angles from the $z$-axis (statements M20-21). In each case, he showed how the $\hat{\phi}$ and/or $\hat{\theta}$ terms he wrote corresponded to a direction one should take if one were attempting to move from the origin to the point of interest. Moreover, for point B, where he wrote the correct position vector, he further explained there is a dependence on $\hat{\phi}$ and $\hat{\theta}$, but those dependencies are 0, making them unnecessary to write (statement M21). He did not mention, as he had done on SQB, that $\hat{r}$ “tells you everything you need to know.” These statements also seem to indicate he believed the Cartesian form of a position vector is the complete form and that the spherical form does not communicate enough information.

Mark’s responses on the 3DQ are consistent with Hinrichs’s description of pattern-matching [57]. However, Hinrichs’s definition of pattern-matching spherical position vectors to the Cartesian form is purely descriptive and not rooted in a theoretical base. His work did not intend to provide a theoretical explanation for why pattern-matched responses are common. A theoretical framework of resources provides one possible explanation. Mark’s responses to the 3DQ potentially provide some evidence that the pattern-matching response occurs due to the activation of a related set of resources consistent with Cartesian conventions. Given that mathematical meaning is often tied to the coordinate system in which a concept is learned [40, 41, 42] and the primacy of Cartesian coordinates in mathematics instruction [41], it’s not surprising that Mark regularly activated Cartesian-like resources. Additionally, Mark is not the only interview subject in our interview sample to respond with pattern-matched position vectors. In fact, all of the interview subjects answered SQB and/or 3DQ with what Hinrichs would have identified as pattern-matched position vectors in non-Cartesian coordinates [57]. A future paper will delve further into these responses and how the use of a resources framework might further elucidate students’ thinking. This new paper will likely also allow us to expand the number of named resources within the Position Vector Resources group and possibly rename – or discover yet-simpler resources driving the activation of – the resources we named as equations in this paper.

2.4.6. Implications

Despite this paper presenting the initial step in a broader analysis, there are initial implications for instruction and future investigation. When looking at Mark’s ideas and the published
literature, it seems that Cartesian-like thinking and some of the associated resources identified here may be a significant barrier to the use of non-Cartesian coordinates and unit vectors for solving math and physics problems. Even when a student activates resources appropriate for non-Cartesian coordinates and unit vectors, activation of resources that are productive – and likely formed [40,41,42] – in Cartesian coordinates can interfere with the thinking of and confuse that student. Returning to the flow of thinking modeled in Figure 2.11 provides insight into one possible path for instruction. An instructional sequence could elicit ideas about $\vec{v} \xrightarrow{\text{needs}} \hat{r}$ and $\vec{v} \xrightarrow{\text{needs}} \hat{\theta}$. It could further provide students opportunities to reflect on the definitions of the polar unit vectors and reconcile how, unlike Cartesian unit vectors, the time derivative of the polar unit vectors are non-zero. A more complete set of student resources will be identified using our remaining interview subjects and will expand the space of what we know about student thinking in this area. Going forward, additional interviews will attempt to probe student thinking with regard to their use of non-Cartesian coordinate systems in answering other physically relevant questions, such as magnetic fields, as well as in other non-Cartesian coordinate systems, i.e., cylindrical coordinates.

An instructional implication arises when considering the number of times Mark activated resources that conflict with one another. This implication also has immediate relevance for upper-division physics courses. On the surface, the task of writing a position vector for a point in space in spherical coordinates may seem to be a straightforward one. Yet, when asked to write such a position vector, a high-achieving student activated numerous resources. Some of these resources were productive for the coordinate system he was working in and some resources would be productive in other coordinate systems, but were not productive in the polar and spherical coordinate systems with which Mark was working. Given this dichotomy, it’s not surprising Mark struggled to explain how these resources fit together. Mapping these varied activation patterns and interactions of resources is helpful for us as researchers as we investigate what ideas a student activates as such knowledge will inform eventual curricular development. For instance, an instructional sequence that supports students’ ideas about mathematical definitions of unit vectors and students’ physical ideas about the direction of a particular vector of interest – such as a velocity vector or a magnetic field vector – could guide students to reconcile differences in their thinking. Such tasks can be designed on models for curriculum design within a resources framework [88,89]. However, in the interim it is also worthwhile for instructors of upper-division physics courses to be aware that even some of
their best students are going to enter their courses with wide-ranging ideas regarding the essential mathematics of those courses.

2.5. Conclusion

This paper is a crucial step in a larger research study aimed at developing research-based curricular materials for an undergraduate math methods course. Towards this goal, we conducted one-on-one interviews to probe students’ thinking regarding Cartesian and non-Cartesian coordinate systems. We presented a detailed analysis of one student’s work, Mark. Mark was selected due to his thoughtfulness and articulate descriptions of his thinking. Analysis was conducted through the theoretical lens of the resources framework. This theoretical framework provides us the opportunity to develop a coherent description of Mark’s nuanced thinking, and in doing so, provides insight into analysis of additional student interviews in the future. It further provides potential paths for curriculum development based on our findings.

The analysis of Mark’s thinking revealed he activated productive resources about the definitions of non-Cartesian unit vectors. Our previous work had focused on resource clusters of similar kinds of student reasoning on unit vector questions, but Mark demonstrated few of the non-productive resources identified there [3]. We expanded our analysis to identify resources associated with questions on position and velocity vectors. When answering questions that required the use of those unit vectors in the context of position and velocity vectors, Mark activated additional resources, some of which are productive. Mark activated what we identified as the $\vec{v} \xrightarrow{\text{needs}} \hat{r}$ and $\vec{v} \xrightarrow{\text{needs}} \hat{\theta}$ resources. He didn’t activate a resource or provide evidence that he was or should consider that time derivative of polar/spherical unit vectors is non-zero. Mark reconciled these ideas by re-evaluating his productive unit vector resources and led him to redefine polar/spherical unit vectors in terms of incorrectly applied Cartesian conventions.

On subsequent questions in the interview, these resources of the redefined unit vectors were not used when Mark realized that his initially activated unit vector resources would be more convenient in answering position vector questions. Despite this realization, his responses to position vector questions included extra terms consistent with the Cartesian “pattern-matching” described by Hinrichs [57]. Mark’s responses are also consistent with the findings of Paoletti et al. [41] and Sayre and Wittman [56]; that students often have an emerging, but not complete, understanding of non-Cartesian coordinate systems. The findings herein provide the initial insights into a broader
study into students’ use of mathematics in the upper-division upon which additional investigation will continue to build.

2.6. Chapter 2 Appendix

2.6.1. Time Derivative of \( \hat{r} \)

Given polar coordinates \((r, \phi)\), the polar basis unit vectors \(\hat{r}\) and \(\hat{\phi}\) in terms of the Cartesian unit vectors \(\hat{x}\) and \(\hat{y}\) are

\[
\hat{r} = \cos \phi \hat{x} + \sin \phi \hat{y}
\]
\[
\hat{\phi} = -\sin \phi \hat{x} + \cos \phi \hat{y}
\]

When a system modeled in polar coordinates evolves in time, \(r\) and \(\phi\) are both dependent on time, \(t\). Therefore, the time derivative of \(\hat{r}\) is:

\[
\frac{d\hat{r}}{dt} = \frac{d}{dt}(\cos \phi \hat{x} + \sin \phi \hat{y})
\]
\[
= \frac{d}{dt} \cos \phi \hat{x} + \cos \phi \frac{d}{dt} \hat{x} + \frac{d}{dt} \sin \phi \hat{y} + \sin \phi \frac{d}{dt} \hat{y}
\]
\[
= \frac{d}{dt} \cos \phi \hat{x} + \frac{d}{dt} \sin \phi \hat{y}
\]
\[
= -\dot{\phi} \sin \phi \hat{x} + \dot{\phi} \cos \phi \hat{y}
\]
\[
= \dot{\phi}(-\sin \phi \hat{x} + \cos \phi \hat{y})
\]

Here we see that the right-hand side of the equation is equal to \(\dot{\phi} \hat{\phi}\). Thus,

\[
\frac{d\hat{r}}{dt} = \dot{\phi} \hat{\phi}
\]

In physics, \(\dot{\phi} = \omega\), angular speed.

APS/123-QED
3. USING A RESOURCES FRAMEWORK TO ACCOUNT FOR “PATTERN-MATCHING” RESPONSES IN THE CONTEXT OF NON-CARTESIAN COORDINATES AND UNIT VECTORS

3.1. Introduction

3.1.1. Background & Purpose

As part of a broader call to study the math-physics interface in the upper-division [22,39], our research team seeks to understand how students’ reason about mathematics concepts, both in middle-division mathematics and upper-division physics courses, and how those concepts are applied in physics contexts. The ultimate goal of this project is to create research-based instructional materials for use in a Math Methods course, often taken by students in a physics department after completing multivariable calculus and the introductory physics sequence, or other suitable upper-division physics courses. The larger project is still in the early phases of exploring how students think about the mathematics commonly used in courses such as Electromagnetic Theory, Intermediate Mechanics, Thermodynamics, and Quantum Mechanics. This paper focuses on our continued efforts to investigate student reasoning about polar, spherical and cylindrical coordinates and unit vectors using a resources framework. Throughout this paper when we use the terms “non-Cartesian coordinates” or “non-Cartesian unit vectors” we are referring to these three coordinate systems.

The inside front cover of David Griffiths’s Introduction to Electrodynamics textbook contains a table of vector derivatives in Cartesian, spherical, and cylindrical coordinates and unit vectors [1]. The first chapter(s) of many undergraduate mechanics textbooks [24, 25, 26, 27], as well as Griffiths’s EM text [1], provide explicit instruction about unit vectors and vector derivatives of unit vectors in both Cartesian and non-Cartesian coordinates. These textbooks then provide physical contexts and content that require learners to use these mathematical ideas to understand and solve physics problems. The pervasive presence of these mathematical chapters across most upper-division textbooks demonstrates an understanding among textbook authors and the broader physics
community of the importance of these mathematical concepts in the upper-division. What remains unclear, and what we aim to shed light on in this work, is how do individuals within upper-division physics courses and individuals who have completed upper-division and graduate physics courses think about these coordinate and unit vector concepts.

The first work within Physics Education Research that explored physics students’ thinking regarding unit and position vectors in spherical coordinates was by Hinrichs [57]. Hinrichs [57] developed a simple concept test to probe student thinking regarding unit and position vectors in non-Cartesian coordinate systems. The test was given to a total of 46 electromagnetic theory students at the undergraduate and graduate level at multiple universities and university types (small private and public liberal arts schools in the Midwest and a large public research university in the southwest). Hinrichs found that while both undergraduate and graduate physics students seemed to understand spherical coordinates generally, they have difficulty expressing position vectors in terms of the spherical basis unit vectors in 3D space. Specifically, there was difficulty translating the ordered triple defining the coordinates of a point in space – from \((x,y,z)\) in Cartesian and \((r,\theta,\phi)\) in spherical – to the algebraic expression of the associated position vector. A common response was “pattern-matching” spherical position vectors to Cartesian notation. In Cartesian coordinates the relationship between the ordered triple representing the coordinates of a point and the associated position vector expression is, in the notation Hinrichs uses, \((x,y,z) = x\hat{x} + y\hat{y} + z\hat{z}\). Many students wrote spherical position vectors as \((r,\theta,\phi) = r\hat{r} + \theta\hat{\theta} + \phi\hat{\phi}\), which is incorrect yet was the answer for 46% of respondents. Hinrichs also reported responses that appeared as \(5\hat{r}, \frac{\pi}{2}\hat{\theta}, 0\hat{\phi}\) or \((5\hat{r}, \frac{\pi}{2}\hat{\theta}, 0\hat{\phi})\) which he states are possibly combinations of the left and right hand sides of the equation \((x,y,z) = x\hat{x} + y\hat{y} + z\hat{z}\); i.e. that students may be thinking \((x,y,z) = x\hat{x} + y\hat{y} + z\hat{z} = (x\hat{x}, y\hat{y}, z\hat{z})\).

Hinrichs points out these responses are possibly another form of pattern-matching but that is uncertain without more explanation from the students as to their reasoning. If all three of these responses are considered pattern-matching in one form or another, they account for 66% of his respondents. Another 24% of respondents did not include unit vectors in their answers at all, with the remainder giving some type of un-categorizable response. Hinrichs makes four overarching claims: 1) very few students can write a correct position vector from the basis unit vectors in spherical coordinates, 2) the most frequent mistakes made were of the pattern-matching variety,
3) the second most common mistake was to not use unit vectors at all, and 4) the results did not depend on level of student (graduate vs. undergraduate), university, or course text.

Vega, et al. studied student thinking about polar unit vectors [3]. Data consisted of both responses to written questions and video recorded interviews of students after instruction in a Math Methods course and before instruction in an Electromagnetic Theory course at two different universities. Using a theoretical lens of resources [18,35,36,37,38], Vega et al. were able to identify clusters of resources that emerged from the interviews. These clusters represented similar (but not identical) ways of thinking about unit vectors. For example, the Motion Cluster described student thinking based on the motion of an object – seen with interview subjects using words such as ‘traveling’ and ‘moving’ – to reason about the direction of the basis unit vectors. They also identified a Coordinate Cluster consisting of student reasoning about the relationship between the coordinates of a point and the unit vectors at that point. Vega, et al. also identified a Location Dependent resource that they note is basis specific and describes how the directions of unit vectors can change with the location within a coordinate system. The paper also included examples where students appear to have the necessary productive resources to answer a question and, instead, activate other resources which are not productive in the current problem. In some cases, this caused individuals to rethink previously stated productive resources. The research team also identified a list of “expert-like” resources they believed needed to be activated to demonstrate a complete understanding of unit vectors: a unit vector is a vector; has 1-unit length; points in the direction of increasing coordinate; and is dimensionless. These were identified by the authors and not from student data.

Farlow et al. [2] performed a case study on a single student’s responses to questions about unit, position, and velocity vectors in plane polar and spherical coordinates. Utilizing the framework of resources, the researchers were able to name the student’s resources to create a vocabulary for analysis of additional interviews. The interviewed student activated productive unit vector resources such as unit vectors have unit length, basis unit vectors are orthogonal, and basis unit vectors point in the increasing coordinate direction. The student also activated a productive resource for position vectors generally, which the authors named “From origin to point.” This resource describes the behavior of position vectors beginning at the origin of their respective coordinate system and terminating at the point of interest. The authors also named two position vector resources that were activated when the case study subject was working in polar coordinates, “\( \vec{r} \) has only \( \hat{r} \)” and “\( \vec{r} \) has...
form $\mathbf{r} = a\hat{a} + b\hat{b}'$. The authors also mentioned these two resources (which may be polar-specific) could be of a different nature than other resources named in that they could themselves be resource clusters rather than an individual, fine-grain ideas. In the course of the interview, the student also demonstrated examples of rethinking previously stated productive resources in a manner similar to such described by Vega et al. [3] When such questioning of productive resources occurred, the student fell back on thinking that was consistent with Cartesian coordinate conventions. At times, the student made explicit statements about how they were trying to use Cartesian-coordinate rules to construct spherical position vectors.

The student also activated some unique resources on a question about velocity vectors, but given the focus of this present paper, it is not necessary to list all of them. However, the student’s non-activation of an expert-identified resource for polar-coordinate velocity vectors, $\frac{d\hat{r}}{dt} \neq 0$ (meaning the radial unit vector changes in time when there is azimuthal or polar motion), led them to question their previously activated and productive resources for unit and velocity vectors. The case study interview subject did not spend much time talking about position vectors, spending more time on unit and velocity vector questions, making naming of individual position vector resources from that interview difficult. However, the authors were able to identify three resources activated while considering position vectors. This Position Vector Resources group was illustrated with a resource graph [72, 73] and is reproduced here in Figure 3.1. Additionally, a discussion of the relevant mathematics — i.e. the definitions of unit vectors and the behavior of position vector expressions in both Cartesian and non-Cartesian coordinates – was presented [2].

Sayre & Wittman [56] found that the understanding of polar coordinates among intermediate-level mechanics students was still emerging. Through a think-aloud pair interview, they were also able to show examples of how such physics students don’t always stop to think which coordinate system might provide the best model for a given situation. Furthermore, even if students determine that Cartesian coordinates may not provide the most efficient model, they will at times persist in using it. Sayre & Wittman contend that this persistence is a result of their greater familiarity with the Cartesian coordinate system than with other coordinate systems.

Literature from Research in Undergraduate Mathematics Education (RUME) has also reported on student thinking in non-Cartesian coordinates, albeit to a limited degree. General findings are that in math education, the Cartesian coordinate system is the first and most commonly
taught [41]. The literature also describes that students’ mathematical meaning is often tied to the coordinate system in which a given mathematical concept is learned [40,41,42]. For example, the vertical line test for functions works in the Cartesian coordinate system but does not always work in the polar coordinate system. Yet some students will still incorrectly attempt to perform the vertical line test for a function in polar coordinates. It appears that a proportion of students have attached the meaning “passes a vertical line test” to the definition of function because such a test works in the coordinate system in which they learned about functions [40].

Reinforcing the Cartesian-centric education described by Paoletti et al. [41], Dalton, Farlow, and Christensen (in prep) reviewed seven popular Calculus textbooks, specifically the multivariable content commonly presented to students in Calculus III. The first key finding was non-Cartesian basis unit vectors are entirely absent in six of the seven books, with a single mention of polar unit vectors in one book. Furthermore, on average only about 20% of chapters of these common texts contain any non-Cartesian content, and of those chapters, only about 25% of their content is about non-Cartesian coordinate systems. Qualitatively, these texts tend to present non-Cartesian coordinate systems first from their transformations from the Cartesian coordinate system. They
also provide little-to-no opportunity for students to practice selecting which coordinate system would be most beneficial in a given context; opting instead to give explicit instructions as to which coordinate system should be used. What may result is the textbooks used in many Calculus III courses present a hidden curriculum to students that suggests non-Cartesian coordinate systems are infrequently used and helpful for solving a small subset of problems, rather than presenting them as alternate spatial models offering increased utility in contexts with high degrees of circular, spherical, or cylindrical symmetry.

This present paper expands on this prior work and further explores this pattern-matching phenomenon. Hinrichs’s term “pattern-matching” is purely descriptive and atheoretical, making no attempt to explain why students are responding in such a manner [57]. The instantiation of “pattern-matching” Hinrichs described [57] and Farlow et al. replicated [2] in the context of position vectors will be further explored here. Again, our primary interest in this work is exploring student reasoning about non-Cartesian unit vector notation. Thus, we chose to investigate this pattern-matching phenomenon with position vectors because position is an introductory-level physics concept [30,31,32]. We therefore assume that students’ understanding of position – or more importantly their lack of understanding – of position would not create a barrier to their use of unit vector-based language, as opposed to more advanced physics topics that might further impede students’ descriptions of their thinking. We do, however, provide some insight into how a few interview subjects articulate their thinking in the context of velocity and current directions of particular objects in Sec. 3.3.5.

This paper seeks to answer the following research questions:

For student reasoning on non-Cartesian coordinates and unit vectors,

1) What resources (or resources clusters) can be identified across multiple interview subjects as they answer questions requiring the use of unit vector notation?

2) To what extent can these identified resources (or resource clusters) across these our interview sample be understood within the context of our prior research and identified resources and clusters?

3) To what extent are these resources (or resource clusters) present in contexts other than position vectors?
3.1.2. Theoretical Framework

The theoretical framework that guides and informs the analysis of the data is the resources framework [18,35,36,37,38]. Within this framework, a cognitive resource refers to a content specific fine-grain idea that students unconsciously use, or sometimes don’t use, to address a question or problem posed to them. These resources may be closely associated or connected to other resources within a frame or schema that individual students have assembled and are activated when presented with some stimulus. These resources are considered to be nuggets of information or procedure that have been useful to the individual at some time in the past and are called upon when the brain thinks it might be useful or applicable. A set of locally coherent resources that the brain subconsciously or intentionally activates in response to some stimulus is referred to as the frame, i.e. the way in which the subject is framing a given situation. The resources framework affords the opportunity to dissect student reasoning into pieces that can be identified, understood, labeled and described; increasing opportunities for the development of targeted curricular materials. It’s worth noting that the precise naming of individual resources does not need to be “correct” in order to be productive to the researcher. Meaning, if the chosen nomenclature for individual resources was somehow incorrect or non-intuitive, the claims of the research are unaffected because the utility of the framework is in its ability to assign working labels to individual resources which allows the use – or lack of use – of those resources to be tracked and analyzed [37]. A resources framework was chosen as the guiding theoretical framework for this study because as the Results section will show, there were several unique combinations of ideas activated at specific times, and not activated at other times, in the responses. These results were consistent with our previous work [2,3] in which we provide more thorough explanations as to why a resources framework provided the most effective theoretical framing for the work presented in this paper and our work more broadly.

3.1.3. Operationalizing “pattern matching”

The term “pattern matching” and its derivatives are used sparsely in the PER literature [57,73,90]. The usage of “pattern matching” in Hinrichs’ work is purely descriptive; Hinrichs offers no explanation as to why students respond in a pattern-matched way, only that some of their responses in spherical coordinates are morphologically similar to correct responses in Cartesian coordinates [57]. Sabella & Redish add a qualifier and use the term “surface pattern-matching [90].”
The context is of instructors solving physics problems in front of a class with the expectation that students will form powerful knowledge structures for problem solving that can be applied to a variety of problems. However, when asked to solve problems on their own, students often mimic the modeled process instead of forming those complex knowledge structures. Thus, students are matching the pattern of the problem-solving process and possibly the morphological patterns of the written outcomes of that process. Wittman & Black appear to agree with the Sabella & Redish usage by saying that a consequence of expecting students to solve problems quickly is the possibility that students will pattern match a modeled problem-solving process [73]. However, Wittman & Black offer that as only one possibility for students’ cognitive processes in such situations; adding that students might be using a locally stored set of coherent resources instead. In each case, pattern matching is either implicitly or explicitly defined as a surface level cognitive process. In sum, the Hinrichs usage refers to responses given in a novel context that are similar in form, if not isomorphic, to responses given in a more familiar context; i.e. as in the present case, incorrectly writing polar-, spherical-, or cylindrical-coordinate position vectors in the Cartesian-like form of \( \vec{r} = a \hat{a} + b \hat{b} + c \hat{c} \). This present paper will reflect Hinrichs’ use of “pattern matching,” referring to non-Cartesian position vectors written in any form resembling the form \( \vec{r} = a \hat{a} + b \hat{b} + c \hat{c} \) (following notation used in Farlow et al. [2]).

The combination of activation of fine-grain ideas in the resources framework and pattern matching as described by Hinrichs [57] overlaps with the cognitive psychology theory of pattern recognition. Broadly speaking, pattern recognition is the cognitive process of responding to environmental stimuli and using those stimuli to build coherence in the understanding of the local environment. Such coherence includes understanding the way in which that environment operates, how objects and individuals behave within and respond to that environment, and allowing predictions to be made about future events both in that environment and in unfamiliar environments [74,75,76,77,78,79]. In the late 1950’s and early 1960’s two theories about pattern recognition emerged: template-matching [75,80,81] and feature-analysis [82,83,84,85]. Template-matching involves the brain responding to a stimulus by matching the input information to a previously existing template – or prototypical form – stored in memory. Feature-analysis involves the brain recognizing the salient features of the input stimulus. Some features can overlap, but the combination of recognized salient features in the brain allows one to recognize the stimulus. Of these two theories,
feature-analysis is the accepted theory in current cognitive psychology literature [74]. Template-matching was considered problematic shortly after its introduction as it would require memory to store a nearly infinite number of templates. Therefore, template-matching was considered not feasible [74], although it does have some uses in artificial intelligence and computer programming [81]. Feature-analysis requires the existence of some number of feature-detectors. When certain regions of the eye respond to a visual sensory input, associated neurons are activated in the brain. The set of neurons that activate for a given input are considered the feature-detectors for that input. Multiple feature-detectors can be simultaneously activated as the sensory information can activate several receptors simultaneously. The combination of activated feature-detectors forms a “neural code” for the incoming stimulus [74]. While no explicit connection has been found in our literature search, the behavior of feature-detectors and the theoretical descriptions of the resources and framing theoretical framework [18,35,36,37,38,69] appear to be quite analogous, and both are patterned after neurological models of the brain [70]. We further comment on the theoretical connections resources and framing has with pattern matching in discussion Sec. 3.4.3.

3.2. Methodology

The data presented were collected using semi-structured, think-aloud interview protocols [91,92]. Two separate but overlapping think-aloud interview protocols were developed to probe thinking about unit, position, and velocity vectors in various coordinate systems. Both sets of interviews were conducted with volunteer undergraduate and graduate students, and one faculty member, from a north-midwestern, midsize, land grant, public research university. All interview subjects were men. Four of them were Caucasian, one was Native American, one was African-American, and one was an international student from the Indian subcontinent. This sample was representative of the department demographics at the time the interviews were conducted, however we recognize that our findings are limited to those we interviewed and implications to the broader and more diverse persons who study physics are limited.

The first interview protocol explored thinking in plane polar and spherical coordinates and was the protocol used in our case study [2]. This protocol was given to four interview subjects (all names are pseudonyms): Ned, a third-year physics graduate student, Aaron, a first-year physics graduate student, and Mark and Jack, both senior undergraduate physics majors. The second protocol added questions about cylindrical coordinates, and additional physical contexts was given
Figure 3.2. First interview question designed to establish students’ initial level of understanding of polar unit vectors.

to David, a junior undergraduate physics major, Chris, a second-year physics graduate student, and Steve, a physics faculty member. The interview questions from each protocol were projected on whiteboards so interview subjects could write and/or draw their answers directly on the figures. Subjects were frequently prompted to verbalize their reasoning. All interviews were video recorded and transcribed for analysis.

The interview protocol initially included physical contexts of position and velocity. We reasoned such introductory topics would be reasonably well understood by our subjects and therefore not impose any barrier to our subjects articulating their reasoning about the target topic of coordinates and unit vectors.

Our first interview protocol initially presents interview subjects with a two-dimensional spiral with two points given arbitrary names A and B (Figure 3.2) [3]. Subjects are asked to draw the directions of the polar unit vectors \( \hat{r} \) (radial) and \( \hat{\theta} \) (azimuthal) originating at those two points, and to explain their reasoning as to why they drew them in their illustrated directions. This question will henceforth be referred to as Spiral Question A (SQA). A correct student response is shown in Figure 3.3. Part B of the Spiral Question (SQB) instructs subjects to assume the magnitude of the radial distance of the particle from the origin is \( r = R_0 - b\theta \) and the angle \( \theta = \omega t \), where \( R_0 \),
b, and ω are constants, and then asks them to identify the units of each constant and then to use those constants to write a polar position vector \( \mathbf{r} \) as a function of time. The correct position vector is \( \mathbf{r} = (R_0 - b\theta)\hat{r} \). Subjects are then asked to write a velocity vector \( \mathbf{v} \) describing the situation, but that question is beyond the scope of this paper.

The next interview question is based on a question developed by Hinrichs \[57\] and presents students with a three-dimensional Cartesian coordinate system with four points, labeled A, B, C, and D, respectively, with one point each along the positive x-, y-, and z-axes and one along the negative y-axis (Figure 3.4). For each point, subjects are asked to write a spherical polar position vector in terms of \( \hat{r}, \hat{\theta}, \) and \( \hat{\phi} \) that defines that point and to explain their reasoning as they do so. Henceforth this question will be referred to as the 3D Question (3DQ). The correct position vectors are \( \mathbf{r}_A = 4\hat{r}, \) \( \mathbf{r}_B = 3\hat{r}, \) \( \mathbf{r}_C = 3\hat{r}, \) and \( \mathbf{r}_D = 2\hat{r} \). Of note, this question differs from Hinrichs’s version of the question in two ways: 1) in Hinrichs’s question, all four points were 5 units from the origin and in our question the points have varying distances from the origin, 2) Hinrichs did not explicitly instruct students to use the spherical basis unit vectors as our question does. Our reasoning for the first modification was to maintain the possibility of at least two points having symbolically
identical position vector expressions to follow-up on any cognitive conflict that might arise from that. We also wanted to include points at different distances so we had more opportunity to probe the thinking behind constructing position vector expressions rather than introducing the possibility that they could all be written using the same reasoning for all points. The reason for the second modification was to be ensure that subjects were using unit-vector notation, which Hinrichs’s results show students do not always do.

The second interview protocol was designed to see if pattern-matching would occur in cylindrical-coordinate contexts and to gain additional insight into why pattern-matching occurred on the 3DQ of the first interview protocol. It also includes questions with additional physical contexts, specifically magnetic fields surrounding a long, straight, current-carrying wire and radial-only velocity to explore whether additional physical context would help students reason about the unit vectors within the vector expressions modeling such situations. The Cylindrical coordinate system is a hybrid of polar and Cartesian coordinates, being polar in the circular cross section and Cartesian along the length. If pattern-matching of position vectors persisted in cylindrical coordinates, it would appear as azimuthal terms tied to the terms in the expressions intended to model the circular
The first two questions of the protocol, the Cylindrical Question 1 (CQ1) (Figure 3.5) and Cylindrical Question 2 (CQ2) (Figure 3.6), were designed to explore this idea. The third question was the 3DQ from the first protocol, (Figure 3.4). The fourth question uses radial-only motion in spherical coordinates to see if pattern-matching would occur with velocity vectors when objects were moving away from a given point in space, in the context of an exploding planet, the Planet Krypton Question (PKQ) (Figure 3.7).

A correct answer for the CQ2 requires recognition of the cylindrical symmetry of the magnetic field surrounding a long, straight, current-carrying wire, which is still an introductory physics topic [30,31,32]. In that question, the magnetic field is cylindrically symmetric around the wire and pointing in the direction counterclockwise around the circle created by the field with the wire at the center as viewed from above. Therefore, at both points G & H, the magnetic field vector is the magnitude of the field (2.5T, as given in the problem) multiplied by the azimuthal unit vector for cylindrical coordinates \( \hat{\theta} \); i.e. \( \vec{B}_G = 2.5T \hat{\theta} \) and \( \vec{B}_H = 2.5T \hat{\theta} \). For the PKQ, subjects are expected to realize that the velocity vectors at these points have a magnitude equal to the initial speed of the pieces divided by the square of the distance away from the center times the radial unit vector, \( \hat{r} \). So
A very long cylindrical wire whose axis runs along the z-axis carries a current I. At points G and H on Figure [shown above], which are equal distances from the axis, the magnitude of the magnetic field is measured to be 2.5 T. Write down the magnetic field \( \vec{B} \) in terms of the unit vectors \( \hat{\mathbf{r}}, \hat{\mathbf{\theta}}, \hat{\mathbf{z}} \) and indicate it using a vector on the diagram at both points G and H. Please explain your reasoning for why this is the direction.

Figure 3.6. Second problem from the second interview protocol, referred to as Cylindrical Question 2 (CQ2).

The planet Krypton has exploded as a result of a nuclear chain reaction caused by the planet’s unstable radioactive core. Parts of planet are flying away from the former center of the planet with an initial speed of 400 m/s and decrease at a rate of \( K/r^2 \) where \( K \) is a constant. For each of the points \( A, B, \) and \( C \), in the figure below, write the velocity vector \( \vec{v} \) of a planet fragment at that location in terms of the unit vectors \( \hat{\mathbf{r}}, \hat{\mathbf{\theta}}, \hat{\mathbf{\phi}} \). Please explain your reasoning for why this is the velocity vector.

Figure 3.7. Fourth problem from the second interview protocol, referred to as The Planet Krypton Question (PKQ).
Table 3.1. All responses to SQA from the first set of interviews.

<table>
<thead>
<tr>
<th>Subject</th>
<th>Initial Response(s)</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aaron</td>
<td>$\vec{r} = r\hat{r} + \theta \hat{\theta}$</td>
<td></td>
</tr>
</tbody>
</table>
| Jack    | $\vec{r}_A = x\hat{r} \sin \theta$  
           | $\vec{r}_B = y\hat{r} \sin \theta$ | The coefficients x and y are magnitudes of distances from pole to points A and B, respectively |
| Mark    | $\vec{r} = (R_0 - b\theta)\hat{r} + \theta \hat{\theta}$ | Corrected to $\vec{r} = (R_0 - b\theta)\hat{r}$ |
| Ned     | $\vec{r} = (R_0 - b\theta)\hat{r}$ | Questioned if $\theta \hat{\theta}$ terms was necessary |

while all the pieces have different magnitudes of velocity, all velocity vectors point in the increasing radial direction, and only in that direction, hence the unit vector $\hat{r}$ is sufficient.

3.3. Results & Analysis

The results and analysis will be intertwined and separated by significant claims of our investigation. By looking across our interview data we provide four central claims: 1) evidence across subjects supports existing resources and resource clusters identified in our prior work [2,3], 2) a more coherent resource cluster is identified that connects to our previously identified $\vec{r}$ has form $\vec{r} = a\hat{a} + b\hat{b} + c\hat{c}$ resource, 3) a heretofore un-identified resource that subjects use to build a position vector by “navigating” from the origin to a point, and 4) activated resources in more complicated physical scenarios mirror those activated for position vectors. After the evidence has been laid out and the analysis is described, additional synthesis will occur in Section 3.4.

Subject responses to SQA, 3DQ, CQ1, CQ2, and PKQ are shown in Tables 3.1-5, respectively. Tables 3.4 and 3.5 are presented in Sec 3.3.5. Also, any blockquotes from an interview subject will be labeled with a preceding $X#$ where $X$ is the first initial of the speaker’s assigned pseudonym and $# is the statement number assigned for this paper.

3.3.1. Evidence of Previously Identified Unit Vector Resources and Resource Clusters Across Interview Subjects

Three different questions in our two interview protocols explicitly probed thinking about unit vectors. On these questions, Mark, Ned, and Steve were able to draw unit vectors in their correct directions for each point of interest on the questions they were asked to do so – SQA for Mark and Ned and CQ1 and 3DQ for Steve. Chris did so on CQ1 and got 2 of 3 correct on 3DQ, drawing the polar angle in the opposite direction for point A on 3DQ. Mark’s response to SQA was used as the example of a correct answer to SQA in Figure 3.3. Chris’s responses to the unit vector
Table 3.2. All responses to 3DQ from both interviews.

<table>
<thead>
<tr>
<th>Subject</th>
<th>Initial Response(s)</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Aaron</td>
<td>( \vec{r}_A = 4\hat{r} + \frac{\pi}{2}\hat{\theta} + \frac{\pi}{2}\hat{\phi} ) &lt;br&gt;( \vec{r}_B = 3\hat{r} + 0\hat{\theta} + \frac{\pi}{2}\hat{\phi} ) &lt;br&gt;( \vec{r}_C = 3\hat{r} + \pi\hat{\theta} + 0\hat{\phi} ) &lt;br&gt;( \vec{r}_D ) no answer given</td>
<td>Used his recall of Cartesian-to-spherical basis translations to derive the template: &lt;br&gt;( \vec{r}_p = r_p \cos \theta \sin \phi \hat{r} + r_p \sin \phi \sin \theta \hat{\theta} + r_p \cos \phi \hat{\phi} )</td>
</tr>
<tr>
<td>Chris</td>
<td>( \vec{r}_A = r_A\hat{\phi} ) &lt;br&gt;( \vec{r}_B ) no answer given &lt;br&gt;( \vec{r}_C = r_C\hat{r} ) &lt;br&gt;( \vec{r}_D ) no answer given</td>
<td>While explaining these answers, Jack realizes if ( \hat{r} ) points radially toward point, the ( \hat{\theta} ) and ( \hat{\phi} ) terms would not be necessary, but rejects this notion and continues to explain why all three terms should be used.</td>
</tr>
<tr>
<td>David</td>
<td>( \vec{r}_D = 2\hat{r} + 0\hat{\theta} + 0\hat{\phi} ) &lt;br&gt;No other answers given</td>
<td>Immediately wrote ( \vec{r} = _\hat{r} + _\hat{\theta} + _\hat{\phi} ) where underscores represent blank spaces as a template for all 4 points, only answered for point D.</td>
</tr>
<tr>
<td>Jack</td>
<td>( \vec{r}_A = 4\hat{r} + \sin 0\hat{\theta} + \sin 90\hat{\phi} ) &lt;br&gt;( \vec{r}_B = 3\hat{r} + \sin \frac{3\pi}{2}\hat{\theta} + \sin \frac{\pi}{2}\hat{\phi} ) &lt;br&gt;( \vec{r}_C = 3\hat{r} + \sin 0\hat{\theta} + \sin 0\hat{\phi} ) &lt;br&gt;( \vec{r}_D = 2\hat{r} + \sin \pi\hat{\theta} + \sin \frac{\pi}{2}\hat{\phi} )</td>
<td>Used the physics convention of ( \hat{\phi} ) as the azimuthal angle and ( \hat{\theta} ) as the polar angle.</td>
</tr>
<tr>
<td>Mark</td>
<td>( \vec{r}_A = 4\hat{r} + \frac{\pi}{2}\hat{\theta} ) &lt;br&gt;( \vec{r}_B = 3\hat{r} ) &lt;br&gt;( \vec{r}_C = 3\hat{r} + \frac{\pi}{2}\hat{\theta} ) &lt;br&gt;( \vec{r}_D = 2\hat{r} + \frac{3\pi}{2}\hat{\theta} + \frac{\pi}{2}\hat{\phi} )</td>
<td>Extra terms exist due to “completeness”</td>
</tr>
<tr>
<td>Ned</td>
<td>( \vec{r}_A = 4\hat{s} + \frac{\pi}{2}\hat{\phi} ) &lt;br&gt;( \vec{r}_B = 3\hat{s} + \frac{\pi}{2}\hat{\phi} ) &lt;br&gt;( \vec{r}_C = 3\hat{s} + 0\hat{\theta} + 0\hat{\phi} ) &lt;br&gt;( \vec{r}_D = 2\hat{s} + \frac{3\pi}{2}\hat{\theta} + \frac{\pi}{2}\hat{\phi} )</td>
<td>( \hat{s} ) was used as the radial unit vector ( \hat{r} ). Eventually Steve realized that the radial term was all that is necessary, wrote ( \vec{r} = 4\hat{s} ), and stated that would be the answer for all points.</td>
</tr>
<tr>
<td>Steve</td>
<td>( \vec{r}_A = 4\hat{s} + \frac{\pi}{2}\hat{\phi} ) &lt;br&gt;( \vec{r}_B = 3\hat{s} + \frac{\pi}{2}\hat{\phi} ) &lt;br&gt;No other answers given</td>
<td></td>
</tr>
</tbody>
</table>

Table 3.3. All responses to CQ1 from second interview. In cylindrical coordinates \( \hat{s} \) is radial unit vector (analogue of \( \hat{r} \) in polar coordinates).

<table>
<thead>
<tr>
<th>Name</th>
<th>Initial Response(s)</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chris</td>
<td>( \vec{r}_A = r_A \cos \theta \hat{s} + r_A \sin \theta \hat{\phi} ) &lt;br&gt;No other answers given</td>
<td>Used his recall of Cartesian-to-cylindrical basis transformations to derive the template: &lt;br&gt;( \vec{r}_1 = r_1 \cos \theta \hat{s} + r_1 \sin \theta \hat{\theta} + r_1 \hat{z} )</td>
</tr>
<tr>
<td>David</td>
<td>Point A: ( 3r = \sin 90 - \theta \hat{s} + 0\theta + 0\hat{\phi} ) &lt;br&gt;No answer given for point B &lt;br&gt;Point C: ( r = \sin [_] \hat{s} + _\hat{\theta} + 6\hat{z} )</td>
<td>Immediately wrote ( \vec{r} = _\hat{s} + _\hat{\theta} + _\hat{z} ) where the underscores are blank spaces as a template for all points.</td>
</tr>
<tr>
<td>Steve</td>
<td>( \vec{r}_A = 3\hat{s} - \frac{\pi}{2}\hat{\theta} + 0\hat{\phi} ) &lt;br&gt;( \vec{r}_B = 3\hat{s} + \frac{3\pi}{2}\hat{\theta} + 4\hat{z} ) &lt;br&gt;No response given for point C</td>
<td></td>
</tr>
</tbody>
</table>
portion of CQ1 are shown in Figure 3.8. On CQ1, Steve’s response was similar to Chris’s response. Arrows drawn by Mark, Ned, Chris, and Steve are consistent with our previously identified unit vector resource *Increasing coordinate direction* [2]. Furthermore, all four provided verbal evidence of activation of this *Increasing coordinate direction* resource. Mark’s statements were described in our case study analysis [2]. Some other example statements include Ned’s response to the interviewer asking him why he drew the \( \hat{r} \) vector pointing to the left (radially outward) at point A for SQA:

**N1:** “From my understanding the radius hat vector – the unit vector for the radius, ah, since we’re dealing with – what? – polar coordinates here? The radius is just going to be from the origin to wherever our point is and so our unit vector is just going to be along that direction of the radius.”

Here, Ned’s language indicates that he understands the \( \hat{r} \) unit vector points in the same direction as the radius vector and that radius vectors point from the origin to the point of interest. Similarly, Steve explains his reasoning for drawing the directions of the cylindrical unit vectors for point C in CQ1 (same directions as can be seen for Chris in Figure 3.8) with a clear mapping of the unit vectors to each coordinate’s direction:

**S1:** “So this vector; unit length into the y-direction and this is the \( \hat{s} \) vector. And \( \theta \)...the unit vector would be...would be tangential to this point [point C] – will be rather difficult for me to draw [interviewer asks “Into the board?”]...Into the board. And this \( z \), would be in this direction, up.”

Of note, Steve says the \( \hat{s} \) unit vector points in the y-direction, and at point C, the y-direction does point in the radial direction. Given that Steve was able to consistently illustrate the directions of the \( \hat{s} \) unit vectors at all the requested points, we do not interpret this comment as a conflation of the \( \hat{s} \) unit vector with the y-direction generally, but as Steve using the y-direction to give an explanation of the \( \hat{s} \) direction consistent with the projected figure.

Aaron, David, and Jack responded to some of those same unit vector questions in slightly different ways than Chris, Mark, Ned, and Steve. In answering SQA, Jack drew the radial unit vectors pointing along the spiral path and the theta unit vectors pointing towards the origin at both points A and B (see Figure 3.9). On the same question, Aaron drew circles around the origin in a counterclockwise direction and drew an arrowhead pointing along one of those circles also in the counter-clockwise direction. While doing so, he said:

**A1:** “The direction is just anti-clockwise direction. So we can point our vector in the anti-clockwise direction.”
Figure 3.8. Chris’s unit vector sketch for CQ1. Evident in this sketch is Chris’s activation of the *Increasing coordinate direction* resource for cylindrical unit vectors. Also, in this sketch are position vector illustrations for 3 of the 4 points, all drawn under the influence of the activation of the *From origin to point* position vector resource. Also note, Steve’s unit vector sketches on this question were essentially identical to Chris’s.

Figure 3.9. Jack’s response to the unit vector portion of SQA. Here, Jack does not draw the unit vectors’ directions in their mathematically correct directions but does draw them pointing in different directions at the different points. Jack, Aaron, and David all answered this or similar questions in a manner that suggested they understand the directions of non-Cartesian basis unit vectors can change with a point’s location but did show evidence of activation of the more expert-like resource *Increasing coordinate direction*. We interpret responses of this type as activating the *location-dependent* resource described in Vega et al. [3].
Aaron had previously drawn $\hat{r}$ unit vectors starting at the pole and pointing in the same directions as position vectors from the pole to the two points. Thus, we interpret Aaron’s circles sketch and statement A1 to mean that the $\hat{\theta}$ unit vector will point along that circle all the way around that circle. David initially struggled to construct a magnetic field vector expression on CQ2, so the interviewer asked him to draw the directions of the cylindrical unit vectors at points G and H. David didn’t actually sketch them but verbally responded with:

**D1:** “[unit vectors] should always be the same [direction] as the vector you’re talking about – in this case magnetic field.”

Later on 3DQ, David had drawn the position vector for point A correctly and was subsequently asked to draw the directions of the spherical unit vectors at point A. At point A, he drew all three spherical unit vectors pointing radially outward parallel to his position vector sketch. He verbally elaborated his reasoning with the statement:

**D2:** “they’re all in the same direction, aren’t they?”

The interviewer followed up by asking David to draw the spherical unit vectors at point C. David again drew all three spherical unit vectors at point C parallel to each other but pointing radially outward and stated:

**D3:** “They’re all in the same direction again.”

The interviewer asked another follow up question about whether the spherical unit vector would also all point in the same directions at points B and D. David responded that they would. This does indicate an as-yet-unseen version of what we would identify as the *Location Dependent* resource.

Aaron’s, David’s, and Jack’s unit vector responses are similar to Chris’s, Mark’s, Ned’s, and Steve’s in that they indicate activation of a resource that unit vector directions are dependent on the locations of the points of interest. The responses between the two groups differ in that Chris, Mark, Ned, and Steve consistently are clear that the location dependence is specifically in the direction of increasing coordinate. The responses of Aaron, David, and Jack are consistent with the *Location Dependent* resource for non-Cartesian unit vectors described in Vega et al. [3]. In the case study analysis [2] we described *Increasing coordinate direction* as a refinement of the *Location Dependent* resource. After analyzing all the interviews, we can now more clearly articulate that there may in fact be several “location dependent resources.” and, as such, we should consider a recategorizing of the *Location Dependent* resource as a *Location Dependent Cluster* of resources.
Whether such resources are productive can be context dependent, just as Vega et al. had previously identified. For example, a resource of *Increasing coordinate direction* is productive in all coordinate systems – making its classification as “location-dependent” dependent on coordinate system. In Cartesian coordinates, the basis unit vectors, $\hat{x}$, $\hat{y}$, $\hat{z}$, point in the increasing $x$, $y$, and $z$, directions, respectively regardless of location within the Cartesian coordinate system. However, the *Location Dependent Cluster* could be productive in non-Cartesian coordinates where unit vector directions change with location, although Vega et al. [3] offered some instances of it being used unproductively. The *Location Dependent Cluster* may in fact be irrelevant in Cartesian coordinates because $\hat{x}$, $\hat{y}$, $\hat{z}$ directions do not change with location. Thus, it appears from the complete set of this interview data that the *Location Dependent Cluster* is productive – the directions of non-Cartesian unit vectors are location dependent – but not sufficient for producing a mathematically correct answer to questions about the directions of unit vectors in non-Cartesian coordinates. Additionally, it appears that activating the resource of *Increasing coordinate direction* and the *Location Dependent Cluster* together does lead to mathematically correct answers to such questions.

Our case study [2] and initial unit vector analysis [3] identified other resources that apply to unit vector thinking, such as *unit vectors are dimensionless* and *unit vectors are orthogonal*. The responses to unit vector questions so far also show surface-level corroboration that these are resources that can be activated by subjects answering unit vector questions: Steve’s statement S1 specifies the $\hat{s}$ unit vector as being of unit length, and all unit vector sketches shown (Mark, Figure 3.3, Chris Figure 3.8) illustrating unit vectors being orthogonal to each other. As will be explained in more detail in Section 3.4, it is the directions of unit vectors that are of most interest when discussing pattern-matched responses.

### 3.3.2. Further Evidence of the $\vec{r}$ has form $\vec{r} = a\hat{a} + b\hat{b}$ Resource

As mentioned in our methodology, a correct answer for 3DQ is a position vector expression that has only an $\hat{r}$ term. The subject interview results on 3DQ are striking (see Table 3.2). Every individual added additional terms to their position vector at some point in their interview. Through careful examination of Mark’s data, we’ve previously identified Mark as having a $\vec{r}$ has form $\vec{r} = a\hat{a} + b\hat{b}$ resource (which would include a “$+c\hat{c}$” term for coordinate systems with a third dimension); meaning that he seems to possess a generic form into which he places the information about the coordinates with its associated unit vector. Answers consistent with this type of resource are seen
by Aaron and Mark on SQA (Table 3.1), Aaron, David, Jack, Mark, Ned and Steve on 3DQ (Table 3.2), and Chris, David and Steve on CQ1 (Table 3.3).

In Mark’s responses to 3DQ, some answers did not have \( \hat{\theta} \) or \( \hat{\phi} \) terms. Mark reasoned that the magnitudes of the missing terms were zero and unnecessary to write down:

\[ M1: \text{“Well, I mean there is a dependence per se, right? I mean you could write all these out and just say that [missing } \theta \text{ term] is you know plus 0 times } \theta \text{ but uh, for B, just the way it, and I think you’re, basically when I think you’re using this you’re free to define } \phi \text{ equals 0 to be anywhere, but I’ve sorta defined it to be like, ya know, you’re measuring it with respect to } \hat{\phi} \text{ equals 0 to be anywhere, but I’ve sorta defined it to be like, ya know, you’re measuring it with respect to [points to } +x\text{-axis] sort of, } \hat{\phi} \text{ yeah, } \phi \text{ would be like you measure it with respect to } +x \text{ so } \hat{\phi} \text{ equals 0 line”} \]

indicating the unit vectors of \( \hat{\phi} \) and \( \hat{\theta} \) are not unnecessary, but that their magnitude is simply zero for several positions. Ned described his answer to 3DQ as being “complete” with the additional \( \hat{\theta} \) and \( \hat{\phi} \) terms even when their magnitudes are zero. When asked why he included azimuthal and polar terms in his position vector expression for point C on the 3DQ, Ned replied with:

\[ N2: \text{“Completeness, in all honesty. Because I know } \vec{r}_C \text{ has no dependence on these two angles } \theta \text{ and } \phi \text{ but, I wanted to put } [+0\hat{\theta} + 0\hat{\phi}] \text{ down there just so I knew that they were still part of it even though they did nothing of it, or contributed nothing, I should say.”} \]

It should be noted that it is not mathematically incorrect to include specifically “\( +0\hat{\theta} + 0\hat{\phi} \)” in every spherical position vector expression because adding zero to the \( \hat{r} \) term does not change the value or meaning of the expression. However, in statement N2, Ned was referring to his answer for \( \vec{r}_C \) only and, as can be seen in Table 3.2, many of his position vector expressions included nonzero “magnitudes” in the \( \hat{\theta} \) and \( \hat{\phi} \) terms. Such responses together with his “completeness” statement suggest that Ned may also be using a similar resource as Mark’s \( \vec{r} \text{ has form } \vec{r} = a\hat{a} + b\hat{b} \) resource. Based on our previous definitions of specific resources and resource clusters [2, 3], it may be more appropriate to refer to \( \vec{r} \text{ has form } \vec{r} = a\hat{a} + b\hat{b} \) as a resource cluster. This particular resource or resource cluster seems to be activated by many subjects across several problems, though very few subjects give identical answers. This indicates to us that subjects may have different ideas of what this particular resource means, or how it is used, or perhaps some subjects are activating their version of this resource with additional resources, leading to a variety of answers. (This classification is still under the consideration of the authors.)

Given that understanding of mathematical concepts is often tied to the coordinate systems in which they are learned [40, 41, 42], and the primacy of Cartesian coordinates in mathematics education [41], it is not surprising that our interview subjects demonstrate a propensity to follow
the Cartesian model for writing position vectors. In Cartesian coordinates, each of the spatial dimensions is represented in a position vector expression. The Cartesian point (1, 2, 3) has an associated position vector of \( \vec{r} = 1\hat{x} + 2\hat{y} + 3\hat{z} \) where the unit vector of each coordinate has the magnitude of the respective coordinate.

Additional examples of calling on information from Cartesian resources for assembling a position vector with non-Cartesian coordinates and unit vectors come from Aaron and Jack. When writing down the position vector for SQA, Aaron drew what can be seen in Figure 3.10. In that figure he had written \( r\hat{r} \) as labeled by the 1 in Figure 3.10. Then he said:

**A2:** “OK, one thing I can do – with Cartesian coordinates we know we can find the – |while sketching what can be seen in Figure 3.10 and labeled with 2| x this is y – so we can just add \( \theta \) with it like \( b\theta \).”
Aaron then wrote “+θ̂θ” (labeled 3 in Figure 3.10). It appears he was creating a polar position vector using ideas from how he would create a Cartesian position vector. He adds the r and θ terms in the same way the x and y terms are added in Cartesian coordinates. This construction also seems to be a clear use of the ř has form ř = aâ + b̂b resource. Another example can be seen in Jack’s interview when answering 3DQ. The interviewer asked Jack to define a unit vector. As part of responding to that question, Jack explained why it is a problem for spherical unit vectors to be directionally location-dependent and invoked the behavior of Cartesian unit vectors in so doing:

J1: “...But ř would have to be pointed, given a coordinate system you’d need to define which direction the unit vectors would be pointing in initially and then make your measurements based off that.”

J2: “You just need to define ř – and you know someone could define it like I defined it to be pointing one unit out this way [in +y] and then kinda work with that.”

Interviewer: “So in this diagram here you’re saying that ř is always pointing one direction in the y direction – or unit in the y direction?”

J3: “Yes.”

Interviewer: “So even for points B, C, and D that is the case?”

J4: “Yeah. What the hell, I’m not going to try to contradict myself on that one.”

J5: “So I was thinking in Cartesian coordinates for a second and that’s where the discrepancy might be coming up but I’m imagining how similar to how this was in Cartesian coordinates.” [emphasis added]

This exchange reveals two key things: 1) Jack is trying to define ř statically (Statement J2) and then use that static direction as the direction to “work with” – which is how Cartesian unit vectors work – 2) Jack explicitly calls upon Cartesian coordinates as the foundation for his thinking (Statement J5). This exchange, in combination with the responses that Jack gave to SQB and 3DQ that include azimuthal and/or polar terms, suggest that Jack too is activating something similar to the ř has form ř = aâ + b̂b – the Cartesian model for position vector expressions. It’s possible that another framework could be helpful in explaining what the subjects are doing in this situation. Sherin’s symbolic forms [93] is connected with the resources framework, and it identifies equations as generic forms where specific information is added to them. Future analysis of these data using symbolic forms may prove fruitful.
3.3.3. “Navigation” Resource: a Way of Writing the Position Vector at a Point

Through our analysis of the video data, it became apparent that our interview subjects were using finger-pointing and other hand gestures while constructing their position vector expressions or used them to augment their explanations of those expressions. These gestures often followed the establishment of a reference point on the projected image of the relevant question figure – such as saying the $x$-axis is the zero for the azimuthal angle – followed by pointing or sweeping gestures that were frequently explicitly linked to the writing or defense of the magnitude of a term in a position vector expression. Scherr [94] wrote an essay outlining how gestures can be used for analyzing what a student might be thinking in a given context. Scherr posits that gestures either provide additional insight into students’ thinking beyond their written and verbal responses or, in some cases, even communicate essential information that a subject may not be able to explicitly verbalize. In our interview data, gestures were frequently present and provide additional insight into our subjects’ understandings of the mathematical concepts probed. Two examples of gesture sequences – both from 3DQ – will be discussed here.

Steve used gestures while he constructed his position vector for point A on 3DQ. He traced the $y$-axes with his finger from the origin to the point and then wrote “$\vec{r}_A = 4\hat{r}$” (See left image of Figure 3.11). Then, starting from the $x$-axis he swept out an arc from the $x$-axis to the $y$-axis (see right image of Figure 3.11) and then wrote “$+\frac{\pi}{2}\hat{\phi}$” to represent the radial term in his position vector. Jack used gestures to explain his position vector for point B on 3DQ, $\vec{r}_B = 3\hat{r} + \sin\frac{3\pi}{2}\hat{\theta} + \sin\frac{\pi}{2}\hat{\phi}$, as follows:

**J6:** “...3 units out...” [explaining radial term]

**J7:** “...and going around so 270 degrees in the $\theta$ direction [while making gesture depicted in left image of Figure 3.12 in the $x$-$y$ plane]”

**J8:** “...and then 90 degrees straight down [while making gesture depicted in the right image of Figure 3.12 from positive $z$-axis down to $x$-$y$ plane].”

(The coefficients for the $\hat{\theta}$ and $\hat{\phi}$ terms in Jack’s position vector are somewhat unique in our dataset in their inclusion of sines or cosines, we will not address them in the present paper.) In each case, it appears that the interview subjects were using gestures to support their construction of the position vector from the origin to a specific point. Across all interviews, there were frequent and
Figure 3.11. On 3DQ, Steve makes this radial gesture (left image) by tracing his finger along the $y$-axis and writes $\vec{r}_A = 4\hat{r}$. Then, he makes an “arc sweeping” gesture (right image) from the $x$-axis to the $y$-axis and writes “$+\frac{\pi}{2}\hat{\phi}$” to construct the position vector $\vec{r}_A = 4\hat{r} + \frac{\pi}{2}\hat{\phi}$.

direct connections made between which term of the expressions was being constructed or defended and which gesture was being made, as Steve’s construction and Jack’s explanation show.

The strategy of using the coordinates of a point as the magnitude of the corresponding position vector terms is appropriate in Cartesian unit vectors. When writing position vectors in non-Cartesian coordinates and unit vectors, the same strategy does not work. The radial term is all that is necessary in polar and spherical coordinates as it already communicates the distance the point is from the origin and the direction one would have to travel to reach that point from the origin (a $\hat{z}$ term is still necessary in cylindrical coordinates). Our subjects’ gesturing also either explicitly or implicitly used the origin as the starting point for position vectors, which we interpret as evidence for the activation of the From origin to point resource described in our case study [2].

These data support a particular way of thinking about the construction of algebraic position vector expressions. The gestures and accompanied statements could be interpreted as a procedural resource that we call the “Navigation” resource. This resource involves instructions for traveling a path from the origin/pole of a coordinate system to the point of interest. This way of thinking is productive in both Cartesian-coordinate contexts and in non-Cartesian-coordinate contexts insofar as it provides a roadmap for getting to the exact location of a point. However, this way of thinking is
not productive for constructing or interpreting position vector expressions in non-Cartesian coordinates. As discussed in Section 3.3.2, The Cartesian point (1, 2, 3) has an associated position vector of \( \mathbf{r} = 1\hat{x} + 2\hat{y} + 3\hat{z} \). In Cartesian coordinates this expression can be productively interpreted as a navigation map for writing the position vector whereby the vector starts at the origin and travels 1 unit in the positive \( x \)-direction, then moves perpendicularly and travels 2 units in the \( y \)-direction, and then again moves perpendicularly 3 units in the positive \( z \)-direction to end at the point (1, 2, 3). The data seem to indicate that subjects use a Navigation resource to inform their thinking about the same concepts in non-Cartesian coordinates. This Navigation resource also supports a notion that a position vector will model a unique position, something that we identify later as a possible point of confusion for subjects (see Discussion in Sec. 3.4.1).

### 3.3.4. Non-Cartesian Unit Vectors’ Relation to Cartesian Unit Vectors

As mentioned earlier, Mark, Ned, Chris, and Steve were all able to productively define non-Cartesian basis unit vectors. However their productive activation of resources such as Increasing coordinate direction did not predict activation of productive resources during further application of unit vectors while writing position vectors. For example, on 3DQ Chris had drawn what can be seen in Figure 3.13 (he drew \( \hat{\theta} \) in opposite direction of its correct direction), where the position
vector and unit vector point in the same direction. The interviewer then asked him to use his finger to point in the direction of the position vector at point A, and Chris pointed in the azimuthal (\(\hat{\phi}\)) direction. When asked to resolve the discrepancy between the directions of his correctly drawn position vector arrow and the direction he pointed with his finger, Chris replied with:

**C1:** “I would argue that since the \(\hat{r}\) direction itself has a dependence on the \(\phi\) angle itself...”

**C2:** “[in spherical coordinates] it’s one of those weird leaps of logic to where the \(\hat{r}\) vector itself is – should be written as a function of \(\theta\) and \(\phi\).”

Here Chris has correctly drawn the position vector and radial unit vectors pointing in the same direction, but decides to include \(\theta\) and \(\phi\) dependence in his answer. Also of interest, Chris productively recognizes that angular information is embedded into the definition of \(\hat{r}\), yet Chris struggles to productively activate resources about the consequences of those embedded dependencies. Of note in Chris’s responses generally, is that he consistently attempted to use his memory of the translations of the non-Cartesian bases from the Cartesian basis to develop a vector-expression-template that he could use for every point for a given question (see the Comment columns of Tables 3.2, 3.3, and 3.5). Chris clarifies that he was attempting to derive a template model while answering the PKQ by saying:
**C3:** “My original hope was that I could get a general definition for cylindrical coordinates such that I could easily apply that, so what I would do is I would do the exact same analysis [as previous questions] and build from there.”

These data are similar to how Mark attempted to use such translations from Cartesian basis unit vectors to polar basis unit vectors which led to our naming a “\( \hat{r} \text{ depends on } \hat{x} \text{ and } \hat{y} \)” resource in our case study [2]. Similar to how Mark activated \( \hat{r} \text{ depends on } \hat{r} \text{ and } \hat{\theta} \) to justify why \( \hat{r} \text{ alone wasn’t sufficient for describing a position vector to a point.} \) It’s possible that had the interview subjects in these instances been given the appropriate definition of \( \hat{r} \) in non-Cartesian coordinates from Cartesian unit vectors, the subjects might have been able to reconcile their conflicts, however the protocol did not provide that information to the subjects. It may be an important concept to target in instructional materials.

### 3.3.5. Activation of Resources in Contexts Other than Position Vectors

The first interview protocol was designed with the reasoning that the physical concepts of position and velocity would be understood well enough that the understanding of physics concepts would not confound our ability to investigate student use of unit vectors. After the results of that first round of interviews, our second protocol included questions in a richer physical context. We added CQ2 and PKQ to the second protocol to see if additional physical context would impact subjects’ reasoning with non-Cartesian vector expressions. Responses for CQ2 can be seen in Table 3.4. One can determine the direction of the magnetic field due to a current carrying wire using the “Right Hand Rule,” pointing the thumb of one’s right hand in the direction of the current and then curling fingers of the hand to give a general sense of the direction of the magnetic field around that current carrying wire. The three subjects interviewed with this question (Chris, David, and Steve) all used the Right Hand Rule to determine the magnetic field direction. Similar to subjects’ who can identify the direction of a position vector to a point as being in the \( \hat{r} \) direction, knowing the direction of the magnetic field was insufficient for subjects to write the correct magnetic field vector expression for points G and H (See Table 3.4).

Chris used the Right Hand Rule and then wrote an expression that included \( \hat{s} \) and \( \hat{\theta} \) terms. His answer for \( \vec{B}_G \) does not include a \( \hat{\theta} \) term because his magnitudes for the magnetic field included a sin0 term and for the point G, that term gave him a zero quantity (like Mark’s answers to 3DQ above – see Statement M1). The answer mirrors his answer for the position vector at a point in
Table 3.4. Responses to CQ2 from the second set of interviews.

<table>
<thead>
<tr>
<th>Name</th>
<th>Initial Response(s)</th>
<th>Comment</th>
</tr>
</thead>
</table>
| Chris | $\vec{B}_G = B\hat{s}$  
$\vec{B}_H = B \cos \theta \hat{s} + B \sin \theta \hat{\theta}$ | After some thinking, the answer became $\vec{B} = 2.5T\hat{s}$ for both points. |
| David | No responses given |         |
| Steve | $\vec{B}_H = 2.5T\hat{s} + \frac{\pi}{2}\hat{\theta} + 0\hat{z}$  
No response given for point H |         |

CQ1 (See Table 3.3). We identify this as Chris activating a $\vec{r}$ has form $\vec{r} = a\hat{a} + b\hat{b}$ resource. Chris’s activation of this resource in a context that does not involve a position vector also introduces the possibility that this resource may need a different symbol than “$\vec{r}$” to account for a more general form for vectors; the “$\vec{r}$” symbol is typically used for position in physics contexts. It’s possible that having just answered CQ1, and activating the position vector resource, that Chris’s framing of this task was similar and thus CQ1 had primed his thinking. Had CQ2 come first, perhaps a subject would more easily activate a productive resource like Increasing Coordinate Direction. Steve, the lone faculty member, initially responded with both an $\hat{s}$ and a $\hat{z}$ term on CQ2, but upon further questioning was able to successfully reason that the azimuthal term was all that was necessary. Upon this realization, Steve said:

S3: “Yeah, because if my first argument was correct than it should be the same for point H, yes.”

S4: “From a physical point of view it’s obvious. Magnetic field is a physical quantity that points into this direction [gestures in azimuthal direction], the one that I indicated by $\theta$. It can be at any point in space you can say which direction the magnetic field points.”

While providing his answer, Steve showed a great deal of non-verbal and verbal dissatisfaction with his response. His shoulders were slumped, and his words regularly trailed off as he spoke. We believe this was due to Steve activating resources like those he used for 3DQ that were in conflict with his unit vector resources. This is in stark contrast to his cadence later in the interview where he answers succinctly and confidently (see Sec. 3.3.3). David used the Right Hand Rule but was unable to write an expression for the magnetic field.

For the PKQ (responses in Table 3.5), the radial only motion might have helped subjects see that only an $\hat{r}$ term was necessary in the velocity vector expression. For the PKQ, David used the same strategy he employed in the 3DQ and CQ1 (See Tables 3.2 and 3.3) of writing $\vec{v} = _\hat{r} + _\hat{\theta} + _\hat{\phi}$ where the underscores are blank spaces as a template and then filled in the $\hat{r}$ term with
Table 3.5. Responses to PKQ from second set of interviews.

<table>
<thead>
<tr>
<th>Name</th>
<th>Initial Response(s)</th>
<th>Comment</th>
</tr>
</thead>
<tbody>
<tr>
<td>Chris</td>
<td>$\vec{v}_A = v_A \hat{r} + 0$</td>
<td>Used same conversion template equation as he did in 3DQ. Thus, sin0 and cos90 terms appeared, resulting in the written &quot;0's&quot;</td>
</tr>
<tr>
<td>David</td>
<td>$\vec{v}_A = 5r\hat{r} + _\hat{\theta} + _\hat{\phi}$ $\vec{v}_B = 3r\hat{r} + _\hat{\theta} + _\hat{\phi}$ $\vec{v}_C = 2r\hat{r} + _\hat{\theta} + _\hat{\phi}$</td>
<td>Again, initially wrote $\vec{v} = _\hat{r} + _\hat{\theta} + _\hat{\phi}$ as a template, but only filled in coefficients for the raidual terms.</td>
</tr>
<tr>
<td>Steve</td>
<td>$\vec{v} = 400(\text{m/s})\hat{s}$</td>
<td>Again used $\hat{s}$ for radial term. Said this equation would be answer for all points.</td>
</tr>
</tbody>
</table>

the distance the point was from the origin (See Table 3.5). We identify this as a similar resource to Mark’s $\vec{r}$ has form $\vec{r} = a\hat{a} + b\hat{b}$ resource. Chris also initially gave an answer consistent with his answer for CQ1 (See Table 3.3). Again, we identify this as activating the $\vec{r}$ has form $\vec{r} = a\hat{a} + b\hat{b}$ resource. His answers did lead to single-term expressions for the velocity vector but he included additional terms, at times. Steve gave a correct answer to the PKQ almost immediately. He seems to be activating essential unit vector resources, like Increasing coordinate direction and Location Dependent to arrive at an expression that is satisfactory to him. He further stated that that same expression would work for all the other points and did not seem bothered by the lack of uniqueness of his velocity vector directions.

In our second interview sample it is clear that the additional physical context provided by CQ2 and PKQ did not help David or Chris activate productive resources to help them construct correct vector expressions. However, it is possible that the additional physical context did so for Steve, the faculty member. Following Statement S4, Steve’s thinking seems to have shifted. This is not to say that David and Chris did not possess the physical knowledge necessary to answer the question. Their use of the Right Hand Rule indicates that they did, but there appears to be resources for the connections between the physical behaviors of the magnetic field/velocity vectors and the mathematical behaviors of the non-Cartesian unit vectors that they did not show evidence of activation.

3.4. Discussion

3.4.1. All Points Should Have a Symbolically Unique Position Vector Expression

It’s worthwhile to consider why the resource $\vec{r}$ has form $\vec{r} = a\hat{a} + b\hat{b}$ is so prevalent in our data. One of the intents of the interview protocol was to elicit any potential cognitive conflict that
might arise from vector expressions that model different points being symbolically identical and then ask subjects to explain their thinking while in such conflict. For example, the correct position vector expressions for points B and C on 3DQ are both \( \vec{r} = 3\hat{r} \). Similar symbolic non-uniqueness appears for the correct magnetic field vectors for points G and H on CQ2, and in the polar terms of the position vector expressions for all points in CQ1. However, the interview subjects’ responses and the thinking behind those responses described in this paper demonstrated that constructing symbolically identical expressions for different points is not a trivial task. Therefore, we were unable to directly probe any cognitive conflict that might arise in such situations. However, an expectation that all points in any coordinate system have a symbolically unique position vector expression to model them is likely a logical extension of \( \vec{r} \text{ has form } \vec{r} = a\hat{a} + b\hat{b} \) and the Navigation resource.

In Cartesian coordinates, mapping the coordinates for a point to a vector expression to that point would result in each point having a symbolically unique expression as no two points have identical coordinates. The same is not true in non-Cartesian coordinates. Given the manifold examples of how our interview subjects applied resources productive in Cartesian coordinates to non-Cartesian situations, it’s possible that there is an expectation that each point in space be symbolically unique. A future protocol could be developed to elicit resources about symbolic uniqueness to further probe this specific concept.

Subjects regularly included unnecessary azimuthal and/or polar terms in the position vector expressions along with appropriate radial terms. Admittedly, the notation is confusing. In 3DQ, \( \vec{r}_B = 3\hat{r} \) and \( \vec{r}_C = 3\hat{r} \), but because points B and C are on different radial rays with respect to the origin, \( \vec{r}_B \) and \( \vec{r}_C \) point in different directions, meaning \( \vec{r}_B \neq \vec{r}_C \) despite the right hand sides of the two position vector expressions being symbolically identical. The two \( 3\hat{r} \) expressions are unique because the respective \( \hat{r} \) unit vectors are not the same. A consequence of non-Cartesian basis unit vectors being location-dependent is that those basis unit vectors are not directionally static in the way that Cartesian basis unit vectors are. Recall that the absolute directions of increasing \( x \), \( y \) and \( z \) do not change regardless of where a point is in space. By contrast, in spherical coordinates, regardless of where a point is, \( \hat{r} \) will point radially away from the origin. At a different point, \( \hat{r} \) will still point radially away from the origin, which means the directions of all \( \hat{r} \) unit vectors within that coordinate system are not static when moving from one point to another point on a different radial ray. Thus, seeing any non-Cartesian unit vector without any additional information is directionally
ambiguous. Yet, even with the additional spatial information that our interview protocol question figures provide, our subjects included extra spatial information in their position vector expressions. While this extra information is likely expected because Cartesian coordinates communicates that information algebraically, the lack of understanding about the behavior of non-Cartesian basis unit vectors may be a barrier to our subjects’ ability to see why their answers were mathematically incorrect. We hypothesize that subjects may be activating some kind of resource associated with the uniqueness of vectors. If subjects activated a “Unique Vector” resource writing position vectors that look non-unique, might create conflict for them. While this claim is largely conjecture, as we don’t have explicit evidence of its use in our data, it is helpful in explaining the prevalence of the $\vec{r}$ has form $\vec{r} = a\hat{a} + b\hat{b}$ responses. Additionally, addressing such an issue through targeted instruction may prove worthwhile in supporting student activation of productive resources.

3.4.2. Emergent Resource Development Evident

In both the presence and absence of activating productive resources, there is evidence of emergent student understanding of vector concepts in our data. For instance, in Jack’s interview, the interviewer asked him what a unit vector is. Jack replied:

J9: “It’s a vector that is usually defined as one unit long that points in the direction of its given component.”

Statement J9 suggests that despite not being able to correctly draw the correct directions of $\hat{r}$ and $\hat{\theta}$ in SQA, Jack is able to activate the Increasing coordinate direction resource for unit vectors, although it’s unclear if this is an idea Jack could productively activate in non-Cartesian coordinate systems. Jack continues:

J10: “We’re doing spherical um, so the $\hat{r}$ would just be pointing, it would only be one, but it would be pointing in the direction of our given point. Uh, no! No. Then why the [expletive deleted] would you need these [$\theta$ and $\phi$ terms]?”

J11: “Um, no so $\vec{r}$ would be pointing in, uh. Ooohh, that is a good one.”

J12: “If $\hat{r}$ is already pointing directly in that direction then why in the hell would you need all this [$\theta$ and $\phi$]? And like spherical would be amazing.”

In this sequence, Jack recognizes that the correct behavior of $\hat{r}$ makes the angular terms unnecessary and seems to understand the powerful affordances of non-Cartesian unit vectors. However, he does not appropriately apply this idea to position vectors, and instead returns to his other activated resources. The data also show several examples of interview subjects drawing unit vectors in the
proper directions or activating productive resources that indicate understanding of the definitions of the non-Cartesian unit vectors, but then not productively applying those definitions to position vectors, or other vectors. For instance, Chris in statements C1 and C2 and Figure 3.13 where he pointed in the $\hat{\phi}$ direction when asked to point in the $\vec{r}$ direction despite the fact that he had correctly drawn $\hat{r}$ in the same direction as $\vec{r}$. Chris also uses the Right Hand Rule on CQ2 after having just activated the Increasing coordinate direction resource on CQ1 – ostensibly creating a scenario where he could see that the magnetic field around the wire and the $\hat{\theta}$ unit vector point in the same direction but, instead, he activates a resource closely related to $\vec{r}$ has form $\vec{r} = a\hat{a} + b\hat{b}$ resource for the magnetic field vector. Mark and Ned also drew the polar unit vectors in the correct directions on SQA and appropriately explained their behavior (statement N1, also see Farlow et al. [2]). These data suggest that being able to activate productive resources for the definitions of unit vectors is not sufficient to apply unit vector concepts to other vector types. It is possible that there are procedural resources that are also necessary to activate so that a resource such as Increasing coordinate direction would help one realize that the radial term is all that’s necessary for a position vector in polar coordinates. Perhaps activities that elicit our identified unit vector resources and guide students to reconcile when and where different unit vectors would be appropriate might help them frame situations in a way that allows them to activate coordinate system-appropriate resources in other problems.

3.4.3. Comments on the Definition of Pattern Matching

In Section 3.1.3 we presented the two definitions of the term “pattern matching” found in PER literature. One was the Hinrichs definition of using the form of a position vector in Cartesian coordinates and writing spherical position vector expressions to look morphologically similar to the Cartesian case [57]. Our interview data suggests that this definition is quite reasonable as our subjects consistently wrote position vector expressions in polar, spherical, and cylindrical coordinates that were morphologically similar to the same in Cartesian coordinates. The second definition of “pattern matching” discussed was the more procedural pattern matching described by Sabella & Redish [90] in which a problem-solving process forms a pattern that students will attempt to match when solving a problem in a novel context. Our data also strongly suggests that this form of pattern matching happens as well. Our subjects interpreting position vector expressions as directions for traveling a path from origin to point (i.e., the Navigation resource) would be an example of how
a procedural pattern that works in Cartesian coordinates is being “matched” when they construct non-Cartesian position vector expressions.

However, we want to draw attention to the affordances of our theoretical framework provides over these two definitions of the term pattern matching. We claim that the theoretical framework of resources provides a clear theoretical underpinning that explains why subject responses may be characterized as pattern matching. Namely, subjects activating resources that are correct in certain contexts in problems where they aren’t productive. Identifying the specific resource that is being activated gives a clear target for instruction whereas identifying some sense of pattern matching does not.

3.4.4. Connections to Prior Literature

Montiel and various colleagues [40,42] and Paoletti et al.’s [41] findings that understanding of mathematical concepts is tied to the coordinate system in which those concepts are learned, in conjunction with the fact that mathematics instruction is dominated by Cartesian coordinates [41] (also Dalton et al., in prep) provide context to the data that we see in this paper. Learning concepts of coordinates and associated vector concepts in Cartesian coordinates provides a strong support for why we see so many Cartesian-like resources in our subjects’ responses to non-Cartesian questions. Then, when students transition into working in a different coordinate system where coordinate and unit vectors are cued, activation of Cartesian coordinate and unit vector resources are activated.

The activated resource or resource cluster of $\vec{r}$ has form $\vec{r} = a\hat{a} + b\hat{b}$ and Navigation resource are examples of applying resources that work in Cartesian coordinates to non-Cartesian situations. This process as described is consistent with the field of cognitive psychology’s descriptions of pattern-recognition as was summarized in Sec. 3.1.3 [74,75,76,77,78,79]. Our data suggest that students’ first attempts to answer questions with non-Cartesian coordinates and unit vectors will likely activate resources that are appropriate for Cartesian coordinates and unit vector resources are activated.

Our interview subjects activated many resources for unit vectors that would be productive for constructing position vector expressions in non-Cartesian coordinates. The activation of a Location Dependent Cluster for non-Cartesian unit vectors and Increasing coordinate direction for either Cartesian or non-Cartesian unit vectors appear to be activated by most subjects. However, our data suggest that activating productive resources for the definitions of non-Cartesian basis unit vectors is not sufficient for constructing correct position vectors or other vector expressions. Pervasive across
our subjects was the activation of the $\vec{r}$ has form $\vec{r} = a\hat{a} + b\hat{b}$ resource which is possibly paired with a Navigation resource. We hypothesize that the activation of a Unique Vector resource may be influential in guiding subjects to utilize the $\vec{r}$ has form $\vec{r} = a\hat{a} + b\hat{b}$ resource. These resources do not appear to be exclusive to unit vectors and position vectors, as responses given to questions for the magnetic field of a current carrying wire also seemed to trigger a resource that is the same as or very similar to $\vec{r}$ has form $\vec{r} = a\hat{a} + b\hat{b}$.

3.4.5. Limitations

There are two primary limitations to this work: 1) there is not a well-established methodology for naming resources within the resources framework, 2) this analysis is based primarily on the interviews of seven, male subjects and may not be generalizable to all populations of people in upper-division physics courses. Limitation 1 is addressed by our attempts to articulate how we name resources and resource clusters here and in our previous work [2,3]. We hope that these papers will eventually inform the authoring of a methodology paper whereby we might more clearly articulate a step-by-step methodology. However, all of the resources named herein describe mathematical concepts that are in alignment with accepted mathematical definitions. Limitation 2 can be addressed in the future with additional investigation with more diverse populations which could uncover additional resources and ways of thinking about non-Cartesian coordinates and unit vectors and should be explored.

3.5. Conclusions

As part of a broader effort to develop a research-based curriculum for an undergraduate math methods course for physics majors, the phenomenon of “pattern-matching” by Hinrichs [57] was explored through an analysis of think-aloud interviews from the theoretical perspective of the resources framework. Unit-vector notation is a common method of solving problems of a particular geometric symmetry in upper-division physics courses [1, 24, 25, 26, 27, 28]. We find activation of resources that are productive for Cartesian coordinate systems for problems that require non-Cartesian is not surprising given that mathematical meaning is often tied to the coordinate system in which a mathematical concept is first learned [40, 41, 42] and the emergent understanding students have of non-Cartesian coordinate systems such as polar coordinates [41, 56]. Physics students have little training with non-Cartesian coordinate systems in general [41] and frequently no training with non-Cartesian unit vectors within their Calculus courses prior to entering upper-division physics.
courses (Dalton et al., in prep), which accounts for a lack of resources specific to non-Cartesian coordinates and unit vectors.

Our larger interview sample showed a great deal of overlap with our previously identified resources and resource clusters. We found that interview subjects frequently activated resources that are productive within Cartesian coordinates and applied those ideas to non-Cartesian contexts. This cross-coordinate activation was evidenced by the persistent activation the \( \vec{r} \) has form \( \vec{r} = a\hat{a} + b\hat{b} \) resource and the Navigation resource. The former seems to be interpreted by the subjects in our interview sample as the canonical model for position vector expressions in all coordinate systems. The latter appears to be a procedural resource for how vector expressions can be constructed. Both of these resources appear to be informed by how vectors can be constructed in Cartesian coordinates.

The limited physical context of position vectors provided an opportunity to investigate emergent student and faculty resources pertaining to Cartesian and non-Cartesian coordinate and unit vectors with little influence of emergent physics understanding, but there is evidence that similar resources were activated in other physical contexts as well.

A primary utility of the resources framework is the direction it provides for developing research-based curriculum. From our analysis, we identify three key areas for targeted instruction: 1) instruction guiding students to consider how the unit vectors in non-Cartesian coordinate systems relate to the Cartesian unit vectors, 2) instruction guiding students to appreciate the affordances of non-Cartesian unit vectors in for writing simplified expressions, and 3) instruction considering the latent information and hence, the non-uniqueness of some vectors, contained within the notation of \( \hat{r} \) and \( \hat{\theta} \), that can be further supported by considering instruction focus 1).
4. MULTIVARIABLE CALCULUS TEXTBOOK REPRESENTATION OF NON-CARTESIAN COORDINATES: A MISALIGNMENT BETWEEN MULTIVARIABLE CALCULUS TEXTBOOK CONTENT AND UPPER-DIVISION PHYSICS APPLICATION

4.1. Introduction
4.1.1. Motivation and Literature Review

The ability to communicate mathematically and quantitatively are fundamental skills important to success in physics. In fact, students’ success in undergraduate physics courses has been generally correlated with their prior mathematical knowledge and performance [12,13,14,15,16,17]. Such skills become even more important in upper-division physics courses, such as Classical Mechanics, Quantum Mechanics, Electromagnetic Theory, and Thermodynamics, where the mathematics necessary for these courses is much more involved and complex than introductory level coursework [1,24,25,26,27] and mathematical models and physical principles inform each other and students’ understanding of physics in more interwoven ways [18,95,96]. As such, physics students often struggle with the mathematics essential to upper-division coursework [20,21]. Moreover, the manner in which the language of mathematics is used in mathematics courses and in physics courses differ in ways that have been characterized as the two disciplines speaking different dialects of the same language, or even speaking two different languages [18]. While communities of physicists and mathematicians might use the same word when referring to a particular mathematical entity, they may struggle to understand the full definition and implications of the other group’s use of the word. This myriad of factors informs research efforts to consider students’ understanding and employment of mathematics at the upper-division level in physics. Therefore, within both a broader call for additional educational research to be conducted at the upper-division in physics [39] and a larger effort to study students’ use and understanding of math in the upper-division [22], an effort is underway to develop a research-based curriculum for a mathematical methods course for undergraduate
physics majors. This paper focuses on a portion of that effort by investigating Calculus textbooks for the prevalence of non-Cartesian coordinates and unit vector content.

Many upper-division physics textbooks contain explicit instruction on unit vectors and unit vector notation in both Cartesian and non-Cartesian coordinate systems, particularly in the disciplines of electromagnetic theory [1] and classical mechanics [24, 25, 26, 27]. These mathematical tools are later used within those textbooks to describe physical phenomena such as position, velocity, forces, electric and magnetic fields, etc., in both Cartesian and non-Cartesian coordinates with and without the associated unit vectors. The presence of this mathematical instruction early in many upper-division textbooks conveys an understanding by the authors that students need to have an ability to use these tools later in the textbook.

Literature in math contexts have suggested that, in general, student understanding of non-Cartesian coordinate systems is weak. Montiel, Vidakovic, and colleagues have performed several studies specifically on student understanding of functions in polar coordinates, which demonstrated that student understanding of functions and other mathematical concepts are often tied to the coordinate system in which they were initially taught, Cartesian [40, 42]. Moore, Paoletti, and Musgrave [97] studied mathematics students who had previously taken mathematics through Calculus III, and observed difficulties with polar coordinates, specifically student difficulties with the ordered pair $(r, \theta)$, which reads (dependent, independent) in comparison to the Cartesian ordered pair $(x, y)$ which reads (independent, dependent), when there are functional dependencies for each. Further, Moore et al. claimed that when students had difficulty with polar coordinate systems, they would return to employing ideas from Cartesian coordinate systems to their thinking about polar systems.

In physics, Sayre and Wittmann [56] studied students in their junior level Classical Mechanics course and found that students’ understanding of the polar coordinate system was much more plastic, meaning under formation and flexible, than their understanding of Cartesian coordinate systems. Their results demonstrate that students often start a problem presuming a Cartesian coordinate system, and furthermore, persist in using the Cartesian coordinate system even when a polar coordinate system serves the question better. Hinrichs [57] surveyed upper-division undergraduate and graduate physics students at four different institutions asking them to write algebraic expressions for position vectors in spherical coordinates for four points all a distance of 5 units from the origin. The correct answer for all points is $\vec{r} = 5\hat{r}$. Nearly half of respondents gave an
answer in the Cartesian-imitating form $\vec{r} = 5\hat{r} + \theta\hat{\theta} + \phi\hat{\phi}$. Hinrichs referred to this type of response as “pattern-matching.” Farlow, Vega, Loverude, and Christensen [2] replicated Hinrichs’s results through think-aloud interviews on a similar question. All of their interview subjects gave similar pattern-matched responses to questions about polar and spherical position vector expressions. The case study subject in their study showed examples of activating resources consistent with Cartesian coordinates while constructing such expressions.

Work by Pepper, Chasteen, Pollock and Perkins [55, 98] has tied student difficulties with certain mathematical concepts and techniques to their difficulties in upper-division Electricity and Magnetism. Manogue, Browne, Dray, and Edwards [99] has done similar work to Pepper et al., specifically focusing on student difficulties with Ampere’s Law, and how mathematical background informs that particular physical application. Additionally, Schermerhorn & Thompson [100, 101] have done work exploring some of the challenges students face while thinking about differential elements in non-Cartesian coordinates and developed some instructional materials that used vector calculus concepts to help mitigate some of the challenges students face within that domain in the context of Electromagnetic theory.

Work on student thinking about non-Cartesian coordinate systems within the research-based Math Methods project’s development initially involved analysis of written responses to questions about unit and position vectors in polar coordinates from students in a Math Methods course and an upper-division Electricity and Magnetism course [3]. To further understand student thinking about unit and position vectors in polar coordinates, Vega and colleagues conducted think-aloud interviews with upper-division undergraduate physics students, and a smaller number of graduate students [2, 3]. Vega et al. [3] found that student thinking regarding basis unit vectors in polar coordinates was often conflated with or informed by the motion of an object rather than the mathematical rules for basis unit vectors. Farlow et al. [2] performed a case-study analysis on a single student, identifying and organizing the student’s resources into groups thematically based on the question the student was answering when that resource activated. That case study subject demonstrated multiple examples of disagreement with their previously stated and productively activated ideas and demonstrated a still emerging understanding of the mathematical consequences of non-Cartesian basis unit vectors’ directions being location-dependent.
In summary, there has been a body of work, most of which comes from the introductory level, with a few studies at the upper-division, linking success in physics courses to students' mathematical ability and background, both in general and with particular topics and techniques. Further, there has been work that shows that both math and physics students struggle with concepts in non-Cartesian coordinate systems and often employing methodology or resources tied to Cartesian coordinates, which may be due to these concepts being introduced to them in Cartesian coordinates [2].

4.1.2. Research Questions

We have identified a need to further explore the instruction students receive regarding the different types of coordinate systems and their associated vector concepts before they reach upper-division physics courses. Upper-division physics textbooks demonstrate the mathematical concepts and practices expected of physics students, and the literature review describes how well physics students can, or where they struggle to, apply these mathematical skills. The work presented in this paper sets out to both quantitatively and qualitatively characterize Calculus textbooks that are presumed to convey ideas regarding Cartesian, polar, spherical, and cylindrical coordinates and some of their associated vector concepts; content which the above literature review establishes as necessary knowledge and skills for success in upper-division physics courses. An analysis of the multivariable calculus content (typically presented to students in Calculus III) of several popular calculus textbooks was thus conducted guided by the following questions:

1) Compared to Cartesian coordinates, to what extent are non-Cartesian coordinates represented in Multivariable Calculus textbooks?

2) What are the natures of the presentation and application of the various coordinate systems?

3) To what extent are mathematical topics/concepts (i.e. unit vectors, multiple integration, etc) presented in different coordinate systems? How are these topics represented or addressed within various coordinate systems?

The first question quantitatively considers how much multivariable textbook content overall is based in each coordinate system. The second question qualitatively seeks to determine the skills assumed and/or required of certain coordinate systems. The third question examines the initial and general presentation of individual mathematical concepts or topics and their relationship to different coor-
dinate systems, especially those that have been demonstrated to be important mathematical tools for upper division physics coursework.

More generally, this textbook analysis hopes to better describe the mathematical background students entering upper-division physics classes will have regarding coordinate systems. Doing so helps determine the “starting point” for curricular materials in a physics math methods course – meeting students “where they are” – while upper-division physics textbook content represents a facet of the “end point” – getting the students to “where they need to be”.

4.2. Methodology

Seven Multivariable Calculus textbooks [4, 5, 102, 103, 104, 105, 106, 107] (see Table 4.1) were selected due to their prevalence in Multivariable Calculus classrooms. The textbooks were selected based on their popularity among Calculus III textbooks according to a best sellers list on Amazon (during the summer of 2017). Subsequently, these textbooks were analyzed using content analysis techniques [108]. Also, Boas’s Mathematical Methods in the Physical Sciences textbook [4], a commonly used Mathematical Methods in Physics textbook, was analyzed, specifically to serve as a reference of what a Mathematical Methods in Physics course might require with respect to non-Cartesian coordinate systems. The Boas textbook was not used when calculating means and other statistics across textbooks because it serves as a reference of comparison, and its content and purpose are fundamentally different from the other textbooks.

These textbooks were analyzed using content analysis techniques [108]. Several qualitative aspects were examined across the textbooks. To determine where and how non-Cartesian content was presented, both the content and the structure were examined. The structure of the content was examined to identify the particular chapters/topics which employed non-Cartesian coordinates and then to examine how those topics/chapters presented non-Cartesian coordinate systems, in particular, in relation to Cartesian coordinate systems.

To identify the topics where non-Cartesian coordinate systems surfaced, each book was initially reviewed to determine whether there were instances of non-Cartesian coordinate systems with any significance within a chapter. Significance includes any instance where a non-Cartesian coordinate system was explicitly stated, or where the variables associated with a non-Cartesian coordinate system were used. If non-Cartesian content was present, that chapter was tagged for a more in-depth qualitative and quantitative review.
After this initial round of analysis, it was clear that answering the research questions above would require two slightly different analyses. The two different question types – quantitative and qualitative – require different units of analysis. Quantitative questions primarily require a “chapter-based” unit of analysis: in order to determine the proportions of chapters and subsections with non-Cartesian content it was necessary to determine the number of sections/chapters that contained such content out of the total numbers of sections/chapters. The qualitative questions are aimed toward determining the nature of the presentation of content, necessitating a “topic-based” unit of analysis. Both of these approaches will be further described in the next two paragraphs.

For chapter-based analysis, once a chapter was tagged as having non-Cartesian content, an analysis of where and to what extent non-Cartesian coordinates exist within a chapter was completed, specifically studying when non-Cartesian coordinates are introduced, and how the particular behavior and characteristics unique to those coordinate systems was presented. Investigating these characteristics helps work toward a better understanding of textbook representation of non-Cartesian coordinate systems. Results reported in this paper on the proportions of content from each coordinate system came from this chapter-based analysis. This chapter-based analysis also served as a starting point to discern the nature and type of problems in non-Cartesian coordinates mathematics students are expected to solve.
Analyzing the data by topics allowed the specific concepts to stand out more clearly since they might occur within the same chapter in one book, but in different chapters, or not at all, for another textbook. Also, denoting an entire chapter as “non-Cartesian” can be misrepresentative – chapters contain many different topics and subsections, and it varies whether each topic within a given chapter utilizes non-Cartesian coordinates. By distinguishing the data by topic rather than chapters, it is also easier to determine whether particular topics were introduced in Cartesian or non-Cartesian coordinates first, and whether the topics were introduced in distinct sections or in the same section. Moore et al., [97] reported that a mathematical concept may continue to carry some associations from the coordinate system in which it was introduced, therefore, it was noted if a concept/topic was introduced in non-Cartesian coordinates before, coinciding with, or after its introduction in Cartesian coordinate systems (in distinct subsections) – to see in what coordinate systems the foundation of mathematical knowledge of certain topics is introduced. Results reported in this paper on the nature and order of presentation of content primarily come from this topic-based analysis.

Quantitatively, content analysis methods were employed to investigate the prevalence of different coordinate systems. The aim of this quantitative analysis was to determine the proportions of content associated with each coordinate system. Three different types of items were coded: 1) definitions, theorems, and properties; 2) worked example problems; 3) exercises.

Definitions, theorems, and properties that were boxed separately from the main body text were examined, as these are items that textbook authors are attempting to make the most salient. Additionally, example problems were coded, as these examples model procedural approaches to both students and instructors. Finally, textbook problems were coded, because students will be most likely to actually work with these problems.

Each item’s content was evaluated to determine what coordinate system it employed across multiple characteristics. A coding scheme was designed addressing five different characteristics with regard to these particular items. Three of the characteristics were very explicit and were applicable to all items. The last two characteristics were item-specific, and the last item required a slightly higher level of interpretation than the other items.

First, the notation used was assessed: specifically, attempting to understand and record what coordinates were used within each item. If $x$, $y$, and/or $z$ was used, Cartesian notation was marked.
If \( r \) and \( \theta \) were used in 2D problems, polar notation was marked. If \( r, \theta, \) and/or \( z \) were used in 3D problems, cylindrical notation was marked. If \( \rho, \theta, \) and/or \( \phi \), were used, spherical notation was marked.

Next, any explicit mention of a coordinate system within an item is recorded. Specifically, an explicit mention might take the form of a particular definition or property within specific coordinate systems, or instructions to use specific coordinate systems. Some problems specifically requested students to decide between using particular coordinate systems. These problems were recorded separately.

Third, the coordinate system denoted by the notation of any item’s accompanying figure(s) is recorded. No code was made for items without figures.

The fourth characteristic was specifically for worked example problems, and focused on the coordinate system used in the presented solution. If there was only a transformation from one coordinate system within the example, and the second coordinate system was used for the rest of the problem, the second coordinate system would be recorded, since it was the one predominantly used. However, if two coordinate systems were both used beyond a straightforward conversion or transformation, it would be recorded that multiple coordinate systems were used.

The fifth characteristic was for exercises and problems for students to work themselves. It centers around the idea of “cue-ing.” Items can have particular cues that can imply to a student that a particular coordinate system is of better use to solve a problem with. As mentioned earlier, cues requires more interpretation from the coder. The cues that were marked were only those that were very obvious, specifically including forms of the equation for a circle, cylinder, and sphere in Cartesian coordinates, or the words “circle,” “cylinder,” “sphere,” or “cone,” in Cartesian coordinates. If there was explicit mention of a coordinate system to use, cues would not be recorded.

These five characteristics were taken cumulatively when measuring the prevalence of a particular system. Additionally, each characteristic was individually examined to determine more specific implications. Taken together, these five characteristics are meant to be a simple but complete categorization of the ways in which a coordinate system could be represented within a textbook item. The first and the third characteristics specifically target the prevalence of using particular variables. The second and fifth characteristics focus on the level of the problem, and whether the student has to demonstrate a greater understanding of how to apply different coordinate systems rather than
requiring instruction to follow a particular coordinate system. The fourth characteristic targets the specific instructions the textbooks give for students to follow.

After the coding scheme was developed and revised, two of the authors coded a subset of the textbook sections independently. The interrater reliability for that sample was 96%. This high level of interrater reliability assured that a single author could code the remaining sections.

In order to avoid coding all the examples, problems, and graphics from every chapter, several textbooks were compared, and it was determined that many chapters of the textbooks didn’t include non-Cartesian coordinate systems at all. Accordingly, after examining two textbooks, Stewart’s [106] and Rogawski & Adams’s [5], only the chapters including non-Cartesian coordinate systems were chosen, which included Vector Geometry, Vector Functions, and Multiple Integration.

4.3. Results

4.3.1. Representation of Topics in Non-Cartesian Coordinate Systems

Proportional results for inclusion of Cartesian and non-Cartesian problems by topic are shown in Figure 4.1. Note that “Not included” refers to the topics that were not referred to in a textbook at all. Also of note is that many of these concepts arose within the same chapters – the common chapter themes were variants of Coordinate Geometry, Vector Geometry, Vector Calculus, and Multiple Integration.

Figure 4.2 illustrates the percentage of chapters of books which include any instance of non-Cartesian coordinates. Across the multivariable calculus textbooks we assessed, on average, 78.7% of chapters contained Cartesian-only content. Boas, the Mathematical Methods in Physics textbook we assessed had 20% of chapters, or 3 out of 15, that contained Cartesian-only content.

4.3.2. Prevalence of Different Coordinate Systems in Chapters that Include Non-Cartesian Coordinates

Figure 4.3 illustrates the proportion of coordinate systems across textbook items. Note that the graph in Figure 4.3 is not fully representative of each textbook as a whole, as only the chapters that included non-Cartesian coordinate systems were coded qualitatively. Therefore, Figure 4.3 must also be read with an understanding of Figure 4.2, which denotes which chapters do not include any instance of non-Cartesian coordinates. The coverage between Calculus books and the Math Methods book varied significantly, for example, only 1 out of Thomas’s 11 chapters include non-Cartesian coordinates, whereas, in the case of the Math Methods textbook, 12 out of Boas’s
Figure 4.1. Total proportions of coordinate system content inclusion, by topic, across all reviewed textbooks

Figure 4.2. On average over 75% of the chapters in the analyzed calculus books contained only Cartesian-coordinate content. *The Boas Math Methods [4] textbook was included in this chart for comparison and is not used in the data analysis.
Figure 4.3. Of the roughly 25% of textbook chapters that contained any non-Cartesian-coordinate content, the bulk of the content was still Cartesian in nature. Additionally, only one textbook provided any opportunity for students to determine which coordinate system was most advantageous for a given situation.

15 chapters include non-Cartesian coordinates. Additionally, Figure 4.3 provides direct empirical answers for Research Question 1.

By coding category, across the chapters with non-Cartesian content in the three Multivariable Calculus books coded, on average, 81.7% of notation of coded content used was Cartesian, 83.7% of figures used favored Cartesian, and 79.4% of example solutions were Cartesian, 20.7% of instructions stated to use Cartesian coordinates, and 24.7% of problems were coded for using strong cues. These numbers do not include the notation, figures, and solutions which used both Cartesian and non-Cartesian content.

4.3.3. Qualitative Trends

There were a few overarching trends also observable in the data. Among all the Calculus textbooks, mathematical topics (with the exception of Vector Fields) are introduced in separate subsections from the Cartesian discussion of the topic, and are universally introduced later. The Mathematical Methods in Physics book, Boas, follows similar trends of Cartesian coordinates preceding non-Cartesian coordinates, but incorporates non-Cartesian coordinates more organically within subsections and chapters. A second trend observed in the data showed that within all the calculus books, there were 28 subsections with explicit language denoting non-Cartesian coordinate systems or a technique for them, and none explicitly denoting Cartesian coordinate systems.
4.4. Discussion

4.4.1. Prevalence of Non-Cartesian Coordinate Material

The key takeaway from this work is the overwhelming prevalence of Cartesian coordinates in the Calculus textbooks. Across the three books coded, on average, 81.7% of notation of coded content used Cartesian, 82.5% of figures used Cartesian, and 80.5% of example solutions were in Cartesian. The notation and the figures serve to contribute toward a Cartesian-centric presentation of content and the overall coded content establishes Cartesian coordinates as the default coordinate system. This trend was also observed in worked example solutions, which likely suggests to readers that problems are most often expected to be solved in Cartesian coordinates. The relatively small number of instructions and problem cues in Cartesian, 20.7% and 24.7% respectively, further establish Cartesian as the default system, by establishing non-Cartesian coordinate systems as needing specific instructions or cues to employ. In summary, across all seven books, approximately 20% of the chapters contain non-Cartesian content as determined by simple presence or absence. Of those 20% of chapters, on average roughly 26% of that content is non-Cartesian for the three books we analyzed on an item-by-item basis.

The Cartesian-centric nature of these textbooks is inconsistent with the expectations of upper division physics course content, which require students to employ their understandings of symmetry in particular situations to choose the most useful or convenient coordinates and unit vectors for each situation [4, 5, 102, 103, 104, 105, 106, 107].

4.4.2. Presentation of Non-Cartesian Coordinate Material

Not only was the prevalence of Cartesian coordinates demonstrative of the Cartesian-centric nature of these multivariable calculus textbooks, the lack of presentation of non-Cartesian coordinate material further de-emphasized non-Cartesian coordinate systems while prioritizing Cartesian coordinate systems.

Within the Calculus textbooks, non-Cartesian coordinates do not appear until seven or more chapters into each book. This presentation means, of course, that there are seven or more chapters in each book that cannot include non-Cartesian content at all, which strongly affects the prevalence of non-Cartesian coordinate material, which is discussed above. In all books, all topics with the exception of Vector Fields were introduced using Cartesian coordinates. In all
the books which included Vector Fields, they were introduced in a single, stand-alone subsection where all coordinate systems were employed. Previous research demonstrates that learning a topic in one coordinate system can often transfer coordinate system-specific ideas about the topic to a coordinate system in which those transferred ideas are not appropriate [40,42,97]. Therefore, it’s logical that students learning material from these textbooks, which introduce topics in Cartesian coordinate systems, are likely to transfer ideas from Cartesian coordinate systems to non-Cartesian coordinate systems. Such transfer aligns with previous physics education research within upper-division physics courses [2,57]. Such transfers can create barriers to students’ ability to understand and solve problems best suited for non-Cartesian coordinates in upper-division physics courses. For example, students might hold the idea that the time derivatives of unit vectors stay constant over time, something true for unit vectors in Cartesian coordinates but not for unit vectors in non-Cartesian coordinates, both of which can be seen in Farlow et al. [2].

Most topics in non-Cartesian coordinate systems were introduced in distinct subsections from their original introduction. As a result, students are only given explicit opportunities to consider non-Cartesian coordinates in a limited number of cases. Therefore, because of the way the material was presented, students had fewer opportunities to choose between different coordinate systems for particular problems, a skill often required in upper-division physics coursework.

Further, of the mentions of coordinates in subsection titles, there were 28 subsections which referenced non-Cartesian coordinates, and none that referenced a Cartesian coordinate system. This suggests that Cartesian coordinates are the default coordinate system because the Cartesian system is implicit whereas non-Cartesian coordinates require specific tagging or instructions. This demonstrates that students are expected to use Cartesian unless explicitly told to use a different coordinate system. As mentioned previously, this presentation may lead to students developing an understanding of the mathematics that is rooted in Cartesian coordinates. This message is inconsistent with the expectations for coordinate and unit vector application in upper-division physics courses, which require students to apply different coordinate systems depending on the geometries of the exercise or problem without an implicit assumption to use or comfort with Cartesian coordinate systems.

4.4.3. Coordinate System Choice

It’s notable that exercises asking for students to decide between two or more coordinate systems only arose in one textbook, Stewart [106]. Furthermore, even though coordinate choice
was present in this book, these exercises made up only 1.1% of the coded items in chapters which include non-Cartesian content. Additionally, only 3.4% of coded items in chapters which include non-Cartesian content involved a combination of multiple coordinate systems. We take these findings to claim that most of the content presented was on the surface level, as very little content required students to choose between coordinate systems or to consider multiple coordinate systems when reading the book or doing homework exercises from the book. (For an example of this in physics, see Sec. 4.4.7) These results suggest that, in their multivariable calculus coursework, students have very little experience with choosing the best coordinate system to use for particular situations. In contrast, upper-division physics textbooks regularly contain problems that do not provide a particular coordinate system to solve the problem with, and instead ask students to use their knowledge of symmetries to solve the problem [1, 24, 25, 26, 27]. Students’ lack of experience with choosing particular coordinate systems paired with the Cartesian-centric nature of these multivariable calculus textbooks may contribute to students’ reasoning illustrated in Vega et al. [3] and Farlow et al. [2] in which students sometimes resort to Cartesian coordinates when posed with physics problems rather than choosing coordinates based strictly on geometries.

4.4.4. Content of Non-Cartesian Coordinate Material Present

In addition to non-Cartesian content being uncommon in these texts, when students are given an opportunity to practice with non-Cartesian content, the examples and exercises tend to present the material in a surface-level manner. For an example of what we mean by surface-level problems, the content that was included often only consists of problems that may only require a conversion between two coordinate systems, most commonly where the choice of coordinates that are being converted are explicitly named in the problem or use variables associated with a particular system. The example from page 881 of Rogawski & Adams (3rd edition) [5] in Figure 4.4 represents a typical textbook exercise. It explicitly states which coordinate system students should use to solve problems, and indicates through the variables used, that the equation given is in Cartesian coordinates.

These problems are much more surface level than more advanced content often required in upper-division physics courses, requiring only simple conversion and integration rather than a more nuanced understanding of non-Cartesian coordinate systems. In upper-division physics textbooks,
students are expected to not only solve the problem, but choose the coordinates and unit vectors that will allow them to solve the problem. An example of this can be seen below in Sec 4.4.7.

4.4.5. Three Dimensions and Non-Cartesian Coordinate Systems

Three-dimensional non-Cartesian coordinate systems are even less prevalent than polar coordinate systems, with an average of 17.6% of polar content, 5.4% of cylindrical content, and 2.7% of spherical content over the coded content within the chapters that were not solely Cartesian in the Multivariable Calculus books. Additionally, three-dimensional basis unit vectors were not introduced in any of the Multivariable Calculus books. Only one book, Rogawski & Adams [5], introduced polar unit vectors. These proportions contrast starkly to the representation of these topics in upper-division physics courses. As mentioned previously, in upper-division physics courses, students are often faced with three dimensional problems that require or are vastly simplified by their use of non-Cartesian coordinates with many of the relevant expressions written in unit-vector notation. Many of these problems involve certain types of symmetry in the spherical and cylindrical directions due to physical phenomena like fields of charges and solenoids or wires.

4.4.6. Synthesizing Results

This sample of calculus textbooks consistently presents mathematical concepts first in Cartesian coordinates and then presents the same concept in other coordinate systems later, often in separate sections. They also operate under the assumption of Cartesian coordinates, as in when there is no explicit statement of coordinate system usage, Cartesian is used. Non-Cartesian coordinate systems are also primarily used in cases of explicit instruction to do so. This structuring provides answers to the qualitative aspects of Research Questions 2 and 3: non-Cartesian coordinate systems are predominantly presented after and as translations of Cartesian coordinates and as tools to be used when instructed to do so; and individual mathematical concepts, such as unit vectors, are also
first introduced in the Cartesian coordinate system and then later their behavior is explained in non-Cartesian coordinates, if such non-Cartesian behavior is explained at all.

When these quantitative and qualitative findings are considered together, a hidden curriculum emerges. *Hidden Curriculum* refers to a set of educational theories initially developed in the 1960’s and 1970’s [109,110,111,112,113,114]. These theories assert that a hidden curriculum constitutes material which students are expected to learn but are not explicitly taught and/or material of which students learn implicitly through the context of the explicit content: how and when content is presented and/or through the norms and practices of their academic environments [115]. Increasing the awareness of the hidden curricula within explicit curricula has been an ongoing effort within both mathematics [116,117] and physics [118,119]. Given the data presented in this review, the overwhelming Cartesian-centric nature and the concurrent surface-level applications of non-Cartesian coordinate systems do communicate the expectations outlined in multivariable calculus textbooks and set Cartesian coordinates as the default coordinate system. Such expectations often differ from those required in upper-division physics coursework.

### 4.4.7. Implications

The most immediate implication of this work is for instructors of upper-division physics courses. As mentioned previously, non-Cartesian coordinates and non-Cartesian unit vectors are utilized extensively in the upper-division physics curriculum. Many common mechanics textbooks – like Taylor [24], Thornton & Marion [25], Kleppner & Kolenkow [26], and Fowles & Cassiday [27] – and Griffiths’s *Introduction to Electrodynamics* explicitly introduce non-Cartesian coordinates and their associated basis unit vectors in their first chapters. The tasks for students in these courses can convey very different expectations about a student’s familiarity with non-Cartesian coordinates than what’s required in their Multivariable Calculus mathematics coursework. As an example, consider the fifth physics problem in Chapter 2 from Griffith’s EM textbook.

The question asks students to determine the electric field due to a uniform circular charge distribution at a point on the \( z \)-axis above the plane of that distribution (Figure 4.5). The problem statement does not specify which coordinates or basis vectors (unit vectors) are best for solving the problem. Recall that our analysis shows that selecting the best coordinates-type problems are only present in two of the textbooks, and thus represent an exceptionally small number of problems in the Calculus textbook. It is, therefore, a task with which students likely have little preparation.
The “cues” within the problem are also ambiguous. While the solution intends students to use cylindrical coordinates with cylindrical unit vectors or Cartesian coordinates with Cartesian unit vectors, the problem labels the radius with the symbol $r$, a coordinate typically reserved for polar, or polar spherical coordinates in Griffith’s text. Assuming the student does elect to solve the problem using cylindrical coordinates, the problem’s initial mathematical setup is relatively straightforward. However, a correct solution cannot be obtained staying in cylindrical coordinates and unit vectors unless a shortcut that leverages physical principles – in this case the components of the electric field parallel to the plane of the loop summing to zero – is used to eliminate the “$s$ term” of the integral. If a student sets up the integral properly and just “does the math” the answer will be incorrect. If a student did not realize this physical shortcut, then the initial cylindrical model would have to be translated into Cartesian coordinates (or the problem initially set up in Cartesian) where three integrals are evaluated, showing that the $x$- and $y$-components’ integrals equal zero around the loop and the $z$-component integral remains. This problem requires fluency with the various coordinate systems and their limitations, the ability to translate between coordinate systems, and the ability to construct mathematical models using unit vector notation. It also includes an additional possibility of using physical reasoning to simplify the solution process. This problem is offered early in the second chapter of a twelve chapter physics textbook, which demonstrates the gap between the level
of understanding of non-Cartesian coordinate systems expected of students in upper-division physics courses and the understanding conveyed and expected across a number of Calculus textbooks.

Furthermore, as stated earlier, this work will inform the development of research-based curricula for a Mathematical Methods in Physics course. A goal of the developed materials will be to bridge what students have likely been taught and understand with what is expected of them in the upper-division physics level. Additionally, this work lays out a methodological framework for identifying the extent to which other concepts are introduced within mathematics textbooks. Further analysis of additional topics could additionally inform physics instructors about what students are actually being taught and the tools they are more likely to have upon entering a Math Methods course, or an upper-division physics course.

4.5. Limitations and Future Work

A few limitations of this work are worth considering. Categorical content analysis was only performed on four of the eight textbooks reviewed, however, upon examination of the other textbooks, the textbooks we did categorize content from were representative of the larger sample. Moreover, the review is specifically of textbooks, which may not be representative of what instructors specifically taught and what students learned in a Multivariable Calculus classroom. Looking at a number of textbooks gives a broad sense of what resources Multivariable Calculus instructors and students have, but fails to examine the actual use of the textbook as a resource for calculus instructors and students.

These limitations are to be addressed in future work, in which interviews will be conducted with Multivariable Calculus instructors. Moreover, interviews and focus groups will be conducted with Multivariable Calculus students. A survey assessing student thinking regarding Cartesian coordinates and unit vectors to be given after all instruction in Multivariable Calculus is also planned.

4.6. Conclusion

Through reviewing seven Multivariable Calculus textbooks using content analysis techniques, the data show that mathematical topics are predominantly represented in Multivariable Calculus textbooks using the Cartesian coordinate system. This study demonstrates that topics in non-Cartesian coordinate systems are predominantly introduced in textbooks both separately and subsequent to those topics in Cartesian coordinate systems. Moreover, the chapters that do include
non-Cartesian coordinates include very few exercises that ask students to consider multiple coordinate systems or to decide what coordinate system to approach an exercise with. This surface-level presentation demonstrates a difference between the types of problems students have mathematical preparation for and the types of problems students are expected to solve in upper-division physics courses, which require a greater application of non-Cartesian coordinate systems. Cartesian coordinate systems were much less frequently explicitly named or stated to use than non-Cartesian coordinate systems, despite the overwhelming prevalence of Cartesian coordinate systems in textbooks. The explicit naming of non-Cartesian systems and lack thereof for Cartesian systems implies that when no system is characterized, students will expect and/or use Cartesian coordinates. Overall, in the textbooks reviewed, Cartesian coordinate systems were predominantly the first introduced, the most used, and the universally assumed. Thus, students entering upper-division physics courses after having received instruction of this manner are likely to bring still emerging understandings of non-Cartesian coordinate systems in general. Further, their ideas about specific mathematical concepts might be tied to how those concepts behave in Cartesian coordinates. These findings will inform efforts to develop targeted curriculum to help students bridge the gap between how these concepts are used in math courses and how they are used in upper-division physics courses. However, in the interim, it is also important for current upper-division physics course instructors to be aware of the still developing understanding their students are likely to have about coordinates and unit vectors.
5. USING THE IDENTIFICATION OF RESOURCES TO EXPLORE MATHEMATICS STUDENTS’ THINKING ABOUT UNIT VECTORS AND UNIT VECTOR NOTATION AT THE END OF THEIR MULTIVARIABLE CALCULUS COURSES

5.1. Introduction

5.1.1. Background & Purpose

There has been a growing interest within the Physics Education Research (PER) community that seeks to understand student thinking at the math-physics interface in upper-division physics [22,39]. Within that broader effort is a project to understand student reasoning at this interface with an ultimate goal of developing curriculum for a mathematical methods course for undergraduate physics majors and minors. Such a course would be designed to help students who have completed their calculus and introductory physics sequences translate the math they learn in their math courses to how math is used in upper-division physics courses. Initial investigations for this study have been to probe upper-division physics students’ thinking about concepts pertaining to vectors in Cartesian and non-Cartesian coordinates [2,3]. Vector concepts – such as basis unit vectors and other vector quantities that can be written in unit-vector notation – are used frequently in many upper-division physics textbooks and often in the context of coordinate systems [1,24,25,26,27,28]. Despite the frequent presence of such mathematical content in physics coursework, previous work has shown that physics students struggle applying vector and coordinate system concepts in physics contexts [2,3,21,44,45,46,47,55,56,57,120]. Knowing some of the challenges physics students face with using vector concepts in the upper-division, we sought to investigate student reasoning about vector concepts ahead of upper-division physics, namely within Calculus III, a multivariable calculus course. This present paper reports on a portion of that investigation into math students’ preparation for some vector and coordinate systems concepts frequently used in upper-division physics coursework.
Redish & Kuo [18] describe the math used in math courses and the math used in physics courses as two dialects of the same language. This description attempts to characterize the fact that mathematicians and physicists will often ascribe different meaning and value to the same vocabulary terms and concepts in the two disciplines. In other words, even though experts in each discipline may use something physicists would likely call a “unit vector,” mathematicians may call it something different, they might use it with different frequency, or there may be different ways that they utilize it. Through this linguistic framing, we first sought to characterize the mathematics community’s expectations for what students should learn about unit vectors and unit-vector notation in Cartesian and non-Cartesian coordinates. To do so, an analysis of several popular calculus textbooks was conducted (Dalton et al., in prep – see Chapter 4). This analysis will be further discussed below. This present paper pushes beyond this textbook analysis by focusing on what ideas about unit vectors and applications of unit vectors to other vector types – position vectors for example – in Cartesian coordinates that Calculus III students use while answering questions about them at the end of Calculus III courses. Thus, this paper will provide answers to the following research questions:

1) What ideas about unit vectors and their applications to other vector types, such as position vectors, do students activate while answering questions requiring their use in Cartesian coordinates?

2) What proportion of students productively activate ideas that are necessary to getting mathematically correct answers to questions requiring the use of unit vectors and unit vector notation in Cartesian coordinates?

The reason this paper focuses on such concepts within only Cartesian coordinates will be explained in the Methodology section (Sec. 5.2). However, as this study progressed a third useful research question emerged. By identifying the ideas activated and the proportions of students activating each idea, we were able to see examples of some response types could be explained by the combinations of ideas activated and how those ideas provided additional insight into student thinking about unit vectors and their applications. That third research question is:
3) How can combinations of ideas activated while responding to a question further elucidate student thinking about unit vectors and their applications to other vector types in Cartesian coordinates?

The methods and data concerning combinations of ideas are left to latter sections of this paper. An outline of the physics community’s expectations for students’ use of unit vectors and unit vector notation in various coordinates systems, physics students’ actual use of those ideas while solving physics problems requiring them, some previous work about mathematics students’ thinking about coordinate systems, and our investigation into some of the mathematics community’s possible expectations regarding these concepts will all be further discussed in the subsections that follow.

5.1.2. Vector and Coordinate System Content in Upper-Division Physics Courses

Upper-division physics courses – i.e. Intermediate Mechanics, Electromagnetic Theory, Quantum Mechanics, Statistical Mechanics, Thermodynamics – have content that makes heavy use of non-Cartesian coordinate systems. When this paper uses the term “non-Cartesian coordinate(s)” it is referring to polar, spherical, and cylindrical coordinates. The first chapters of David Griffiths’s *Introduction to Electrodynamics* textbook and several common mechanics textbooks [24,25,26,27] include explicit instruction on non-Cartesian coordinates, the basis unit vectors within the various coordinate systems, vector expressions using those unit vectors, and often derivatives of those unit vectors. Griffiths’s textbook makes heavy use of spherical and cylindrical coordinates with and without the associated unit vectors in many chapters of the textbook. Griffiths’s *Introduction to Quantum Mechanics* [28] uses spherical coordinates and associated unit vectors in the fourth chapter, including in the derivation of the allowed energies of a hydrogen atom. Mechanics texts use polar coordinates when describing rotational motion and rotating reference frames, angular momentum, and the motion of pendula [24,25,26,27]. Taken together, the existence of explicit instruction about such content in the first chapters of all these upper-division texts and the return to such content while providing instruction about physics content later in the texts provides a strong statement that the physics community believes that a certain level of fluency with this aspect of mathematical language is essential in being able to understand and communicate physics principles.
5.1.3. Physics Students’ Thinking About Vector and Coordinate System Concepts

Some PER studies have shown that physics students experience difficulties applying the mathematics of non-Cartesian coordinates to electromagnetic fields [55, 98, 120]. Another study found that students in Classical Mechanics often have an emerging understanding of the polar coordinate system and often default to using Cartesian coordinates to solve physics problems [56]. The same study also found that students will sometimes persist in using Cartesian coordinates even if they realize that polar coordinates are more effective at modeling the situation at hand.

Barniol & Zavala [50] found that physics students’ thinking about unit vectors is problematic in some ways. They looked specifically at some of the difficulties introductory physics students have with unit vectors in the Cartesian coordinate system. Students were asked to draw the unit vector for a vector of length $2\sqrt{2}$ with tail at the origin and pointing at a $45^\circ$ angle between the $x$ and $y$ axes. They found students can know a unit vector is of length 1 but still not draw it to proper length. Difficulties with students reasoning the correct length and direction of the unit vector from the basis unit vectors $\hat{i}$ and $\hat{j}$ were also found.

Our previous work has found that upper-division undergraduate and graduate physics students bring a widely varying array of cognitive resources [18, 35, 36, 37, 38] to bear when solving problems involving the use of unit vectors, position vectors, and velocity vectors in Cartesian and non-Cartesian coordinates [2, 3] (Farlow et al., in prep – see Chapter 3). The initial study into student thinking about such mathematical concepts found that students, when working in polar coordinates, will often conflate the definition of basis unit vectors at a point with position vectors to a point. It also found some students will use the direction of motion of a particle at a given point to inform their judgment as to the directions the polar basis unit vectors point at that point. Furthermore, even students who activate productive resources – meaning good and mathematically correct ideas for the given situation – will often activate resources that are mathematically productive in other contexts but unproductive for the question they are answering [2, 3].

Farlow, Vega, Loverude, and Christensen [2] built upon previous work [3] by doing a case study of one high-achieving senior undergraduate physics major’s responses to questions about unit vectors, position vectors, and velocity vectors in Cartesian, polar, and spherical coordinates. Findings supported some of Vega et al.’s conclusions, particularly in regard to the activation of
productive resources not always being transferred to new contexts [3], and pushed beyond Vega et al.’s findings, particularly in regard to position and velocity vectors. The case study found that this high-achieving student activated several productive resources about the definitions and behavior of polar basis unit vectors yet struggled to productively activate those ideas in the contexts of position and velocity vectors in both polar and spherical coordinates. Once the student’s struggles began, they were unable to reconcile how two productive resources worked together to provide a mathematically correct answer. In an attempt to resolve this struggle, the student demonstrated a new approach to solving the relevant problem and explicitly sought to use a Cartesian-like way of thinking – using resources that are productive and useful in Cartesian coordinates – and inappropriately applying those resources to polar and spherical situations. The case study subject was one of seven interview subjects, the entire sample in the study, to do such activation of resources productive in Cartesian coordinates in non-Cartesian contexts.

Farlow, Bradley, Vega, Loverude, and Christensen (Farlow et al., in prep – see Chapter 3) expanded upon Farlow et al.’s work by applying and expanding the resource vocabulary described in their case study analysis to the rest of the interview sample, building on our identification of resources and resource clusters. The intent of that work was to shed light on the phenomena of “pattern-matching” the algebraic expressions of position vectors in polar, spherical, and cylindrical coordinates to resemble the form of the same in Cartesian coordinates. Hinrichs [57] defined this type of pattern-matching as students writing position vectors, which have the Cartesian form \( \vec{r} = x\hat{x} + y\hat{y} + z\hat{z} \), in the incorrect form \( \vec{r} = r\hat{r} + \theta\hat{\theta} + \phi\hat{\phi} \) in spherical coordinates. The correct form of a position vector to a point in spherical coordinates is \( \vec{r} = r\hat{r} \), where the magnitude \( r \) is the distance from the origin to the point of interest, because the angular information is included in the definition of \( \hat{r} \) (see Mathematics Discussion in Farlow et al. [2]). Hinrichs found that approximately half to two-thirds of upper-division physics undergraduates and physics graduate students will do this type of pattern-matching regardless of university type, university location, or course textbook used in their instruction. Farlow et al. [2] (Farlow et al., in prep – see Chapter 3) were able to replicate Hinrichs’s findings in not only spherical coordinates, but polar and cylindrical as well, and utilized our identified resources, such as \( \vec{r} \) has form \( \vec{r} = a\hat{a} + b\hat{b} + c\hat{c} \), to offer possible explanations as to why such responses are common. That study found that while some students can productively activate resources regarding non-Cartesian basis unit vectors – and in particular resources
regarding the location-dependent directionality of those unit vectors – they will frequently activate those resources while writing position vector expressions in non-Cartesian coordinates, which is not appropriate. In fact, activation of \( \vec{r} \) has form \( \vec{r} = a \hat{a} + b \hat{b} + c \hat{c} \) resource occurred whether interview subjects productively activated resources regarding non-Cartesian basis unit vectors or not. The consistency with which this activation occurred revealed that there could be a cluster of resources that lead to students thinking that \( \vec{r} \) has form \( \vec{r} = a \hat{a} + b \hat{b} + c \hat{c} \) is the general form for vector expressions regardless of coordinate system. Additionally, a procedural resource, the Navigation Resource, was identified and represents the thought process that students used to construct or interpret position vector expressions as directions for traveling a path from the origin to the point of interest – another resource that is productive in Cartesian coordinates, but not in non-Cartesian coordinates. Finally, our resources-framework-guided analysis showed that students’ development and/or productive activation of resources necessary to arrive at mathematically correct answers for questions about these vector concepts in non-Cartesian coordinates is emerging.

5.1.4. Mathematics Students’ Thinking About Coordinate Systems

The Research in Undergraduate Mathematics Education (RUME) community has also done some research on student thinking about and understanding of non-Cartesian coordinate systems. Paoletti, Moore, and Musgrave [41] agreed with Sayre & Wittman’s physics study [56] in that students have an emerging understanding of polar coordinates and further explained students spend most of their educational career working with an implicit assumption of Cartesian coordinates. Montiel and various colleagues [40,42] found that student understanding of mathematical concepts is often tied to the coordinate system in which they learn those concepts; meaning that how a student understands a concept is at least partly informed by how that mathematical idea behaves or operates within the coordinate system in which it is first learned. An example they describe is the vertical line test for functions. If a curve on a Cartesian graph is intersected by a vertical line at one and only one \( y \)-coordinate for each \( x \)-coordinate the curve represents a function. This vertical line test frequently fails in polar coordinates, but many students attempt to use a version of it to test if a curve models a function in polar coordinates nonetheless.
5.1.5. Investigating Students’ Preparation for Coordinate System and Associated Vector Concepts

Synthesizing the findings in literature described above with our own findings, we realized it was necessary to more fully investigate the instruction students receive about coordinate systems and the vector concepts within them before they get to upper-division physics courses. This investigation had a three-pronged approach: 1) gain some insight into the mathematics community’s expectations for students regarding coordinate systems and their associated basis unit vectors, position vectors, and velocity vectors by analyzing popular calculus textbooks, 2) interview Calculus III instructors to see how closely their instruction matched the textbooks and each other, and 3) survey Calculus III students to see to what extent they have learned the content presented in the calculus textbooks. Findings from the first prong will be summarized in the next paragraph with full details available in Dalton et al. (in prep – see Chapter 4). Work in the second prong is ongoing. The summary of findings for the third prong are presented in this paper.

Dalton, Farlow, and Christensen (in prep – see Chapter 4) did a review of seven Calculus textbooks [4, 5, 102, 103, 104, 105, 106, 107] and found that on average approximately 20% of the chapters of those books contained any non-Cartesian-coordinate content. Only one textbook contained a single mention of polar unit vectors [5]. A more detailed analysis of the relevant chapters within three of those texts [5, 106, 107] found that within those about 20% of chapters, about 26% of the bolded definitions, worked example problems, and end-of-chapter exercises involved non-Cartesian-coordinate content. Qualitatively, non-Cartesian content was heavily scaffolded, often involving explicit instructions on when to use a particular non-Cartesian coordinate system. Only one textbook [106] included any exercises where a student/reader was expected to determine which coordinate system was most appropriate for a given situation, and at that only 1.1% of the exercises within the 20% of relevant chapters did so. Furthermore, all of the textbooks “assumed” Cartesian coordinates. In other words, the expository text, the definitions, worked examples, and exercises operated as if Cartesian coordinates is the default coordinate system. When no coordinate system was named, the reader was to use Cartesian coordinates. When non-Cartesian coordinates were expected to be used, it was near-universally stated so. Both the quantitative and qualitative results of the textbook analysis showed that there is a “Hidden Curriculum” [109, 110, 111, 112, 113, 114, 115] in
those texts. This hidden curriculum suggests that Cartesian coordinates is the default and possibly even the most important coordinate system.

While the textbook analysis is useful in at least partially reflecting the mathematics community’s expectations for students regarding coordinate systems and vector concepts within them, it doesn’t shed any light on what ideas students do or do not activate when answering questions about this content at the end of their Calculus III courses. This present paper provides some answers to that question. Two rounds of surveys were developed to probe student thinking about unit vectors and position vectors in various coordinate systems. Taken together, the findings of these surveys and the results of the textbook analysis will inform the starting point for the research-based instructional materials for our broader study.

5.1.6. Theoretical Frameworks

This study’s research questions imply the need for a theoretical framework that hybridizes two theoretical frameworks commonly used in PER: the identifying student difficulties theoretical framework [62,63,64,65,121] and the resources and framing theoretical framework [18,35,36,37,38]. Our initial investigations into physics students’ thinking about unit vectors in coordinate systems were guided by resources and framing [2,3] (also see Chapter 3). Research Question 1 for this present study focuses on the ideas students use when answering questions about the mathematics of interest – which fits in well with the resources and framing theoretical framework. Research Question 2 has elements more consistent with an identifying student difficulties framework, but arguably within the structure of identifying ideas – or resources – activated. The following three paragraphs will further define resources and framing, identifying student difficulties, and further articulate the hybridization of the two theoretical frameworks necessary for this paper, respectively.

The resources framework emerged from a knowledge-in-pieces approach [67,68] and is rooted in results from broader education research and both behavioral and cognitive psychology [18,35,36,37,38]. A resources framework posits that knowledge can be modeled as fine-grain ideas that exist in an individual’s brain – broadly defined as resources – that are gained through experience. These resources can be activated or not activated depending on a given stimulus [36]. Resources that are closely connected to other resources form networks which can become hierarchical cognitive structures known as schemas. Those clusters and schemas which activate together eventually become strongly tied together such that they always activate together. This model is thus explicitly built
on well-established neurological models describing the interconnection of neurons [70]. As of yet, the resources framework does not provide a means for empirical models or quantitative predictions [38]. Instructional material development informed by a resources framework is less common in the published literature than that of identifying student difficulties [39], but what work does exist relies on encouraging students to analyze and be critical of their underlying epistemologies rather than the cognitive conflict model at the core of difficulties-based materials [37]. In PER, there is a practice of identifying the resources students activate while thinking about a problem and then describing how those resources connect or don’t connect to each other. These interconnected resources can be represented in a diagram known as a resource graph [72,73].

The framework of identifying student difficulties emerged as the guiding framework of the Physics Education Group (PEG) at the University of Washington and has been widely adopted as a guide for transforming undergraduate physics curricula [63,66]. Given that course transformation is a stated goal of the PEG, they advocate a two-step approach that first identifies common student errors in thinking – difficulties [62,64] – and then elicits and addresses those difficulties with targeted instructional materials [65,121]. The instructional materials rely on a cognitive conflict model by creating situations where students are made aware of the inconsistency between their own thinking and observations of observable phenomena. Then, they are forced to reconcile the conflict [63,121]. This two-step process has resulted in reformed curricula that has led to substantial learning gains and/or improved conceptual understanding of physics concepts. A notable example of such curricula is Tutorials in Introductory Physics [66]. The difficulties framework also takes an inherently quantitative approach to researching student thinking. It affords the ability to predict what approximate proportions of a group of students will exhibit various relevant difficulties and allows instructors to prepare for such group compositions. As such, the difficulties framework takes a lab-bench-like approach to researching student thinking and designing curriculum [62,121]. Making this quantitative nature possible is that, within a given context, a difficulty is persistent and consistently applied. A common example of this is the thinking that electric current is “used up” in an electrical circuit. This difficulty is observed across demographics and in a wide variety of electrical circuit related questions [64].

Research question 2 of this study has the goal of determining the proportion of students who activate each of the individual ideas that are identified as part of answering Research Question 1.
As such this work hybridizes the frameworks of resources and identifying student difficulties. The activation of ideas is consistent with resources framework, the proportions of students who activate those ideas is more consistent with the quantitative nature of identifying student difficulties. To our knowledge, this hybridization or similar has not been utilized before in studies described in PER literature. We posit that knowing how the rough proportions of students who activate each idea found herein is helpful in determining which ideas are most fruitful for identifying targets for future instructional materials. Since such instructional materials will ultimately be presented to groups of students, we consider it wise to know how prevalent each idea might be within a population of students as designing instruction for an individual student is likely to be inefficient.

5.2. Methodology

There were two primary stages of development of the survey. The first stage ended with relatively little useful data but did more fully inform the development of the survey used in the second stage. So, this Methodology section is divided into two parts. The first takes a more narrative approach to describing the development of and reasoning behind the survey ultimately analyzed. The second part will discuss the coding process, some of the codes, and rater agreement of the survey for which the results provide the bulk of this paper’s analysis. There will also be a standalone Survey Analysis section following this Methodology section. As will be explained in more detail in that section, the coding process used was both methodological and filled with results. Since Research Question 1 seeks to determine the ideas students activate, the codes developed represent ideas of interest and are therefore results.

5.2.1. Survey Round 1: Initial Survey

The first-stage survey primarily adapted questions that had been given in our physics student interviews [2,3]. A complete copy of this initial survey is included in Appendix A. The survey was administered to a single section of Calculus III (n = 68) during the last week of a spring semester at a north midwestern United States, public, land grant, PhD granting university. While the survey was being distributed the instructor of the course informed the first author (Farlow) that the students would not successfully answer the survey questions. The instructor explained that he did not teach them the terminology of “unit vectors” or “coordinate vectors” used in the survey, saying that they used “standard basis vectors” instead. Furthermore, there was no instruction about polar standard basis vectors. Additionally, the vector notation they used was not standard basis vector notation,
i.e. in the form of $\vec{n} = a\hat{x} + b\hat{y}$, rather $\vec{n} = < x, y, z >$. We chose to continue administering the survey to see if anything useful would come of it.

The instructor was quite accurate in their prediction that the students would struggle to answer the survey questions. For example, while the first question explicitly asks for a drawing of unit vectors, less than half the students drew arrows at all. It was quite clear that there was very little understanding of what the questions were trying to ask, and only 3 students out of 68 gave a correct answer. It was also quite clear that the researchers’ expectations about what Calculus III students should know were quite disparate from reality. One clear manifestation of that lack of alignment was the difference in vocabulary we use as physics researchers and that used in Calculus courses. It is likely that this difference in vocabulary is an instantiation of what Redish & Kuo [18] describe as a difference in math and physics “dialects”. Thus, we realized that a revised survey was necessary and that the previously described textbook analysis (Dalton et al., in prep – see Chapter 4) should be done first so that we could more closely align the “dialect” of our questions to the one Calculus III students are more likely to use.

5.2.2. Revised Survey

After completing the textbook analysis (Dalton et al., in prep – see Chapter 4) and using the feedback from the first Calculus III instructor, we were able to revise the Calculus III student survey. The revised survey had several key changes from the initial. The first was the decision to drop any questions asking students to work in polar coordinates. This decision was made due to the realizations that such a small portion of the students’ instruction was about polar coordinates [41](Dalton et al., in prep – see Chapter 4), that students have an emerging understanding of polar coordinates generally [41,56], and that student understanding of the rules for the vector concepts of interest were strongly informed by those concepts’ behaviors in Cartesian coordinates [2,40,41,42] (also see Chapter 3). Thus, the survey became strictly about unit vectors and their application to position vectors in Cartesian coordinates. Our thinking was that by removing the added variable of different coordinate systems, we could get a clearer picture of what ideas students actually have about unit vectors and unit-vector notation. The second decision was to do our best to align the verbiage used in our questions to the vocabulary presented to students by the textbook used in their course – which in this case at the previously described university is Rogawski & Adams’s textbook [5] – and to what the instructor had verbally communicated to the first author during
the initial survey administration. An example of this vocabulary alignment was to use the term “standard basis vector” in lieu of “unit vector” or “coordinate vector”. Additionally, bold text such as \( \mathbf{i} \) and \( \mathbf{j} \) were used as representations of Cartesian basis vectors in this revised survey. These boldface vectors reflect how Rogawski & Adams [5] print basis vectors in their textbook. It should be noted that \( \mathbf{i} \) and \( \mathbf{j} \) are equivalent ways of writing \( \mathbf{\hat{i}} \) and \( \mathbf{\hat{j}} \), respectively. They are also equivalent to \( \mathbf{\hat{x}} \) and \( \mathbf{\hat{y}} \), respectively.

The revised survey can be seen in Appendix B. It was administered to a total of three sections of Calculus III in the final week of two consecutive fall semesters for a total of \( n = 94 \). The three sections each had different instructors but used the same textbook. A quick note is that the first semester of this revised survey’s administration had 2 slight differences from what is seen in Appendix B, both to question 1. This first round did not include the “and label” instruction in the written part of question 1 and did not included the “6” labels on the axes of the associated figure. Two sections for a total of \( n = 47 \) took the survey in that form, one section (\( n = 47 \)) took the survey as seen in Appendix B a year later. The two edits were added to be more explicit in our instructions and that the grid lines in the figure each represented an interval of 1 unit. After the initial 47 surveys were analyzed, we realized that 20/47 students did not label their sketched arrows at all and that 24/47 did not draw the arrows 1 box in length. The 47 surveys the following year had 23 unlabeled arrow sketches and 34 arrow sketches not drawn 1 box in length. Thus, we argue that the additional clarity provided by the wording of the question did not unduly affect the ability of the second round of 47 students and we include all 94 in the aggregate results reported in this paper. The final survey has 7 total questions, but only the first 5 are discussed in this paper as they were deemed the most relevant to the overall trajectory or our research. It should also be noted that even though the questions were revised to match the language the students were likely to understand as closely as possible, the questions were still written by physicists. Given that mathematicians and physicists use different math dialects [18], it stands to reason that these questions may still not align with students’ math question expectations. That could potentially lead to a lower rate of activation of productive ideas than if the questions had been written by mathematicians more familiar with the conventions of math question writing.

There was a limited validation process for this second survey. The first author wrote the questions using the feedback from the instructor for the first class surveyed and the findings from the
calculus textbook analysis (Dalton et al., in prep – see Chapter 4) as a guide for determining question content. The third author reviewed and edited those questions. This process established a limited degree of face validity with two of the authors agreeing that the questions were appropriate for both the targeted population and for answering the research questions. The survey was also intended to be anonymous. The anonymity was intended to remove any response suppression that might occur if the students were worried about having their names attached to their work. Therefore, there was not a means to track down individual students to perform follow-up interviews. As a result, establishing robust content validity was not a research objective. However, as the survey analysis in Sec. 5.3 will show, the questions were able to draw out student responses that had similarities to and were consistent with some of the results of our Case Study [2] and Pattern-Matching (Farlow et al., in prep – see Chapter 3) analyses. In other words, there was noticeable overlap in some of both of the productively and non-productively applied resources in the survey analysis and those other two previous analyses. This consistency suggests that the questions were of sufficient clarity that the calculus students had reasonable understanding of what the questions were asking. This consistency also suggests the questions were appropriate for the target population to a degree sufficient to be operationally useful to this study.

5.2.3. Coding Scheme

Given the primary research objective of these surveys was to ascertain what ideas – which in this context we interpret as resources – students activate when asked about unit vectors and position vectors, we developed a coding scheme to help with the identification and description of the underlying ideas present in the responses. This coding scheme accounted for both productively applied resources – resources that are productive in the given context and necessary for providing a mathematically correct answer – and non-productively applied resources – resources that are productive in other contexts but are not productively activated in the given context. The coding scheme developed used the same general process for every question, but the individual ideas coded varied by question, as each question targeted different content or the same content in a different way than a previous question. Each question had its own set of resources that were coded on the basis of presence/absence, and not the level of quality in the response. There was some overlap of resources between some questions, however. For example, some of the same resources were coded
for in questions 1 and 2 since they were both about unit vectors. However, each question had some ideas that were not relevant or present in the other questions as well.

The first two authors (Farlow & Brainard) coded all responses and were the primary developers of the coding scheme. The last author (Christensen) served in an advisory and consultation role. The coding scheme was done in separate stages; one stage for questions 1 and 2 and a second stage for questions 3-5. Different stages were used because questions 1 and 2 focus on unit vectors, questions 3 and 4 on position vectors, and question 5 on algebraic notation, creating a thematic divide in the questions for relatively straightforward grouping. The following describes the coding scheme development and coding processes for questions 1 and 2. The process used for questions 3-5 used the same basic procedural steps but had different numbers for initial agreement, number of codes, and a number of iterations of coding scheme development. The first two authors initially read through all of the survey responses before coding began, through an inductive analysis approach, then discussed what qualitative trends they were seeing. After those discussions, it was clear that each survey response contained some combination of mathematical ideas. Those mathematical ideas either were productive toward answering the question on which they appeared or would have been better suited for a different mathematical question. Furthermore, many of the productive ideas appeared to align quite well with many of the unit vector and position vector resources identified from our previous work [2,3] (Farlow et al., in prep – see Chapter 3). Thus, an initial coding scheme was developed that combined looking for a-priori codes that are some of the resources from that previous work, and emergent ideas for those mathematical concepts that were not productive for a given question. This initial scheme then had a-priori codes for most of the productively activated resources, and emergent codes – resulting from an emergent coding process – for the non-productively activated resources (and a small number of productively activated resources). The first two authors then performed an initial round of coding using this initial scheme and then met again. Coding agreement between them on this initial scheme was rather high. For example, on questions 1 and 2 the initial coding scheme contained 17 total codes with the two coders having over 90% agreement on 14 of those codes. Through discussion of the disagreements that were present, codes were redefined or renamed if necessary, or new codes were created, or redundant/unnecessary removed. Interpretations of codes were also discussed. A new coding scheme was developed based on those discussions and both coders re-coded the entire sample, and met again to go through a similar pro-
cess of discussion of code definitions, interpretations, and resolving disagreements. This iterative process was repeated until the coders agreed that the codes developed accurately represented the students’ responses. The final coding scheme was agreed upon after a third round of coding and coding scheme development. Thus, the fourth iteration of the coding scheme became the final coding scheme for the data analysis to be presented here. After coding, the agreement was over 75% for all codes, with the majority of codes having agreement over 90%. All disagreements were resolved through discussion and 100% agreement on all coded items was reached for all coded questions. The following Survey Analysis section will define each of the codes for the 5 coded questions in the order of the questions on the survey.

5.3. Survey Analysis

5.3.1. Use of Both A-Priori and Emergent Codes

As mentioned above, the coding scheme developed used a mix of a-priori codes and emergent codes. The a-priori codes are resources that were identified and described in our initial work into students’ unit vector thinking [3], our case study [2], and our analysis of pattern-matched responses (in prep, see Chapter 3). Emergent codes were identified from the written data in this survey. Because some codes were a-priori, their presence constitutes a result; the presence of any a-priori code is itself a partial answer to Research Question 1. Because some codes were emergent, their presence is both methodological — we used an emergent coding process to identify them — and a result. The presence of an emergent code represents a student’s activation of a resource while answering a question about unit vectors or position vectors. This activation of an emergent code is also a result, being a partial answer to Research Question 1. It should also be noted that for all questions, we did not code for a correct answer, i.e. there was no “Correct” code. A response was classified as mathematically correct if it had all the necessary productive resources and none of the non-productively applied resources for that question. The following subsections will describe each code in detail organized by survey question.

5.3.2. Codes for Question 1

Question 1 presents students with a Cartesian-coordinate grid with two points labeled A and B. Students are asked to draw and label the standard basis vectors \( \mathbf{i} \) and \( \mathbf{j} \) at those two points. Figure 5.1 shows a correct response for question 1. In the sketch, there are two vector arrows drawn at each point. When two arrows are present at or associated with each point, we coded the response
Figure 5.1. A correct response to survey question 1. This response is mathematically correct because it contains all four productive resources codes for question 1: COMP, LAB, UL, ICD, and none of the non-productively activated resources codes for question 1.

as complete or “COMP”. The arrows in the Figure 5.1 response are also labeled. When the arrows are labeled – regardless of whether they are productively drawn – we coded the response as labeled or “LAB”. The arrows are also drawn one box in length. When arrows were drawn with a length of one box, we coded the response as illustrating unit length or “UL”. The sketch also shows the arrows labeled “i” and “j” pointing in the directions of increasing $x$ and $y$, respectively. We coded such responses as illustrating the vector arrows pointing in the increasing coordinate direction, or “ICD”. The UL and ICD codes are a-priori codes because they are resources first described by our work in Vega et al. [3] and then observed and refined in our case study [2]. It should be noted that the COMP and LAB codes are not content specific in the way that UL and ICD are. Rather, they show whether the student was being thorough and clear as to which arrow had what meaning.

Other identified codes for question 1 attempted to capture additional student resources that were emergent from the data. One such code describes sketches that appear to conflate the definitions of component vector and unit vector, or “CCV”. In Figure 5.2, the arrows labeled $i$ and $j$ are drawn as the component vectors of what would be the position vectors for the two points. The sketch shown in Figure 5.2 was also coded as “COMP” and “LAB” as there are two vector arrows associated with each point and all arrows are labeled. Of note on this CCV code is that
it encompasses two possibilities of conflation. In Cartesian coordinates, the component vectors of what would be a position vector to a point have a magnitude equal to the absolute value of the coordinates of that point. It is possible that some of these responses are the results of students thinking that what they think are the unit vector arrows need to be of length equal to the absolute values of the coordinates for points A and B. This question is unable to tease apart whether a student is conflating the definitions of unit vectors with component vectors or if they are not thinking about component vectors at all and simply mapping the points’ coordinate values to the arrow lengths. However, both ways of thinking show the conflation of two ideas that result in non-productively applied resources leading to mathematically incorrect answers. This duality also implies that this CCV code also happens because students are doing some “hidden thinking” insofar as it is necessary to first construct – at least mentally – the position vectors to points A and B so that their component vectors can be drawn. However, sketches that received the CCV code were different in nature than those that appeared to conflate the definitions of unit vector and position vector. Responses that did this position vector type of conflation received their own code to be discussed in the next paragraph. Finally, because it’s not possible to distinguish between those responses that were based on vector thinking from those based on coordinate thinking, the code CCV was applied to both response types. Thus, a response coded CCV means a response that either conflated the definitions of component vector with unit vector or conflated unit vector with the coordinates of the points.

Another emergent code describes a second non-productively applied resource for sketches that appear to conflate the definitions of position vector and unit vector. One such example can be seen in Figure 5.3. In this sketch, the two arrows drawn appear to be position vectors as they begin at the origin and terminate at the respective points as position vectors do. This response would also not be labeled as “COMP” as there are single arrows associated with each point. It would also not be coded as “LAB” because there are no labels. The CPV code is also consistent with a finding presented in our earlier work [3] that students will sometimes conflate unit vectors at a point with position vectors to a point. This consistency is additionally meaningful because the CPV code here is an emergent code. Thus, both this present work and the Vega et al. work [3] observed similar student thinking independently.

A third non-productively activated resource was not a-priori in the way that UL and ICD are, but is also strongly connected to the Motion Cluster described by Vega et al. [3]. This code is
Figure 5.2. An example response of our emergent code “CCV” for question 1. The drawing shows the definitions of component vector and unit vector being conflated in the form of the response by labeling what appears to be the components of what would be the position vectors to the 2 points being labeled as the unit vectors.

Figure 5.3. An example response of our emergent “CPV” for question 1. The drawing shows what appears to be the position vectors for the two points instead of showing the unit vectors.
called “physical reasoning” or “PR” because the physical concepts of displacement or motion appear to be illustrated. These physical concepts appeared in two primary ways. The first (Figure 5.4) are sketches of what looks like the displacement vector from one point to the other. The second, Figure 5.5, shows what looks like the component vectors of what would be the displacement vector from point A to point B. An argument could be made that the response in Figure 5.5 could also be coded as CCV. However, we determined that the use of PR was distinct enough from CCV that we did not code such responses as CCV.

If a student did not provide a sketch for question 1, the question was coded as “NR” for no response. This code existed as a possibility for all questions. Thus, we will not repeat it for the code descriptions of subsequent questions.

**5.3.3. Codes for Question 2**

Question 2 asks students to provide a definition for $\mathbf{i}$ using words. Figure 5.6 shows a correct response for question 2. As a result of the second question asked students to describe $\mathbf{i}$ using words, the ICD and UL codes were again applicable. The UL code was seen in the use of the phrases “length of 1”, “magnitude of 1”, or similar. Some responses included “$< 1, 0 >$” and received a UL code because such notation does communicate a length of 1. A response was coded ICD if it
Figure 5.5. An example student response that was also coded as “PR” for showing evidence of physical reasoning. In this sketch, the component vectors of what would be the displacement vector from point A to point B are drawn.

explicitly linked \( i \) with the specifically positive \( x \) direction. This linkage could be done in a handful of ways. One would be to explicitly use the phrases “positive \( x \)” or “positive in the horizontal direction” or similar. Another would be to use the vector notation “\(< 1, 0 >\)”. During the coding process, another code emerged that illustrated the need to distinguish between responses that linked \( i \) with the specifically positive \( x \) direction and those that more generally linked \( i \) with the horizontal direction. In the response shown in Figure 5.7, the unit vector is linked to the horizontal direction, but it is not clear if that means the positive or negative direction. Since “horizontal direction” could mean either positive or negative, we decided to code such responses as “ix”. This code represents a productive resource because it is mathematically correct that \( i \) is linked to the horizontal direction. The response shown in Figure 5.6 was coded with all three of these productive codes.

The only identifiable non-productively activated resource on question 2 was again physical reasoning, PR. If responses included verbiage such as “moves”, “shifts”, or “motion” it was coded as PR (Figures 5.8-9). The presence of only one non-productively activated resource does not imply a high rate of correct responses on question 2. In this question, the absence of the productively activated resources played a large role in our using combinations of codes to describe response types. The results of that will be discussed in Section 5.4.
Figure 5.6. A student response to question 2 deemed mathematically correct. This response contains all of the necessary productive codes – ix, UL, ICD, – and none of the codes non-productively activated resources for question 2.

Figure 5.7. An example student response for question 2 that shows activation of the “ix” code. The response links i to the x direction but does not specify the positive x direction. Thus, this code is productive but not sufficient to say that questions with this code are “correct” without other productive codes present.

Figure 5.8. An example of student that used physical reasoning, code “PR”, in their response. The word “shift” was interpreted to imply motion — a physical concept.
5.3.4. Codes for Question 3

Question 3 asks students to provide a definition of ‘position vector’ using words. Productively activated resources for question 3 are a further refinement of the resource named *From origin to point* in our case study [2] and again observed in our resource analysis across multiple subjects (Farlow et al., in prep – see Chapter 3). In those studies, our interview subjects were consistent in their understanding that position vectors begin at the origin and terminate at the point of interest. The survey responses analyzed here revealed that this resource is possibly a grouping of two distinct resources: *position vectors start at the origin* and *position vectors end at the point of interest*. Some survey responses showed the presence of the former but not the latter and vice-versa. Thus, two of the codes for productively activated resources on the third question are “Starts at origin” (“SO”) for responses that communicate that the position vector does so, and “Ends at point P” (“EP”) for responses that indicate so. We interpret the response shown in Figure 5.10 as doing both of these. Also productive, but in a similar way that “ix” is productive but not sufficient to get a fully correct answer for question 2, was a code for responses that implied the origin as a starting point but did not explicitly name a starting point. We gave such responses the code “OI?”.

Figure 5.11 is an example of such a response. In our interpretation, stating that the position vector gives the location of a point implies that there is some reference point. The use of the term “location” further implies that that point is the origin because an accurate location cannot be given without knowledge of the position of the origin. We acknowledge that the level of interpretation required for the OI? code is high and thus there are likely other reasonable interpretations of responses of this type. However, we believe our interpretation and code is consistent with the definitions of “position vector” and “location” and reflective of what the students were trying to communicate.
Figure 5.10. An example of a student response deemed “correct” for question 3. This response activates all of the productive codes – SO, EP – for question 3.

Figure 5.11. A student response for question 3 that illustrates the activation of the “origin implied?” – OI? – code. The use of the term “location” implies that there is some reference point from which to define a point’s position, but the origin is not specifically mentioned.

Codes for non-productively activated resources on question 3 were rather numerous with three of them also contributing to our separating the productive ideas of SO and EP out of the From origin to point resource. Those three are 1) the position vector pointing to but not ending at the point of interest (“PT”); 2) there being no clearly defined starting point for a position vector (“NS”); and 3) there being no clearly defined end point for a position vector (“NE”). These three codes appear to be distinct ideas from merely not activating SO and/or EP. The PT code was usually used for responses that used explicit language about the position vector pointing to the point such as is the case in the response shown in Figure 5.12. Figure 5.12 is also an example of the NS code as there is no clear starting point for the vector indicated either. The NE code was used when the response was not clear about an ending point for a position vector, such as in the response in Figure 5.13. Figure 5.13 also provides an example of why we separated the From origin to point resource into the two productive ideas SO and EP. In this response we see a clear activation of the SO code in the clause “originating from origin” but there is no clear end point included in the definition. Thus, it is clearly possible to activate a portion of what we had previously called the From origin to point resource, implying that said resource is in fact a group of resources itself.
Figure 5.12. A student response that shows activations of the codes “PT” – because the wording states that the position vector points to a point – and “NS” — because there is no clear starting point for the vector arrow indicated.

Figure 5.13. A student response that shows activation of the productive SO code, but does not indicate a clear ending point, thus it is coded “NE” for “no end point specified.” This response is also one example of how it is possible to activate a resource for a starting point for a position vector, but not an ending point. Thus, this response was one that lead us to refine our previous From origin to point resource [2] into separate resources for position vectors beginning at origin and ending at the point of interest.

There were also responses where EP was activated but not SO. In Figure 5.14 there is an example of EP without SO. The student indicates that the point of interest is the end point of a position vector but is not clear about a starting point.

There were two other codes for non-productively activated resources that emerged from the responses on question 3. The first is again the use of physical reasoning (“PR”) in the definition, similar to that seen in question 2. These responses typically used motion-based language. Figure 5.14 is again a useful example as the student uses the term “goes” in the response. The other is for responses that use the specific point (4,3) in the definition (“(4,3”)). Some responses used this specific point as the end point. It is quite possible that this response is an artifact of the survey construction as the figure to be used for question 4 has the point (4,3) labeled and is placed in what can be interpreted as the writing area for question 3. Figure 5.15 contains both an example of a
Figure 5.14. An example student response to question 3 that provides another example of why we separated our previous resource \textit{From origin to point} \cite{2} for position vectors into separate resources. In this case, the idea of end point “EP” is activate, but no observable resource for a clear starting point “SO” is activated.

student response using the point (4,3) and also illustrates the survey construction that may have lead to such responses. It is also true that the response shown in Figure 5.15 is mathematically correct for the point (4,3). However, as the caption under the survey’s second figure was intended to make clear, that figure was meant for use with question 4. We wrote question 3 seeking a general definition of the term “position vector”. Thus, we consider a response to include non-productively activated resources if it is dependent on the point (4,3) and does not indicate a more general understanding that position vectors terminate at the point of interest – whatever that point may be. It could be argued that under the resources framework that (4,3) would be a productively activated resource because it is productive in the context of this question. However, since the wording of both questions 3 and 4 make it clear that Figure 2 of the survey is not connected to question 3, we interpret responses that specifically reference the point (4,3) as the end point for the vector definition in question 3 in lieu of language that communicates a more general understanding that the end point of a position vector is the point of interest as activating a resource non-productively.

5.3.5. Codes for Question 4

Question 4 asks students to draw a position vector on a Cartesian-coordinate grid for a point labeled “P” with coordinates (4,3). As will be further discussed in the results section, question 4 had the highest rate of mathematically correct responses on the survey. The codes for productive resources SO and EP were used for question 4 for sketches that started at the origin and had arrowhead pointing to and touching the point (4,3), respectively. The code for non-productively activated resource for the vector arrow pointing to but not ending at the point (PT) returns as well, with the response in Figure 5.16 being an example. A new non-productively activated resource
Figure 5.15. This figure illustrates two important things. The first is the student’s response to question 3 being an example of a response that was coded “(4,3)” because it used to point (4,3) in the definition of position vector. The second is how the structure of the survey could have influenced some students to use that point in their definition. The figure for question 4 in the survey is in a physical space that could be interpreted as the provided writing area for question 3. So, even though the caption on that figure specifies it is to be used for question 4, its proximity to question 3 may have influenced some into thinking that the point (4,3) was relevant to question 3.

also emerged and was given a code that describes responses in which the vector arrow is sketched such that it begins at the point (4,3) and points away from the origin (“PA” – Figure 5.17). These sketches at least superficially mirror the directions of what would be the \( \hat{r} \) unit vector for polar coordinates at the point (4,3). Other codes for non-productively activated resources are for sketches that include multiple arrowheads (“MA” – Figure 5.18) and for sketches that appear to show the component vectors of the position vector that would terminate at the point (4,3) (“DC” – Figure 5.19). Question 4 was also the first question in the survey that produced singular responses for which there was not usable evidence for a useful code. In other words, we were not able to meaningfully interpret these responses and these responses were not not demonstrate any evidence of activation of any of the resources discussed herein. These responses resulted in an “other” code, which were unique and accounted for only 4 total responses.

5.3.6. Codes for Question 5

Question 5 asks students to write an algebraic expression, in standard basis vector notation, for the position vector for a point at the coordinates (4,3) in a Cartesian-coordinate plane. A
Figure 5.16. An example student response to question 4 that illustrates a response coded “PT” – or “points to” – for a position vector arrow sketch that points to the P but does not terminate there.

Figure 5.17. An example student response for question 4 that was coded “PA” for a position vector arrow sketch that shows the arrow starting at point P and “pointing away” from the origin.
Figure 5.18. An example student response for question 4 that was coded “MA” for the vector arrow sketch having “multiple arrowheads”.

Figure 5.19. An example student response for question 4 that was coded “DC” for a vector arrow sketch that appears to be the component vector arrows for what would be the position vector.
mathematically correct response for question 5 would simply be $4\mathbf{i} + 3\mathbf{j}$. That expression may or may not be set equal to $\vec{r}$. Since all the question requests is an expression, the “$\vec{r} =$” is not necessary, and it also does not change the correctness of the response. A correct expression was coded as showing activation productive resources for the expression having the form “$\_\mathbf{i} + \_\mathbf{j}$” (“$\mathbf{i} + \mathbf{j}$”) and the expression including unit vectors (“UV”).

Additional codes for non-productively activated resources for question 5 were emergent. One such code represents the use of the forms of linear equations, either slope-intercept form $y = mx + b$ or point-slope form $(y - y_1) = m(x - x_1)$, and a slope of $4/3$. We gave such responses the code “r/r” for “rise over run”. An example response is “$y = \frac{4}{3}x$” since it follows the form of $y = mx + b$. We also observed responses that appeared to use the notational conventions – such as braces, brackets, introduction of a new variable, and/or matrix operations – of linear algebra, “LA”; examples of which can be seen in Figures 5.20 and 5.21. A third such code is both emergent and connected to the results of our textbook analysis (Dalton et al., in prep – see Chapter 4) and the feedback from the instructor of the course where we administered the very first survey. That is, the presence of responses that used a vector notation that matched that of what is presented in the Rogawski & Adams calculus textbook [5]: $\overrightarrow{n} = < x, y, z >$. We call this code “TVN” for “textbook vector notation”. Responses that earned this code often appeared as “$<4, 3>$” or “$<4\mathbf{i}, 3\mathbf{j}>$”. The latter would also be coded with UV as it contains unit vectors. A fourth code for a non-productively activated resource is for responses that use mathematical operations (“MO”) different than those associated with linear algebra and the LA code. These responses typically had simple arithmetic operations embedded somewhere in the expression. Sometimes such operations were also contained within bracket vector notation, and thus also being coded TVN (Figure 5.22) or without bracket vector notation (Figure 5.23). A fifth code for a non-productively activated resource describes responses that used ordered pair notation, “OP”. These responses often appeared as “(4,3)” or “(4\mathbf{i}, 3\mathbf{j})”. Much like the TVN response $<4\mathbf{i}, 3\mathbf{j}>$, a response of (4\mathbf{i}, 3\mathbf{j}) would also earn a UV code for using unit vectors. Finally, there was again an “other” category for question 5 for responses that we could not confidently interpret. This “other” code applied to only 5 total responses.

5.3.7. Methods for Combinations of Codes

The descriptions of codes and student examples above show that each response on each question has the potential to earn multiple codes, thus demonstrating the activations of multiple
5) Write an algebraic expression point \( P \) in Figure 2.

\[
\begin{align*}
\vec{r}(t) &= \langle 0,0 \rangle + t \langle 9,3 \rangle \\
&= \langle 0+9t, 0+3t \rangle \\
&= \langle 9t, 3t \rangle \\
B &= \{ \langle 9t, 3t \rangle \mid t \in \mathbb{R}^3 \}
\end{align*}
\]

Figure 5.20. An example student response for question 5 that was coded “LA” for using notational conventions consistent with linear algebra.

\[
\begin{bmatrix} 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 1 \\ 3 \end{bmatrix}
\]

Figure 5.21. An example student response for question 5 that was coded “LA” for using matrix notation frequently associated with linear algebra.

\[
\langle 2, 2 \rangle + \langle 2, 1 \rangle = \langle 4, 3 \rangle
\]

Figure 5.22. An example student response for question 5 that was coded “MO” for having a mathematical operation included. In this case, that operation is inside the bracket vectors on the left-hand side of the equation.
resources. As a consequence of our interest in the ideas present in each response, we did not collapse any response into a single code. In any individual example of a single-code response, it was because that single code represented the only resource for which we could see clear evidence of activation. This coding approach affords the opportunity to explore how combinations of codes can be used to characterize different response types. In other words, we can start to map different combination of codes – meaning combinations of resources activated – to actual responses allowing us to make predictions of the form “(code combination X) predicts (response type Y)”. For example, as was discussed for question 1, the combination of codes COMP + LAB + UL + ICD + 0×(non-productive codes) describes an unambiguously mathematically correct response. We also will argue that describing different response types as an emergent outcome of combinations of codes provides increased resolution with which student thinking can be targeted for instructional intervention. Such intervention could leverage productively activated resources and resolving conflicts between activated resources. Methodologically, we have not yet developed a robust procedure for identifying which code combinations should be analyzed more deeply and will in the future work to explore more combinations. At this time, we have identified two convenient examples of code combinations which will be more fully articulated in Section 5.4.2. However, one of these examples potentially sheds more light on what students actually understand – or don’t understand – about unit vectors and unit-vector notation. The other example is one that demonstrates how our coding-for-activated resources scheme did increase the resolution with which we can analyze student thinking and how that increased resolution resulted in a likely fruitful instructional target with an accompanying learning objective. Thus, from a methodological standpoint this paper presents how we interpret combinations of codes once they are selected and not necessarily how to select meaningful and illustrative combinations of codes. A future paper could delve into such identification.
Table 5.1. Question 1 number and proportion of students activating each resource.

<table>
<thead>
<tr>
<th>Productively Activated Resources</th>
<th>Non-Productively Activated Resources</th>
</tr>
</thead>
<tbody>
<tr>
<td>Correct</td>
<td>COMP</td>
</tr>
<tr>
<td>Number</td>
<td>19</td>
</tr>
<tr>
<td>Percent</td>
<td>20.2%</td>
</tr>
</tbody>
</table>

5.4. Results & Discussion

The resources identified in the Survey Analysis section are the answer to Research Question 1. Those resources are the ideas that students activate about unit vectors and their applications to other vector types, such as position vectors, in Cartesian coordinates in this survey. A complete reference table of those resources, their associated codes, abbreviated definitions for them, and select examples of each are provided in Appendix C.

The remainder of this Results & Discussion section will contain two main sections further divided into subsections. The first main section will provide results and discussion answering Research Question 2. The second main section will provide results and discussion for our first foray into exploring what combinations of codes can elucidate about student thinking about unit-vector notation, thus providing preliminary answers for Research Question 3. Each of the two main sections will also provide discussion about some of the implications for the results in those sections. Finally, a summary will be provided at the end of this broader Results & Discussion section to tie together any commonalities between the two main sections, point out any important contrasts, and comment about the implications of this work as a whole.

5.4.1. Prevalence of Code Activations

Tables 5.1-5 show the proportions of students as a percentage of 94 total students who activate each of the resources described in the Survey Analysis section. Recall that for a response to be deemed “correct” it must show evidence of activation of all the productive resources for that question and no evidence of activation of all of the non-productively applied resources for that question. Thus, the presence of a “correct” column in each table does not indicate that “correct” is itself a code, but that the associated percentage is the proportion of responses that had the code combination necessary to be considered mathematically correct.
Table 5.2. Question 2 number and proportion of students activating each resource.

<table>
<thead>
<tr>
<th></th>
<th>Productively Activated Resources</th>
<th>Non-Productively Activate Resources</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Correct</td>
<td>UL</td>
</tr>
<tr>
<td>Number</td>
<td>22</td>
<td>32</td>
</tr>
<tr>
<td>Percent</td>
<td>23.4%</td>
<td>34.0%</td>
</tr>
</tbody>
</table>

Table 5.3. Question 3 number and proportion of students activating each resource.

<table>
<thead>
<tr>
<th></th>
<th>Productively Activated Resources</th>
<th>Non-Productively Activated Resources</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Correct</td>
<td>SO</td>
</tr>
<tr>
<td>Number</td>
<td>29</td>
<td>42</td>
</tr>
<tr>
<td>Percent</td>
<td>30.9%</td>
<td>44.7%</td>
</tr>
</tbody>
</table>

The first notable trend in the data is the low rates of activation, <50%, of productive resources on questions 1, 2, 3, and 5 with the exception of ix on question 2, which was quite high. Question 4 had high activation rates of productive ideas SO and EP and an overall high correctness rate approaching 80%. However, given the low rates of productive resource activation and correctness on the other four questions, a notable finding is that the ability to correctly draw a position vector for a specified point does not predict activation of productive resources or correct answers on the other 4 questions.

Rates of individual non-productively activated resources were also quite low, rarely exceeding 25%. There are also a larger total number of non-productively activated resources than productively activated resources. These two trends taken together suggest that students have more varied ways of activating non-productive resources for these unit-vector concepts than they do for activating productive resources. That said, the sum of the activation rates of the CCV and CPV codes for question 1 approaches 40%, suggesting that students don’t activate the set of resources necessary to

Table 5.4. Question 4 number and proportion of students activating each resource.

<table>
<thead>
<tr>
<th></th>
<th>Productively Activated Resources</th>
<th>Non-Productively Activated Resources</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>Correct</td>
<td>SO</td>
</tr>
<tr>
<td>Number</td>
<td>72</td>
<td>85</td>
</tr>
<tr>
<td>Percent</td>
<td>76.6%</td>
<td>90.4%</td>
</tr>
</tbody>
</table>
Table 5.5. Question 5 number and proportion of students activating each resource.

<table>
<thead>
<tr>
<th>Correct</th>
<th>i+j</th>
<th>UV</th>
<th>r/r</th>
<th>LA</th>
<th>TVN</th>
<th>MO</th>
<th>OP</th>
<th>Other</th>
<th>NR</th>
</tr>
</thead>
<tbody>
<tr>
<td>Number</td>
<td>32</td>
<td>44</td>
<td>45</td>
<td>12</td>
<td>14</td>
<td>20</td>
<td>7</td>
<td>5</td>
<td>5</td>
</tr>
<tr>
<td>Percent</td>
<td>40.8%</td>
<td>46.8%</td>
<td>47.9%</td>
<td>12.8%</td>
<td>14.9%</td>
<td>21.3%</td>
<td>7.4%</td>
<td>5.3%</td>
<td>5.3%</td>
</tr>
</tbody>
</table>

Resolving to disentangle the concepts of unit vectors, position vectors, and component vectors when answering a question on basis vectors. We have more to say about students conflating the definitions of component vector and unit vector in the discussion of combinations of codes forthcoming (Section 5.4.2). Rates of non-productively activated resources on questions 3 and 4 also suggest that students find it challenging to translate their position vector sketches – which often do show activation of productive resources – to words that describe those sketches, or that they activate other resources when writing out a description of a position vector.

As our long-term goal is to develop curriculum that addresses the thinking we see in this survey and in our other work [2,3] (Farlow et al., in prep – see Chapter 3), we must also comment on the implications this data has for developing instructional materials. The first implication is that the ix resource – demonstrating the link between i and the x-direction – is quite strong and might be leveraged within instructional materials. Beyond that, instructional materials will also need to help students either create or activate productive resources such as UL and ICD in appropriate contexts. The activation rates presented here also suggest that students need further assistance in being able to describe the starting and ending points of a position vector and then translating a position vector sketch or description into an algebraic model.

5.4.2. Results for Combinations of Codes

Two examples of how we used combinations of codes to further understand some student responses will be discussed in this paper. The first code combination is more of a classification of code combinations. We called this classification “Likely Correct”. This class of code combinations arose from wondering if activating all four of the productive resources for question 1 was necessary to sufficiently determine if students understood unit vectors. The second example of code combinations was discovered during the coding process for question 2 and sheds additional light on how some
students were activating combinations of resources that likely led to their conflation of the definitions of *component vector* and *unit vector*. We will discuss both of these examples in turn.

### 5.4.2.1. “Likely Correct” Responses

There were a total of 21 responses to question 1 that contained three of the four productive resources and none of the non-productively activated resources. Eight of those responses were coded with COMP, LAB, and ICD. Responses with these three codes consistently had vector arrows without unit length, typically longer than one box length, such as can be seen in Figure 5.24. We called this the *too long* group. Nine responses had unit vector drawings that were illustrated productively and coded COMP + UL + ICD but were not labeled (Figure 5.25). This group is called the *no labels* group. The remaining 4 responses showed evidence of activation of UL, ICD, and were labeled (code LAB), but were incomplete. These appeared as having 1 arrow each originating from the two points of interest (Figure 5.26). These four formed the *incomplete* group. These three response types – and thus the combinations of codes that describe them – were of interest because on the surface they look like nearly correct answers. Furthermore, they appear to be the types of responses that might happen if a student who did have somewhat robust understanding of the content would give if they were in a hurry and/or not reading the directions carefully. Also, in the case of the *too long* group, a finding of Barniol & Zavala [50] suggested that at least some students who draw unit vector arrows in the correct directions but of incorrect length are able to write the proper length in words or give the correct length in an interview setting. Additionally, some of the wording and construction of the first iteration of the coded survey did not label the grid for question 1 and also did not specify that the unit vector sketches should be labeled. It’s reasonable to assume that some students may therefore have not felt the need to label their arrows or be sure to draw them at one box in length. Therefore, we looked at responses to other questions to see if the resources activated elsewhere could be used to fill in some of the “gaps” in these “Likely Correct” responses. In other words, we wanted to know if these “Likely Correct” responses could reasonably be counted as fully correct – or at least demonstrative of all of the productive thinking necessary to get a correct answer. As a test case to explore those questions, this paper will consider the *too long* responses in detail, and then comment briefly on the other two.
Figure 5.24. An example of a too long response for question 1. This response type was coded COMP+LAB+ICD and was one of our test cases for how using combinations of codes could improve our understanding of student thinking.

Figure 5.25. An example of a response to question 1 that was correct in all ways except for not being labeled. This response type was coded COMP+UL+ICD and was one our test cases for how using combinations of codes could improve our understanding of student thinking.
Figure 5.26. An example response from question 1 where the question directions may have been interpreted to mean “draw $\mathbf{i}$ at A and $\mathbf{j}$ at B”. This response type could be a test case to see if combinations of codes can be useful in shedding additional light on student understanding. However, this type of response occurred too infrequently in this dataset for further analysis.

5.4.2.1.1 The Too Long Response  The *too long* response came with the codes COMP, LAB, and ICD on question 1. The *too long* response lacks the UL code for question 1. Question 2 provides another opportunity for students to state in words that the unit vector should be of length 1. If a *too long* student had activated UL in question 2, their code combination would be COMP + LAB + ICD + UL(Q2). We reasoned that such a combination of codes across questions would be evidence that the student(s) understood the concept and were in fact being too quick or careless with their sketches. However, of the 8 students in the *too long* group, 7 did not show evidence of activation of UL on question 2. That also means 7 of 8 students in the *too long* group did not show evidence of activation of UL anywhere on the survey. It thus appears that our survey did not prompt the activation of the UL resource for the bulk of these eight students. The resources and framing theoretical framework is not equipped to say if a person does not possess a given resource, just that a given resource doesn’t activate. Our data shows that 7 of 8 students in the *too long* group do not provide evidence of activation of the UL code/resource in response to the questions on this survey. We cannot say if the UL resource is not there. Either way, it is true that in our data 7 of 8 students did not show evidence of activation of a resource we believe to be a necessary
piece in fully understanding the definition of unit vector, increasing doubt as to whether the *too long* response is representative of a high level of understanding with insufficient attention to detail.

It is also possible to treat the survey like a quiz. Scoring for such a ‘quiz’ would be the number of question responses deemed correct out of five. Such a score gives at least a rough indication as to the overall level of understanding demonstrated on any given student’s survey. Then, the average score of any subgroup of students – such as the *too long* group – can be compared to the average score of the whole sample. Such comparisons then give at least a rough indication of how the subgroup’s overall level of understanding compares to the overall level of understanding of the whole sample. The average score for all 94 students on this survey is 1.85/5 with a standard deviation of 1.26. For the eight *too long* students, the average score is a 1.13/5 with a standard deviation also of 1.13. Thus, the average score of the *too long* group is more than half a standard deviation below the average score for the whole sample. A parametric statistical test here is likely not appropriate given the sample size of 8 for the *too long* group. However, this difference in average scores is suggestive of a lower overall level of understanding of the survey content among the *too long* group. In an even higher contrast, the 19 students who gave a question 1 response deemed correct – meaning activating the combination of resources COMP + LAB + ICD + UL – averaged 3.26/5 with a standard deviation of 1.10. Therefore, we argue that the *too long* response, which again is the activated code combination of COMP + LAB + ICD on question 1, is not a “Likely Correct” response. The evidence in this survey sample suggests that such a response is actually indicative of a lower-than-average level of understanding of unit vectors in Cartesian coordinates. In retrospect, this result should not be surprising because that unit vectors are of unit length is of fundamental importance to the concept of unit vector. The non-activation of that idea could therefore be likely to lead to more general confusion about unit vectors.

5.4.2.1.2 The No Label Response  The *no label* response – code combination COMP + ICD + UL on question 1 – is more of a different nature than the *too long* response because the missing code, LAB, is not a content resource like UL and ICD. The LAB – and COMP – code are more ‘bookkeeping’ codes. They are codes for ideas that reduce the ambiguity of interpreting a response by making it clear which arrow is for $\mathbf{i}$ and which arrow is for $\mathbf{j}$, for example, but don’t provide additional insight into which ideas are present for the mathematical behavior of $\mathbf{i}$ and $\mathbf{j}$. Therefore,
the no label response actually shows activation of the key content resources UL and ICD for question 1. It would therefore be reasonable to hypothesize that the no label response is more representative of productive overall thinking on question 1 than the too long response. Unfortunately, there are no other unit vector questions on the survey that give students another opportunity to label vector sketches. However, using the approach of comparing average scores between the no label group and the whole sample, there is very little difference between the no label group and the whole sample. The nine students in the no label group averaged 1.78 with a standard deviation of 0.83. Again, the sample size of 9 for the no label group makes statistical testing unwise, but the average scores are close enough that in our sample there is likely to be very little difference in overall understanding of surveyed content between the no label group and the sample as a whole. However, this average is likely meaningfully lower than the 3.26 average seen by the 19 students who answered question 1 unambiguously correctly. So, does the no label response indicate a “Likely Correct” response? This code combination, from this data, does not appear to be indicative of a lower level of overall understanding of unit vector concepts than the sample as a whole, but is indicative of a lower level of understanding than the group that answered question 1 unambiguously correctly. However, given that performance on the survey – as measured by the number of correct responses – is generally low and the no label group scores at roughly the same level as the whole sample, it is probably not the case that the no label answer is “Likely Correct.”

5.4.2.1.3 The Incomplete Response The sample size on this code combination makes it extremely difficult to ascertain meaningful conclusions. Our comment will be that we hypothesize this code combination arises because of a particular reading of the wording of the question. The instructions are to draw \( i \) and \( j \) at points A and B. A small number of students may interpret that instruction as draw \( i \) at A and \( j \) at B. We are curious if this response type would persist in future surveys at the same institution and other institutions.

5.4.2.2. The Conflation Responses

In response to question 2, many students were stating that \( i \) is the “x component” of a vector or using a phrase to the effect of \( i \) being the “number of units in the x direction”. We initially had a CCV code similar to question 1 to account for these responses on question 2. Through discussions we realized that there were more ideas either present or absent in these responses than a single
Figure 5.27. A flowchart showing how the activation of the ideas UL and ICD on questions 1 and 2 work together to predict whether a response conflated the definitions of “unit vector” and “component vector”.

CCV code could capture. Upon that realization, we began to suspect that a response conflating the definition of *unit vector* with either the definition of *component vector* or the coordinates of a point is an emergent property of the activation of a certain combination or combinations of resources. One of those resources was the association of \( i \) with the \( x \)-direction, giving rise to the ix code. As coding progressed, it became increasingly clear that question 2 responses that were coded ix but did not show activation of ICD – which for question 2 means there is language stating that \( i \) is in the specifically positive \( x \)-direction – were at least the first step toward identifying a combination of resources to model a conflation response. There are 53 such responses for question 2. This group of 53 further broke down into 2 subgroups; a group of 26 students who also did not show evidence of activation of ICD on question 1, and 27 students who did show evidence of activation of ICD in their response to question 1. We will discuss each of these groups in the following two paragraphs. For clarity, a flowchart has been provided in Figure 5.27 to assist with breakdown of each of these two subgroups into yet smaller groups.

*Conflation Subgroup 1 – Code Combination ix + NO ICD(Q2) + NO ICD(Q1)*

This subgroup consisted of 26 total responses, or 27.7% of the whole sample. Figure 5.28 shows two examples of responses from this subgroup. In these two responses we see language “direction and size of a vector in the x direction” and “component of a vector w.r.t. [with respect to] the x-axis”. We interpret such statements to be conflating the definition of *unit vector* with the
Figure 5.28. Two example responses from the subgroup of 26 students who activated \textit{ix} on question 2 but did show evidence of activation of ICD on both questions 1 and 2. Both responses demonstrate a conflation of the definition of a unit vector with either the definition of a component vector or with the coordinates of a point.

The definition of \textit{component vector}. These two responses are also representative of 23 of the 26 responses in this subgroup, i.e. 23 of the 26 responses that had the code combination \textit{ix+NO ICD(Q2)+NO ICD(Q1)}, accounting for 24.5\% of the entire survey sample. Of the three that did not provide an answer consistent with this specific set of codes, two were coded CPV on question 1 and the other simply labeled the points A and B as \textit{i} and \textit{j} respectively on question 1.

\textbf{Conflation Subgroup 2 – Code Combination \textit{ix + NO ICD(Q2) + ICD(Q1)}}

This subgroup consists of 27 students for 28.7\% of the whole sample. Within this subgroup of 27, there were 16 students (17.0\% of whole sample) who gave a response very similar in nature to those conflationary responses of subgroup 1, as can be seen Figure 5.29. That leaves 11 students (11.7\% of whole sample) who did not give a conflationary response. Two example responses from these 11 can be seen in Figure 5.30. In the responses from these 11 students, we consistently see language that leads to the \textit{ix} code – “\textit{i} is a vector in the \textit{x}-direction...”, but there is no statement specifically saying that the important \textit{x}-direction is the positive \textit{x}-direction, thus meaning they would not be coded as ICD for question 2. However, there is one key apparent difference between these non-conflationary responses and the 16 conflationary responses from the subgroup: the presence of the resource that \textit{i} is of unit length, or code UL. The two examples shown in Figure 5.30 both clearly specify \textit{i} has having a magnitude of one. In fact, 10 of the 11 non-conflationary responses in this subgroup had that or similar language. By contrast, 15 of the 16 conflationary responses in this subgroup did not have any language leading to the activation of UL on question 2. These activation patterns suggest two things: 1) the activation of ICD in question 1 is necessary but not sufficient to be a predictor of a conflationary responses in question 2; and 2) once ICD has been activated, it is the activation or non-activation of UL in question 2 that predicts a conflationary
provide a definition of i using words.

\[
\begin{align*}
\hat{i} \text{ is the part of } \\
\text{the vector that goes to the left and right.}
\end{align*}
\]

provide a definition of i using words.

\[
\begin{align*}
\hat{i} \text{ is the } x \text{-component or } \\
\text{vector component, } \begin{pmatrix} x \\ y \\ z \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \\ 0 \end{pmatrix}.
\end{align*}
\]

provide a definition of i using words.

\[
\begin{align*}
\hat{i} \text{ is a vector in the } x \text{-direction that has a magnitude of one.}
\end{align*}
\]

provide a definition of i using words.

\[
\begin{align*}
The \text{ unit vector (with magnitude } 1) \text{ in the } x \text{-direction.}
\end{align*}
\]

2) Provide a definition of i using words.

Figure 5.29. Two examples of student responses from the subgroup of students with question 2 code combination ix + NO ICD(Q2) + ICD(Q1). This group consisted of 16 students who gave responses like these two. These responses are also nearly identical in nature to the 23 of 26 students who gave conflationary responses to question 2 with code combination ix + NO ICD(Q2) + NO ICD(Q1).

Figure 5.30. Two example student responses from the subgroup of students with question 2 code combination ix + NO ICD(Q2) + ICD(Q1) who did not give a conflationary response to question 2. There were 11 such students. The key difference between these responses and those from the same ix + NO ICD(Q2) + ICD(Q1) subgroup who did provide a conflationary response appears to be the activation of the code UL with the explicit statements of I having length/magnitude of 1.

Looking at both subgroups together, we see that there are combinations of codes that describe response types quite well. The question 2 combinations of codes ix + NO ICD(Q2) + NO ICD(Q1) + NO UL(Q2) and ix + NO ICD(Q2) + ICD(Q1) + NO UL(Q2) resulted in responses that conflated the definition of unit vector with the definition of component vector for over 90% of opportunities for each combination. The combination of codes ix + NO ICD(Q2) + ICD(Q1) + UL(Q2) represented a response type that did not conflate the definition of unit vector with the definition of component vector for over 90% of the relevant cases.

5.4.3. Comments on Combinations of Codes

This discussion of the 53 survey responses that had ix activated for question 2 but not ICD is important for 2 reasons. The first is that it validates our coding scheme and process to some degree. The second is that it has implications for future curriculum development. The validation
of our coding scheme is found within the consistency with which the UL resource either appears in those responses that don’t conflate definitions or does appear in those that do conflate definitions. Since UL is a key resource in fully understanding the definitions of unit vectors, it makes sense that its presence is an indicator of whether students are able to disentangle the definitions of these various vector types. In light of that, we interpret this consistency as our coding scheme effectively drawing out the relevant resources that either are or are not present in responses of various types. As far as curriculum development, the conflation of the definitions of different vector types is an area of curricular emphasis. The total number of students who provided a conflationary response to question 2 is 39, which is nearly 40% of the total survey sample. That’s a rather large proportion of students who are confusing some key resources that are essential for understanding unit vectors and their applications. That being able to explicitly state in words that unit vectors should be of length 1 – code UL on question 2 – appears to be the key predictor of a conflationary response within the 53 students who activated the code combination ix + NO ICD for question 2 suggests a learning objective toward targeting this conflationary thinking: use words to define a unit vector as having magnitude 1.

The focus of this combinations of codes discussion is not to develop a procedure for identifying combinations of codes, but to provide examples of how combinations of codes can represent certain response types and be used to further elucidate student thinking. In other words, the methodology in this paper does not focus on how to find which combinations of codes are common on a given question or group of questions, but the methodology here does provide some insight on how to interpret known or easily-identifiable combinations of codes. In the two examples of code combinations in this paper – the “ Likely Correct” response from question 1 and the conflation of the definitions of unit vector and component vector in question 2 – each offered interesting insights into student thinking. The “ Likely Correct” responses suggested that answers that look like they represent good thinking but with insufficient attention to detail may in fact not be indicative of overall productive thinking on a question or of understanding of the content generally. The conflation responses data were able to show that the resource of unit vectors being of unit length is of vital importance in helping a student disentangle the definitions of component vector and unit vector. Future work could include administering more surveys to students from the same university and other colleges and universities to bolster sample size to sufficient levels that computer programs could be used
to find patterns in code combinations both within each survey question and across all five of the coded survey questions. A larger sample size would also lend itself better to doing more in-depth statistical analyses to compare the overall survey performance of groups of students with certain code combinations to performance of the whole sample. It should also be noted that such quantitative data is more consistent with the methodologies of the identifying student difficulties framework than the resources and framing framework [62, 121].

5.5. Implications, Limitations, and Conclusions

5.5.1. Implications

This work defines a set of resources that late-semester multivariable calculus students activate—both productively and non-productively—when they think about problems involving unit vectors and position vectors in Cartesian coordinates. This set of resources will be useful in informing the development of instructional materials that leverage the productive activation of resources and guide students toward an epistemological shift regarding non-productively activated resources. Such instructional material development will be guided by similar efforts within the resources and framing theoretical framework [71, 89]. The findings herein are also helpful for current upper-division physics course instructors. Such instructors should be aware that students entering middle- and upper-division physics courses may struggle to productively activate resources necessary to successfully use unit vectors and unit vector notation while learning about physics.

5.5.2. Limitations

As was mentioned in a couple places but is worthy of repeating, one limitation of this study is sample size. The n of 94 for the overall sample is reasonable on its own. The sample size limitation starts to assert itself when comparing a subset of that 94 to the overall trends of the 94, such as was attempted with the too long and no label responses. As such, we do not make claims of statistical significance and we are not attempting to make general claims about what all math students are thinking. Rather, we are looking for ideas that may or may not be useful in informing future curricular development on this content.

A second limitation is found in choosing what the codes for each question should be. Data analysis for the broader project has been guided by the theoretical framework of resources and framing. One aspect of that framework that needs further development is the process of identifying individual resources. Our case study paper [2] and, to a lesser extent, our pattern-matching paper
(Farlow et al., in prep – see Chapter 3) identified individual resources through a combination of content expertise – knowing what content ideas were important in understanding a topic – and repeated viewings of interviews to see what the underlying ideas were upon which students were basing the rest of their thinking. The process of choosing which ideas to code in this survey analysis was similar. As the Methodology section states, some of the codes for the questions were connected to or in fact identical to resources identified in the case study [2]. Some codes were further refinement of some of the resources identified in the case study and observed in the pattern-matching paper (Farlow et al., in prep – see Chapter 3). Other codes were emergent and required content expertise on our part. It was left to us to determine what “nuggets of mathematical knowledge” were fine-grain enough for identification as a resource, and thus a code. These resources appeared to us, after reading numerous student responses, to be the key ideas that were either leading a particular student’s thinking in a productive or non-productive direction for any individual survey question. That process requires a fair amount of interpretation. We acknowledge that other researchers with other teaching and research experiences and perhaps deeper levels of content expertise might observe different ideas that lead the analysis in a different direction. However, our resource identification process resulted in showing that a canonical idea important to unit vectors is a predictive factor in whether unit vector resources are conflated with on another. This result gives us some confidence that our interpretations are reasonable enough to provide useful information for designing instructional materials.

The third limitation of note is the relatively novel methodology of our analysis. In some sense we have combined the coding process of the identifying student difficulties framework [62,63,64,121] with the activation/non-activation of productive and non-productive ideas that is fundamental to a resources and framing theoretical framework [18,35,36,37,38], and then piloted the use of patterns of activated/non-activated resources to characterize certain response types. This methodology is one that we had to thoughtfully form and reform through our analysis. While we were as systematic as possible, it is also true that in developing a new methodology, we likely missed some steps or incorporated some steps that should not belong. That said, the test case of conflationary responses discussed in this paper are also some indication that our assembling and interpretation of code combinations may prove useful. The code combination shows how evidence for either the activation or non-activation of a canonical resource for understanding unit vectors – UL – affects student
responses in both productive and non-productive ways. That means that our coding methodology identified an idea of fundamental mathematical important to unit vector concepts was the result of our coding methodology.

5.5.3. Conclusions

As part of broader effort to develop research-based instructional materials to build a curriculum for a math methods course for undergraduate physics students, we surveyed 94 multivariable calculus students during the last week of their Calculus III courses about their understanding of basis unit vectors and unit vector notation in Cartesian coordinates. Unit-vector notation is an important aspect of the mathematical language used in upper-division physics courses [1,24,25,26,27,28]. We used a coding scheme guided by a theoretical framework of resources and framing [18,35,36,37,38] to identify fine-grain ideas students bring to bear while answering questions about unit vectors and some applications of unit vectors, both resources that were productively activated and resources that were non-productively activated. We also used methods consistent with the theoretical framework of identifying student difficulties [62,63,64,121] to determine the proportions of students who activated all of those resources. It was found that less than 50% of students activate most of the productive resources identified. Students showed evidence of non-productively activating individual resources at rates typically around 25% or less, but there are generally far more resources that could be non-productively activated for each question than resources that could be productively activated. This coding scheme also afforded the opportunity to characterize different response types with combinations of activated resources. We piloted this process in two areas: determining if responses that on a surface level look like representations of good thinking but with poor attention to detail — such as drawing unit vector arrows in the correct directions but more than 1 x-y coordinate grid box in length – are in fact representations of good thinking and showing how combinations of resources can predict whether a student is likely to conflate the definition of component vector and unit vector. Our data suggests that those responses that look reasonably good on a surface level are in fact indicative of a lower-level of understanding than the drawing appears to show. Our data also suggests that certain combinations of activated resources are in fact predictive of certain response types. We did not develop a procedure for identifying combinations of codes because the sample size was not large enough to do so sufficiently. However, we have demonstrated a procedure by which known combinations of activated resources can characterize response types and even suggest
a learning objective. In this case, the suggested learning objective was that students should be able to state in words that unit vectors have a magnitude of 1.

This work is an important step in shedding light on students’ thinking about unit vectors and some applications of unit vectors. Such content is vital in communicating mathematically about physical quantities such as position, velocity, acceleration, forces, and electromagnetic fields; content which forms core concepts in upper-division physics [1, 24, 25, 26, 27, 28]. The results herein will inform the future development of instructional materials. In the interim, current instructors of upper-division physics courses should be aware that there is a reasonably high probably that their students will not be as comfortable with unit vectors and unit-vector notation as they typically seem expected to be.
6. CONCLUSIONS

6.1. Conclusions

Mathematics is often described as a language [6,7,8,9,18]. Mathematics is also a critical aspect of communication within physics [10,11,12,13,14,15,16,17]. Yet, the math used in mathematics and the math used in physics often functions as two different dialects of the same language, or as two different languages [18]. The research presented in this dissertation describes work that is part of a broader effort to develop research-based instructional materials to help upper-division physics students translate across that math-physics interface. To that end, the work described herein was an investigation of physics students’ thinking about vector concepts in Cartesian and non-Cartesian coordinates and their preparation for the same in their mathematics courses taken prior to upper-division physics. Two studies probed physics students’ thinking, a third study looked at the relevant mathematical instruction students receive in calculus textbooks and a fourth study investigated student thinking at the end of a Calculus III course. The physics studies used think-aloud interviews to deeply explore students’ in-the-moment thinking as they solved math and physics problems requiring the use of vector concepts in primarily non-Cartesian coordinates, with some inquiries into Cartesian thinking for comparison purposes. The mathematics studies were a textbook analysis and an analysis of students’ written responses to math questions about vector concepts in Cartesian coordinates.

6.1.1. Summary of Findings by Chapter

In Chapter 2, a case study of a high-achieving, senior undergraduate physics major was presented. A resources framework allowed us to identify cognitive resources that the student activated while solving the problems during a semi-structured, think-aloud interview protocol. For our analysis, we grouped those resources by the context of the questions being asked, creating resource groups for unit vectors, position vectors, and velocity vectors. The Velocity Vectors Resources group also allowed us to track, chronologically, how activation of resources led to the activation of other resources. Doing so provided an opportunity to see how the non-activation of a key and expert-identified resource led the case study subject to abandon previously activated productive resources and attempt to redefine unit vector concepts in spherical coordinates in a way that was consis-
tent with how unit vectors behave in Cartesian coordinates. More generally, the identification and grouping of resources in the case study gave us a starting point for analyzing the other six physics interviews in our sample.

In Chapter 3, we conducted a detailed analysis of what has been previously identified as the “pattern-matching” response type. “Pattern-matching” was first described by Hinrichs [57] as writing algebraic position vector expressions, using unit-vector notation, in the morphological form used in Cartesian coordinates. All seven of our physics interview subjects answered multiple position vector questions in a manner consistent with that definition. We found that resources that are productive in Cartesian coordinates are frequently activated and applied to non-Cartesian contexts, often unproductively. The activated resources include a cluster of resources that we call \( \vec{r} \) has form \( \vec{r} = a\hat{a} + b\hat{b} \) as well as a Navigation Resource that we observed being used to write a position vector by tracking a component-like path from the origin to a point. Emergent understanding of unit vectors and their applications was also observed in the interview data. This emergence often appeared as interview subjects activating the \( \vec{r} \) has form \( \vec{r} = a\hat{a} + b\hat{b} \) resource regardless of whether they had previously activated productive unit vector resources, such as Increasing coordinate direction. The consistency with which resources productive in Cartesian coordinates were inappropriately activated in non-Cartesian contexts was somewhat surprising. This surprise prompted our investigation into the mathematical preparation about vector concepts in various coordinate systems students receive before taking upper-division physics courses.

In Chapter 4, we transitioned from exploring physics students’ thinking to investigating the Cartesian and non-Cartesian content of textbooks for Calculus courses. A detailed analysis of seven popular calculus textbooks identified the proportions of the content – including bolded definitions, worked example problems, and end-of-chapter exercises – that were based in each of Cartesian, polar, spherical, cylindrical, and any combination of those coordinates. The overwhelming majority of that content was either explicitly based in Cartesian coordinates or assumed Cartesian coordinates as the default coordinate systems. Mention of non-Cartesian unit vectors was only identified in a single instance in one Calculus book [5]. Qualitatively, non-Cartesian content was heavily scaffolded and explicitly cued. In general, if the texts expected non-Cartesian coordinates to be used, the instructions would explicitly say so, or would provide a problem that was very convenient for a single coordinate system. By contrast, rarely did textbooks explicitly cue the use of Cartesian
content, it was implied that this was the default and only deviations from that default warranted special attention. These results established a hidden curriculum [109,110,111,112,113,114,115] that painted Cartesian coordinates as the default coordinate system and non-Cartesian coordinates as something to be used far less often and helpful in a small number of situations.

In Chapter 5, we surveyed multivariable calculus students during the last weeks of their Calculus III courses about unit vector and position vector concepts in Cartesian coordinates. We coded their written responses to those surveys for the activation of both productive resources and non-productively activated resources [18,35,36,37,38] for each survey question. Some of those resources were resources that had been previously identified in our students’ unit vector thinking [3], case study [2] and pattern-matching analyses (Farlow et al., in prep – see Chapter 3). Other resources emerged from the survey responses themselves. We found that the proportions of students who activated productive resources for each question were generally low and that there were many more non-productively activated resources than productive ones. The coding for both productively and non-productively activated resources also allowed us to synthesize research methodologies consistent with both a student difficulties theoretical framework [62,63,64,121] and the resources and framing theoretical framework [18,35,36,37,38]. This synthesis and coding methodology allowed us to pilot a methodology for using combinations of codes – i.e., combinations of productively and/or non-productively activated resources – to see how different combinations of codes can describe different response types. We used combinations of codes to show that within our survey sample, students who drew unit vectors in their correct directions and correctly labeled them but drew them longer than one unit in length generally had a more emerging understanding of unit vectors than students who drew them in their correct directions, correctly labeled them, and drew them one unit in length. Combinations of codes also showed that students who realize that the unit vector $\mathbf{i}$ is linked to $x$-direction but did not specify that direction as the positive $x$-direction were quite likely to conflate the definitions of ‘unit vector’ and ‘component vector’, unless they also specified in words that $\mathbf{i}$ is of unit length. If such students did specify in words that $\mathbf{i}$ is of unit length, they were much more likely to effectively disentangle the definitions of ‘unit vector’ and ‘component vector.’ These conflation-predictive responses suggest that a learning goal for future instructional materials is for students to be able to explicitly define unit vectors as having unit length when asked to define a unit vector in words.
6.1.2. Synthesis of Findings

A synthesis of the all of the results just described led to the first clause in this dissertation’s title: *Square Peg Thinking, Round Hole Problems*. This title appropriates the colloquialism of fitting square pegs into round holes to illustrate the type of thinking that we observed from physics students. Students use multiple ways of activating square peg thinking – meaning ideas or cognitive resources productive in Cartesian coordinates and unit vectors – and applying that thinking to round hole problems – situations effectively modeled by polar, spherical, or cylindrical coordinates and unit vectors. The calculus textbook analysis in Chapter 4 further supports that description by showing that the vast majority of material presented to students from those calculus textbooks is in Cartesian coordinates. This notion of square-peg thinking is also consistent with some research findings in the Research in Undergraduate Mathematics Education community that show that understanding of mathematical concepts is often tied to the coordinate system in which those concepts are learned [40,41,42]. Furthermore, the cognitive psychology theory of pattern recognition demonstrates that it is a common cognitive process for the human brain to use the familiar to make sense of and make predictions about the unfamiliar [74,75,76,77,78,79]. Our results show that when students were uncertain of how unit vectors behaved in non-Cartesian coordinates, they often used the rules of Cartesian coordinates to inform their thinking. In sum, the totality of the work presented in this dissertation shows that physics majors are often coming into their upper-division physics courses with wide-ranging ideas about vector concepts that are both productive and non-productive for various contexts but heavily based in the rules and conventions of Cartesian coordinates. Then, by the time they reach the upper-division undergraduate level and beyond, and are asked to use non-Cartesian coordinates, their first instinct is often to apply the rules of Cartesian coordinates whether or not applying those rules is productive. These findings have implications for both research and teaching.

6.2. Implications

6.2.1. Implications for Research

The research on student reasoning utilizes a resources framework [18,35,36,37,38] as a lens for investigating physics students’ thinking and their preparation for the mathematical concepts they will encounter in upper-division physics coursework. By developing think-aloud interview
protocols and interviewing physics students we were able to construct a picture of what physics students might be thinking about some vector concepts in Cartesian and non-Cartesian coordinates. Through a calculus textbook analysis and a survey we were also able to investigate their mathematics instruction about those ideas as well as the outcomes of that instruction that are most applicable to the upper-division content. This same basic procedure can be applied to other mathematical ideas essential to upper-division physics curricula. A resources framework also incorporates cognitive psychology theories to identify resources and resource clusters as being toy models for neurological models of the brain [18,38]. Thus, the research presented herein serves as a methodological guide for identifying individual resources and grouping them.

6.2.2. General Implications for Teaching

The end goal of this research project is to develop research-based instructional materials to help upper-division physics students translate across the math-physics interface. The work described in this dissertation identifies some areas of curricular emphasis in the context of unit vectors and unit vector notation in Cartesian and non-Cartesian coordinates. One such area is the wide-ranging ideas, both productive and non-productive, students have about these concepts before and after taking upper-division physics courses. Another area is the student propensity to apply the rules and conventions of Cartesian coordinates to non-Cartesian contexts. The interview results described in Chapters 2 and 3 suggest that current instruction about the relevant vector concepts in the various coordinate systems is insufficient in helping students bridge the gap of content between middle-division mathematics courses and how such content is used in upper-division physics courses. Thus, in the interim, instructors of upper-division physics courses should be aware that their students are likely coming into their courses with wide-ranging ideas about the essentially mathematics of those course. These wide ranging ideas are particularly applicable to some of the mathematics of Electromagnetic Theory and Classical Mechanics. Those instructors should also be aware that when students are uncomfortable working with non-Cartesian coordinates, they are likely to first try applying the rules and conventions of Cartesian coordinates and unit vectors in those non-Cartesian contexts.

Another possible instructional implication is that individual instructors could consider using more specific notation. By convention and across upper-division physics texts [1,24,25,26,27,28] and current math methods texts [4,23], \( \hat{r} \) is the standard notation for radial unit vector in polar
and spherical coordinates (a different symbol is sometimes used for the radial term in cylindrical coordinates). This symbol is used for all radial unit vectors. As the Mathematics Discussion points out, this notation is ambiguous (see Sec. 1.2). Without any additional contextual information, \( \hat{r} \) has infinite potential meanings but can only have one meaning for a given expression. We hypothesize that individual instructors could ameliorate some of their students’ likely confusion with \( \hat{r} \) or other location-dependent unit vector symbols by using more notational specificity in their instruction. Such specificity could be as simple as using the names of points as a subscript for the associated unit vector(s). For example, in the 3DQ used in Chapters 2 and 3, the position vector expressions for points B and C are \( \vec{r}_B = 3\hat{r} \) and \( \vec{r}_C = 3\hat{r} \), respectively. The right hand sides of the expressions are identical. However, a quick notational addition such as \( \vec{r}_B = 3\hat{r}_B \) and \( \vec{r}_C = 3\hat{r}_C \) might help students more immediately understand that the respective \( \hat{r} \) symbols represent two different basis vectors. Such a notational addition does not require the underlying curricula to change and could be implemented in both traditional lecture and active-learning-based instructional methodologies. Again, this notational addition would be performed by individual instructors and their students; textbooks still use the \( \hat{r} \) without subscripts.

### 6.2.3. Research Informing Curriculum Development

Given the extent of our investigation into student reasoning and work on a calculus textbook analysis about Cartesian and non-Cartesian coordinate systems we can identify several foci for curriculum development. There are numerous models for curriculum development in PER [122] that are categorized as research-validated active learning strategies and curricula. The framework of resources and framing emphasizes a strategy of “refining intuitions” and promoting student sense-making, and is most notably described by Elby [71] and Hammer [35]. There are also a significant source of materials for the introductory sequence named Open Source Tutorials based on this model available at http://umdperg.pbworks.com/w/page/10511239/Tutorials%20in%20Physics%20Sense-Making [88].

Using this model, instructional materials present tasks where particular resources are known to be elicited. This eliciting is done in situations where the students’ activated resources may or may not be productive. The materials then guide students to consider features of the problem under consideration and their activated resources. Materials might then ask students to refine their activated resources within the given context, or perhaps to consider additional resources that might
prove useful. A primary goal of this strategy is to avoid conveying the notion that ideas are “wrong,” but rather, ideas are potentially productive, but perhaps not in the existing situation. We’ve used this language within the previous chapters, attempting to classify the activation of resources that may have been used productively or non-productively depends on the specifics of the problem. We elaborate here on what instructional materials for this content might address.

Chapter 4, the analysis of calculus textbook content, does not feature a resources theoretical lens. However, it does provide insight into the kind of instruction and problems students see in mathematics courses before instruction in upper-division physics courses. Curriculum should be written with an understanding that students likely have little experience with non-Cartesian coordinates, and almost zero experience with using non-Cartesian unit vectors. Students will likely have limited practice with selecting an appropriate coordinate system for a problem (our study did not investigate this particular aspect of student thinking), however it’s reasonable to expect that students would need support in considering a problem that asks them to do such a task. The results from Chapter 5 show that many students struggle to draw or write out Cartesian unit vectors and positions vectors in Cartesian coordinates. The proportions of students who exhibit such struggles give us a baseline of what to expect from students coming into a Math Methods course. We can’t expect that most students will activate productive resources when answering questions about Cartesian unit vectors. First steps in any curricular sequence should begin with fundamental ideas of Cartesian unit vectors and giving students opportunities to explore their utility in physics contexts.

Results from Chapter 3 give us additional insight on students’ activated resources in this content area. Several students attempted to write down non-Cartesian unit vectors in terms of their translations from Cartesian unit vectors. For those that did, the resources they activated were insufficient to derive a mathematically correct definition, such as \( \vec{r} \) has form \( \vec{r} = a\hat{a} + b\hat{b} \) for planar polar coordinates. However, this might be a fruitful resource to explore through curriculum development. This is a resource that students activated without an explicit prompt from the interview protocols. If these questions prompt activation of these ideas for some students, it might be worth prompting it as part of a curriculum.

From Chapters 2 and 3, we find a number of other resources and resource clusters that students regularly activate. It would seem helpful for curricular materials to elicit the \( \vec{r} \) has form
\[ \vec{r} = a\hat{a} + b\hat{b} \] resource and guide students to consider why they might be using such a resource, because it’s great for Cartesian vectors, and then to refine that resource for non-Cartesian vectors. For instance, asking students to identify the directions of the spherical unit vectors at a point could help students start engaging with unit vector behaviors. Next, students could write down a position vector in terms of \( \hat{r}, \hat{\theta}, \) and \( \hat{\phi} \), as our interview questions 3DQ and SQB do, to that point (likely activating the \( \vec{r} = a\hat{a} + b\hat{b} \) resource). Final steps could potentially be asking students to reconcile, perhaps though pointing with their fingers, what direction the position vector is pointing, and how that aligns with \( \hat{r} \). Useful follow-up questions could include asking about what portion of this radial position vector points the direction of \( \hat{\theta} \) or \( \hat{\phi} \).

A series of instructional tasks could focus on the Navigation Resource we identified in Chapter 3. This resource was not apparent in all interview subjects, so some care must be taken in drafting these tasks. Perhaps a sequence of tasks could follow the section that elicits the \( \vec{r} \) has form \( \vec{r} = a\hat{a} + b\hat{b} \) resource. Students could then be asked to reason through how a Cartesian position vector expression can be read as directions for traveling a path from the origin to the point of interest. Once the connections between each component vector in the position vector expression and the “movement” they define are made, students could be asked to do a similar exercise for a polar or spherical position vector. At this point, students might activate resources that conflict, much the way our interview subject Jack activated conflicting resources when he began to realize that the behavior of \( \hat{r} \) makes \( \hat{\theta} \) and \( \hat{\phi} \) terms unnecessary (see Section 3.4.2). Through resolving those conflicts, it is possible students might see why a polar/spherical position vector does not provide navigation instructions in the same way that Cartesian position vector expressions can. However, a challenge for developing curriculum in this instance is finding tasks that engage students with the navigation resources whether or not they activate it initially.

Considerations from an expert perspective could also be considered. For instance, asking students to consider the units of a \( \theta\hat{\theta} \) term in a position vector might also be productive in guiding students to refine their intuitions, because position vectors have units of distance, and \( \theta \) is unitless. This type of pedagogical step is informed from the work of Heron [121] and the identifying student difficulties framework and does not rely on documented student activated resources.
Research-based curriculum development is informed by a theoretical framework and/or a theory of learning and requires an iterative process of creation and revision informed by either classroom, clinical testing, or both. The development of curriculum will proceed in the near future.

6.3. Future Work

The above outlines three clear avenues of future work. The first is to develop the instructional materials that leverage the findings herein to help students translate across the math-physics interface of vector concepts in various coordinate systems. The second is to continue using the basic methodology described in this dissertation to identify areas of curricular emphasis for other mathematical concepts frequently used in upper-division physics course curricula. Some of those other mathematical concepts include, but are not limited to, complex numbers, linear algebra, or differential equations. The third avenue is already underway and is to define the other side of the apparent instructional gap between middle-division mathematics courses and upper-division physics courses within the context of vector concepts in Cartesian and non-Cartesian coordinates. There is a discrepancy between the overwhelming Cartesian-centric nature of calculus textbook content and the fact that Griffiths’s *Introduction to Electrodynamics* textbook [1] and multiple classical mechanics textbooks [24, 25, 26, 27] provide explicit instruction on unit vector behavior in non-Cartesian coordinates in their first chapters and return to that content later in the books. This discrepancy certainly suggests an instructional gap between the two disciplines and the results of our physics student interviews suggests that said gap is not being sufficiently bridged by current instruction within the two disciplines. However, the presence of such instruction in the physics texts does not directly provide a clear picture of the expectations these textbooks have for how unit vectors in their respective coordinate systems are to be used and understood. Therefore, an analysis of physics textbooks using a similar approach to how our research team analyzed calculus textbooks is also warranted.

Analysis of upper-division physics textbooks is currently underway. Preliminary results into this effort were presented in a poster at the American Association of Physics Teachers conference in summer 2019 [123]. Preliminary results show that even within the early chapters of three common upper-division physics textbooks – the first through third chapters of Taylor’s mechanics textbook [24] and Thornton & Marion’s mechanics [25] and the first through fifth chapters of Griffiths’s EM – the proportions of content in the various coordinate systems take a substantial shift toward
using non-Cartesian coordinates. This shift is most pronounced in Griffiths’s EM where nearly half of the content that uses a coordinate system is presented in either spherical or cylindrical coordinates. Furthermore, explicit cues for coordinate system use become quite rare in the physics texts, when compared to the explicit cues and instructions in the Calculus textbooks, (see Chapter 4), representing a significant shift in what students are expected to do when solving a problem when moving from math courses to physics courses. Additional analysis of upper-division textbooks and instruction are required before any definitive claims can be made. Once this analysis is complete the physics side of the apparent instructional gap between mathematics and physics courses regarding unit vectors and unit vector notation in Cartesian and non-Cartesian coordinates will be more clearly defined.

6.4. Limitations

The primary data for the first two studies in this dissertation were individual think-aloud interviews [91, 92]. An inherent limitation to such interviews is that verbalizing one’s thought processes takes a portion of one’s available thinking capacity at any given time. As a result, there is less cognitive capacity available to think about the problem being solved or question being asked. This limitation was present in all of the interviews analyzed here. However, we hope the impacts of this limitation are minimal given that there was remarkable thematic consistency in our interview data of bringing “square peg thinking to round hole problems.” Such consistency is suggestive of a robust outcome despite this cognitive load limitation. Another limitation is the sample sizes of both our interview dataset for chapters 2 and 3 and survey dataset for chapter 5. In the interview case it is a near certainty that only seven interviews from an all-male and white-majority dataset did not capture all of the ways in which students or subjects could approach these problems. As such it is likely there will be additional student thinking not identified in this work that could be investigated and might further inform the development of curricular materials. In the survey study, the overall sample size of 94 is a small concern when considering some categories of combinations of codes had a total of less than 10 respondents. These small samples make statistical tests less useful due to issues of significance, and make generalizable claims difficult. However, as a demonstration of student thinking coming out of middle-division mathematics, and as example of how combinations of codes can further illuminate student thinking, we feel that both the findings and the approach are of worth to the PER community.
A point of consideration in the use of a resources framework. As a theoretical framework, resources is widely accepted within physics education research but a set of methodologies for identifying resources and their activations has yet to be formally established and widely embraced [38]. Thus, work in this dissertation represents an early effort to name individual resources and how they may or may not be activated individually or in clusters or groups. We therefore acknowledge that there are likely other ways to describe and define the ideas observed in these data. Such alternate descriptions, such as Sherin’s symbolic forms [93], could shed additional light on or reframe the conclusions of this work. That said our identification efforts in this work were rigorous and we contend that our descriptions of the resources identified herein are an accurate representation of students' thinking. Those resources we identified as productive each describe one aspect of thinking about mathematical or physical concepts that are consistent with the accepted mathematical and physical definitions of those concepts. This consistency provides some evidence that our identifications of resources are reasonable.

6.5. Funding Acknowledgements

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[34] “High School: Number and Quantity » Vector & Matrix Quantities » Represent and model with vector quantities. » 3 | Common Core State Standards Initiative.”


APPENDIX A.  CALCULUS III STUDENT SURVEY

VERSION 1
1.) Consider the motion of a particle in a plane as shown as it spirals inward toward the origin. For each motion, answer the question given.

A. Please draw arrows on points A & B to indicate the directions of the polar coordinate vectors $\mathbf{r}$ and $\mathbf{\theta}$ on the diagram below and label them clearly. (you may draw your arrows originating at points A & B). Please explain why you drew your arrows in the direction you did.

B. Assume the magnitude of the radial distance of the particle shown above is $r = R_0 - b\theta$ and the angle $\theta = \omega t$, where $R_0$, $b$, and $\omega$ are constants.

Determine expressions for the following in terms of the given variables, constants and the unit vectors $\mathbf{r}$ and $\mathbf{\theta}$.

What is the position vector $\mathbf{r}$? Please explain.

What is the velocity vector $\mathbf{\dot{r}}$? Please explain.

What reasoning would an expert mathematician give for the existence and use of non-Cartesian coordinate systems, such as polar coordinates? Do you agree with this reasoning? Why or why not? *Remember,* this is anonymous so please be honest.
APPENDIX B.  CALCULUS III STUDENT SURVEY
VERSION 2
1) Draw and label the standard basis vectors \( \mathbf{i} \) and \( \mathbf{j} \) at points \( A \) & \( B \) in the rectangular coordinate plane in Figure 1.

2) Provide a definition of \( \mathbf{i} \) using words.

3) Provide a definition of 'position vector' using words.

4) Draw the position vector for point \( P \) in Figure 2. Do this drawing on Figure 2.

5) Write an algebraic expression, in standard basis vector notation, for the position vector for point \( P \) in Figure 2.

There are more questions on the reverse side.
6) Transform the rectangular coordinate expression, $x^2 + y^2 = 4$ into polar coordinates.

7) Transform the polar coordinate expression, $r = \sin \theta$ into rectangular coordinates.
## Question 1 Codes

### Productively Activated Resources

<table>
<thead>
<tr>
<th>Code</th>
<th>Code Meaning</th>
<th>Code Criteria</th>
<th>Example Response(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>COMP</td>
<td>Complete</td>
<td>There are 2 vector arrows originating at or associated with each point. If there is only one arrow for each point or only one point has 2 arrows and the other has 1 arrow or 0 arrows, COMP is not marked.</td>
<td><img src="example_images1.png" alt="Example Images" /></td>
</tr>
<tr>
<td>LAB</td>
<td>Labeled</td>
<td>The vector arrows are labeled with unit vectors. A response does not have to be coded COMP to be coded LAB.</td>
<td><img src="example_images2.png" alt="Example Images" /></td>
</tr>
<tr>
<td>UL</td>
<td>Unit Length</td>
<td>The vector arrows are drawn with length equal to 1 box on the grid.</td>
<td><img src="example_images3.png" alt="Example Images" /></td>
</tr>
<tr>
<td>ICD</td>
<td>Increasing Coordinate Direction</td>
<td>The vector arrows are drawn such that the i arrows point in the positive x direction and the j arrows point in the positive y direction at each point</td>
<td><img src="example_images4.png" alt="Example Images" /></td>
</tr>
</tbody>
</table>
### Question 1 Codes

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<thead>
<tr>
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<tbody>
<tr>
<td>CCV</td>
<td>Conflation with Component Vectors</td>
<td>The sketch shows what looks like the component vectors of what would be the position vectors to the points A &amp; B</td>
<td><img src="image1.png" alt="Diagram" /></td>
</tr>
<tr>
<td>CPV</td>
<td>Conflation with Position Vectors</td>
<td>The sketch has arrows that originate at the origin and end at the points A &amp; B, appearing to be position vector sketches for the two points</td>
<td><img src="image2.png" alt="Diagram" /></td>
</tr>
<tr>
<td>PR</td>
<td>Physical Reasoning</td>
<td>The arrow sketches appear to be a displacement vector sketch from one point to another, or the component vectors of what would be such a displacement vector are drawn</td>
<td><img src="image3.png" alt="Diagram" /></td>
</tr>
<tr>
<td>NR*</td>
<td>No Response</td>
<td>The question is not answered; i.e. answer space is blank, there is only a question mark, or something like “no idea” or “no clue” is written.</td>
<td><img src="image4.png" alt="Diagram" /></td>
</tr>
</tbody>
</table>

*NR is a possible code for all questions and will not repeated in subsequent tables.*
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<tbody>
<tr>
<td>UL</td>
<td>Unit Length</td>
<td>The response uses language that indicates clear understanding that $i$ is of unit length. This could include statements like “unit length”, “length of 1”, “magnitude of 1”, or vector notation such as $&lt;1,0&gt;$</td>
<td>$i$ is the unit vector, or vector of length 1 that points in the positive $x$ direction.</td>
</tr>
<tr>
<td>ICD</td>
<td>Increasing Coordinate Direction</td>
<td>The response uses language that indicates that $i$ points in the $+x$ direction. Phrases such as “in positive $x$” work, or vector notation such as $&lt;1,0&gt;$</td>
<td>$&lt;1,0&gt;$ for vectors.</td>
</tr>
<tr>
<td>ix</td>
<td>$i$ is linked to $x$ direction</td>
<td>The response uses language that $i$ is associated with the $x$ or horizontal direction but does not distinguish between the positive or negative $x$/horizontal directions</td>
<td>$i$ represents a vector in the horizontal direction.</td>
</tr>
</tbody>
</table>

### Non-Productively Activated Resources

<table>
<thead>
<tr>
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</tr>
</thead>
<tbody>
<tr>
<td>PR</td>
<td>Physical Reasoning</td>
<td>The response indicates reasoning based on physical principles often indicated by terms such as “moves” or “shifts”</td>
<td>$i$ is a vector that moves in the $x$ direction of one unit, $i$ is the vector in the $x$ direction needed to shift from $A$ to $B$.</td>
</tr>
</tbody>
</table>
## Question 3 Codes

<table>
<thead>
<tr>
<th>Code</th>
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<th>Code Criteria</th>
<th>Example Response(s)</th>
</tr>
</thead>
<tbody>
<tr>
<td>SO</td>
<td>Starts at Origin</td>
<td>Language of response indicates that the origin is the starting point for a position vector. Key phrases include “starts at origin” or “begins at (0,0).”</td>
<td>A position vector is the vector from the origin to a point ((x, y)). A position vector is a vector with direction and magnitude originating from origin.</td>
</tr>
<tr>
<td>EP</td>
<td>Ends at Point of interest</td>
<td>Language of response indicates that the position vector terminates at the point of interest. Key phrases include “ends at a point” or “to a point (P)”</td>
<td>A position vector is the vector from the origin to a point ((x, y)). A vector that goes to an exact point on a graph or 3D space.</td>
</tr>
<tr>
<td>OI?</td>
<td>Origin is Implied</td>
<td>Language implies – but does not explicitly state – that the origin is the starting point for a position vector. An example is “a vector that gives the location of the point”. To give a location requires a reference point, which in coordinate systems is typically understood to be the origin/pole.</td>
<td>Vector that gives location of a point.</td>
</tr>
</tbody>
</table>
### Question 3 Codes

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</tr>
</thead>
<tbody>
<tr>
<td>PT</td>
<td>Points To</td>
<td>The position vector is said to “point to” but not explicitly end at the location of</td>
<td>a vector pointing to where a point is</td>
</tr>
<tr>
<td>PR</td>
<td>Physical Reasoning</td>
<td>The response uses physical principles in the explanation of position vector. Key</td>
<td>A vector that goes to an exact point on a graph or 3D space</td>
</tr>
<tr>
<td></td>
<td></td>
<td>terms such as “moves” or “goes” are used.</td>
<td></td>
</tr>
<tr>
<td>NS</td>
<td>No Start point defined</td>
<td>The response does not clearly indicate a starting point for a position vector.</td>
<td>The position vector represents a direction and magnitude necessary to reach a certain point in space.</td>
</tr>
<tr>
<td>NE</td>
<td>No End point defined</td>
<td>The response does not clearly indicate an ending point for a position vector.</td>
<td>position vector is a vector with direction and magnitude originating from origin.</td>
</tr>
<tr>
<td>(4,3)</td>
<td>The Point (4,3) is used in the definition</td>
<td>The point (4,3) informs the response. Examples include “to the point (4,3)” or over 4 units and up 3 units</td>
<td>Position vector would be the direction from the axis (0,0) a vector goes in. (4,3) would be 4 in the i direction &amp; 3 in the j direction.</td>
</tr>
</tbody>
</table>
### Question 4 Codes

<table>
<thead>
<tr>
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</tr>
</thead>
<tbody>
<tr>
<td>SO</td>
<td>Start at Origin</td>
<td>The sketch shows the position vector starting at the origin</td>
<td><img src="image1" alt="Diagram" /></td>
</tr>
<tr>
<td>EP</td>
<td>Ends at Point</td>
<td>The sketch shows the position vector ending at the point (4,3)</td>
<td><img src="image2" alt="Diagram" /></td>
</tr>
</tbody>
</table>

**Diagram labels:**
- **SO**: Start at Origin
- **EP**: Ends at Point
### Question 4 Codes

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<tbody>
<tr>
<td>PT</td>
<td>Points To</td>
<td>The sketch is of an arrow that points to but does not end at the point (4,3)</td>
<td>![Image]</td>
</tr>
<tr>
<td>DC</td>
<td>Drawn Components</td>
<td>The sketch appears to illustrate the component vectors of the position vector for the point (4,3)</td>
<td>![Image]</td>
</tr>
<tr>
<td>MA</td>
<td>Multiple Arrowheads</td>
<td>There are multiple arrowheads along the same line segment in the sketch</td>
<td>![Image]</td>
</tr>
</tbody>
</table>

Table continues on next page.
### Question 4 Codes

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</tr>
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<tbody>
<tr>
<td>PA</td>
<td>Points Away</td>
<td>The sketch is of an arrow that begins at the point (4,3) and points radially away from the origin</td>
<td><img src="image1.png" alt="Diagram" /></td>
</tr>
<tr>
<td>Other*</td>
<td>Non-codeable</td>
<td>Response is difficult to determine the meaning of and is mathematically incorrect. These responses are unique.</td>
<td><img src="image2.png" alt="Diagram" /></td>
</tr>
</tbody>
</table>

*The ‘Other’ code is also of use for question 5 but will not be repeated for the question 5 table.*
### Question 5 Codes

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<table>
<thead>
<tr>
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</thead>
<tbody>
<tr>
<td>i+j</td>
<td>Unit vector notation used</td>
<td>The response includes a mathematical expression of the form ( _i + _j )</td>
<td>( \vec{i} + 3\vec{j} )</td>
</tr>
<tr>
<td>UV</td>
<td>Unit Vectors</td>
<td>Unit vectors are used in the mathematical expression in the response.</td>
<td>( \langle 4\hat{i}, 3\hat{j} \rangle )</td>
</tr>
</tbody>
</table>

#### Non-Productively Activated Resources

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<tbody>
<tr>
<td>r/r</td>
<td>Rise over Run</td>
<td>The expression is written in the form of a linear equation and the slope is specified as ( \frac{4}{3} )</td>
<td>( y = \frac{3}{4}x )</td>
</tr>
<tr>
<td>LA</td>
<td>Linear Algebra</td>
<td>The response uses the notation and symbolic conventions of Linear Algebra, such as matrices, braces and brackets, and the introduction of a dummy variable</td>
<td>( \vec{y} = \frac{3}{4} \vec{x} )</td>
</tr>
<tr>
<td>TVN</td>
<td>Textbook Vector Notation</td>
<td>The expression is written in the form ( &lt;x, y&gt; ) or ( &lt;4i, 3j&gt; ), which matches the vector notation taught in the textbook used in the surveyed class</td>
<td>( &lt;4, 3&gt; )</td>
</tr>
<tr>
<td>MO</td>
<td>Mathematical Operation</td>
<td>An arithmetic operation different from the notational conventions of Linear Algebra is shown.</td>
<td>( \langle x-4 \rangle + \langle y-3 \rangle = \langle 4, 3 \rangle )</td>
</tr>
<tr>
<td>OP</td>
<td>Ordered Pair</td>
<td>The expression takes the form of an ordered pair. Examples include ( (4,3) ) and ( (4i,3j) ).</td>
<td>( (4,3) \in \mathbb{R} )</td>
</tr>
</tbody>
</table>

\[ \]