ANALYSIS OF VARIANCE BASED FINANCIAL INSTRUMENTS AND TRANSITION PROBABILITY DENSITIES: SWAPS, PRICE INDICES, AND ASYMPTOTIC EXPANSIONS

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The supervisory committee certifies that this dissertation complies with North Dakota State University's regulations and meets the accepted standards for the degree of

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ABSTRACT

This dissertation studies a couple of variance-dependent instruments in the financial market. Firstly, a number of aspects of the variance swap in connection to the Barndorff-Nielsen and Shephard model are studied. A partial integro-differential equation that describes the dynamics of the arbitrage-free price of the variance swap is formulated. Under appropriate assumptions for the first four cumulants of the driving subordinator, a Večeř-type theorem is proved. The bounds of the arbitrage-free variance swap price are also found. Finally, a price-weighted index modulated by market variance is introduced. The large-basket limit dynamics of the price index and the "error term" are derived. Empirical data driven numerical examples are provided in support of the proposed price index.

We implemented Feynman path integral method for the analysis of option pricing for certain Lévy process-driven financial markets. For such markets, we find closed form solutions of transition probability density functions of option pricing in terms of various special functions. Asymptotic analysis of transition probability density functions is provided. We also find expressions for transition probability density functions in terms of various special functions for certain Lévy process-driven markets where the interest rate is stochastic.

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DEDICATION

I dedicate this doctoral dissertation to the Almighty God, to my mother Mrs. Jamilatu Issaka and to the soul of my father Mr. Issaka Osumanu. All I have and will accomplish are only possible due to their love and sacrifices.

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1. INTRODUCTION

In modern markets swaps are becoming increasingly useful for hedging and speculation of volatility. A swap is a financial derivative in which two counterparties exchange cash flows of two securities, interest rates, or other financial instruments for the mutual benefit of the exchangers. The benefit depends on the type of financial instruments involved. There are various types of swap in a market. In this study, we focus mainly on the variance swap. The variance swap is a forward contract on the square of future realized volatility, which is referred to as the variance. Since swaps are relatively recent financial instruments that can be used by traders for volatility hedging and speculation. The variance swap is usually very profitable when the traders have some insight on the level of the future fluctuation of the underlying stock price.

The literature devoted to the variance swap is developing rapidly. In [19] the authors provided an analytical approximation for the valuation of volatility swaps and analyzed other options with such analytic estimation. In [41] the authors discussed the valuation and hedging of volatility swaps within the frame of a GARCH(1,1) stochastic volatility model: A general partial differential equation approach was used to determine the first two moments of the realized variance in a continuous or discrete context. In [61] a new probabilistic approach using the Heston model is proposed to study various swaps for financial markets. In [62, 63] variance swaps for financial markets with underlying asset and stochastic volatilities with delay were considered; and additionally some analytical approximate formal asymptotic forms were obtained for expectation and variance of the realized continuously sampled variance for stochastic volatility with delay. The variance swap was evaluated with delay both in a risk-neutral world and in the physical world. An upper bound for delay as a measure of risk was obtained and two numerical examples as applications using S&P 60 Canada Index (1998-2002) and S&P 500 Index (1990-1993) were provided to price variance swaps with delay. As observed in [62], variance swaps for stochastic volatility with delay is similar to variance swaps for stochastic volatility in the Heston model. However, for stochastic volatility models with delay, more parameters are present compared to the Heston model. In [64], the Heston model is presented with a variance drift-adjusted version that leads to a significant improvement of the market volatility surface fitting compared to the Heston model.

In order to study swaps it is essential to model the "riskiness" of the underlying assets. Classical Black-Scholes model assumes that the volatility of stock, which is a measure of riskiness of the underlying asset, is a constant. Obviously such assumptions for financial models are not compatible with derivative prices observed in the market. One of the most popular approaches in recent literature to address this issue is connected with the stochastic volatility scenario. Financial time series of different assets share many common features such as heavy tailed distributions of log-returns, aggregational Gaussianity, and quasi long-range dependence. Many such facts are successfully captured by models in which stochastic volatility of log-returns is constructed through Ornstein-Uhlenbeck (OU) type stationary stochastic process driven by a subordinator, where a subordinator is a Lévy process with no Gaussian component and positive increments. Using Lévy processes as driving noise, a large family of mean reverting jump processes with linear dynamics can be constructed. On these processes various properties such as positiveness or choice of marginal distribution can be imposed. These Lévy-driven processes are known as non-Gaussian Ornstein-Uhlenbeck processes or simply Ornstein-Uhlenbeck processes. Non-Gaussian processes of OU type are one of the most significant candidates for being the building blocks of models of financial economics. These models offer the possibility of capturing important distributional deviations from Gaussianity and thus are more practical models of dependence structures. This model is introduced in various works (see [9, 12, 13]) of Barndorff-Nielsen and Shephard and is known in modern literature as the BN-S model. In [16] the authors investigate swaps written on powers of realized volatility in the stochastic volatility model proposed by Barndorff-Nielsen and Shephard. In [34] the arbitrage free pricing of variance and volatility swaps for Barndorff-Nielsen and Shephard type Lévy process driven financial markets are studied. One of the major challenges in arbitrage free pricing of swap is to obtain an accurate pricing expression which can be used with good computational accuracy. In [34], the authors obtain various approximate expressions for the pricing of volatility and variance swaps. It is shown that with the approximate formulas obtained from the Barndorff-Nielsen and Shephard model the error estimation in fitting the delivery price is much less than the existing models with comparable parameters. Numerical results are provided in support of the accuracy of approximate formulas. A similar analysis for the covariance swap is provided in [33].

In the Chapter 3 of the present dissertation we study various aspects of the variance swap in connection to the BN-S model.

A stock index or stock market index is a market statistic of the value of a section of the stock market. It is typically computed using some weighted average of selected stocks. In general, a price index can be thought of as a weighted sum of the prices of stocks in the index basket. It is a tool used by investors and financial managers to describe the market, and to compare the return on specific investments. Two of the most popular index types are price-weighted and capitalisation-weighted indices. There has been an increasing trend in recent years to create index funds, a passively managed mutual funds that are based on market indices.

Some indices, such as the S&P 500, have multiple versions. These versions can differ based on the weights and dividends. For example, there are three versions of the S&P 500 index: price return, which only considers the price of the components, total return, which accounts for dividend reinvestment, and net total return, which accounts for dividend reinvestment after the deduction of a withholding tax. In the Section 4.1 of the present dissertation we introduce a price-weighted index modulated by market variance/volatility. As variance swaps can be used by traders for volatility speculation, the presented price index is strongly connected to the pricing of variance swap. We assume a BN-S type asset-price model for the component stocks with stochastic volatility. This is a generalized model compared to the existing simple single-sector model in [20]. We consider the index as a weighted sum of the product of these stock-price processes with the square of the volatility. The main results focus on the large-basket limit dynamics of the price index and limit behavior of the "error term" for the large basket dynamics. In [35] the price index is derived for the processes with jumps in the asset-prices. In that work each asset follows a jump diffusion model with constant drift and instantaneous volatility. The asset-prices are correlated via a single market factor capturing global economic effects and each asset has its own idiosyncratic noise consisting of a Brownian component and a jump component. The model presented in this dissertation considers stochastic volatility, and the asset-price dynamics is more general compared to the model in [35].

In the Chapter 4 of the present dissertation we introduce and analyze a new price index that is dependent on the market variance.

The path integral method is proposed by R. Feynman in [30] and nowadays it becomes one of the most powerful methods in theoretical physics. A path integral is defined as a limit of the sequence of finite-dimensional integrals, in a similar way as the Riemannian integral is defined as a limit of the sequence of finite sums. Over the last few decades it finds its application in various other disciplines such as statistics, polymer physics, financial markets etc. (see [45, 54, 57]). In financial markets path integral methodology has been successfully implemented in options pricing for reasonably simple models. In [47] a review of some applications of the path integral methodology of quantum mechanics to financial modeling and options pricing is provided. In the papers [4, 5], describing physical analogies, the path integral method is applied to a series of financial problems. In [17] using Feynman path integral the evolution operator kernel for the Merton-Garman Hamiltonian is constructed. Based on this calculation option pricing formula, which generalizes the Black-Scholes result, is obtained. In [18] a general formula to price European path-dependent options on multidimensional assets is obtained and implemented by means of various flexible and efficient algorithms. In a recent paper [44] explicit formulas are given for computing the bond pricing function in Black-Karasinski model in the analog of quantum mechanical "semiclassical" approximation.

The organization of this dissertation is as follows: The remainder of Chapter 1 includes some preliminaries of Lévy processes, a brief introduction to path integrals and a brief summary of the Barndorff-Nielsen and Shephard model aslo known in literature as the BNS-model. In Chapter 2, we present some already known results in the literature. We present the works of B. M. Hambly and J. Vaicenavicius [35], J. Večeř and M. Xu 2.2, J. Večeř [65, 66, 60], S. Habtemicael and I. SenGupta [34], N. Bellamy and M. Jeanblanc [15] and B. Baaquie 2.5. In Chapter 3, after giving a brief overview of pricing procedure of variance swap and the Barndorff-Nielsen and Shephard (BN-S) model. We formulate a partial intrgro-differential equation that describes the dynamics of the arbitrage-free price of the variance swap. Under appropriate assumptions for the first four cumulants of the driving subordinator, we prove a Večeř-type theorem that gives the arbitrage-free price of the variance swap. Finally we find the bounds of the arbitrage-free variance swap price. In Chapter 4, we introduce a price-weighted index modulated by market variance and study the index dynamics for the large basket limit case. We also study numerical examples based on empirical data in support of the proposed price index [37]. In Chapter 5 of this dissertation, we implement the method of Feynman path integral for the analysis of option pricing for certain Lévy process driven financial markets. For a Lévy process driven financial market, we find closed form solution of the transition probability density function (or, the pricing kernel) of option pricing in terms of various special functions. Asymptotic analysis of transition probability density function is provided to represent the option pricing formulas for "sufficiently large" horizon date. We also provide formulas for transition probability density function for certain Lévy process driven markets where the interest rate is stochastic [38]. Chapter 6 is devoted the conclusion of the present dissertation as well some recommendations for further research.

1.1. Lévy processes: definitions and properties

In this section we introduce Lévy processes together with some definitions and properties. Lévy processes are much like Brownian motion (a process with stationary and independent increments) but have discontinuous paths or have jumps. In financial mathematics, Lévy processes are becoming extremely useful because they can describe the observed reality of financial markets in a more accurate way than models based on classical Brownian motion. Such processes have been proposed to incorporate many empirical features in the return of financial stock prices. In this section we follow the excellent textbooks on Lévy processes by R. Cont and P. Tankov [23], W. Schoutens [56], D. Applebaum [3] and J. Jacod and A. N. Shiryaev [39].

Definition 1.1.1 (Lévy process [23])**.** *A cádlág (sample paths are almost surely right continuous with left limits) stochastic process* $X = (X_t)_{t\geq 0}$ *on* $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{t\geq 0}, P)$ *with values in* $\mathbb{R}^d$ *such that* $X_0 = 0$ *is called a Lévy process if it possesses the following properties:*

1. *Independent increments: for every increasing sequence of times* $t_0, \dots, t_n$ *the random variables* X_{t_0}, $X_{t_1} - X_{t_0}$, $\dots$, $X_{t_n} - X_{t_{n-1}}$ *are independent.*

2. *Stationary increments: the law of* $X_{t+h} - X_t$ *does not depend on* t.

3. *Stochastic continuity:* $\forall \epsilon > 0$, $\lim_{h\to 0} P\left(|X_{t+h} - X_t| \geq \epsilon\right) = 0$.

Definition 1.1.2 (Lévy measure [23])**.** *Let* $(X_t)_{t\geq 0}$ *be a Lévy process on* $\mathbb{R}^d$. *The measure* ν *on* $\mathbb{R}$ *defined by:*

$$\nu(A) = \mathbb{E}\left[\#\left\{t \in [0,1] : \Delta X_t \neq 0, \Delta X_t \in A\right\}\right],$$

for $A \in \mathcal{B}(\mathbb{R})$, *is called the Lévy measure of* X. $\nu(A)$ *can be interpreted as the expected number of jumps whose size is an element of* A *per unit time, see [23].*

Definition 1.1.3 (Poisson process [23]). *Let $(\tau_i)_{i\geq 1}$ be a sequence of independent exponential random variables with parameter λ that is, with cumulative distribution function $\mathbb{P}[\tau_i \geq x] = e^{-\lambda x}$ and $T_n = \sum_{i=1}^{n} \tau_i$. The process $(N_t)_{t\geq 0}$ defined by*

$$N_t = \sum_{n\geq 1} 1_{t\geq T_n}$$

is called a Poisson process with intensity λ.

The sample paths of a Poisson process [23] are piecewise constant, almost surely right continuous with left limits and with jump size of 1. The jumps [23] occur at times T_i and the interval between jumps are exponential distributed. Poisson process have independent and stationary increments. For every $t > 0$, N_t follows the Poisson distribution with parameter λt, that is

$$\mathbb{P}[N_t = n] = e^{-\lambda} \frac{(\lambda t)^n}{n!}.$$

The characteristic function of a Poisson process is given by

$$\phi_{N_t}(u) = \mathbb{E}\left[e^{iuN_t}\right] = \exp\{\lambda t \left(e^{iu} - 1\right)\}.$$

Definition 1.1.4 (Compound Poisson process [23]). *Let $(N_t)_{t\geq 0}$ be a Poisson process with intensity $\lambda > 0$ and $(Y_i)_{i\geq 1}$ be a sequence of i.i.d random variables with distribution f, and which are independent of $(N_t)_{t\geq 0}$. The stochastic process X_t defined as*

$$X_t = \sum_{i=1}^{N_t} Y_i$$

is called compound Poisson process.

The sample paths of a Poisson process are piecewise constant, almost surely right continuous with left limits but the jump sizes are random with distribution f. Compound Poisson process has independent and stationary increments. The distribution of compound Poisson process is known explicitly but the characteristic function [23] is given by

$$\phi_{X_t}(u) = \mathbb{E}\left[e^{iuX_t}\right] = \exp\{\lambda t \int_{\mathbb{R}} \left(e^{iu} - 1\right) f(dx)\}.$$

To every cádlág process $(X_t)_{t\geq 0}$ one can define an integer-valued random measure $J_X(\omega; dt, dx)$ as

$$J_X(\omega; dt, dx) = \sum_s 1_{\{\Delta X_s(\omega)\neq 0\}} \delta_{(s,\Delta X_s(\omega))}(dt, dx),$$

where δ denotes the Dirac measure. The measure J_X, can be interpreted as a counter which increase whenever within a time increment dt a jump occurs whose size falls into dt. For any Borel measurable function f, one can define

$$\sum_{0<s\leq t} f(\Delta X_s) = \int_0^t \int_{\mathbb{R}} f(x) J_X(dt, dx).$$

Definition 1.1.5 ([52])**.** *Let $(X_t)_{t\geq 0}$ be a Lévy process and let $D := \{z \in \mathbb{C} : \mathbb{E}\left[e^{\mathcal{R}(z)X_1}\right] < \infty\}$. The cumulant function $\kappa : D \to \mathbb{C}$ for $t > 0$ is defined as*

$$e^{\kappa(z)t} = \mathbb{E}\left[e^{zX_1}\right].$$

The cumulant function exists for $z \in \mathbb{C}$ with $\mathcal{R}(z) = 0$ and in which case $\kappa(iu)$ agrees with the characteristic exponent of X_1

$$\psi(u) := \log\left[e^{zX_1}\right].$$

The characteristic exponent of X_1 usually has a simpler form than the distribution of X_1 for which it determines uniquely. For further details, we refer the reader to [23] and [52].

Theorem 1.1.6 (Lévy-Khintchine formula [23])**.** *Let $(X_t)_{t\geq 0}$ be a Lévy process. Let $h : \mathbb{R} \to \mathbb{R}$ be a bounded measurable function such that $h(x) = x$ in a neighbourhood of zero. Then there exists a triplet (γ, σ^2, ν) such that the cumulant function can be written for $z \in D$ as*

$$\kappa(z) = \gamma z + \frac{1}{2}\sigma^2 z^2 + \int_{\mathbb{R}} \left(e^{zx} - 1 - zh(x)\right) \nu(dx),$$

whereγ and σ^2 are constants, and ν is the Lévy measure. (σ^2, ν, γ) is called the characteristic triplet of the Lévy process $(X_t)_{t\geq 0}$. Different choices of h do not affect σ^2 and ν but γ depends on the choice of h.

If X is a a Lévy process with characteristic triplet (σ^2, ν, γ), then [23]

$$[X, X]_t = \sigma^2 t + \sum_{\substack{s \in [0,t] \\ \Delta X_s \neq 0}} |\Delta X_s|^2 = \sigma^2 t + \int_0^t \int_{\mathbb{R}} y^2 J_X(dt, dy)$$

is called the quadratic variation X.

Theorem 1.1.7 (Itô formula for multidimensional Lévy process [23])**.** *Let $X_t = \left(X_t^1, \ldots, X_t^d\right)$ be a multidimensional Lévy process with characteristic $(A, \nu\gamma)$. Then for any $C^{1,2}$ function $f : [0, T] \times \mathbb{R}^d \to \mathbb{R}$,*

$$\begin{aligned} f(t, X_t) - f(0, 0) = & \int_0^t \sum_{i=1}^d \frac{\partial f}{\partial x_i}(s, X_{s-}) dX_s^i + \int_0^t \frac{\partial f}{\partial s}(s, X_s) ds \\ & + \frac{1}{2} \int_0^t \sum_{i=1, j=1}^d A_{ij} \frac{\partial^2 f}{\partial x_i \partial x_j}(s, X_{s-}) ds \\ & + \sum_{0 \leq s \leq t}^{\Delta X_s \neq 0} \left[f(s, X_{s-} + \Delta X_s) - f(s, X_{s-}) - \sum_{i=1}^d \Delta X_s^i \frac{\partial f}{\partial x_i}(s, X_{s-}) \right]. \end{aligned}$$

1.2. Path integrals

In this section, we introduce path integrals following an excellent book of S. Albeverio, R. Høegh-Krohn, S. Mazzucchi [2]. In quantum mechanics the state of the particle at time t is described by a function $\psi(x, t)$ which belongs to $L_2(\mathbb{R}^n)$ for every t and satisfies Schrödinger equation of motion

$$-\frac{\partial \psi}{\partial t} = H\psi, \tag{1.1}$$

with $\psi(x, t) = g(x)$, where H is Hamiltonian of the quantum particle. In [2, 43] M. Kac provided a solution to (1.1) when $H = -\frac{1}{2}\Delta + V(x)$, where Δ is the Laplacian operator on $\mathbb{R}^n$. The solution is the celebrated Feynman-Kac formula:

$$\psi(x, t) = \int_{W_{t,x}} \exp\left\{ - \int_0^t V(w(s)) ds \right\} g(w(t)) d\mathbb{P}_{t,x}(w) \tag{1.2}$$

where $W_{t,x} = \{w \in [C[0,t];\mathbb{R}] : w(0) = x\}$ and $\mathbb{P}_{t,x}$ is the Wiener measure on $W_{t,x}$. Now explain how the Wiener measure $\mathbb{P}_{t,x}$ [22] can be constructed on the space of all continuous paths $w : [0,t] \to \mathbb{R}$ such that $w(0) = x$. Consider a cylinder set of paths defined by time $0 \leq t_1 < t_2 < \cdots < t_n$ and real intervals $I_i = (a_i, b_i)$, $(i = 1, 2, \ldots, n)$ as $C(t_1, t_2, \ldots, t_n; I_1, \ldots, I_n) = \{w(t) \in W_{t,x} : w(t_i) \in I_i \text{ for all } 1 \leq i \leq n\}$. The cylinder $C(t_1, t_2; I_1, I_2)$ consists of all continuous functions $w(.)$ such that $a_1 < w(t_1) < b_1$ and $a_2 < w(t_2) < b_2$. That is, $C(t_1, t_2; I_1, I_2)$ consists of all continuous paths that are observed at t_1 to be between the levels a_1 and b_1 and at t_2 to be between a_2 and b_2. The collection $\mathcal{I}$ of finite disjoint unions of cylinder sets is an algebra which generates the product sigma-algebra $\mathcal{F}$. We can define a measure μ on a cylinder set and then extend μ to all sets in $\mathcal{I}$ such that μ is finitely-additive on $\mathcal{I}$. By Caratheodory Extension Theorem, it can be shown that μ can be extended to a unique countably additive measure $\mathbb{P}_{t,x}$ on $\mathcal{F}$ called the Wiener measure. For further detail on the construction of the Wiener measure, we refer the reader to [22].

The integral (1.2) is called a path integral. For a thorough investigation of Feynman path integral, we refer the reader to [2].

1.3. Barndorff-Nielsen and Shephard model

Consider a frictionless financial market where a riskless asset with constant return rate r and a stock are traded up to a fixed horizon date T. Assume that (see [12, 13]) the price process of the stock $S = (S_t)_{t\geq 0}$ is defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, P)$ and is given by:

$$S_t = S_0 \exp(X_t), \tag{1.3}$$

$$dX_t = (\mu + \beta\sigma_t^2)\, dt + \sigma_t\, dW_t + \rho\, dZ_{\lambda t}, \tag{1.4}$$

$$d\sigma_t^2 = -\lambda\sigma_t^2\, dt + dZ_{\lambda t}, \quad \sigma_0^2 > 0, \tag{1.5}$$

where the parameters $\mu, \beta, \rho, \lambda \in \mathbb{R}$ with $\lambda > 0$ and $\rho \leq 0$. $W = (W_t)$ is a Brownian motion and the process $Z = (Z_{\lambda t})$ is a subordinator. Barndorff-Nielsen and Shephard refer to Z as the *background driving Lévy process* (BDLP). Also W and Z are assumed to be independent and $(\mathcal{F}_t)$ is assumed to be the usual augmentation of the filtration generated by the pair (W, Z). This model is known in literature as Barndorff-Nielsen and Shephard model (BN-S model). In (1.4) the Brownian motion and the Lévy process appear as a linear combination and therefore the dynamics of the process is linear. Also, the negative sign appearing in (1.5) makes the associated process mean-reverting. Observe that the fact that (1.5) is driven by Z (instead of W) makes the process non-Gaussian.

We denote the interval $(0, \infty)$ by $\mathbb{R}+$. We assume that Z satisfies the assumptions as described in [48]. The assumptions are as follows.

Assumption 1.3.1. *Z has no deterministic drift and its Lévy measure has density $w(x)$.*

From Assumption 1.3.1 it follows from [55] (Theorem 19.3) that the cumulant transform $\kappa(\theta) = \log E[e^{\theta Z_1}]$, where it exists, takes the form

$$\kappa(\theta) = \int_{\mathbb{R}+} (e^{\theta x} - 1) w(x)\, dx.$$

Assumption 1.3.2. *Letting $\hat{\theta} = \sup\{\theta \in \mathbb{R} : \kappa(\theta) < +\infty\}$, then $\hat{\theta} > 0$.*

Assumption 1.3.3. $\lim_{\theta \to \hat{\theta}} \kappa(\theta) = +\infty$.

An important concept that will be useful for the next theorem is that of a stochastic exponential of a Lévy process $((X)_{t \geq 0}$ also known as a Doléans-Dade exponential. For the proof of the following proposition, see [23].

Proposition 1.3.4 (Doléans-Dade exponential [23])**.** *Let $(X)_{t \geq 0}$ is a Lévy process with Lévy triplet (σ^2, ν, γ). There exists a unique cádlág process $(Z_t)_{t \geq 0}$ such that*

$$dZ_t = Z_{t-}\, dX_t, \quad Z_0 = 1.$$

Z is given by:

$$Z_t = e^{X_t - \frac{\sigma^2 t}{2}} \prod_{0 \leq s \leq t} (1 + \Delta X_s)\, e^{-\Delta X_s}.$$

If $\int_{-1}^{1} |x| \nu(dx) < \infty$ then the jumps of X have finite variation and the stochastic exponential of X can be expressed as

$$Z_t = e^{X_t^c - \frac{\sigma^2 t}{2}} \prod_{0 \le s \le t} (1 + \Delta X_s)\, e^{-\Delta X_s},$$

where X_t^c is the continuous martingale part of X_t.

Z is called the stochastic exponential or Doléans-Dade exponential of X and is denoted by $Z = \mathcal{E}(X)$. It is shown in [48] that there exists an equivalent martingale measure (EMM) under which equations (1.4) and (1.5) preserve their structures. We summarize the related theorem from [48] (Theorem 3.2).

Theorem 1.3.5 (E. Nicolato and E. Venardos [48])**.** *Let $y \in \mathcal{Y}'$ where $\mathcal{Y}' := \{y : \mathbb{R}_+ \to \mathbb{R}_+ | \int_{\mathbb{R}_+} (\sqrt{y(x)} - 1)^2 w(x)\, dx < +\infty\}$. Then the process*

$$\psi_t = \frac{1}{\sigma_t}(r - \mu - (\beta + \frac{1}{2}\sigma_t^2 - \lambda\kappa^y(\rho)),$$

where $\kappa^y(\theta) = \int_{\mathbb{R}_+} (e^{\theta x} - 1) w^y(x)\, dx$, for real part of $\theta < 0$, and $w^y(x) = y(x)w(x)$, are such that

$$P\left(\int_0^T \psi_s^2\, ds < \infty\right) = 1,$$

and

$$L_t^y = \mathcal{E}(\psi \cdot W + (y-1) \star (\mu_Z - \nu_Z))_t, \quad 0 \le t \le T$$

is a density process.

The probability measure Q^y defined by $dQ^y = L_T^y\, dP$ is an EMM under which equations (1.4) and (1.5) can be written as:

$$dX_t = b_t\, dt + \sigma_t\, dW_t + \rho\, dZ_{\lambda t}, \tag{1.6}$$

$$d\sigma_t^2 = -\lambda\sigma_t^2\, dt + dZ_{\lambda t}, \quad \sigma_0^2 > 0, \tag{1.7}$$

where

$$b_t = (r - \lambda\kappa^y(\rho) - \frac{1}{2}\sigma_t^2), \tag{1.8}$$

where W_t and $Z_{\lambda t}$ are Q^y-Brownian motion and Q^y-Lévy process respectively. The processes W and Z are independent under Q^y.

For the rest of this section we assume that the risk-neutral dynamics of the stock price and volatility are given by (1.6), (1.6) and (1.7). Let the random measure associated with the jumps of Z, and Lévy density of Z be given by J_Z and ν_Z respectively. The compensator for $J_Z(\lambda dt, dx)$ is $\lambda\nu(dx)\,dt$ and we define $\tilde{J}_Z(\lambda dt, dx) = J_Z(\lambda dt, dx) - \lambda\nu_Z(dx)\,dt$. Clearly, with respect to the risk-neutral measure, the dynamics of S_t is given by

$$\frac{dS_t}{S_t} = r dt + \sigma_t\, dW_t + \int_{\mathbb{R}+} (e^{\rho x} - 1)\tilde{J}_Z(\lambda dt, dx). \tag{1.9}$$

2. MOTIVATION AND ALREADY KNOWN RESULTS

The aim of this chapter which consist of five main sections, is to present some already known results in the literature. The chapter consist of four main sections. In section 2.1, we present results of B. M. Hambly and J. Vaicenavicius [35] concerning a simple single-sector model. In section 2.2, we present some results of J. Večeř and M. Xu [67] for the arithmetic Asian options when the stock is driven by special semimartingale processes and results of J. Večeř [65, 66, 60] for arithmetic Asian options when the stock is driven by geometric Brownian motion. Section 2.2 is devoted to the results of S. Habtemicael and I. SenGupta[34], where the authors studied the variance swap for Gaussian models such as the Hull-White model [36] and non-Gaussian model such the Barndorff-Nielsen and Shephard Model. In section 2.4, we present N. Bellamy and M. Jeanblanc [15] for range of the European claim prices. Section 2.5 is devoted to the results of B. Baaquie for path integrals and Hamiltonian for financial markets. The material covered in this chapter forms the basis of our dissertation.

2.1. Price index approximations and weak convergence theorems

In this section, we present some results of B. M. Hambly and J. Vaicenavicius who studied the price-weighted index for simple single-sector model where all assets have the same constant drift, instantaneous volatility, and are correlated via a single market factor capturing global economic effects. In [35] B. M. Hambly and J. Vaicenavicius considered the following problem:

- Consider a probability space $(\Omega^n, \mathcal{F}^n, \mathbb{P}^n)$ corresponding to a market with n risky assets and a riskless asset.
- Suppose the prices process $S_i(t)$ where $i = 1, \ldots, n$ for the risky assets evolve under a measure $\mathbb{P}^n$ according to the jump diffusion process

$$\frac{dS_i(t)}{S_i(t-)} = \alpha dt + \sigma\rho dM(t) + \sigma\sqrt{1-\rho^2}dW_i(t) + \int_{\mathbb{R}\setminus\{0\}}(e^x - 1)N_i(dt, dx). \tag{2.1}$$

- M, W_i are Brownian motions, N_i is a Poisson random measure with a finite Lévy intensity measure $\nu = \lambda g$, where λ is the intensity of the Poisson counting process $N^i(t)$ and g is the probability density function of the jumps $\mathcal{J}_k^i$ occurring at random times τ_k^i in the compound Poisson process

$$R_i(t) = \int_0^t \int_{\mathbb{R}\setminus\{0\}} xN_i(du, dx) = \sum_{k=1}^{N^i(t)} \mathcal{J}_k^i.$$

  with the jumps occurring at random times τ_k^i. The process $M, W_1, \ldots, W_n, R_1, \ldots, R_n$ are independent. α is a drift coefficient, the total instantaneous expected return per unit time is given by $\mu = \alpha + \int_{\mathbb{R}\{0\}}(e^x - 1)\nu(dx)$ where $\mu > 0$.

- The instantaneous volatility, arising from the Brownian motion terms, is denoted by $\sigma > 0$; and $\rho \in (0, 1)$ is the correlation coefficient which is assumed to be strictly positive (although provided $\rho = 0$ we could take $\rho < 0$).

- The riskless money market account is assumed to pay a constant rate of interest r satisfying $0 < r < \mu$.

- Define the price-weighted market index

$$I_n(t) = \frac{1}{n}\sum_{i=1}^{n} S_i(t). \tag{2.2}$$

- Combining model (2.1) and (2.2), the index process has dynamics

$$I_n(t) = \int_0^t I_n(u-)[(\alpha + \beta_1)du + \rho\sigma dM(u)] + \frac{1}{\sqrt{n}}\Pi_n(t)$$

  where the process $\Pi_n(t)$ is given by

$$\Pi_n(t) = \frac{\sigma\sqrt{1-\rho^2}}{\sqrt{n}}\sum_{i=1}^{n}\int_0^t S_i(u-)dW_i(u) + \frac{1}{\sqrt{n}}\sum_{i=1}^{n}\int_{\mathbb{R}\setminus\{0\}} S_i(u-)(e^x - 1)\tilde{N}_i(dt, dx).$$

- More generally, define the k-th empirical moment process

$$I_n^{(k)}(t) = \frac{1}{n}\sum_{i=1}^{n} S_i^k(t).$$

- In [35], B. M. Hambly and J. Vaicenavicius found an approximation process driven by fewer randomness which approximates the index process I_n for large values of n. Hambly and Vaicenavicius showed that the process $\Pi_n(t) = \sqrt{n}\left(I_n(t) - I_n^{(1)}(t)\right)$ converges weakly to a non-trivial process Π.

- The following theorem is a summary of the main result of Hambly and Vaicenavicius [35].

Theorem 2.1.1 (B.M. Hambly, J. Vaicenavicius [35])**.** *Let $k, i \in \mathbb{N}$ and suppose that $E[S_i(0)^{4k}] < \infty$ and $\int_{\mathbb{R}} e^{4kx}\nu(dx) < \infty$. Then $I_n^{(k)} \Rightarrow I^{(k)}$ as $n \to \infty$, where the process $I^{(k)}$ is given by*

$$\begin{aligned} dI^{(k)}(t) &= \left(k\alpha + \frac{k(k-1)}{2}\sigma^2 + \beta_k\right) I^{(k)}(t)dt + k\sigma\rho I^{(k)}(t)dM, \\ I^{(k)}(0) &= E[S_1(0)^k], \end{aligned}$$

and

$$\Pi_n \Rightarrow \Pi := \int_0^t \xi\sqrt{I^{(2)}(u)}dB(u) \quad \text{as } n \to \infty,$$

where $\mu = \alpha + \beta_1$, $\gamma = \sigma^2(1-\rho^2)$, $\kappa = \beta_2 - 2\beta_1 = \int_{\mathbb{R}\setminus\{0\}}(e^x - 1)^2\nu(dx)$, $\beta_k = \int_{\mathbb{R}\setminus\{0\}}(e^{kx} - 1)^2\nu(dx)$ $\xi = \sqrt{\gamma + \kappa}$ and where B and M are independent Brownian.

2.2. Pricing Asian options

In this section, we present some results of J. Večeř and M. Xu [67] concerning the arithmetic Asian options when the stock is driven by special semimartingale processes. J. Večeř and M. Xu showed that the inherently path dependent problem of pricing Asian options can be transformed into a problem without path dependency in the payoff function. J. Večeř and M. Xu also showed that the price satisfies a simpler integro-differential equation when the stock price is driven by a process with independent increments, where Lévy processes are a special case. J. Večeř [60, 66] studied a unifying approach for pricing Asian options when the underlying stocks follows a geometric Brownian motion. In [67, 66] J. Večeř and M. Xu considered the following problem:

- Let S be a real-valued, strictly positive semimartingale on the stochastic basis $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ that satisfies the usual conditions.

- Assume that $e^{rt}S_t$ is a martingale under $\mathbb{P}$, where r is constant interest rate and $\mathbb{P}$ is a risk-neutral measure .

- Define a new measure $\mathbb{Q}$ by

$$\frac{d\mathbb{Q}}{d\mathbb{P}} = \frac{S_t}{S_0 e^{rt}} \tag{2.3}$$

  and a numeraire process $Z_t = \frac{X_t}{S_t}$. This change of numeraire technique was introduced by H. Geman, N. El Jaroui, J.-C. Rochet [31].

  Theorem 2.2.1 (J. Večeř and M. Xu [67]). *Let $V^\lambda(0, S_0, K_1, K_2)$, the price of the Asian option be defined as*

$$V^\lambda(0, S_0, K_1, K_2) = \mathbb{E}^{\mathbb{P}}\left[e^{-rT}\left(\int_0^T S_t d\lambda(t) - K_1 S_T - K_2\right)^+\right].$$

  Then

$$V^\lambda(0, S_0, K_1, K_2) = S_0 \cdot \mathbb{E}^{\mathbb{Q}}\left[Z_T - K_1)^+\right],$$

  where $\mathbb{Q}$ is defined (2.3), X_t is the self-financing portfolio

$$dX_t = q_{t-}dS_t + r(X_{t-} - q_{t-}S_{t-})dt,$$

  with the initial condition $X_0 = q_0 S_0 - e^{-rT}K_2$ and trading strategy $q_t = e^{-rT}\int_t^T e^{rs}d\lambda(s)$, where $\lambda(t)$ is the averaging factor and $Z_t = \frac{X_t}{S_t}$.

- J. Večeř and M. Xu [67] considered the stock price with the following dynamics:

$$dS_t = S_{t-}dH_t, \tag{2.4}$$

where H is a semimartingale. Using the notation in J. Jacod and A. N. Shiryaev [39] H has the canonical decomposition :

$$H_t = rt + H_t^c + \int_0^t \int_{-\infty}^{\infty} x\left(\mu(ds, dx) - \nu(ds, dx)\right),$$

with $H_0 = 0$, H_t^c is the continuous martingale part, $\mu(dt, dx)$ is the random measure associated with the jumps of H and $\nu(dt, dx)$ is the compensator.

- J. Večeř and M. Xu [67] proved the following integro-differential equation for the price of the Asian option.

Theorem 2.2.2 (J. Večeř and M. Xu [67]). *Suppose that H is a process with independent increment given by* (2.4). *The value of the Asian option is a function of t and Z_t, denoted by $v(t, Z_t)$, such that $V^\gamma(0, S_0, K_1, K_2) = S_0 v(0, Z_0)$. Assume v_t, v_z and v_{zz} exist and are continuous. Then the following integro-differential equation holds:*

$$\begin{aligned}\int_0^t \int_{-\infty}^{\infty} &\left\{ v_s\left(s, Z_{s-} + (q_{s-} - Z_{s-})\frac{x}{1+x}\right)\right\} v(ds, dx)\\ &- \int_0^t \int_{-\infty}^{\infty} \left\{ v(s, Z_{s-}) + v_z(s, Z_{s-})(q_{s-} - Z_{s-})\frac{x}{1+x}\right\} v(ds, dx)\\ &+ \int_0^t v_s(s, Z_{s-})ds + \frac{1}{2}v_{zz}(s, Z_{s-})(q_{s-} - Z_{s-})^2 d\left\langle H^c\right\rangle_s = 0\end{aligned}$$

for $0 \le t \le T$.

- J. Večeř [60, 65, 66] considered an Asian call option whose payoff at T is given

$$V(T) = \left(\frac{1}{T}\int_0^T S(t)dt - K\right)^+,$$

where K is the strike price and the underlying asset $S(t)$ follows a geometric Brownian motion:

$$dS(t) = rS(t)dt + \sigma S(t)d\tilde{W}(t),$$

where $\tilde{W}(t)$, $0 \le t \le T$, is a Brownian motion under the risk-neutral measure $\tilde{\mathbb{P}}$.

- It is shown in S. E. Shreve [60] that the value of the Asian option

$$v(t, S(t), Y(t)) = \tilde{\mathbb{E}}\left[e^{-r(T-t)}V(T)|\mathcal{F}_t(t)\right],$$

where $Y(t) = \int_0^t S(u)du$ satisfies the following partial differential equation:

$$v_t(t,x,y) + rxv_x(t,x,y) + xv_y(t,x,y) + \frac{1}{2}\sigma^2 x^2 v_{xx}(t,x,y) = rv_x(t,x,y),$$

and the boundary conditions

$$v(t,0,y) = e^{-r(T-t)}\left(\frac{y}{T} - K\right)^+, 0 \le t \le T, y \in \mathbb{R},$$

$$\lim_{y \downarrow -\infty} v(t,x,y) = 0, 0 \le t \le T, x \ge 0,$$

and

$$v(T,x,y) = \left(\frac{y}{T} - K\right)^+, x \ge 0, y \in \mathbb{R}.$$

- J. Večeř [60, 65, 66] showed in the following theorem that the dimensionality of pricing the Asian option can be reduced with simplified boundary conditions.

Theorem 2.2.3 (J. Večeř [60, 65, 66])**.** *For $0 \le t \le T$, the risk-neutral pricing*

$$V(t) = \tilde{\mathbb{E}}\left[e^{-r(T-t)}V(T)|\mathcal{F}(t)\right]$$

at time t of the Asian call option is

$$V(t) = S(t)g\left(t, \frac{X(t)}{S(t)}\right),$$

where $g(t,y)$ satisfies

$$g_t(t,y) + \frac{1}{2}\sigma^2\left(\gamma(t) - y\right)^2 g_{yy}(t,y) = 0, \quad 0 \le t \le T, \quad y \in \mathbb{R},$$

and $X(t)$ is given by

$$X(t) = \begin{cases} \frac{1}{rc}\left(1-e^{-rc}\right)S(t) - e^{-r(T-t)}K, & 0 \le t \le T-c, \\ \frac{1}{rc}\left(1-e^{-rc}\right)S(t) + e^{-r(T-t)}\frac{1}{c}\int_{T-c}^{t} S(u)du - e^{-r(T-t)}K, & T-c \le t \le T. \end{cases}.$$

The boundary conditions for $g(t,y)$ are

$$g(T,y) = y^+, y \in \mathbb{R},$$

$$\lim_{y\to-\infty} g(t,y) = 0, 0 \le t \le T$$

and

$$\lim_{y\to-\infty} (g(t,y) - y) = 0, 0 \le t \le T.$$

2.3. Pricing variance swap for stochastic volatility model

In [34] S. Habtemicael and I. SenGupta studied the variance swap for Gaussian models such as the Hull-White model [36] and for non-Gaussian model such the Barndorff-Nielsen and Shephard Model. S. Habtemicael and I. SenGupta [34] considered the following problem:

- Consider a probability space $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\ge0}, \mathbb{P})$ with a risky asset $(S_t)_{t\ge0}$ and riskless asset with constant interest rate r.

- Assume that the stock price process $(S_t)_{t\in\mathbb{R}^+}$ satisfies the following dynamics:

$$dS_t = rS_t dt + \sigma_t S_t dW_t^1 \tag{2.5}$$

$$d\sigma_t^2 = \kappa\sigma_t^2\, dt + \zeta\sigma_t^2\, dW_t^2 \tag{2.6}$$

  where r is the risk-free interest rate, $\kappa < 0$ and ζ are real constant, W_t^1 and W_t^2 are independent Wiener processes and the variance process σ_t follows Hull-White model [36].

- S. Habtemicael and I. SenGupta [34] proved the following theorem concerning the arbitrage of the variance swap $P_{Var} = \mathbb{E}\left[e^{-rT}\left(\sigma_R^2(S) - K_{Var}\right)\right]$, where K_{Var} is the delivery price or exercise price for the variance swap and $\sigma_R^2(S)$ is the realized variance defined as the average of the instantaneous variance which is given by:

$$\sigma_R^2(S) = \frac{1}{T}\int_0^T \sigma_s^2 ds.$$

Theorem 2.3.1 (S. Habtemicael and I. SenGupta [34]). *The arbitrage free price of variance swap for the stock dynamics* (2.5) *and volatility dynamics* (2.6) *is given by*

$$P_{Var} = e^{-rT}\left(\frac{\sigma_0^2}{\kappa T}(e^{\kappa T}-1) - K_{Var}\right).$$

- S. Habtemicael and I. SenGupta [34] prove the following theorem related to the arbitrage-free pricing of variance swaps when the underlying stock price process follows the Barndorff-Nielsen and Shephard Model .

Theorem 2.3.2 (S. Habtemicael and I. SenGupta [34]). *The arbitrage free price of the variance swap* $P_{Var} = \mathbb{E}\left[e^{-rT}\left(\sigma_R^2(S) - K_{Var}\right)\right]$ *for the BNS-Model* (1.6), (1.7) *is given by*

$$P_{Var} = e^{-rT}\left[\frac{1}{T}\left(\lambda^{-1}\left(1-e^{-\lambda T}\right)\left(\sigma_0^2-\kappa_1\right)+\kappa_1 T\right)+\rho^2\lambda\kappa_2 - K_{Var}\right],$$

where κ_1 *and* κ_2 *are the first cumulant (i.e., the expected value) and the second cumulant (i.e., the variance) of* Z_1 *respectively.*

2.4. Range of prices

In this section, we present some results of N. Bellamy and M. Jeanblanc [15]. It is well known that for an incomplete market, there are several equivalent martingale measures, which means it is not always possible perfectly hedge every contingent claim. Therefore to price options, one has to choose a particular martingale measure and any choice of an equivalent martingale measure will correspond to an arbitrage free price. In [15] N. Bellamy and M. Jeanblanc determined the range of European and American claim prices. Here I present the following problem for the range of the European claim prices, for more detail see N. Bellamy and M. Jeanblanc [15].

- Consider a financial market $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with riskless asset with deterministic return r and a risky asset $(S_t)_{t\geq 0}$ with the following dynamics under the historical probability:

$$dS_t = S_{t-}\left(b(t)dt + \sigma(t)dW_t + \phi(t)dM_t\right)$$

  where b, σ and ϕ are deterministic bounded functions with $|\sigma(t)| > c$, $\phi(t) > -1$, $\frac{1}{c} < |\phi(t)| < c$ where c is a strictly positive constant. W is a Brownian motion and $M_t = N_t - \lambda t$ is the compensated martingale associated with a Poisson process with deterministic intensity λ.

- Since the market is incomplete, it is not possible to hedge a price for every contingent claim $H \in \mathcal{F}_T$. Consider the set of values of $\mathbb{E}^{\mathbb{Q}}[R(T)H|\mathcal{F}_t]$, where $R(T) = e^{-\int_0^T r(s)}$ and $\mathbb{Q}$ describes the set of risk-neutral measures. N. Bellamy and M. Jeanblanc denote this set as the set of viable prices, see [15, 25, 42] .

- Let $V^{\gamma}(t)$ be a time t viable price for the contingent claim H be defined by

$$R(t)V^{\gamma}(t) = \mathbb{E}^{\gamma}[R(T)H|\mathcal{F}_t].$$

  under the martingale measure $\mathbb{P}^{\gamma}$, where the set of equivalent martingale measures is parametrized by mean of a process γ valued in $(-1, \infty)$.

- The range of viable prices is an interval given by $]\inf_{t\in\Gamma} V^{\gamma}(t), \sup_{t\in\Gamma} V^{\gamma}(t)[$.

- N. Bellamy and M. Jeanblanc [15, 25] studied the range of viable prices which is an interval given by $[\inf_{t\in\Gamma} V^{\gamma}(t),\ \sup_{t\in\Gamma} V^{\gamma}(t)]$, where Γ is the set of predictable processes γ such that $L^{\gamma} := \frac{d\mathbb{P}^{\gamma}}{d\mathbb{P}}\big|_{\mathcal{F}_t}$ is a $\mathbb{P}^{\gamma}$-square integrable strictly positive martingale. For more detail definition of L^{γ}, see Proposition 3.1 in [15].

- Consider the Black-Scholes function C such that

$$R(t)C(t, X_t) = \mathbb{E}[R(T)(X_T - K)^+|X_t], \;\; C(T, x) = (x - K)^+,$$

when the dynamics of X are given by

$$dX_t = X_t(r(t)dt + \sigma(t)dW_t), \quad X_0 = x.$$

- C is a convex function of x with $\partial_x C(t,x) \leq 1$ and satisfies

$$\mathcal{L}(C)(t,x) = rC(t,x),$$

where

$$\mathcal{L}(f)(t,x) = \frac{\partial f}{\partial t}(t,x) + rx\frac{\partial f}{\partial x}(t,x) + \frac{1}{2}x^2\sigma^2\frac{\partial^2 f}{\partial x^2}(t,x).$$

- N. Bellamy and M. Jeanblanc [15, 25] proved the following theorem related to the time t viable price $V^\gamma(t)$.

Theorem 2.4.1 (N. Bellamy, M. Jeanblanc, [15, 25])**.** *Let $\mathbb{P}^\gamma \in \mathbb{Q}$. Then the associated viable price is bounded below by the Black-Scholes function, evaluated at the underlying asset value, and bounded above by the underlying asset value, i.e.,*

$$R(t)C(t,S_t) \leq \mathbb{E}^\gamma\left[R(T)(S_T - K)^+|\mathcal{F}_t\right] \leq R(t)S_t,$$

where $\mathcal{R}_t^\gamma = \mathbb{E}^\gamma\left[\int_t^T R(s)(1+\gamma_s)\lambda(s)\Lambda\mathcal{H}(s,S_s)ds|\mathcal{F}_t\right]$.

- The range of viable prices $V^\gamma(t) = \frac{R(T)}{R(t)}\mathbb{E}^\gamma\left[R(T)(S_T - K)^+|\mathcal{F}_t\right]$ is exactly the interval $]C(t,S_t), S_t[$, for more details see [15, 25].

2.5. Path integrals for financial markets

In this section, we present some results of B. Baaquie [4, 5] concerning path integrals formulation for pricing of options. Path integrals have many applications, among which are in the financial markets, quantum mechanics and polymer physics [45]. In [4, 5, 6, 7] B. Baaquie studied applied concepts of quantum mechanics to the modeling of interest rates and the theory of option pricing. B. Baaquie [4] studied path integrals and hamiltonians for options and interest rates. In [5] B. Baaquie studied path integral approach to option pricing with stochastic volatility. In [4] B.

Baaquie considered applied the path integral approach to pricing a European call option for the Black-Scholes model.

- Consider a financial market $(\Omega, \mathcal{F}, \mathbf{F} = (\mathcal{F}_t)_{t\geq 0}, \mathbb{P})$ with a riskless asset paying a constant rate of return r and a stock $S = (S_t)_{t\geq 0}$ modeled by the geometric Brownian motion [60]:

$$dS(t) = S(t)\left(\mu dt + \sigma dW(t)\right),$$

  where the constant μ is the expected rate of return, the constant σ is the volatility of the stock price process and $W(t)$ is a Brownian motion.

- Consider the price of a European call option $\tilde{C}(t, S)$ on underlying security S that pays $(S(T) - K)^+$. The strike price K is some nonnegative constant.

- The famous Black-Scholes equation for the option $\tilde{C}(t, S)$ is given by [60, 4]:

$$\frac{\partial \tilde{C}(t,S)}{\partial t} + rS\frac{\partial \tilde{C}(t,S)}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \tilde{C}(t,S)}{\partial S^2} = r\tilde{C}(t,S), \tag{2.7}$$

  for all $t \in [0, T)$ and satisfies the final condition $\tilde{C}(T, S) = (S(T) - K)^+$.

- Consider a change of variable in (2.7) with $S = e^x$, where $-\infty \leq x \leq \infty$ and denote $\tilde{C}(t, e^x) = C(t, x)$ and $C(T, x) = (x - K)^+$. This yield the Schrodinger type-equation for the Black-Scholes equation (2.7):

$$\frac{\partial C}{\partial t} = H_{BS}C \tag{2.8}$$

$$C(t,x) = (x - K)^+ \tag{2.9}$$

  where H_{BS} is called the Black-Scholes Hamiltonian and is given by

$$H_{BS} = -\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \left(\frac{1}{2}\sigma^2 - r\right)\frac{\partial}{\partial x} + r. \tag{2.10}$$

- Introducing a quantum mechanical formalism, one can interpret the option price $C(t, x)$ as a ket $|C\rangle$ in the basis $|x\rangle$, the logarithm of the underlying stock price.

- Using Dirac notation, one can reinterpret the option price $C(t,x) = \langle x|C(t,x)\rangle$ as a wave function $|C\rangle$ in the position space.

- Using Dirac notation [27], (2.8) can be expressed as

$$\frac{\partial |C(t,x)\rangle}{\partial t} = H_{BS}|C(t,x)\rangle. \tag{2.11}$$

- Using the final value condition at $t = T$, (2.11) can be solved explicitly as

$$|C(t,x)\rangle = e^{-(T-t)H}|C(T,x)\rangle$$

  where $C(0,x) = g(x)$ is the final condition.

- Hence using the completeness equation $\int_{-\infty}^{\infty} |x'\rangle\langle x'|dx' = I$, where I is the identity operator, B. Baaquie [4, 5] obtained

$$\begin{aligned} C(t,x) &= \langle x|C(t,x)\rangle \\ &= \langle x|e^{-(T-t)H_{BS}}|C(T,x)\rangle \\ &= \int_{-\infty}^{\infty} \langle x|e^{-(T-t)H_{BS}}|x'\rangle\langle x'|C(T,x)\rangle dx' \\ &= \int_{-\infty}^{\infty} \langle x|e^{-(T-t)H_{BS}}|x'\rangle C(T,x')dx'. \end{aligned} \tag{2.12}$$

- The expression $\langle x|e^{-(T-t)H_{BS}}|x'\rangle$ describes the probability of transition from a security price x' at time T to a security price x at time t.

- The completeness equation for the momentum space basis $|p\rangle$ is given by

$$\frac{1}{2\pi}\int_{-\infty}^{\infty} |p\rangle\langle p|dp = I, \tag{2.13}$$

  with scalar product

$$\langle x|p\rangle = e^{ipx} \;\; ; \;\; \langle p|x\rangle = e^{-ipx}. \tag{2.14}$$

- To compute $\langle x|e^{-(T-t)H}|x'\rangle$, one needs to first find the eigenfunctions of H_{BS}. For this, B. Baaquie [4, 5] considered the matrix elements of H_{BS} which is given by

$$\begin{aligned}
\langle x|H_{BS}|p\rangle &= H_{BS}\langle x|p\rangle \\
&= H_{BS}e^{ipx} \\
&= \left(-\frac{\sigma^2}{2}\frac{\partial^2}{\partial x^2} + \left(\frac{1}{2}\sigma^2 - r\right)\frac{\partial}{\partial x} + r\right)e^{ipx} \\
&= \left(-\frac{\sigma^2}{2}\frac{\partial^2 e^{ipx}}{\partial x^2} + \left(\frac{1}{2}\sigma^2 - r\right)\frac{\partial e^{ipx}}{\partial x} + re^{ipx}\right) \\
&= \left(\frac{1}{2}\sigma^2 p^2 + i\left(\frac{1}{2}\sigma^2 - r\right)p + r\right)e^{ipx}. \qquad (2.15)
\end{aligned}$$

- B. Baaquie [4, 5] observed from (2.15) that e^{ipx} is an eigenfunction of H_{BS} with corresponding eigenvalue of $\left(\frac{1}{2}\sigma^2 p^2 + i\left(\frac{1}{2}\sigma^2 - r\right)p + r\right)$.

- From this observation, one can see that e^{ipx} is also an eigenfunction of $e^{H_{BS}}$ with a corresponding eigenvalue of $e^{\left\{\frac{1}{2}\sigma^2 p^2 + i\left(\frac{1}{2}\sigma^2 - r\right)p + r\right\}}$. Hence

$$\begin{aligned}
\langle x|e^{-\tau H_{BS}}|p\rangle &= e^{-\tau H_{BS}}\langle x|p\rangle \\
&= e^{-\tau H_{BS}}e^{ipx} \\
&= e^{-\tau\left(\frac{1}{2}\sigma^2 p^2 + i\left(\frac{1}{2}\sigma^2 - r\right)p + r\right)}e^{ipx}. \qquad (2.16)
\end{aligned}$$

- To compute $\langle x|e^{-(T-t)H}|x'\rangle$, B. Baaquie [4, 5] used (2.13), (2.14) and (2.16) with $\tau = T - t$ to obtain

$$\begin{aligned}
\langle x|e^{-(T-t)H}|x'\rangle &= \frac{1}{2\pi}\int_{-\infty}^{\infty}\langle x|e^{-(T-t)H}|p\rangle\langle p|x'\rangle dp \\
&= \frac{1}{2\pi}e^{-r(T-t)}\int_{-\infty}^{\infty} e^{-(T-t)\frac{\sigma^2}{2}p^2 - (T-t)\left(\frac{1}{2}\sigma^2 - r\right)ip}e^{ip(x-x')}dp \\
&= \frac{1}{2\pi}e^{-r(T-t)}\int_{-\infty}^{\infty} e^{-\frac{1}{2}(T-t)\sigma^2 p^2}e^{ip\left((x-x')+(T-t)(r-\frac{1}{2}\sigma^2)\right)}dp \\
&= e^{-r(T-t)}\frac{1}{\sqrt{2\pi(T-t)\sigma^2}}e^{-\frac{1}{2(T-t)\sigma^2}\left\{x-x'+(T-t)(r-\sigma^2/2)\right\}^2} \qquad (2.17)
\end{aligned}$$

- Expression (2.17) means that $x - x'$ has a normal distribution with mean $-(T-t)(r-\frac{\sigma^2}{2})$ and variance $(T-t)\sigma^2$, which further implies that $\log S(T)$ has a normal distribution with mean $\log S(t) + (T-t)(r-\frac{\sigma^2}{2})$ and variance $(T-t)\sigma^2$ and this is exactly expected for the Black-Scholes model with constant volatility and where the underlying asset follows a geometric Brownian motion, for further details see [60].

- Finally, plugging (2.17) into (2.12), B. Baaquie [4, 5] obtained the price European call option:

$$C(t,x) = e^{-r(T-t)} \int_{-\infty}^{\infty} \frac{1}{\sqrt{2\pi(T-t)\sigma^2}} e^{-\frac{1}{2(T-t)\sigma^2}\left\{x-x'+(T-t)(r-\sigma^2/2)\right\}^2} C(T,x')dx',$$

  for more details see [4, 5, 6, 7, 8].

3. ANALYSIS OF VARIANCE SWAP FOR THE BN-S MODEL

In this chapter, we present our main results for the price of variance swap for the Barndorff-Nielsen and Shephard Model. Firstly, we prove a theorem related to the dynamics of arbitrage free price of variance swap. Secondly, we prove a Večeř-type theorem for the price of variance swap. Finally, we prove a theorem related to the range of prices for variance swap.

3.1. Properties of the variance swap price with respect to the BN-S model

Realized volatility $\sigma_R(S)$ is a statistical quantity which is the annualized standard deviation of the stock returns during a fixed period of time, which is called the exercise date of the option. The subscript R denotes the observed or realized volatility for some given underlying asset S. When the underlying asset is clear from the context, realized volatility is denoted simply as σ_R. If σ_t, $0 \leq t \leq T$ is a stochastic volatility for a given underlying asset S, then the realized volatility σ_R over the life time of a contract is given by

$$\sigma_R = \sqrt{\frac{1}{T}\int_0^T \sigma_t^2 dt}. \tag{3.1}$$

Usually σ_R is quoted in annual terms. The realized variance is σ_R^2 over the life of the contract is defined as

$$\sigma_R^2 = \frac{1}{T}\int_0^T \sigma_t^2\, dt. \tag{3.2}$$

Definition 3.1.1. *A variance swap is a forward contract on realized variance. The payoff of variance swap at the maturity T is given by $N(\sigma_R^2 - K_{Var})$, where K_{Var} is the annualized delivery price or exercise price of the variance swap, and N is the notional amount of the dollars per annualized volatility point squared.*

The holder of the variance swap at expiration receives N dollars for every point by which the stock's realized variance σ_R^2 has exceeded the variance delivery price K_{Var}. Without loss of generality we take $N = 1$. The arbitrage free price of the variance swap is the expectation of the present value of the payoff in the risk-neutral world and it is given by $E\left[e^{-r(T-t)}(\sigma_R^2 - K_{\text{Var}})|\mathcal{F}_t\right]$, $0 \leq t \leq T$, where $E(\cdot)$ is the expectation with respect to some equivalent martingale measure and $\mathcal{F}_t$ is the σ-field generated by the history of the process up to time t. Note that for calculating arbitrage free variance swap price it is sufficient to compute $E(\sigma_R^2)$. If $V_t = \int_0^t \sigma_t^2\, dt$, then by above expression, given a fixed horizon date T, we consider $P_{\text{Var}}(t, S_t, V_t)$ as a function of t, S_t and V_t with the final condition (independent of S) given by

$$P_{\text{Var}}(T, S_T, V_T) = \sigma_R^2 - \text{K}_{\text{Var}} = \frac{V_T}{T} - \text{K}_{\text{Var}}.$$

We make the following assumptions related to the integrated volatility V_t.

Assumption 3.1.2. *We assume the Lévy measure ν satisfies $\int_{y>1} e^{2y}\nu(dy) < \infty$. Also, assume when $V_t = 0$, there exists $\zeta \in (0,2)$ such that $\liminf_{\epsilon\to 0} \epsilon^{-\zeta} \int_0^\epsilon x^2\nu(dx) > 0$.*

With Assumption 3.1.2 we prove the following theorem related to the dynamics of the arbitrage-free price of variance swap. For the rest of this section we denote the price of variance swap $P_{\text{Var}}(t, S_t, V_t)$ by $P(t, S_t, V_t)$.

Theorem 3.1.3. *Consider the BN-S model given by* (1.3), (1.6) *and* (1.7). *Then, the arbitrage free value of $P(t, S_t, V_t)$, with respect to the equivalent martingale measure Q^y (defined in Theorem 1.3.5), is almost surely given by*

$$\begin{aligned} &-rP + \frac{\partial P}{\partial t} + rS\frac{\partial P}{\partial S} + \frac{1}{2}\sigma_t^2 S^2 \frac{\partial^2 P}{\partial S^2} + \sigma_t^2 \frac{\partial P}{\partial V} \\ &+ \int_{\mathbb{R}+} \left(P(t, S_{t-}e^{\rho x}, V_t) - P(t, S_{t-}, V_t) - \frac{\partial P}{\partial S} S_{t-}(e^{\rho x} - 1) \right) \nu_Z(dx) = 0, \end{aligned} \tag{3.3}$$

with final condition

$$P(T, S_T, V_T) = \frac{V_T}{T} - K_{Var}. \tag{3.4}$$

Proof. Suppose $\hat{P}(t, S_t, V_t) = e^{r(T-t)}P(t, S_t, V_t)$. Then by construction,

$$\hat{P}(t, S_t, V_t) = \tilde{E}\left[\left(\frac{V_T}{T} - \text{K}_{\text{Var}}\right)|\mathcal{F}_t\right]$$

is a martingale, where the expectation is taken with respect to the equivalent martingale measure Q^y (defined in Theorem 1.3.5). Denote the continuous part of the stochastic processes S and V by S^c and V^c respectively and denote the quadratic variation/covariation by the notation $[\cdot,\cdot]$. Using the two-dimensional Itô formula for $\hat{P}$, we obtain

$$d\hat{P}(t,S,V) = e^{r(T-t)}\Bigg[\left(-rP + \frac{\partial P}{\partial t}\right)dt + \frac{\partial P}{\partial S}dS + \frac{\partial P}{\partial V}dV + \frac{1}{2}\frac{\partial^2 P}{\partial S^2}d[S^c,S^c](t) + \frac{1}{2}\frac{\partial^2 P}{\partial V^2}d[V^c,V^c](t) + \frac{\partial^2 P}{\partial S \partial V}d[S^c,V^c](t) + P(t,S_t,V_t) - P(t,S_{t-},V_{t-}) - \frac{\partial P}{\partial S}\Delta S - \frac{\partial P}{\partial V}\Delta V\Bigg].$$

For the present context

$$d[V^c,V^c](t) = 0, \;\; d[S^c,V^c](t) = 0 \;\text{ and }\; \Delta V = 0,$$

and

$$P(t,S_t,V_t) - P(t,S_{t-},V_{t-}) = P(t,S_{t-}e^{\Delta X},V_t) - P(t,S_{t-},V_{t-}).$$

Also, as V is continuous $V_{t-} = V_t$. Therefore we obtain $d\hat{P}_t = a(t)\,dt + dR_t$, where

$$a(t) = e^{r(T-t)}[-rP + \frac{\partial P}{\partial t} + rS\frac{\partial P}{\partial S} + \frac{1}{2}\sigma_t^2 S^2 \frac{\partial^2 P}{\partial S^2} + \sigma_t^2\frac{\partial P}{\partial V} + \int_{\mathbb{R}+}\left(P(t,S_{t-}e^{\rho x},V_t) - P(t,S_{t-},V_t) - \frac{\partial P}{\partial S}S_{t-}(e^{\rho x}-1)\right)\nu_Z(dx)]$$

and

$$dR_t = e^{r(T-t)}\left[\sigma_t S\frac{\partial P}{\partial S}dW_t + \int_{\mathbb{R}+}\left(P(t,S_{t-}e^{\rho x},V_t) - P(t,S_{t-},V_t)\right)\tilde{J}_Z(\lambda dt,dx)\right].$$

With the use of Assumption 3.1.2 and procedures in [24], it is clear that R_t is a martingale and hence $\hat{P}_t - R_t$ is a (square integrable) martingale. But $\hat{P}_t - R_t = \int_0^t a(u)\,du$ is a continuous process with finite variation. Hence $a(t) = 0$ almost surely with respect to the equivalent martingale measure Q^y. This gives (3.3). □

We denote $\kappa_n = \int_{\mathbb{R}+} y^n \nu(dy)$, $n = 1, 2, \ldots$. Note that $\kappa_n > 0$ for all n when subordinators are considered. The following theorem can be considered as a Večeř- type theorem (see [60, 66, 67]) for the variance swap. Contrary to Theorem 3.1.3, for the next theorem we assume that at time t, the price P is explicitly dependent on σ_t^2 and not explicitly dependent on S_t. We denote the price of variance swap $P(t, \sigma_t^2, V_t)$.

Theorem 3.1.4. *Suppose that the cumulants of Z satisfy the following conditions:*

$$\frac{\kappa_1}{\kappa_3} < 1, \tag{3.5}$$

and

$$\frac{\kappa_4}{\kappa_1}\left(\frac{\kappa_1}{\kappa_3}\right)^{3/2} = 1. \tag{3.6}$$

Define

$$\theta_2 = -\sqrt{\frac{\kappa_1}{\kappa_3}}. \tag{3.7}$$

Then

$$P(t, \sigma_t^2, V_t) = e^{rt+\lambda\theta_2\kappa_1 T}\frac{X(t)}{M(t)}, \quad 0 \le t \le T,$$

where $X(t)$ is a stochastic process given by

$$dX(t) = rX(t)\,dt + \gamma(t)(d\Lambda(t) - r\Lambda(t)\,dt), \tag{3.8}$$

where $\Lambda(t) = \mu(t)\sigma_t^2$, and $\gamma(t) = \frac{e^{-rT}}{rT}(e^{-\lambda t} - e^{-\lambda T})$, and

$$X(T) = \left(\frac{1}{T}\int_0^T \sigma_t^2 dt - K_{Var}\right) = \left(\frac{V_T}{T} - K_{Var}\right), \tag{3.9}$$

and

$$M(t) = \exp\left[t(r + \lambda\theta_2\kappa_1) + \lambda t\int_{\mathbb{R}+}[\ln(1+\theta_2 y) - \theta_2 y]\nu(dy) + \int_0^{\lambda t}\int_{\mathbb{R}+}\ln(1+\theta_2 y)\tilde{J}_Z(ds, dy)\right]. \tag{3.10}$$

Proof. We consider a portfolio process $X(t)$ with value at time T given by (3.9). We begin with a deterministic function of time $\gamma(t)$, $0 \leq t \leq T$, which will be the number of shares of a proxy of variance $\Lambda(t) = \mu(t)\sigma_t^2$ held by the portfolio, where $\mu(t)$ is a deterministic function to be determined later.

An agent who holds $\gamma(t)$ shares of $\Lambda(t)$ at each time t and finances by investing or borrowing at the interest rate r will have a portfolio whose value evolves according to the stochastic differential equation (3.8). We choose $\mu(t) = e^{t(r+\lambda)}$ to obtain

$$d\Lambda(t) - r\Lambda(t)\,dt = e^{t(r+\lambda)} \int_{\mathbb{R}+} y J_Z(\lambda dt, dy). \tag{3.11}$$

Consequently

$$d(e^{r(T-t)}X(t)) = \gamma(t)e^{rT+\lambda t} \int_{\mathbb{R}+} y J_Z(\lambda dt, dy). \tag{3.12}$$

To study the variance swap with payoff (3.9) we take $\gamma(t) = \frac{e^{-rT}}{rT}(e^{-\lambda t} - e^{-\lambda T})$, $0 \leq t \leq T$, and

$$X(0) = \gamma(0)\sigma_0^2 - e^{-rT}K_{\text{Var}}. \tag{3.13}$$

From (3.12) we obtain (using $\gamma(T) = 0$)

$$X(T) - e^{rT}X(0) = e^{rT} \int_0^T \gamma(t)e^{\lambda t} dZ_{\lambda t}.$$

But for the present model

$$\frac{1}{T}\int_0^T \sigma_t^2\,dt = \frac{1}{\lambda T}(1 - e^{-\lambda T})\sigma_0^2 + \frac{1}{\lambda T}\int_0^T (1 - e^{-\lambda(T-t)})\,dZ_{\lambda t}.$$

Thus using the initial value of $X(0)$ from (3.13) we obtain,

$$X(T) = \left(\frac{1}{T}\int_0^T \sigma_t^2 dt - K_{\text{Var}}\right). \tag{3.14}$$

The price of the variance swap at time t prior to expiration is

$$P(t, S_t, V_t) = \tilde{E}[e^{-r(T-t)} X(T) | \mathcal{F}_t], \tag{3.15}$$

where $\tilde{E}$ is the expectation with respect to the risk-neutral measure Q^y (defined in Theorem 1.3.5). To calculate the right hand side of (3.15) we use a change-of-numéraire argument. We define

$$Y(t) = \frac{X(t)}{M(t)} = \frac{e^{-rt} X(t)}{e^{-rt} M(t)},$$

where $M(t)$ is defined as the solution of

$$\begin{aligned} dM(t) &= M(t)(r\,dt + \int_{\mathbb{R}+} \theta_2 y J_Z(\lambda dt, dy)), \quad M(0) = 1, \\ &= M(t)((r + \lambda\theta_2\kappa_1)\,dt + \int_{\mathbb{R}+} \theta_2 y \tilde{J}_Z(\lambda dt, dy)), \end{aligned} \tag{3.16}$$

where θ_2 is a constants which will be chosen later. Solution of this equation is given by (3.10). We proceed to compute the differential of $Y(t)$. We find

$$\begin{aligned} d(e^{-rt} X(t)) &= -re^{-rt} X(t)\,dt + e^{-rt}\,dX(t) \\ &= \gamma(t) e^{\lambda t} \int_{\mathbb{R}+} y J_Z(\lambda dt, dy) \\ &= \gamma(t) e^{\lambda t} \left(\lambda\kappa_1\,dt + \int_{\mathbb{R}+} y \tilde{J}_Z(\lambda dt, dy) \right). \end{aligned}$$

Similarly,

$$d(e^{-rt} M(t)) = e^{-rt} M(t) \left[\lambda\theta_2\kappa_1\,dt + \int_{\mathbb{R}+} \theta_2 y \tilde{J}_Z(\lambda dt, dy) \right],$$

and

$$d(e^{-rt} M(t))^{-1} = -(e^{-rt} M(t))^{-1} \left[\lambda(\theta_2\kappa_1 - \theta_2^2\kappa_2)\,dt + \int_{\mathbb{R}+} \theta_2 y (1 - \theta_2 y) \tilde{J}_Z(\lambda dt, dy) \right].$$

Therefore

$$
\begin{aligned}
dY(t) &= d[(e^{-rt}X(t))(e^{-rt}M(t))^{-1}] \\
&= -\frac{X(t)}{M(t)}\left[\lambda(\theta_2\kappa_1 - \theta_2^2\kappa_2)\,dt + \int_{\mathbb{R}+}\theta_2 y(1-\theta_2 y)\tilde{J}_Z(\lambda dt, dy)\right] \\
&+ \frac{\gamma(t)e^{(r+\lambda)t}}{M(t)}\left[\lambda(-\theta_2\kappa_2 + \theta_2^2\kappa_3 + \kappa_1)\,dt + \int_{\mathbb{R}+}(y - y^2\theta_2(1-\theta_2 y))\tilde{J}_Z(\lambda dt, dy)\right].
\end{aligned}
$$

Define

$$
\begin{aligned}
Z_1(t) &= \exp\left[\int_0^t\int_{\mathbb{R}+}\ln(1-\theta y)\tilde{J}_Z(dt,dy) + \int_0^t\int_{\mathbb{R}+}(\ln(1-\theta y)+\theta y)\nu(dy)dt\right] \\
&= \exp\left[\int_0^t\int_{\mathbb{R}+}\ln(1-\theta y)\tilde{J}_Z(dt,dy) + t\int_{\mathbb{R}+}(\ln(1-\theta y)+\theta y)\nu(dy)\right], \qquad (3.17)
\end{aligned}
$$

where $\theta < 1$ will be chosen later. Clearly (by [23], Proposition 8.23, page- 288) $Z_1(t)$ is martingale and hence $\tilde{E}(Z_1(\lambda T)) = \tilde{E}(Z_1(T)) = 1$. We choose $-1 < \theta_2 < 0$ such that $\theta = -\theta_2$. The quantity θ_2 is chosen as the solution of

$$
-\int_{\mathbb{R}+}\theta_2 y^2\left(1-\theta_2 y(1-\theta_2 y)\right)\nu(dy) = -\theta_2\kappa_2 + \theta_2^2\kappa_3 + \kappa_1, \qquad (3.18)
$$

which implies

$$
-\theta_2^3\kappa_4 = \kappa_1. \qquad (3.19)
$$

Define a new measure Q by $dQ(\omega) = Z_1(\lambda T)d\tilde{P}(\omega)$, where $\tilde{P}$ is the risk neutral measure used so far in the proof. With respect to Q, the dynamics of $dY(t)$ becomes (see [49], Chapter 1),

$$
dY(t) = -\frac{X(t)}{M(t)}\left[\lambda(\theta_2\kappa_1 - \theta_2^3\kappa_3)\,dt + \int_{\mathbb{R}+}\theta_2 y(1-\theta_2 y)\tilde{J}_{ZQ}(\lambda dt, dy)\right] + M_1(t),
$$

where $\tilde{J}_{ZQ}$ is the compensated Poisson measure and $M_1(t)$ is a martingale with respect to Q. If $\theta_2\kappa_1 - \theta_2^3\kappa_3 = 0$, then clearly we have

$$
\kappa_1 = \theta_2^2\kappa_3. \qquad (3.20)
$$

Then $Y(t)$ is martingale with respect to Q- measure. Hence the condition we need are (3.5) and (3.6). In that case we choose θ_2 by (3.7). Then both (3.19) and (3.20) are satisfied and $\theta < 1$. We observe with the value of θ_2 defined in (3.7)

$$e^{-rT}M(T) = e^{\lambda\theta_2\kappa_1 T}Z_1(\lambda T).$$

Then clearly $Y(t)$ is a martingale with respect to the Q measure. Now we have

$$\begin{aligned}
P(t, S_t, V_t) = \tilde{E}[e^{-r(T-t)}X(T)|\mathcal{F}_t] &= e^{-r(T-t)}\tilde{E}[M(T)Y(T)|\mathcal{F}_t] \\
&= e^{rt}\tilde{E}\left[e^{\lambda\theta_2\kappa_1 T}Z_1(\lambda T)Y(T)|\mathcal{F}_t\right] \\
&= e^{rt+\lambda\theta_2\kappa_1 T}E^Q\left[Y(T)|\mathcal{F}_t\right] \\
&= e^{rt+\lambda\theta_2\kappa_1 T}Y(t) = e^{rt+\lambda\theta_2\kappa_1 T}\frac{X(t)}{M(t)}.
\end{aligned}$$

□

We conclude this section with a lower and upper bound on the set of prices spanned by the value of a claim with respect to various equivalent martingale measures (EMM) of the BN-S model. This analysis is motivated by [15, 40]. Note that the set of EMMs for the BN-S model is derived in [48]. For generalized BN-S model the EMMs are derived in [58]. We restrict our analysis to contracts with payoff $H(X_T, Y_T)$, where H is the function expressing the payoff in terms of the underlying stock. We define the corresponding Black-Scholes type function $\mathcal{H}^f(t, x, y)$ by

$$\mathcal{H}^f(t, x, y) = \mathbb{E}\left[e^{-r(T-t)}H(X_T, Y_T)|X_t = x, Y_t = y\right], \quad \mathcal{H}^f(T, x, y) = H(x, y),$$

where the dynamics of X and Y are given by

$$\begin{aligned}
&dX_t = X_t(rdt + f_t dW_t), \quad X_0 = x, \\
&dY_t = f_t^2 dt, \quad Y_0 = y,
\end{aligned}$$

where f_t is a deterministic and continuous functions. Let $\mathcal{H}^m(t, x, y)$ be the Black-Scholes function corresponding to $f_s = m(s)$, where $m(s) = \sigma_t \exp\left(-\frac{\lambda}{2}(s-t)\right)$, $s \geq t$. Then using (1.5) it is easy to show that $\sigma_s^2 \geq m(s)^2$ for $s \geq t$. We make the following assumptions related to $\mathcal{H}^m$ and the pay-off function H.

Assumption 3.1.5. *We assume that H is convex with respect to the first variable and satisfies $0 \leq H(x,y) \leq x$ for $x > 0$. Also, the "delta" of $\mathcal{H}^m$ is bounded, i.e., $\left|\frac{\partial \mathcal{H}^m}{\partial x}\right| < C$, for some $C > 0$.*

First we prove the following theorem related to the dynamics of the Black-Scholes function corresponding to the deterministic and continuous f_s.

Theorem 3.1.6. *If $\mathcal{H}^f(t,x,y)$ belongs to $C^{1,2}$ then*

$$\frac{\partial \mathcal{H}^f(t,x,y)}{\partial t} + f_t^2 \frac{\partial \mathcal{H}^f(t,x,y)}{\partial x} + ry\frac{\partial \mathcal{H}^f(t,x,y)}{\partial y} + \frac{1}{2}y^2 f_t^2 \frac{\partial^2 \mathcal{H}^f(t,x,y)}{\partial y^2} - r\mathcal{H}^f(t,x,y) = 0. \quad (3.21)$$

Proof. Applying Ito's formula to $\mathcal{H}^f(t,x,y)$, we obtain

$$\begin{aligned}
d\mathcal{H}^f(t,x,y) &= \frac{\partial \mathcal{H}^f(t,x,y)}{\partial t}dt + \frac{\partial \mathcal{H}^f(t,x,y)}{\partial x}dX_t + \frac{\partial \mathcal{H}^f(t,x,y)}{\partial y}dY_t + \frac{1}{2}\frac{\partial^2 \mathcal{H}^f(t,x,y)}{\partial x^2}d[X,X](t) \\
&+ \frac{1}{2}\frac{\partial^2 \mathcal{H}^f(t,x,y)}{\partial y^2}d[Y,Y](t) + \frac{1}{2}\frac{\partial^2 \mathcal{H}^f(t,x,y)}{\partial x \partial y}d[X,Y](t) \\
&= \frac{\partial \mathcal{H}^f(t,x,y)}{\partial t}dt + rx\frac{\partial \mathcal{H}^f(t,x,y)}{\partial x}dt + xf\frac{\partial \mathcal{H}^f(t,x,y)}{\partial x}dW_t + g_t\frac{\partial \mathcal{H}^f(t,x,y)}{\partial y}dt \\
&+ \frac{1}{2}x^2 f_t^2 \frac{\partial^2 \mathcal{H}^f(t,x,y)}{\partial x^2}dt \\
&= \left[\frac{\partial \mathcal{H}^f(t,x,y)}{\partial t} + rx\frac{\partial \mathcal{H}^f(t,x,y)}{\partial x} + g_t\frac{\partial \mathcal{H}^f(t,x,y)}{\partial y} + \frac{1}{2}x^2 f_t^2 \frac{\partial^2 \mathcal{H}^f(t,x,y)}{\partial x^2}\right]dt \\
&+ xf\frac{\partial \mathcal{H}^f(t,x,y)}{\partial x}dW_t.
\end{aligned}$$

By letting $\hat{\mathcal{H}} = e^{r(T-t)}\mathcal{H}^f(t,x,y)$, we have $d\hat{\mathcal{H}}_t = a(t)\,dt + dR_t$, where

$$a(t) = e^{r(T-t)}\left[\frac{\partial \mathcal{H}^f(t,x,y)}{\partial t} + rx\frac{\partial \mathcal{H}^f(t,x,y)}{\partial x} + g_t\frac{\partial \mathcal{H}^f(t,x,y)}{\partial y} + \frac{1}{2}x^2 f_t^2 \frac{\partial^2 \mathcal{H}^f(t,x,y)}{\partial x^2} - r\mathcal{H}^f(t,x,y)\right],$$

and

$$dR_t = e^{r(T-t)}\left[xf\frac{\partial \mathcal{H}^f(t,x,y)}{\partial x}dW_t\right].$$

It is clear that R_t is a martingale and hence $\hat{\mathcal{H}}_t - R_t$ is a (square integrable) martingale. But $\hat{\mathcal{H}}_t - R_t = \int_0^t a(u)\,du$ is a continuous process with finite variation. Hence $a(t) = 0$ almost surely with respect to the some equivalent martingale measure. This gives (3.21). □

We proceed to prove the last main result for this section related to the arbitrage-free price of the variance swap $P(t, S_t, V_t)$ with pay-off at T given by (3.4),

$$H(S_T, V_T) = \frac{V_T}{T} - \mathrm{K}_{\mathrm{Var}}.$$

Theorem 3.1.7. *Let Q be an arbitrary EMM for the BN-S model. Then the arbitrage-free price of the variance swap at time t corresponding to Q is bounded above by S_t and is bounded below by $e^{rt}\mathcal{R}(t,T) + \mathcal{H}^m(t, S_t, V_t)$, where*

$$\mathcal{R}(t,T) = Ce^{-(r+2\lambda)T}\left(\frac{e^{\lambda T}\left(\kappa_1 e^{\lambda t} - \sigma_0^2 + \kappa_1\right)}{r+\lambda} + \frac{e^{\lambda t}(\sigma_0^2 - \kappa_1)}{r+2\lambda} - \frac{\kappa_1 e^{2\lambda T}}{r}\right) \\ + C\frac{e^{-(r+\lambda)t}\left(r\lambda(\sigma_0^2 - \kappa_1) + \lambda\kappa_1(r+2\lambda)e^{\lambda t}\right)}{r(r+\lambda)(r+2\lambda)}. \tag{3.22}$$

Proof. The arbitrage-free price of the variance swap at time t corresponding to Q is given by

$$\mathbb{E}^Q\left[e^{-r(T-t)}\mathcal{H}^m(T, S_T, V_T)|\mathcal{F}_t\right] = \mathbb{E}^Q\left[e^{-r(T-t)}H(S_T, V_T)|\mathcal{F}_t\right].$$

Proving the upper bound is trivial with the application of Assumption 5.35 and martingale property of the process $e^{-r(T-t)}S_t$ with respect to Q. We proceed to prove the result related to the lower bound. Applying Itô 's formula to $(\mathcal{H}^m(T, S_T, V_T))_{T\geq t}$ we obtain the following.

$$
\begin{aligned}
e^{-rT}\mathcal{H}^m(T,S_T,V_T) &= e^{-rt}\mathcal{H}^m(t,S_t,V_t) + \int_t^T \frac{\partial \mathcal{H}^m}{\partial x} d\left(e^{-rs}S_s\right) \\
&+ \int_t^T e^{-rs}\left(\frac{\partial \mathcal{H}^m}{\partial t} + rS_{s-}\frac{\partial \mathcal{H}^m}{\partial x} + \sigma_s^2\frac{\partial \mathcal{H}^m}{\partial y} + \frac{1}{2}\sigma_s^2 S_{s-}^2\frac{\partial^2 \mathcal{H}^m}{\partial x^2} - r\mathcal{H}^m\right) ds \\
&+ \sum_{t\le s\le T} e^{-rs}\left(\mathcal{H}^m(s,S_s,V_s) - \mathcal{H}^m(s,S_{s-},V_{s-}) - \frac{\partial \mathcal{H}^m}{\partial x}\Delta S_s\right) \\
&= e^{-rt}\mathcal{H}^m(t,S_t,V_t) + \int_t^T \frac{\partial \mathcal{H}^m}{\partial x} d\left(e^{-rs}S_s\right) \\
&+ \int_t^T e^{-rs}\left[\frac{\partial \mathcal{H}^m}{\partial t} + rS_{s-}\frac{\partial \mathcal{H}^m}{\partial x} + m(s)^2\frac{\partial \mathcal{H}^m}{\partial y} + \frac{1}{2}m(s)^2 S_{s-}^2\frac{\partial^2 \mathcal{H}^m}{\partial x^2} - r\mathcal{H}^m\right] ds \\
&+ \int_t^T e^{-rs}\frac{1}{2}\left(\sigma_s^2 - m^2(s)\right) S_{s-}^2\frac{\partial^2 \mathcal{H}^m}{\partial x^2} ds + \int_t^T e^{-rs}\left(\sigma_s^2 - m^2(s)\right)\frac{\partial \mathcal{H}^m}{\partial y} ds \\
&+ \sum_{t\le s\le T} e^{-rs}\left(\mathcal{H}^m(s,S_s,V_s) - \mathcal{H}^m(s,S_{s-},V_{s-}) - \frac{\partial \mathcal{H}^m}{\partial x}\Delta S_s\right). \qquad (3.23)
\end{aligned}
$$

Using Assumption 5.35 and the fact that $\sigma_s^2 \ge m^2(s)$ for $s \ge t$, we obtain

$$
\int_t^T e^{-rs}\frac{1}{2}\left(\sigma_s^2 - m^2(s)\right) S_{s-}^2\frac{\partial^2 \mathcal{H}^m}{\partial x^2} ds \ge 0.
$$

For any convex and differentiable function f we have

$$
f(x) - f(y) - f'(x)(x-y) \ge 0.
$$

Consequently

$$
\sum_{t\le s\le T} e^{-rs}\left(\mathcal{H}^m(s,S_s,V_s) - \mathcal{H}^m(s,S_{s-},V_{s-}) - \frac{\partial \mathcal{H}^m}{\partial y}\Delta S_s\right) \ge 0.
$$

Note that by (3.21)

$$
\frac{\partial \mathcal{H}^m(t,x,y)}{\partial t} + rx\frac{\partial \mathcal{H}^m(t,x,y)}{\partial x} + m(t)^2\frac{\partial \mathcal{H}^m(t,x,y)}{\partial y} + \frac{1}{2}m(t)^2x^2\frac{\partial^2 \mathcal{H}^m(t,x,y)}{\partial x^2} - r\mathcal{H}^m(t,x,y) = 0.
$$

Using all these results in (3.23) we obtain

$$e^{-rT}\mathcal{H}^m(T,S_T,V_T) \geq e^{-rt}\mathcal{H}^m(t,S_t,V_t) + \int_t^T \frac{\partial \mathcal{H}^m}{\partial x} d\left(e^{-rs}S_s\right) + \int_t^T e^{-rs}\left(\sigma_s^2 - m^2(s)\right)\frac{\partial \mathcal{H}^m}{\partial y} ds.$$

Therefore

$$\int_t^T \frac{\partial \mathcal{H}^m}{\partial x} d\left(e^{-rs}S_s\right) + \int_t^T e^{-rs}\left(\sigma_s^2 - m^2(s)\right)\frac{\partial \mathcal{H}^m}{\partial y} ds \leq e^{-rT}\mathcal{H}^m(T,S_T,V_T) - e^{-rt}\mathcal{H}^m(t,S_t,V_t).$$

Taking the expectation on both sides we have

$$\begin{aligned}\mathbb{E}^Q\left[\int_t^T \frac{\partial \mathcal{H}^m}{\partial x} d\left(e^{-rs}S_s\right)|\mathcal{F}_t\right] + \mathbb{E}^Q\left[\int_t^T e^{-rs}\left(\sigma_s^2 - m^2(s)\right)\frac{\partial \mathcal{H}^m}{\partial y} ds|\mathcal{F}_t\right]& \\ \leq \mathbb{E}^Q\left[e^{-rT}\mathcal{H}^m(T,S_T,V_T)|\mathcal{F}_t\right] - \mathbb{E}^Q\left[e^{-rt}\mathcal{H}^m(t,S_t,V_t)|\mathcal{F}_t\right].&\end{aligned}$$

Clearly $\mathbb{E}^Q\left[\int_t^T \frac{\partial \mathcal{H}^m}{\partial y} d\left(e^{-rs}S_s\right)|\mathcal{F}_t\right] = 0$. Therefore

$$\begin{aligned}\mathbb{E}^Q\left[\int_t^T e^{-rs}\left(\sigma_s^2 - m^2(s)\right)\frac{\partial \mathcal{H}^m}{\partial x} ds|\mathcal{F}_t\right] + e^{-rt}\mathcal{H}^m(t,S_t,V_t) &\leq \mathbb{E}^Q\left[e^{-rT}\mathcal{H}^m(T,S_T,V_T)|\mathcal{F}_t\right] \\ &= \mathbb{E}^Q\left[e^{-rT}H(S_T,V_T)|\mathcal{F}_t\right].\end{aligned}$$

From (1.7) we can derive $\sigma_s^2 = e^{-\lambda s}\sigma_0^2 + \int_0^s e^{-\lambda(s-u)}dZ_{\lambda u}$ and consequently

$$\mathbb{E}^Q[\sigma_s^2] = e^{-\lambda s}\sigma_0^2 + \lambda\kappa_1 \int_0^s e^{-\lambda(s-u)}du = e^{-\lambda s}\sigma_0^2 + \kappa_1\left(1 - e^{-\lambda s}\right).$$

Hence

$$\begin{aligned}&\mathbb{E}^Q\left[\int_t^T e^{-rs}\left(\sigma_s^2 - m^2(s)\right)\frac{\partial \mathcal{H}^m}{\partial y} ds|\mathcal{F}_t\right] \leq C\int_t^T e^{-rs}\mathbb{E}^Q\left[\left(\sigma_s^2 - m^2(s)\right)|\mathcal{F}_t\right] ds \\ &= C\int_t^T e^{-rs}\left(1 - e^{-\lambda(s-t)}\right)\left[e^{-\lambda s}\sigma_0^2 + \kappa_1\left(1 - e^{-\lambda s}\right)\right] ds,\end{aligned}$$

where the integral above is given by $\mathcal{R}(t,T)$ in (3.22). □

4. ANALYSIS OF VARIANCE DEPENDENT PRICE INDEX

In this Chapter, we introduce a price-weighted index modulated by market variance and study the index dynamics for the large basket limit case. We also study numerical examples based on empirical data in support of the proposed price index.

4.1. Variance-dependent price index and large-basket limit analysis

The VIX is used as an indicator of the S&P 500 market. However, it is well known that the VIX is much more of a short-term than a long-term market indicator. The VIX, which is officially known as the Chicago Board Options Exchange (CBOE) Volatility Index, is considered by many to be a gauge of fear and greed in the stock market. More precisely, the VIX measures the implied volatility in S&P 500 options. Through the use of a wide variety of option prices, the index gives an estimation of thirty-day implied volatility as priced by the S&P 500 index option market. This index can be used to estimate the nature of market movement that the option prices are projecting on the S&P 500 over the next 30-day (or may be shorter) period. Empirical evidence shows that a good statistic that captures the performance of the S&P 500 should depend on the VIX index (see [53]). Based on such empirical evidences, in this section, we introduce a new price index that is dependent on the market volatility/variance.

In this section we formulate a BN-S type market model with stochastic volatility. Then we proceed to prove the main convergence theorem describing the behavior of the volatility (or, variance) dependent price index in the large-basket limit. Under additional assumptions for the model we prove a convergence theorem related to the behavior of the "error term" in the large-basket limit. We conclude this section with empirical data driven numerical examples.

Consider a probability space $(\Omega^n, \mathcal{F}^n, \mathbb{P}^n)$ corresponding to a market with n assets whose prices $S_i(t)$ for $i = 1, \ldots, n$, evolve, under the measure $\mathbb{P}^n$, according to the following equations. We denote the expectation with respect to this probability measure by $\mathbb{E}$. Note that we are not restricting jump processes to subordinators.

$$\frac{dS_i(t)}{S_i(t)} = \alpha_1\, dt + \sigma(t)\left(\rho_1 dM + \sqrt{1-\rho_1^2}\,dW_i\right) + \int_{\mathbb{R}} (e^{\alpha_2 x} - 1) N_i(dt, dx), \tag{4.1}$$

with

$$d\sigma^2(t) = -\lambda\sigma(t)^2\,dt + \frac{1}{n}\sum_{j=1}^{n}\int_{\mathbb{R}} xN_j(dt,dx), \tag{4.2}$$

where the parameters $\lambda > 0$, $-1 \leq \rho_1 \leq 1$, $\alpha_1, \alpha_2 \in \mathbb{R}$, M, W_i are Brownian motions, and N_i a Poisson random measures related to the jump of the i-th asset price for $i = 1, \ldots, n$. We assume that N_i, $i = 1, \ldots, n$ are identically distributed with Lévy density ν. Denote the set of natural numbers by $\mathbb{N}$. Also, we assume that $\{S_i(0)\}_{i=1}^n$ is a family of independent identically distributed $(0, \infty)$-valued random variables and this family is independent of M, W_i and N_i, for $i \in \mathbb{N}$. As in [35] we denote the compound Poisson process

$$R_i(t) = \int_0^t \int_{\mathbb{R}} xN_i(du,dx) = \sum_{p=1}^{N^i(t)} \mathcal{J}_p^i, \quad i \in \mathbb{N},$$

where $\mathcal{J}_p^i$ are jumps occurring at random times τ_p^i. As described in the beginning of this section, empirical evidence shows the dependence of price indices on the volatility of the market. We define a market index modulated by volatility by

$$I_n(t) = \frac{1}{n}\sum_{i=1}^{n} \sigma^2(t)S_i(t). \tag{4.3}$$

In fact, the index defined above depends on the square of the volatility- i.e., on the variance. We also define a "k-th empirical moment process modulated by volatility/variance" as

$$I_n^{(k)}(t) = \frac{1}{n}\sum_{i=1}^{n} \sigma^2(t)S_i^k(t), \quad k \in \mathbb{N}. \tag{4.4}$$

Note that $I_n^{(1)}(t) = I_n(t)$. We can find

$$d\left(\sigma^2(t)S_i(t)\right) = S_i(t)\sigma^2(t)\left(\alpha_1 - \lambda\right)dt + S_i(t)\sigma^3(t)\rho_1 dM + S_i(t)\sigma^3(t)\sqrt{1-\rho_1^2}dW_i$$
$$+\,\sigma^2(t)S_i(t)\int_{\mathbb{R}}(e^{\alpha_2 x} - 1)N_i(dt,dx) + \frac{S_i(t)}{n}\sum_{j=1}^{n}\int_{\mathbb{R}} xN_j(dt,dx) + \frac{S_i(t)}{n}\int_{\mathbb{R}} x(e^{\alpha_2 x} - 1)N_i(dt,dx),$$

and from this we can derive

$$
\begin{aligned}
dI_n(t) = I_n(t)\left(\alpha_1 - \lambda\right)dt + \sigma(t)\rho_1 I_n(t)dM(t) + \frac{\sigma^3(t)\sqrt{1-\rho_1^2}}{n}\sum_{i=1}^n S_i(t)dW_i(t) \\
+ \frac{\sigma^2(t)}{n}\sum_{i=1}^n S_i(t)\int_{\mathbb{R}}(e^{\alpha_2 x}-1)N_i(dt,dx) + \frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n S_i(t)\int_{\mathbb{R}} xN_j(dt,dx) \\
+ \frac{1}{n^2}\sum_{i=1}^n S_i(t)\int_{\mathbb{R}} x(e^{\alpha_2 x}-1)N_i(dt,dx).
\end{aligned}
$$

Similarly,

$$
\begin{aligned}
dI_n^{(2)}(t) = I_n^{(2)}(t)\left(2\alpha_1 + \sigma^2(t) - \lambda\right)dt + 2\sigma(t)\rho_1 I_n^{(2)}(t)dM(t) \\
+ \frac{2\sigma^3(t)\sqrt{1-\rho_1^2}}{n}\sum_{i=1}^n S_i^2(t)dW_i(t) + \frac{\sigma^2(t)}{n}\sum_{i=1}^n S_i^2(t)\int_{\mathbb{R}}(e^{2\alpha_2 x}-1)N_i(dt,dx) \\
+ \frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n S_i^2(t)\int_{\mathbb{R}} xN_j(dt,dx) + \frac{1}{n^2}\sum_{i=1}^n S_i^2(t)\int_{\mathbb{R}} x(e^{2\alpha_2 x}-1)N_i(dt,dx).
\end{aligned}
$$

In general, for $k \in \mathbb{N}$, the k-th empirical moment process modulated by volatility/variance satisfies the following stochastic differential equation.

$$
\begin{aligned}
dI_n^{(k)}(t) =& I_n^{(k)}(t)\left(k\alpha_1 + \frac{k(k-1)}{2}\sigma^2(t) - \lambda\right)dt + k\sigma(t)\rho_1 I_n^{(k)}(t)dM + \frac{k\sigma^3(t)\sqrt{1-\rho_1^2}}{n}\sum_{i=1}^n S_i^k(t)dW_i \\
&+ \frac{\sigma^2(t)}{n}\sum_{i=1}^n S_i^k(t)\int_{\mathbb{R}}(e^{k\alpha_2 x}-1)N_i(dt,dx) + \frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n S_i^k(t)\int_{\mathbb{R}} xN_j(dt,dx) \\
&+ \frac{1}{n^2}\sum_{i=1}^n S_i^k(t)\int_{\mathbb{R}} x(e^{k\alpha_2 x}-1)N_i(dt,dx).
\end{aligned}
$$

With

$$
\gamma = \int_{\mathbb{R}} x\nu(dx),
$$

and

$$
\beta_k = \int_{\mathbb{R}}(e^{k\alpha_2 x}-1)\nu(dx), \quad \mu_k = \int_{\mathbb{R}} x(e^{k\alpha_2 x}-1)\nu(dx), \quad k \in \mathbb{N},
$$

we obtain

$$
\begin{aligned}
dI_n^{(k)}(t) &= \left(k\alpha_1 + \frac{k(k-1)}{2}\sigma^2(t) - \lambda + \beta_k + \frac{\gamma}{\sigma^2(t)} + \frac{\mu_k}{n\sigma^2(t)}\right) I_n^{(k)}(t)dt \\
&\quad + k\sigma(t)\rho_1 I_n^{(k)}(t)dM(t) + \frac{k\sigma^3(t)\sqrt{1-\rho_1^2}}{n}\sum_{i=1}^{n} S_i^k(t)dW_i(t) \\
&\quad + \frac{\sigma^2(t)}{n}\sum_{i=1}^{n} S_i^k(t)\int_{\mathbb{R}}(e^{k\alpha_2 x}-1)\tilde{N}_i(dt,dx) + \frac{1}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n} S_i^k(t)\int_{\mathbb{R}} x\tilde{N}_j(dt,dx) \\
&\quad + \frac{1}{n^2}\sum_{i=1}^{n} S_i^k(t)\int_{\mathbb{R}} x(e^{k\alpha_2 x}-1)\tilde{N}_i(dt,dx). \qquad (4.5)
\end{aligned}
$$

Note that, from (4.1), with respect to a risk-neutral measure we have $\alpha_1 + \beta_1 = r$. At first, we proceed to prove a result concerning the weak convergence of the stochastic process $I_n^{(k)}$ as $n \to \infty$. We denote the limit process by $I^{(k)}$ for $k \in \mathbb{N}$. The following Lemma is obtained for a simple model in [35]. It is straight forward to show that this can be generalized for the present model.

Lemma 4.1.1. *Suppose that $\mathbb{E}[S_i(t)^{2k}] < \infty$ for all $t \geq 0$. Then for any $T > 0$ and $k \in \mathbb{N}$,*

$$\mathbb{E}\left[\left(\sup_{t\leq T}\frac{1}{\sqrt{n}}\int_0^t I_n^{(k)}(u)\,du\right)^2\right] \to 0, \quad \text{as} \quad n \to \infty.$$

The main tool that we use for the convergence analysis is the following theorem due to R. Rebolledo [51]. The present version can be found in [29, 35]. An important concept that will be useful for the next theorem is that of weak convergence of sequence of random variables [21]. Consider a sequence of probabilities $(\mathbb{P}_n)_{n\in\mathbb{N}}$ on $(\mathbb{R}, \mathcal{B}_{\mathbb{R}})$. The sequence of probabilities $(\mathbb{P}_n)_{n\in\mathbb{N}}$ converges weakly to $\mathbb{P}$ if

$$\lim_{n\to\infty}\int f d\mathbb{P}_n = \int f d\mathbb{P}$$

for all $f : \mathbb{R} \to \mathbb{R}$ continuous and bounded [21].

Definition 4.1.2 (Weak convergence [21])**.** *Let $(X_n)_{n\in\mathbb{N}}$ be a sequence o of random variables the probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and X a random variable defined on the same probability space $(\Omega, \mathcal{F}, \mathbb{P})$. $(X_n)_{n\in\mathbb{N}}$ converges in distribution to X if the sequence $\mathbb{P}\circ X_n^{-1}$ converges weakly to $\mathbb{P}\circ X^{-1}$ and is* *denoted by*

$$X_n \Longrightarrow X$$

as $n \to \infty$.

Theorem 4.1.3 (R. Rebolledo [51, 29, 35])**.** *Let* $a = ((a_{ij}))$ *be a continuous, symmetric, non-negative definite,* $d \times d$ *matrix-valued function on* $\mathbb{R}^d$ *and let* $b : \mathbb{R}^d \to \mathbb{R}^d$ *be continuous. Let*

$$A(a, b) = \left\{ \left(f, Gf \equiv \frac{1}{2} \sum a_{ij} \partial_i \partial_j f + \sum b_i \partial_i f \right) : f \in C_c^\infty(\mathbb{R}^d) \right\},$$

where $C_c^\infty(\mathbb{R}^d)$ *denotes the class of compactly supported infinitely differentiable functions on* $\mathbb{R}^d$*, and suppose that the* $C_{\mathbb{R}^d}[0, \infty)$ *martingale problem for* A *is well-posed. For* $n \in \{1, 2, \dots\}$*, let* X_n *and* B_n *be processes with sample paths in* $D_{\mathbb{R}^d}[0, \infty)$*, and let* $A_n = ((A_n^{ij}))$ *be a symmetric* $d \times d$ *matrix-valued process such that* A_n^{ij} *has sample paths in* $D_{\mathbb{R}^d}[0, \infty)$ *and* $A_n(t) - A_n(s)$ *is non-negative definite for* $t > s \geq 0$*. Set* $\mathcal{F}_t^n = \sigma(X_n(s), B_n(s), A_n(s) : s \leq t)$*.*

Let $\tau_n^r = \inf\{t \geq 0 : |X_n(t)| \geq r \quad \text{or} \quad |X_n(t-)| \geq r\}$*, and suppose that*

$$M_n \equiv X_n - B_n,$$

and

$$M_n^i M_n^j - A_n^{ij},\ i, j = 1, \dots, d,$$

are $\{F_t^n\}$*-local martingales, and that for each* $r > 0$*,* $T > 0$ *and* $i, j = 1, \dots, d$*,*

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{t \leq T \wedge \tau_n^r} |X_n(t) - X_n(t-)|^2 \right] = 0,$$
$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{t \leq T \wedge \tau_n^r} |B_n(t) - B_n(t-)|^2 \right] = 0,$$
$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{t \leq T \wedge \tau_n^r} |A_n^{ij}(t) - A_n^{ij}(t-)| \right] = 0,$$

$$\sup_{t\leq T\wedge\tau_n^r}\left|B_n^i(t)-\int_0^t b_i(X_n(s))ds\right|\to 0,\quad \textit{in probability,}$$

$$\sup_{t\leq T\wedge\tau_n^r}\left|A_n^{ij}(t)-\int_0^t a_{ij}(X_n(s))ds\right|\to 0,\quad \textit{in probability.} \tag{4.6}$$

Suppose that $\mathbb{P}\circ X_n(0)^{-1}\Longrightarrow\eta\in\mathcal{P}(\mathbb{R}^d)$. *Then* $\{X_n\}$ *converges in distribution to the solution of the martingale problem for* (A,η).

We now proceed to state a weak convergence theorem describing the behavior of the index process in the large-basket limit.

Theorem 4.1.4. *Let* $k,i\in\mathbb{N}$ *and suppose that* $E[S_i(0)^{4k}]<\infty$, $\int_{\mathbb{R}}e^{4k\alpha_2 x}\nu(dx)<\infty$, *and for* $t\in[0,T]$, $|\sigma(t)|^2\leq C$, *for some* $C>0$. *Further assume that* $\mu_k<\infty$ *for* $k\in\mathbb{N}$. *Then for* $t\in[0,T]$, $I_n^{(k)}\Rightarrow I^{(k)}$ *as* $n\to\infty$, *where the process* $I^{(k)}$ *is given by*

$$dI^{(k)}(t)=I^{(k)}(t)\left(k\alpha_1+\frac{k(k-1)}{2}\sigma^2(t)-\lambda+\beta_k+\frac{\gamma}{\sigma^2(t)}\right)dt+k\sigma(t)\rho_1 I^{(k)}(t)dM,$$

with $I^{(k)}(0)=E[S_1(0)^k]$.

Proof. The dynamics of $I_n^{(k)}$ is given by (4.5). Let

$$B_n(t)=\int_0^t I_n^{(k)}(u)\left(k\alpha_1+\frac{k(k-1)}{2}\sigma^2(u)-\lambda+\beta_k+\frac{\gamma}{\sigma^2(u)}\right)du.$$

Clearly, with $X_n=I_n^{(k)}$ we have $M_n=X_n-B_n$ a local martingale, where

$$\begin{aligned}M_n(t)=&\int_0^t k\sigma(u)\rho_1 I_n^{(k)}(u)dM(u)+\frac{k\sqrt{1-\rho_1^2}}{n}\sum_{i=1}^n\int_0^t\sigma^3(u)S_i^k(u)dW_i(u)\\&+\sum_{i=1}^n\int_0^t\int_{\mathbb{R}}S_i^k(u)(e^{k\alpha_2 x}-1)\left(\frac{\sigma^2(u)}{n}+\frac{x}{n^2}\right)\tilde{N}_i(du,dx)+\frac{1}{n^2}\sum_{i=1}^n\sum_{j=1}^n\int_0^t\int_{\mathbb{R}}S_i^k(u)x\tilde{N}_j(du,dx).\end{aligned}$$

Next, we define $A_n(t)=[M_n,M_n](t)$. By construction, clearly $A_n(t)-A_n(s)$ is non-negative definite for $t\geq s\geq 0$. By Doob-Meyer decomposition $M_n^2-A_n$ is a local martingale. Since the jumps occur at distinct times almost surely, we have

$$\mathbb{E}\left[\sup_{t\le T}|X_n(t)-X_n(t-)|^2\right] = \mathbb{E}\left[\sup_{1\le i\le n}\sup_{t\le T}\left(\frac{1}{n}(\sigma^2(t)S_i^k(t)-\sigma^2(t-)S_i^k(t-))\right)^2\right]$$
$$\le \frac{C^2}{n^2}\mathbb{E}\left[\sup_{1\le i\le n}\sup_{t\le T}S_i^{2k}(t)\right].$$

Since the jump sizes $|A_n(t)-A_n(t-)|$ are essentially same as $|X_n(t)-X_n(t-)|^2$,

$$\mathbb{E}\left[\sup_{t\le T}|A_n(t)-A_n(t-)|\right] \le \frac{C^2}{n^2}\mathbb{E}\left[\sup_{1\le i\le n}\sup_{t\le T}S_i^{2k}(t)\right].$$

Assumptions of this theorem ($E[S_i(0)^{4k}]<\infty$, $\int_{\mathbb{R}} e^{4k\alpha_2 x}\nu(dx)<\infty$, and for $t\in[0,T]$, $|\sigma(t)|^2\le C$) imply (see [35]) that for $t\in[0,T]$, $E[S_i(t)^k]<\infty$. Hence by [35] (Lemma 4.3) we obtain

$$\mathbb{E}\left[\sup_{t\le T}|X_n(t)-X_n(t-)|^2\right]\to 0,$$

and

$$\mathbb{E}\left[\sup_{t\le T}|A_n(t)-A_n(t-)|^2\right]\to 0,$$

as $n\to\infty$. Also, since B_n is continuous $\lim_{n\to\infty}\mathbb{E}\left[\sup_{t\le T}|B_n(t)-B_n(t-)|^2\right]=0$. We observe that

$$B_n(t)-\int_0^t b(X_n(u))du = \int_0^t I_n^{(k)}(u)\left(k\alpha_1+\frac{k(k-1)}{2}\sigma^2(u)-\lambda+\beta_k+\frac{\gamma}{\sigma^2(u)}+\frac{\mu_k}{n\sigma^2(u)}\right)du$$
$$-\int_0^t I_n^{(k)}(u)\left(k\alpha_1+\frac{k(k-1)}{2}\sigma^2(u)-\lambda+\beta_k+\frac{\gamma}{\sigma^2(u)}\right)dt$$
$$=\frac{1}{n}\int_0^t\frac{\mu_k}{\sigma^2(u)}I_n^{(k)}(u)du \;\to\; 0,\quad \text{in probability},$$

by Lemma 4.1.1 and the assumption of the theorem. Thus all the conditions except (4.6) are verified for Theorem 4.1.3. Next, we proceed to verify (4.6). We denote

$$
\begin{aligned}
G_n(t) &:= \frac{1}{n}\int_0^t k^2(1-\rho_1^2)\sigma^4(u)I_n^{(2k)}(u)du \\
H_n^1(t) &:= \frac{1}{n^4}\sum_{p=1}^n\sum_{q=1}^n\sum_{i=1}^n\int_0^t\int_{\mathbb{R}} S_p^k(u)S_q^k(u)x^2N_i(du,dx), \\
H_n^2(t) &:= \frac{2}{n^3}\sum_{i=1}^n\int_0^t\int_{\mathbb{R}} S_i^{2k}(u)\sigma^2(u)x(e^{k\alpha_2 x}-1)^2N_i(du,dx), \\
H_n^3(t) &:= \frac{1}{n^4}\sum_{i=1}^n\int_0^t\int_{\mathbb{R}} S_i^{2k}(u)x^2(e^{k\alpha_2 x}-1)^2N_i(du,dx), \\
H_n^4(t) &:= \frac{1}{n^2}\sum_{i=1}^n\int_0^t\int_{\mathbb{R}} S_i^{2k}(u)\sigma^4(u)(e^{k\alpha_2 x}-1)^2N_i(du,dx), \\
H_n^5(t) &:= \frac{2}{n^4}\sum_{i=1}^n\sum_{j=1}^n\int_0^t\int_{\mathbb{R}} S_i^k(u)S_j^k(u)x(e^{k\alpha_2 x}-1)N_j(du,dx), \\
H_n^6(t) &:= \frac{2}{n^3}\sum_{i=1}^n\sum_{j=1}^n\int_0^t\int_{\mathbb{R}} S_i^k(u)S_j^k(u)\sigma^2(u)x(e^{k\alpha_2 x}-1)N_j(du,dx).
\end{aligned}
$$

Consider

$$
\begin{aligned}
\mathbb{E}\left[\sup_{t\le T}\left|A_n(t)-\int_0^t a(X_n(t))\right|^2\right] &\le \mathbb{E}\left[\left(\sup_{t\le T}|G_n(t)|+\sum_{i=1}^6\sup_{t\le T}|H_n^i(t)|\right)^2\right] \\
&\le 8\mathbb{E}\left[\sup_{t\le T}|G_n(t)|^2\right]+8\sum_{i=1}^6\mathbb{E}\left[\sup_{t\le T}|H_n^i(t)|^2\right]. \qquad (4.7)
\end{aligned}
$$

To verify (4.6), it is sufficient to show that $\mathbb{E}\left[\sup_{t\le T}|G_n(t)|^2\right]$ and $\mathbb{E}\left[\sup_{t\le T}|H_n^i(t)|^2\right]$ for $1\le i\le 6$ converge to 0 as $n\to\infty$. Clearly, by Lemma 4.1.1 and boundedness of σ^2, we obtain

$$
\mathbb{E}\left[\sup_{t\le T}|G_n(t)|^2\right] = \frac{k^4(1-\rho_1^2)^2}{n}\mathbb{E}\left[\sup_{t\le T}\left|\int_0^t\frac{\sigma^4(u)}{\sqrt{n}}I_n^{(2k)}(u)du\right|^2\right]\to 0, \quad \text{as} \quad n\to\infty.
$$

Also,

$$
\begin{aligned}
\mathbb{E}\left[\sup_{t\le T}|H_n^1(t)|^2\right] &= \mathbb{E}\left[H_n^1(T)^2\right] = \frac{1}{n^6}\mathbb{E}\left[\left(\sum_{p=1}^{n}\sum_{q=1}^{n}\sum_{r=1}^{N^1(T)} S_p^k(\tau_r^1)S_q^k(\tau_r^1)(\mathcal{J}_r^1)^2\right)^2\right] \\
&= \frac{1}{n^4}\mathbb{E}\left[\left(\sum_{r=1}^{N^1(T)} S_1^{2k}(\tau_r^1)(\mathcal{J}_r^1)^2\right)^2\right] \le \frac{1}{n^4}\mathbb{E}\left[N^1(T)\sum_{r=1}^{N^1(T)} S_1^{4k}(\tau_r^1)(\mathcal{J}_r^1)^4\right] \\
&\le \frac{1}{n^4}\sum_{N=1}^{\infty} N^2\left(\sup_{t\le T}\mathbb{E}[S_1^{4k}(t)]\right)\mathbb{E}[(\mathcal{J}_j^1)^4]\mathbb{P}(N^1(T)=N) \\
&\le \frac{1}{n^4}\left(\sup_{t\le T}\mathbb{E}[S_1^{4k}(t)]\right)\mathbb{E}[(\mathcal{J}_j^1)^4]\mathbb{E}[N^1(T)^2] \to 0, \quad \text{as} \quad n\to\infty.
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\left[\sup_{t\le T}|H_n^2(t)|^2\right] &= \mathbb{E}\left[H_n^2(T)^2\right] = \frac{4}{n^6}\mathbb{E}\left[\left(\sum_{i=1}^{n}\int_0^T\int_{\mathbb{R}} S_i^{2k}(u)\sigma^2(u)x(e^{k\alpha_2 x}-1)^2 N_i(du,dx)\right)^2\right] \\
&\le \frac{4C}{n^5}\mathbb{E}\left[\sum_{i=1}^{n}\left(\int_0^T\int_{\mathbb{R}} S_i^{2k}(u)x(e^{k\alpha_2 x}-1)^2 N_i(du,dx)\right)^2\right] \\
&\le \frac{4C}{n^4}\mathbb{E}\left[N^1(T)\sum_{j=1}^{N^1(T)} S_1^{4k}(\tau_j^1)(\mathcal{J}_j^1)^2(e^{k\alpha_2\mathcal{J}_j^1}-1)^4\right] \\
&\le \frac{4C}{n^4}\left(\sup_{t\le T}\mathbb{E}[S_1^{4k}(t)]\right)\mathbb{E}[(\mathcal{J}_j^1)^2(e^{k\alpha_2\mathcal{J}_j^1}-1)^4]\mathbb{E}[N^1(T)^2] \to 0, \quad \text{as} \quad n\to\infty.
\end{aligned}
$$

$$
\begin{aligned}
\mathbb{E}\left[\sup_{t\le T}|H_n^3(t)|^2\right] &= \mathbb{E}\left[H_n^3(T)^2\right] = \mathbb{E}\left[\left(\frac{1}{n^4}\sum_{i=1}^{n}\int_0^T\int_{\mathbb{R}} S_i^{2k}(u)x^2(e^{k\alpha_2 x}-1)^2 N_i(du,dx)\right)^2\right] \\
&\le \frac{1}{n^7}\mathbb{E}\left[\sum_{i=1}^{n}\left(\int_0^T\int_{\mathbb{R}} S_i^{2k}(u)x^2(e^{k\alpha_2 x}-1)^2 N_i(du,dx)\right)^2\right] \\
&\le \frac{1}{n^7}\mathbb{E}\left[\sum_{i=1}^{n}\left(\sum_{j=1}^{N^i(T)} S_i^{2k}(\tau_j^i)x^2(e^{k\alpha_2\mathcal{J}_j^i}-1)^2\right)^2\right] \\
&\le \frac{1}{n^6}\sum_{N=1}^{\infty} N^2\left(\sup_{t\le T}\mathbb{E}[S_1^{4k}(t)]\right)\mathbb{E}[(\mathcal{J}_j^1)^4(e^{k\alpha_2\mathcal{J}_j^1}-1)^4]\mathbb{P}(N^1(T)=N) \\
&\le \frac{1}{n^6}\left(\sup_{t\le T}\mathbb{E}[S_1^{4k}(t)]\right)\mathbb{E}[(\mathcal{J}_j^1)^4(e^{k\alpha_2\mathcal{J}_j^1}-1)^4]\mathbb{E}[N^1(T)^2] \to 0, \quad \text{as} \quad n\to\infty.
\end{aligned}
$$

In a similar procedure as in the case of $H_n^2(t)$, we can show

$$\mathbb{E}\left[\sup_{t\le T}|H_n^4(t)|^2\right] \le \frac{C}{n^3}\left(\sup_{t\le T}\mathbb{E}[S_1^{4k}(t)]\right)\mathbb{E}[(\mathcal{J}_j^1)^2(e^{k\alpha_2\mathcal{J}_j^1}-1)^4]\mathbb{E}[N^1(T)^2] \to 0, \quad \text{as} \quad n\to\infty.$$

Next,

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\le T}|H_n^5(t)|^2\right] &= \mathbb{E}\left[H_n^5(T)^2\right] = \mathbb{E}\left[\left(\frac{2}{n^4}\sum_{i=1}^n\sum_{j=1}^n\int_0^T\int_{\mathbb{R}} S_i^k(u)S_j^k(u)x(e^{k\alpha_2 x}-1)N_j(du,dx)\right)^2\right] \\
&\le \frac{4}{n^8}\mathbb{E}\left[n^2\sum_{i=1}^n\sum_{j=1}^n\left(\int_0^T\int_{\mathbb{R}} S_i^k(u)S_j^k(u)x(e^{k\alpha_2 x}-1)N_j(du,dx)\right)^2\right] \\
&\le \frac{4}{n^4}\mathbb{E}\left[N^1(T)\sum_{j=1}^{N^1(T)} S_1^{4k}(\tau_r^1)(\mathcal{J}_r^1)^2(e^{k\alpha_2\mathcal{J}_r^1}-1)^2\right] \\
&\le \frac{4}{n^4}\sum_{N=1}^{\infty}N^2\left(\sup_{t\le T}\mathbb{E}[S_1^{4k}(t)]\right)\mathbb{E}[(\mathcal{J}_j^1)^2(e^{k\alpha_2\mathcal{J}_j^1}-1)^2]\mathbb{P}(N^1(T)=N) \\
&\le \frac{4}{n^4}\left(\sup_{t\le T}\mathbb{E}[S_1^{4k}(t)]\right)\mathbb{E}[(\mathcal{J}_j^1)^2(e^{k\alpha_2\mathcal{J}_j^1}-1)^2]\mathbb{E}[N^1(T)^2] \to 0, \quad \text{as} \quad n\to\infty.
\end{aligned}$$

Finally,

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\le T}|H_n^6(t)|^2\right] &= \mathbb{E}\left[H_n^6(T)^2\right] \\
&= \mathbb{E}\left[\left(\frac{2}{n^3}\sum_{i=1}^n\sum_{j=1}^n\int_0^T\int_{\mathbb{R}} S_i^k(u)S_j^k(u)\sigma^2(u)x(e^{k\alpha_2 x}-1)N_j(du,dx)\right)^2\right] \\
&\le \frac{4C}{n^6}\mathbb{E}\left[n^2\sum_{i=1}^n\sum_{j=1}^n\left(\int_0^T\int_{\mathbb{R}} S_i^k(u)S_j^k(u)x(e^{k\alpha_2 x}-1)N_j(du,dx)\right)^2\right] \\
&\le \frac{4C}{n^2}\mathbb{E}\left[N^1(T)\sum_{j=1}^{N^1(T)} S_1^{4k}(\tau_r^1)(\mathcal{J}_r^1)^2(e^{k\alpha_2\mathcal{J}_r^1}-1)^2\right] \\
&\le \frac{4C}{n^2}\sum_{N=1}^{\infty}N^2\left(\sup_{t\le T}\mathbb{E}[S_1^{4k}(t)]\right)\mathbb{E}[(\mathcal{J}_j^1)^2(e^{k\alpha_2\mathcal{J}_j^1}-1)^2]\mathbb{P}(N^1(T)=N) \\
&\le \frac{4C}{n^2}\left(\sup_{t\le T}\mathbb{E}[S_1^{4k}(t)]\right)\mathbb{E}[(\mathcal{J}_j^1)^2(e^{k\alpha_2\mathcal{J}_j^1}-1)^2]\mathbb{E}[N^1(T)^2] \to 0, \quad \text{as} \quad n\to\infty.
\end{aligned}$$

Combining all these results we obtain from (4.7) that $\mathbb{E}\left[\sup_{t\leq T}\left|A_n(t)-\int_0^t a(X_n(t))\right|^2\right]\to 0$ as $n\to\infty$. Hence (4.6) is verified and consequently all the assumptions in Theorem 4.1.3 are verified. Hence the proof is complete. □

We define the "error term" by

$$\Pi_n(t)=\sqrt{n}(I_n(t)-I^{(1)}(t)), \tag{4.8}$$

i.e.,

$$\Pi_n(t)=\frac{1}{n^{\frac{3}{2}}}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^t\int_{\mathbb{R}}S_i(u)x\tilde{N}_j(dx,du)+\sum_{i=1}^{n}\int_0^t\int_{\mathbb{R}}S_i(u)(e^{\alpha_2 x}-1)\left(\frac{\sigma^2(u)}{\sqrt{n}}+\frac{x}{n^{\frac{3}{2}}}\right)\tilde{N}_i(du,dx). \tag{4.9}$$

As observed in [35], Π_n can be thought of as a scaled fluctuation of $I_n(t)$ around the approximation $I(t)=I^{(1)}(t)$. The final result in this section is the large n limit behavior of Π_n. For the simplicity of computation we derive the result when $\rho_1=1$ in (4.1). We also assume $\alpha_2\geq 0$ and N_i $(i=1,2,\ldots,n)$ are subordinators. Suppose that

$$X_n(t):=\begin{pmatrix}\Pi_n(t)\\ \xi^2 I_n^{(2)}(t)\end{pmatrix}, \tag{4.10}$$

and

$$X(t):=\begin{pmatrix}\int_0^t \xi\sigma(u)\sqrt{I^{(2)}(u)}dB(u)\\ \xi^2 I^{(2)}(t)\end{pmatrix}, \tag{4.11}$$

where B is a standard Brownian motion independent of M and

$$\xi^2=\int_{\mathbb{R}+}(e^{\alpha_2 x}-1)^2\nu(dx). \tag{4.12}$$

By Theorem 4.1.4 we obtain the dynamics of $I^{(2)}$ as

$$dI^{(2)}(t) = I^{(2)}(t)\left(2\alpha_1 + \sigma^2(t) - \lambda + \beta_2 + \frac{\gamma}{\sigma^2(t)}\right)dt + 2\sigma(t)I^{(2)}(t)dM(t).$$

From (4.11) we obtain

$$dX(t) = \begin{pmatrix} 0 \\ \xi^2\left(2\alpha_1 + \sigma^2(t) - \lambda + \beta_2 + \frac{\gamma}{\sigma^2(t)}\right)I^{(2)}(t) \end{pmatrix} dt$$
$$+ \begin{pmatrix} \xi\sigma(t)\sqrt{I^{(2)}(t)} & 0 \\ 0 & 2\xi^2\sigma(t)I^{(2)}(t) \end{pmatrix}\begin{pmatrix} dB(t) \\ dM(t) \end{pmatrix}.$$

We use Theorem 4.1.3 to show that X_n converges weakly to the solution of a well-posed martingale problem solved by X with generator

$$b(x,y) = \begin{pmatrix} 0 \\ \left(2\alpha_1 + \sigma^2(t) - \lambda + \beta_2 + \frac{\gamma}{\sigma^2(t)}\right)y \end{pmatrix},$$
$$a(x,y) = \begin{pmatrix} \sigma^2(t)|y| & 0 \\ 0 & 4\sigma^2(t)y^2 \end{pmatrix}.$$

We define

$$B_n(t) = \begin{pmatrix} 0 \\ \xi^2\int_0^t \left(2\alpha_1 + \sigma^2(u) - \lambda + \beta_2 + \frac{\gamma}{\sigma^2(t)}\right)I_n^{(2)}(u)du \end{pmatrix}.$$

Clearly $M_n = X_n - B_n$ has no drift part and is a local martingale. Also, since $B_n(t)$ is continuous,

$$\lim_{n\to\infty}\mathbb{E}\left[\sup_{t\le T}|B_n(t) - B_n(t-)|^2\right] = 0, \tag{4.13}$$

and trivially for $i = 1, 2$,

$$\sup_{t\le T}\left|B_n^i(t) - \int_0^t b_i(X_n(s))ds\right| \to 0 \quad \text{in probability.} \tag{4.14}$$

By the Doob-Meyer decomposition we choose

$$A_n^{ij}(t) = [M_n^i, M_n^j](t), \quad 1 \le i, j \le 2,$$

where

$$\begin{aligned} M_n^1(t) = & \frac{1}{n^{\frac{3}{2}}} \sum_{i=1}^{n} \sum_{j=1}^{n} \int_0^t \int_{\mathbb{R}+} S_i(u) x \tilde{N}_j(dx, du) \\ & + \sum_{i=1}^{n} \int_0^t \int_{\mathbb{R}+} S_i(u)(e^{\alpha_2 x} - 1) \left(\frac{\sigma^2(u)}{\sqrt{n}} + \frac{x}{n^{\frac{3}{2}}} \right) \tilde{N}_i(du, dx), \end{aligned}$$

$$\begin{aligned} M_n^2(t) = & \xi^2 \int_0^t 2\sigma(u) I_n^{(2)}(u) dM(u) \\ & + \xi^2 \sum_{i=1}^{n} \int_0^t \int_{\mathbb{R}+} S_i^2(u)(e^{2\alpha_2 x} - 1) \left(\frac{\sigma^2(u)}{n} + \frac{x}{n^2} \right) \tilde{N}_i(dx, du) \\ & + \frac{\xi^2}{n^2} \sum_{i=1}^{n} \sum_{j=1}^{n} \int_0^t \int_{\mathbb{R}+} S_i^2(u) x \tilde{N}_j(du, dx). \end{aligned}$$

We write

$$A_n^{ij}(t) = G_n^{ij}(t) + H_n^{ij}(t), \quad 1 \le i, j \le 2,$$

where

$$G_n^{11}(t) = 0,$$

$$\begin{aligned} H_n^{11}(t) = & \frac{1}{n^3} \sum_{p=1}^{n} \sum_{q=1}^{n} \sum_{i=1}^{n} \int_0^t \int_{\mathbb{R}+} S_p(u) S_q(u) x^2 N_i(du, dx) \\ & + \sum_{i=1}^{n} \int_0^t \int_{\mathbb{R}+} S_i^2(u)(e^{\alpha_2 x} - 1)^2 \left(\frac{\sigma^2(u)}{\sqrt{n}} + \frac{x}{n^{\frac{3}{2}}} \right)^2 N_i(du, dx) \\ & + \frac{2}{n^{\frac{3}{2}}} \sum_{i=1}^{n} \sum_{j=1}^{n} \int_0^t \int_{\mathbb{R}+} S_i(u) S_j(u) x (e^{\alpha_2 x} - 1) \left(\frac{\sigma^2(u)}{\sqrt{n}} + \frac{x}{n^{\frac{3}{2}}} \right) N_j(du, dx), \end{aligned}$$

$$G_n^{12}(t) = G_n^{21}(t) = 0,$$

$$\begin{aligned}
H_n^{12}(t) = H_n^{21}(t) = & \frac{\xi^2}{n^{\frac{7}{2}}} \sum_{p=1}^{n}\sum_{q=1}^{n}\sum_{i=1}^{n} \int_0^t \int_{\mathbb{R}+} S_p(u) S_q^2(u) x^2 N_i(du, dx) \\
& + \frac{\xi^2}{n^{\frac{3}{2}}} \sum_{i=1}^{n}\sum_{j=1}^{n} \int_0^t \int_{\mathbb{R}+} S_i(u) S_j^2(u) x (e^{2\alpha_2 x} - 1) \left(\frac{\sigma^2(u)}{n} + \frac{x}{n^2} \right) N_j(du, dx) \\
& + \frac{\xi^2}{n^2} \sum_{i=1}^{n}\sum_{j=1}^{n} \int_0^t \int_{\mathbb{R}+} S_i(u) S_j^2(u) x (e^{\alpha_2 x} - 1) \left(\frac{\sigma^2(u)}{\sqrt{n}} + \frac{x}{n^{\frac{3}{2}}} \right) N_j(du, dx) \\
& + \xi^2 \sum_{i=1}^{n} \int_0^t \int_{\mathbb{R}+} S_i^3(u) (e^{2\alpha_2 x} - 1)(e^{\alpha_2 x} - 1) \left(\frac{\sigma^2(u)}{\sqrt{n}} + \frac{x}{n^{\frac{3}{2}}} \right) \left(\frac{\sigma^2(u)}{n} + \frac{x}{n^2} \right) N_i(du, dx),
\end{aligned}$$

$$G_n^{22}(t) = \xi^4 \int_0^t 4\sigma^2(u) (I_n^{(2)}(u))^2 \, du,$$

$$\begin{aligned}
H_n^{22}(t) = & \frac{\xi^4}{n^4} \sum_{p=1}^{n}\sum_{q=1}^{n}\sum_{i=1}^{n} \int_0^t \int_{\mathbb{R}+} S_p^2(u) S_q^2(u) x^2 N_i(du, dx) \\
& + \xi^4 \sum_{i=1}^{n} \int_0^t \int_{\mathbb{R}+} S_i^4(u) (e^{2\alpha_2 x} - 1)^2 \left(\frac{\sigma^2(u)}{n} + \frac{x}{n^2} \right)^2 N_i(du, dx) \\
& + \frac{2\xi^4}{n^2} \sum_{i=1}^{n}\sum_{j=1}^{n} \int_0^t \int_{\mathbb{R}+} S_i^2(u) S_j^2(u) x (e^{2\alpha_2 x} - 1) \left(\frac{\sigma^2(u)}{n} + \frac{x}{n^2} \right) N_j(du, dx).
\end{aligned}$$

We state two lemmas that are essential in proving Theorem 4.1.7, the convergence theorem describing the behavior of the "error term" in the large-basket limit. Note that Lemma 4.1.5 holds irrespective of the assumptions $\alpha_2 \geq 0$ and N_i $(i = 1, 2, \ldots, n)$ are subordinators. We provide a general proof for Lemma 4.1.5. However, for Lemma 4.1.6 we need those assumptions.

Lemma 4.1.5. *Suppose that $E[S_i(0)^k] < \infty$ and $\int_{\mathbb{R}} e^{k\alpha_2 x} \nu(dx) < \infty$, for $1 \leq k \leq 8$. Also, suppose that for $t \in [0, T]$, $|\sigma(t)|^2 \leq C$, for some $C > 0$ and $\mu_l < \infty$ for $l = 1, 2$. Then for $i, j \in \{1, 2\}$,*

$$\lim_{n \to \infty} \mathbb{E} \left[\sup_{t \leq T} |X_n(t) - X_n(t-)|^2 \right] = 0, \tag{4.15}$$

$$\lim_{n\to\infty} \mathbb{E}\left[\sup_{t\le T}|A_n^{ij}(t) - A_n^{ij}(t-)|\right] = 0, \tag{4.16}$$

and

$$\sup_{t\le T}\left|A_n^{ij}(t) - \int_0^t a_{ij}(X_n(s))ds\right| \to 0, \quad \textit{in probability.} \tag{4.17}$$

Proof. By (4.8) jumps of $\Pi_n(t)$ are same as jumps of $\sqrt{n}I_n(t)$. Hence,

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\le T}|X_n(t) - X_n(t-)|^2\right] = \\
&\mathbb{E}\left[\sup_{t\le T}\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n(\sigma^2(t)S_i(t) - \sigma^2(t-)S_i(t-))\right)^2 + \left(\frac{\xi^2}{n}\sum_{i=1}^n(\sigma^2(t)S_i^2(t) - \sigma^2(t-)S_i^2(t-))\right)^2\right] \\
&\le C\mathbb{E}\left[\sup_{t\le T}\left(\left(\frac{1}{\sqrt{n}}\sum_{i=1}^n(S_i(t) - S_i(t-))\right)^2 + \left(\frac{\xi^2}{n}\sum_{i=1}^n(S_i^2(t) - S_i^2(t-))\right)^2\right)\right] \\
&\le \frac{C}{n}\mathbb{E}\left[\sup_{t\le T}\sum_{i=1}^n(S_i(t) - S_i(t-))^2\right] + \frac{C\xi^4}{n^2}\mathbb{E}\left[\sup_{t\le T}\sum_{i=1}^n(S_i^2(t) - S_i^2(t-))^2\right] \\
&\le \frac{C}{n}\mathbb{E}\left[\sup_{1\le i\le n}\sup_{t\le T}S_i^2(t)\right] + \frac{C\xi^4}{n^2}\mathbb{E}\left[\sup_{1\le i\le n}\sup_{t\le T}S_i^4(t)\right],
\end{aligned}$$

where we have used repeatedly the fact that no two jumps occur at the same time almost surely. Hence (4.15) is proved. Proof of (4.16) is similar and is as follows,

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\le T}|A_n^{11}(t) - A_n^{11}(t-)|\right] &= \mathbb{E}\left[\sup_{t\le T}\frac{1}{n}\sum_{i=1}^n(\sigma^3(t)S_i(t) - \sigma^3(t-)S_i(t-))^2\right] \\
&\le \frac{C}{n}\mathbb{E}\left[\sup_{1\le i\le n}\sup_{t\le T}S_i^2(t)\right],
\end{aligned}$$

and

$$\begin{aligned}
&\mathbb{E}\left[\sup_{t\le T}|A_n^{12}(t)-A_n^{12}(t-)|\right]\\
&\quad=\mathbb{E}\left[\sup_{t\le T}\xi^2 n^{-\frac{3}{2}}\sum_{i=1}^n|\sigma^3(t)S_i^2(t)-\sigma^3(t-)S_i^2(t-)||\sigma^3(t)S_i(t)-\sigma^3(t-)S_i(t-)|\right]\\
&\quad\le C\xi^2 n^{-\frac{3}{2}}\mathbb{E}\left[\sup_{t\le T}\sum_{i=1}^n|S_i^2(t)-S_i^2(t-)||S_i(t)-S_i(t-)|\right]\\
&\quad\le C\xi^2 n^{-\frac{3}{2}}\mathbb{E}\left[\sup_{1\le i\le n}\sup_{t\le T}|S_i^2(t)-S_i^2(t-)||S_i(t)-S_i(t-)|\right]\\
&\quad\le C\xi^2 n^{-\frac{3}{2}}\mathbb{E}\left[\sup_{1\le i\le n}\sup_{t\le T}S_i^3(t)\right],
\end{aligned}$$

and

$$\begin{aligned}
\mathbb{E}\left[\sup_{t\le T}|A_n^{22}(t)-A_n^{22}(t-)|\right]&=\mathbb{E}\left[\sup_{t\le T}\frac{\xi^4}{n^2}\sum_{i=1}^n(\sigma^3(t)S_i^2(t)-\sigma^3(t-)S_i^2(t-))^2\right]\\
&\le\frac{C\xi^4}{n^2}\mathbb{E}\left[\sup_{t\le T}\sum_{i=1}^n(S_i^2(t)-S_i^2(t-))^2\right]\\
&\le\frac{C\xi^4}{n^2}\mathbb{E}\left[\sup_{1\le i\le n}\sup_{t\le T}S_i^4(t)\right].
\end{aligned}$$

Hence (4.16) is proved. Now we proceed to prove (4.17).

(i) Case: $i=j=1$. We define

$$\begin{aligned}
&U_n(t):=A_n^{11}-\int_0^t a_{11}(X_n(u))du\\
&=G_n^{11}+H_n^{11}-\int_0^t a_{11}(X_n(u))du\\
&=-\xi^2\int_0^t\sigma^2(u)I_n^{(2)}(u)du+\frac{1}{n^3}\sum_{p=1}^n\sum_{q=1}^n\sum_{i=1}^n\int_0^t\int_{\mathbb{R}}S_p(u)S_q(u)x^2N_i(du,dx)\\
&+\sum_{i=1}^n\int_0^t\int_{\mathbb{R}}S_i^2(u)(e^{\alpha_2 x}-1)^2\left(\frac{\sigma^2(u)}{\sqrt{n}}+\frac{x}{n^{\frac{3}{2}}}\right)^2N_i(du,dx)\\
&+\frac{2}{n^{\frac{3}{2}}}\sum_{i=1}^n\sum_{j=1}^n\int_0^t\int_{\mathbb{R}}S_i(u)S_j(u)x(e^{\alpha_2 x}-1)\left(\frac{\sigma^2(u)}{\sqrt{n}}+\frac{x}{n^{\frac{3}{2}}}\right)N_j(du,dx).
\end{aligned}$$

After simplification of the above expression, and using (4.12), we obtain the following expression for $U_n(t)$.

$$\begin{aligned}
&U_n(t) = \\
&= \frac{1}{n^3}\sum_{p=1}^{n}\sum_{q=1}^{n}\sum_{i=1}^{n}\int_0^t\int_{\mathbb{R}} S_p(u)S_q(u)x^2 N_i(du,dx) \\
&+ \frac{1}{n^3}\sum_{i=1}^{n}\int_0^t\int_{\mathbb{R}} S_i^2(u)x^2(e^{\alpha_2 x}-1)^2 N_i(du,dx) \\
&+ \frac{1}{n}\sum_{i=1}^{n}\int_0^t\int_{\mathbb{R}} S_i^2(u)\sigma^4(u)(e^{\alpha_2 x}-1)^2 \tilde{N}_i(du,dx) \\
&+ \frac{2}{n^2}\sum_{i=1}^{n}\int_0^t\int_{\mathbb{R}} S_i^2(u)\sigma^2(u)x(e^{\alpha_2 x}-1)^2 N_i(du,dx) \\
&+ \frac{2}{n^4}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^t\int_{\mathbb{R}} S_i(u)S_j(u)x^2(e^{\alpha_2 x}-1) N_j(du,dx) \\
&+ \frac{2}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^t\int_{\mathbb{R}} S_i(u)S_j(u)\sigma^2(u)x(e^{\alpha_2 x}-1) N_j(du,dx). \qquad (4.18)
\end{aligned}$$

A similar proof for Theorem 4.1.4 (in particular, the analysis related to $\sup_{t\le T}|H_n^i(t)|^2$, for $i = 1, 2, 3, 4, 5, 6$) can be used to show that for $0 \le t \le T$, each of the terms in the right hand side of (4.18) is converging to 0 in probability as $n \to \infty$. Consequently we conclude that $\mathbb{E}[\sup_{t\le T}|U_n(t)|^2] \to 0$ as $n \to \infty$.

(ii) Case: $i = 1$, $j = 2$. In this case clearly $\int_0^t a_{12}(X_n(u))du = 0$.

$$\begin{aligned}
&A_n^{12}(t) - \int_0^t a_{12}(X_n(u))du = G_n^{12}(t) + H_n^{12}(t) - \int_0^t a_{12}(X_n(u))du \\
&= \frac{\xi^2}{n^{\frac{7}{2}}}\sum_{p=1}^{n}\sum_{q=1}^{n}\sum_{i=1}^{n}\int_0^t\int_{\mathbb{R}} S_p(u)S_q^2(u)x^2 N_i(du,dx) \\
&+ \frac{\xi^2}{n^{\frac{3}{2}}}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^t\int_{\mathbb{R}} S_i(u)S_j^2(u)x(e^{2\alpha_2 x}-1)\left(\frac{\sigma^2(u)}{n}+\frac{x}{n^2}\right) N_j(du,dx) \\
&+ \frac{\xi^2}{n^2}\sum_{i=1}^{n}\sum_{j=1}^{n}\int_0^t\int_{\mathbb{R}} S_i(u)S_j^2(u)x(e^{\alpha_2 x}-1)\left(\frac{\sigma^2(u)}{\sqrt{n}}+\frac{x}{n^{\frac{3}{2}}}\right) N_j(du,dx) \\
&+ \xi^2\sum_{i=1}^{n}\int_0^t\int_{\mathbb{R}} S_i^3(u)(e^{2\alpha_2 x}-1)(e^{\alpha_2 x}-1)\left(\frac{\sigma^2(u)}{\sqrt{n}}+\frac{x}{n^{\frac{3}{2}}}\right)\left(\frac{\sigma^2(u)}{n}+\frac{x}{n^2}\right) N_i(du,dx),
\end{aligned}$$

and this can be simplified to

$$A_n^{12}(t) - \int_0^t a_{12}(X_n(u))du = \frac{\xi^2}{n^{\frac{7}{2}}} \sum_{p=1}^n \sum_{q=1}^n \sum_{i=1}^n \int_0^t \int_{\mathbb{R}} S_p(u) S_q^2(u) x^2 N_i(du, dx)$$
$$+ \frac{2\xi^2}{n^{\frac{5}{2}}} \sum_{i=1}^n \sum_{j=1}^n \int_0^t \int_{\mathbb{R}} S_i(u) S_j^2(u) \sigma^2(u) x (e^{2\alpha_2 x} - 1) N_j(du, dx)$$
$$+ \frac{2\xi^2}{n^{\frac{7}{2}}} \sum_{i=1}^n \sum_{j=1}^n \int_0^t \int_{\mathbb{R}} S_i(u) S_j^2(u) x^2 (e^{2\alpha_2 x} - 1) N_j(du, dx)$$
$$+ \frac{\xi^2}{n^{\frac{3}{2}}} \sum_{i=1}^n \int_0^t \int_{\mathbb{R}} S_i^3(u) \sigma^4(u) (e^{2\alpha_2 x} - 1)(e^{\alpha_2 x} - 1) N_i(du, dx)$$
$$+ \frac{2\xi^2}{n^{\frac{5}{2}}} \sum_{i=1}^n \int_0^t \int_{\mathbb{R}} S_i^3(u) \sigma^2(u) x (e^{2\alpha_2 x} - 1)(e^{\alpha_2 x} - 1) N_i(du, dx)$$
$$+ \frac{\xi^2}{n^{\frac{7}{2}}} \sum_{i=1}^n \int_0^t \int_{\mathbb{R}} S_i^3(u) x^2 (e^{2\alpha_2 x} - 1)(e^{\alpha_2 x} - 1) N_i(du, dx).$$

Once again, a similar procedure as in Theorem 4.1.4 (in particular, the analysis related to $\sup_{t \leq T} |H_n^i(t)|^2$, for $i = 1, 2, 3, 4, 5, 6$) can be used to show that for $0 \leq t \leq T$, each of the terms in the right hand side of the above expression is converging to 0 in probability as $n \to \infty$.

(iii) Case: $i = j = 2$. We have $\int_0^t a_{22}(X_n(u))du = \xi^4 \int_0^t 4\sigma^2(u)(I_n^{(2)}(u))^2 du$.

$$A_n^{22}(t) - \int_0^t a_{22}(X_n(u))du = G_n^{22}(t) + H_n^{22}(t) - \int_0^t a_{22}(X_n(u))du$$
$$= \xi^4 \int_0^t 4\sigma^2(u)(I_n^{(2)}(u))^2 du - \xi^4 \int_0^t 4\sigma^2(u)(I_n^{(2)}(u))^2 du$$
$$+ \frac{\xi^4}{n^4} \sum_{p=1}^n \sum_{q=1}^n \sum_{i=1}^n \int_0^t \int_{\mathbb{R}} S_q^2(u) S_q^2(u) x^2 N_i(du, dx)$$
$$+ \xi^4 \sum_{i=1}^n \int_0^t \int_{\mathbb{R}} S_i^4(u-)(e^{2\alpha_2 x} - 1)^2 \left(\frac{\sigma^2(u)}{n} + \frac{x}{n^2} \right)^2 N_i(du, dx)$$
$$+ \frac{2\xi^4}{n^2} \sum_{i=1}^n \sum_{j=1}^n \int_0^t \int_{\mathbb{R}} S_i^2(u) S_j^2(u) x (e^{2\alpha_2 x} - 1) \left(\frac{\sigma^2(u)}{n} + \frac{x}{n^2} \right) N_j(du, dx).$$

After expanding the above expression, a similar procedure as in Theorem 4.1.4 (in particular, the analysis related to $\sup_{t \leq T} |H_n^i(t)|^2$, for $i = 1, 2, 3, 4, 5, 6$) can be implemented to show that for $0 \le t \le T$, each of the terms in the right hand side of the above expression is converging to 0 in probability as $n \to \infty$.

Combining all the above three cases we complete the proof of (4.17). □

Lemma 4.1.6. *For any $t > s \geq 0$, $A_n(t) - A_n(s)$ is non-negative definite.*

Proof. For any $t > s \geq 0$, it is enough to show that $G_n(t) - G_n(s)$ and $H_n(t) - H_n(s)$ are non-negative definite. Since

$$G_n^{11}(t) = G_n^{21}(t) = G_n^{12}(t) = 0,$$

therefore

$$G_n(t) - G_n(s) = \begin{pmatrix} 0 & 0 \\ 0 & G_n^{22}(t) - G_n^{22}(s) \end{pmatrix}.$$

Since by construction G^{22} is increasing hence $G_n(t) - G_n(s)$ is trivially non-negative definite. It remains to show that

$$H_n(t) - H_n(s) = \begin{pmatrix} H_n^{11}(t) - H_n^{11}(s) & H_n^{12}(t) - H_n^{12}(s) \\ H_n^{21}(t) - H_n^{21}(s) & H_n^{22}(t) - H_n^{22}(s) \end{pmatrix},$$

is non-negative definite. Thus it is sufficient to show that all the principal minors are nonnegative. However this is obvious from the expressions of H_n^{ij} $(i, j \in \{1, 2\})$ given that $\alpha_2 \geq 0$ and N_i $(i = 1, 2, \ldots, n)$ are subordinators. □

We conclude this section with the weak convergence theorem describing the behavior of the "error term" in the large-basket limit.

Theorem 4.1.7. *Suppose $X_n(t)$ and $X(t)$ are given by (4.10) and (4.11) respectively. Also, suppose that $\rho_1 = 1$, $\alpha_2 \geq 0$ and N_i $(i = 1, 2, \ldots, n)$ are subordinators in (4.1). Suppose that $E[S_i(0)^k] < \infty$ and $\int_{\mathbb{R}} e^{k\alpha_2 x}\nu(dx) < \infty$, for $1 \leq k \leq 8$. Also, suppose that for $t \in [0, T]$, $|\sigma(t)|^2 \leq C$, for some $C > 0$. Further assume that $\mu_l < \infty$, for $l = 1, 2$. Then $X_n \Rightarrow X$ as $n \to \infty$.*

Proof. All the conditions of the Theorem 4.1.3 are checked in (4.13), (4.14), Lemma 4.1.5, and Lemma 4.1.6. Hence the proof follows from Theorem 4.1.3. □

We conclude this section with the following analysis based on the S&P 500 data. We use the data for the S&P 500 index from January 23, 2017 to March 3, 2017. We compute the characteristic function from this empirical data. We use the variance independent price index model proposed in [35] for the fitting of the characteristic function of the empirical data. The root-mean-square error (RMSE) is obtained to be 1.30728. Finally, we use the variance dependent price index model proposed in this dissertation for the fitting of the characteristic function of the empirical data. The RMSE in this case is 0.000915226. The plots are shown in Figure 1 and Figure 2 respectively. In the plots (Figure 1 and Figure 2), the red and green dots indicate the characteristic functions of the data and the model fit respectively. This shows an empirical evidence of the usefulness of a variance dependent price index model. Figure 3 is the combined plot of Figure 1 and Figure 2. In Figure 3, red, green, and blue dots indicate the characteristic functions of the data, the variance independent model fit, and the variance dependent model fit, respectively.

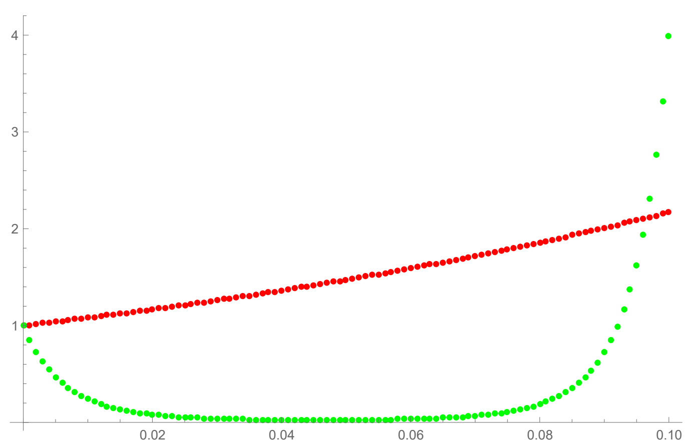

Figure 4.1. Fitting of the characteristic function of the empirical data (red) by the variance-independent model (green).

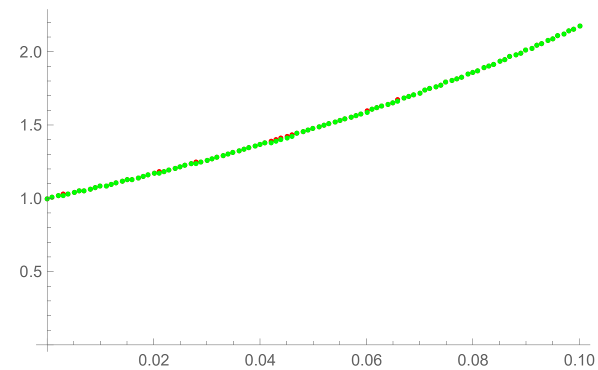

Figure 4.2. Fitting of the characteristic function of the empirical data (red) by the variance-dependent model (green).

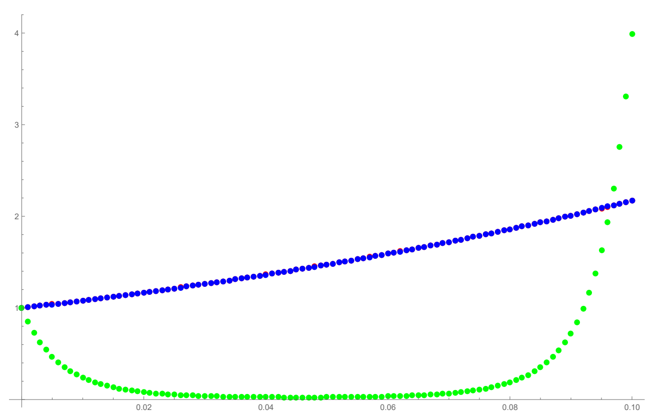

Figure 4.3. Fitting of the characteristic function of the empirical data (red) by the variance-independent model (green) and the variance-dependent model (blue).

5. ANALYSIS OF TRANSITION PROBABILITY DENSITIES FOR SOME LÉVY DRIVEN FINANCIAL MARKETS

Chapter 5 has five main sections. Section 5.1 is devoted to the formulation of the Feynman path integral for Lévy driven markets.In section 5.2, we present a brief introduction of some special functions. In section 5.3, we implement the method of the Feynman path integral for the analysis of option pricing for certain Lévy process driven financial markets. For such a Lévy process driven financial market, we find closed form solution of the transition probability density function (or, the pricing kernel) of option pricing in terms of various special functions. In section 5.4, we provide asymptotic analysis of transition probability density function which represent the option pricing formulas for "sufficiently large" horizon date. In section 5.5, we provide formulas for transition probability density function for certain Lévy process driven markets where the interest rate is stochastic.

5.1. Formulation of Feynman path integral for Lévy-driven markets

In this section we consider exponential Lévy models where at time t the risk-neutral dynamics of the stock price S_t is given by

$$S_t = S_0 e^{rt+X_t}, \tag{5.1}$$

where r is the risk free interest rate and X_t is a Lévy process under the risk neutral measure with characteristic triplet (σ, γ, ν). Arbitrage-free condition in financial market implies that $\int_{|y|>1} e^y \nu(dy) < \infty$, and $\gamma = -\frac{\sigma^2}{2} - \int_{-\infty}^{\infty}(e^y - 1 - y1_{|y|<1})\nu(dy)$. It is shown in [24] (Proposition 2) that under appropriate conditions the option price value $\tilde{C}(t,S)$ is given by

$$\begin{aligned}\frac{\partial \tilde{C}(t,S)}{\partial t} = -\Bigg[& rS\frac{\partial \tilde{C}(t,S)}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 \tilde{C}(t,S)}{\partial S^2} - r\tilde{C}(t,S) \\ & + \int_{\mathbb{R}} \left(\tilde{C}(t, Se^y) - \tilde{C}(t,S) - S(e^y - 1)\frac{\partial \tilde{C}(t,S)}{\partial S} \right) \nu(dy) \Bigg].\end{aligned}$$

With the transformation $S = e^x$, and denoting $\tilde{C}(t, e^x) = C(t, x)$, we obtain

$$\begin{aligned}\frac{\partial C(t,x)}{\partial t} = &-\Bigg[\frac{\sigma^2}{2}\frac{\partial^2 C(t,x)}{\partial x^2} + (r - \frac{\sigma^2}{2})\frac{\partial C(t,x)}{\partial x} - rC(t,x) \\ &+ \int_{\mathbb{R}}\left(C(t,y+x) - C(t,x) - (e^y - 1)\frac{\partial C(t,x)}{\partial x}\right)\nu(dy)\Bigg]. \end{aligned} \tag{5.2}$$

Following Dirac's notation (see [27]) we denote the "ket" vector by $|\cdot\rangle$ and corresponding "bra" vector by $\langle\cdot|$. Also, if k_n is an eigenvalue of some operator $\hat{K}$, the corresponding eigenvector (when unique) is denoted as $|k_n\rangle$. We work in the units for which Planck's constant $\hbar = 1$ (see [26, 27]). The one-dimensional momentum operator is given by $\hat{P} = -i\frac{\partial}{\partial x}$ and thus $\hat{P}^2 = -\frac{\partial^2}{\partial x^2}$. It is well known that if $|x\rangle$ and $|p\rangle$ are state vectors corresponding to position operator $\hat{X}$ and momentum operator $\hat{P}$ respectively, then $\int_{-\infty}^{\infty} |x\rangle\langle x|\, dx = \int_{-\infty}^{\infty} \frac{1}{2\pi}|p\rangle\langle p|\, dp = I$, where I is the identity operator (see e.g., [45]). The scalar products are given by $\langle x|p\rangle = e^{ipx}$ and $\langle p|x\rangle = e^{-ipx}$.

We denote $|C\rangle = |C(t,x)\rangle$ to be the state vector with associated "cost function" $C(t,x)$. We use "cost function" in the present financial setting to represent the same thing as "wave function" in quantum mechanics. Note that the "shift" in position by amount y is given by the operator $U(y) = e^{-i\hat{P}y}$. In general, if $S(x)$ is the cost function corresponding to the state vector $|S(x)\rangle$, then for a fixed $y \in \mathbb{R}$, $S(x-y)$ is the cost function corresponding to the state vector $|S(x+y)\rangle$. Thus the state vector corresponding to the cost function $S(x+y)$ is given by $|S(x-y)\rangle = U(-y)|S(x)\rangle = e^{i\hat{P}y}|S(x)\rangle$.

Returning to (5.2), we observe the state vector corresponding to the cost function $C(t, y+x)$, for a fixed y, is given by $|C(t, x-y)\rangle$ and

$$e^{i\hat{P}y}|C(t,x)\rangle = |C(t,x-y)\rangle.$$

With these notations the dynamics of $|C\rangle$ is given by

$$\frac{\partial |C\rangle}{\partial t} = \left[\frac{\sigma^2}{2}\hat{P}^2 - i\beta\hat{P} + r - \int_{\mathbb{R}}(e^{i\hat{P}y} - 1)\nu(dy)\right]|C\rangle, \tag{5.3}$$

where $\beta = r - \frac{\sigma^2}{2} - \int_{\mathbb{R}}(e^y - 1)\nu(dy)$. We denote

$$\hat{H} = \left[\frac{\sigma^2}{2}\hat{P}^2 - i\beta\hat{P} + r - \int_{\mathbb{R}}(e^{i\hat{P}y} - 1)\nu(dy)\right], \tag{5.4}$$

and therefore (5.3) can be written as

$$\frac{\partial|C\rangle}{\partial t} = \hat{H}|C\rangle. \tag{5.5}$$

Given the end time $t = T$, (5.5) can be solved as

$$|C(t,x)\rangle = e^{-(T-t)\hat{H}}|C(T,x)\rangle.$$

Therefore we may find the cost function $C(t,x)$ associated with the state vector $|C(t,x)\rangle$ as

$$\begin{aligned} C(t,x) &= \langle x|C(t,x)\rangle = \langle x|e^{-(T-t)\hat{H}}|C(T,x)\rangle \\ &= \int_{-\infty}^{\infty} \langle x|e^{-\tau\hat{H}}|x'\rangle C(T,x')\,dx', \quad \text{where} \quad \tau = T - t. \end{aligned} \tag{5.6}$$

Clearly the transition probability density function is given by $\langle x|e^{-\tau\hat{H}}|x'\rangle$. Given τ, we divide the time interval $t_a = t_0 = 0$ to $t_b = t_{N+1} = \tau$ in $N+1$ equally spaced subintervals $\{t_1, t_2, \ldots, t_N\}$, such that the spacing is given by $\epsilon = t_n - t_{n-1} = \frac{(t_b - t_a)}{N+1}$, $n = 2, 3, \ldots, N$. We also set $x_0 = x_a = x'$ and $x_{N+1} = x_b = x$.

Note that in the present case we may consider $\hat{H} = H(\hat{P}, t_n)$. Therefore

$$\begin{aligned} \langle x_n|e^{-\epsilon\hat{H}}|x_{n-1}\rangle &= \int_{-\infty}^{\infty} \langle x_n|e^{-\epsilon H(\hat{P},t_n)}|p_n\rangle\langle p_n|x_{n-1}\rangle \frac{dp_n}{2\pi} \\ &= \int_{-\infty}^{\infty} \exp[ip_n(x_n - x_{n-1}) - \epsilon H(p_n, t_n)]\frac{dp_n}{2\pi}. \end{aligned}$$

Consequently, we obtain the following:

$$\begin{aligned} \langle x_b|e^{-\tau\hat{H}}|x_a\rangle &= \prod_{n=1}^{N}\int_{-\infty}^{\infty} dx_n \prod_{n=1}^{N+1}\langle x_n|e^{-\epsilon\hat{H}}|x_{n-1}\rangle \\ &= \prod_{n=1}^{N}\left[\int_{-\infty}^{\infty} dx_n\right]\prod_{n=1}^{N+1}\left[\int_{-\infty}^{\infty}\frac{dp_n}{2\pi}\right]\exp(iA^N), \end{aligned}$$

where

$$A^N = \sum_{n=1}^{N+1} \left[p_n(x_n - x_{n-1}) + i\epsilon H(p_n, t_n) \right]. \tag{5.7}$$

As $N \to \infty$, $A^N \to \mathcal{A}[p]$, where

$$\mathcal{A}[p] = \int_{t_a=0}^{t_b=\tau} \left[p(t)\dot{x}(t) + iH(p(t), t) \right] dt.$$

We use the notation of Feynman path integral as follows:

$$\lim_{N\to\infty} \prod_{n=1}^{N} \left[\int_{-\infty}^{\infty} dx_n \right] \prod_{n=1}^{N+1} \left[\int_{-\infty}^{\infty} \frac{dp_n}{2\pi} \right] = \int_{x(t_a)}^{x(t_b)} \mathfrak{D}'x \int \frac{\mathfrak{D}p}{2\pi}. \tag{5.8}$$

Therefore we can write the transition probability in terms of Feynman path integral as

$$\langle x_b | e^{-\tau \hat{H}} | x_a \rangle = \int_{x(t_a)=x_a}^{x(t_b)=x_b} \mathfrak{D}'x \int \frac{\mathfrak{D}p}{2\pi} e^{i\mathcal{A}[p]}. \tag{5.9}$$

In the present case

$$A^N = \sum_{n=1}^{N+1} \left[p_n(x_n - x_{n-1}) + i\epsilon \left(\frac{\sigma^2}{2} p_n^2 - i\beta p_n + r - \int_{\mathbb{R}} (e^{iyp_n} - 1)\nu(dy) \right) \right]. \tag{5.10}$$

At this point, we consider some special cases of the Lévy density ν.

1. For Inverse-Gaussian (IG) process,

$$\nu(dx) = \frac{1}{\sqrt{2\pi}} a x^{-3/2} e^{-\frac{1}{2}b^2 x}\, dx,$$

   where $a, , b > 0$, and $x > 0$. In this case

$$\int_{\mathbb{R}} (e^{iyp_n} - 1)\nu(dy) = a \left(b - \sqrt{b^2 - 2ip_n} \right).$$

2. For Gamma (Γ) process,

$$\nu(dx) = a \frac{e^{-bx}}{x}\, dx,$$

where $a, , b > 0$, and $x > 0$. In this case

$$\int_{\mathbb{R}} (e^{iyp_n} - 1)\nu(dy) = a \log \left(\frac{b}{b - ip_n} \right).$$

3. For Variance-Gamma process,

$$\nu(dx) = \begin{cases} -C\frac{e^{Gx}}{x}\, dx, & x < 0 \\ C\frac{e^{-Mx}}{x}\, dx, & x > 0, \end{cases}$$

where $C, G, M > 0$. In this case

$$\int_{\mathbb{R}} (e^{iyp_n} - 1)\nu(dy) = C \log \left[\left(\frac{G}{G + ip_n} \right) \left(\frac{M}{M - ip_n} \right) \right].$$

4. For CGMY process,

$$\nu(dx) = \begin{cases} Ce^{Gx}(-x)^{-1-Y}\, dx, & x < 0 \\ Ce^{-Mx}x^{-1-Y}\, dx, & x > 0, \end{cases}$$

where $C, G, M > 0$ and $Y < 2$. In this case

$$\int_{\mathbb{R}} (e^{iyp_n} - 1)\nu(dy) = C\Gamma(-Y) \left[-G^Y - M^Y + (G + ip_n)^Y + (M - ip_n)^Y \right].$$

5. For Tempered Stable process,

$$\nu(dx) = a2^{\kappa} \frac{\kappa}{\Gamma(1-\kappa)} x^{-\kappa-1} \exp\left(-\frac{1}{2} b^{1/\kappa} x \right) dx,$$

where $a, b > 0$, and $0 < \kappa < 1$. In this case

$$\int_{\mathbb{R}} (e^{iyp_n} - 1)\nu(dy) = a\kappa \frac{\Gamma(-\kappa)}{\Gamma(1-\kappa)} \left(-b + (b^{1/\kappa} - 2ip_n)^{\kappa} \right).$$

Note that Tempered Stable process becomes IG process when $\kappa = \frac{1}{2}$.

5.2. Special functions

For $\Re(\alpha) > 0$, the Gamma function $\Gamma(\alpha)$ can be defined as

$$\Gamma(\alpha) = \int_0^\infty e^{-1} t^{\alpha-1} dt,$$

when $\Re(\alpha) \leq 0$, $\Gamma(\alpha)$ can be defined by analytic continuation. It is a meromorphic function the complex plane with simple poles at $\alpha = 0, -1, -2, \ldots,$ (see, [50, 5.2.i]).

The incomplete gamma functions are defined by the integral

$$\gamma(\alpha, x) = \int_0^x e^{-t} t^{\alpha-1} dt,$$

and

$$\Gamma(a, x) = \int_x^\infty e^{-t} t^{\alpha-1} dt.$$

The definition of $\gamma(\alpha, x)$ requires that $\Re(\alpha) > 0$. It is well-known that if $x = \lambda\alpha$, $\lambda \in (0, 1)$, the incomplete gamma function $\gamma(\alpha, x)$ has the following asymptotic expansion:

$$\gamma(\alpha, x) \sim -x^\alpha e^{-x} \sum_{j=0}^\infty \frac{(-\alpha)^j b_j(\lambda)}{(x-\alpha)^{2j+1}}$$

as $\alpha \to \infty$ in the sector $|\arg(\alpha)| \leq \frac{\pi}{2} - \delta < \frac{\pi}{2}$ (see, e.g., [50, 8.11.iii]). It is also well-known in [50, 8.11.iii] that, if $x = \lambda\lambda$, $\lambda > 1$, the incomplete gamma function $\Gamma(\alpha, x)$ has an asymptotic expansion

$$\Gamma(\alpha, x) \sim x^\alpha e^{-x} \sum_{j=0}^\infty \frac{(-\alpha)^j b_j(\lambda)}{(x-\alpha)^{2j+1}}$$

as $\alpha \to \infty$ in the sector $|\arg(\alpha)| \leq \frac{3\pi}{2} - \delta < \frac{3\pi}{2}$ (see, e.g., [50, 8.11.iii]). The first few coefficients $b_j(\lambda)$ are $b_0(\lambda) = 1$, $b_1(\lambda) = \lambda$, $b_2(\lambda) = \lambda(2\lambda^2 + 1)$ [50, Eq. 8.11.8]. Computations of higher coefficients $b_j(\lambda)$ for $j \geq 1$ can be found using the recurrence relation [50, Eq. 8.11.9]

$$b_j(\lambda) = \lambda(1-\lambda)b'_{j-1}(\lambda) + (2j-1)\lambda b_{j-1}(\lambda).$$

An interesting recurrence relation in [50, Eq. 8.8.7] for the incomplete gamma function $\gamma(\alpha, x)$ is given by

$$\gamma(\alpha+m, x)=\frac{\Gamma(\alpha+m)}{\Gamma(\alpha)}\gamma(\alpha, x)-x^{\alpha}e^{-x}\sum_{j=0}^{m-1}\frac{\Gamma(\alpha)}{\Gamma(\alpha+j+1)},$$

and in [50, Eq. 8.8.9] for the incomplete gamma function $\Gamma(\alpha, x)$ is given by

$$\Gamma(\alpha+m, x)=\frac{\Gamma(\alpha+m)}{\Gamma(\alpha)}\Gamma(\alpha, x)+x^{\alpha}e^{-x}\sum_{j=0}^{m-1}\frac{\Gamma(\alpha)}{\Gamma(\alpha+j+1)},$$

where $m = 0, 1, 2, 3, \ldots$.

Consider the following differential equation [1, Eq. 19.1.2]:

$$\frac{d^2y}{dx^2}-\left(\frac{1}{4}x^2+a\right)y=0. \tag{5.11}$$

The solutions to (5.11) are called parabolic cylinder functions. One of the standard solutions [1, Eq. 19.3.1] to (5.11) is denoted by $U(a, x)$. Another notation for the parabolic cylinder function is in terms of the well-known Whittaker and Watson's function $D_\nu(x)$ [1, 68, 69] :

$$D_\nu(x)=U\left(-\nu-\frac{1}{2}, x\right).$$

The parabolic cylinder function $U(a, x)$ may be expressed as

$$U(a, x)=D_{-a-\frac{1}{2}}(x)=\cos\pi\left(\frac{1}{4}+\frac{1}{2}a\right)Y_1-\sin\pi\left(\frac{1}{4}+\frac{1}{2}a\right)Y_2,$$

where

$$Y_1=\frac{1}{\sqrt{\pi}}\frac{\Gamma\left(\frac{1}{4}-\frac{1}{2}a\right)}{2^{\frac{a}{2}+\frac{1}{4}}}y_1,$$

$$Y_2=\frac{1}{\sqrt{\pi}}\frac{\Gamma\left(\frac{3}{4}-\frac{1}{2}a\right)}{2^{\frac{a}{2}-\frac{1}{4}}}y_2,$$

where

$$y_1=1+a\frac{x^2}{2!}+\left(a^2+\frac{1}{2}\right)\frac{x^2}{4!}+\left(a^3+\frac{7}{2}a\right)\frac{x^6}{6!}+\cdots,$$

and

$$y_2 = x + a\frac{x^3}{3!} + \left(a^2 + \frac{3}{2}\right)\frac{x^5}{5!} + \left(a^4 + \frac{13}{2}a\right)\frac{x^7}{7!} + \cdots .$$

It is known that for large values of x and a fixed, the parabolic cylinder function $U(a,x)$ has the following asymptotic expansion [1, Eq. 19.8.1]

$$U(a,x) \sim e^{-\frac{1}{4}x^2} x^{-a-\frac{1}{2}} \left\{1 - \frac{(a+\frac{1}{2})(a+\frac{3}{2})}{2x^2} + \frac{(a+\frac{1}{2})(a+\frac{3}{2})(a+\frac{5}{2})(a+\frac{7}{2})}{2\cdot 4x^4} + \cdots \right\}.$$

5.3. Computation of Feynman path integrals

The objective of this section is to compute (5.9) when the Lévy densities are in the form as described at the end of Section 5.1. We start this section with the following Lemma.

Lemma 5.3.1. *For a sufficiently smooth function L,*

$$\lim_{N\to\infty} \prod_{n=1}^{N} \left[\int_{-\infty}^{\infty} dx_n\right] \prod_{n=1}^{N+1} \left[\int_{-\infty}^{\infty} \frac{dp_n}{2\pi}\right] \exp\left(\sum_{n=1}^{N+1} (ip_n(x_n - x_{n-1}) + \epsilon L(p_n))\right)$$
$$= \int_{-\infty}^{\infty} \exp\left(ip(x_b - x_a) + \tau L(p)\right) \frac{dp}{2\pi}.$$

Proof.

$$\begin{aligned}
&\lim_{N\to\infty} \prod_{n=1}^{N} \left[\int_{-\infty}^{\infty} dx_n\right] \prod_{n=1}^{N+1} \left[\int_{-\infty}^{\infty} \frac{dp_n}{2\pi}\right] \exp\left(\sum_{n=1}^{N+1} (ip_n(x_n - x_{n-1}) + \epsilon L(p_n))\right) \\
&= \lim_{N\to\infty} \prod_{n=1}^{N} \left[\int_{-\infty}^{\infty} dx_n\right] \prod_{n=1}^{N+1} \left[\int_{-\infty}^{\infty} \frac{dp_n}{2\pi}\right] \exp\left(i\left(p_{N+1}x_b - p_1 x_a - \sum_{n=1}^{N} x_n(p_{n+1} - p_n)\right) + \epsilon L(p_n)\right) \\
&= \lim_{N\to\infty} \prod_{n=1}^{N} \left[\int_{-\infty}^{\infty} dx_n\right] \prod_{n=1}^{N+1} \left[\int_{-\infty}^{\infty} \frac{dp_n}{2\pi}\right] \exp(-i\sum_{n=1}^{N} x_n(p_{n+1} - p_n)) \exp\left(i(p_{N+1}x_b - p_1 x_a) + \epsilon L(p_n)\right).
\end{aligned} \tag{5.12}$$

We note that

$$\prod_{n=1}^{N} \int_{-\infty}^{\infty} \exp\left(-i\sum_{n=1}^{N} x_n(p_{n+1} - p_n)\right) dx_n = \prod_{n=1}^{N} \delta\left(-\frac{p_{n+1} - p_n}{2\pi}\right) = \prod_{n=1}^{N} 2\pi\delta\left(p_{n+1} - p_n\right). \tag{5.13}$$

Using (5.13) in (5.12) we obtain

$$\lim_{N\to\infty}\prod_{n=1}^{N}\left[\int_{-\infty}^{\infty}dx_n\right]\prod_{n=1}^{N+1}\left[\int_{-\infty}^{\infty}\frac{dp_n}{2\pi}\right]\exp\left(\sum_{n=1}^{N+1}(ip_n(x_n-x_{n-1})+\epsilon L(p_n))\right)$$
$$=\lim_{N\to\infty}\int_{-\infty}^{\infty}\exp\left(ip(x_b-x_a)+\epsilon(N+1)L(p)\right)\frac{dp}{2\pi}$$
$$=\int_{-\infty}^{\infty}\exp\left(ip(x_b-x_a)+\tau L(p)\right)\frac{dp}{2\pi}.$$

□

We consider three special cases as described below:

Case I:

$$\int_{\mathbb{R}}(e^{iyp_n}-1)\nu(dy)=a\log\left(\frac{b}{b-ip_n}\right),$$

where $a,b>0$. Gamma process is an example of this case. In such case, from (5.10) we obtain

$$A^N=\sum_{n=1}^{N+1}\left[p_n(x_n-x_{n-1})+i\epsilon\left(\frac{\sigma^2}{2}p_n^2-i\beta p_n+r-a\log\left(\frac{b}{b-ip_n}\right)\right)\right].$$

In this case using (5.9) and Lemma 5.3.1 we obtain

$$\langle x_b|e^{-\tau\hat{H}}|x_a\rangle=\int_{-\infty}^{\infty}\exp\left[ip(x_b-x_a)-\tau\left(\frac{\sigma^2}{2}p^2-i\beta p+r-a\log\left(\frac{b}{b-ip}\right)\right)\right]\frac{dp}{2\pi}$$
$$=e^{-\tau r}\int_{-\infty}^{\infty}\left(\frac{b}{b-ip}\right)^{a\tau}\exp\left[-\frac{\tau\sigma^2}{2}p^2-iqp\right]\frac{dp}{2\pi},\tag{5.14}$$

where

$$q=-(x_b-x_a)-\tau\beta.\tag{5.15}$$

With $p=-\tilde{p}-ib$, we can obtain

$$\langle x_b|e^{-\tau\hat{H}}|x_a\rangle=e^{-\tau r-qb+\frac{\tau\sigma^2b^2}{2}}\frac{b^{a\tau}}{2\pi}\int_{-\infty}^{\infty}(i\tilde{p})^{-a\tau}\exp\left[-\frac{\tau\sigma^2}{2}\tilde{p}^2-i(\tau\sigma^2b-q)\tilde{p}\right]d\tilde{p}$$
$$=e^{-\tau r-qb+\frac{\tau\sigma^2b^2}{2}}\frac{b^{a\tau}}{\sqrt{2\pi}}\left(\sqrt{\tau}\sigma\right)^{a\tau-1}\exp\left(-\frac{(\tau\sigma^2b-q)^2}{4\tau\sigma^2}\right)D_{-a\tau}\left(\frac{\tau\sigma^2b-q}{\sqrt{\tau}\sigma}\right),\tag{5.16}$$

where $a\tau < 1$ and $D_\beta(\cdot)$ is the parabolic cylinder function of order β (see [69]). The last equality is obtained from [32].

Case II:

$$\int_{\mathbb{R}} (e^{iyp_n} - 1)\nu(dy) = c_1 + c_2(c_3 + ip_n)^{\nu_1} + c_4(c_5 - ip_n)^{\nu_1},$$

where c_1, c_2, c_3, c_4, c_5 are some real constants and $\nu_1 > 0$. Inverse-Gaussian, Tempered Stable, CGMY processes are the examples of this case. In such case, from (5.10) we obtain

$$A^N = \sum_{n=1}^{N+1} \left[p_n(x_n - x_{n-1}) + i\epsilon \left(\frac{\sigma^2}{2} p_n^2 - i\beta p_n + (r - c_1) - c_2(c_3 + ip_n)^{\nu_1} - c_4(c_5 - ip_n)^{\nu_1} \right) \right].$$

In this case using (5.9) and Lemma 5.3.1 we obtain

$$\begin{aligned}
&\langle x_b | e^{-\tau \hat{H}} | x_a \rangle \\
&= \int_{-\infty}^{\infty} \exp \left[ip(x_b - x_a) - \tau \left(\frac{\sigma^2}{2} p^2 - i\beta p + (r - c_1) - c_2(c_3 + ip)^{\nu_1} - c_4(c_5 - ip)^{\nu_1} \right) \right] \frac{dp}{2\pi} \\
&= e^{-\tau(r - c_1)} \int_{-\infty}^{\infty} \exp \left[-\frac{\tau\sigma^2 p^2}{2} - iqp + \tau c_2(c_3 + ip)^{\nu_1} + \tau c_4(c_5 - ip)^{\nu_1} \right] \frac{dp}{2\pi} \\
&= \frac{e^{-\tau(r - c_1)}}{2\pi} \int_{-\infty}^{\infty} \exp \left[-\frac{\tau\sigma^2 p^2}{2} - iqp \right] \sum_{l=0}^{\infty} \frac{\tau^l}{l!} [c_2(c_3 + ip)^{\nu_1} + c_4(c_5 - ip)^{\nu_1}]^l \, dp \\
&= \frac{e^{-\tau(r - c_1)}}{2\pi} \sum_{l=0}^{\infty} \sum_{m=0}^{l} \binom{l}{m} \frac{\tau^l}{l!} c_2^m c_4^{l-m} \int_{-\infty}^{\infty} \exp \left[-\frac{\tau\sigma^2 p^2}{2} - iqp \right] (c_3 + ip)^{m\nu_1} (c_5 - ip)^{\nu_1(l-m)} \, dp,
\end{aligned} \tag{5.17}$$

where q is given by (5.15). In Theorem 5.3.3 we provide an expression for the computation of the integral in (5.17).

Interesting subcases of Case II can be obtained when $c_2 = 0$. In this case as shown below we can obtain a much simpler expression than the result provided in Theorem 5.3.3. For example, IG process falls into this category with $c_1 = ab$, $c_4 = -\sqrt{2}a$, $c_5 = \frac{b^2}{2}$, and $\nu_1 = \frac{1}{2}$. Tempered Stable process also falls into this category with $c_1 = -ba\kappa\frac{\Gamma(-\kappa)}{\Gamma(1-\kappa)}$, $c_4 = 2^\kappa a\kappa\frac{\Gamma(-\kappa)}{\Gamma(1-\kappa)}$, $c_5 = \frac{b^{1/\kappa}}{2}$, and $\nu_1 = \kappa$.

In this case using (5.9) and Lemma 5.3.1 we obtain

$$
\begin{aligned}
\langle x_b|e^{-\tau\hat{H}}|x_a\rangle &= e^{-\tau(r-c_1)}\int_{-\infty}^{\infty}\exp\left[-\frac{\tau\sigma^2p^2}{2}-iqp+\tau c_4(c_5-ip)^{\nu_1}\right]\frac{dp}{2\pi}\\
&= \frac{e^{-\tau(r-c_1)}}{2\pi}\int_{-\infty}^{\infty}\exp\left[-\frac{\tau\sigma^2p^2}{2}-iqp\right]\sum_{l=0}^{\infty}\frac{\tau^l}{l!}c_4^l(c_5-ip)^{l\nu_1}\,dp\\
&= \frac{e^{-\tau(r-c_1)}}{2\pi}\sum_{l=0}^{\infty}\frac{\tau^l}{l!}c_4^l\int_{-\infty}^{\infty}\exp\left[-\frac{\tau\sigma^2p^2}{2}-iqp\right](c_5-ip)^{l\nu_1}\,dp, \qquad (5.18)
\end{aligned}
$$

where q is given by (5.15). With $p=-\tilde{p}-ic_5$, we can obtain

$$
\begin{aligned}
\langle x_b|e^{-\tau\hat{H}}|x_a\rangle &= \frac{e^{-\tau(r-c_1)-qc_5+\frac{\tau\sigma^2c_5^2}{2}}}{2\pi}\sum_{l=0}^{\infty}\frac{\tau^l}{l!}c_4^l\int_{-\infty}^{\infty}(i\tilde{p})^{l\nu_1}\exp\left[-\frac{\tau\sigma^2}{2}\tilde{p}^2-i(\tau\sigma^2c_5-q)\tilde{p}\right]d\tilde{p}\\
&= \frac{e^{-\tau(r-c_1)-qc_5+\frac{\tau\sigma^2c_5^2}{2}}}{2\pi}\sum_{l=0}^{\infty}\frac{\tau^l}{l!}c_4^l\frac{(\sqrt{\tau}\sigma)^{-l\nu_1-1}}{\sqrt{2\pi}}\exp\left(-\frac{(\tau\sigma^2c_5-q)^2}{4\tau\sigma^2}\right)D_{l\nu_1}\left(\frac{\tau\sigma^2c_5-q}{\sqrt{\tau}\sigma}\right), \qquad (5.19)
\end{aligned}
$$

where $\nu_1>0$ and $D_\beta(\cdot)$ is the parabolic cylinder function of order β (see [69]). The last equality is obtained from [32].

Case III:

$$
\int_{\mathbb{R}}(e^{iyp_n}-1)\nu(dy)=C\log\left[\left(\frac{G}{G+ip_n}\right)\left(\frac{M}{M-ip_n}\right)\right],
$$

where $C,G,M>0$. Variance-Gamma process is an example of this case. In such case, from (5.10) we obtain

$$
A^N=\sum_{n=1}^{N+1}\left[p_n(x_n-x_{n-1})+i\epsilon\left(\frac{\sigma^2}{2}p_n^2-i\beta p_n+r-C\log\left[\left(\frac{G}{G+ip_n}\right)\left(\frac{M}{M-ip_n}\right)\right]\right)\right].
$$

In this case using (5.9) and Lemma 5.3.1 we obtain

$$
\begin{aligned}
&\langle x_b|e^{-\tau\hat{H}}|x_a\rangle\\
&= \int_{-\infty}^{\infty}\exp\left[ip(x_b-x_a)-\tau\left(\frac{\sigma^2}{2}p^2-i\beta p+r\right)\right]\left[\left(\frac{G}{G+ip}\right)\left(\frac{M}{M-ip}\right)\right]^{\tau C}\frac{dp}{2\pi}\\
&= \frac{e^{-r\tau}(GM)^{\tau C}}{2\pi}\int_{-\infty}^{\infty}\exp\left[-\frac{\tau\sigma^2p^2}{2}-iqp\right](G+ip)^{-\tau C}(M-ip)^{-\tau C}\,dp, \qquad (5.20)
\end{aligned}
$$

where q is given by (5.15). In Theorem 5.3.4 we provide an expression for the computation of the integral in (5.20).

We conclude this section with two theorems for the computation of the expressions in (5.17) and (5.20). For the next theorems we use the incomplete gamma functions as defined by:

$$\gamma(\alpha, x) = \int_0^x e^{-t} t^{\alpha-1}\, dt, \quad \Re(\alpha) > 0,$$

and

$$\Gamma(\alpha, x) = \int_x^\infty e^{-t} t^{\alpha-1}\, dt.$$

We start with the following lemma.

Lemma 5.3.2. *For* $\Re(\nu_1), \Re(\nu_2) \geq 0$,

$$\int_{-\infty}^{\infty} (\alpha_1 + ix)^{\nu_1} (\alpha_2 + ix)^{\nu_2} e^{-ax^2 - bix} dx = e^{ac^2 - \frac{b^2}{4a}} \sum_{n=0}^{\infty} (-1)^n \frac{(2ac)^n}{n!} \sum_{k=0}^{\infty} \binom{\nu_2}{k} \left[\alpha_3^{\nu_2 - k} I_1 + \alpha_3^k I_2 \right],$$

where

$$\begin{aligned} I_1 = &\left(\frac{i}{\sqrt{a}}\right)^{\nu_1+n+k} \frac{1}{2\sqrt{a}} \gamma\left(\frac{\nu_1+n+k}{2} + \frac{1}{2}, a|\alpha_3|^2\right) \\ &+ \left(\frac{-i}{\sqrt{a}}\right)^{\nu_1+n+k} \frac{1}{2\sqrt{a}} \gamma\left(\frac{\nu_1+n+k}{2} + \frac{1}{2}, a|\alpha_3|^2\right), \end{aligned} \tag{5.21}$$

and

$$\begin{aligned} I_2 = &\left(\frac{i}{\sqrt{a}}\right)^{\nu_1+\nu_2+n+k} \frac{1}{2\sqrt{a}} \Gamma\left(\frac{\nu_1+\nu_2+n+k}{2} + \frac{1}{2}, a|\alpha_3|^2\right) \\ &+ \left(\frac{-i}{\sqrt{a}}\right)^{\nu_1+\nu_2+n+k} \frac{1}{2\sqrt{a}} \Gamma\left(\frac{\nu_1+\nu_2+n+k}{2} + \frac{1}{2}, a|\alpha_3|^2\right), \end{aligned} \tag{5.22}$$

and $\alpha_3 = \alpha_2 - \alpha_1$, $c = \alpha_1 + \frac{b}{2a}$.

Proof. It is easy to show that

$$\begin{aligned}
&\int_{-\infty}^{\infty} (\alpha_1 + ix)^{\nu_1} (\alpha_2 + ix)^{\nu_2} e^{-ax^2 - bix} dx \\
&= e^{ac^2 - \frac{b^2}{4a}} \sum_{n=0}^{\infty} (-1)^n \frac{(2ac)^n}{n!} \int_{-\infty}^{\infty} (ix)^{\nu_1 + n} (\alpha_3 + ix)^{\nu_2} e^{-ax^2} dx, \qquad (5.23)
\end{aligned}$$

where $\alpha_3 = \alpha_2 - \alpha_1$, $c = \alpha_1 + \frac{b}{2a}$. Using the binomial expansion from (5.23) we obtain:

$$\begin{aligned}
&\int_{-\infty}^{\infty} (\alpha_1 + ix)^{\nu_1} (\alpha_2 + ix)^{\nu_2} e^{-ax^2 - bix} dx \\
&= e^{ac^2 - \frac{b^2}{4a}} \sum_{n=0}^{\infty} (-1)^n \frac{(2ac)^n}{n!} \left[\int_{|x|<|\alpha_3|} (ix)^{\nu_1 + n} (\alpha_3 + ix)^{\nu_2} e^{-ax^2} dx \right. \\
&\left. + \int_{|x|>|\alpha_3|} (ix)^{\nu_1 + n} (\alpha_3 + ix)^{\nu_2} e^{-ax^2} dx \right] \\
&= e^{ac^2 - \frac{b^2}{4a}} \sum_{n=0}^{\infty} (-1)^n \frac{(2ac)^n}{n!} \sum_{k=0}^{\infty} \binom{\nu_2}{k} \alpha_3^{\nu_2 - k} \int_{|x|<|\alpha_3|} (ix)^{\nu_1 + n + k} e^{-ax^2} dx \\
&+ e^{ac^2 - \frac{b^2}{4a}} \sum_{n=0}^{\infty} (-1)^n \frac{(2ac)^n}{n!} \sum_{k=0}^{\infty} \binom{\nu_2}{k} \alpha_3^{k} \int_{|x|>|\alpha_3|} (ix)^{\nu_1 + \nu_2 + n - k} e^{-ax^2} dx \\
&= e^{ac^2 - \frac{b^2}{4a}} \sum_{n=0}^{\infty} (-1)^n \frac{(2ac)^n}{n!} \sum_{k=0}^{\infty} \binom{\nu_2}{k} \left[\alpha_3^{\nu_2 - k} \int_{|x|<|\alpha_3|} (ix)^{\nu_1 + n + k} e^{-ax^2} dx \right. \\
&\left. + \alpha_3^{k} \int_{|x|>|\alpha_3|} (ix)^{\nu_1 + \nu_2 + n - k} e^{-ax^2} dx \right]. \qquad (5.24)
\end{aligned}$$

We define

$$\begin{aligned}
I_1 &= \int_{|x|<|\alpha_3|} (ix)^{\nu_1 + n + k} e^{-ax^2} dx = \int_{-|\alpha_3|}^{|\alpha_3|} (ix)^{\nu_1 + n + k} e^{-ax^2} dx \\
&= \int_0^{|\alpha_3|} (ix)^{\nu_1 + n + k} e^{-ax^2} dx + (-1)^{(\nu_1 + n + k)} \int_0^{|\alpha_3|} (ix)^{\nu_1 + n + k} e^{-ax^2} dx.
\end{aligned}$$

Clearly

$$\begin{aligned}
\int_0^{|\alpha_3|} (ix)^{\nu_1 + n + k} e^{-ax^2} dx &= \left(\frac{i}{\sqrt{a}} \right)^{\nu_1 + n + k} \frac{1}{2\sqrt{a}} \int_0^{a|\alpha_3|^2} x^{\frac{\nu_1 + n + k}{2} + \frac{1}{2} - 1} e^{-x} dx \\
&= \left(\frac{i}{\sqrt{a}} \right)^{\nu_1 + n + k} \frac{1}{2\sqrt{a}} \gamma \left(\frac{\nu_1 + n + k}{2} + \frac{1}{2}, a|\alpha_3|^2 \right).
\end{aligned}$$

Therefore

$$\begin{aligned} I_1 = &\left(\frac{i}{\sqrt{a}}\right)^{\nu_1+n+k} \frac{1}{2\sqrt{a}}\gamma\left(\frac{\nu_1+n+k}{2}+\frac{1}{2}, a|\alpha_3|^2\right) \\ &+\left(\frac{-i}{\sqrt{a}}\right)^{\nu_1+n+k} \frac{1}{2\sqrt{a}}\gamma\left(\frac{\nu_1+n+k}{2}+\frac{1}{2}, a|\alpha_3|^2\right). \end{aligned}$$

Similarly we define

$$\begin{aligned} I_2 &= \int_{|x|>|\alpha_3|} (ix)^{\nu_1+\nu_2+n-k}\, e^{-ax^2} dx \\ &= \int_{|\alpha_3|}^{\infty} (ix)^{\nu_1+\nu_2+n-k}\, e^{-ax^2} dx + (-1)^{(\nu_1+\nu_2+n-k)} \int_{|\alpha_3|}^{\infty} (ix)^{\nu_1+\nu_2+n-k}\, e^{-ax^2} dx. \end{aligned}$$

Clearly

$$\begin{aligned} \int_{|\alpha_3|}^{\infty} (ix)^{\nu_1+\nu_2+n-k}\, e^{-ax^2} dx &= \left(\frac{i}{\sqrt{a}}\right)^{\nu_1+\nu_2+n+k} \frac{1}{2\sqrt{a}} \int_{a|\alpha_3|^2}^{\infty} x^{\frac{\nu_1+\nu_2+n+k}{2}+\frac{1}{2}-1} e^{-x} dx \\ &= \left(\frac{i}{\sqrt{a}}\right)^{\nu_1+\nu_2+n+k} \frac{1}{2\sqrt{a}}\Gamma\left(\frac{\nu_1+\nu_2+n+k}{2}+\frac{1}{2}, a|\alpha_3|^2\right), \end{aligned}$$

and therefore

$$\begin{aligned} I_2 = &\left(\frac{i}{\sqrt{a}}\right)^{\nu_1+\nu_2+n+k} \frac{1}{2\sqrt{a}}\Gamma\left(\frac{\nu_1+\nu_2+n+k}{2}+\frac{1}{2}, a|\alpha_3|^2\right) \\ &+\left(\frac{-i}{\sqrt{a}}\right)^{\nu_1+\nu_2+n+k} \frac{1}{2\sqrt{a}}\Gamma\left(\frac{\nu_1+\nu_2+n+k}{2}+\frac{1}{2}, a|\alpha_3|^2\right). \end{aligned}$$

From the expressions (5.24), (5.21), and (5.22), the required result follows immediately. □

Theorem 5.3.3. *For* $\Re(\nu_1), \Re(\nu_2) \geq 0$,

$$\int_{-\infty}^{\infty} (\alpha_1+ix)^{\nu_1}\, (\alpha_2-ix)^{\nu_2} e^{-ax^2-bix} dx = e^{ac^2-\frac{b^2}{4a}} \sum_{n=0}^{\infty} (-1)^{n+\nu_2} \frac{(2ac)^n}{n!} \sum_{k=0}^{\infty} \binom{\nu_2}{k} \left[\alpha_3^{\nu_2-k} I_1 + \alpha_3^k I_2\right],$$

where I_1 *and* I_2 *are given by* (5.21) *and* (5.22) *respectively, and* $\alpha_3 = -\alpha_2 - \alpha_1$, $c = \alpha_1 + \frac{b}{2a}$.

Proof. From Lemma 5.3.2 it is clear that for $\Re(\nu_1), \Re(\nu_2) \geq 0$ and $A \in \mathbb{C}$,

$$\int_{-\infty}^{\infty} (\alpha_1 + ix)^{\nu_1} (A + ix)^{\nu_2} e^{-ax^2 - bix} dx = e^{ac^2 - \frac{b^2}{4a}} \sum_{n=0}^{\infty} (-1)^n \frac{(2ac)^n}{n!} \sum_{k=0}^{\infty} \binom{\nu_2}{k} \left[\alpha_3^{\nu_2 - k} I_1 + \alpha_3^k I_2\right], \tag{5.25}$$

where I_1 and I_2 are given by (5.21) and (5.22) respectively, and $\alpha_3 = A - \alpha_1$, $c = \alpha_1 + \frac{b}{2a}$. Hence the theorem follows immediately by taking $A = -\alpha_2$ in (5.25). □

Theorem 5.3.4. *For $\Re(\nu) \geq 0$,*

$$\int_{-\infty}^{\infty} (\alpha_1 + ix)^{-\nu} (\alpha_2 - ix)^{-\nu} e^{-ax^2 - bix} dx = e^{ac^2 - \frac{b^2}{4a}} \sum_{n=0}^{\infty} (-1)^{n+\nu} \frac{(2ac)^n}{n!} \sum_{k=0}^{\infty} \binom{-\nu}{k} \left[\alpha_3^{-\nu - k} I_3 + \alpha_3^k I_4\right],$$

where

$$\begin{aligned} I_3 = &\left(\frac{i}{\sqrt{a}}\right)^{-\nu+n+k} \frac{1}{2\sqrt{a}} \gamma\left(\frac{-\nu+n+k}{2} + \frac{1}{2}, a|\alpha_3|^2\right) \\ &+ \left(\frac{-i}{\sqrt{a}}\right)^{-\nu+n+k} \frac{1}{2\sqrt{a}} \gamma\left(\frac{-\nu+n+k}{2} + \frac{1}{2}, a|\alpha_3|^2\right), \end{aligned}$$

and

$$\begin{aligned} I_4 = &\left(\frac{i}{\sqrt{a}}\right)^{-2\nu+n+k} \frac{1}{2\sqrt{a}} \Gamma\left(\frac{-2\nu+n+k}{2} + \frac{1}{2}, a|\alpha_3|^2\right) \\ &+ \left(\frac{-i}{\sqrt{a}}\right)^{-2\nu+n+k} \frac{1}{2\sqrt{a}} \Gamma\left(\frac{-2\nu+n+k}{2} + \frac{1}{2}, a|\alpha_3|^2\right), \end{aligned}$$

and $\alpha_3 = \alpha_2 + \alpha_1$, $c = \alpha_1 + \frac{b}{2a}$.

Proof.

$$\begin{aligned} &\int_{-\infty}^{\infty} (\alpha_1 + ix)^{-\nu} (\alpha_2 - ix)^{-\nu} e^{-ax^2 - bix} dx \\ &\qquad = e^{-\frac{b^2}{4a}} e^{ac^2} \sum_{n=0}^{\infty} (-1)^{n+\nu} \frac{(2ac)^n}{n!} \int_{-\infty}^{\infty} (iy)^{-\nu+n} (\alpha_3 + iy)^{-\nu} e^{-ay^2} dy, \end{aligned}$$

where $\alpha_3 = \alpha_2 + \alpha_1$, $c = \alpha_1 + \frac{b}{2a}$. Using the following well known identity

$$(x+y)^r = \sum_{k=0}^{\infty} \binom{r}{k} x^{r-k} y^k$$

for $|x| > |y|$, we have that

$$\int_{-\infty}^{\infty} (\alpha_1 + ix)^{-\nu} (\alpha_2 - ix)^{-\nu} e^{-ax^2 - bix} dx = e^{ac^2 - \frac{b^2}{4a}} \sum_{n=0}^{\infty} (-1)^{n+\nu} \frac{(2ac)^n}{n!} \sum_{k=0}^{\infty} \binom{-\nu}{k} \alpha_3^{-\nu-k} I_3 + e^{ac^2 - \frac{b^2}{4a}} \sum_{n=0}^{\infty} (-1)^{n+\nu} \frac{(2ac)^n}{n!} \sum_{k=0}^{\infty} \binom{-\nu}{k} \alpha_3^{k} I_4,$$

where

$$I_3 = \int_{|x|<|\alpha_3|} (ix)^{-\nu+n+k} e^{-ax^2} dx = \left(\frac{i}{\sqrt{a}}\right)^{-\nu+n+k} \frac{1}{2\sqrt{a}} \gamma\left(\frac{-\nu+n+k}{2} + \frac{1}{2}, a\alpha_3\right) - \left(\frac{-i}{\sqrt{a}}\right)^{-\nu+n+k} \frac{1}{2\sqrt{a}} \gamma\left(\frac{-\nu+n+k}{2} + \frac{1}{2}, a\alpha_3\right), \quad (5.26)$$

and

$$I_4 = \int_{|x|>|\alpha_3|} (ix)^{-2\nu+n-k} e^{-ax^2} dx = \left(\frac{i}{\sqrt{a}}\right)^{-2\nu+n+k} \frac{1}{2\sqrt{a}} \Gamma\left(\frac{-2\nu+n+k}{2} + \frac{1}{2}, a\alpha_3\right) - \left(\frac{-i}{\sqrt{a}}\right)^{-2\nu+n+k} \frac{1}{2\sqrt{a}} \Gamma\left(\frac{-2\nu+n+k}{2} + \frac{1}{2}, a\alpha_3\right). \quad (5.27)$$

□

5.4. Asymptotic expansions of transition probability density

In this section we find asymptotic expansions for the formulas derived in Theorem 5.3.3 and Theorem 5.3.4 in Section 5.3. These formulas will correspond to the cases when τ is very large. From the option pricing point of view, therefore, the formulas in this section correspond to the expressions of transition probability densities when the exercise date (or, horizon date) of option is sufficiently large.

We remark that though the goal of this section is what we stated in the last paragraph, for the sake of generalization, we extend the settings of the last section. In this section we also provide a framework of finding asymptotic expansions for integrals which are similar to the ones derived in Theorem 5.3.3 and Theorem 5.3.4 of Section 5.3. We start this section by Watson's Lemma for Complex Integral [68].

Theorem 5.4.1. *Let*

$$I(t) = \int_0^\infty e^{-zt} f(t) dt.$$

Let $f(t)$ be such that

$$f(t) \sim \sum_{n=0}^\infty a_n t^{\alpha n + \beta}$$

as $t \to 0^+$, $\alpha > 0$ and $\mathrm{Re}(\beta) > -1$. Then

$$\int_0^\infty e^{-zt} f(t) dt \sim \sum_{n=0}^\infty \frac{a_n \Gamma(\alpha n + \beta + 1)}{z^{\alpha n + \beta + 1}},$$

as $z \to +\infty$ in the sector $|\arg z| \le \frac{\pi}{2} - \delta < \frac{\pi}{2}$ (with $0 < \delta \le \frac{\pi}{2}$ being fixed), provided that the abscissa of convergence is not $+\infty$, where $z^{\alpha n+\beta+1}$ has its principal value.

Theorem 5.4.1 is essential to prove the following theorems.

Theorem 5.4.2. *If $|\arg(\gamma_i)| < \pi$ and $\sigma_i > 0$ for $i = 1, 2$, then as $\mu \to +\infty$ with $|\arg(\mu)| < \frac{\pi}{2}$,*

$$\int_0^\infty (\gamma_1 + ix)^{\sigma_1} (\gamma_2 + ix)^{\sigma_2} e^{-\mu x^2} dx \sim \frac{\gamma_1^{\sigma_1} \gamma_2^{\sigma_2}}{2} \sum_{n=0}^\infty a_n \frac{\Gamma\left(\frac{n+1}{2}\right)}{\mu^{\frac{n+1}{2}}},$$

where the coefficients are given by

$$a_n = \sum_{k=0}^n \binom{\sigma_1}{k} \binom{\sigma_2}{n-k} \left(\frac{i}{\gamma_1}\right)^k \left(\frac{i}{\gamma_2}\right)^{n-k}.$$

Proof. We write the given integral as

$$\begin{aligned}\int_0^\infty (\gamma_1 + ix)^{\sigma_1} (\gamma_2 + ix)^{\sigma_2} e^{-\mu x^2} dx &= \gamma_1^{\sigma_1} \gamma_2^{\sigma_2} \int_0^\infty \left(1 + i\frac{x}{\gamma_1}\right)^{\sigma_1} \left(1 + i\frac{x}{\gamma_2}\right)^{\sigma_2} e^{-\mu x^2} dx \\ &= \gamma_1^{\sigma_1} \gamma_2^{\sigma_2} \int_0^\infty \frac{1}{2\sqrt{t}} \left(1 + i\frac{\sqrt{t}}{\gamma_1}\right)^{\sigma_1} \left(1 + i\frac{\sqrt{t}}{\gamma_2}\right)^{\sigma_2} e^{-\mu t} dt.\end{aligned}$$

Now for small t, we have

$$\frac{1}{\sqrt{t}}\left(1+i\frac{\sqrt{t}}{\gamma_1}\right)^{\sigma_1}\left(1+i\frac{\sqrt{t}}{\gamma_2}\right)^{\sigma_2} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{\sigma_1}{k}\binom{\sigma_2}{n-k}\left(\frac{i}{\gamma_1}\right)^{k}\left(\frac{i}{\gamma_2}\right)^{n-k}\right)t^{\frac{n-1}{2}}$$
$$= \sum_{n=0}^{\infty} a_n t^{\frac{n-1}{2}},$$

where

$$a_n = \sum_{k=0}^{n}\binom{\sigma_1}{k}\binom{\sigma_2}{n-k}\left(\frac{i}{\gamma_1}\right)^{k}\left(\frac{i}{\gamma_2}\right)^{n-k}.$$

Therefore, the required result is clear from with the application of Watson's lemma. □

Proof of the following theorem is similar to the proof of Theorem 5.4.2 and is as follows,

Theorem 5.4.3. *If* $|\arg(\gamma_i)| < \pi$ *and* $\sigma_i > 0$ *for* $i = 1, 2$*, then as* $\mu \to +\infty$ *with* $|\arg(\mu)| < \frac{\pi}{2}$*,*

$$\int_0^{\infty}(\gamma_1+ix)^{-\sigma_1}(\gamma_2-ix)^{-\sigma_2}e^{-\mu x^2}dx \sim \frac{\gamma_1^{-\sigma_1}\gamma_2^{-\sigma_2}}{2}\sum_{n=0}^{\infty}a_n\frac{\Gamma\left(\frac{n+1}{2}\right)}{\mu^{\frac{n+1}{2}}},$$

where the coefficients is given

$$a_n = \sum_{k=0}^{n}\binom{-\sigma_1}{k}\binom{-\sigma_2}{n-k}\left(\frac{i}{\gamma_1}\right)^{k}\left(-\frac{i}{\gamma_2}\right)^{n-k}.$$

Proof. To prove Theorem 5.4.3, write the above integral as

$$\int_0^{\infty}(\gamma_1+ix)^{-\sigma_1}(\gamma_2-ix)^{-\sigma_2}e^{-\mu x^2}dx = \gamma_1^{-\sigma_1}\gamma_2^{-\sigma_2}\int_0^{\infty}\left(1+i\frac{x}{\gamma_1}\right)^{-\sigma_1}\left(1-i\frac{x}{\gamma_2}\right)^{-\sigma_2}e^{-\mu x^2}dx$$
$$= \gamma_1^{-\sigma_1}\gamma_2^{-\sigma_2}\int_0^{\infty}\frac{1}{2\sqrt{t}}\left(1+i\frac{\sqrt{t}}{\gamma_1}\right)^{-\sigma_1}\left(1-i\frac{\sqrt{t}}{\gamma_2}\right)^{-\sigma_2}e^{-\mu t}dt.$$

Observe that the quantity $\frac{1}{\sqrt{t}}\left(1+i\frac{\sqrt{t}}{\gamma_1}\right)^{-\sigma_1}\left(1-i\frac{\sqrt{t}}{\gamma_2}\right)^{-\sigma_2}$ with $\sigma_i > 0$ for $i = 1, 2$ is continuous for $t > 0$. Now for small t, we have

$$\left(1+i\frac{\sqrt{t}}{\gamma_1}\right)^{-\sigma_1}\left(1-i\frac{\sqrt{t}}{\gamma_2}\right)^{-\sigma_2} = \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{-\sigma_1}{k}\binom{-\sigma_2}{n-k}\left(\frac{i}{\gamma_1}\right)^{k}\left(-\frac{i}{\gamma_2}\right)^{n-k}\right)t^{\frac{t}{2}}$$
$$= \sum_{n=0}^{\infty} a_n t^{\frac{t}{2}},$$

where

$$a_n = \sum_{k=0}^{n} \binom{-\sigma_1}{k} \binom{-\sigma_2}{n-k} \left(\frac{i}{\gamma_1}\right)^k \left(-\frac{i}{\gamma_2}\right)^{n-k}.$$

Whence, by Watson's lemma, we deduce

$$\begin{aligned}\gamma_1^{-\sigma_1}\gamma_2^{-\sigma_2} \int_0^\infty \frac{1}{2\sqrt{t}} \left(1 - i\frac{\sqrt{t}}{\gamma_1}\right)^{-\sigma_1} \left(1 - i\frac{\sqrt{t}}{\gamma_2}\right)^{\sigma_2} e^{-\mu t} dt &\sim \frac{\gamma_1^{-\sigma_1}\gamma_2^{-\sigma_2}}{2} \sum_{n=0}^{\infty} a_n \int_0^\infty t^{\frac{n-1}{2}} e^{-\mu t} dt \\ &\sim \frac{\gamma_1^{-\sigma_1}\gamma_2^{-\sigma_2}}{2} \sum_{n=0}^{\infty} a_n \frac{\Gamma\left(\frac{n+1}{2}\right)}{\mu^{\frac{n+1}{2}}}.\end{aligned}$$

□

Proof of the following theorem is similar to the proof of Theorem 5.4.2 and is as follows,

Theorem 5.4.4. *If* $|\arg(\gamma_i)| < \pi$ *and* $\sigma_i > 0$ *for* $i = 1, 2$, *then as* $\mu \to +\infty$ *with* $|\arg(\mu)| < \frac{\pi}{2}$,

$$\int_0^\infty (\gamma_1 + ix)^{\sigma_1} (\gamma_2 - ix)^{\sigma_2} e^{-\mu x^2} dx \sim \frac{\gamma_1^{\sigma_1}\gamma_2^{\sigma_2}}{2} \sum_{n=0}^{\infty} a_n \frac{\Gamma\left(\frac{n+1}{2}\right)}{\mu^{\frac{n+1}{2}}},$$

where the coefficients are given by

$$a_n = \sum_{k=0}^{n} \binom{\sigma_1}{k} \binom{\sigma_2}{n-k} \left(\frac{i}{\gamma_1}\right)^k \left(-\frac{i}{\gamma_2}\right)^{n-k}.$$

Proof. To prove Theorem 5.4.4, write the above integral as

$$\gamma_1^{\sigma_1}\gamma_2^{\sigma_2} \int_0^\infty \left(1 + i\frac{x}{\gamma_1}\right)^{\sigma_1} \left(1 - i\frac{x}{\gamma_2}\right)^{\sigma_2} e^{-\mu x^2} dx = \gamma_1^{\sigma_1}\gamma_2^{\sigma_2} \int_0^\infty \frac{1}{2\sqrt{t}} \left(1 + i\frac{\sqrt{t}}{\gamma_1}\right)^{\sigma_1} \left(1 - i\frac{\sqrt{t}}{\gamma_2}\right)^{\sigma_2} e^{-\mu t} dt.$$

Now for small t, we have

$$\begin{aligned}\left(1 + i\frac{\sqrt{t}}{\gamma_1}\right)^{\sigma_1} \left(1 - i\frac{\sqrt{t}}{\gamma_2}\right)^{\sigma_2} &= \sum_{n=0}^{\infty} \left(\sum_{k=0}^{n} \binom{\sigma_1}{k} \binom{\sigma_2}{n-k} \left(\frac{i}{\gamma_1}\right)^k \left(-\frac{i}{\gamma_2}\right)^{n-k} \right) t^{\frac{t}{2}} \\ &= \sum_{n=0}^{\infty} a_n t^{\frac{t}{2}},\end{aligned}$$

where

$$a_n = \sum_{k=0}^{n} (-1)^n \binom{\sigma_1}{k} \binom{\sigma_2}{n-k} \left(\frac{i}{\gamma_1}\right)^k \left(-\frac{i}{\gamma_2}\right)^{n-k}.$$

Whence, by Watson's lemma, we deduce

$$\begin{aligned}\gamma_1^{\sigma_1}\gamma_2^{\sigma_2}\int_0^\infty \frac{1}{2\sqrt{t}}\left(1-i\frac{\sqrt{t}}{\gamma_1}\right)^{\sigma_1}\left(1-i\frac{\sqrt{t}}{\gamma_2}\right)^{\sigma_2} e^{-\mu t}dt &\sim \frac{\gamma_1^{\sigma_1}\gamma_2^{\sigma_2}}{2}\sum_{n=0}^{\infty} a_n \int_0^\infty t^{\frac{n-1}{2}} e^{-\mu t}dt \\ &\sim \frac{\gamma_1^{\sigma_1}\gamma_2^{\sigma_2}}{2}\sum_{n=0}^{\infty} a_n \frac{\Gamma\left(\frac{n+1}{2}\right)}{\mu^{\frac{n+1}{2}}}.\end{aligned}$$

□

Proof of the following theorem is similar to the proof of Theorem 5.4.2 and is as follows,

Theorem 5.4.5. *If* $|\arg(\gamma_i)| < \pi$ *and* $\sigma_i > 0$ *for* $i = 1, 2$*, then as* $\mu \to +\infty$ *with* $|\arg(\mu)| < \frac{\pi}{2}$*,*

$$\int_0^\infty (\gamma_1 - ix)^{\sigma_1}(\gamma_2 - ix)^{\sigma_2} e^{-\mu x^2}dx \sim \frac{\gamma_1^{\sigma_1}\gamma_2^{\sigma_2}}{2}\sum_{n=0}^{\infty} a_n \frac{\Gamma\left(\frac{n+1}{2}\right)}{\mu^{\frac{n+1}{2}}},$$

where the coefficients are given by

$$a_n = \sum_{k=0}^{n}(-1)^n \binom{\sigma_1}{k}\binom{\sigma_2}{n-k}\left(\frac{i}{\gamma_1}\right)^k\left(\frac{i}{\gamma_2}\right)^{n-k}.$$

Proof. To prove Theorem 5.4.5, write the above integral as

$$\gamma_1^{\sigma_1}\gamma_2^{\sigma_2}\int_0^\infty \left(1-i\frac{x}{\gamma_1}\right)^{\sigma_1}\left(1-i\frac{x}{\gamma_2}\right)^{\sigma_2} e^{-\mu x^2}dx = \gamma_1^{\sigma_1}\gamma_2^{\sigma_2}\int_0^\infty \frac{1}{2\sqrt{t}}\left(1-i\frac{\sqrt{t}}{\gamma_1}\right)^{\sigma_1}\left(1-i\frac{\sqrt{t}}{\gamma_2}\right)^{\sigma_2} e^{-\mu t}dt.$$

Now for small t, we have

$$\begin{aligned}\left(1-i\frac{\sqrt{t}}{\gamma_1}\right)^{\sigma_1}\left(1-i\frac{\sqrt{t}}{\gamma_2}\right)^{\sigma_2} &= \sum_{n=0}^{\infty}\left(\sum_{k=0}^{n}\binom{\sigma_1}{k}\binom{\sigma_2}{n-k}\left(-\frac{i}{\gamma_1}\right)^k\left(-\frac{i}{\gamma_2}\right)^{n-k}\right)t^{\frac{t}{2}} \\ &= \sum_{n=0}^{\infty} a_n t^{\frac{t}{2}},\end{aligned}$$

where

$$a_n = \sum_{k=0}^{n}(-1)^n \binom{\sigma_1}{k}\binom{\sigma_2}{n-k}\left(\frac{i}{\gamma_1}\right)^k\left(\frac{i}{\gamma_2}\right)^{n-k}.$$

Whence, by Watson's lemma, we deduce

$$\gamma_1^{\sigma_1}\gamma_2^{\sigma_2}\int_0^\infty \frac{1}{2\sqrt{t}}\left(1-i\frac{\sqrt{t}}{\gamma_1}\right)^{\sigma_1}\left(1-i\frac{\sqrt{t}}{\gamma_2}\right)^{\sigma_2} e^{-\mu t}dt \sim \frac{\gamma_1^{\sigma_1}\gamma_2^{\sigma_2}}{2}\sum_{n=0}^{\infty} a_n \int_0^\infty t^{\frac{n-1}{2}} e^{-\mu t}dt$$
$$\sim \frac{\gamma_1^{\sigma_1}\gamma_2^{\sigma_2}}{2}\sum_{n=0}^{\infty} a_n \frac{\Gamma\left(\frac{n+1}{2}\right)}{\mu^{\frac{n+1}{2}}}.$$

□

We now state and prove the theorems related to the asymptotic expansions related to integrals of the form of Theorem 5.3.3.

Theorem 5.4.6. *If $|\arg(\alpha_i)| < \pi$ and $\nu_i > 0$ for $i = 1, 2$, then as $a \to +\infty$ with $|\arg(a)| < \frac{\pi}{2}$,*

$$\int_{-\infty}^{\infty} (\alpha_1 + ix)^{\nu_1} (\alpha_2 - ix)^{\nu_2} e^{-ax^2 - bix} dx \sim e^{-\frac{b^2}{4a}} \frac{\left(\alpha_1 + \frac{b}{2a}\right)^{\nu_1} \left(\alpha_2 - \frac{b}{2a}\right)^{\nu_2}}{2} \sum_{n=0}^{\infty} c_n \frac{\Gamma\left(\frac{n+1}{2}\right)}{a^{\frac{n+1}{2}}},$$

where the coefficients are given by

$$c_n = 0, \tag{5.28}$$

when n is odd, and

$$c_n = 2i^n \sum_{k=0}^{n} (-1)^k \binom{\nu_1}{k}\binom{\nu_2}{n-k}\left(\frac{1}{\alpha_1 + \frac{b}{2a}}\right)^k \left(\frac{1}{\alpha_2 - \frac{b}{2a}}\right)^{n-k}, \tag{5.29}$$

when n is even.

Proof. We have

$$\begin{aligned}
\int_{-\infty}^{\infty}(\alpha_1+ix)^{\nu_1}(\alpha_2-ix)^{\nu_2}e^{-ax^2-bix}dx &= e^{-\frac{b^2}{4a}}\int_{-\infty}^{\infty}(\alpha_1+ix)^{\nu_1}(\alpha_2-ix)^{\nu_2}e^{-a\left(x+\frac{b}{2a}i\right)^2}dx\\
&= e^{-\frac{b^2}{4a}}\int_{-\infty}^{\infty}\left(\alpha_1+\frac{b}{2a}+iy\right)^{\nu_1}\left(\alpha_2-\frac{b}{2a}-iy\right)^{\nu_2}e^{-ay^2}dy\\
&= e^{-\frac{b^2}{4a}}\int_{0}^{\infty}\left(\alpha_1+\frac{b}{2a}-iy\right)^{\nu_1}\left(\alpha_2-\frac{b}{2a}+iy\right)^{\nu_2}e^{-ay^2}dy\\
&+ e^{-\frac{b^2}{4a}}\int_{0}^{\infty}\left(\alpha_1+\frac{b}{2a}+iy\right)^{\nu_1}\left(\alpha_2-\frac{b}{2a}-iy\right)^{\nu_2}e^{-ay^2}dy\\
&\sim e^{-\frac{b^2}{4a}}\frac{\left(\alpha_1+\frac{b}{2a}\right)^{\nu_1}\left(\alpha_2-\frac{b}{2a}\right)^{\nu_2}}{2}\sum_{n=0}^{\infty}a_n\frac{\Gamma\left(\frac{n+1}{2}\right)}{a^{\frac{n+1}{2}}}\\
&+ e^{-\frac{b^2}{4a}}\frac{\left(\alpha_1+\frac{b}{2a}\right)^{\nu_1}\left(\alpha_2-\frac{b}{2a}\right)^{\nu_2}}{2}\sum_{n=0}^{\infty}b_n\frac{\Gamma\left(\frac{n+1}{2}\right)}{a^{\frac{n+1}{2}}}\\
&= e^{-\frac{b^2}{4a}}\frac{\left(\alpha_1+\frac{b}{2a}\right)^{\nu_1}\left(\alpha_2-\frac{b}{2a}\right)^{\nu_2}}{2}\sum_{n=0}^{\infty}c_n\frac{\Gamma\left(\frac{n+1}{2}\right)}{a^{\frac{n+1}{2}}},
\end{aligned}$$

where a_n and b_n can be obtained from Theorem 5.4.4, with

$$\begin{aligned}
c_n = a_n+b_n &= \sum_{k=0}^{n}\binom{\nu_2}{k}\binom{\nu_1}{n-k}\left(\frac{i}{\alpha_2-\frac{b}{2a}}\right)^{k}\left(-\frac{i}{\alpha_1+\frac{b}{2a}}\right)^{n-k}\\
&+\sum_{k=0}^{n}\binom{\nu_1}{k}\binom{\nu_2}{n-k}\left(\frac{i}{\alpha_1+\frac{b}{2a}}\right)^{k}\left(-\frac{i}{\alpha_2-\frac{b}{2a}}\right)^{n-k}.
\end{aligned}$$

After relabelling the intex we thus obtain

$$c_n = i^n\sum_{k=0}^{n}(-1)^k\binom{\nu_1}{k}\binom{\nu_2}{n-k}\left(\frac{1}{\alpha_1+\frac{b}{2a}}\right)^{k}\left(\frac{1}{\alpha_2-\frac{b}{2a}}\right)^{n-k}\left((-1)^n+1\right).$$

The required expressions for c_n are clear from the above result. □

Theorem 5.4.7. *If $|\arg(\alpha_i)|<\pi$ and $\nu_i>0$ for $i=1,2$, then as $a\to+\infty$ with $|\arg(a)|<\frac{\pi}{2}$,*

$$\int_{-\infty}^{\infty}(\alpha_1+ix)^{\nu_1}(\alpha_2+ix)^{\nu_2}e^{-ax^2-bix}dx \sim e^{-\frac{b^2}{4a}}\frac{\left(\alpha_1+\frac{b}{2a}\right)^{\nu_1}\left(\alpha_2+\frac{b}{2a}\right)^{\nu_2}}{2}\sum_{n=0}^{\infty}c_n\frac{\Gamma\left(\frac{n+1}{2}\right)}{a^{\frac{n+1}{2}}}$$

where the coefficients are given by

$$c_n = 0, \tag{5.30}$$

when n is odd, and

$$c_n = 2i^n \sum_{k=1}^{n} \binom{\nu_1}{k} \binom{\nu_2}{n-k} \left(\frac{1}{\alpha_1 + \frac{b}{2a}} \right)^k \left(\frac{1}{\alpha_2 + \frac{b}{2a}} \right)^{n-k}, \tag{5.31}$$

when n is even.

Proof. We have

$$\begin{aligned}
\int_{-\infty}^{\infty} (\alpha_1 + ix)^{\nu_1} (\alpha_2 + ix)^{\nu_2} e^{-ax^2 - bix} dx &= e^{-\frac{b^2}{4a}} \int_{-\infty}^{\infty} (\alpha_1 + ix)^{\nu_1} (\alpha_2 + ix)^{\nu_2} e^{-a\left(x + \frac{b}{2a} i\right)^2} dx \\
&= e^{-\frac{b^2}{4a}} \int_0^{\infty} \left(\alpha_1 + \frac{b}{2a} - iy \right)^{\nu_1} \left(\alpha_2 + \frac{b}{2a} - iy) \right)^{\nu_2} e^{-ay^2} dy \\
&+ e^{-\frac{b^2}{4a}} \int_0^{\infty} \left(\alpha_1 + \frac{b}{2a} + iy \right)^{\nu_1} \left(\alpha_2 + \frac{b}{2a} + iy) \right)^{\nu_2} e^{-ay^2} dy \\
&\sim e^{-\frac{b^2}{4a}} \frac{\left(\alpha_1 + \frac{b}{2a}\right)^{\nu_1} \left(\alpha_2 + \frac{b}{2a}\right)^{\nu_2}}{2} \sum_{n=0}^{\infty} a_n \frac{\Gamma\left(\frac{n+1}{2}\right)}{a^{\frac{n+1}{2}}} \\
&+ e^{-\frac{b^2}{4a}} \frac{\left(\alpha_1 + \frac{b}{2a}\right)^{\nu_1} \left(\alpha_2 + \frac{b}{2a}\right)^{\nu_2}}{2} \sum_{n=0}^{\infty} b_n \frac{\Gamma\left(\frac{n+1}{2}\right)}{a^{\frac{n+1}{2}}} \\
&= e^{-\frac{b^2}{4a}} \frac{\left(\alpha_1 + \frac{b}{2a}\right)^{\nu_1} \left(\alpha_2 + \frac{b}{2a}\right)^{\nu_2}}{2} \sum_{n=0}^{\infty} c_n \frac{\Gamma\left(\frac{n+1}{2}\right)}{a^{\frac{n+1}{2}}},
\end{aligned}$$

where

$$\begin{aligned}
c_n = \sum_{k=0}^{n} (-1)^k \binom{\nu_1}{k} \binom{\nu_2}{n-k} \left(\frac{i}{\left(\alpha_1 + \frac{b}{2a}\right)} \right)^k \left(\frac{i}{\left(\alpha_2 + \frac{b}{2a}\right)} \right)^{n-k} \\
+ \sum_{k=0}^{n} \binom{\nu_1}{k} \binom{\nu_2}{n-k} \left(\frac{i}{\left(\alpha_1 + \frac{b}{2a}\right)} \right)^k \left(\frac{i}{\left(\alpha_2 + \frac{b}{2a}\right)} \right)^{n-k}.
\end{aligned}$$

The required expressions for c_n are clear from the above result. □

We conclude this section with a theorem of asymptotic expansions related to integrals of the form of Theorem 5.3.4.

Theorem 5.4.8. *If $|\arg(\alpha_i)| < \pi$ and $\nu > 0$ for $i = 1, 2$, then as $a \to +\infty$ with $|\arg(a)| < \frac{\pi}{2}$,*

$$\int_{-\infty}^{\infty} (\alpha_1 + ix)^{-\nu} (\alpha_2 - ix)^{-\nu} e^{-ax^2 - bix} dx \sim e^{-\frac{b^2}{4a}} \frac{\left(\alpha_1 - \frac{b}{2a}\right)^{-\nu} \left(\alpha_2 + \frac{b}{2a}\right)^{-\nu}}{2} \sum_{n=0}^{\infty} c_n \frac{\Gamma\left(\frac{n+1}{2}\right)}{a^{\frac{n+1}{2}}},$$

where the coefficients are given by

$$c_n = 0,$$

when n is odd, and

$$c_n = 2\sum_{k=0}^{n}(-1)^k \binom{-\nu}{k}\binom{-\nu}{n-k}\left(\frac{i}{\alpha_1+\frac{b}{2a}}\right)^k \left(\frac{i}{\alpha_2-\frac{b}{2a}}\right)^{n-k},$$

when n is even.

Proof. We have

$$\begin{aligned}
\int_{-\infty}^{\infty}(\alpha_1+ix)^{-\nu}(\alpha_2-ix)^{-\nu}e^{-ax^2-bix}dx &= e^{-\frac{b^2}{4a}}\int_{-\infty}^{\infty}(\alpha_1+ix)^{-\nu}(\alpha_2-ix)^{-\nu}e^{-a\left(x+\frac{b}{2a}i\right)^2}dx \\
&= e^{-\frac{b^2}{4a}}\int_{-\infty}^{\infty}\left(\alpha_1+\frac{b}{2a}+iy\right)^{-\nu}\left(\alpha_2-\frac{b}{2a}-iy\right)^{-\nu}e^{-ay^2}dy \\
&= e^{-\frac{b^2}{4a}}\int_{0}^{\infty}\left(\alpha_1+\frac{b}{2a}+iy\right)^{-\nu}\left(\alpha_2-\frac{b}{2a}-iy\right)^{-\nu}e^{-ay^2}dy \\
&+ e^{-\frac{b^2}{4a}}\int_{0}^{\infty}\left(\alpha_1+\frac{b}{2a}-iy\right)^{-\nu}\left(\alpha_2-\frac{b}{2a}+iy\right)^{-\nu}e^{-ay^2}dy \\
&\sim e^{-\frac{b^2}{4a}}\frac{\left(\alpha_1+\frac{b}{2a}\right)^{-\nu}\left(\alpha_2-\frac{b}{2a}\right)^{-\nu}}{2}\sum_{n=0}^{\infty}a_n\frac{\Gamma\left(\frac{n+1}{2}\right)}{a^{\frac{n+1}{2}}} \\
&+ e^{-\frac{b^2}{4a}}\frac{\left(\alpha_1+\frac{b}{2a}\right)^{-\nu}\left(\alpha_2-\frac{b}{2a}\right)^{-\nu}}{2}\sum_{n=0}^{\infty}b_n\frac{\Gamma\left(\frac{n+1}{2}\right)}{a^{\frac{n+1}{2}}} \\
&= e^{-\frac{b^2}{4a}}\frac{\left(\alpha_1+\frac{b}{2a}\right)^{-\nu}\left(\alpha_2-\frac{b}{2a}\right)^{-\nu}}{2}\sum_{n=0}^{\infty}c_n\frac{\Gamma\left(\frac{n+1}{2}\right)}{a^{\frac{n+1}{2}}}.
\end{aligned}$$

After applying the above theorem, we collect the coefficients a_n and b_n given by:

$$\begin{aligned}
c_n &= a_n + b_n \\
&= \sum_{k=0}^{n}\binom{-\nu}{k}\binom{-\nu}{n-k}\left(\frac{i}{\alpha_1+\frac{b}{2a}}\right)^k\left(-\frac{i}{\alpha_2-\frac{b}{2a}}\right)^{n-k} \\
&+ \sum_{k=0}^{n}\binom{-\nu}{k}\binom{-\nu}{n-k}\left(-\frac{i}{\alpha_1+\frac{b}{2a}}\right)^k\left(\frac{i}{\alpha_2-\frac{b}{2a}}\right)^{n-k} \\
&= \sum_{k=0}^{n}\binom{-\nu}{k}\binom{-\nu}{n-k}\left(\frac{i}{\alpha_1+\frac{b}{2a}}\right)^k\left(\frac{i}{\alpha_2-\frac{b}{2a}}\right)^{n-k}((-1)^{n-k}+(-1)^k).
\end{aligned}$$

Clearly $c_n = 0$ for all n odd. When n is even

$$c_n = 2\sum_{k=0}^{n}(-1)^k \binom{-\nu}{k}\binom{-\nu}{n-k}\left(\frac{i}{\alpha_1 + \frac{b}{2a}}\right)^k \left(\frac{i}{\alpha_2 - \frac{b}{2a}}\right)^{n-k}.$$

□

5.5. Option pricing with stochastic interest rate

Let the dynamics of stock price be given by (5.1). In addition to that, we assume that the interest rate dynamics is given by the stochastic differential equation

$$dr = \alpha_1 r dt + \beta_1 r dX_2, \tag{5.32}$$

where X (in (5.1)) and X_2 are independent Brownian motions and $\mu, \sigma, \alpha_1, \beta_1$ are constants. If $|\tilde{C}(t,S,r)\rangle$ is the state vector of arbitrage free option price, then a similar computation as in [59] can be used to show that the option price $\tilde{C}(t,S,r)$ follows the equation

$$\begin{aligned}\frac{\partial \tilde{C}(t,S,r)}{\partial t} &= -\Bigg[rS\frac{\partial \tilde{C}(t,S,r)}{\partial S} + \frac{\sigma^2 S^2}{2}\frac{\partial^2 \tilde{C}(t,S,r)}{\partial S^2} - r\tilde{C}(t,S,r) \\ &\quad + \int_{\mathbb{R}} (\tilde{C}(t,Se^y,r) - \tilde{C}(t,S,r) - S(e^y-1)\frac{\partial \tilde{C}(t,S,r)}{\partial S})\nu(dy)\Bigg] \\ &\quad + \frac{1}{2}\beta_1^2 r^2 \frac{\partial^2 \tilde{C}(t,S,r)}{\partial r^2} + (\alpha_1 - \lambda\beta_1) r \frac{\partial \tilde{C}(t,S,r)}{\partial r}.\end{aligned}$$

With the transformation $S = e^x$, $r = e^z$, and denoting $\tilde{C}(t,e^x,e^z) = C(t,x,z)$, we obtain

$$\begin{aligned}\frac{\partial C(t,x,z)}{\partial t} &= -\Bigg[\frac{\sigma^2}{2}\frac{\partial^2 C(t,x,z)}{\partial x^2} + (e^z - \frac{\sigma^2}{2})\frac{\partial C(t,x,z)}{\partial x} - e^z C(t,x,z) \\ &\quad + \int_{\mathbb{R}} (C(t,y+x,z) - C(t,x,z) - (e^y-1)\frac{\partial C(t,x,z)}{\partial x})\nu(dy)\Bigg] \\ &\quad + \frac{1}{2}\beta_1^2 \frac{\partial^2 C(t,x,z)}{\partial z^2} + \beta_2 \frac{\partial C(t,x,z)}{\partial z},\end{aligned} \tag{5.33}$$

where $\beta_2 = \alpha_1 - \lambda\beta_1 - \frac{1}{2}\beta_1^2$.

We denote $\hat{P}^{(x)} = -i\frac{\partial}{\partial x}$ and $\hat{P}^{(z)} = -i\frac{\partial}{\partial z}$. We also denote $|C\rangle = |C(t,x,z)\rangle$ to be the state vector with associated cost function $C(t,x,z)$.

Thus the evolution of the state vector can be written as

$$\frac{\partial |C\rangle}{\partial t} = \hat{H}|C\rangle, \tag{5.34}$$

where $\hat{H} = (\hat{H}_1 + \hat{H}_2)$, and from the momentum perspective $\hat{H}_1$ and $\hat{H}_2$ depend on $\hat{P}^{(x)}$ and $\hat{P}^{(z)}$ respectively. The quantities are given by

$$\hat{H}_1 = \left[\frac{\sigma^2}{2}(\hat{P}^{(x)})^2 - i\beta(\hat{Z})\hat{P}^{(x)} + e^{\hat{Z}} - \int_{\mathbb{R}} (e^{iy\hat{P}^{(x)}} - 1)\nu(dy)\right], \tag{5.35}$$

with $\beta(\hat{Z}) = e^{\hat{Z}} - \frac{\sigma^2}{2} - \int_{\mathbb{R}}(e^y - 1)\nu(dy)$, and

$$\hat{H}_2 = \left[\frac{\beta_1^2}{2}(\hat{P}^{(z)})^2 - i\beta_2\hat{P}^{(z)}\right]. \tag{5.36}$$

In the present case we may consider $\hat{H} = H(\hat{P}, t_n)$. For a given τ, we divide the time interval $t_a = t_0 = 0$ to $t_b = t_{N+1} = \tau$ in $N + 1$ equally spaced subintervals $\{t_1, t_2, \ldots, t_N\}$, such that the spacing is given by $\epsilon = t_n - t_{n-1} = \frac{(t_b - t_a)}{N+1}$, $n = 2, 3, \ldots, N$. We also set $x_0 = x_a$ and $x_{N+1} = x_b$ and $z_0 = z_a$ and $z_{N+1} = z_b$. A similar computation as in the last section can be used to show that

$$\begin{aligned}
&\langle x_n, z_n|e^{-\epsilon\hat{H}}|x_{n-1}, z_{n-1}\rangle \\
&\approx \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[ip_n^{(x)}(x_n - x_{n-1}) + ip_n^{(z)}(z_n - z_{n-1}) - \epsilon H(p_n^{(x)}, p_n^{(z)}, z_n, t_n)]\frac{dp_n^{(x)}}{2\pi}\frac{dp_n^{(z)}}{2\pi} \\
&= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[ip_n^{(x)}(x_n - x_{n-1}) + ip_n^{(z)}(z_n - z_{n-1}) - \epsilon(H_1(p_n^{(x)}, z_n, t_n) + H_2(p_n^{(z)}, t_n))]\frac{dp_n^{(x)}}{2\pi}\frac{dp_n^{(z)}}{2\pi}.
\end{aligned}$$

Therefore we obtain the following:

$$\begin{aligned}
\langle x_b, z_b|e^{-\tau\hat{H}}|x_a, z_a\rangle &\approx \prod_{n=1}^{N}\int_{-\infty}^{\infty} dx_n \prod_{n=1}^{N}\int_{-\infty}^{\infty} dz_n \prod_{n=1}^{N+1}\langle x_n, z_n|e^{-\epsilon\hat{H}}|x_{n-1}, z_{n-1}\rangle \\
&= \prod_{n=1}^{N}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} dx_n dz_n\right]\prod_{n=1}^{N+1}\left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{dp_n^{(x)}}{2\pi}\frac{dp_n^{(z)}}{2\pi}\right]\exp(iA^N),
\end{aligned} \tag{5.37}$$

where

$$A^N = \sum_{n=1}^{N+1} \left[p_n^{(x)}(x_n - x_{n-1}) + p_n^{(z)}(z_n - z_{n-1}) + i\epsilon \left(H_1(p_n^{(x)}, z_n, t_n) + H_2(p_n^{(z)}, t_n) \right) \right]. \tag{5.38}$$

As $N \to \infty$, $A^N \to \mathcal{A}[p^{(x)}, p^{(z)}, x, z]$, where

$$\mathcal{A}[p^{(x)}, p^{(z)}, x, z] = \int_{t_a=0}^{t_b=\tau} \left[p^{(x)}(t)\dot{x}(t) + p^{(z)}(t)\dot{z}(t) + i \left(H_1(p^{(x)}(t), z(t), t) + H_2(p^{(z)}(t), t) \right) \right] dt.$$

We use the notation of Feynman path integral as follows:

$$\lim_{N\to\infty} \prod_{n=1}^{N} \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx_n dz_n \right] \prod_{n=1}^{N+1} \left[\int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{dp_n^{(x)}}{2\pi} \frac{dp_n^{(z)}}{2\pi} \right] = \int_{x(t_a),z(t_a)}^{x(t_b),z(t_b)} \mathfrak{D}'x \mathfrak{D}'z \int \frac{\mathfrak{D}p^{(x)}}{2\pi} \frac{\mathfrak{D}p^{(z)}}{2\pi}. \tag{5.39}$$

Therefore we can write the transition probability density in terms of Feynman path integral as

$$\langle x_b, z_b | e^{-\tau \hat{H}} | x_a, z_a \rangle = \int_{x(t_a)=x_a, z(t_a)=z_a}^{x(t_b)=x_b, z(t_b)=z_b} \mathfrak{D}'x \mathfrak{D}'z \int \frac{\mathfrak{D}p^{(x)}}{2\pi} \frac{\mathfrak{D}p^{(z)}}{2\pi} e^{i\mathcal{A}[p^{(x)}, p^{(z)}, x, z]}. \tag{5.40}$$

Theorem 5.5.1. *For the stock dynamics* (5.1) *and interest rate dynamics* (5.32)*, the transition probability density given by Feynman path integral* (5.40) *can be computed as*

$$\langle x_b, z_b | e^{-\tau \hat{H}} | x_a, z_a \rangle = \lim_{N\to\infty} \frac{1}{\sqrt{2\pi\epsilon}\beta_1} \prod_{n=1}^{N} \left[\int_{-\infty}^{\infty} \frac{dz_n}{\sqrt{2\pi\epsilon}\beta_1} \right] \exp(iC^N), \tag{5.41}$$

where

$$C^N = i \sum_{n=1}^{N+1} \left[\frac{\epsilon}{2\beta_1^2} \left(\frac{z_n - z_{n-1}}{\epsilon} + \beta_2 \right)^2 - F(z_n) \right], \tag{5.42}$$

with $\epsilon = \frac{(t_b - t_a)}{N+1}$. *In the above expression*

$$F(\hat{Z}) = \log \left[\lim_{N\to\infty} \prod_{n=1}^{N} \left[\int_{-\infty}^{\infty} dx_n \right] \prod_{n=1}^{N+1} \left[\int_{-\infty}^{\infty} \frac{dp_n^{(x)}}{2\pi} \right] \exp(iB_2^N(\hat{Z})) \right], \tag{5.43}$$

where

$$B_2^N(\hat{Z}) = \sum_{n=1}^{N+1} \left[p_n^{(x)}(x_n - x_{n-1}) + i\epsilon H_1(p_n^{(x)}, \hat{Z}, t_n) \right].$$

Proof. We note that from (5.37) and (5.38) it is possible to obtain

$$\langle x_b, z_b | e^{-\tau \hat{H}} | x_a, z_a \rangle \approx \tag{5.44}$$

$$= \prod_{n=1}^{N} \left[\int_{-\infty}^{\infty} dz_n \right] \prod_{n=1}^{N+1} \left[\int_{-\infty}^{\infty} \frac{dp_n^{(z)}}{2\pi} \right] \exp(iB_1^N) \prod_{n=1}^{N} \left[\int_{-\infty}^{\infty} dx_n \right] \prod_{n=1}^{N+1} \left[\int_{-\infty}^{\infty} \frac{dp_n^{(x)}}{2\pi} \right] \exp(iB_2^N(z_n)), \tag{5.45}$$

where

$$B_1^N = \sum_{n=1}^{N+1} \left[p_n^{(z)}(z_n - z_{n-1}) + i\epsilon H_2(p_n^{(z)}, t_n) \right], \tag{5.46}$$

and

$$B_2^N(z_n) = \sum_{n=1}^{N+1} \left[p_n^{(x)}(x_n - x_{n-1}) + i\epsilon H_1(p_n^{(x)}, z_n, t_n) \right]. \tag{5.47}$$

Using (5.43) we can write the transition probability density $\langle x_b, z_b | e^{-\tau \hat{H}} | x_a, z_a \rangle$ as

$$\langle z_b | \langle x_b | e^{-\tau \hat{H}} | x_a \rangle | z_a \rangle = \lim_{N \to \infty} \prod_{n=1}^{N} \left[\int_{-\infty}^{\infty} dz_n \right] \prod_{n=1}^{N+1} \left[\int_{-\infty}^{\infty} \frac{dp_n^{(z)}}{2\pi} \right] \exp(iB_1^N + \sum_{n=1}^{N+1} F(z_n))$$

$$= \lim_{N \to \infty} \prod_{n=1}^{N} \left[\int_{-\infty}^{\infty} dz_n \right] \prod_{n=1}^{N+1} \left[\int_{-\infty}^{\infty} \frac{dp_n^{(z)}}{2\pi} \right] \exp\left(\sum_{n=1}^{N+1} (ip_n^{(z)}(z_n - z_{n-1} + \epsilon\beta_2) - \frac{\epsilon\beta_1^2}{2}(p_n^{(z)})^2 + F(z_n)) \right).$$

We use Gauss' formula

$$\int_{-\infty}^{\infty} e^{-\frac{\alpha}{2}p^2} \frac{dp}{\sqrt{2\pi}} = \frac{1}{\sqrt{\alpha}}, \quad \Re(\alpha) > 0,$$

to obtain (5.41) and (5.42). Hence the theorem is proved. □

Note that as H_1 is independent of x_n, for reasonable Lévy density ν, we can compute

$$\exp(F(\hat{Z})) = \lim_{N \to \infty} \prod_{n=1}^{N} \left[\int_{-\infty}^{\infty} dx_n \right] \prod_{n=1}^{N+1} \left[\int_{-\infty}^{\infty} \frac{dp_n^{(x)}}{2\pi} \right] \exp(iB_2^N(\hat{Z})), \tag{5.48}$$

by the methods as described in the Section 4.1. All the results in that section produce the quantity (5.48) when r (in the formulas of Section 5.3) is replaced by $e^{\hat{Z}}$.

Remark 5.5.2. *A very similar derivation for transition probability can be obtained when the volatility σ is considered to be stochastic*

6. CONCLUSIONS AND RECOMMENDATIONS

In this dissertation, we have presented a new approach based on the Barndorff-Nielsen and Shephard model to obtain the arbitrage-free pricing equation and an analytical solution for the variance swap. We have also presented a new variance driven price index for the financial market. The stochastic models that are used for analysis are empirically reasonable as well as having many appealing features from a theoretical finance perspective. The results derived in this dissertation are potentially important given the empirical fact that the considered financial instruments are variance driven. The improvement of numerical results in the analysis is very significant over the existing (non-variance driven) model. More generally, the results obtained in this dissertation have important implications for their use in, for example, energy markets. Crude oil and natural gas are one of the most liquid option markets among all commodities. Variance or volatility risk premia for energy commodities, crude oil and natural gas, is becoming increasingly popular and the approach considered in this dissertation can be further developed to analyze such markets. The price index proposed in this dissertation can be considered to be an effective indicator of such markets.

We have also shown that the Feynman path integral method can be used for the analysis of transition probability density functions for option pricing for certain Lévy process driven financial markets. Implementing the close connection of certain integrals with special functions, various interesting results are derived in terms of parabolic cylinder functions and incomplete gamma functions. A very similar derivation for transition probability can be obtained when the volatility σ is considered to be stochastic. In our future work we plan to extend this method to more complicated option pricing models in financial market and obtain asymptotic expansions of solutions in those cases. We also plan to implement this method to other Lévy process driven financial instruments-for example-variance, volatility, and covariance swaps.

Financial institutions such as banks face the risk of losing their earnings from investing in various financial products. Management of a portfolio of asset is risky and a central problem for financial industry. Therefore it is extremely important for financial managers to develop a hedging strategies which can fully eliminate or minimize this risk associated with investing in volatile financial products. Financial companies have adopted sophisticated hedging strategies for hedging risks associated with investing in a financial products. In energy market, investors sell futures to hedge the risk associate with holding a commodity. The Barndorff-Nielsen and Shephard differs from other models because it incorporates jumps in the model which destroys market completeness. This makes BNS-model more efficient for modelling financial instruments since market completeness is not a robust property. In an incomplete market, it is not possible to replicate every contingent claim even though it is possible to price claim with respect to an equivalent martingale measure. This is also means that in an incomplete market one cannot construct a heding strategy that can fully elimate risks associate with a volatile asset. There are several approaches to pricing and hedging in an incomplete market and the most commonly used approaches are: Merton's approach, utility maximization, and quadratic hedging [23]. Quadratic hedging can be defined as the choice of a hedging strategy which minimizes the hedging error in a mean-square sense [23]. Further research will be to implement Barndorff-Nielsen and Shephard (BNS) model with variance swaps to find optimal hedging strategies for for the energy market. An optimal amount of the underlying commodity that hast to be held for minimizing the hedging error can be determined.

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