EMPIRICAL COMPARISON OF STATISTICAL TESTS OF DENSE SUBGRAPH IN NETWORK DATA

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ABSTRACT

Network analysis is useful in modeling the structures of different phenomena. A fundamental question in the analysis of network data is whether a network contains community structure. One type of community structure of interest is a dense subgraph. Statistically deciding whether a network contains a dense subgraph can be formulated as a hypothesis test where under the null hypothesis, there is no community structure, and under the alternative hypothesis, the network contains a dense subgraph. One method in performing this hypothesis test is by counting the frequency of shapes created by all three-node subgraphs. In this study, three different test statistics based on the frequency of three-node subgraph shapes will be compared in their ability to detect a dense subgraph in simulated networks of varying size and characteristics.

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CHAPTER 1. INTRODUCTION

Network analysis is useful in modeling the structures of different phenomena. These models are created using graphs consisting of a set of nodes and a set of edges that represent some type of connection between a pair of nodes. Graphs of networks are denoted $\mathcal{G} = (\mathcal{E}, \mathcal{V})$, where $\mathcal{E}$ is the set of edges and $\mathcal{V}$ is the set of nodes. A common example of a network comes from social media where each node represents a user and an edge is present if two nodes are associated or connected in some way, such as having similar interests, following the same celebrities, or being members of the same social group. Given this example, one possible goal would be to determine whether there is a subset, or subsets, of users that are more interconnected to each other than to other users outside of the subset. In a social media setting, this subset of highly connected users could indicate a group of close friends or colleagues. This simple example also leads to one of the most fundamental problems in network analysis: determining whether a set of network data displays a community structure.

A network displaying community structure contains one or more subsets of nodes that are more highly connected to nodes within a particular subset of $\mathcal{V}$ than to nodes outside of that subset. This could also be explained as there being more edges between a certain subset of nodes. Networks not displaying any community structure come from what is known as an Erdős-Rényi model. An Erdős-Rényi network graph is commonly denoted $G(n, p)$, where n is the number of nodes and p is the probability of an edge between any two nodes. Under the Erdős-Rényi network model, properties known as the within-group and between-group edge probabilities are equal (Fortunato 2010), implying the probability two nodes having an edge connecting them is the same regardless of the community membership of either node; the edges are randomly spread among all nodes. Real-world networks, including social media networks, are rarely random as the nodes are typically organized in some way (Fortunato 2010). This is clear as the affiliations people have on social media are rarely random.

For a network that is known to display a community structure, a stochastic block model(SBM) is more appropriate than an Erdős-Rényi model (Holland et al. 1983). Networks derived from an SBM are denoted by $G(n, \frac{a}{n}, \frac{b}{n})$, where the first parameter, n, is the number of nodes, and the second and third parameters are, respectively, the within-group and between-group edge probabilities of two nodes, and where $a \geq b$ which implies nodes in the same group have a higher or equal probability of being connected. A useful property of SBMs is that this model reduces to an Erdős-Rényi model when a is equal to b (Mossel et al. 2015). A real-world network where an SBM model may be appropriate is shown later with the political authors dataset in Chapter 3.

Selecting the most appropriate model for a network is not always obvious. Most times, the underlying community structure, or even whether a network has a community structure, is unknown. One way to determine whether a network displays a community structure is through statistical hypothesis testing with the null hypothesis indicating the network comes from an Erdős-Rényi model and lacks community structure, and an alternative hypothesis indicating the network contains a community structure. Hypothesis testing in community detection problems involves comparing the observed network properties to what is expected under an Erdős-Rényi model. This technique is common, and it has been used in several studies(Bubeck et al. 2016 , Gao & Lafferty 2017a; Gao & Lafferty 2017b; Verzelen & Arias-Castro 2014; Verzelen & Arias-Castro 2015).

Dense subgraphs are one type of community structure that is detectable using a hypothesis test. A dense subgraph is a subset of nodes in a graph that have a higher density of edges within the subset when compared to the rest of the graph(Lee et al. 2010). Dense subgraphs are often thought of as a unique type of community structure(Chen & Saad 2012). The difference between a network containing a typical community structure and a dense subgraph community is shown in Figure 1.1 where the network in the left panel shows a network which has three distinct communities where every node is part a single community. The right panel is a network that contains two dense subgraphs, one denoted by green nodes and the other denoted by yellow. The community membership of the purple nodes is not obvious. This study is focused on detecting the presence of planted dense subgraphs.

Detecting a dense subgraph using hypothesis testing can be accomplished through studying the frequency of shapes created by subgraphs of differing sizes in the network. Studying subgraphs of networks has applications in many fields from biochemistry, computer science, and the social sciences (Saha et al. 2010; Milo et al. 2002; Jin et al. 2018). Often the frequency of shapes created by all possible combinations of three-node(Gao & Lafferty 2017a; Jin et al. 2018; Bubeck et al. 2016) or four-node(Jin et al. 2018; Milo et al. 2002) subgraphs are studied.

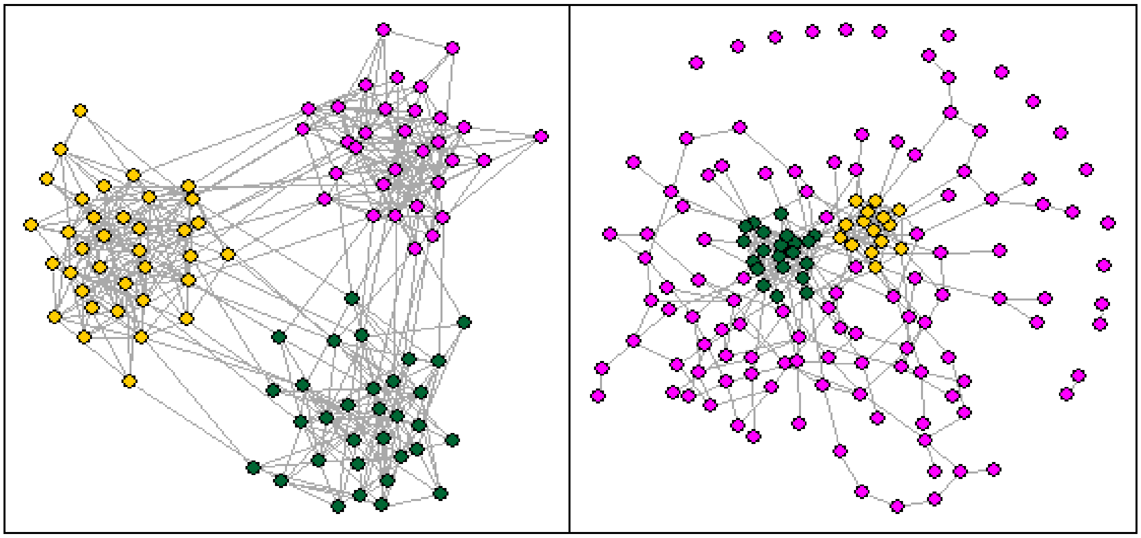

Figure 1.1. Plot comparing a dense subgraph and a standard community.

Subgraphs of a particular size can only take on a finite number of shapes based on the number of edges present and whether the network is directed or undirected. A directed network has edges that have a direction, or a start node and an end node(Fortunato 2010). A good example of a directed graph is hyperlinks(edges) on webpages(nodes). Many webpages link to other webpages, but the hyperlinks are not always reciprocated between web pages(Fortunato 2010). For example, it may be possible to go from page A to page B by clicking on a hyperlink, but page B does not have a hyperlink to get back to page A. However, for this paper, only undirected networks are used, so the direction of the relationship between nodes will not matter.

A three-node subgraph in an undirected network can have four possible shapes: no edges, one edge, two edges(vee shape), or three edges(triangle shape). The number of possible shapes increases as the number of nodes in the subgraph increases. A four-node subgraph can have 11 possible shapes(Ugander et al. 2013). The possible shapes for undirected three and four node subgraphs are shown in Figures 1.2 and 1.3.

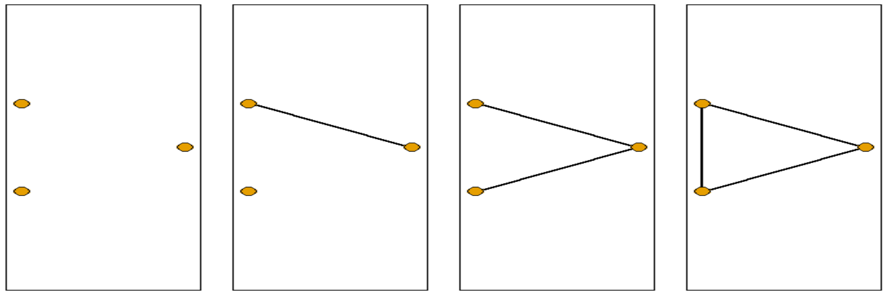

Figure 1.2. Three-node subgraph shapes.

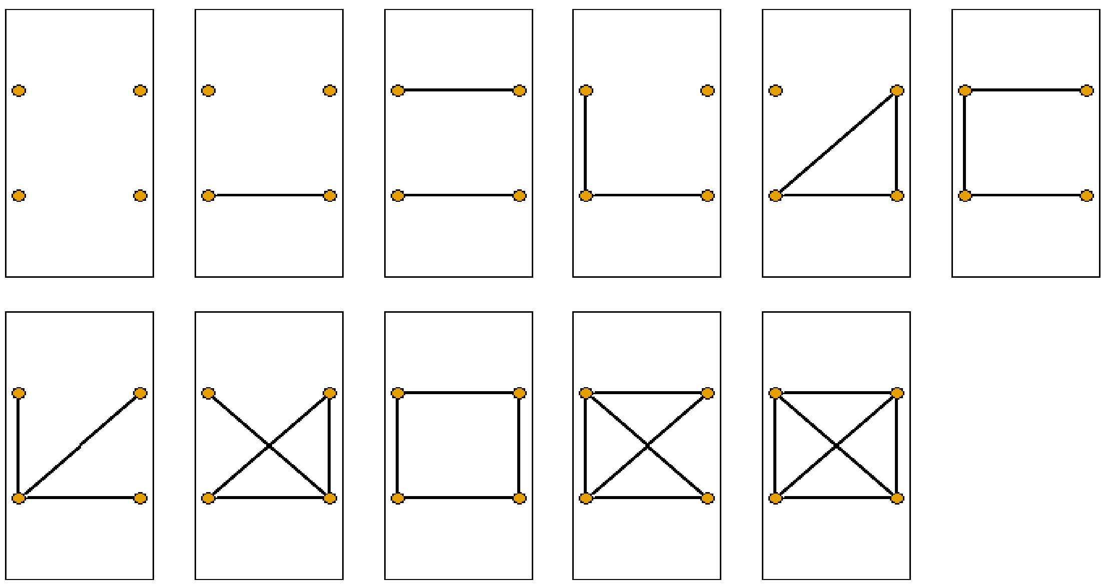

Figure 1.3. Four node subgraph shapes.

Statistically testing whether a dense subgraph is present in a given network is equivalent to the following hypothesis test:

$$H_0\colon G\left(n, p_0\right) \quad vs \quad H_1\colon G\left(n, p, \frac{a}{n}, \frac{b}{n}\right) \tag{1.1}$$

where the null hypothesis indicates the network comes from an Erdős-Rényi model of n nodes with an edge probability of p_0 for all pairs of nodes, and thus the network does not display community structure. A rejection of the null hypothesis in favor of the alternative suggests the network was generated from a model where n is the number of nodes, p is the probability of node belonging to the dense subgraph, $\frac{a}{n}$ is the edge probability between two nodes both belonging to the dense subgraph, and $\frac{b}{n}$ is the edge probability otherwise. Graphs of networks generated under both H_0 and H_1 are shown in Figure 1.4. In Figure 1.4, the graph on the left is from $G(n, p_0)$, and the graph on the right is from $G\left(n, p, \frac{a}{n}, \frac{b}{n}\right)$. The yellow nodes were randomly assigned the dense subgraph. In the null model, the yellow nodes are spread out randomly because the edge probability is the same for all nodes. In the alternative model, the yellow nodes are concentrated together, because there is an increased edge probability between two nodes in the dense subgraph.

The goal of this study is to compare the effectiveness of three different test statistics in their ability to detect a single dense subgraph in simulated networks. All three test statistics are based on the empirical number of each shape generated by three-node subgraphs from undirected networks. The comparison of test statistics will be carried out through a simulation study. The simulation study will estimate the size and power of each test statistic for numerous randomly generated networks with varying parameters using R software. Following the simulation study, the three test statistics will also be used to determine the presence of community structure in two real-world networks: a network consisting of over 4000 Facebook users, and a network consisting of political authors.

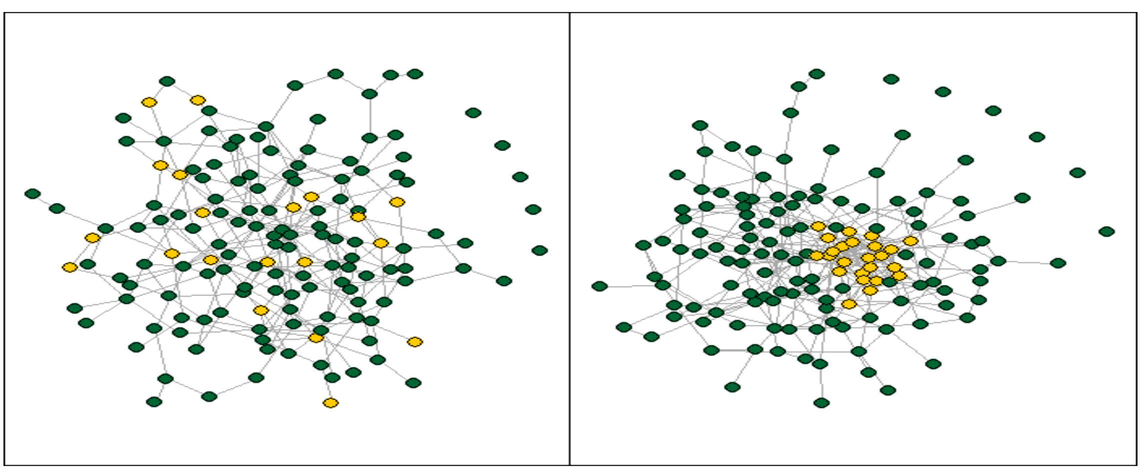

Figure 1.4. Graphs generated under H_0 and H_1.

This study is based in large part on two papers by Arias-Castro and Verzelen(2014) and Gao and Lafferty (2017a). These two papers present the formulas for all three test statistics provided in Section 2.1.2. Gao and Lafferty(2017a) also performed a simulation study on the power of the test statistics when the networks were simulated from SBM with two communities. This study differs from Gao and Lafferty(2017a) as the networks simulated here will be based on networks containing a dense subgraph rather than two bipartite communities. This study will also provide a better understanding of how the power changes as the underlying parameters are altered as a larger number of networks will be simulated.

CHAPTER 2. METHODOLOGY

This chapter is divided into two main sections. The first section describes the test statistics and the procedures used in the evaluation of powers in the simulation portion of this study. The second section describes the datasets and the procedures used in the analysis of two real-world networks.

2.1. Test Statistics

2.1.1. Description of Adjacency Matrix

The analysis of the test statistics is based on calculations derived from the adjacency matrix for each network. The adjacency matrix of an undirected network is a symmetric n x n matrix where all entries are either 0 or 1 with $diag(A) = 0$. To generate a simulated adjacency matrix, first an $n \times 1$ vector, Z, is randomly generated where $Z_i \sim bernoulli(p)$ and $0 < p < 1$ and n is the number of nodes. The Z vector indicates which nodes were randomly assigned membership to the dense subgraph where a 1 indicates membership in the dense subgraph. Using the Z vector, the n x n adjacency matrix, A, was constructed where

$$A_{ij} = A_{ji} = \begin{cases} 0 \; if \; i = j, \\ bernoulli\left(\dfrac{a}{n}\right) \; if \; Z_i = Z_j = 1, \\ bernoulli\left(\dfrac{b}{n}\right) \; otherwise \end{cases}$$

where $\frac{a}{n}$ is the edge probability for two nodes belonging to the dense subgraph, and $\frac{b}{n}$ is edge probability otherwise. Here, a 1 indicates an edge between the i^{th} and j^{th} nodes, and a 0 indicates no edge.

2.1.2. Description of Test Statistics

All three of the test statistics studied in this paper are based on observing the frequency of shapes over all three node subgraphs of a network. Specifically, the test statistics are based on the comparing the difference between the expected number and empirical number of triangles and vees in each network. This difference can be found by observing the adjacency matrix for each network. Formulas for calculating the empirical frequency of triangles and vees created by all combinations of three node subgraphs are given in Equations 2.1 and 2.2 provided below. These equations are presented in Gao and Lafferty (2017a).

$$\hat{F}_3 = \frac{1}{\binom{n}{3}} \sum_{1 \leq i < j < k \leq n} A_{ij} A_{jk} A_{ki} \tag{2.1}$$

$$\hat{F}_2 = \frac{1}{\binom{n}{3}} \sum_{1 \leq i < j < k \leq n} \{(1 - A_{ij}) A_{jk} A_{ki} + A_{ij}(1 - A_{jk}) A_{ki} + A_{ij} A_{jk}(1 - A_{ki})\} \tag{2.2}$$

Equation 2.1 is the relative frequency of triangles and Equation 2.2 is the relative frequency of vee shapes. Equation 2.2 is similar to the Degree variance test presented by Arias-Castro and Verzelen(2014). The formulas presented below in Equations 2.3 and 2.4 provide the difference between the empirical number of triangles and vees and their respective estimated values(Gao & Lafferty 2017a).

$$T_2 = 3\hat{p}^2(1 - \hat{p}) - \hat{F}_2 \tag{2.3}$$

$$T_3 = \hat{p}^3 - \hat{F}_3 \tag{2.4}$$

where $\hat{p}$ is the estimated overall edge probability of the network and is given by

$$\hat{p} = \frac{1}{\binom{n}{2}} \sum_{1 \leq i < j \leq n} A_{ij} \tag{2.5}$$

The first of the three test statistics presented will be referred to as T_{ez}, and is also known as the Erdos-Zuckerberg(EZ) test(Gao & Lafferty 2017b). T_{ez} is based on the frequency of triangles in a particular network. The formula for calculating T_{ez} is given by

$$T_{ez} = \frac{\sqrt{\binom{n}{3}} T_3}{\sqrt{\hat{p}(1 - \hat{p})^3 + 3\hat{p}^4(1 - \hat{p})^2}} \tag{2.6}$$

An alternative method for calculating T_{ez} can be achieved by counting the number of edges, vees, and triangles in a network and using matrix multiplication to obtain the number of each subgraph shape. This method is more computationally efficient for adjacency matrices generated by sparse networks(Gao & Lafferty 2017b). The matrix multiplication method is what will be used for the remainder of this study. As previously mentioned this method is dependent on calculating the number of edges, vees, and triangles, for which the formulas using matrix multiplication, as described by Gao and Lafferty (2017b) are displayed below.

$$\hat{E} = \frac{1}{2\binom{n}{2}} \langle \mathbb{1}, A \rangle, \tag{2.7}$$

$$\hat{V} = \frac{1}{6\binom{n}{3}} \left(\langle \mathbb{1}, A^2 \rangle - tr(A^2) \right), \tag{2.8}$$

$$\hat{T} = \frac{1}{6\binom{n}{3}} tr(A^3) \tag{2.9}$$

where tr(.) is the matrix trace and $\langle A, B \rangle = tr(A^T B)$ is the inner product of the matrix(Gao & Lafferty 2017b). Now, using the formulas shown in Equations 2.7, 2.8, and 2.9, Gao and Lafferty(2017b) show T_{ez} can be calculated using the formula

$$T_{ez} = 2\sqrt{\binom{n}{3}} \left(\sqrt{\hat{T}} - \left(\frac{\hat{V}}{\hat{E}} \right)^{3/2} \right) \tag{2.10}$$

It has been shown(Gao & Lafferty 2017b) that when the between-group and within-group edge probability are equal, that is the network is simulated under H_0 in Equation 1.1, the asymptotic distribution of T_{ez} is the standard normal distribution. This implies that the significance level of the test can be calculated using the Gaussian distribution.

Proving T_{ez} has the ability to detect community structure in networks can be achieved by calculating the expected difference between P_3 and P_0^3 where P_3 and P_0^3 are the probabilities of a triangle in a single three node subgraph under H_1 and H_0, respectively.

Proposition 2.1.1. *Under H_1,*

$$P_3 - P_0^3 = \frac{p^3(a-b)^2}{n^3} [(a-b)(1-p^3)(a+2b)]$$

Proof. First we examine P_3. Under H_1,

$$
\begin{aligned}
P_3 &= Pr\left(A_{ij}A_{ik}A_{jk} = 1\right) \\
&= \mathbb{E}\left[Pr\left(A_{ij}A_{ik}A_{jk} = 1|Z\right)\right] \\
&= \mathbb{E}\left[\frac{(a-b)Z_iZ_j + b}{n} * \frac{(a-b)Z_iZ_k + b}{n} * \frac{(a-b)Z_jZ_k + b}{n}\right]
\end{aligned}
$$

If we let $\mathbb{E}[Z_i] = p$, where p is the probability of being a member of the dense subgraph, the above simplifies to

$$
P_3 = \frac{1}{n^3}\left[p^3(a-b)^3 + 3p^3b(a-b)^2 + 3p^2b^2(a-b) + b^3\right]
$$

Now, using the result above, and subtracting P_0^3 we get,

$$
\begin{aligned}
P_3 - P_0^3 =& \frac{1}{n^3}[p^3(a-b)^3 + 3pr^3(a-b)^2 + 3p^2r^2(a-b) + b^3 \\
&- p^6(a-b)^3 - 3bp^4(a-b)^2 - 3b^2p^2(a-b) - b^3] \\
=& \frac{p^3(a-b)^2}{n^3}[(a-b)(1-p^3)(a+2b)]
\end{aligned}
$$

□

Because the quantity $(a - b)$ appears in this difference, T_{ez} should have some power in detecting dense subgraphs when a network is generated under a model where $a \neq b$. That is, the network is generated under H_1.

The second test statistic to be presented in this paper will be denoted as $\tilde{T}$. The formula for $\tilde{T}$ as described by Gao & Lafferty (2017a) is based on the statistics presented in Equations 2.4 and 2.5 and is given below in Equation 2.11.

$$
\tilde{T} = \frac{\sqrt{\binom{n}{3}}T_2}{\sqrt{3\hat{p}^2(1-\hat{p})^2(1-3\hat{p})^2 + 9\hat{p}^3(1-\hat{p})^3}} \tag{2.11}
$$

$\tilde{T}$ is derived from the difference between the expected number and empirical number of vees as evidenced by the presence of T_2 in the formula. $\tilde{T}$ also has some some properties that make it a useful test statistic. One of these properties, as demonstrated by Gao and Lafferty (2017a), is that its asymptotic distribution is the standard normal distribution, so, similar to T_{ez}, significance levels can easily be found using the Gaussian distribution.

Proving $\tilde{T}$ has the ability to detect dense subgraphs in networks can be achieved in a similar fashion to the approach used for T_{ez}. In this case, P_2 will be the probability a single three node subgraph has a vee shape under H_1 and P_0^2 will be the probability a single three-node subgraph has a vee shape under H_0. Then, examining the difference between P_2 and P_0^2 will show that $\tilde{T}$ has the ability to detect community structure.

Proposition 2.1.2. *Under* H_1,

$$P_2 - P_0^2 = \frac{(a-b)^2 p^3 (1-p)}{n^2}$$

Proof. First P_2 is examined. Under H_1,

$$\begin{aligned} P_2 &= Pr\left(A_{ij}A_{ik} = 1\right) \\ &= \mathbb{E}\left[Pr\left(A_{ij}A_{ik} = 1 | Z\right)\right] \\ &= \mathbb{E}\left[\frac{(a-b)Z_iZ_j + b}{n} * \frac{(a-b)Z_iZ_k + b}{n}\right] \end{aligned}$$

If we let $\mathbb{E}[Z_i] = p$ the above simplifies to

$$P_2 = \frac{1}{n^2}\left[p^3(a-b)^2 + 2bp^2(a-b) + b^2\right]$$

Now using the result above and subtracting P_0^2 we get,

$$\begin{aligned} P_2 - P_0^2 = &\frac{1}{n^2}[p^3(a-b)^2 + 2bp^2(a-b) + b^2 \\ &- p^4(a-b)^2 - 2bp^2(a-b) + b^2] \\ = &\frac{(a-b)^2 p^3 (1-p)}{n^2} \end{aligned}$$

□

Again, because the difference above is dependent on the quantity $(a-b)$, $\tilde{T}$ has some power in detecting community structure when a network is generated under a model where $a \neq b$. That is the network is generated under H_1.

The third test statistic, T^2, is based on the values of both T_{ez} and $\tilde{T}$, defined by Gao and Lafferty(2017a) as

$$\begin{aligned} T^2 &= \binom{n}{3}\left(\frac{T_2^2}{3\hat{p}^2(1-\hat{p})^2(1-3\hat{p})^2+9\hat{p}^3(1-\hat{p})^3}+\frac{T_3^2}{\hat{p}(1-\hat{p})^3+3\hat{p}^4(1-\hat{p})^2}\right) \\ &= \tilde{T}^2 + T_{ez}^2 \end{aligned} \tag{2.12}$$

As previously stated, both T_{ez} and $\tilde{T}$ have standard normal distributions under an Erdős-Rényi network. It has also been shown that T_3 and T_2 are asymptotically independent (Gao & Lafferty 2017a). If T^2 is the sum of the squared values of T_{ez} and $\tilde{T}$, then based on the properties described above, it is clear that T^2 will follow a chi-square distribution with 2 degrees of freedom when the network is generated under H_0.

2.2. Simulation Methodology

2.2.1. Estimation of Size and Power

The estimation of size and power was done through the use of R software where the networks and their adjacency matrices were simulated using the process described in Section 2.1.1. In this study, a significance level of $\alpha = 0.05$ is used throughout, and the estimation of each size and power is based on 500 repetitions. The networks are simulated using various parameters for the model

$$G\left(n, p, \frac{a}{n}, \frac{b}{n}\right) \tag{2.13}$$

Equation 2.13 represents a stochastic block model with an induced dense subgraph where n is the number of nodes, p is the probability a node is in the dense subgraph, $\frac{a}{n}$ is the within dense subgraph edge probability, and $\frac{b}{n}$ is the edge probability otherwise. Based on these four parameters a density, d, for each network was calculated using the formula

$$d = ap^2 + b \tag{2.14}$$

where d can also be thought of as the expected degree, or average number of edges, for each node.

In total 75 different power tables are to be created. Each table will represent a particular combination of d, n, and p with the values of a and b changing in a way which keeps d constant. The values of n will be 50, 100, and 150, and the values for p will range from 0.10 to 0.30 in increments of 0.05.

The first step in creating the power tables involves estimating the size for each table, that is the networks are simulated under H_0 with $a = b$. When estimating the size, the values of $a, b \in \{1, 2, 3, 4, 5\}$ where $a = b$. Each test statistic is used to decide whether H_0 should be rejected for each repetition. The percentage of rejections is the estimated size when the network is generated under H_0. Based on a significance level of .05, H_0 is rejected for both T_{ez} and $\tilde{T}$ when the absolute value of their calculated value is larger than 1.96 the 97.5^{th} percentile of a standard normal distribution. T^2 rejects H_0 when its calculated value is larger than 5.99 the 95^{th} percentile of a chi-square distribution with 2 degrees of freedom.

After an estimated size was calculated, the value of b was decreased by a predetermined value, typically 0.05 or 0.1 depending on the table, and the value of a was increased by an amount to keep d constant. Once the values for a and b are changed the network is now simulated under H_1, and now the number of repetitions that cause a rejection of H_0 is the power for each test. This process of changing a and b is repeated until b had decreased by one from its original value in the estimation of the size or the value of a is larger than n. The values of a and b will be changed 10 or 11 times depending on the table. Once a or b reaches the stopping point, the values for d, n, p were changed and the whole process was repeated for another power table.

2.3. Real Data Analysis Methods

In addition to the simulation study described above, this study also applies the three tests to two real world networks. A description of each network is described in the following sections.

2.3.1. Political Authors Network

The first network to be studied contains information on political authors. This dataset contains information for 104 different political books by numerous authors. An edge exists between books if the book references one of the other books in the data set. In this network the underlying community structure is known. One community represents conservative authors and the other represents liberal authors. There is also a small number of neutral authors. The three tests are used to determine if they can detect the underlying community structure.

2.3.2. Facebook Network

The second real network studied contains information on users of the social media site Facebook. This dataset was retrieved from the Stanford Network Analysis Project(SNAP). This dataset was originally used in work by McAuley and Leskovec(2012). This dataset contains anonymous information on 4039 Facebook users. In this data set the ground truth of each nodes membership was not given in the edge list of the network. The edge list was converted into an adjacency matrix and all three tests were used to analyze the dataset to determine whether there is an underlying community structure.

CHAPTER 3. RESULTS

In this study, both simulations and real data were used as a way to evaluate the power and overall usefulness of three different test statistics in the detection of dense subgraphs. This chapter is divided into two main sections presenting the results of the analyses. The first section discusses the validity and estimated powers of the three tests used in the simulation portion of the study. The second section discusses the results of the tests when applied to the political authors and Facebook networks.

3.1. Simulation Results

3.1.1. Validity of Tests: Estimating Size

A test is valid if the size, also called type 1 error rate (α), can be estimated and is in an acceptable range. For this study, a size was considered acceptable if $0.04 \leq \alpha \leq 0.06$. The sizes for all three tests are estimated by simulating 500 random networks as described by the null hypothesis in equation 1.1, and then calculating the percentage of simulated networks which cause H_0 to be rejected.

The validity of each test was dependent on the values a and b. The size of T_{ez} was never acceptable when a and b were set to values 1 or 2. Thus, T_{ez} was not an appropriate test when the networks were relatively sparse. As a result of T_{ez} not having an acceptable size, T^2 was also not valid when T_{ez} was not valid. An acceptable value for the size of $\tilde{T}$ was always in the acceptable range regardless of the values of the parameters. Sizes for all simulations can be found in the first columns of the tables of the Appendix starting with table A.1.

3.1.2. Trends of Powers

Estimation of the powers of the tests was done in a similar fashion as estimating the sizes. Similar to estimating the size, 500 simulations were used for each network model. However, when estimating the power, the networks were simulated using the model described by the alternative hypothesis in Equation 1.1. Now, the percentage of simulated networks that cause H_0 to be rejected is the power of the test.

A number of trends were apparent across all three of the tests. As expected the power of each test increased as n was increased. In general, the power of the tests increased as the difference between a and b was increased. Exceptions to this rule can be seen in multiple simulations most notably when examining $\tilde{T}$ when $p \leq 0.15$. Figure 3.1 shows a particular instance of this behavior. It can be seen that the power of $\tilde{T}$ decreases even though one expects the power to continue to rise. It is not apparent why this decrease in power occurs even though the difference between a and b is quite large. The power does appear to increase again when the difference becomes even larger, which can also be seen in Figure 3.1, which is based on data from Table A.8.

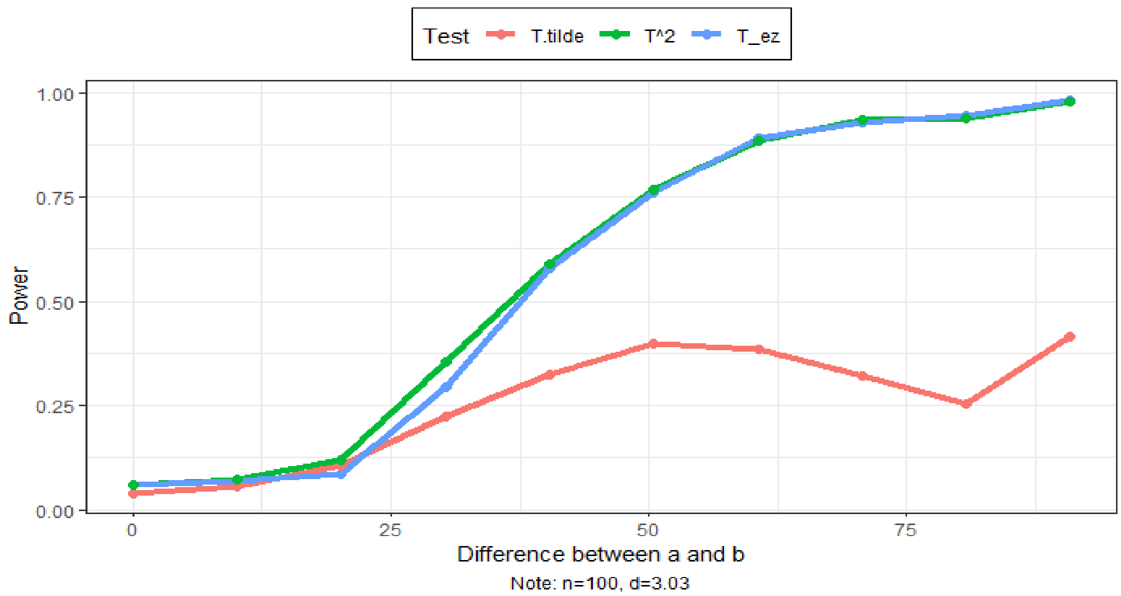

Figure 3.1. Power plot illustrating the unexpected drop in power for $\tilde{T}$.

Changes in the density of the network also provide interesting trends in the powers. One way of examining the impact the density has on the powers is by holding p, n, and $a - b$ constant (a and b are changing, but their difference is not). When this is the case, all tests see a decrease in power as the density increases. Two cases of this occurring are displayed in Figure 3.2, which is based on data taken from Appendix tables A.6 - A.10 and A.36 - A.40 This trend is essentially showing that as $\frac{b}{n}$ is getting closer to $\frac{a}{n}$ the tests can not distinguish the community structure as effectively. In Figure 3.2, some points are not connected because, the size was not close to 0.05 so the test was considered invalid.

The second way the density can change is through changes in p. When p increases and the difference between a and b remain constant, the tests can more effectively distinguish community

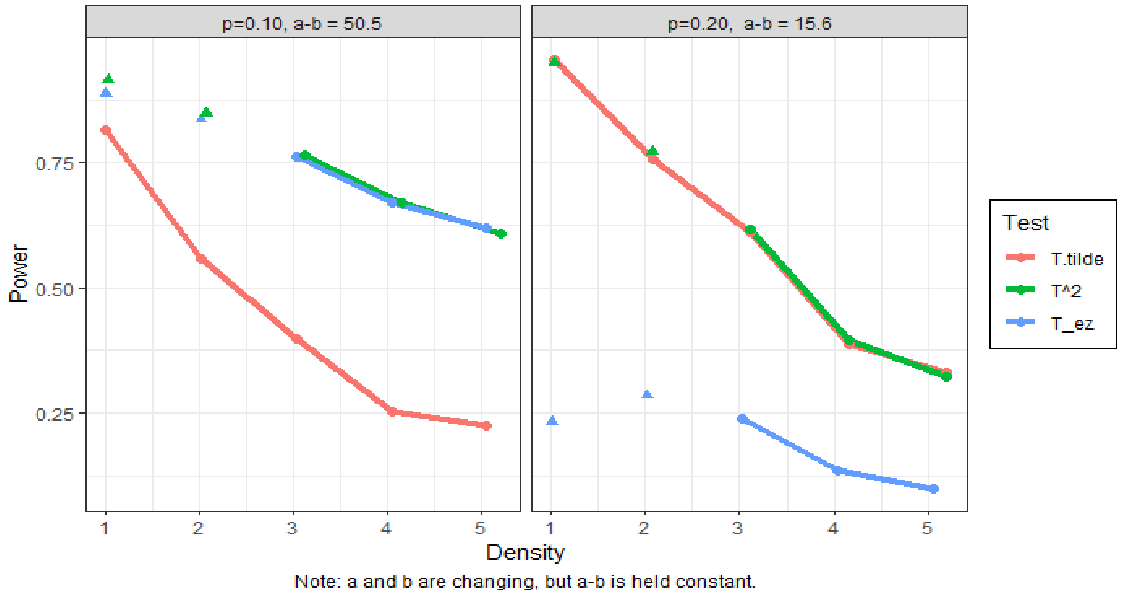

Figure 3.2. Power plots illustrating decreasing power as the density increases due to changes in a and b.

structure even when the difference between a and b is relatively small. For example, the power of a test when $p = 0.30$ and $a - b = 10$ is greater than when $p = 0.10$ and $a - b = 10$. Thus, when p is large the powers of the test statistics are converging to 1.00 quicker with respect to smaller differences in a and b. This behavior is illustrated in Figure 3.3 below, which is based on data taken from Appendix tablesA.9, A.24, A.39, A.54, and A.69. In figure 3.3, the density is a function of p, a, and b, so when $a - b$ is held constant, we can inspect the effect that p has on power of the test. These plots demonstrate that as density increases, due to an increase in p, the difference between a and b needed to effectively detect community structure is decreased.

3.1.3. Comparison of Powers

Determining which of the three tests performs the best is dependent on the parameters of the network model. Based on the simulation results, it is obvious that if the network is overly sparse, that is $a, b < 3$, $\tilde{T}$ is the most appropriate. This is due to the fact that an acceptable size was unable to be estimated in these sparse cases for either T_{ez} or T^2.

Comparing the powers of the tests when all three tests are valid presents more interesting patterns. T^2 seems to be the most consistent of the three test statistics. The power of T^2 is typically comparable to either $\tilde{T}$ or T_{ez}, whichever one performs better based on the given parameters. T^2 seldomly has the highest power, although there are few a instances, specifically when $n = 50$, that

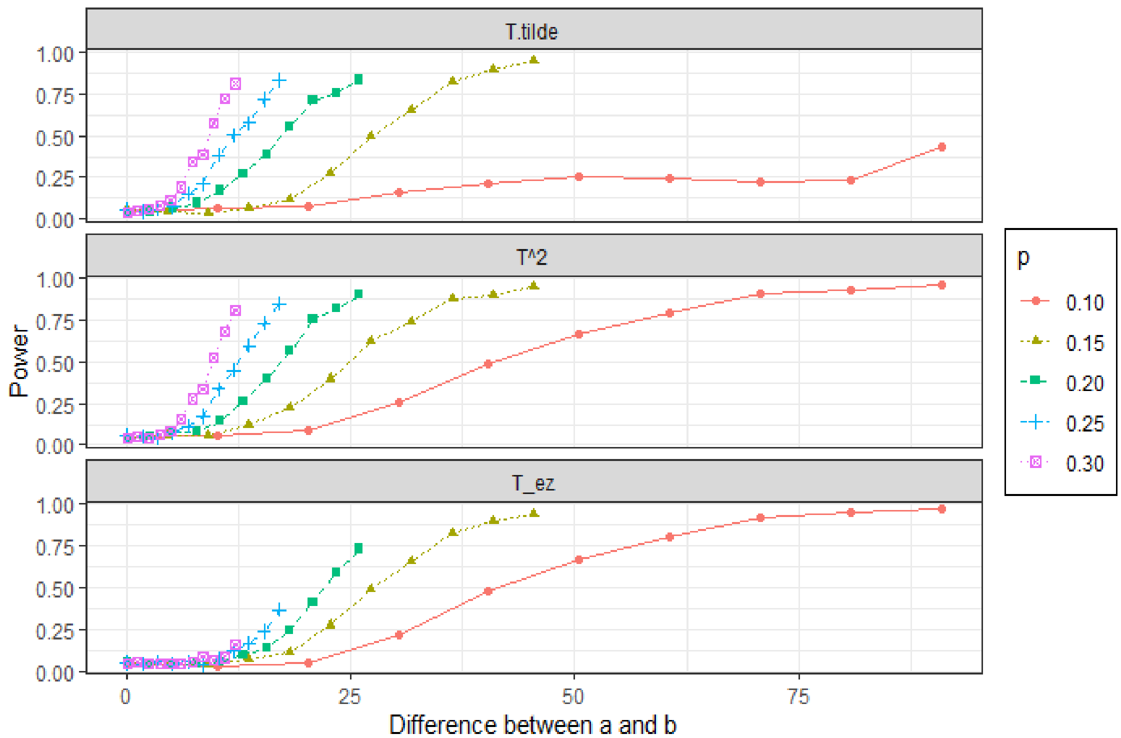

Figure 3.3. Power plots showing the rate at which the powers converge to 1 with respect to the difference between a and b.

it provides a slight increase in power over the other two tests. Examples of this occurring are shown in Appendix Tables A.33, A.34 and, A.35.

$\tilde{T}$ was the most powerful test in the majority of the simulations that were run. In particular, $\tilde{T}$ was most effective when the value of p was relatively large($p \geq 0.20$). At these large values of p, $\tilde{T}$ was consistently the most powerful of the three tests; however, when p was relatively small there were numerous simulations where the power of $\tilde{T}$ stayed quite low in comparison to the other tests. For simulations when both n and p were small $\tilde{T}$ was not effective at detecting the underlying dense subgraph.

In contrast to $\tilde{T}$, T_{ez} performed better in the simulations where $\tilde{T}$ struggled. T_{ez} was typically the most powerful of three tests when p was smaller than 0.20 and $n < 150$. When the value of p was set to values larger than 0.20, the power of T_{ez} quickly dropped to levels that were substantially lower compared to the other tests, and once $p = 0.30$, T_{ez} was by far the least powerful. The Appendix shows this decrease in power of T_{ez} starting in Table A.48.

It is clear that choosing the most powerful test is highly dependent on the situation and the values of the network parameters. Power plots for a selected number of simulations are presented in Figure 3.4 which illustrate how the most appropriate test differs for a number of scenarios. The panels in Figure 3.4 correspond to data taken from Tables A.15, A.45 and, A.75, respectively, from the Appendix. In general, $\tilde{T}$ would be the most appropriate test when the network is overly sparse (when $d < 3$), as it was the only test that had a valid size for those parameters. $\tilde{T}$ would also be the preferred test when p is relatively large. T_{ez} would be preferred when p is relatively small, and T^2 is typically comparable to whichever of the other tests is most powerful for the given situation, assuming T_{ez} was appropriate.

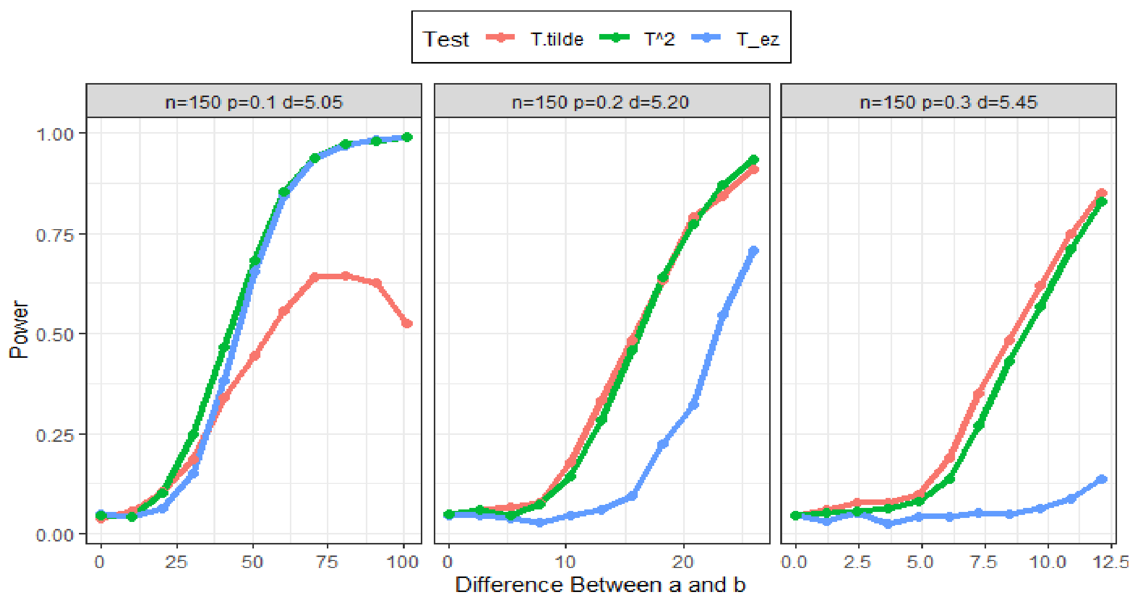

Figure 3.4. Power plots showing the change in power of tests over multiple sets of network parameters.

Power plots based on Appendix tables A.9, A.24, A.39, A.54,and A.69, are presented below in Figures 3.5, 3.7, 3.9, 3.11, 3.13, respectively. Below each power plot is a figure consisting of a set of network graph visualizations which correspond to the power plot directly above it. These graphs are shown in Figures 3.6, 3.8, 3.10, 3.12, 3.14. Each graph was randomly generated using the same d and n as its respective power plot. In these graph visualizations a node that is colored yellow is considered a member of the dense subgraph. The graphs in the left panel are a realization of an Erdős-Rényi network. In the left panels the edges are approximately evenly spread out between all nodes, but in the center and right panels the edges become more concentrated within the dense subgraph of yellow nodes. This changing structure is because the between-group and within-group edge probabilities are diverging, observing the graphs from left to right is analogous to reading it's respective power plot from left to right.

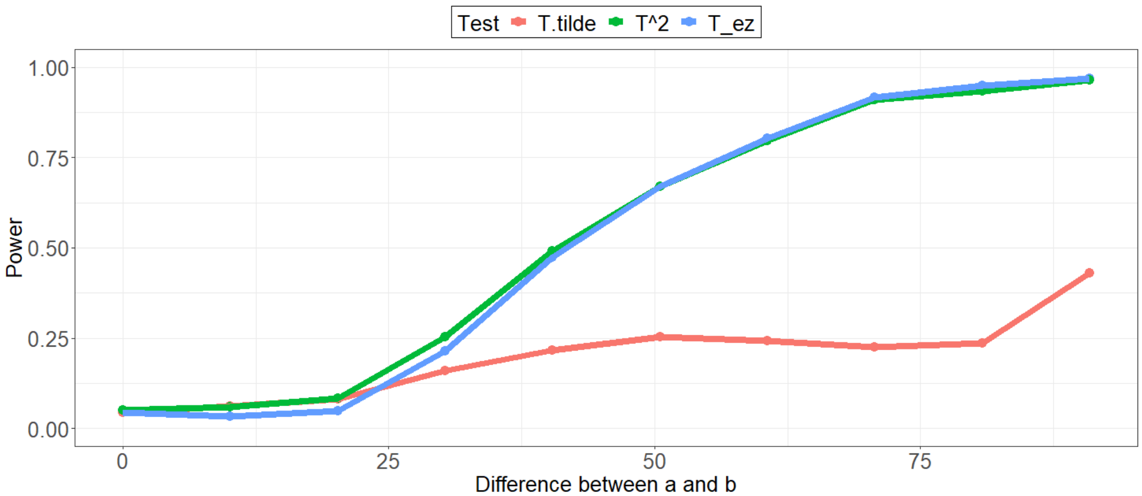

Figure 3.5. Power plots for simulated networks with $p = 0.10$, $n = 100$, and $d = \frac{404}{100}$.

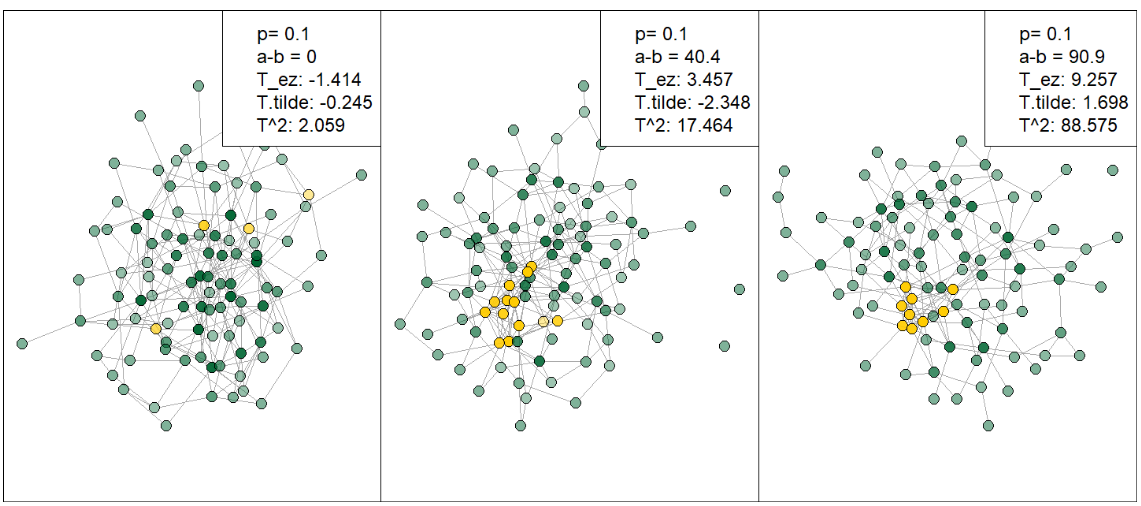

Figure 3.6. Simulated graphs for networks with $p = 0.10$, $n = 100$, and $d = \frac{404}{100}$.

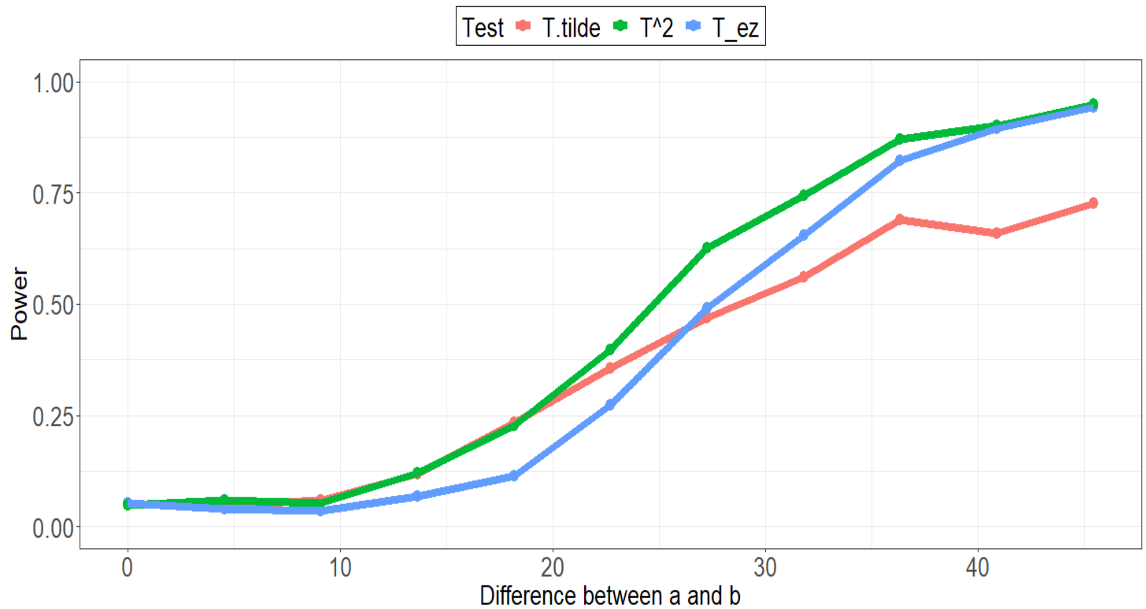

Figure 3.7. Power plots for simulated networks with $p = 0.15$, $n = 100$, and $d = \frac{409}{100}$.

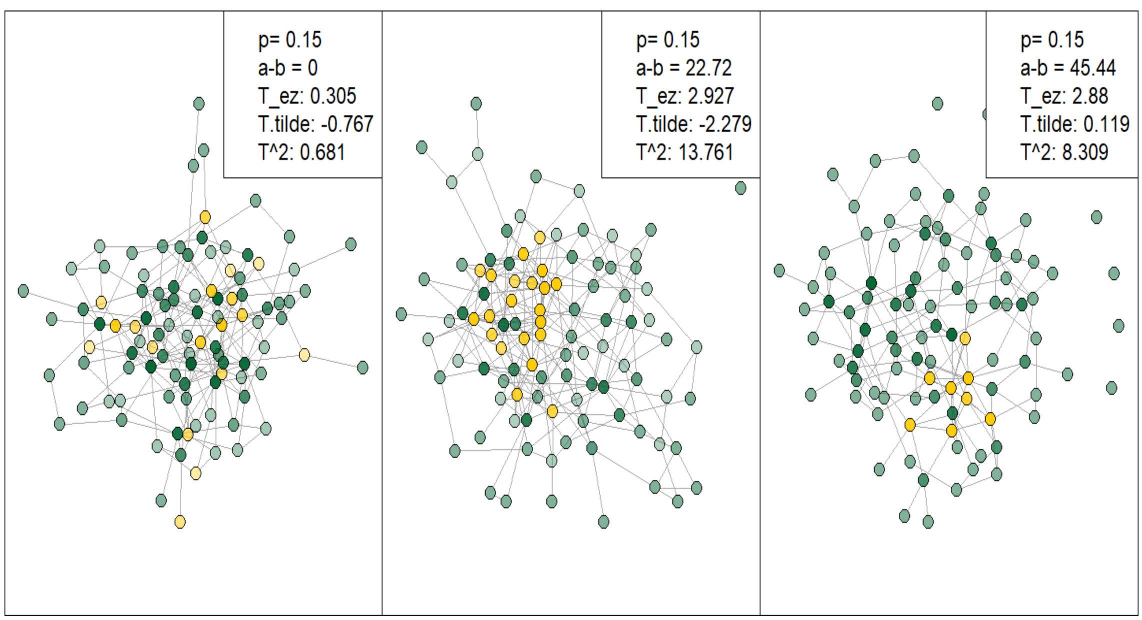

Figure 3.8. Simulated graphs for networks with $p = 0.15$, $n = 100$, and $d = \frac{409}{100}$.

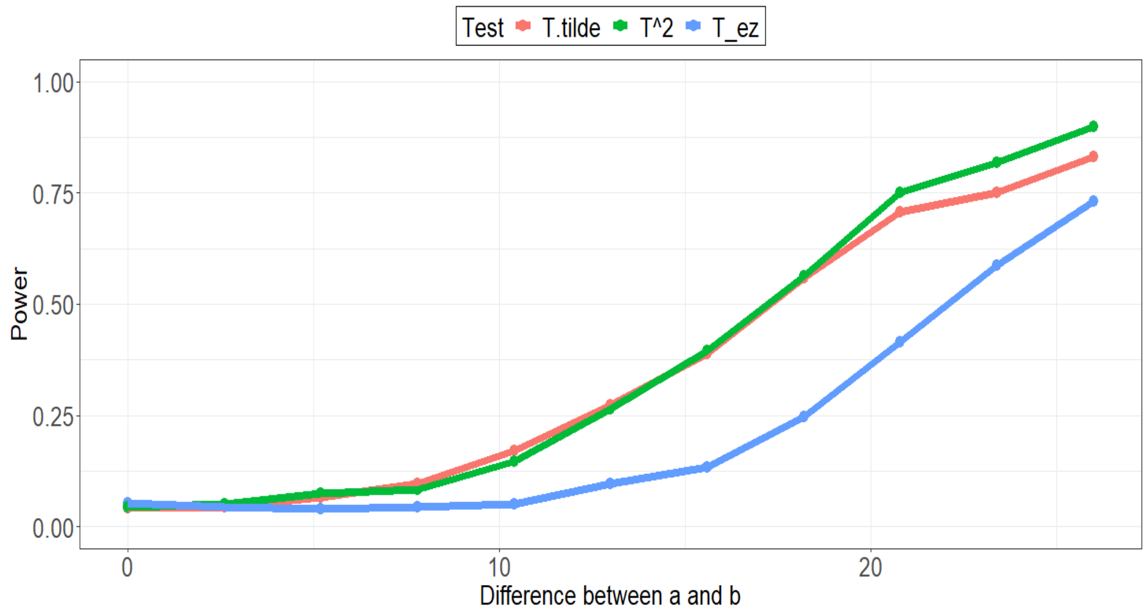

Figure 3.9. Power plots for for simulated networks with $p = 0.20$, $n = 100$, and $d = \frac{416}{100}$.

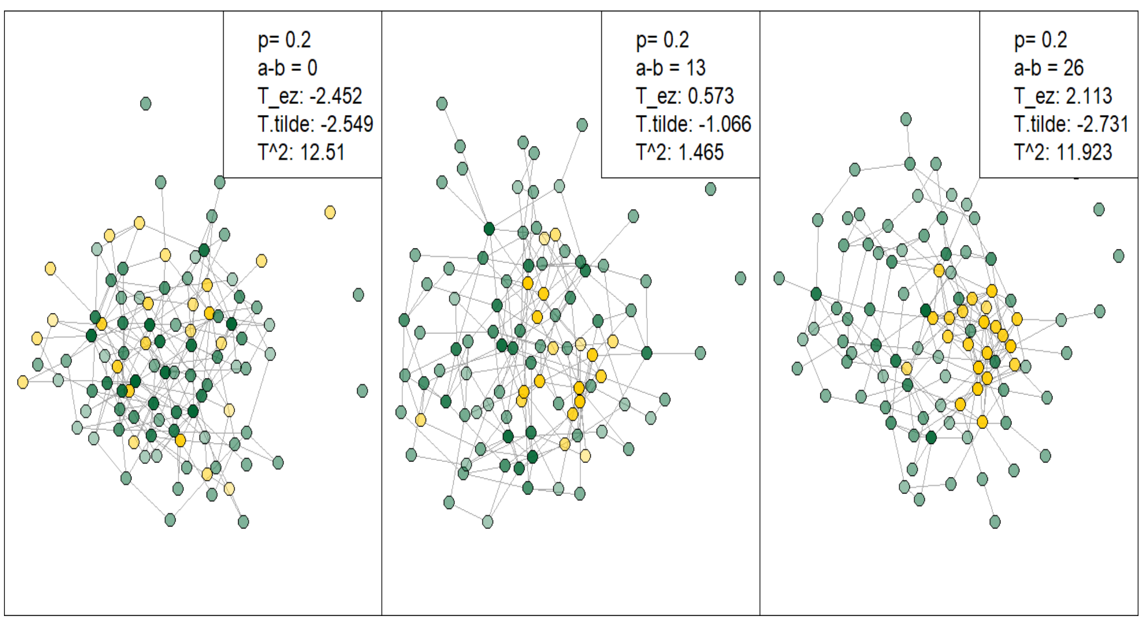

Figure 3.10. Simulated graphs for networks with $p = 0.20$, $n = 100$, and $d = \frac{416}{100}$.

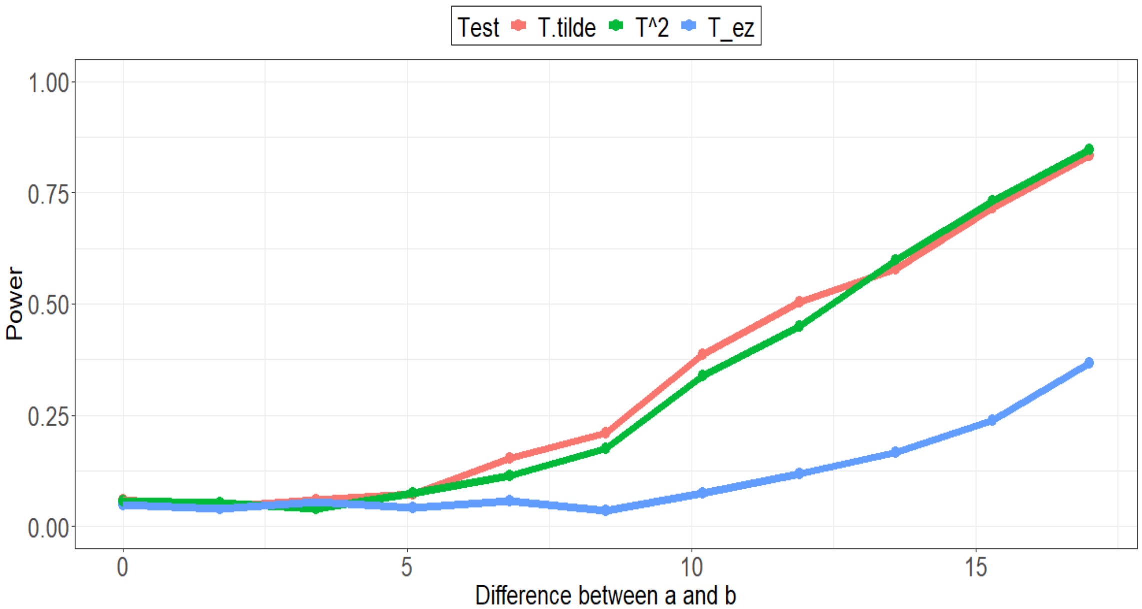

Figure 3.11. Power plots for for simulated networks with $p = 0.25$, $n = 100$, and $d = \frac{17}{4}$.

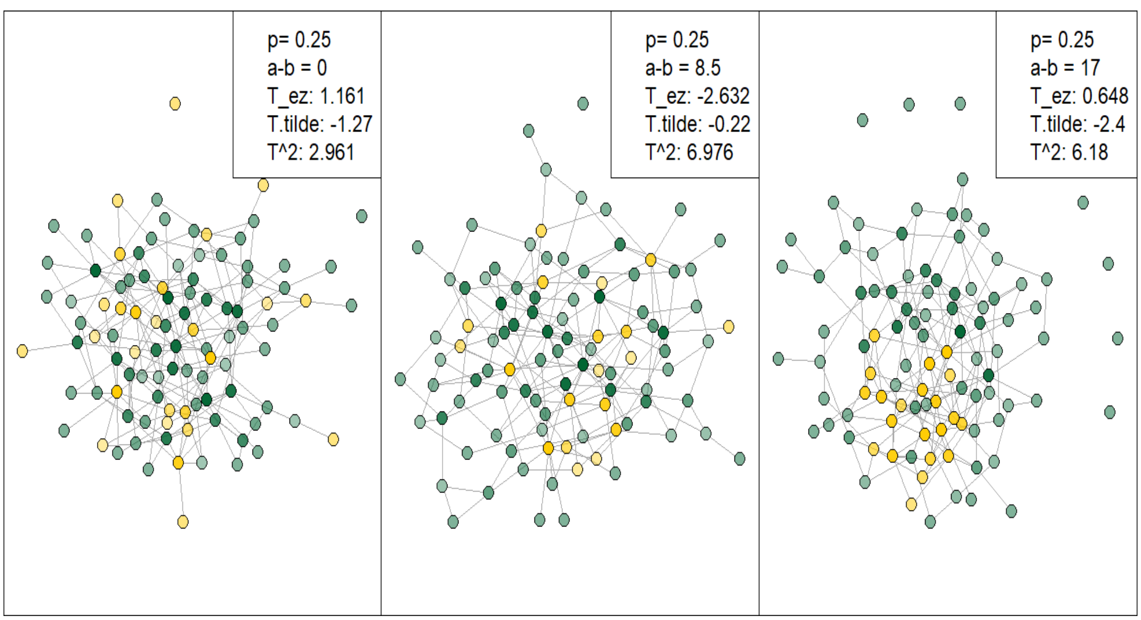

Figure 3.12. Simulated graphs for networks with $p = 0.25$, $n = 100$, and $d = \frac{17}{4}$.

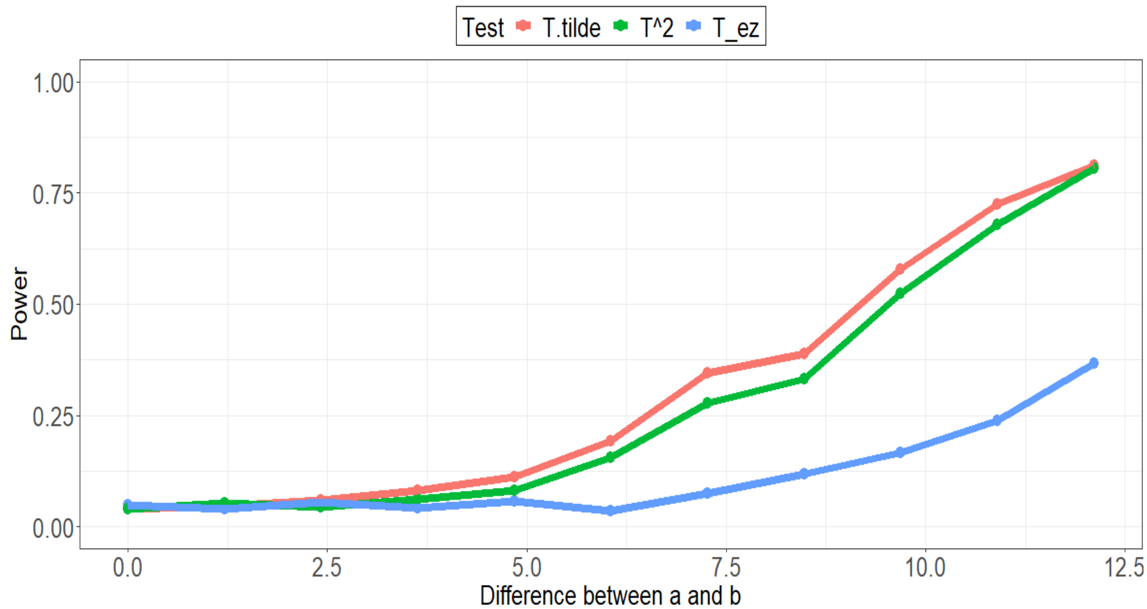

Figure 3.13. Power plots for for simulated networks with $p = 0.30$, $n = 100$, and $d = \frac{109}{25}$.

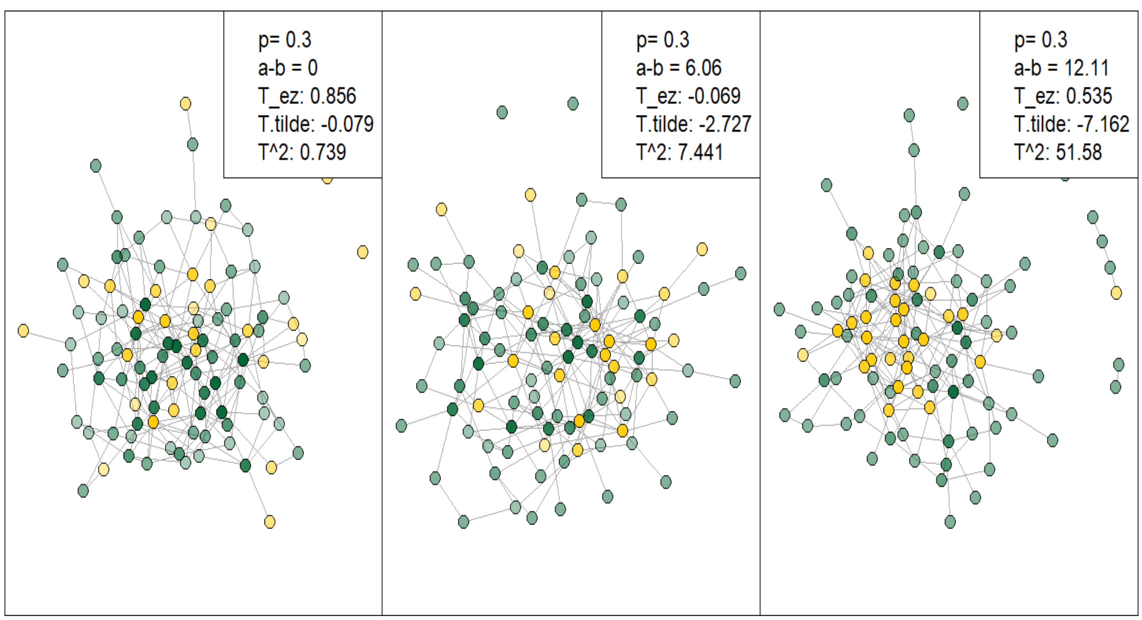

Figure 3.14. Simulated graphs for networks with $p = 0.30$, $n = 100$, and $d = \frac{109}{25}$.

3.2. Real Data Analysis Results

3.2.1. Political Authors Network

Testing for community structure in the political authors network using the three tests produced a definitive outcome. All three tests rejected the null hypothesis of the network having no community structure. All tests had p-values < 0.0001 indicating highly significant results. A plot of the network, along with the test statistics, is shown below in Figure 3.15. This plot clearly shows that there are two distinct communities: one community for conservative authors and another for liberal authors. The cluster of red nodes represents conservative authors, and the cluster of blue nodes represents liberal authors. Also of note is the presence of neutral authors, indicated by purple nodes, that often appear between the two communities. These results suggest that a political author is typically more likely to reference other authors that have political beliefs that align with their own.

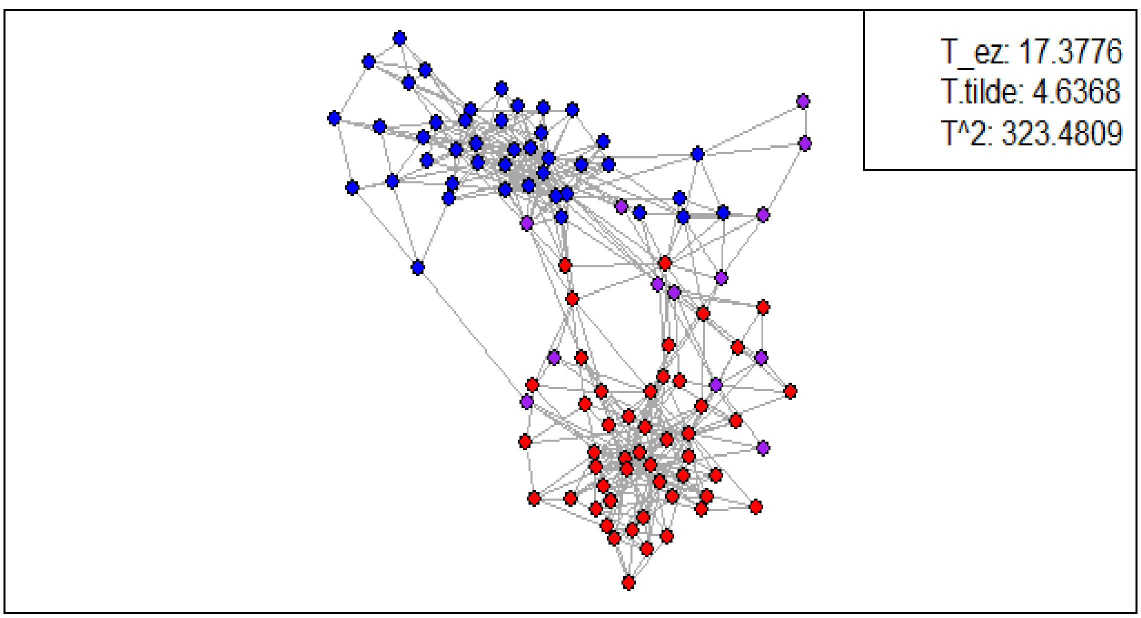

Figure 3.15. Visualization of the Authors Network.

3.2.2. Facebook Network

Applying the three tests to the larger Facebook network consisting of 4039 nodes, resulted in similar conclusions as the political authors network. Again, the network can reasonably be assumed to contain community structures. All three tests had highly significant results with all the p-values < 0.0001. A visualization and the calculated test statistics of this network are presented in Figure 3.16. The visualization of this network is less straight forward. First, there appears to be severe overlapping of points. This is due to the larger number of nodes. Second, the nodes are also not colored as the underlying community membership was not given. Regardless, the visualization clearly demonstrates that there are a number of different communities represented by a high concentration of nodes in a certain area.

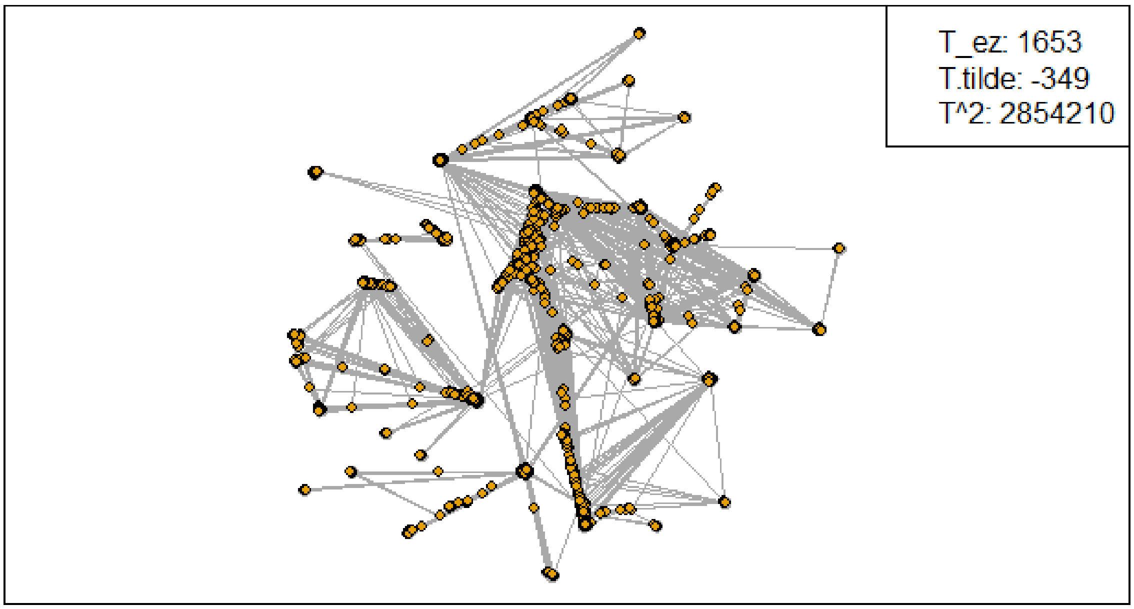

Figure 3.16. Visualization of the Facebook Network.

CHAPTER 4. CONCLUSIONS

This paper provided a simulation study that demonstrated the usefulness of three different test statistics based on three node subgraphs in a number of simulated networks containing a dense subgraph. The results of the simulation study show that the most appropriate test is highly dependent on the network parameters from which the network is generated. The simulation showed that a test based on the number of vees in three node subgraphs, is most effective when the size of the dense subgraph is relatively large. Where as a test based on the the number of triangles in three node subgraphs is most effective when the dense subgraph is relatively small. A good compromise between the two would be to use a test based on both vees and triangles which was demonstrated to have a reasonable power in most situations presented in this study.

However, one issue that was observed was that not all tests converge in every situation. The tests that rely on the number of triangles do not always converge especially when the networks are relatively sparse, and have small probabilities for the occurrence of edges. Therefore, the choice of test should be considered carefully when very little is known about the underlying structure of the network.

This paper also presented a theoretic proof showing why the tests based on triangles and vees have power in at least some of the simulated networks. Proofs were provided for both tests based on the number of triangles and the number of vees. These proofs were based on the difference between the probabilities of observing each shape under the null and alternative hypothesis. Both proofs demonstrated the the difference was based on the difference between the within-group and between-group edge probability, thus showing the tests had power for networks generated under the null hypothesis.

Along with the simulation study, this study also showed that these tests could be applied to real life networks. All tests presented were able to detect community structure in both the political authors and Facebook networks demonstrating the usefulness of these tests are more than than just theoretical.

Researchers in future studies in this field may interested investigating the effectiveness of these test statistics on larger networks, or networks that display a community structure other than the one dense subgraph regime such as networks having more than two communities or dense subgraphs, or in networks that allow for nodes to have membership in multiple communities.

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APPENDIX

Table A.1. Simulated(size)power, repeat 500 times. $d = \frac{101}{100}$, $n = 50$

P=.10	(a, b) (1, 1)	(a, b) (6, 0.95)	(a, b) (11, 0.90)	(a, b) (16, 0.85)	(a, b) (21, 0.80)	(a, b) (26, 0.75)	(a, b) (31, 0.70)	(a, b) (36, 0.65)	(a, b) (41, 0.60)	(a, b) (46, 0.55)
T_{ez}	0.004	0.004	0.010	0.028	0.150	0.282	0.358	0.472	0.626	0.704
$\tilde{T}$	0.050	0.062	0.064	0.108	0.170	0.198	0.180	0.146	0.170	0.200
T^2	0.034	0.040	0.044	0.106	0.232	0.323	0.406	0.486	0.612	0.702

Table A.2. Simulated(size)power, repeat 500 times. $d = \frac{202}{100}$, $n = 50$

P=.10	(a, b) (2, 2)	(a, b) (7, 1.95)	(a, b) (12, 1.90)	(a, b) (17, 1.85)	(a, b) (22, 1.80)	(a, b) (27, 1.75)	(a, b) (32, 1.70)	(a, b) (37, 1.65)	(a, b) (42, 1.60)	(a, b) (47, 1.55)
T_{ez}	0.144	0.160	0.130	0.144	0.134	0.278	0.365	0.046	0.578	0.642
$\tilde{T}$	0.048	0.040	0.054	0.074	0.086	0.116	0.102	0.126	0.150	0.210
T^2	0.090	0.092	0.082	0.132	0.142	0.254	0.336	0.436	0.556	0.612

Table A.3. Simulated(size)power, repeat 500 times. $d = \frac{303}{100}$, $n = 50$

P=.10	(a, b) (3, 3)	(a, b) (8, 2.95)	(a, b) (13, 2.90)	(a, b) (18, 2.85)	(a, b) (23, 2.80)	(a, b) (28, 2.75)	(a, b) (33, 2.70)	(a, b) (38, 2.65)	(a, b) (43, 2.60)	(a, b) (48, 2.55)
T_{ez}	0.060	0.078	0.064	0.076	0.082	0.156	0.240	0.386	0.454	0.546
$\tilde{T}$	0.042	0.056	0.034	0.060	0.066	0.072	0.074	0.106	0.140	0.202
T^2	0.058	0.084	0.066	0.096	0.100	0.154	0.218	0.376	0.420	0.502

Table A.4. Simulated(size)power, repeat 500 times. $d = \frac{404}{100}$, $n = 50$

P=.10	(a, b) (4, 4)	(a, b) (9, 3.95)	(a, b) (14, 3.90)	(a, b) (19, 3.85)	(a, b) (24, 3.80)	(a, b) (29, 2.75)	(a, b) (34, 3.70)	(a, b) (39, 3.65)	(a, b) (44, 3.60)	(a, b) (49, 3.55)
T_{ez}	0.040	0.036	0.024	0.048	0.052	0.088	0.180	0.296	0.390	0.490
$\tilde{T}$	0.046	0.060	0.054	0.062	0.058	0.054	0.064	0.090	0.142	0.210
T^2	0.042	0.056	0.054	0.062	0.076	0.094	0.170	0.294	0.370	0.466

Table A.5. Simulated(size)power, repeat 500 times. $d = \frac{505}{100}$, $n = 50$

P=.10	(a, b) (5, 5)	(a, b) (10, 4.95)	(a, b) (15, 4.90)	(a, b) (20, 4.85)	(a, b) (25, 4.80)	(a, b) (30, 4.75)	(a, b) (35, 4.70)	(a, b) (40, 4.65)	(a, b) (45, 4.60)	(a, b) (50, 4.55)
T_{ez}	0.038	0.020	0.016	0.026	0.048	0.060	0.126	0.228	0.334	0.430
$\tilde{T}$	0.044	0.038	0.052	0.054	0.050	0.054	0.054	0.068	0.130	0.212
T^2	0.054	0.034	0.048	0.052	0.050	0.070	0.132	0.198	0.304	0.426

Table A.6. Simulated(size)power, repeat 500 times. $d = \frac{101}{100}$, $n = 100$

P=.10	(a, b) (1, 1)	(a, b) (11, 0.9)	(a, b) (21, 0.8)	(a, b) (31, 0.7)	(a, b) (41, 0.6)	(a, b) (51, 0.5)	(a, b) (61, 0.4)	(a, b) (71, 0.3)	(a, b) (81, 0.2)	(a, b) (91, 0.1)
T_{ez}	0.006	0.004	0.156	0.432	0.742	0.888	0.944	0.976	0.982	0.988
$\tilde{T}$	0.050	0.108	0.374	0.602	0.736	0.816	0.838	0.832	0.594	0.500
T^2	0.026	0.074	0.386	0.648	0.832	0.916	0.948	0.976	0.988	0.988

Table A.7. Simulated(size)power, repeat 500 times. $d = \frac{202}{100}$, $n = 100$

P=.10	(a, b) (2, 2)	(a, b) (12, 1.9)	(a, b) (22, 1.8)	(a, b) (32, 1.7)	(a, b) (42, 1.6)	(a, b) (52, 1.5)	(a, b) (62, 1.4)	(a, b) (72, 1.3)	(a, b) (82, 1.2)	(a, b) (92, 1.1)
T_{ez}	0.218	0.172	0.200	0.390	0.704	0.836	0.938	0.946	0.972	0.984
$\tilde{T}$	0.050	0.054	0.190	0.314	0.458	0.558	0.564	0.480	0.365	0.458
T^2	0.114	0.112	0.236	0.462	0.714	0.848	0.932	0.934	0.974	0.986

Table A.8. Simulated(size)power, repeat 500 times. $d = \frac{303}{100}$, $n = 100$

P=.10	(a, b) (3, 3)	(a, b) (13, 2.9)	(a, b) (23, 2.8)	(a, b) (33, 2.7)	(a, b) (43, 2.6)	(a, b) (53, 2.5)	(a, b) (63, 2.4)	(a, b) (73, 2.3)	(a, b) (83, 2.2)	(a, b) (93, 2.1)
T_{ez}	0.060	0.068	0.086	0.294	0.580	0.762	0.890	0.928	0.944	0.982
$\tilde{T}$	0.040	0.054	0.106	0.224	0.324	0.398	0.386	0.320	0.252	0.414
T^2	0.058	0.074	0.120	0.365	0.590	0.766	0.886	0.934	0.940	0.980

Table A.9. Simulated(size)power, repeat 500 times. $d = \frac{404}{100}$, $n = 100$

P=.10	(a, b) (4, 4)	(a, b) (14, 3.9)	(a, b) (24, 3.8)	(a, b) (34, 3.7)	(a, b) (44, 3.6)	(a, b) (54, 3.5)	(a, b) (64, 3.4)	(a, b) (74, 3.3)	(a, b) (84, 3.2)	(a, b) (94, 3.1)
T_{ez}	0.044	0.032	0.048	0.214	0.474	0.670	0.804	0.916	0.950	0.970
$\tilde{T}$	0.044	0.062	0.080	0.160	0.216	0.254	0.242	0.224	0.236	0.430
T^2	0.050	0.060	0.084	0.254	0.492	0.670	0.796	0.910	0.934	0.964

Table A.10. Simulated(size)power, repeat 500 times. $d = \frac{505}{100}$, $n = 100$

P=.10	(a, b) (5, 5)	(a, b) (15, 4.9)	(a, b) (25, 4.8)	(a, b) (35, 4.7)	(a, b) (45, 4.6)	(a, b) (55, 4.5)	(a, b) (65, 4.4)	(a, b) (75, 4.3)	(a, b) (85, 4.2)	(a, b) (95, 4.1)
T_{ez}	0.044	0.050	0.054	0.142	0.340	0.620	0.742	0.858	0.928	0.948
$\tilde{T}$	0.050	0.056	0.078	0.148	0.162	0.224	0.172	0.202	0.248	0.460
T^2	0.048	0.048	0.082	0.192	0.338	0.608	0.722	0.846	0.922	0.942

Table A.11. Simulated(size)power, repeat 500 times. $d = \frac{101}{100}$, $n = 150$

P=.10	(a, b) (1, 1)	(a, b) (11, 0.9)	(a, b) (21, 0.8)	(a, b) (31, 0.7)	(a, b) (41, 0.6)	(a, b) (51, 0.5)	(a, b) (61, 0.4)	(a, b) (71, 0.3)	(a, b) (81, 0.2)	(a, b) (91, 0.1)	(a, b) (101, 0.0)
T_{ez}	0.004	0.016	0.180	0.494	0.800	0.934	0.980	0.990	0.998	0.986	0.742
$\tilde{T}$	0.056	0.158	0.506	0.764	0.914	0.964	0.982	0.992	1.000	0.998	1.000
T^2	0.116	0.116	0.496	0.808	0.930	0.984	0.986	1.000	1.000	1.000	1.000

Table A.12. Simulated(size)power, repeat 500 times. $d = \frac{202}{100}$, $n = 150$

P=.10	(a, b) (2, 2)	(a, b) (12, 1.9)	(a, b) (22, 1.8)	(a, b) (32, 1.7)	(a, b) (42, 1.6)	(a, b) (52, 1.5)	(a, b) (62, 1.4)	(a, b) (72, 1.3)	(a, b) (82, 1.2)	(a, b) (92, 1.1)	(a,b) (102, 1.0)
T_{ez}	0.224	0.206	0.194	0.462	0.732	0.886	0.970	0.976	0.994	0.998	0.994
$\tilde{T}$	0.054	0.108	0.276	0.510	0.714	0.802	0.900	0.930	0.946	0.942	0.928
T^2	0.116	0.156	0.310	0.618	0.822	0.906	0.966	0.976	0.996	0.998	0.994

Table A.13. Simulated(size)power, repeat 500 times. $d = \frac{303}{100}$, $n = 150$

P=.10	(a, b) (3, 3)	(a, b) (13, 2.9)	(a, b) (23, 2.8)	(a, b) (33, 2.7)	(a, b) (43, 2.6)	(a, b) (53, 2.5)	(a, b) (63, 2.4)	(a, b) (73, 2.3)	(a, b) (83, 2.2)	(a, b) (93, 2.1)	(a, b) (103, 2.0)
T_{ez}	0.046	0.052	0.084	0.288	0.622	0.836	0.936	0.964	0.986	1.000	0.994
$\tilde{T}$	0.040	0.076	0.168	0.388	0.588	0.650	0.760	0.818	0.838	0.852	0.810
T^2	0.058	0.060	0.180	0.454	0.734	0.856	0.934	0.964	0.988	1.000	0.996

Table A.14. Simulated(size)power, repeat 500 times. $d = \frac{404}{100}$, $n = 150$

P=.10	(a, b) (4, 4)	(a, b) (14, 3.9)	(a, b) (24, 3.8)	(a, b) (34, 3.7)	(a, b) (44, 3.6)	(a, b) (54, 3.5)	(a, b) (64, 3.4)	(a, b) (74, 3.3)	(a, b) (84, 3.2)	(a, b) (94, 3.1)	(a,b) (94, 3.0)
T_{ez}	0.052	0.036	0.064	0.244	0.496	0.718	0.872	0.926	0.976	0.976	0.996
$\tilde{T}$	0.042	0.064	0.094	0.290	0.436	0.532	0.620	0.694	0.758	0.710	0.662
T^2	0.052	0.058	0.088	0.350	0.606	0.764	0.886	0.926	0.974	0.972	0.994

Table A.15. Simulated(size)power, repeat 500 times. $d = \frac{505}{100}$, $n = 150$

P=.10	(a, b) (5, 5)	(a, b) (15, 4.9)	(a, b) (25, 4.8)	(a, b) (35, 4.7)	(a, b) (45, 4.6)	(a, b) (55, 4.5)	(a, b) (65, 4.4)	(a, b) (75, 4.3)	(a, b) (85, 4.2)	(a, b) (95, 4.1)	(a, b) ((105, 4.0)
T_{ez}	0.048	0.042	0.064	0.150	0.382	0.654	0.842	0.938	0.970	0.982	0.992
$\tilde{T}$	0.040	0.056	0.106	0.186	0.340	0.444	0.558	0.640	0.644	0.626	0.526
T^2	0.046	0.042	0.100	0.250	0.464	0.682	0.854	0.938	0.972	0.980	0.990

Table A.16. Simulated(size)power, repeat 500 times. $d = \frac{409}{400}$, $n = 50$

P=.15	(a, b) (1, 1)	(a, b) ($\frac{49}{9}$, 0.9)	(a, b) ($\frac{89}{9}$, 0.8)	(a, b) ($\frac{43}{3}$, 0.7)	(a, b) ($\frac{169}{9}$, 0.6)	(a, b) ($\frac{209}{9}$, 0.5)	(a, b) ($\frac{83}{3}$, 0.4)	(a, b) ($\frac{289}{9}$, 0.3)	(a, b) ($\frac{329}{9}$, 0.2)	(a, b) (41, 0.1)	(a, b) ($\frac{409}{9}$, 0.0)
T_{ez}	0.004	0.008	0.024	0.080	0.210	0.402	0.552	0.678	0.696	0.744	0.656
$\tilde{T}$	0.040	0.058	0.130	0.252	0.420	0.504	0.500	0.484	0.364	0.330	0.562
T^2	0.026	0.040	0.100	0.252	0.458	0.618	0.748	0.800	0.816	0.736	0.620

Table A.17. Simulated(size)power, repeat 500 times. $d = \frac{409}{200}$, $n = 50$

P=.15	(a, b) (2, 2)	(a, b) ($\frac{58}{9}$, 1.9)	(a, b) ($\frac{98}{9}$, 1.8)	(a, b) ($\frac{46}{3}$, 1.7)	(a, b) ($\frac{178}{9}$, 1.6)	(a, b) ($\frac{218}{9}$, 1.5)	(a, b) ($\frac{86}{3}$, 1.4)	(a, b) ($\frac{298}{9}$, 1.3)	(a, b) ($\frac{338}{9}$, 1.2)	(a, b) (42, 1.1)	(a, b) ($\frac{418}{9}$, 1.0)
T_{ez}	0.132	0.114	0.128	0.116	0.228	0.454	0.604	0.758	0.824	0.888	0.908
$\tilde{T}$	0.042	0.042	0.074	0.134	0.184	0.218	0.236	0.234	0.194	0.316	0.520
T^2	0.078	0.094	0.112	0.156	0.268	0.472	0.626	0.770	0.824	0.882	0.900

Table A.18. Simulated(size)power, repeat 500 times. $d = \frac{1227}{400}$, $n = 50$

P=.15	(a, b) (3, 3)	(a, b) ($\frac{67}{9}$, 2.9)	(a, b) ($\frac{107}{9}$, 2.8)	(a, b) ($\frac{49}{3}$, 2.7)	(a, b) ($\frac{187}{9}$, 2.6)	(a, b) ($\frac{227}{9}$, 2.5)	(a, b) ($\frac{89}{3}$, 2.4)	(a, b) ($\frac{307}{9}$, 2.3)	(a, b) ($\frac{347}{9}$, 2.2)	(a, b) (43, 2.1)	(a, b) ($\frac{427}{9}$, 2.0)
T_{ez}	0.054	0.064	0.078	0.090	0.184	0.314	0.524	0.654	0.780	0.864	0.888
$\tilde{T}$	0.040	0.034	0.066	0.098	0.106	0.130	0.130	0.120	0.174	0.314	0.456
T^2	0.058	0.066	0.098	0.120	0.194	0.350	0.508	0.638	0.776	0.852	0.878

Table A.19. Simulated(size)power, repeat 500 times. $d = \frac{409}{100}$, $n = 50$

P=.15	(a, b) (4, 4)	(a, b) ($\frac{76}{9}$, 3.9)	(a, b) ($\frac{116}{9}$, 3.8)	(a, b) ($\frac{52}{3}$, 3.7)	(a, b) ($\frac{196}{9}$, 3.6)	(a, b) ($\frac{236}{9}$, 3.5)	(a, b) ($\frac{92}{3}$, 3.4)	(a, b) ($\frac{316}{9}$, 3.3)	(a, b) ($\frac{356}{9}$, 3.2)	(a, b) (43, 3.1)	(a, b) ($\frac{436}{9}$, 3.0)
T_{ez}	0.040	0.018	0.036	0.038	0.128	0.238	0.368	0.546	0.644	0.790	0.866
$\tilde{T}$	0.042	0.050	0.062	0.080	0.090	0.096	0.092	0.106	0.168	0.332	0.488
T^2	0.044	0.062	0.070	0.066	0.148	0.242	0.358	0.514	0.624	0.756	0.854

Table A.20. Simulated(size)power, repeat 500 times. $d = \frac{409}{80}$, $n = 50$

P=.15	(a, b) (5, 5)	(a, b) ($\frac{85}{9}$, 4.9)	(a, b) ($\frac{125}{9}$, 4.8)	(a, b) ($\frac{55}{3}$, 4.7)	(a, b) ($\frac{205}{9}$, 4.6)	(a, b) ($\frac{245}{9}$, 4.5)	(a, b) ($\frac{95}{3}$, 4.4)	(a, b) ($\frac{325}{9}$, 4.3)	(a, b) ($\frac{365}{9}$, 4.2)	(a, b) (45, 4.1)	(a, b) ($\frac{445}{9}$, 4.0)
T_{ez}	0.048	0.018	0.022	0.030	0.046	0.208	0.310	0.490	0.636	0.744	0.796
$\tilde{T}$	0.050	0.058	0.032	0.058	0.074	0.094	0.078	0.150	0.184	0.332	0.502
T^2	0.056	0.050	0.040	0.058	0.086	0.210	0.262	0.468	0.586	0.708	0.786

Table A.21. Simulated(size)power, repeat 500 times. $d = \frac{409}{400}$, $n = 100$

P=.15	(a, b) (1, 1)	(a, b) ($\frac{49}{9}$, 0.9)	(a, b) ($\frac{89}{9}$, 0.8)	(a, b) ($\frac{43}{3}$, 0.7)	(a, b) ($\frac{169}{9}$, 0.6)	(a, b) ($\frac{209}{9}$, 0.5)	(a, b) ($\frac{83}{3}$, 0.4)	(a, b) ($\frac{289}{9}$, 0.3)	(a, b) ($\frac{329}{9}$, 0.2)	(a, b) (41, 0.1)	(a, b) ($\frac{409}{9}$, 0.0)
T_{ez}	0.006	0.012	0.034	0.116	0.276	0.458	0.576	0.668	0.638	0.522	0.028
$\tilde{T}$	0.042	0.070	0.266	0.556	0.796	0.900	0.962	0.986	0.988	1.000	1.000
T^2	0.020	0.050	0.244	0.544	0.800	0.900	0.958	0.990	0.994	1.000	1.000

Table A.22. Simulated(size)power, repeat 500 times. $d = \frac{409}{200}$, $n = 100$

P=.15	(a, b) (2, 2)	(a, b) ($\frac{58}{9}$, 1.9)	(a, b) ($\frac{98}{9}$, 1.8)	(a, b) ($\frac{46}{3}$, 1.7)	(a, b) ($\frac{178}{9}$, 1.6)	(a, b) ($\frac{218}{9}$, 1.5)	(a, b) ($\frac{86}{3}$, 1.4)	(a, b) ($\frac{298}{9}$, 1.3)	(a, b) ($\frac{338}{9}$, 1.2)	(a, b) (42, 1.1)	(a, b) ($\frac{418}{9}$, 1.0)
T_{ez}	0.214	0.160	0.162	0.146	0.270	0.476	0.686	0.816	0.920	0.958	0.984
$\tilde{T}$	0.054	0.050	0.112	0.348	0.484	0.668	0.756	0.860	0.916	0.928	0.926
T^2	0.114	0.104	0.150	0.334	0.530	0.728	0.830	0.926	0.960	0.982	0.986

Table A.23. Simulated(size)power, repeat 500 times. $d = \frac{1227}{400}$, $n = 100$

P=.15	(a, b) (3, 3)	(a, b) ($\frac{67}{9}$, 2.9)	(a, b) ($\frac{107}{9}$, 2.8)	(a, b) ($\frac{49}{3}$, 2.7)	(a, b) ($\frac{187}{9}$, 2.6)	(a, b) ($\frac{227}{9}$, 2.5)	(a, b) ($\frac{89}{3}$, 2.4)	(a, b) ($\frac{307}{9}$, 2.3)	(a, b) ($\frac{347}{9}$, 2.2)	(a, b) (43, 2.1)	(a, b) ($\frac{427}{9}$, 2.0)
T_{ez}	0.060	0.060	0.064	0.084	0.174	0.344	0.598	0.732	0.876	0.932	0.970
$\tilde{T}$	0.050	0.044	0.058	0.182	0.348	0.478	0.606	0.702	0.782	0.810	0.856
T^2	0.056	0.058	0.084	0.170	0.368	0.526	0.722	0.824	0.920	0.944	0.976

Table A.24. Simulated(size)power, repeat 500 times. $d = \frac{409}{100}$, $n = 100$

P=.15	(a, b) (4, 4)	(a, b) ($\frac{76}{9}$, 3.9)	(a, b) ($\frac{116}{9}$, 3.8)	(a, b) ($\frac{52}{3}$, 3.7)	(a, b) ($\frac{196}{9}$, 3.6)	(a, b) ($\frac{236}{9}$, 3.5)	(a, b) ($\frac{92}{3}$, 3.4)	(a, b) ($\frac{316}{9}$, 3.3)	(a, b) ($\frac{356}{9}$, 3.2)	(a, b) (43, 3.1)	(a, b) ($\frac{436}{9}$, 3.0)
T_{ez}	0.052	0.040	0.034	0.068	0.114	0.274	0.492	0.656	0.822	0.896	0.944
$\tilde{T}$	0.048	0.048	0.058	0.118	0.234	0.356	0.470	0.562	0.690	0.660	0.726
T^2	0.048	0.060	0.052	0.120	0.228	0.398	0.626	0.744	0.872	0.902	0.950

Table A.25. Simulated(size)power, repeat 500 times. $d = \frac{409}{80}$, $n = 100$

P=.15	(a, b) (5, 5)	(a, b) ($\frac{85}{9}$, 4.9)	(a, b) ($\frac{125}{9}$, 4.8)	(a, b) ($\frac{55}{3}$, 4.7)	(a, b) ($\frac{205}{9}$, 4.6)	(a, b) ($\frac{245}{9}$, 4.5)	(a, b) ($\frac{95}{3}$, 4.4)	(a, b) ($\frac{325}{9}$, 4.3)	(a, b) ($\frac{365}{9}$, 4.2)	(a, b) (45, 4.1)	(a, b) ($\frac{445}{9}$, 4.0)
T_{ez}	0.048	0.052	0.044	0.064	0.086	0.226	0.364	0.522	0.712	0.836	0.904
$\tilde{T}$	0.046	0.044	0.070	0.092	0.180	0.298	0.356	0.454	0.572	0.574	0.568
T^2	0.056	0.046	0.062	0.098	0.170	0.360	0.460	0.628	0.776	0.842	0.920

Table A.26. Simulated(size)power, repeat 500 times. $d = \frac{409}{400}$, $n = 150$

P=.15	(a, b) (1, 1)	(a, b) ($\frac{49}{9}$, 0.9)	(a, b) ($\frac{89}{9}$, 0.8)	(a, b) ($\frac{43}{3}$, 0.7)	(a, b) ($\frac{169}{9}$, 0.6)	(a, b) ($\frac{209}{9}$, 0.5)	(a, b) ($\frac{83}{3}$, 0.4)	(a, b) ($\frac{289}{9}$, 0.3)	(a, b) ($\frac{329}{9}$, 0.2)	(a, b) (41, 0.1)	(a, b) ($\frac{409}{9}$, 0.0)
T_{ez}	0.004	0.004	0.050	0.144	0.286	0.472	0.614	0.678	0.648	0.396	0.024
$\tilde{T}$	0.048	0.072	0.368	0.752	0.896	0.968	0.990	1.000	1.000	1.000	1.000
T^2	0.032	0.050	0.346	0.748	0.876	0.970	0.992	1.000	1.000	1.000	1.000

Table A.27. Simulated(size)power, repeat 500 times. $d = \frac{409}{200}$, $n = 150$

P=.15	(a, b) (2, 2)	(a, b) ($\frac{58}{9}$, 1.9)	(a, b) ($\frac{98}{9}$, 1.8)	(a, b) ($\frac{46}{3}$, 1.7)	(a, b) ($\frac{178}{9}$, 1.6)	(a, b) ($\frac{218}{9}$, 1.5)	(a, b) ($\frac{86}{3}$, 1.4)	(a, b) ($\frac{298}{9}$, 1.3)	(a, b) ($\frac{338}{9}$, 1.2)	(a, b) (42, 1.1)	(a, b) ($\frac{418}{9}$, 1.0)
T_{ez}	0.230	0.206	0.160	0.164	0.278	0.484	0.724	0.866	0.984	0.980	0.990
$\tilde{T}$	0.044	0.082	0.182	0.426	0.646	0.818	0.910	0.962	0.980	0.998	0.992
T^2	0.130	0.138	0.188	0.432	0.678	0.836	0.934	0.972	0.994	1.000	0.998

Table A.28. Simulated(size)power, repeat 500 times. $d = \frac{1227}{400}$, $n = 150$

P=.15	(a, b) (3, 3)	(a, b) $(\frac{67}{9}, 2.9)$	(a, b) $(\frac{107}{9}, 2.8)$	(a, b) $(\frac{49}{3}, 2.7)$	(a, b) $(\frac{187}{9}, 2.6)$	(a, b) $(\frac{227}{9}, 2.5)$	(a, b) $(\frac{89}{3}, 2.4)$	(a, b) $(\frac{307}{9}, 2.3)$	(a, b) $(\frac{347}{9}, 2.2)$	(a, b) (43, 2.1)	(a, b) $(\frac{427}{9}, 2.0)$
T_{ez}	0.048	0.048	0.038	0.060	0.178	0.384	0.598	0.794	0.918	0.956	0.990
$\tilde{T}$	0.050	0.056	0.108	0.250	0.476	0.658	0.780	0.888	0.940	0.962	0.994
T^2	0.054	0.052	0.074	0.192	0.464	0.706	0.848	0.924	0.964	0.990	0.998

Table A.29. Simulated(size)power, repeat 500 times. $d = \frac{409}{100}$, $n = 150$

P=.15	(a, b) (4, 4)	(a, b) $(\frac{76}{9}, 3.9)$	(a, b) $(\frac{116}{9}, 3.8)$	(a, b) $(\frac{52}{3}, 3.7)$	(a, b) $(\frac{196}{9}, 3.6)$	(a, b) $(\frac{236}{9}, 3.5)$	(a, b) $(\frac{92}{3}, 3.4)$	(a, b) $(\frac{316}{9}, 3.3)$	(a, b) $(\frac{356}{9}, 3.2)$	(a, b) (43, 3.1)	(a, b) $(\frac{436}{9}, 3.0)$
T_{ez}	0.046	0.050	0.024	0.066	0.094	0.280	0.474	0.680	0.854	0.956	0.966
$\tilde{T}$	0.046	0.054	0.076	0.188	0.356	0.550	0.678	0.796	0.874	0.930	0.928
T^2	0.050	0.062	0.064	0.164	0.350	0.582	0.750	0.858	0.940	0.982	0.982

Table A.30. Simulated(size)power, repeat 500 times. $d = \frac{409}{80}$, $n = 150$

P=.15	(a, b) (5, 5)	(a, b) $(\frac{85}{9}, 4.9)$	(a, b) $(\frac{125}{9}, 4.8)$	(a, b) $(\frac{55}{3}, 4.7)$	(a, b) $(\frac{205}{9}, 4.6)$	(a, b) $(\frac{245}{9}, 4.5)$	(a, b) $(\frac{95}{3}, 4.4)$	(a, b) $(\frac{325}{9}, 4.3)$	(a, b) $(\frac{365}{9}, 4.2)$	(a, b) (45, 4.1)	(a, b) $(\frac{445}{9}, 4.0)$
T_{ez}	0.044	0.036	0.046	0.042	0.080	0.184	0.392	0.582	0.774	0.894	0.932
$\tilde{T}$	0.046	0.058	0.088	0.144	0.252	0.430	0.552	0.692	0.780	0.876	0.870
T^2	0.054	0.048	0.084	0.114	0.224	0.436	0.630	0.784	0.874	0.950	0.968

Table A.31. Simulated(size)power, repeat 500 times. $d = \frac{26}{25}$, $n = 50$

P=.20	(a, b) (1, 1)	(a, b) (3.5, 0.9)	(a, b) (6, 0.8)	(a, b) (8.5, 0.7)	(a, b) (11, 0.6)	(a, b) (13.5, 0.5)	(a, b) (16, 0.4)	(a, b) (18.5, 0.3)	(a, b) (21, 0.2)	(a, b) (23.5, 0.1)	(a, b) (26, 0.0)
T_{ez}	0.000	0.000	0.018	0.028	0.072	0.124	0.164	0.240	0.234	0.134	0.008
$\tilde{T}$	0.050	0.028	0.110	0.262	0.408	0.550	0.732	0.798	0.860	0.932	0.952
T^2	0.028	0.024	0.198	0.236	0.390	0.546	0.756	0.832	0.880	0.940	0.952

Table A.32. Simulated(size)power, repeat 500 times. $d = \frac{52}{25}$, $n = 50$

P=.20	(a, b) (2, 2)	(a, b) (4.5, 1.9)	(a, b) (7, 1.8)	(a, b) (9.5, 1.7)	(a, b) (12, 1.6)	(a, b) (14.5, 1.5)	(a, b) (17, 1.4)	(a, b) (19.5, 1.3)	(a, b) (22, 1.2)	(a, b) (24.5, 1.1)	(a, b) (27, 1.0)
T_{ez}	0.128	0.136	0.126	0.092	0.110	0.152	0.268	0.408	0.576	0.696	0.796
$\tilde{T}$	0.044	0.032	0.044	0.096	0.212	0.284	0.382	0.464	0.532	0.490	0.556
T^2	0.090	0.084	0.094	0.100	0.196	0.336	0.472	0.614	0.736	0.808	0.882

Table A.33. Simulated(size)power, repeat 500 times. $d = \frac{78}{25}$, $n = 50$

P=.20	(a, b) (3, 3)	(a, b) (5.5, 2.9)	(a, b) (8, 2.8)	(a, b) (10.5, 2.7)	(a, b) (13, 2.6)	(a, b) (15.5, 2.5)	(a, b) (18, 2.4)	(a, b) (20.5, 2.3)	(a, b) (23, 2.2)	(a, b) (25.5, 2.1)	(a, b) (28, 2.0)
T_{ez}	0.046	0.056	0.044	0.068	0.078	0.112	0.218	0.354	0.468	0.644	0.766
$\tilde{T}$	0.046	0.048	0.070	0.078	0.126	0.224	0.230	0.248	0.294	0.268	0.270
T^2	0.060	0.058	0.074	0.078	0.130	0.232	0.306	0.440	0.540	0.682	0.764

Table A.34. Simulated(size)power, repeat 500 times. $d = \frac{104}{25}$, $n = 50$

P=.20	(a, b) (4, 4)	(a, b) (6.5, 3.9)	(a, b) (9, 3.8)	(a, b) (12.5, 3.7)	(a, b) (14, 3.6)	(a, b) (16.5, 3.5)	(a, b) (19, 3.4)	(a, b) (21.5, 3.3)	(a, b) (24, 3.2)	(a, b) (26.5, 3.1)	(a, b) (29, 3.0)
T_{ez}	0.042	0.024	0.040	0.024	0.036	0.072	0.114	0.280	0.394	0.528	0.658
$\tilde{T}$	0.048	0.044	0.052	0.046	0.084	0.124	0.128	0.202	0.178	0.184	0.192
T^2	0.056	0.038	0.050	0.044	0.076	0.112	0.176	0.352	0.440	0.556	0.684

Table A.35. Simulated(size)power, repeat 500 times. $d = \frac{130}{25}$, $n = 50$

P=.20	(a, b) (5, 5)	(a, b) (7.5, 4.9)	(a, b) (10, 4.8)	(a, b) (13.5, 4.7)	(a, b) (15, 4.6)	(a, b) (17.5, 4.5)	(a, b) (20, 4.4)	(a, b) (22.5, 4.3)	(a, b) (25, 4.2)	(a, b) (27.5, 4.1)	(a, b) (30, 4.0)
T_{ez}	0.044	0.010	0.014	0.018	0.036	0.034	0.082	0.206	0.286	0.440	0.574
$\tilde{T}$	0.044	0.042	0.034	0.052	0.076	0.080	0.098	0.110	0.126	0.124	0.148
T^2	0.044	0.034	0.036	0.038	0.066	0.068	0.126	0.192	0.298	0.436	0.554

Table A.36. Simulated(size)power, repeat 500 times. $d = \frac{26}{25}$, $n = 100$

P=.20	(a, b) (1, 1)	(a, b) (3.5, 0.9)	(a, b) (6, 0.8)	(a, b) (8.5, 0.7)	(a, b) (11, 0.6)	(a, b) (13.5, 0.5)	(a, b) (16, 0.4)	(a, b) (18.5, 0.3)	(a, b) (21, 0.2)	(a, b) (23.5, 0.1)	(a, b) (26, 0.0)
T_{ez}	0.002	0.002	0.016	0.056	0.094	0.180	0.232	0.998	0.214	0.118	0.012
$\tilde{T}$	0.046	0.070	0.204	0.506	0.750	0.882	0.956	0.986	0.860	1.000	1.000
T^2	0.024	0.042	0.172	0.440	0.734	0.882	0.950	0.988	0.880	1.000	1.000

Table A.37. Simulated(size)power, repeat 500 times. $d = \frac{52}{25}$, $n = 100$

P=.20	(a, b) (2, 2)	(a, b) (4.5, 1.9)	(a, b) (7, 1.8)	(a, b) (9.5, 1.7)	(a, b) (12, 1.6)	(a, b) (14.5, 1.5)	(a, b) (17, 1.4)	(a, b) (19.5, 1.3)	(a, b) (22, 1.2)	(a, b) (24.5, 1.1)	(a, b) (27, 1.0)
T_{ez}	0.200	0.194	0.190	0.126	0.134	0.192	0.284	0.474	0.600	0.762	0.856
$\tilde{T}$	0.046	0.048	0.096	0.238	0.422	0.600	0.758	0.842	0.920	0.958	0.980
T^2	0.110	0.132	0.150	0.232	0.402	0.588	0.772	0.870	0.924	0.974	0.986

Table A.38. Simulated(size)power, repeat 500 times. $d = \frac{78}{25}$, $n = 100$

P=.20	(a, b) (3, 3)	(a, b) (5.5, 2.9)	(a, b) (8, 2.8)	(a, b) (10.5, 2.7)	(a, b) (13, 2.6)	(a, b) (15.5, 2.5)	(a, b) (18, 2.4)	(a, b) (20.5, 2.3)	(a, b) (23, 2.2)	(a, b) (25.5, 2.1)	(a, b) (28, 2.0)
T_{ez}	0.056	0.048	0.062	0.056	0.078	0.102	0.240	0.372	0.566	0.690	0.810
$\tilde{T}$	0.042	0.062	0.090	0.112	0.232	0.396	0.612	0.708	0.802	0.886	0.898
T^2	0.058	0.062	0.072	0.112	0.240	0.386	0.618	0.748	0.856	0.918	0.954

Table A.39. Simulated(size)power, repeat 500 times. $d = \frac{104}{25}$, $n = 100$

P=.20	(a, b) (4, 4)	(a, b) (6.5, 3.9)	(a, b) (9, 3.8)	(a, b) (12.5, 3.7)	(a, b) (14, 3.6)	(a, b) (16.5, 3.5)	(a, b) (19, 3.4)	(a, b) (21.5, 3.3)	(a, b) (24, 3.2)	(a, b) (26.5, 3.1)	(a, b) (29, 3.0)
T_{ez}	0.052	0.044	0.040	0.044	0.050	0.096	0.134	0.246	0.414	0.588	0.732
$\tilde{T}$	0.042	0.044	0.066	0.096	0.170	0.274	0.388	0.558	0.708	0.750	0.832
T^2	0.044	0.050	0.074	0.082	0.146	0.264	0.396	0.564	0.752	0.818	0.900

Table A.40. Simulated(size)power, repeat 500 times. $d = \frac{130}{25}$, $n = 100$

P=.20	(a, b) (5, 5)	(a, b) (7.5, 4.9)	(a, b) (10, 4.8)	(a, b) (13.5, 4.7)	(a, b) (15, 4.6)	(a, b) (17.5, 4.5)	(a, b) (20, 4.4)	(a, b) (22.5, 4.3)	(a, b) (25, 4.2)	(a, b) (27.5, 4.1)	(a, b) (30, 4.0)
T_{ez}	0.044	0.028	0.038	0.028	0.036	0.060	0.098	0.218	0.340	0.458	0.652
$\tilde{T}$	0.046	0.038	0.060	0.080	0.160	0.198	0.332	0.430	0.578	0.638	0.688
T^2	0.050	0.038	0.052	0.072	0.128	0.182	0.322	0.462	0.632	0.706	0.824

Table A.41. Simulated(size)power, repeat 500 times. $d = \frac{26}{25}$, $n = 150$

P=.20	(a, b) (1, 1)	(a, b) (3.5, 0.9)	(a, b) (6, 0.8)	(a, b) (8.5, 0.7)	(a, b) (11, 0.6)	(a, b) (13.5, 0.5)	(a, b) (16, 0.4)	(a, b) (18.5, 0.3)	(a, b) (21, 0.2)	(a, b) (23.5, 0.1)	(a, b) (26, 0.0)
T_{ez}	0.004	0.004	0.022	0.064	0.100	0.194	0.244	0.230	0.216	0.102	0.012
$\tilde{T}$	0.044	0.074	0.274	0.670	0.868	0.974	0.990	0.998	1.000	1.000	1.000
T^2	0.022	0.044	0.232	0.636	0.852	0.974	0.990	0.998	1.000	1.000	1.000

Table A.42. Simulated(size)power, repeat 500 times. $d = \frac{52}{25}$, $n = 150$

P=.20	(a, b) (2, 2)	(a, b) (4.5, 1.9)	(a, b) (7, 1.8)	(a, b) (9.5, 1.7)	(a, b) (12, 1.6)	(a, b) (14.5, 1.5)	(a, b) (17, 1.4)	(a, b) (19.5, 1.3)	(a, b) (22, 1.2)	(a, b) (24.5, 1.1)	(a, b) (27, 1.0)
T_{ez}	0.208	0.194	0.166	0.142	0.142	0.220	0.348	0.476	0.658	0.738	0.882
$\tilde{T}$	0.046	0.048	0.116	0.312	0.530	0.738	0.924	0.952	0.974	0.992	0.996
T^2	0.132	0.116	0.130	0.290	0.514	0.726	0.918	0.952	0.976	0.992	0.994

Table A.43. Simulated(size)power, repeat 500 times. $d = \frac{78}{25}$, $n = 150$

P=.20	(a, b) (3, 3)	(a, b) (5.5, 2.9)	(a, b) (8, 2.8)	(a, b) (10.5, 2.7)	(a, b) (13, 2.6)	(a, b) (15.5, 2.5)	(a, b) (18, 2.4)	(a, b) (20.5, 2.3)	(a, b) (23, 2.2)	(a, b) (25.5, 2.1)	(a, b) (28, 2.0)
T_{ez}	0.054	0.068	0.052	0.046	0.078	0.140	0.236	0.386	0.596	0.752	0.886
$\tilde{T}$	0.048	0.060	0.070	0.208	0.368	0.580	0.750	0.858	0.926	0.960	0.990
T^2	0.058	0.080	0.050	0.178	0.356	0.550	0.762	0.880	0.942	0.976	0.992

Table A.44. Simulated(size)power, repeat 500 times. $d = \frac{104}{25}$, $n = 150$

P=.20	(a, b) (4, 4)	(a, b) (6.5, 3.9)	(a, b) (9, 3.8)	(a, b) (12.5, 3.7)	(a, b) (14, 3.6)	(a, b) (16.5, 3.5)	(a, b) (19, 3.4)	(a, b) (21.5, 3.3)	(a, b) (24, 3.2)	(a, b) (26.5, 3.1)	(a, b) (29, 3.0)
T_{ez}	0.050	0.038	0.044	0.050	0.058	0.086	0.132	0.284	0.420	0.682	0.786
$\tilde{T}$	0.044	0.050	0.072	0.118	0.262	0.426	0.604	0.792	0.866	0.924	0.946
T^2	0.050	0.050	0.082	0.100	0.228	0.402	0.594	0.790	0.894	0.948	0.972

Table A.45. Simulated(size)power, repeat 500 times. $d = \frac{130}{25}$, $n = 150$

P=.20	(a, b) (5, 5)	(a, b) (7.5, 4.9)	(a, b) (10, 4.8)	(a, b) (13.5, 4.7)	(a, b) (15, 4.6)	(a, b) (17.5, 4.5)	(a, b) (20, 4.4)	(a, b) (22.5, 4.3)	(a, b) (25, 4.2)	(a, b) (27.5, 4.1)	(a, b) (30, 4.0)
T_{ez}	0.044	0.044	0.038	0.028	0.044	0.060	0.096	0.224	0.322	0.546	0.708
$\tilde{T}$	0.050	0.060	0.066	0.078	0.180	0.332	0.482	0.634	0.790	0.844	0.910
T^2	0.050	0.060	0.046	0.072	0.144	0.284	0.460	0.640	0.774	0.870	0.934

Table A.46. Simulated(size)power, repeat 500 times. $d = \frac{17}{16}$, $n = 50$

P=.25	(a, b) (1, 1)	(a, b) (2.6, 0.9)	(a, b) (4.2, 0.8)	(a, b) (5.8, 0.7)	(a, b) (7.4, 0.6)	(a, b) (9, 0.5)	(a, b) (10.6, 0.4)	(a, b) (12.2, 0.3)	(a, b) (13.8, 0.2)	(a, b) (15.4, 0.1)	(a, b) (17, 0.0)
T_{ez}	0.002	0.006	0.010	0.026	0.038	0.078	0.064	0.092	0.086	0.040	0.012
$\tilde{T}$	0.048	0.040	0.094	0.172	0.406	0.580	0.710	0.814	0.890	0.970	0.992
T^2	0.026	0.030	0.076	0.146	0.344	0.536	0.684	0.810	0.902	0.962	0.990

Table A.47. Simulated(size)power, repeat 500 times. $d = \frac{17}{8}$, $n = 50$

P=.25	(a, b) (2, 2)	(a, b) (3.6, 1.9)	(a, b) (5.2, 1.8)	(a, b) (6.8, 1.7)	(a, b) (8.4, 1.6)	(a, b) (10, 1.5)	(a, b) (11.6, 1.4)	(a, b) (13.2, 1.3)	(a, b) (14.8, 1.2)	(a, b) (16.4, 1.1)	(a, b) (18, 1.0)
T_{ez}	0.148	0.148	0.102	0.110	0.102	0.088	0.156	0.152	0.254	0.336	0.464
$\tilde{T}$	0.044	0.024	0.066	0.074	0.178	0.236	0.376	0.464	0.574	0.702	0.714
T^2	0.096	0.086	0.070	0.084	0.184	0.218	0.392	0.472	0.646	0.762	0.832

Table A.48. Simulated(size)power, repeat 500 times. $d = \frac{51}{16}$, $n = 50$

P=.25	(a, b) (3, 3)	(a, b) (4.6, 2.9)	(a, b) (6.2, 2.8)	(a, b) (7.8, 2.7)	(a, b) (9.4, 2.6)	(a, b) (11, 2.5)	(a, b) (12.6, 2.4)	(a, b) (14.2, 2.3)	(a, b) (15.8, 2.2)	(a, b) (17.4, 2.1)	(a, b) (19, 2.0)
T_{ez}	0.048	0.048	0.056	0.054	0.058	0.048	0.092	0.110	0.218	0.296	0.440
$\tilde{T}$	0.060	0.034	0.050	0.066	0.112	0.152	0.224	0.284	0.370	0.424	0.468
T^2	0.058	0.060	0.060	0.062	0.120	0.104	0.240	0.288	0.464	0.556	0.666

Table A.49. Simulated(size)power, repeat 500 times. $d = \frac{17}{4}$, $n = 50$

P=.25	(a, b) (4, 4)	(a, b) (5.6, 3.9)	(a, b) (7.2, 3.8)	(a, b) (8.8, 3.7)	(a, b) (10.4, 3.6)	(a, b) (12, 3.5)	(a, b) (13.6, 3.4)	(a, b) (15.2, 3.3)	(a, b) (16.8, 3.2)	(a, b) (18.4, 3.1)	(a, b) (20, 3.0)
T_{ez}	0.044	0.060	0.046	0.024	0.024	0.044	0.062	0.076	0.150	0.244	0.350
$\tilde{T}$	0.054	0.072	0.048	0.066	0.080	0.110	0.132	0.212	0.270	0.282	0.310
T^2	0.050	0.068	0.058	0.056	0.060	0.086	0.134	0.202	0.298	0.384	0.472

Table A.50. Simulated(size)power, repeat 500 times. $d = \frac{85}{16}$, $n = 50$

P=.25	(a, b) (5, 5)	(a, b) (6.6, 4.9)	(a, b) (8.2, 4.8)	(a, b) (9.8, 4.7)	(a, b) (11.4, 4.6)	(a, b) (13, 4.5)	(a, b) (14.6, 4.4)	(a, b) (16.2, 4.3)	(a, b) (17.8, 4.2)	(a, b) (19.4, 4.1)	(a, b) (21, 4.0)
T_{ez}	0.044	0.028	0.018	0.022	0.022	0.024	0.022	0.058	0.090	0.166	0.244
$\tilde{T}$	0.056	0.040	0.040	0.042	0.078	0.076	0.094	0.136	0.164	0.194	0.200
T^2	0.060	0.046	0.040	0.038	0.060	0.066	0.078	0.122	0.180	0.258	0.322

Table A.51. Simulated(size)power, repeat 500 times. $d = \frac{17}{16}$, $n = 100$

P=.25	(a, b) (1, 1)	(a, b) (2.6, 0.9)	(a, b) (4.2, 0.8)	(a, b) (5.8, 0.7)	(a, b) (7.4, 0.6)	(a, b) (9, 0.5)	(a, b) (10.6, 0.4)	(a, b) (12.2, 0.3)	(a, b) (13.8, 0.2)	(a, b) (15.4, 0.1)	(a, b) (17, 0.0)
T_{ez}	0.002	0.002	0.010	0.032	0.060	0.062	0.118	0.124	0.094	0.068	0.020
$\tilde{T}$	0.040	0.048	0.150	0.378	0.640	0.874	0.930	0.994	1.000	1.000	1.000
T^2	0.024	0.030	0.116	0.336	0.600	0.858	0.932	0.990	0.998	1.000	1.000

Table A.52. Simulated(size)power, repeat 500 times. $d = \frac{17}{8}$, $n = 100$

P=.25	(a, b) (2, 2)	(a, b) (3.6, 1.9)	(a, b) (5.2, 1.8)	(a, b) (6.8, 1.7)	(a, b) (8.4, 1.6)	(a, b) (10, 1.5)	(a, b) (11.6, 1.4)	(a, b) (13.2, 1.3)	(a, b) (14.8, 1.2)	(a, b) (16.4, 1.1)	(a, b) (18, 1.0)
T_{ez}	0.180	0.206	0.186	0.124	0.110	0.100	0.146	0.222	0.258	0.392	0.472
$\tilde{T}$	0.056	0.042	0.072	0.142	0.338	0.516	0.742	0.798	0.928	0.968	0.988
T^2	0.108	0.128	0.126	0.144	0.300	0.606	0.744	0.798	0.926	0.966	0.986

Table A.53. Simulated(size)power, repeat 500 times. $d = \frac{51}{16}$, $n = 100$

P=.25	(a, b) (3, 3)	(a, b) (4.6, 2.9)	(a, b) (6.2, 2.8)	(a, b) (7.8, 2.7)	(a, b) (9.4, 2.6)	(a, b) (11, 2.5)	(a, b) (12.6, 2.4)	(a, b) (14.2, 2.3)	(a, b) (15.8, 2.2)	(a, b) (17.4, 2.1)	(a, b) (19, 2.0)
T_{ez}	0.058	0.062	0.072	0.056	0.032	0.068	0.094	0.124	0.334	0.222	0.444
$\tilde{T}$	0.044	0.036	0.062	0.078	0.222	0.348	0.502	0.610	0.872	0.762	0.922
T^2	0.060	0.056	0.070	0.084	0.174	0.286	0.472	0.610	0.876	0.744	0.922

Table A.54. Simulated(size)power, repeat 500 times. $d = \frac{17}{4}$, $n = 100$

P=.25	(a, b) (4, 4)	(a, b) (5.6, 3.9)	(a, b) (7.2, 3.8)	(a, b) (8.8, 3.7)	(a, b) (10.4, 3.6)	(a, b) (12, 3.5)	(a, b) (13.6, 3.4)	(a, b) (15.2, 3.3)	(a, b) (16.8, 3.2)	(a, b) (18.4, 3.1)	(a, b) (20, 3.0)
T_{ez}	0.048	0.040	0.054	0.042	0.056	0.036	0.074	0.118	0.166	0.238	0.366
$\tilde{T}$	0.058	0.044	0.058	0.072	0.152	0.210	0.386	0.504	0.578	0.716	0.834
T^2	0.056	0.052	0.040	0.074	0.114	0.174	0.338	0.450	0.598	0.732	0.848

Table A.55. Simulated(size)power, repeat 500 times. $d = \frac{85}{16}$, $n = 100$

P=.25	(a, b) (5, 5)	(a, b) (6.6, 4.9)	(a, b) (8.2, 4.8)	(a, b) (9.8, 4.7)	(a, b) (11.4, 4.6)	(a, b) (13, 4.5)	(a, b) (14.6, 4.4)	(a, b) (16.2, 4.3)	(a, b) (17.8, 4.2)	(a, b) (19.4, 4.1)	(a, b) (21, 4.0)
T_{ez}	0.040	0.042	0.038	0.038	0.042	0.034	0.068	0.062	0.110	0.166	0.330
$\tilde{T}$	0.046	0.054	0.056	0.070	0.100	0.192	0.228	0.382	0.486	0.598	0.696
T^2	0.044	0.056	0.050	0.062	0.088	0.138	0.210	0.354	0.450	0.594	0.718

Table A.56. Simulated(size)power, repeat 500 times. $d = \frac{17}{16}$, $n = 150$

P=.25	(a, b) (1, 1)	(a, b) (2.6, 0.9)	(a, b) (4.2, 0.8)	(a, b) (5.8, 0.7)	(a, b) (7.4, 0.6)	(a, b) (9, 0.5)	(a, b) (10.6, 0.4)	(a, b) (12.2, 0.3)	(a, b) (13.8, 0.2)	(a, b) (15.4, 0.1)	(a, b) (17, 0.0)
T_{ez}	0.008	0.002	0.012	0.026	0.060	0.106	0.098	0.104	0.092	0.070	0.024
$\tilde{T}$	0.042	0.058	0.206	0.490	0.850	0.940	0.990	0.998	1.000	1.000	1.000
T^2	0.016	0.042	0.166	0.440	0.826	0.930	0.988	0.992	1.000	1.000	1.000

Table A.57. Simulated(size)power, repeat 500 times. $d = \frac{17}{8}$, $n = 150$

P=.25	(a, b) (2, 2)	(a, b) (3.6, 1.9)	(a, b) (5.2, 1.8)	(a, b) (6.8, 1.7)	(a, b) (8.4, 1.6)	(a, b) (10, 1.5)	(a, b) (11.6, 1.4)	(a, b) (13.2, 1.3)	(a, b) (14.8, 1.2)	(a, b) (16.4, 1.1)	(a, b) (18, 1.0)
T_{ez}	0.212	0.212	0.210	0.150	0.116	0.130	0.172	0.198	0.322	0.450	0.468
$\tilde{T}$	0.046	0.054	0.080	0.220	0.490	0.688	0.852	0.960	0.982	1.000	1.000
T^2	0.114	0.138	0.164	0.228	0.450	0.660	0.830	0.958	0.980	0.996	1.000

Table A.58. Simulated(size)power, repeat 500 times. $d = \frac{51}{16}$, $n = 150$

P=.25	(a, b) (3, 3)	(a, b) (4.6, 2.9)	(a, b) (6.2, 2.8)	(a, b) (7.8, 2.7)	(a, b) (9.4, 2.6)	(a, b) (11, 2.5)	(a, b) (12.6, 2.4)	(a, b) (14.2, 2.3)	(a, b) (15.8, 2.2)	(a, b) (17.4, 2.1)	(a, b) (19, 2.0)
T_{ez}	0.056	0.060	0.048	0.056	0.058	0.074	0.100	0.152	0.242	0.346	0.528
$\tilde{T}$	0.040	0.048	0.058	0.106	0.286	0.450	0.664	0.784	0.922	0.972	0.980
T^2	0.042	0.050	0.046	0.096	0.242	0.402	0.622	0.752	0.908	0.970	0.982

Table A.59. Simulated(size)power, repeat 500 times. $d = \frac{17}{4}$, $n = 150$

P=.25	(a, b) (4, 4)	(a, b) (5.6, 3.9)	(a, b) (7.2, 3.8)	(a, b) (8.8, 3.7)	(a, b) (10.4, 3.6)	(a, b) (12, 3.5)	(a, b) (13.6, 3.4)	(a, b) (15.2, 3.3)	(a, b) (16.8, 3.2)	(a, b) (18.4, 3.1)	(a, b) (20, 3.0)
T_{ez}	0.052	0.048	0.028	0.044	0.042	0.058	0.080	0.102	0.178	0.262	0.384
$\tilde{T}$	0.046	0.040	0.046	0.094	0.196	0.302	0.504	0.682	0.792	0.900	0.962
T^2	0.060	0.062	0.050	0.100	0.148	0.256	0.476	0.644	0.774	0.900	0.964

Table A.60. Simulated(size)power, repeat 500 times. $d = \frac{85}{16}$, $n = 150$

P=.25	(a, b) (5, 5)	(a, b) (6.6, 4.9)	(a, b) (8.2, 4.8)	(a, b) (9.8, 4.7)	(a, b) (11.4, 4.6)	(a, b) (13, 4.5)	(a, b) (14.6, 4.4)	(a, b) (16.2, 4.3)	(a, b) (17.8, 4.2)	(a, b) (19.4, 4.1)	(a, b) (21, 4.0)
T_{ez}	0.058	0.030	0.036	0.044	0.020	0.054	0.056	0.078	0.114	0.228	0.308
$\tilde{T}$	0.046	0.038	0.048	0.070	0.172	0.254	0.398	0.532	0.718	0.864	0.886
T^2	0.042	0.034	0.056	0.052	0.116	0.208	0.338	0.486	0.664	0.840	0.888

Table A.61. Simulated(size)power, repeat 500 times. $d = \frac{109}{100}$, $n = 50$

P=.30	(a, b) (1, 1)	(a, b) $(\frac{19}{9}, 0.9)$	(a, b) $(\frac{29}{9}, 0.8)$	(a, b) $(\frac{13}{3}, 0.7)$	(a, b) $(\frac{49}{9}, 0.6)$	(a, b) $(\frac{59}{9}, 0.5)$	(a, b) $(\frac{23}{3}, 0.4)$	(a, b) $(\frac{79}{9}, 0.3)$	(a, b) $(\frac{89}{9}, 0.2)$	(a, b) (11, 0.1)	(a, b) $(\frac{109}{9}, 0.0)$
T_{ez}	0.002	0.002	0.008	0.020	0.036	0.044	0.042	0.052	0.044	0.032	0.030
$\tilde{T}$	0.046	0.034	0.068	0.174	0.320	0.514	0.690	0.830	0.926	0.964	0.982
T^2	0.024	0.024	0.046	0.146	0.274	0.440	0.638	0.796	0.906	0.958	0.980

Table A.62. Simulated(size)power, repeat 500 times. $d = \frac{109}{50}$, $n = 50$

P=.30	(a, b) (2, 2)	(a, b) $(\frac{28}{9}, 1.9)$	(a, b) $(\frac{38}{9}, 1.8)$	(a, b) $(\frac{16}{3}, 1.7)$	(a, b) $(\frac{58}{9}, 1.6)$	(a, b) $(\frac{68}{9}, 1.5)$	(a, b) $(\frac{26}{3}, 1.4)$	(a, b) $(\frac{88}{9}, 1.3)$	(a, b) $(\frac{98}{9}, 1.2)$	(a, b) (12, 1.1)	(a, b) $(\frac{118}{9}, 1.0)$
T_{ez}	0.128	0.116	0.144	0.112	0.088	0.090	0.098	0.064	0.082	0.162	0.188
$\tilde{T}$	0.040	0.054	0.042	0.072	0.126	0.230	0.360	0.452	0.568	0.644	0.790
T^2	0.066	0.102	0.082	0.100	0.128	0.216	0.336	0.402	0.566	0.680	0.814

Table A.63. Simulated(size)power, repeat 500 times. $d = \frac{327}{100}$, $n = 50$

P=.30	(a, b) (3, 3)	(a, b) $(\frac{37}{9}, 2.9)$	(a, b) $(\frac{47}{9}, 2.8)$	(a, b) $(\frac{19}{3}, 2.7)$	(a, b) $(\frac{67}{9}, 2.6)$	(a, b) $(\frac{77}{9}, 2.5)$	(a, b) $(\frac{29}{3}, 2.4)$	(a, b) $(\frac{97}{9}, 2.3)$	(a, b) $(\frac{107}{9}, 2.2)$	(a, b) (13, 2.1)	(a, b) $(\frac{127}{9}, 2.0)$
T_{ez}	0.044	0.054	0.040	0.038	0.062	0.044	0.050	0.046	0.092	0.120	0.184
$\tilde{T}$	0.052	0.038	0.064	0.064	0.086	0.112	0.212	0.276	0.378	0.438	0.570
T^2	0.054	0.068	0.076	0.082	0.100	0.102	0.190	0.232	0.356	0.456	0.620

Table A.64. Simulated(size)power, repeat 500 times. $d = \frac{109}{25}$, $n = 50$

P=.30	(a, b) (4, 4)	(a, b) $(\frac{46}{9}, 3.9)$	(a, b) $(\frac{56}{9}, 3.8)$	(a, b) $(\frac{22}{3}, 3.7)$	(a, b) $(\frac{76}{9}, 3.6)$	(a, b) $(\frac{86}{9}, 3.5)$	(a, b) $(\frac{32}{3}, 3.4)$	(a, b) $(\frac{106}{9}, 3.3)$	(a, b) $(\frac{116}{9}, 3.2)$	(a, b) (14, 3.1)	(a, b) $(\frac{136}{9}, 3.0)$
T_{ez}	0.044	0.036	0.020	0.036	0.020	0.034	0.038	0.046	0.076	0.094	0.166
$\tilde{T}$	0.042	0.066	0.048	0.056	0.086	0.096	0.126	0.182	0.230	0.306	0.372
T^2	0.050	0.072	0.042	0.054	0.068	0.076	0.100	0.150	0.202	0.286	0.404

Table A.65. Simulated(size)power, repeat 500 times. $d = \frac{109}{20}$, $n = 50$

P=.30	(a, b) (5, 5)	(a, b) $(\frac{55}{9}, 4.9)$	(a, b) $(\frac{65}{9}, 4.8)$	(a, b) $(\frac{25}{3}, 4.7)$	(a, b) $(\frac{85}{9}, 4.6)$	(a, b) $(\frac{95}{9}, 4.5)$	(a, b) $(\frac{35}{3}, 4.4)$	(a, b) $(\frac{115}{9}, 4.3)$	(a, b) $(\frac{125}{9}, 4.2)$	(a, b) (15, 4.1)	(a, b) $(\frac{145}{9}, 4.0)$
T_{ez}	0.040	0.016	0.028	0.022	0.024	0.016	0.028	0.014	0.054	0.044	0.084
$\tilde{T}$	0.054	0.050	0.048	0.046	0.038	0.078	0.088	0.098	0.198	0.174	0.228
T^2	0.056	0.034	0.056	0.042	0.042	0.068	0.076	0.064	0.154	0.146	0.204

Table A.66. Simulated(size)power, repeat 500 times. $d = \frac{109}{100}$, $n = 100$

P=.30	(a, b) (1, 1)	(a, b) $(\frac{19}{9}, 0.9)$	(a, b) $(\frac{29}{9}, 0.8)$	(a, b) $(\frac{13}{3}, 0.7)$	(a, b) $(\frac{49}{9}, 0.6)$	(a, b) $(\frac{59}{9}, 0.5)$	(a, b) $(\frac{23}{3}, 0.4)$	(a, b) $(\frac{79}{9}, 0.3)$	(a, b) $(\frac{89}{9}, 0.2)$	(a, b) (11, 0.1)	(a, b) $(\frac{109}{9}, 0.0)$
T_{ez}	0.006	0.000	0.006	0.032	0.044	0.070	0.062	0.072	0.074	0.052	0.046
$\tilde{T}$	0.042	0.060	0.110	0.274	0.530	0.836	0.934	0.990	0.994	1.000	1.000
T^2	0.022	0.032	0.088	0.240	0.488	0.786	0.924	0.988	0.990	1.000	1.000

Table A.67. Simulated(size)power, repeat 500 times. $d = \frac{109}{50}$, $n = 100$

P=.30	(a, b)	(a, b)	(a, b)	(a, b)	(a, b)	(a, b)	(a, b)	(a, b)	(a, b)	(a, b)	(a, b)
	(2, 2)	($\frac{28}{9}$, 1.9)	($\frac{38}{9}$, 1.8)	($\frac{16}{3}$, 1.7)	($\frac{58}{9}$, 1.6)	($\frac{68}{9}$, 1.5)	($\frac{26}{3}$, 1.4)	($\frac{88}{9}$, 1.3)	($\frac{98}{9}$, 1.2)	(12, 1.1)	($\frac{118}{9}$, 1.0)
T_{ez}	0.192	0.184	0.180	0.150	0.134	0.116	0.108	0.122	0.152	0.172	0.226
$\tilde{T}$	0.040	0.062	0.058	0.170	0.268	0.434	0.630	0.782	0.914	0.960	0.986
T^2	0.118	0.114	0.112	0.178	0.264	0.408	0.620	0.766	0.904	0.968	0.982

Table A.68. Simulated(size)power, repeat 500 times. $d = \frac{327}{100}$, $n = 100$

P=.30	(a, b)	(a, b)	(a, b)	(a, b)	(a, b)	(a, b)	(a, b)	(a, b)	(a, b)	(a, b)	(a, b)
	(3, 3)	($\frac{37}{9}$, 2.9)	($\frac{47}{9}$, 2.8)	($\frac{19}{3}$, 2.7)	($\frac{67}{9}$, 2.6)	($\frac{77}{9}$, 2.5)	($\frac{29}{3}$, 2.4)	($\frac{97}{9}$, 2.3)	($\frac{107}{9}$, 2.2)	(13, 2.1)	($\frac{127}{9}$, 2.0)
T_{ez}	0.050	0.052	0.052	0.050	0.048	0.054	0.060	0.060	0.120	0.142	0.218
$\tilde{T}$	0.040	0.038	0.046	0.080	0.180	0.278	0.416	0.566	0.714	0.824	0.906
T^2	0.056	0.054	0.064	0.080	0.164	0.234	0.364	0.534	0.696	0.820	0.906

Table A.69. Simulated(size)power, repeat 500 times. $d = \frac{109}{25}$, $n = 100$

P=.30	(a, b)	(a, b)	(a, b)	(a, b)	(a, b)	(a, b)	(a, b)	(a, b)	(a, b)	(a, b)	(a, b)
	(4, 4)	($\frac{46}{9}$, 3.9)	($\frac{56}{9}$, 3.8)	($\frac{22}{3}$, 3.7)	($\frac{76}{9}$, 3.6)	($\frac{86}{9}$, 3.5)	($\frac{32}{3}$, 3.4)	($\frac{106}{9}$, 3.3)	($\frac{116}{9}$, 3.2)	(14, 3.1)	($\frac{136}{9}$, 3.0)
T_{ez}	0.046	0.052	0.044	0.044	0.044	0.040	0.050	0.082	0.064	0.080	0.162
$\tilde{T}$	0.040	0.046	0.060	0.080	0.112	0.192	0.346	0.388	0.578	0.724	0.812
T^2	0.040	0.052	0.044	0.062	0.080	0.154	0.278	0.332	0.524	0.678	0.806

Table A.70. Simulated(size)power, repeat 500 times. $d = \frac{109}{20}$, $n = 100$

P=.30	(a, b) (5, 5)	(a, b) ($\frac{55}{9}$, 4.9)	(a, b) ($\frac{65}{9}$, 4.8)	(a, b) ($\frac{25}{3}$, 4.7)	(a, b) ($\frac{85}{9}$, 4.6)	(a, b) ($\frac{95}{9}$, 4.5)	(a, b) ($\frac{35}{3}$, 4.4)	(a, b) ($\frac{115}{9}$, 4.3)	(a, b) ($\frac{125}{9}$, 4.2)	(a, b) (15, 4.1)	(a, b) ($\frac{145}{9}$, 4.0)
T_{ez}	0.040	0.026	0.046	0.034	0.044	0.036	0.048	0.046	0.058	0.086	0.116
$\tilde{T}$	0.046	0.058	0.042	0.052	0.038	0.120	0.214	0.312	0.460	0.556	0.656
T^2	0.040	0.046	0.044	0.050	0.042	0.074	0.158	0.262	0.404	0.522	0.634

Table A.71. Simulated(size)power, repeat 500 times. $d = \frac{109}{100}$, $n = 150$

P=.30	(a, b) (1, 1)	(a, b) ($\frac{19}{9}$, 0.9)	(a, b) ($\frac{29}{9}$, 0.8)	(a, b) ($\frac{13}{3}$, 0.7)	(a, b) ($\frac{49}{9}$, 0.6)	(a, b) ($\frac{59}{9}$, 0.5)	(a, b) ($\frac{23}{3}$, 0.4)	(a, b) ($\frac{79}{9}$, 0.3)	(a, b) ($\frac{89}{9}$, 0.2)	(a, b) (11, 0.1)	(a, b) ($\frac{109}{9}$, 0.0)
T_{ez}	0.002	0.008	0.004	0.022	0.048	0.072	0.092	0.088	0.074	0.054	0.030
$\tilde{T}$	0.040	0.058	0.144	0.448	0.770	0.940	0.934	1.000	1.000	1.000	1.000
T^2	0.014	0.036	0.114	0.390	0.712	0.922	0.924	0.998	1.000	1.000	1.000

Table A.72. Simulated(size)power, repeat 500 times. $d = \frac{109}{50}$, $n = 150$

P=.30	(a, b) (2, 2)	(a, b) ($\frac{28}{9}$, 1.9)	(a, b) ($\frac{38}{9}$, 1.8)	(a, b) ($\frac{16}{3}$, 1.7)	(a, b) ($\frac{58}{9}$, 1.6)	(a, b) ($\frac{68}{9}$, 1.5)	(a, b) ($\frac{26}{3}$, 1.4)	(a, b) ($\frac{88}{9}$, 1.3)	(a, b) ($\frac{98}{9}$, 1.2)	(a, b) (12, 1.1)	(a, b) ($\frac{118}{9}$, 1.0)
T_{ez}	0.216	0.208	0.190	0.170	0.144	0.126	0.104	0.126	0.130	0.176	0.218
$\tilde{T}$	0.048	0.038	0.082	0.194	0.354	0.618	0.846	0.930	0.972	0.992	0.996
T^2	0.126	0.104	0.158	0.214	0.350	0.596	0.816	0.916	0.962	0.994	0.994

Table A.73. Simulated(size)power, repeat 500 times. $d = \frac{327}{100}$, $n = 150$

P=.30	(a, b) (3, 3)	(a, b) ($\frac{37}{9}$, 2.9)	(a, b) ($\frac{47}{9}$, 2.8)	(a, b) ($\frac{19}{3}$, 2.7)	(a, b) ($\frac{67}{9}$, 2.6)	(a, b) ($\frac{77}{9}$, 2.5)	(a, b) ($\frac{29}{3}$, 2.4)	(a, b) ($\frac{97}{9}$, 2.3)	(a, b) ($\frac{107}{9}$, 2.2)	(a, b) (13, 2.1)	(a, b) ($\frac{127}{9}$, 2.0)
T_{ez}	0.050	0.040	0.034	0.050	0.048	0.038	0.078	0.066	0.112	0.140	0.216
$\tilde{T}$	0.042	0.032	0.048	0.128	0.208	0.352	0.606	0.770	0.876	0.952	0.982
T^2	0.048	0.042	0.050	0.118	0.188	0.274	0.540	0.720	0.864	0.934	0.978

Table A.74. Simulated(size)power, repeat 500 times. $d = \frac{109}{25}$, $n = 100$

P=.30	(a, b) (4, 4)	(a, b) ($\frac{46}{9}$, 3.9)	(a, b) ($\frac{56}{9}$, 3.8)	(a, b) ($\frac{22}{3}$, 3.7)	(a, b) ($\frac{76}{9}$, 3.6)	(a, b) ($\frac{86}{9}$, 3.5)	(a, b) ($\frac{32}{3}$, 3.4)	(a, b) ($\frac{106}{9}$, 3.3)	(a, b) ($\frac{116}{9}$, 3.2)	(a, b) (14, 3.1)	(a, b) ($\frac{136}{9}$, 3.0)
T_{ez}	0.054	0.042	0.050	0.058	0.054	0.052	0.062	0.060	0.094	0.080	0.222
$\tilde{T}$	0.046	0.058	0.040	0.092	0.140	0.244	0.416	0.622	0.750	0.724	0.940
T^2	0.060	0.056	0.040	0.086	0.140	0.208	0.350	0.568	0.708	0.832	0.932

Table A.75. Simulated(size)power, repeat 500 times. $d = \frac{109}{20}$, $n = 150$

P=.30	(a, b) (5, 5)	(a, b) ($\frac{55}{9}$, 4.9)	(a, b) ($\frac{65}{9}$, 4.8)	(a, b) ($\frac{25}{3}$, 4.7)	(a, b) ($\frac{85}{9}$, 4.6)	(a, b) ($\frac{95}{9}$, 4.5)	(a, b) ($\frac{35}{3}$, 4.4)	(a, b) ($\frac{115}{9}$, 4.3)	(a, b) ($\frac{125}{9}$, 4.2)	(a, b) (15, 4.1)	(a, b) ($\frac{145}{9}$, 4.0)
T_{ez}	0.046	0.030	0.054	0.024	0.042	0.042	0.054	0.048	0.064	0.088	0.138
$\tilde{T}$	0.044	0.060	0.076	0.078	0.098	0.190	0.350	0.482	0.618	0.748	0.852
T^2	0.044	0.052	0.056	0.062	0.080	0.136	0.270	0.432	0.566	0.710	0.828