STOCHASTIC PROCESSES, AND DEVELOPMENT OF THE BARNDORFF-NIELSEN AND SHEPHARD MODEL FOR FINANCIAL MARKETS

A Paper
Submitted to the Graduate Faculty
of the
North Dakota State University
of Agriculture and Applied Science

By

Austin Uden

In Partial Fulfillment of the Requirements
for the Degree of
MASTER OF SCIENCE

Major Department:
Mathematics

April 2022

Fargo, North Dakota

NORTH DAKOTA STATE UNIVERSITY

Graduate School

Title

STOCHASTIC PROCESSES, AND DEVELOPMENT OF THE BARNDORFF-NIELSEN AND SHEPHARD MODEL FOR FINANCIAL MARKETS

By

Austin Uden

The supervisory committee certifies that this paper complies with North Dakota State University's regulations and meets the accepted standards for the degree of

MASTER OF SCIENCE

SUPERVISORY COMMITTEE:

Prof. Indranil SenGupta
Chair

Prof. Artem Novozhilov

Prof. Nikita Barabanov

Prof. Somnath Banerjee

Approved:

04/14/2022
Date

Friedrich Littmann
Department Chair

ABSTRACT

In this paper, we introduce Brownian motion, and some of its drawbacks in connection to the financial modeling. We then introduce geometric Brownian motion as the basis for European call option pricing as we navigate our way through the Black-Scholes-Merton equation. Lévy Processes round out the background information of the paper as we discuss Poisson and compound Poisson processes and the pricing of European call options using the stochastic calculus of jump processes. Ornstein-Uhlenbeck processes are then constructed. Finally we review and analyze the Barndorff-Nielsen and Shepard model. We provide its application to price European call options using the fast Fourier transform and the direct integration method.

ACKNOWLEDGEMENTS

I want to thank my family for being my encouragement through graduate school. My advisor, Dr. SenGupta, for his dedication to all of his graduate students and making sure we have everything we need to be successful. My committee for their valuable input to my final paper. Finally, to my church family who have lifted me up through the trials and tribulations over the past four years.

I will never be able to thank you all enough for the person you have helped mold me into and for that I am deeply appreciative of each and every one of you.

TABLE OF CONTENTS

LIST OF FIGURES

1. BROWNIAN MOTION

1.1. Introduction

In 1827 Robert Brown witnessed the random movement of pollen molecules suspended in fluid and decided he needed to write about his observations. His writing lead Louis Bachelier to apply this observed randomness to the stock market around 1900 and it was expanded on by Norbert Wiener in the 1920s. Wiener created a rigorous framework using probability to describe the movements of pollen suspended in fluid. The contributions of Brown and Wiener to this framework has lead to Wiener's framework being called Brownian motion/Wiener process. Thanks to the probability framework that Wiener created, Brownian motion is widely used in option modeling today.

1.2. Definition and properties

In this section Brownian motion (Wiener Process), which we will denote W_t, will be introduced. We will then go through important properties of Brownian motion and corresponding results.

Definition. *Brownian motion: A stochastic process* $\{W_t\}_{t\geq 0}$, *is called a Brownian motion if:*

1) $W_0 = 0$.

2) It has continuous sample paths.

3) It has independent, stationary increments.

4) $W_t \sim N(0,t)$.

The above definition gives Brownian motion the following probability density with x denoting the value of W_t:

$$f(x) = \frac{1}{\sqrt{t}\sqrt{2\pi}} \exp(\frac{-1}{2}(\frac{x}{\sqrt{t}})^2), \tag{1.1}$$

where as the probability distribution of $W_{t+u} - W_t$ is:

$$P[W_{t+u} - W_t \leq a] = \int_{-\infty}^{a} \frac{1}{\sqrt{u}\sqrt{2\pi}} \exp(\frac{-1}{2}(\frac{x}{\sqrt{u}})^2)\, dx. \tag{1.2}$$

Since our probability distribution depends solely on the amount of time our process has been running, and we set $W_0 = 0$, using the fact that W_t is normally distributed, we get $Var[W_t] = E[W_t^2] - E[W_t]^2 = t$, where $E[W_t]^2 = 0$ and the standard deviation is $\sqrt{t}$. If we have a short increment of time, we expect the process to be close to the mean and as our increment of time increases we expect a more spread out density.

Theorem 1. [Covariance of two Brownian motions] For $s < t$, $C\text{ov}[W_s,W_t]=s$.

Proof. From the definition of covariance and the expected values of Brownian motion, we get the following:

$$\text{C}ov[W_s,W_t]=\text{E}[\{W_s\text{-}E[W_s]\}\{W_t\text{-E}[W_t]\}]$$
$$=\text{E}[W_sW_t].$$

Since our time intervals are $[0, s]$ and $[0, t]$, they are overlapping so we can consider the Brownian motion on the interval $[0, t]$ as the sum of the Brownian motion on $[0, s]$ and the Brownian motion on $[s, t]$, which means they are independent time increments. This means that $W_t = W_s + W_t - W_s$. Therefore, we get the following:

$$\text{E}[W_sW_t]=\text{E}[W_s]^2 + E[W_s(W_t\text{-}W_s)]$$
$$=\text{E}[{W_s}^2] + E[W_s(W_t\text{-}W_s)]$$
$$=\text{E}[{W_s}^2] + E[W_s]\text{E}[W_t\text{-}W_s]\text{=s}.$$

□

1.3. Brownian motion and the stock market

Most of us, when we buy stocks, really only want to know if we will make or lose money on the stock with the former being preferred. One way to do this is to evaluate the dynamics of the stock, will it go up or down, and how much is will change in either direction so we can buy other stocks to balance out our portfolios to increase on average. We do need to realize that generally this is a long term decision and our portfolios might not always indicate that we made the correct decision even if they will long term. We can use Brownian motion to model this evolving stock price through the following:

$$\frac{S(t+\Delta t)-S(t)}{S(t)}=\mu\Delta t+\sigma\Delta W_t, \tag{1.3}$$

where Δt is our time interval, $S(t)$ and $S(t+\Delta t)$ are our current and future time stock prices, ΔW_t is the Brownian motion increment over Δt, μ and σ are constants. This equation denotes a growth of our stock value at a set rate μ per time increment as well as a random change proportionate to our Brownian motion over the same time increment and proportionality parameter σ.

However, for modeling stock price the above equation does not make sense as it can take on negative values. The reason this does not make sense is a stock price bottoms out at a value of zero which is when a company has a market evaluation that says there will be no return on that company and the limited liability of shareholders says that a shareholder can not owe a company money for its own failings.

1.4. Building Brownian motion from a random walk

Let us consider a symmetric random walk on the time interval $[0, T]$ and partition the walk into n intervals of length $\Delta t = T/n$. At each node we have two options. The first option is to increase by $\sqrt{\Delta t}$ and the second option is to decrease by $\sqrt{\Delta t}$. In order for this to be a symmetric walk, there is an equal probability for each outcome at every node of our walk. This means that previous data does not affect the outcome of the next step.

We will consider a particle that starts at position 0 at time 0 that follows the rules set forth for this walk. At the first time point, our particle is either at $\sqrt{\Delta t}$ or -$\sqrt{\Delta t}$ and at our second time point the particle is at $2\sqrt{\Delta t}$, 0, or -$2\sqrt{\Delta t}$. Note, at each time point j, there are 2^j paths to the possible nodes so some paths lead to the same node increasing the probability of landing on that node. For example, in our path above, at the second time interval, there are 4 paths with two paths leading to the value 0 so based on the equal probabilities, we have a 1/2 chance of reaching 0. Below we can see the sample path system with $n = 5$ and the probability of reaching each node. We notice that the nodes of these walks follow Pascal's triangle for the partition of how many paths reach individual nodes.

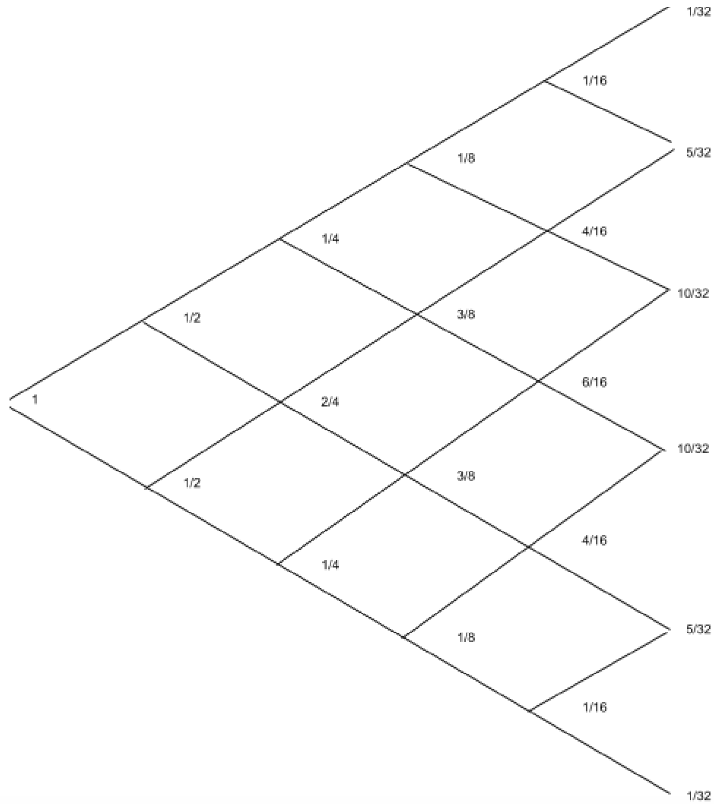

Figure 1.1. A random walk with 5 time steps and the probability of reaching each node

We will now show that our random walk does produce a Brownian motion. We will do this by first assuming a finite number of time steps and show the expected value and variance hold true to the requirements of Brownian motion. We will then consider our random walk as we approach an infinite number of time steps and show that we are still in the confines of Brownian motion.

Between time points t_k and t_{k+1} we will consider a two valued random variable X_k. As noted above we know that we have equal probability of moving up or down so the expected value is:

$$\mathrm{E}[X_k]=\tfrac{1}{2}\sqrt{\Delta t}+\tfrac{1}{2}(-\sqrt{\Delta t})=0.$$

We will now check that the variance between consecutive time points is Δt:

$$V\mathrm{ar}[X_k]=E[{X_k}^2]\text{-}(E[X_k])^2$$
$$=\mathrm{E}[{X_k}^2]=\tfrac{1}{2}(\sqrt{\Delta t})^2+\tfrac{1}{2}(-\sqrt{\Delta t})^2$$
$$=\Delta t.$$

In order to view our entire process from time 0 to time T, we can add n of the above processes together such that our random walk is $S_n = X_0 + X_1 + \ldots + X_{n-1}$. Note, each X_i is independent as the time intervals are independent, this will allow the manipulations in the following calculations of the expected value and variance of S_n.

The expected value of S_n is:

$$E[S_n]{=}E[X_0{+}X_1{+}\ldots{+}X_{n-1}]$$
$$=E[X_0]{+}E[X_1]{+}\ldots{+}E[X_{n-1}]{=}\text{n}0{=}0.$$

The variance is as follows:

$$V\text{ar}[S_n]{=}Var[\textstyle\sum_{k=0}^{n-1} X_k]$$
$$={\textstyle\sum_{k=0}^{n-1}} Var[X_k]$$
$$=n\Delta t{=}\text{n(T/n)}{=}\text{T}.$$

Theorem 2. *As $n \to \infty$, the probability distribution of S_n converges to the probability distribution of Brownian motion at time T.*

Proof. See [38] pg. 9-10 □

We will now use the random walk formulation of Brownian motion to show that Brownian motion is not differentiable using $\Delta t = 1/n$ and starting at time t. Let,

$$X_n = \frac{W_{t+\Delta t} - W_t}{\Delta t} = \frac{W_{t+1/n} - W_t}{1/n} = n[W_{t+1/n} - W_t]. \tag{1.4}$$

Thus, X_n is normally distributed as it is a scalar of the difference of normal distributions with the same mean. Also,

$$E[X_n]{=}n^2[W_{t+1/n} - W_t] = n0 = 0,$$
$$V\text{ar}[X_n]{=}n^2\, Var[W_{t+1/n} - W_t] = n^2(\tfrac{1}{n}){=}\text{n}.$$

Based on the expected value and variance, X_n has the same probability distribution as $\sqrt{n}Z$, where Z is the standard normal distribution. Also, since the definition of X_n is the difference quotient, we can look at what happens as $\Delta t \to 0$ which occurs when $n \to \infty$ to determine the differentiability of X_n. A stochastic process is said to be differentiable if its sample paths are almost surely differentiable. Let $K > 0$, then:

$$P[|X_n| > \text{K}] = \text{P}[|\sqrt{n}Z| > K]$$
$$= P[\sqrt{n}|Z| > K] = P[|Z| > \tfrac{K}{\sqrt{n}}]$$
$$\to P[|Z| > 0] = 1.$$

Since K can be chosen arbitrarily large, we have an infinite rate of change which implies that the path of our Brownian motion is not differentiable at time t. Since t is an arbitrary time, Brownian motion is nowhere differentiable. This can be seen in the random walk as there is always an equal chance that our walk increases or decreases which means we never know which direction the walk is moving.

1.5. Correlated Brownian motions

Thus far, we have only looked at Brownian motions that are independent and therefore a lot of the probability equations can be split using this independence. What do we do, if instead, our Brownian motions are correlated? This section will focus on this question by building a Brownian motion using two independent Brownian motions and looking at what happens when we use our newly constructed Brownian motion and one of the two Brownian motions that are combined to produce it.

Let us consider two independent Brownian motions W_t and $W_t{}^*$ and a given number $-1 \leq \rho \leq 1$. For $0 \leq t \leq T$ we will define a new process,

$$Z_t = \rho W_t + \sqrt{1-\rho^2} W_t^*. \tag{1.5}$$

Theorem 3. *The process Z_t is a Brownian motion with respect to the filtration on W_t and W_t^*.*

Proof. See [38] pg 14-15. □

Now we will discuss the correlation between the Brownian motions W_t and Z_t. Since Z_t was built with W_t as one of the foundational Brownian motions, the correlation between them is nonzero.

Theorem 4. *$Corr[Z_t,\, W_t] = \rho$.*

Proof. $Corr[Z_t, W_t] = \frac{Cov[Z_t, W_t]}{\sqrt{Var[Z_t]}\sqrt{Var[W_t]}}$

Substituting for Z_t and using independence of our Brownian motions, we get:

$$\begin{aligned}
Cov[Z_t, W_t] &= Cov[\rho W_t + \sqrt{1-\rho^2}W_t^*, W_t] \\
&= Cov[\rho W_t, W_t] + Cov[\sqrt{1-\rho^2}W_t^*, W_t] \\
&= \rho Cov[W_t, W_t] + \sqrt{1-\rho^2}\,Cov[W_t^*, W_t] \\
&= \rho Var[W_t, W_t] + \sqrt{1-\rho^2}0 \\
&= \rho t.
\end{aligned}$$

Therefore,

$$Corr[Z_t, W_t] = \frac{\rho t}{\sqrt{t}\sqrt{t}} = \rho.$$

□

1.6. Brownian motion as a martingale

Given a sequence of random variables $X_1, X_2, \ldots, X_n, \ldots$ on a probability space $(\Omega, \mathcal{F}, P)$, which, at successive time points, measures some phenomena. Then we have a random process with values that will become known in order as time passes. When X_1 becomes known, we obtain a set of information about our process, we will denote this information as $\mathcal{F}_1$. Similarly, X_2 will become known and the accumulated information that became known with X_1 and X_2 will be denoted $\mathcal{F}_2$. The process of our random variables becoming known will continue and at each step we know the information that became known when the random variable was realized as well as all previous information that had been obtained. This preservation of information is written as $\mathcal{F}_1 \subseteq \mathcal{F}_2 \subseteq \ldots \subseteq \mathcal{F}_n$ and this sequence of information is called a filtration. With the information that becomes known at each time instance, we have a filtered probability space $(\Omega, \mathcal{F}_{t\geq 0}, P)$.

Let $|X|$ be the sum of the positive and negative parts of a random variable X, then X is said to be integrable if the unconditional expected value $E[|X|]$ exists and is finite. What happens if we know X is integrable and we have a conditional expectation $Y = E[X|\mathcal{F}]$? Then we get:

$$E[Y]{=}E[E[X|\mathcal{F}]]{=}E[X].$$

This allows us to also convert a normal expected value into a conditional expectation. When we need to find the expected value of the product of two random variables, we will do this conversion. Given a random variable Z that is known given $\mathcal{F}$, then:

$$E[XZ]{=}E[E[XZ|\mathcal{F}]]{=}E[ZE[X|\mathcal{F}]].$$

Typically we will know $E[X|\mathcal{F}]$ and substitute it into the above equation. This leads to the tower property when a random variable is conditioned on the history up to time s and then conditioning the resulting variable on the history up to time r, where $r < s < t$:

$$E[E[X_t|\mathcal{F}_s]|\mathcal{F}_r]=E[X_t|\mathcal{F}_r].$$

Definition. *(martingale) A sequence of random variables* $\{X_n\}$ *on the probability space* $(\Omega, \{\mathcal{F}_t\}_{t\geq 0}, P)$ *is a martingale if the following hold:*

a) $E[|X_n|] < \infty$ *for all* n*,*

b)$E[X_{n+1}|\mathcal{F}_n] = X_n$ *for all* n*.*

Theorem 5. *Brownian motion is a martingale on* $(\Omega, \mathcal{F}, P)$*.*

Proof. Let W_t be our future time Brownian motion and W_s be the Brownian motion that we know with $\mathcal{F}_s$ as the information determined through time s. We can then decompose W_t into two independent time intervals of Brownian motion in the same way we did previously: $W_t = W_s + (W_t - W_s)$. Therefore, our conditional probability for our martingale is:

$$\begin{aligned} E[W_t|\mathcal{F}_s]&=E[W_s + (W_t - W_s)|\mathcal{F}_s] \\ &=E[W_s|\mathcal{F}_s] + E[W_t - W_s|\mathcal{F}_s]. \end{aligned}$$

Since our Brownian motion is known at time s, W_s is a known value and not a random variable. From the definition of Brownian motion, the time increment from s to t is independent of the time increment prior to s. Thus, the expected value of the Brownian motion from time s to t is independent of the information up until time s. Therefore, $E[W_t - W_s|\mathcal{F}_s]=E[W_t - W_s]=0$. So we get the following:

$$\begin{aligned} E[W_s|\mathcal{F}_s] + E[W_t - W_s|\mathcal{F}_s] &= W_s + E[W_t - W_s] \\ &=W_s + 0=W_s. \end{aligned}$$

Therefore, Brownian motion is a martingale. □

1.7. Construction of a non-negative Brownian motion

The main drawback of the previous sections is that nothing stops a Brownian motion, in the traditional sense, from being negative. In this section, we will introduce stochastic differential equations (SDEs) and use them to build a type of Brownian motion that is always positive and therefore, is a better representation of the price of a stock.

Given a variable X, which is driven by one or more random processes, for our purposes, these random processes will be Brownian motions, the equation that describes the increment of X is called a SDE. We will specifically focus on X when only one Brownian motion is present which gives us the specification with μ and σ as given continuous functions:

$$dX_t = \mu(t, X_t)dt + \sigma(t, X_t)dW_t. \tag{1.6}$$

Our initial time is taken as $t = 0$ and X_0 is known. We will also utilize the integral representation of the SDE given as:

$$\int_{t=0}^{T} dX = X_T - X_0 = \int_{t=0}^{T} \mu(t, X_t)dt + \int_{t=0}^{T} \sigma(t, X_t)dW_t,$$

$$\mathrm{X}_T = X_0 + \int_{t=0}^{T} \mu(t, X_t)dt + \int_{t=0}^{T} \sigma(t, X_t)dW_t. \tag{1.7}$$

This latter integral expression is the exact specification of the SDE. The continuous function μ is called the drift coefficient and σ is called the diffusion coefficient which measures the scaling of the randomness presented by the Brownian motion. At times, the values of X in the coefficients of the above function might only depend on the history of X. The first integral in our representation is an ordinary pathwise integral whereas the second integral is an Itô stochastic integral. Due to the randomness in the second integral, the solution to a SDE is a random process which has a stochastic differential of the same form as the SDE when the Brownian motion process is given. A unique pathwise solution X to a SDE exists when the following two conditions are met for positive constants K and L for any time $0 \leq t \leq T$:

Growth condition: $\mu(t, x)^2 + \sigma(t, x)^2 \leq K(1 + x^2)$.

Lipschitz condition: $|\mu(t, x_1) - \mu(t, x_2)| + |\sigma(t, x_1) - \sigma(t, x_2)| \leq L|x_1 - x_2|$.

1.7.1. Arithmetic Brownian motion

Arithmetic Brownian motion is a specification of a SDE when both of the continuous functions μ and σ are constant. This produces the following SDE and its corresponding integral form:

$$dX_t = \mu dt + \sigma dW_t.$$

$$\textstyle\int_{t=0}^{T} dX_t = \int_{t=0}^{T} \mu dt + \int_{t=0}^{T} \sigma dW_t. \tag{1.8}$$

Since there are no unknowns on the right hand side of the integral equation, we get the solution:

$$X_t = X_0 + \mu T + \sigma W_T.$$

The first two terms in our solution are non-random and the third term is a normally distributed random variable, thus our solution X is normally distributed. This leads to the expected value and variance, to be linear scalings of T and be represented by:

$$E[X_T] = X_0 + \mu T.$$
$$Var[X_T] = \sigma^2 T.$$

The solution X in this model can still take on negative values which means that it cannot be used in full effectiveness for stock prices. However, we can use these results to build our model in the next section called Geometric Brownian motion (GBM).

1.7.2. Geometric Brownian motion

In GBM, we look into a proportional change of X in comparison to its current value. The SDE governing this proportional change, when μ and σ are known constants with $\sigma > 0$ is:

$$\frac{dX_t}{X_t} = \mu dt + \sigma dW_t.$$

$$\mathrm{d}X_t = \mu X_t dt + \sigma X_t dW_t. \tag{1.9}$$

Both our drift and diffusion coefficients are proportional to the latest known value X_t. Since X_t is constantly changing, so are our drift and diffusion coefficients. As the latest X becomes larger, the drift and diffusion also grows. This leads to a Brownian motion with a greater variance being needed to generate the random term in the solution. GBM is now considered the standard model for stock prices and is thus denoted S. We will now solve $dS_t = \mu S_t dt + \sigma S_t dW_t$ for S, we will assume S is deterministic so $dS(t)/S(t)$ would be the derivative of $\ln[S(t)]$ with respect to S. This leads to the following equation:

$$d\ln[S] = \frac{d\ln[S]}{dS}dS + \frac{1}{2}\frac{d^2\ln[S]}{dS^2}(dS)^2.$$

Where the following are the substitutions that will be made:

$$\frac{d\ln[S]}{dS} = \frac{1}{S}.$$
$$\frac{d^2\ln[S]}{dS^2} = \frac{-1}{S^2}.$$
$$(dS)^2 = \sigma^2 S^2 dt.$$

With these substitutions, we get the following:

$$\begin{aligned} d\ln[S] =& \frac{1}{S}(\mu S dt + \sigma S dW_t) + \frac{1}{2}\frac{-1}{S^2}\sigma^2 S^2 dt \\ =& \mu dt + \sigma dW_t - \frac{1}{2}\sigma^2 dt \\ =& (\mu - \frac{1}{2}\sigma^2)dt + \sigma dW_t. \end{aligned}$$

Solving this equation for $S(T)$ obtains:

$$\int_{t=0}^{T} d\ln[S(t)] = \int_{t=0}^{T}(\mu - \frac{1}{2}\sigma^2)dt + \int_{t=0}^{T}\sigma dW_t$$
$$\ln[S(T)] - ln[S(0)] = (\mu - \frac{1}{2}\sigma^2)T + \sigma W_T$$
$$\ln\left[\frac{S(T)}{S(0)}\right] = (\mu - \frac{1}{2}\sigma^2)T + \sigma W_T$$
$$S(T) = S(0)\exp((\mu - \frac{1}{2}\sigma^2)T + \sigma W_T).$$

Thus $\ln[S(T)/S(0)]$ is a constant plus a normally distributed variable and therefore is normally distributed. Since W_T is our Brownian motion at $t = T$, the mean and variance are:

$$E\left\{\ln\left[\frac{S(T)}{S(0)}\right]\right\} = (\mu - \frac{1}{2}\sigma^2)T.$$
$$Var\left\{\ln\left[\frac{S(T)}{S(0)}\right]\right\} = \sigma^2 T.$$

Since $S(T)$ is an exponential, it cannot become negative and is therefore a suitable candidate for a stock price. GBM is the traditional model for the stock price due to overcoming this obstacle of Brownian motion.

1.8. Ornstein-Uhlenbeck SDE

The description of the bombardment of pollen particles by molecules was postulated by Ornstein-Uhlenbeck (OU). Given two positive constants λ and σ, the representation is:

$$dX(t) = -\lambda X(t)dt + \sigma dW_t. \tag{1.10}$$

In this equation, $X(t)$ is the one-dimensional velocity of the particle and $dX(t)$ is the acceleration of the particle modelled by a frictional force proportional to velocity $-\lambda$X(t), with the addition of a random turbulence W_t with intensity σ caused by the molecules interacting with the particle.

We will use the transformation $Y(t) = X(t)\exp(\lambda t)$ and apply Itô's formula to Y as a function of X and t. The dynamics of Y are as follows:

$$dY = \frac{\partial Y}{\partial t}dt + \frac{\partial Y}{\partial X}dX + \frac{1}{2}\frac{\partial^2 Y}{\partial X^2}(dX)^2 + \frac{\partial^2 Y}{\partial X \partial t}dtdX. \tag{1.11}$$

To simplify this expression we can use the following substitutions:

$$\begin{gathered} \tfrac{\partial Y}{\partial t} = X\exp(\lambda t)\lambda = Y\lambda. \\ \tfrac{\partial Y}{\partial X} = \exp(\lambda t). \\ \tfrac{\partial^2 Y}{\partial X^2} = 0. \\ dtdX = dt(-\lambda X dt + \sigma dW) = 0. \end{gathered}$$

Which result in the simplified equation:

$$dY = Y\lambda dt + \exp(\lambda t)dX$$

$$\begin{aligned} &= \mathrm{Y}\lambda dt + \exp(\lambda t)(-\lambda X dt + \sigma dW) \\ &= \mathrm{Y}\lambda dt - \lambda Y dt + \sigma\exp(\lambda t)dW \\ &= \sigma\exp(\lambda t)dW_t \end{aligned} \tag{1.12}$$

Solving this equation and reverting back to $X(T)$ using $X(T) = \exp(-\lambda T)Y(T)$, we get:

$$\begin{gathered} Y(T) = Y(0) + \sigma \textstyle\int_{t=0}^{T} \exp(\lambda t)dW_t \\ X(T) = \exp(-\lambda T)X(0) + \exp(-\lambda T)\sigma \textstyle\int_{t=0}^{T} \exp(\lambda t)dW_t \\ E[X(T)] = \exp(-\lambda T)X(0). \end{gathered}$$

1.9. Mean-reversion SDE

When a process fluctuates around a mean level, the random process used to model it is called a mean-reversion process. The canonical example is continuously compounded interest rate given by the following SDE with λ, σ, and $\bar{r}$ known positive constants:

$$dr_t = -\lambda[r_t - \bar{r}]dt + \sigma dW_t. \tag{1.13}$$

We will now show that the long-term mean is $\bar{r}$, so r reverts to the mean. Using the transformation $X_t = r_t - \bar{r}$, we get from Itô's formula $dX = dr$, thus $dX = -\lambda X dt + \sigma dW$ with the solution:

$$X_T = \exp(-\lambda T)X_0 + \exp(-\lambda T)\sigma \textstyle\int_{t=0}^{T} \exp(\lambda t)dW_t.$$

Since $r_T = X_T + \bar{r}$ and $X_0 = r_0 - \bar{r}$, we obtain:

$$r_T = \exp(-\lambda T)[r_0 - \bar{r}] + \exp(-\lambda T)\sigma \textstyle\int_{t=0}^{T} \exp(\lambda t)dW_t + \bar{r}$$
$$= r_0 \exp(-\lambda T) + \bar{r}[1 - \exp(-\lambda T)] + \exp(-\lambda T)\sigma \textstyle\int_{t=0}^{T} \exp(\lambda t)dW_t.$$

With the expected value:

$$E[r_T] = r_0 \exp(-\lambda T) + \bar{r}[1 - \exp(-\lambda T)].$$

As T increases, the expected value approaches $\bar{r}$.

For further study of the material presented in this section, see [38].

2. STOCHASTIC CALCULUS

2.1. Introduction

Now that we have developed Brownian motion, the next logical question is what can be done with this development? We have discussed how Brownian motions are used in stock and asset pricing and in this section we will discuss the calculus techniques developed by Kiyosi Itô. These techniques differ from ordinary calculus because while we will be performing the same techniques, integrating with respect to a non-differentiable function had to be developed.

2.2. Itô's integral for simple integrands

How can we make sense of the following function:

$$\int_0^T \Delta_t W_t? \tag{2.1}$$

We have our Brownian motion $W_t, t \geq 0$ and its corresponding filtration, as well as our integrand Δ_t which we will let be an adapted (F_t-measurable $\forall t$) stochastic process so that it can take on the value of our asset at time t. As with Brownian motion, we cannot determine anything about Δ_t in the future as it is also a random variable, but at each time instant, we have enough information to evaluate its value at that time. Also, due to $\Delta_t's$ measurability, it must be independent of future increments of the Brownian motion driving the prices of our assets.

In order to construct Itô integration, consider a simple integrand Δ_t and a partition $P = \{t_0, t_1, \ldots, t_n\}$ of $[0, T]$ with Δ_t constant in t on each interval $[t_i, t_{i+1})$ so Δ_t is a simple process. We will choose these processes to take a value at t_i and retain that value until, but not necessarily including, time t_{i+1}. Our process Δ_t depends on the same ω as our Brownian motion and while each process for differing ω might be different at time t, all processes have the same value at $t = 0$ as there is no past information to influence the current value, but each future time can depend on past observations.

How do the processes discussed interact with our Brownian motion? If we consider our Brownian motion as the price per share of an asset as before, our partition can be thought about as the trading dates for the asset with $\Delta_{t_0}, \Delta_{t_1}, \ldots, \Delta_{t_{n-1}}$ as the number of shares the asset takes in during each trading date that is held until the next trading date. We will let I_t denote the gain from trading at each time t where if $t_k \leq t \leq t_{k+1}$, I_t is given by:

$$I_t = \sum_{i=0}^{k-1} \Delta_{t_i}[W_{t_{i+1}} - W_{t_i}] + \Delta_{t_k}[W_t - W_{t_k}]. \tag{2.2}$$

The process given by I_t is the Itô integral of the process Δ_t. Also stated as:

$$I_t = \int_0^t \Delta_u dW_u. \tag{2.3}$$

If we take $t = t_n = T$, then we have a definition for our original integral.

2.2.1. Properties of Itô's integral

The way we have defined the Itô integral, it is the gain from trading in W_t. This means we have Itô's integral defined by a martingale which has no tendency to change and it should be expected that by its definition, I_t also does not have a tendency to change.

Theorem 6. *I_t is a martingale with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ of the underlying Brownian motion.*

Proof. Let $0 \leq s \leq t \leq T$ be given where s and t are in different subintervals of the partition P. We then get Itô's integral as:

$$I_t = \sum_{j=0}^{l-1} \Delta_{t_j}[W_{t_{j+1}} - W_{t_j}] + \Delta_{t_l}[W_{t_{l+1}} - W_{t_l}] + \sum_{j=l+1}^{k-1} \Delta_{t_j}[W_{t_{j+1}} - W_{t_j}] + \Delta_{t_k}[W_t - W_{t_k}].$$

For I_t to be a martingale, we need to show that the expected value at time t given the information up to time s, is the value of I_s. Taking the conditional probability of I_t with respect to the information known by time s is the same as taking the conditional probability of each term on the right hand side. We will begin by taking the conditional expectation of the first two terms and showing that it is equal to I_s. We will then proceed with the final two terms and show that the conditional probability is 0. For the first two terms we get:

$$E[\sum_{j=0}^{l-1} \Delta_{t_j}[W_{t_{j+1}} - W_{t_j}] + \Delta_{t_l}[W_{t_{l+1}} - W_{t_l}]|\mathcal{F}_s]$$
$$= E[\sum_{j=0}^{l-1} \Delta_{t_j}[W_{t_{j+1}} - W_{t_j}]|\mathcal{F}_s] + E[\sum_{j=0}^{l-1} \Delta_{t_l}[W_{t_{l+1}} - W_{t_l}]|\mathcal{F}_s].$$

Notice that the terms in the first expression happen before time s as $t_l \leq s$ as is the Δ_{t_l} and the final term in the second expression so we can simplify the above equation to:

$$= \sum_{j=0}^{l-1} \Delta_{t_j}[W_{t_{j+1}} - W_{t_j}] + \Delta_{t_l}[E[W_{t_{l+1}}|\mathcal{F}_s] - W_{t_l}].$$

Now using the martingale property of Brownian motion, we obtain:

$$= \sum_{j=0}^{l-1} \Delta_{t_j}[W_{t_{j+1}} - W_{t_j}] + \Delta_{t_l}[W_s - W_{t_l}]$$
$$= I_s.$$

Since $t_j \geq t_{l+1} > s$ conditioning with our filtration, that is known up to time s, our third expression is:

$$E[\Delta_{t_j}(W_{t_{j+1}} - W_{t_j})|\mathcal{F}_s] = E[(E[\Delta_{t_j}(W_{t_{j+1}} - W_{t_j})]|\mathcal{F}_{t_j})|\mathcal{F}_s]$$
$$= E[\Delta_{t_j}((E[(W_{t_{j+1}})]|\mathcal{F}_{t_j}) - W_{t_j})|\mathcal{F}_s]$$
$$= E[\Delta_{t_j}(W_{t_j} - W_{t_j})|\mathcal{F}_s] = 0.$$

Using these same techniques for the fourth expression we obtain:

$$E[\Delta_{t_k}(W_t - W_{t_k})|\mathcal{F}_s] = 0.$$

Summing these together we get that I_s is a martingale. □

Theorem 7. *(Itô isometry) Itô's integral satisfies*

$$E(I_t^2) = E(\int_0^t \Delta_u^2 du). \tag{2.4}$$

Proof. See [37] page 130. □

From the previous two theorems and $I_0 = 0$, we get the following for the expected value and variance:

$$E(I_t) = 0.$$
$$Var(I_t) = E(I_t^2).$$

Now we can discuss the quadratic variation of I_t which is a process in its upper limit of integration t.

Definition. *(Quadratic Variation) Let $f(t)$ be a function defined for $0 \leq t \leq T$. The quadratic variation of f up to time T is*

$$[f,f](T) = lim_{\|\Pi\|\to 0} \sum_{j=0}^{n-1} [f_{t_j+1} - f_{t_j}]^2,$$

where $\Pi = t_0, t_1, \ldots, t_n$ *and* $0 = t_0 < t_1 < \ldots < t_n = T$.

We know that our Brownian motion W_t accumulates quadratic variation at the rate of one per unit time. Considering how I_t is defined, we get the quadratic variation of Brownian motion scaled by a factor of $\Delta^2(u)$.

Theorem 8. *The quadratic variation accumulated up to time* t *by* I_t *is:*

$$[I,I]_t = \int_0^t \Delta_u^2 du.$$

Proof. See [37] pg 131. □

Another way to view the same result for the quadratic variation of I_t is using the following setup:

$$dI_t = \Delta_t dW_t.$$
$$dI_t dI_t = \Delta_t^2 dW_t dW_t.$$
$$= \Delta_t^2 dt.$$

The previous theorems and results, we have a process in which the variance and quadratic variation are different. This is due to the fact that the quadratic variation, sometimes regarded as a measure of risk, depends on how large of positions ($\Delta(u)$) we choose in its calculation. On the other hand, the variance of I_t depends on all of the possible position sizes that we can choose for quadratic variation and is taken as the average over all of these possible paths and thus, cannot be random.

2.3. Itô's integral for general integrands

How does Itô's integral change if Δ_t is not simple (changes continuously with time and/or has jumps)? We will do a very common idea in probability theory and approximate Δ_t by simple processes. First, it is assumed that $\Delta_t, t \geq 0$ is adapted to some filtration for $t \geq 0$ and that the square-integrability condition holds:

$$E(\int_0^T \Delta_t^2 dt) < \infty. \tag{2.5}$$

First, we will partition the time interval where $0 = t_0 < t_1 < t_2 < t_3 < \ldots < t_n = T$ and form our approximation on the intervals $[t_j, t_{j+1})$. The simple processes that we will choose to approximate Δ_t will be assumed to be constant and equal to Δ_{t_j} on the interval $[t_j, t_{j+1})$. This can be seen in figure 2.1:

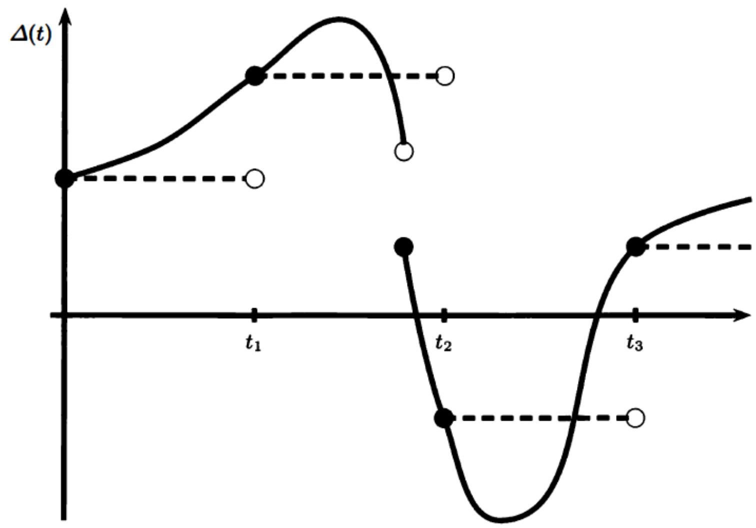

Figure 2.1. Approximating the integrand [38].

As $n \to \infty$, the sequence Δ_t^n of simple processes will converge to Δ_t., i.e.:

$$\lim_{n\to\infty} E(\int_0^T |\Delta_t^n - \Delta_t|^2 dt) = 0,$$

in probability. Therefore, since the Itô integral is defined for simple processes, we now define the Itô integral for the general integrand as:

$$\int_0^t \Delta_u dW_u = \lim_{n\to\infty} \int_0^t \Delta_u^n dW_u, \tag{2.6}$$

where, again, the limit can be shown to converge in probability. Since the right hand side is the limit of an Itô integral of a simple process, the Itô integral of a general process inherits the same properties.

Theorem 9. *Let T be a positive constant and let $\Delta_t, 0 \le t \le T$, be an adapted stochastic process satisfying the square-integrability condition. Then I_t has the following properties.*

a) As a function of the upper limit of integration t, the paths of I_t are continuous.

b) For each t, I_t is $\mathcal{F}_t-measurable$.

c) If $I_t = \int_0^t \Delta_u dW_u$ and $J_t = \int_0^t \Gamma_u dW_u$, then $I_t \pm J_t = \int_0^t (\Delta_u \pm \Gamma_u) dW_u$ and for every constant c, $cI_t = \int_0^t c\Delta_u dW_u$.

d) I_t is a martingale.

e) $E(I_t^2) = E(\int_0^t \Delta_u^2 du)$.

f) $[I, I]_t = \int_0^t \Delta_u^2 du$.

For an example see [37] pages 134-137.

2.4. Itô's formula

What occurs if we want to differentiate expressions of the form $f(W_t)$? One's first thought is perform the chain rule as in ordinary calculus, but as discussed earlier, Brownian motion is not differentiable. To coincide with the integration that he created, Itô also formed the chain rule for Brownian motion as follows:

$$df(W_t) = f'(W_t)dW_t + \frac{1}{2}f''(W_t)dt. \tag{2.7}$$

This is known as the Itô formula. As with previous formulas, there is an analogous integral form:

$$f(W_t) - f(W_0) = \int_0^t f'(W_u)dW_u + \frac{1}{2}\int_0^t f''(W_u)du. \tag{2.8}$$

In the integral form, we have talked in the previous couple of sections how to compute the first term on the right hand side and the second term on the right hand side is an ordinary integral with respect to time. Therefore, we know how to calculate the entire integral form of the Itô formula even though there are not precise definitions of some of the terms in the differential form.

Theorem 10. *(Itô formula for Brownian motion) Let $f(t, x)$ be a function with partial derivatives $f_t(t, x)$, $f_x(t, x)$, and $f_{xx}(t, x)$ that are defined and continuous and with W_t a Brownian motion. Then, $\forall T \geq 0$,*

$$f(T, W_T) = f(0, W_0) + \int_0^T f_t(t, W_t)dt + \int_0^T f_x(t, W_t)dW_t + \tfrac{1}{2}\int_0^T f_{xx}(t, W_t)dt.$$

Proof. For a sketch of the proof see [37] pages 138-141. □

We will now extend Itô's formula to an Itô process.

Definition. *(Itô Process) Let $W_t, t \geq 0$ be a Brownian motion, and let $\mathcal{F}, t \geq 0$ be an associated filtration. An Itô process is a stochastic process of the following form where X_0 is nonrandom and Δ_u and Θ_u are adapted stochastic processes.*

$$X_t = X_0 + \int_0^t \Delta_u dW_u + \int_0^t \Theta_u du.$$

Theorem 11. *The quadradic variation of the Itô process is:*

$$[X, X]_t = \int_0^t \Delta_u^2 du.$$

Proof. See [37] pages 143-144. □

The conclusion of this theorem can be remembered by computing $dX_t dX_t$ when:

$$dX_t = \Delta_t dW_t + \Theta_t dt.$$

We can then use the following to find $dX_t dX_t$:

$$dW_t dW_t = dt.$$
$$dt dW_t = dW_t dt = 0.$$
$$dt dt = 0.$$

Therefore, we obtain:

$$\begin{aligned} dX_t dX_t &= \Delta_t^2 dW_t dW_t + 2\Delta_t \Theta_t dW_t dt + \Theta_t^2 dt dt \\ &= \Delta_t^2 dt. \end{aligned}$$

So at each instant of time, the Itô process is accumulating Δ_t^2 quadratic variation and hence we get the conclusion of the previous theorem. We are now ready to generalize Itô's formula for any Itô process.

Definition. *Let $X_t, t \geq 0$ be an Itô process and let $\Gamma_t, t \geq 0$ be an adapted process. Then the integral with respect to an X_t is:*

$$\int_0^t \Gamma_u dX_u = \int_0^t \Gamma_u \Delta_u dW_u + \int_0^t \Gamma_u \Theta_u du. \tag{2.9}$$

We can now generalize Itô's formula for any Itô process.

Theorem 12. *Let $X_t, t \geq 0$ be an Itô process and let $f(t,x)$ be a function whose partial derivatives $f_t(t,x)$, $f_x(t,x)$, and $f_{xx}(t,x)$ are defined and continuous. Then, $\forall T \geq 0$:*

$$f(T, X_t) = f(0, X_0) + \int_0^T f_t(t, X_t)dt + \int_0^T f_x(t, X_t)dX_t + \tfrac{1}{2}\int_0^T f_{xx}(t, X_t)d[X, X]_t$$
$$= f(0, X_0) + \int_0^T f_t(t, X_t)dt + \int_0^T f_x(t, X_t)\Delta_t dW_t + \int_0^T f_x(t, X_t)\Theta_t dt + \tfrac{1}{2}\int_0^T f_{xx}(t, X_t)\Delta_t^2 dt.$$

The differential form is:

$$df(t, X_t) = f_t(t, X_t)dt + f_x(t, X_t)dX_t + \tfrac{1}{2}f_{xx}(t, X_t)dX_t dX_t$$
$$= f_t(t, X_t)dt + f_x(t, X_t)\Delta_t dW_t + f_x(t, X)t)\Theta_t dt + \tfrac{1}{2}f_{xx}(t, X_t)\Delta_t^2 dt.$$

For further study of the material presented in this section, see [37].

3. BLACK-SCHOLES-MERTON EQUATION

3.1. Option Types

Definition. *Options are financial derivatives that give the buyer the right to buy or sell the underlying asset at a stated price within a specified period.*

In this section, we introduce the following vanilla options [39]: call options and put options. How do they differ and why choose one over the other?

3.1.1. European call option

The European call option is the simplest of the financial options available and is a contract with the following conditions:

At a prescribed time in the future, known as the expiry date, the holder of the option can either

1) purchase a prescribed asset, known as the underlying asset, for a prescribed amount, or
2) pass on the purchase of the underlying asset.

This implies that for the person who owns the European call option, the contract is a right and not an obligation. As this is a contract, there are two parties, the holder, who decides whether or not to execute the option, and the writer, whom does have the obligation to sell the asset if the holder decided to buy. Due to this obligation to sell if the holder chooses to buy, contracts for options have a premium in order to purchase them. This helps ensure that the writer does not assume all of the risk for the underlying asset. We will use European call options throughout this paper.

What happens if the holder wants to sell before the expiry date due to a massive profit margin? For a European call option, this is not allowed as the holder can only exercise the option on the expiry date. This leads to the next type of option, the American call option.

3.1.2. American call option

An American call option is similar to the holder/writer description of the European call option, but the holder can choose to exercise the option at any time prior to the expiry date. We still need to be able to assign a value to this option, but the holder must determine when it is best to exercise the option. This final aspect is unique to the American call option. For further study of pricing and exercising American call options, see [39] chapters 7-9.

3.1.3. European put option

We have discussed the option to buy assets above, namely call options. The right to sell an asset is known as a put option and flips the payoff properties of a call option. A European put option allows the holder to sell assets on a specified date for a predetermined (at the time of buying the option) amount. The writer would then be obligated to buy the assets that they have agreed to in the option contract. Unlike a European call option, the holder of a European put option wants the price of the asset to diminish in value. This allows the holder to buy the asset at a lower price on the day of expiry and sell it to the writer at the contracted price and that difference in price is the holder's profit.

3.1.4. American put option

An American put option is similar to the holder/writer description of the European put option, but the holder can choose to exercise the option at any time prior to the expiry date. We still need to be able to assign a value to this option, but the holder must determine when it is best to exercise the option.

3.2. Portfolio value evaluation

How should we approach investing between a money market account and the stock market based on the amount of money available to us? We will consider that at time t, our portfolio has a value of X_t and we invest into a money market account with a constant interest rate r and a stock modeled by geometric Brownian motion:

$$dS_t = \alpha S_t dt + \sigma S_t dW_t,$$

where α is the drift coefficient and σ is the diffusion coefficient. We will assume for each time t, we hold Δ_t shares of the aforementioned stock which can be random as long as it is adapted to the same filtration as our Brownian motion W_t. We then invest the remaining value of our portfolio, $X_t - \Delta_t S_t$, into the money market account. As with most investments, we want to know how the value of our portfolio is changing with time which in our setup relies on two factors, the change in the stock price and the interest earned on our money market account. With these being the two factors that affect our portfolio value, we arrive at the differential for our portfolio value being:

$$dX_t = \Delta_t dS_t + r(X_t - \Delta_t S_t)dt$$

$$
\begin{aligned}
&= \Delta_t(\alpha S_t dt + \sigma S_t dW_t) + r(X_t - \Delta_t S_t)dt \\
&= \mathrm{rX}_t dt + \Delta_t(\alpha - r)S_t dt + \Delta_t \sigma S_t dW_t.
\end{aligned} \tag{3.1}
$$

To make sense of this equation, we need to understand what the terms on the right hand side measure. The first term, $rX_t dt$, is an underlying return on the portfolio. This return does not take into consideration how much of the portfolio is put into the stock and how much is in the money market account so we are guaranteed a set return rate based solely on the amount of money we have invested. The second term, $\Delta_t(\alpha - r)S_t dt$, takes into account the risk premium, $\alpha - r$ for investing in the stock. We consider this a risk because their is always the chance that the stock does worse than just a straight investment into the money market account. In this scenario, it would have been more beneficial to us to have invested solely in the money market. There is no way to know if this is the case until after we know the stock price at the later time at which point it is too late to switch our investment which is the risk we take on. The final term, $\Delta_t S_t dW_t$, represents the volatility of our investments proportional to the size of the stock investment. This volatility is due to our stock being modeled by a geometric Brownian motion which as we know is a random process. What this means is that since our money market investment has a fixed return rate r, then the sole piece of our investment that is driving our portfolio's price volatility is our investments into the stock market. This can be understood because if we solely invested in the money market, we would know exactly how much money is in our portfolio at any time due to the fixed return rate.

The discounted stock price, $e^{-rt}S_t$, and the discounted portfolio value, $e^{-rt}X_t$ are often considered. The reason for this consideration is that the mean rate of return will be reduced from α to $\alpha - r$ or the risk premium term from the original consideration. We will also see that considering the discounted stock price will also remove the underlying rate of return so that we get our return as a function of S_t, t, and Δ_t. To determine the differential of the discounted stock price, let $f(t, S_t) = e^{-rt}S_t$. We will now use Itô's formula to determine the differential:

$$
\begin{aligned}
d(e^{-rt}S_t) &= -re^{-rt}S_t dt + e^{-rt}dS_t \\
&= (\alpha - r)e^{-rt}S_t dt + \sigma e^{-rt}S_t dW_t.
\end{aligned} \tag{3.2}
$$

We can do a similar procedure for the discounted portfolio value and we obtain:

$$d(e^{-rt}X_t) = -re^{-rt}X_t dt + e^{-rt}dX_t$$

$$=\Delta_t(\alpha - r)e^{-rt}S_t dt + \Delta_t \sigma e^{-rt}S_t dW_t$$

$$=\Delta_t d(e^{-rt}S_t). \tag{3.3}$$

3.3. Option value evolution

We will now consider a European call option that pays out $(S_t - K)^+ = max(0, S_t - K)$ at time T where K is the strike price which is a nonnegative constant. Black, Scholes, and Merton argued that the value of this option relies on the time remaining until expiration and the value of the stock at the time we are evaluating the value. It should also depend on the model parameters r and σ, and finally the strike price K. If we let $c(t, S_t)$ denote the value of the call at time t, then we have a stochastic process as the stock price is random. Therefore, as with the stock price discussed early, we do not know any future value of the stock price and therefore, we do not know any future value of $c(t, S_t)$. Even though we cannot calculate an exact value, our goal is to find a formula of the future call option for when we know the future stock price.

We will proceed as in the previous section with our European call option price in place of our stock price. Using Itô's formula on the call price, we obtain:

$$dc(t, S_t) = c_t(t, S_t)dt + c_x(t, S_t)(\alpha S_t dt + \sigma S_t dW_t) + \frac{1}{2}c_{xx}(t, S_t)\sigma^2 S_t^2 dt$$

$$=\Big[c_t(t, S_t) + \alpha S_t c_x(t, S_t) + \tfrac{1}{2}\sigma^2 S_t^2 c_{xx}(t, S_t)\Big]dt + \sigma S_t c_x(t, S_t)dW_t. \tag{3.4}$$

We will now consider the discounted European call price $e^{-rt}c(t, S_t)$ and let $f(t, x) = e^{-rt}x$, then using Itô's formula, we obtain:

$$d(e^{-rt}c(t, S_t)) = -re^{-rt}c(t, S_t)dt + e^{-rt}dc(t, S_t)$$

$$=e^{-rt}\Big[-rc(t, S_t) + c_t(t, S_t) + \alpha S_t c_x(t, S_t) + \tfrac{1}{2}\sigma^2 S_t^2 c_{xx}(t, S_t)\Big]dt + e^{-rt}\sigma S_t c_x(t, S_t)dW_t. \tag{3.5}$$

3.4. Equality of the evolutions

A hedging portfolio starts with an initial capital X_0 and invests in the stock and money markets such that X_t and $c(t, S_t)$ agree $\forall t \in [0, T]$. This occurs if and only if the discounted portfolio value is equal to the discounted call option value for all t. One way to ensure this equality is to have the following equivalence:

$$d(e^{-rt}X_t) = d(e^{-rt}c(t, S_t), \forall t \in [0, T), \tag{3.6}$$

and $X_0 = c(0, S_0)$. If we integrate (3.6) from 0 to t, we obtain:

$$e^{-rt}X_t - X_0 = e^{-rt}c(t, S_t) - c(0, S_0), \forall t \in [0, T). \tag{3.7}$$

Since we know that the equality only holds if $X_0 = c(0, S_0)$, we can make a cancellation and obtain our desired equality. Making the comparison between (3.3) and (3.5), we know that (3.6) holds if and only if the following equality holds:

$$\Delta_t(\alpha - r)e^{-rt}S_t dt + \Delta_t \sigma e^{-rt} S_t dW_t$$

$$= \mathrm{e}^{-rt}\Big[-rc(t, S_t) + c_t(t, S_t) + \alpha S_t c_x(t, S_t) + \tfrac{1}{2}\, \sigma^2 S_t^2 c_{xx}(t, S_t)\Big] dt + e^{-rt}\sigma S_t c_x(t, S_t) dW_t. \tag{3.8}$$

We will now examine what is required for the above equation to hold by equating parts of the equation. First, equating the dW_t terms gives:

$$\Delta_t = c_x(t, S_t), \quad \forall t \in [0, T). \tag{3.9}$$

This is referred to as the delta-hedging rule. The number of shares of stock held by the hedge at each time t prior to expiration is the partial derivative of the option value with respect to the stock's price. The quantity $c_x(t, S_t)$ is called the delta of the option. We will now equate the dt terms of (3.8), using (3.9), to obtain

$$(\alpha - r)S_t c_x(t, S_t) = -rc(t, S_t) + c_t(t, S_t) + \alpha S_t c_x(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 c_{xx}(t, S_t), \quad \forall t \in [0, T) \tag{3.10}$$

Cancelling the $\alpha S_t c_x(t, S_t)$ on both sides of the equation we obtain:

$$rc(t, S_t) = c_t(t, S_t) + rS_t c_x(t, S_t) + \frac{1}{2}\sigma^2 S_t^2 c_{xx}(t, S_t), \quad \forall t \in [0, T). \tag{3.11}$$

Therefore, we seek a continuous function $c(t, x)$ that solves the Black-Scholes-Merton partial differential equation:

$$c_t(t, x) + rxc_x(t, x) + \frac{1}{2}\sigma^2 x^2 c_{xx}(t, x) = rc(t, x), \forall t \in [0, T), x \geq 0, \tag{3.12}$$

and satisfies:

$$c(T, x) = (x - K)^+. \tag{3.13}$$

Once found, the function that solves this equation will meet all requirements for our portfolio equation as well as the requirements for our option equation. Therefore, this function will give us equality $\forall t \in [0, T)$. If we take the limit as $t \uparrow T$ and the fact that both X_t and $c(t, S_t)$ are continuous, we can conclude that $X_T = c(T, S_T) = (S_t - K)^+$. Therefore, our portfolio has a value that agrees with the option payoff.

3.5. Solution to the Black-Scholes-Merton equation

In this section, rather than deriving the solution to the Black-Scholes-Merton equation, we will present a solution and check that it meets all criteria. We want the Black-Scholes-Merton equation to hold for all $x \geq 0$ and $t \in [0, T)$ so (3.12) holds regardless of the stock's price path. If the stock starts at a value of zero, then it will remain at a value of zero, but if the starting price is positive, then it remains positive and can take on any value, both of these cases are covered by requiring (3.12) to hold for $x \geq 0$.

The Black-Scholes-Merton equation (3.12), is a backward parabolic partial differential equation. Therefore, we need boundary conditions at $x = 0$ and $x = \infty$ to determine the solution. Substituting $x = 0$ into (3.12), we obtain:

$$c_t(t, 0) = rc(t, 0). \tag{3.14}$$

This is an ordinary differential equation with solution:

$$c(t, 0) = e^{rt} c(0, 0).$$

Since $c(T, 0) = (0 - K)^+ = 0$, we have:

$$c(t, 0) = 0 \quad \forall t \in [0, T]. \tag{3.15}$$

One way to specify the boundary condition at $x = \infty$ is:

$$lim_{x \to \infty}[c(t, x) - (x - e^{-r(T-t)}K)] = 0, \quad \forall t \in [0, T]. \tag{3.16}$$

Which means that $c(t, x)$ grows at the same rate as x as $x \to \infty$.

Taking into consideration the terminal condition (3.13) along with the boundary conditions (3.15) and (3.16), the solution to the Black-Scholes-Merton equation is [37], pg 159:

$$c(t,x) = xN(d_+(T-t,x)) - Ke^{-r(T-t)}N(d_-(T-t,x)), \quad 0 \le t < T, \quad x > 0 \tag{3.17}$$

where

$$d_\pm(\tau, x) = \frac{1}{\sigma\sqrt{\tau}}\left[log(\frac{x}{K}) + \left(r \pm \frac{\sigma^2}{2}\right)\tau\right], \tag{3.18}$$

and N is the cumulative standard normal distribution

$$N(y) = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y} e^{-x^2/2}dz = \frac{1}{\sqrt{2\pi}}\int_{-y}^{\infty} e^{-x^2/2}dz. \tag{3.19}$$

Note that (3.17) does not define $c(t,x)$ when $t = T$ or when $x = 0$ as we divide by 0 and have log(0) respectively. However, (3.17) defines $c(t,x)$ such that $lim_{t\to T}c(t,x) = (x-K)^+$ and $lim_{x\downarrow 0}c(t,x) = 0$.

For further study of the material presented in this section, see [37].

4. LÉVY PROCESSES

4.1. Introduction

As we look at data for asset pricing, especially stock pricing, we see jumps or spikes that occur that need to be accounted for. These spikes can happen for various reasons, companies getting major contracts, massive stock dumping of a companies stocks, etc, but no matter why they happen, models based off of Brownian motion are not the most accurate to describe these instances. In order to overcome these situations, we will introduce Lévy processes which have independent and stationary increments.

Definition. *[Lévy process] A càdlàg stochastic process* $\{X_t\}_{t\geq 0}$ *on* $(\Omega, \mathcal{F}, P)$ *with values in* R^d *such that* $X_0 = 0$ *is called a Lévy process if it satisfies the following properties:*

1) Independent increments: for every increasing sequence of times $t_0, \ldots, t_n$*, the random variables* $X_{t_0}, X_{t_1} - X_{t_0}, \ldots, X_{t_n} - X_{t_{n-1}}$ *are independent.*

2) Stationary increments: the law of $X_{t+h} - X_t$ *does not depend on* t*.*

3) Stochastic continuity: $\forall \varepsilon > 0$, $lim_{h\to 0} P(|X_{t+h} - X_t| \geq \varepsilon) = 0$.

In the following sections we will discuss two processes, the Poisson process and the Compound Poisson process. We will then discuss how the stochastic calculus we have done previously changes to include processes with jumps.

4.2. Poisson process

4.2.1. Construction

Consider a sequence of independent exponential random variables $\tau_1, \tau_2, \ldots$, all with the same mean $\frac{1}{\lambda}$. For each τ_i, the density function is:

$$f(t) = \begin{cases} 0 & t < 0 \\ \lambda e^{-\lambda t} & t \geq 0. \end{cases}$$

Now we will build a process that allows jumps to occur. The first jump will occur at time τ_1, there will then be a time span of τ_2 units before the second jump, and similarly, a time span of τ_i units between the $(i-1)^{th}$ jump and the i^{th} jump. In this process, the τ_i are called interarrival times and the arrival times are:

$$S_n = \sum_{i=1}^{n} \tau_i.$$

The Poisson process N_t counts the number of jumps prior to time t:

$$N_t = \begin{cases} 0 & S_0 \leq t < S_1 \\ 1 & S_1 \leq t < S_2 \\ \dots & \\ n & S_n \leq t < S_{n+1} \\ \dots & \end{cases} \tag{4.1}$$

We will denote the σ-algebra of information given by N_t for $0 \leq s \leq t$ by $\mathcal{F}_t$. Also note, the jumps of the Poisson process average λ jumps per time unit and we say that N_t has intensity λ.

4.2.2. Poisson process increments

In order to begin to understand the increments of the Poisson process, we need to understand the distribution of the jump times.

Theorem 13. *For $n \geq 1$, the random variable S_n has the gamma density*

$$g_n(s) = \frac{(\lambda s)^{n-1}}{(n-1)!} \lambda e^{-\lambda s}, \quad s \geq 0.$$

Proof. See [37] pg. 464. □

Theorem 14. *The Poisson process N_t with intensity λ has the distribution:*

$$P[N_t = k] = \frac{(\lambda t)^k}{k!} e^{-\lambda t}, \quad k = 0, 1, \dots.$$

Proof. For $k \geq 1$, $N_t \geq k$ if and only if there are at least k jumps prior to time t, thus:

$$P[N_t \geq k] = P[S_k \leq t] = \int_0^t \frac{(\lambda s)^{k-1}}{(k-1)!} \lambda e^{-\lambda s} ds.$$

Similarly,

$$P[N_t \geq k+1] = P[S_{k+1} \leq t] = \int_0^t \frac{(\lambda s)^k}{(k)!} \lambda e^{-\lambda s} ds.$$

Integration by parts of the previous expression yields:

$$P[N_t \geq k+1] = -\frac{(\lambda s)^k}{(k)!} e^{-\lambda s}|_{s=0}^{s=t} + \int_0^t \frac{(\lambda s)^{k-1}}{(k-1)!} \lambda e^{-\lambda s} ds$$
$$= -\frac{(\lambda t)^k}{(k)!} e^{-\lambda t} + P[N_t \geq k].$$

This implies for $k \geq 1$,

$$P[N_t = k] = P[N_t \geq k] - P[N_t \geq k+1] = \frac{(\lambda t)^k}{k!} e^{-\lambda t}.$$

For $k = 0$, we get,

$$P[N_t = 0] = P[S_1 > t] = P[\tau_1 > t] = e^{-\lambda t},$$

which is the needed equation with $k = 0$. □

What can be determined about the distribution of $N_{t+s} - N_s$ conditioned on the information up to and including at time s? Since N_{t+s} tells us how many jumps have occurred up to and including time $t+s$ and N_s tells us how many jumps occurred up to time s, then $N_{t+s} - N_s$ tells us the number of jumps in an interval of length t. Therefore, the interval does not depend on s and keeps the same exponential distribution with mean $\frac{1}{\lambda}$. Thus, $N_{t+s} - N_s$ is independent of $\mathcal{F}_s$ and is the same as the distribution of N_t.

Theorem 15. *Let N_t be a Poisson process with intensity $\lambda > 0$ and let $0 = t_0 < t_1 < \ldots < t_n$ be given. Then $N_{t_1} - N_{t_0}$, $N_{t_2} - N_{t_1}, \ldots,$ $N_{t_n} - N_{t_{n-1}}$ are stationary and independent. Also,*

$$P[N_{t_{j+1}} - N_{t_j} = k] = \frac{\lambda^k (t_{j+1} - t_j)^k}{k!} e^{-\lambda(t_{j+1} - t_j)}, \quad k = 0, 1, \ldots.$$

Proof. For an outline of this proof, see [37] pg 466. □

4.2.3. Poisson process as a martingale

The expected value of the poison increment $N_t - N_s$ is as follows:

$$E[N_t - N_s] = \sum_{k=0}^{\infty} k e^{-\lambda(t-s)} \frac{\lambda^k (t-s)^k}{k!}$$
$$= \lambda(t-s) e^{-\lambda(t-s)} \sum_{k=1}^{\infty} \frac{\lambda^{k-1} (t-s)^{k-1}}{(k-1)!}$$
$$= \lambda(t-s) e^{-\lambda(t-s)} e^{\lambda(t-s)}$$
$$= \lambda(t-s)$$

Consider the compensated Poisson process $M_t = N_t - \lambda t$ where N_t is a Poisson process with intensity λ.

Theorem 16. *M_t is a martingale with respect to the filtration $\{\mathcal{F}_t\}_{t\geq 0}$ on N_t.*

Proof. Let $0 \leq s < t$ be given, then $N_t - N_s$ is independent of F_s and has expected value $\lambda(t-s)$. Then,

$$\begin{aligned} E[M_t|F_s] &= E[M_t - M_s|F_s] + E[M_s|F_s] \\ &= E[N_t - N_s - \lambda(t-s)|F_s] + M_s \\ &= E[N_t - N_s] - \lambda(t-s) + M_s \\ &= M_s. \end{aligned}$$

□

4.3. Compound Poisson process

Let N_t be a Poisson process with intensity λ, and let $Y_1, Y_2, \ldots$ be a sequence of independent identically distributed random variables with $\beta = E(Y_i)$ that are also independent of N_t. A compound Poisson process is defined as,

$$Q_t = \sum_{i=1}^{N_t} Y_i, \quad t \geq 0. \tag{4.2}$$

Comparing N_t and Q_t, the jumps occur at the same, but the jumps in Q_t are random in size. The i^{th} jump of Q_t has size Y_i. Also, the increments of the compound Poisson process are independent, for $0 < t$,

$$Q_s = \textstyle\sum_{i=1}^{N_s} Y_i$$

Which sums up the first N_s jumps, thus

$$Q_t - Q_s = \textstyle\sum_{i=N_s+1}^{N_t} Y_i,$$

which sums up jumps $N_s + 1$ to N_t, are independent. Since $N_t - N_s$ has the same distribution as N_{t-s}, $Q_t - Q_s$ has the same distribution as Q_{t-s}.

4.3.1. Compound Poisson process as a martingale

The expected value of the compound Poisson process is as follows:

$$\begin{aligned} E[Q_t] = &\textstyle\sum_{k=0}^{\infty} E\left[\sum_{i=1}^{k} Y_i \middle| N_t = k\right] P[N_t = k] \\ =&\textstyle\sum_{k=0}^{\infty} \beta k e^{-\lambda t} \frac{(\lambda t)^k}{k!} \end{aligned}$$

$$=\beta\lambda t e^{-\lambda t} \sum_{k=1}^{\infty} \frac{(\lambda t)^{k-1}}{(k-1)!}$$
$$=\beta\lambda t.$$

Consider the compensated compound Poisson process $Q_t - \beta\lambda t$.

Theorem 17. *$Q_t - \beta\lambda t$ is a martingale.*

Proof. Let $0 \leq s < t$ be given, then $Q_t - Q_s$ is independent of F_s and has expected value $\beta\lambda(t-s)$. Then,

$$E[Q_t - \beta\lambda t | F_s] = E[Q_t - Q_s | F_s] + Q_s - \beta\lambda t$$
$$=\beta\lambda(t-s) + Q_s - \beta\lambda t$$
$$=Q_s - \beta\lambda s.$$

□

4.4. The calculus of jump processes

Our jump processes in this section will consist of Brownian motion, a Poisson process, and a compound Poisson process all adapted to the same filtration $\mathcal{F}$. This means that for $t \geq 0$ W_t, N_t, and Q_t are $\mathcal{F}_t - measurable$ and for every $u > t$ the increments $W_u - W_t$, $N_u - N_t$, and $Q_u - Q_t$ are independent of $\mathcal{F}_t$. Using this, we will determine the stochastic integral:

$$\int_0^t \phi_s dX_s, \tag{4.3}$$

where X can have jumps and is right-continuous. We will consider integrators X that have the following form:

$$X_t = X_0 + I_t + R_t + J_t$$

where X_0 is a nonrandom initial condition. The process

$$I_t = \int_0^t \Gamma_s dW_s$$

is an Itô integral where W_s is a Brownian motion and Γ_s is a process adapted to W_s. The process

$$R_t = \int_0^t \Theta_s ds$$

is a Riemann integral for some adapted process Θ_t. Therefore, the continuous part of X_t is defined to be

$$X_t^c = X_0 + I_t + R_t = X_0 + \int_0^t \Gamma_s dW_s + \int_0^t \Theta_s ds.$$

The quadratic variation of this process is:

$$dX_t^c dX_t^c = \Gamma_t^2 dt.$$

The final part of X_t is J_t which is a right-continuous pure jump process with $J_0 = 0$. If J has a jump at time t, then J_t is the value of J immediately after the jump, whereas J_{t-} is the value immediately before the jump. We will assume that J does not jump at time zero, has finitely many jumps in each finite time interval $(0, T]$, and is constant between jumps.

We will call the process X_t as defined above a jump process. Since I_t and R_t are continuous, the left-continuous version of X_t is:

$$X_{t-} = X_0 + I_t + R_t + J_{t-}. \tag{4.4}$$

Definition. *Let X_t be a jump process and ϕ_s an adapted process. Then the stochastic integral of ϕ with respect to X is defined as:*

$$\int_0^t \phi_s dX_s = \int_0^t \phi_s \Gamma_s dW_s + \int_0^t \phi_s \Theta_s ds + \sum_{0<s\leq t} \phi_s \Delta J_s. \tag{4.5}$$

In differential notation we have:

$$\phi_t dX_t = \phi_t dI_t + \phi_t dR_t + \phi_t dJ_t$$

$$= \phi_t dX_t^c + \phi_t dJ_t, \tag{4.6}$$

where

$$\phi_t dI_t = \phi_t \Gamma_t dW_t,\ \phi_t dR_t = \phi_t \Theta_t dt.$$

Similar to Brownian motion, we want to ensure that the integral for a jump process is a martingale so that we can replace the integrand by a position in an asset. Consider an investor in the compensated Poisson process M_t who chooses his position according to the formula $\phi_s = \Delta N_s$. This means the only time his investment could change value is when jumps occur and since all of the jumps of M_s

are always positive, then the investor would become extremely rich. At the same time however, if any investor were able to do this, then they would have to have insider knowledge and would be caught for insider trading so we want to find a way to manipulate this scenario such that the investor can still be successful while remaining out of prison. Note that ϕ_s only depends on the path of the underlying process M up to and including time s and does not depend on anything that happens after. The following theorem is given without proof.

Theorem 18. *Assume that the jump process X_s is a martingale, the integrand ϕ_s is left-continuous and adapted, and*

$$E[\int_0^t \Gamma_s^2 \phi_s^2 ds] < \infty \quad \forall t \geq 0.$$

Then the stochastic integral $\int_0^t \phi_s dX_s$ is also a martingale.

4.4.1. Itô's formula for a one jump process

If we had a continuous-path process, then the Itô formula is as follows:

$$X_t^c = X_0^c + \int_0^T \Gamma_s dW_s + \int_0^t \Theta_s ds,$$

with the differential notation being:

$$dX_s^c = \Gamma_s dW_s + \Theta_s ds,$$
$$dX_s^c dX_s^c = \Gamma_s^2 ds.$$

Now, let $f(x)$ be a function with continuous first and second derivatives. Then,

$$df(X_s^c) = f'(X_s^c) dX_s^c + \frac{1}{2} f''(X_s^c) dX_s^c dX_s^c$$

$$=\text{f'(X}_s^c)\Gamma_s dW_s + f'(X_s^c)\Theta_s ds + +\tfrac{1}{2} f''(X_s^c)\Gamma_s^2 ds. \tag{4.7}$$

The integral notation is:

$$f(X_t^c) = f(X_0^c) + \int_0^t f'(X_s^c)\Gamma_s dW_s + \int_0^t f'(X_s^c)\Theta_s ds + \frac{1}{2}\int_0^t f''(X_s^c)\Gamma_s^2 ds. \tag{4.8}$$

Now let $X_t = X_0 + I_t + R_t + J_t$, where J is a right-continuous pure jump term and $I_t = \int_0^t \Gamma_s dW_s$ and $R_t = \int_0^t \Theta_s ds$. Between jumps of J, X is continuous and thus we can use the above, but if there is a jump in X from X_{s-} to X_s, then there typically will be a resultant jump in $f(X)$ from

$f(X_{s-})$ to $f(X_s)$. Therefore, we need to add the jumps in when we integrate both sides from 0 to t. Thus, we get the following theorem.

Theorem 19. *(Itô formula for a one jump process) Let X_t be a jump process and $f(x)$ a function for which $f'(x)$ and $f''(x)$ are defined and continuous. Then,*

$$f(X_t) = f(X_0) + \int_0^t f'(X_s)dX_s^c + \frac{1}{2} \int_0^t f''(X_s)dX_s^c dX_s^c + \sum_{0<s\leq t}[f(X_s) - f(X_{s-})].$$

Proof. See [37] page 485. □

4.4.2. Itô-Doeblin formula for multiple jump processes

In this section, we will give the two-dimensional version of the Itô-Doeblin formula for processes with jumps. To extend this to higher dimensions, it will follow the same pattern.

Theorem 20. *Let X_t^1 and X_t^2 be jump processes, and let $f(t, x_1, x_2)$ be a function whose first and second partial derivatives appearing in the following are defined and continuous. Then,*

$$\begin{aligned} f(t, X_t^1, X_t^2) &= f(0, X_0^1, X_0^2) + \int_0^t f_t(s, X_s^1, X_s^2)ds + \int_0^t f_{x_1}(s, X_s^1, X_s^2)dX_s^{1c} \\ &+ \int_0^t f_{x_2}(s, X_s^1, X_s^2)dX_s^{2c} + \frac{1}{2}\int_0^t f_{x_1,x_1}(s, X_s^1, X_s^2)dX_s^{1c}dX_s^{1c} \\ &+ \int_0^t f_{x_1,x_2}(s, X_s^1, X_s^2)dX_s^{1c}dX_s^{2c} + \frac{1}{2}\int_0^t f_{x_2,x_2}(s, X_s^1, X_s^2)dX_s^{2c}dX_s^{2c} \\ &+ \sum_{0<s\leq t}[f(s, X_s^1, X_s^2) - f(s, X_{s-}^1, X_{s-}^2)]. \end{aligned}$$

Theorem 21. *(Itô's product rule for jump processes). Let X_t^1 and X_t^2 be jump processes. Then,*

$$\begin{aligned} X_t^1 X_t^2 &= X_0^1 X_0^2 + \int_0^t X_s^2 dX_s^{1c} + \int_0^t X_s^1 dX_s^{2c} \\ &+ [X^{1c}, X^{2c}](t) + \sum_{0<s\leq t}[X_s^1 X_s^2 - X_{s-}^1 X_{s-}^2] \\ &= X_0^1 X_0^2 + \int_0^t X_{s-}^2 dX_s^1 + \int_0^t X_{s-}^1 dX_s^2 + [X^1, X^2](t). \end{aligned}$$

Proof. Take $f(x_1, x_2) = x_1 x_2$ so that

$$f_{x_1} = x_2,\ f_{x_2} = x_1,\ f_{x_1 x_1} = 0,\ f_{x_1 x_2} = 1,\ f_{x_2 x_2} = 0.$$

Then, the two-dimensional It$\hat{0}$-Doeblin formula implies,

$$X_t^1 X_t^2 = X_0^1 X_0^2 + \int_0^t X_s^2 dX_s^{1c} + \int_0^t X_s^1 dX_s^{2c} + \int_0^t 1 dX_s^{1c} dX_s^{2c} + \sum_{0<s\leq t}[X_s^1 X_s^2 - X_{s-}^1 X_{s-}^2].$$

In the above equation, $\int_0^t 1 dX_s^{1c} dX_s^{2c} = [X^{1c}, X^{2c}](t)$, thus the first equality of the theorem is true. We will denote the pure jump parts of X_t^1 and X_t^2 as $J_t^1 = X_t^1 - X_t^{1c}$ and $J_t^2 = X_t^2 - X_t^{2c}$ respectively. Thus, we can use the last line of the theorem to obtain:

$$X_0^1 X_0^2 + \int_0^t X_{s-}^2 dX_s^1 + \int_0^t X_{s-}^1 dX_s^2 + [X^1, X^2](t)$$
$$= X_0^1 X_0^2 + \int_0^t X_{s-}^2 dX_s^{1c} + \int_0^t X_{s-}^2 dJ_s^1 + \int_0^t X_{s-}^1 dX_s^{2c} + \int_0^t X_{s-}^1 dJ_s^2 + [X^{1c}, X^{2c}](t) + \sum_{0<s\leq t} \Delta J_s^1 \Delta J_s^2$$
$$= X_0^1 X_0^2 + \int_0^t X_s^2 dX_s^{1c} + \int_0^t X_s^1 dX_s^{2c} + [X^{1c}, X^{2c}](t) + \sum_{0<s\leq t} [X_{s-}^2 \Delta X_s^1 + X_{s-}^1 \Delta X_s^2 + \Delta X_s^1 \Delta X_s^2].$$

We now need to show that

$$\sum_{0<s\leq t} [X_{s-}^2 \Delta X_s^1 + X_{s-}^1 \Delta X_s^2 + \Delta X_s^1 \Delta X_s^2] = \sum_{0<s\leq t} [X_s^1 X_s^2 - X_{s-}^1 X_{s-}^2].$$

Expanding the term inside of the sum on the right hand side we obtain:

$$\begin{aligned} X_s^1 X_s^2 - X_{s-}^1 X_{s-}^2 &= (X_{s-}^1 + \Delta X_s^1)(X_{s-}^2 + \Delta X_s^2) - X_{s-}^1 X_{s-}^2 \\ &= X_{s-}^1 X_{s-}^2 + X_{s-}^1 \Delta X_s^2 + \Delta X_s^1 X_{s-}^2 + \Delta X_s^1 \Delta X_s^2 - X_{s-}^1 X_{s-}^2 \\ &= X_{s-}^1 \Delta X_s^2 + \Delta X_s^1 X_{s-}^2 + \Delta X_s^1 \Delta X_s^2. \end{aligned}$$

□

For a process without jumps, we can use Girsanov's Theorem to change the measure using the Radon-Nikodým derivative process

$$Z_t = \exp\left\{ -\int_0^t \Gamma_s dW_s - \frac{1}{2}\int_0^t \Gamma_s^2 ds \right\}.$$

This process satisfies the stochastic differential equation

$$dZ_t = -\Gamma_t Z_t dW_t = Z_t dX_t^c$$

where $X_t^c = -\int_0^t \Gamma_s dW_s$ and $[X^c, X^c](t) = \int_0^t \Gamma_s^2 ds$. Therefore,

$$Z_t = \exp\left\{ X_t^c - \frac{1}{2}[X^c, X^c](t) \right\}.$$

This can be used for processes with jumps using the analogous stochastic differential equation:

$$dZ_t^X = Z_{t-}^X dX_t.$$

The integrator X is allowed to have jumps and whenever there is a jump in X, Z^X also has a jump of size

$$\Delta Z_s^X = Z_{s-}^X \Delta X_s.$$

Thus,

$$Z_s^X = Z_{s-}^X + \Delta Z_s^X = Z_{s-}^X(1 + \Delta X_s).$$

Theorem 22. *Let X_t be a jump process. The Doleans-Dade exponential of X is defined to be the process*

$$Z_t^X = \exp\left\{X_t^c - \tfrac{1}{2}[X^c, X^c](t)\right\} \prod_{0<s\leq t}(1 + \Delta X_s)$$

This process is the solution to $dZ_t^X = Z_{t-}^X dX_t$ with $Z_0^X = 1$. In integral form we get,

$$Z_t^X = 1 + \int_0^t Z_{s-}^X dX_s.$$

Proof. See [37] page 492. □

4.5. Change of measure

4.5.1. Change of measure for a Poisson process

Consider the Poisson process N_t with filtration $\mathcal{F}_t, t \geq 0$ on the probability space $(\Omega, \mathcal{F}, P)$. We will let N_t have intensity λ. Then, $M_t = N_t - \lambda t$ is a martingale. Let $\widetilde{\lambda}$ be a positive number and define Z_t as follows:

$$Z_t = e^{(\lambda - \widetilde{\lambda})t}(\frac{\widetilde{\lambda}}{\lambda})^{N_t}. \tag{4.9}$$

Z_t allows us to change to a measure $\widetilde{P}$ where N_t has intensity $\widetilde{\lambda}$.

Theorem 23. *The process Z_t satisfies:*

$$dZ_t = \tfrac{\widetilde{\lambda} - \lambda}{\lambda}\ Z_{t-} dM_t.$$

Particularly, Z_t is a martingale under P and $E(Z_t) = 1, \forall t$.

Proof. Define $X_t = \frac{\widetilde{\lambda}-\lambda}{\lambda} M_t$, which is martingale with continuous part $X_t^c = (\lambda - \widetilde{\lambda})t$ and pure jump part $J_t = \frac{\widetilde{\lambda}-\lambda}{\lambda} N_t$. Then $[X^c, X^c](t) = 0$, and if there is a jump at time t, then $\Delta X_t = \frac{\widetilde{\lambda}-\lambda}{\lambda}$, so:

$$1 + \Delta X_t = \frac{\widetilde{\lambda}}{\lambda}.$$

Thus, Z_t can be rewritten as:

$$Z_t = e^{X_t^c - \frac{1}{2}[X_t^c, X_t^c](t)} \prod_{0<s\leq t}(1 + \Delta X_s).$$

Therefore, Z_t is the Doleans-Dade exponential, particularly:

$$Z_t = 1 + \int_0^t Z_{s-} dX_s.$$

Since X is a martingale and Z_{s-} is left-continuous, Z_t is a martingale. Also, $Z(0) = 1$, therefore,

$$E(Z_t) = 1, \forall t \geq 0.$$

□

Fixing time T, we can now use Z_T to change the measure by the following:

$$\widetilde{P}_A = \int_A Z_T dP, \quad \forall A \in \mathcal{F}.$$

Theorem 24. *Under the new probability measure, $\widetilde{P}$, N_t, $0 \leq t \leq T$, is Poisson with intensity $\widetilde{\lambda}$.*

Proof. See [37] page 494. □

4.5.2. Change of measure for a compound Poisson process

Consider the compound Poisson process $Q_t = \sum_{i=1}^{N_t} Y_i$. We will change the measure so that the intensity of N_t and the distribution of jump sizes both change. We will consider the case where each Y_i takes one of the nonzero values $y_1, y_2, \ldots, y_M$ and $p(y_m)$ is the probability that a jump is of size y_m. We will assume that $p(y_m) > 0, \forall m$, and $\sum_{m=1}^{M} p(y_m) = 1$. Let N_t^m denote the number of jumps in Q_t of size y_m up to and including time t. Thus,

$$N_t = \sum_{m=1}^{M} N_t^m \text{ and } Q_t = \sum_{m=1}^{M} y_m N_t^m.$$

Thus, $N_1, \ldots, N_M$ are independent Poisson processes and each N_m has intensity $\lambda_m = \lambda p(y_m)$. Let $\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_M$ be given positive numbers, then, similar to the Poisson process,

$$Z_t^m = e^{(\lambda_m - \widetilde{\lambda}_m)t} (\frac{\widetilde{\lambda}_m}{\lambda_m})^{N_t^m} \text{ and } Z_t = \prod_{m=1}^{M} Z_t^m.$$

Theorem 25. *The process Z_t is a martingale with $E(Z_t) = 1, \forall t$.*

Proof. From the martingale proof for the Poisson process, we have:

$$dZ_t^m = \frac{\widetilde{\lambda}_m - \lambda_m}{\lambda_m} Z_{t-}^m dM_t^m,$$

where

$$M_t^m = N_t^m - \lambda_m dt.$$

Thus, Z^m is a martingale. Also, by construction, if $m \neq n$, then N^m and N^n have no simultaneous jumps, thus $[Z^m, Z^n] = 0$. Itô's product rule implies:

$$d(Z_t^1 Z_t^2) = Z_{t-}^2 dZ_t^1 + Z_{t-}^1 dZ_t^2.$$

Since both Z^1 and Z^2 are martingales and the integrands are left-continuous, then $Z^1 Z^2$ is a martingale. We can then repeat this process with Z^3 to get $Z^1 Z^2 Z^3$ is a martingale. Therefore, by continuing this process, $Z_t = Z_t^1 Z_t^2 \cdots Z_t^m$ is a martingale. □

Fixing time T, we can now use Z_T to change the measure by the following:

$$\widetilde{P}_A = \int_A Z_T dP, \quad \forall Z \in \mathcal{F}.$$

Theorem 26. *Under the new probability measure, $\widetilde{P}$, Z_t is a compound Poisson process with intensity $\widetilde{\lambda} = \sum_{m=1}^{M} \widetilde{\lambda}_m$ and $Y_1, Y_2, \ldots$ are independent, identically distributed random variables with:*

$$P[Y_i = y_m] = \widetilde{p}(y_m) = \frac{\widetilde{\lambda}_m}{\widetilde{\lambda}}.$$

Proof. See [37] page 497. □

The Radon-Nikodým derivative of Z_t is,

$$Z_t = \exp\left\{ \sum_{m=1}^{M} (\lambda_m - \widetilde{\lambda}_m) t \right\} \prod_{m=1}^{M} \left(\frac{\widetilde{\lambda}\widetilde{p}(y_m)}{\lambda p(y_m)} \right)^{N_t^m} = e^{(\lambda - \widetilde{\lambda})t} \prod_{i=1}^{N_t} \frac{\widetilde{\lambda}\widetilde{p}(Y_i)}{\lambda p(Y_i)}. \tag{4.10}$$

Therefore, if $Y_1, Y_2, \ldots$ are not discrete and instead all have density $f(y)$, then we could change the measure such that Q_t has intensity $\widetilde{\lambda}$ and $Y_1, Y_2, \ldots$ have density $\widetilde{f}(y)$ using the Radon-Nikodým process

$$Z_t = e^{(\lambda - \widetilde{\lambda})t} \prod_{i=1}^{N_t} \frac{\widetilde{\lambda}\widetilde{f}(Y_i)}{\lambda f(Y_i)}.$$

We will assume that if $f(y) = 0$, then $\widetilde{f}(y) = 0$ in order to not have a division by zero.

Theorem 27. *The process Z_t is a martingale with $E(Z_t) = 1$ $\forall t \geq 0$.*

Proof. See [37] pages 498-499. □

We can again fix a positive T and define a new probability measure,

$$\widetilde{P}_A = \int_A Z_T dP \quad \forall A \in \mathcal{F}.$$

Theorem 28. *Under the probability measure $\widetilde{E}$, Q_t, $0 \leq t \leq T$, is a compound Poisson process with intensity $\widetilde{\lambda}$. The jumps in Q_t are independent and identically distributed with density $\widetilde{f}(y)$.*

Proof. For the key steps in the proof, see [37] pages 500-502. □

4.5.3. Change of measure for a compound Poisson process with a Brownian motion

Consider a probability space $(\Omega, \mathcal{F}, P)$ and a Brownian motion W_t defined on the space. Also, let a compound Poisson process,

$$Q_t = \sum_{i=1}^{N_t} Y_i$$

be defined on the same space with intensity λ and jumps with density $f(y)$. Let the filtration $\mathcal{F}_t$, $t \geq 0$ be used for both the Brownian motion and the compound Poisson process so that they are independent.

Consider $\widetilde{\lambda} > 0$, $\widetilde{f}(y)$ is another density function such that if $f(y) = 0$, then $\widetilde{f}(y) = 0$, and let Θ_t be an adapted process. We define the following:

$$Z_t^1 = \exp\Big\{ - \int_0^t \Theta_u dW_u - \tfrac{1}{2} \int_0^t \Theta_u^2 du \Big\},$$
$$Z_t^2 = e^{(\lambda - \widetilde{\lambda})t} \prod_{i=1}^{N_t} \frac{\widetilde{\lambda}\widetilde{f}(Y_i)}{\lambda f(Y_i)},$$
$$Z_t = Z_t^1 Z_t^2.$$

Theorem 29. *The process Z_t is a martingale (with respect to the filtration of the Brownian motion) with $E(Z_t) = 1, \forall t \geq 0$.*

Proof. We know that Z_t^1 is a martingale from stochastic calculus for continuous processes. Also, from the previous section, Z_t^2 is a martingale. According to Itô's product rule, since Z_t^1 is continuous and Z_t^2 has no Itô integral part, $[Z^1, Z^2](t) = 0$, therefore,

$$Z_t^1 Z_t^2 = Z_0^1 Z_0^2 + \int_0^t Z_{s-}^1 dZ_s^2 + \int_0^t Z_{s-}^2 dZ_s^1.$$

Since both integrals are martingales, we know Z_t is a martingale. Also, since $Z_0 = 1$, we know

$$E(Z_t) = 1, \forall t \geq 0.$$

□

Fix a positive T and define $\widetilde{P}_A = \int_A Z_T dP$, $\forall A \in \mathcal{F}$. We then have the following.

Theorem 30. *Under the probability measure $\widetilde{P}$, the process*

$$\widetilde{W}_t = W_t + \int_0^t \Theta_s ds$$

is a Brownian motion, Q_t is a compound Poisson process with intensity $\widetilde{\lambda}$ and independent, identically distributed jump sizes having density $\widetilde{f}(y)$, and the processes $\widetilde{W}_t$ and Q_t are independent.

Proof. For the key steps in the proof, see [37] pages 503-504. □

We will now consider a compound Poisson process Q_t whose jumps take on finitely many nonzero values $y_1, y_2, \ldots, y_M$, with $p(y_m) = P[Y_i = y_m]$ such that $p(y_m) > 0$ and $\sum_{m=1}^M p_m = 1$. Let $\widetilde{\lambda} > 0$ and let $\widetilde{p}(y_1), \ldots, \widetilde{p}(y_M) > 0$ such that $\sum_{m=1}^M \widetilde{p}_m = 1$. We now replace Z_t^2 with,

$$Z_t^2 = e^{(\lambda - \widetilde{\lambda})t} \prod_{i=1}^{N_t} \frac{\widetilde{\lambda}\widetilde{p}(Y_i)}{\lambda p(Y_i)}.$$

We will let Z_t^1 and Z_t be defined the same as above. Then we get the following modification of the previous theorem.

Theorem 31. *Under the probability measure $\widetilde{P}$, the process*

$$\widetilde{W}_t = W_t + \int_0^t \Theta_s ds$$

is a Brownian motion, Q_t is a compound Poisson process with intensity $\widetilde{\lambda}$ and independent, identically distributed jump sizes satisfying $\widetilde{P}[Y_i = y_m] = \widetilde{p}(y_m)$, $\forall i$ and $m = 1, \ldots, M$, and the processes $\widetilde{W}_t$ and Q_t are independent.

4.6. Pricing a European call in a jump model

In section 3, we looked at the pricing of a European Call Option done by Black, Scholes, and Merton. How does this pricing change when we change our driving process to be a single Poisson process? What about a Brownian motion and a compound Poisson process?

4.6.1. A Poisson process driven asset

Let the underlying asset price be given by:

$$\begin{aligned} S_t &= S_0 \exp(\alpha t N_t log(\sigma + 1) - \lambda\sigma t) \\ &= S_0 e^{(\alpha-\lambda\sigma)t}(\sigma+1)^{N_t}, \end{aligned}$$

which has the differential form,

$$dS_t = \alpha S_t dt + \sigma S_{t-} dM_t.$$

Let us fix a time T and say that we want our option's payoff at time T to be,

$$V_T = (S_T - K)^+.$$

To avoid arbitrage, assume $\lambda > \frac{\alpha-r}{\sigma}$ which gives, $\widetilde{\lambda} = \lambda - \frac{\alpha-r}{\sigma} > 0$ and a unique risk-neutral measure,

$$\widetilde{P}_A = \int_A Z_T dP, \quad \forall A \in \mathcal{F},$$

where

$$Z_t = e^{(\lambda-\widetilde{\lambda})t}(\tfrac{\widetilde{\lambda}}{\lambda})^{N_t}.$$

Under our new risk-neutral measure, the compensated Poisson process $M_t = \widetilde{N_t} - \widetilde{\lambda}t$ is a martingale. Our differential for the stock price is now,

$$dS_t = rS_t dt + \sigma S_{t-} d\widetilde{M}_t$$

or,

$$d(e^{-rt}S_t) = \sigma e^{-rt} S_{t-} d\widetilde{M}_t.$$

Under $\widetilde{P}$, we can rewrite S_t as follows,

$$S_t = S_0 e^{(r-\widetilde{\lambda}\sigma)t}(\sigma+1)^{N_t}.$$

The discounted European call price is a martingale under the risk-neutral measure. Therefore, V_t satisfies,

$$e^{-rt}V_t = \widetilde{E}[e^{-rT}V_T|\mathcal{F}_t] = \widetilde{E}[e^{-rT}(S_T-K)^+|\mathcal{F}_t].$$

We have,

$$S_T = S_t e^{(r-\widetilde{\lambda}\sigma)(T-t)}(\sigma+1)^{N_T-N_t},$$

thus,

$$V_t = \widetilde{E}[e^{-r(T-t)}(S_t e^{(r-\widetilde{\lambda}\sigma)(T-t)}(\sigma+1)^{N_T-N_t} - K)^+|\mathcal{F}_t].$$

Note that S_t is $\mathcal{F}_t$-measurable and $e^{(r-\widetilde{\lambda}\sigma)(T-t)}(\sigma+1)^{N_T-N_t}$ is independent of $\mathcal{F}_t$. Now, let $V_t = c(t, S_t)$, where,

$$\begin{aligned} c(t,x) &= \widetilde{E}[e^{-r(T-t)}(xe^{(r-\widetilde{\lambda}\sigma)(T-t)}(\sigma+1)^{N_T-N_t} - K)^+] \\ &= \textstyle\sum_{j=0}^{\infty} e^{-r(T-t)}(xe^{(r-\widetilde{\lambda}\sigma)(T-t)}(\sigma+1)^j - K)^+ \frac{\widetilde{\lambda}^j (T-t)^j}{j!} e^{-\widetilde{\lambda}(T-t)} \\ &= \textstyle\sum_{j=0}^{\infty}(xe^{-\widetilde{\lambda}\sigma(T-t)}(\sigma+1)^j - Ke^{-r(T-t)})^+ \frac{\widetilde{\lambda}^j (T-t)^j}{j!} e^{-\widetilde{\lambda}(T-t)}. \end{aligned}$$

The first term in the expansion of the sum, i.e. when $j=0$ is,

$$(xe^{-\widetilde{\lambda}\sigma(T-t)} - Ke^{-r(T-t)})^+ e^{-\widetilde{\lambda}(T-t)}.$$

When $t = T$, this term becomes $(x-K)^+$, and is the only non-zero term. Therefore, the terminal condition is,

$$c(T,x) = (x-K)^+, \quad \forall x \geq 0.$$

Also,

$$e^{-rt} = \widetilde{E}[e^{-rT}(S_T-K)^+|\mathcal{F}_t]$$

is a martingale under $\widetilde{P}$. We can then compute the derivative and set the dt term equal to zero. We can rewrite dS_t as,

$$dS_t = (r - \widetilde{\lambda}\sigma)S_t dt + \sigma S_{t-} dN_t,$$

thus the continuous part of the stock price satisfies,

$$dS_t^c = (r - \widetilde{\lambda}\sigma)S_t dt.$$

If there is a jump at time t, then

$$\Delta S_t = S_t - S_{t-} = \sigma S_{t-}, \quad S_t = (\sigma + 1)S_{t-}.$$

The Itô-Doeblin formula implies (for all steps see [37] page 508),

$$\begin{aligned} e^{-rt}c(t, S_t) = c(0, S_0) &+ \textstyle\int_0^t e^{-ru}[-rc(u, S_u) + c_t(u, S_u) + (r - \widetilde{\lambda}\sigma)S_u c_x(u, S_u)]du \\ &+ \textstyle\int_0^t e^{-ru}[c(u, (\sigma + 1)S_{u-}) - c(u, S_{u-})]\widetilde{\lambda} du \\ &+ \textstyle\int_0^t e^{-ru}[c(u, (\sigma + 1)S_{u-}) - c(u, S_{u-})]d\widetilde{M}_u. \end{aligned}$$

However, we can use the following equality to combine the first two integrals,

$$\textstyle\int_0^t e^{-ru}[c(u, (\sigma + 1)S_{u-}) - c(u, S_{u-})]\widetilde{\lambda} du = \int_0^t e^{-ru}[c(u, (\sigma + 1)S_u) - c(u, S_u)]\widetilde{\lambda} du.$$

Therefore,

$$\begin{aligned} e^{-rt}c(t, S_t) = c(0, S_0) &+ \textstyle\int_0^t e^{-ru}[-rc(u, S_u) + c_t(u, S_u) + (r - \widetilde{\lambda}\sigma)S_u c_x(u, S_u) \\ &+\widetilde{\lambda}(c(u, (\sigma + 1)S_u) - c(u, S_u))]du \\ &+ \textstyle\int_0^t e^{-ru}[c(u, (\sigma + 1)S_{u-}) - c(u, S_{u-})]d\widetilde{M}_u. \end{aligned}$$

Since $\widetilde{M}_u$ is a martingale and the integrand is left-continuous as well as the left hand side is a martingale, we can solve for,

$$\begin{aligned} c(0, S_0) &+ \textstyle\int_0^t e^{-ru}[-rc(u, S_u) + c_t(u, S_u) + (r - \widetilde{\lambda}\sigma)S_u c_x(u, S_u) \\ &+\widetilde{\lambda}(c(u, (\sigma + 1)S_u) - c(u, S_u))]du. \end{aligned}$$

Since this is this is the difference of two martingales, it is itself a martingale. Thus, the integrand must be zero,

$$-rc(u, S_u) + c_t(u, S_u) + (r - \widetilde{\lambda}\sigma)S_u c_x(u, S_u) + \widetilde{\lambda}(c(u, (\sigma + 1)S_u) - c(u, S_u)) = 0.$$

We will proceed by replacing the stock price with a dummy variable x. Therefore,

$$-rc(u,x) + c_t(u,x) + (r - \widetilde{\lambda}\sigma)xc_x(u,x) + \widetilde{\lambda}(c(u,(\sigma+1)x) - c(u,x)) = 0,$$

which must hold for $0 \leq t < T$ and $x \geq 0$. Since $e^{-rt}c(t,S_t)$ is a martingale under P by construction, then $c(t,x)$ satisfies the above equation. Using this equation, we get:

$$e^{-rt}c(t,S_t) = c(0,S_0) + \int_0^t e^{-ru}[c(u,(\sigma+1)S_{u-}) - c(u,S_{u-})]d\widetilde{M}_u.$$

We want to know what happens at the end of the hedging time, i.e. at time $t = T$. At time T, we get:

$$\begin{aligned} e^{-rT}(S_T - K)^+ &= e^{-rT}c(T,S_T) \\ &= c(0,S_0) + \int_0^T e^{-ru}[c(u,(\sigma+1)S_{u-}) - c(u,S_{u-})]d\widetilde{M}_u. \end{aligned}$$

What happens if we sell the European call at time zero in exchange for initial capital $X_0 = c(0,S_0)$? We want $X_t = c(t,S_t)$ for all t or for X_t to satisfy,

$$e^{-rt}X_t = e^{-rt}c(t,S_t), \quad \forall t \in [0,T].$$

We will accomplish this task by matching differentials. We can see that the differential of $e^{-rt}c(t,S_t)$ is,

$$d(e^{-rt}c(t,S_t) = e^{-rt}[c(t,(\sigma+1)S_{t-}) - c(t,S_{t-})]d\widetilde{M}_t.$$

The differential of the value of the portfolio (X_t) which holds Γ_t shares of stock at any time t is,

$$dX_t = \Gamma_{t-}dS_t + r[X_t - \Gamma_t S_t]dt.$$

Therefore,

$$\begin{aligned} d(e^{-rt}X_t) &= e^{-rt}[-rX_t dt + dX_t] \\ &= e^{-rt}[\Gamma_{t-}dS_t - r\Gamma_t S_t dt] \\ &= e^{-rt}\sigma\Gamma_{t-}S_{t-}d\widetilde{M}_t. \end{aligned}$$

Since we cannot predict when jumps will occur, we want to know what happens prior to a jump taking place. We will determine the value of Γ_{t-} using the fact,

$$\sigma\Gamma_{t-}S_{t-} = c(t,(\sigma+1)S_{t-}) - c(t,S_{t-})$$
$$\to \Gamma_{t-} = \frac{c(t,(\sigma+1)S_{t-} - c(t,S_{t-})}{\sigma S_{t-}}.$$

We should hold this hedging position at all times since we do not know when the jumps will occur. More specifically, if we define

$$\Gamma_t = \frac{c(t,(\sigma+1)S_t - c(t,S_t)}{\sigma S_t}, \quad \forall t \in [0,T],$$

then, we obtain the following:

$$e^{-rt}X_t = X_0 + \int_0^t e^{-ru}[c(u,(\sigma+1)S_{u-}) - c(u,S_{u-})]d\widetilde{M}_u.$$

Therefore, $X_t = c(t,S_t)$, $\forall t$ and specifically, $X_T = (S_T - K)^+$, so the short position has been hedged.

4.6.2. Brownian motion and a compound Poisson process driven asset

Consider the probability space $(\Omega, \mathcal{F}, P)$ on which a Brownian motion W_t is defined for $0 \le t \le T$, and M independent Poisson processes $N_t^1, \ldots, N_t^M$ are defined on the same interval. Let $\mathcal{F}_t$, $0 \le t \le T$, be the filtration generated by the Brownian motion and the M Poisson processes. Also, we will let $\lambda_m > 0$ be the intensity of the mth Poisson process and $-1 < y_1 < \ldots < y_M$ be nonzero numbers. Define the following,

$$N_t = \sum_{m=1}^M N_t^m, \quad Q_t = \sum_{m=1}^M y_m N_t^m.$$

Then, as seen before, N is a Poisson process with intensity $\lambda = \sum_{m=1}^M \lambda_m$ and Q is a compound Poisson process. As done previously, let Y_i denote the size of the ith jump of Q such that the Y_i random variables take the values $y_1, \ldots, y_M$. Then Q_t can be written as

$$Q_t = \sum_{i=1}^{N_t} Y_i.$$

Define $p(y_m) = \frac{\lambda_m}{\lambda}$. Then the random variables $Y_1, Y_2, \ldots$ are independent and identically distributed, with $P[Y_i = y_m] = p(y_m)$.

Set $\beta = E[Y_i] = \sum_{m=1}^M y_m p(y_m) = \frac{1}{\lambda}\sum_{m=1}^M \lambda_m y_m$. Then the following is a martingale,

$$Q_t - \beta\lambda t = Q_t - t\sum_{m=1}^M \lambda_m y_m.$$

For this section, we will model the stock price using the following stochastic differential equation:

$$
\begin{aligned}
dS_t &= \alpha S_t dt + \sigma S_t dW_t + S_{t-} d(Q_t - \beta\lambda t) \\
&= (\alpha - \beta\lambda) S_t dt + \sigma S_t dW_t + S_{t-} dQ_t.
\end{aligned}
$$

The mean rate of return for the stock, under the original probability measure P, is α. The restriction that $y_i > -1$ allows the stock price to jump down, but if the price is positive, it will remain positive. Thus, we will begin with a positive stock price S_0 and the stock price will remain positive at all future times. If $S_0 = 0$, then $S_t = 0$ for all t.

Theorem 32. *The solution to the stock price model is,*

$$
S_t = S_0 \exp[\sigma W_t + (\alpha - \beta\lambda - \tfrac{1}{2}\sigma^2)t] \prod_{i=1}^{N_t} (Y_i + 1).
$$

Proof. See [37] pages 513-514. □

Now our goal is to construct a measure that is risk-neutral. Let θ be a constant and let $\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_M$ be positive constants. Define the following:

$$
\begin{gathered}
Z_t^0 = \exp[-\theta W_t - \tfrac{1}{2}\theta^2 t], \\
Z_t^m = e^{(\lambda_m - \widetilde{\lambda}_m)t} (\tfrac{\widetilde{\lambda}_m}{\lambda_m})^{N_t^m}, \quad m = 1, \ldots, M, \\
Z_t = Z_t^0 \prod_{m=1}^{M} Z_t^m, \\
\widetilde{P}_A = \int_A Z_T dP, \quad \forall A \in \mathcal{F}.
\end{gathered}
$$

Therefore, under the probability measure $\widetilde{P}$, the following hold:

i) the process $\widetilde{W}_t = W_t + \theta t$ is a Brownian motion,

ii) each N^m is a Poisson process with intensity $\widetilde{\lambda}_m$, and

iii) $\widetilde{W}$ and $N^1, \ldots, N^m$ are independent.

Define

$$
\widetilde{\lambda} = \sum_{m=1}^{M} \widetilde{\lambda}_m, \quad \widetilde{p}(y_m) = \frac{\widetilde{\lambda}_m}{\widetilde{\lambda}}.
$$

As previously for P, under $\widetilde{P}$, the process $N_t = \sum_{m=1}^{M} N_t^m$ is Poisson with intensity $\widetilde{\lambda}$, the jump-size random variables $Y_1, Y_2, \ldots$ are independent and identically distributed with $\widetilde{P}[Y_i = y_m] = \widetilde{p}(y_m)$, and $Q_t - \widetilde{\beta}\widetilde{\lambda}t$ is a martingale, with

$$\widetilde{\beta} = \widetilde{E}[Y_i] = \frac{1}{\widetilde{\lambda}} \sum_{m=1}^{M} \widetilde{\lambda}_m y_m.$$

The only way for $\widetilde{P}$ to be risk-neutral is for the mean rate of return of the stock under $\widetilde{P}$ to be the interest rate r. This only occurs if,

$$\begin{aligned} dS_t &= (\alpha - \beta\lambda) S_t dt + \sigma S_t dW_t + S_{t-} dQ_t \\ &= r S_t dt + \sigma S_t d\widetilde{W}_t + S_{t-} d(Q_t - \widetilde{\beta}\widetilde{\lambda} t). \end{aligned}$$

This is equivalent to:

$$\alpha - \beta\lambda = r + \sigma\theta - \widetilde{\beta}\widetilde{\lambda},$$

which is known as the market price of risk equation for the model. We can rewrite this equation and use the definitions of β and $\widetilde{\beta}$ to obtain:

$$\begin{aligned} \alpha - r &= \sigma\theta + \beta\lambda - \widetilde{\beta}\widetilde{\lambda} \\ &= \sigma\theta + \sum_{m=1}^{M} (\lambda_m - \widetilde{\lambda}_m) y_m. \end{aligned}$$

Let us choose some θ and $\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_M$ satisfying the market price of risk equation. Then, we obtain:

$$\begin{aligned} dS_t &= r S_t + \sigma S_t d\widetilde{W}_t + S_{t-} d(Q_t - \widetilde{\beta}\widetilde{\lambda} t) \\ &= (r - \widetilde{\beta}\widetilde{\lambda}) dt + \sigma S_t d\widetilde{W}_t + S_{t-} dQ_t. \end{aligned}$$

This is in the same form of the equation with solution given by the previous theorem and therefore, has solution:

$$S_t = S_0 \exp[\sigma \widetilde{W}_t + (r - \widetilde{\beta}\widetilde{\lambda} - \tfrac{1}{2}\sigma^2) t] \prod_{i=1}^{N_t} (Y_i + 1).$$

Now, we will compute the risk-neutral price of an option on the stock with this price process. Let us choose $\widetilde{\lambda}_1, \ldots, \widetilde{\lambda}_M$ to be positive constants such that our process is risk-neutral and then choose θ so the market price of risk equation is satisfied. It is assumed moving forward that some choice was made for our parameters which allows calibration of our model to market data.

We now need some new notation. Define

$$\kappa(\tau, x) = x N_{d_+(\tau,x)} - K e^{-r\tau} N_{d_-(\tau,x)},$$

where

$$d_{\pm}(\tau,x) = \frac{1}{\sigma\sqrt{\tau}}\Big[log\frac{x}{K} + (r \pm \tfrac{1}{2}\sigma^2)\tau\Big]$$

and N_y is the cumulative standard normal distribution,

$$N_y = \frac{1}{\sqrt{2\pi}}\int_{\infty}^{y} e^{-\frac{1}{2}z^2} dz.$$

Thus, $\kappa(\tau,x)$ is the standard Black-Scholes-Merton European call price on a geometric Brownian motion. Therefore, we have:

$$\kappa(\tau,x) = \widetilde{E}\Big[e^{-r\tau}\Big(x\exp\Big[-\sigma\sqrt{\tau}Y + \Big(r - \tfrac{1}{2}\sigma^2\Big)\tau\Big] - K\Big)^+\Big],$$

where Y is a standard normal random variable under $\widetilde{P}$.

Theorem 33. *For $0 \le t < T$, the risk-neutral price of a call,*

$$V_t = \widetilde{E}[e^{-r(T-t)}(S_T - K)^+|\mathcal{F}_t],$$

is given by $V_t = c(t, S_t)$, where

$$c(t,x) = \sum_{j=0}^{\infty} e^{-\widetilde{\lambda}(T-t)}\frac{\widetilde{\lambda}^j(T-t)^j}{j!}\widetilde{E}(\kappa(T-t, xe^{-\widetilde{\beta}\widetilde{\lambda}(T-t)}\prod_{i=1}^{j}(Y_i+1)).$$

Proof. See [37] pages 517-519. □

Theorem 34. *The European call price $c(t,x)$ satisfies the following equation,*

$$-rc(t,x) + c_t(t,x) + (r - \widetilde{\beta}\widetilde{\lambda})xc_x(t,x) + \tfrac{1}{2}\sigma^2x^2c_{xx}(t,x)$$
$$+\widetilde{\lambda}\Big[\sum_{m=1}^{M}\widetilde{p}(y_m)c(t,(y_m+1)x) - c(t,x)\Big] = 0, \quad 0 \le t < T, x \ge 0,$$

with terminal condition,

$$c(T,x) = (x-K)^+, \quad x \ge 0.$$

Proof. See [37] pages 520-521. □

Furthermore, as in the previous section, we want to hedge the short position of our option whose discounted stock price satisfies,

$$d(e^{-rt}c(t,S-t)) = e^{-rt}\sigma S_t c_x(t,S_t)d\widetilde{W}_t$$
$$+\textstyle\sum_{m=1}^{M} e^{-rt}[c(t,(y_m+1)S_{t-}) - c(t,S_{t-})]d(N_t^m - \widetilde{\lambda}_m t)$$
$$= e^{-rt}\sigma S_t c_x(t,S_t)d\widetilde{W}_t + e^{-rt}[c(t,S_t) - c(t,S_{t-})]dN_t$$
$$-e^{-rt}\widetilde{\lambda}\Big[\textstyle\sum -m = 1^M \widetilde{p}(y_m)c(t,(y+1)S_{t-}) - c(t,S_{t-})\Big]dt.$$

We will start with a portfolio that has initial capital $X_0 = c(0,S_0)$ and compare the differential of the discounted European call price with the differential of the discounted portfolio value. We will once again let Γ_t represent the number of shares of stock that are held at each time t. We then get,

$$dX_t = \Gamma_{t-}dS_t + r[X_t - \Gamma_t S_t]dt.$$

Therefore,

$$d(e^{-rt}X_t) = e^{-rt}[-rX_t dt + dX_t]$$
$$= e^{-rt}[\Gamma_{t-}dS_t - r\Gamma_t S_t dt]$$
$$= e^{-rt}[\Gamma_t \sigma S_t d\widetilde{W}_t + \Gamma_{t-}S_{t-}d(Q_t - \widetilde{\beta}\lambda t)]$$
$$= e^{-rt}[\Gamma_t \sigma S_t d\widetilde{W}_t + \Gamma_{t-}S_{t-}\textstyle\sum_{m=1}^{M} y_m(dN_t^m - \widetilde{\lambda}_m dt)].$$

We will look at the typical delta-hedging strategy,

$$\Gamma_t = c_x(t,S_t).$$

This equates the $d\widetilde{W}_t$ terms so we hedge perfectly against the risk introduced by the Brownian motion. However, we are left with:

$$d[e^{-rt}c(t,S_t) - e^{-rt}X_t]$$
$$= \textstyle\sum_{m=1}^{M} e^{-rt}[c(t,(y_m+1)S_{t-}) - y_m S_{t-}c_x(t,S_{t-})] \times (dN_t^m - \widetilde{\lambda}_m dt).$$

Since $\kappa(\tau,x)$ is convex, $c(t,x)$ is strictly convex. Therefore, we have

$$c(t,x_2) - c(t,x_1) > (x_2 - x_1)c_x(t,x_1), \quad \forall x_1 \geq 0, x_2 \geq 0 \text{ such that } x_1 \neq x_2.$$

Thus,

$$c(t,(y_m+1)S_{t-}) - c(t,S_{t-}) > y_m S_{t-}c_x(t,S_{t-})$$

where the strict inequality is due to $y_m + 1 > 0$ from our restrictions on y_m. Also, between jumps

$$d[e^{-rt}c(t,S_t) - e^{-rt}X_t] < 0.$$

We are therefore in a scenario in which our hedging outperforms the option between jumps, but the option outperforms our hedging at the time of jumps.

Since both $e^{-rt}c(t,S_t)$ and $e^{-rt}X_t$ are martingales, then their difference is also a martingale. At $t = 0$, the difference is $c(0,S_0) - X_0 = 0$, so the expected value of the difference is always zero, which gives us:

$$E[e^{-rt}c(t,S_t)] = E[e^{-rt}X_t], \quad 0 \leq t \leq T.$$

So, we have hedged the option on average using the delta-hedging formula under the risk-neutral measure. By taking $\widetilde{\lambda}_m = \lambda_m$, as far as the jumps are concerned, the average under the risk-neutral measure coincides with the average under the actual measure.

For further study of the material presented in this section, see [37].

5. ORNSTEIN-UHLENBECK PROCESS CONSTRUCTION

5.1. Introduction

We can now construct jump processes, which are driven by Lévy processes, that stay around their long term mean (mean-reverting) with linear dynamics on which we can impose various properties. These contructed processes are known as non-Gaussian Ornstein-Uhlenbeck processes or OU processes. Consider the following model of a stock-price process S_t:

$$S_t = S_0 e^{X_t},$$

such that,

$$dX_t = (\mu + \beta\sigma_t^2)dt + \sigma_t dW_t + \rho dZ_{\lambda t},$$

where W_t is a standard Brownian motion, Z_t is a Lévy process, $\mu, \beta, \rho, \lambda$ are constants with $\lambda > 0$ and $\rho \leq 0$. The dynamics are said to be linear since the Brownian motion and the Lévy process appear as a linear combination above. We now need a way to deal with the volatility (σ_t) since it changes overtime and sometimes in unpredictable ways. We will accomplish this by making σ_t stochastic and finding a process σ_t^2 that will describe the shifts in the market. In particular, we will let the following hold:

$$d\sigma_t^2 = -\lambda\sigma_t^2 dt + dZ_{\lambda t}.$$

This is an example of a Lévy driven OU process. The reason for the negative sign is to make the process mean-reverting.

5.2. Self-decomposability

Definition. *(Self-decomposability, [7])* A probability distribution $\mathcal{F}$ on R is said to be self-decomposable or belonging to Lévy class $\mathcal{L}$ if, for each $\lambda > 0$, there exists a probabiility distribution $\mathcal{F}_\lambda$ on R such that

$$\phi(u) = \phi(e^{-\lambda}u)\phi_\lambda(u), \quad u \in R$$

where ϕ and ϕ_λ denote the characteristic functions of $\mathcal{F}$ and $\mathcal{F}_\lambda$ respectively. A random variable X is self-decomposable if, for each $\lambda > 0$, there exists an independent random variable Y_λ such that

$$X := e^{\lambda} X + Y_{\lambda}.$$

If $\mathcal{F}$ is self-decomposable, then $\mathcal{F}_{\lambda}$ is infinitely divisible. The following theorem gives a characterization of the class $\mathcal{L}$ as a subclass of all infinitely divisible distributions.

Theorem 35. *(Schoutens, [31] pg. 47) Let $\nu(dx)$ denote the Lévy measure of an infinitely divisible distribution $\mathcal{F}$ on R. Then the following statements are equivalent:*

(1) $\mathcal{F}$ is self-decomposable

(2) The functions $\nu((-\infty, -e^s])$ and $\nu([e^s, \infty))$, $s > 0$ are both convex.

(3) $\nu(dx)$ is of the form $\nu(dx) = u(x)dx$ with $|x|u(x)$ increasing on $(-\infty, 0)$ and decreasing on $(0, \infty)$.

If u is differentiable, (2) is equivalent to,

$$u(x) + xu'(x) \leq 0, \text{ for } x \neq 0.$$

A proof of this theorem can be found in [5]. If $u(x)$ is known, then using the equivalence of (2) is useful for checking whether the distribution is self-decomposable. See [24] pages 29-30 to see the derivation of the equivalence of (2).

5.3. Ornstein-Uhlenbeck processes

The general OU process σ_t^2 is the solution to the following stochastic differential equation, [24] pg 30,

$$d\sigma_t^2 = -\lambda \sigma_t^2 dt + dZ_t, \quad \sigma_0^2 > 0. \tag{5.1}$$

If Z is a Brownian motion, then we have the usual Gaussian OU process, but if Z is a pure jump Lévy process, then we have a non-Gaussian analogue of the usual Gaussian OU process. We will say that σ^2 is the OU process driven by Z, i.e. Z is the background driving Lévy process of σ^2. The positive Lévy process Z, (if Z is not positive, we can simply exponentiate or square the σ^2 process), has bounded variation on finite intervals so if f is a deterministic continuous function, integrals of the form $\int_0^t f_s dZ_t$ are well defined as Riemann-Stieljes integrals.

We will now solve the equation explicitly when $t > 0$ and $\lambda > 0$.

$$d\sigma_t^2 + \lambda \sigma_t^2 dt = dZ_t$$
$$e^{\lambda t}(d\sigma_t^2 + \lambda \sigma_t^2 dt) = e^{\lambda t} dZ_t$$

$$d(e^{\lambda t}\sigma_t^2) = e^{\lambda t} dZ_t.$$

We can now integrate both sides and divide by $e^{\lambda t}$ to obtain the solution:

$$\sigma_t^2 = e^{-\lambda t}\sigma_0^2 + \int_0^t e^{-\lambda(t-s)} dZ_s.$$

Since Z is an increasing process and $\sigma_0^2 > 0$, σ^2 is strictly positive. We also know that σ^2 is bounded below by $\sigma_0^2 e^{-\lambda t}$. Based on the equation for σ^2, the process has the same jump times as Z_t, but exponentially decays between jumps. The result is a closer approximation of real world volatility that gradually decreases when the process it is measuring is steady.

As suggested by Barndorff-Nielsen and Shephard, we will now use positive OU processes to directly represent σ^2. Thus, the model for the squared volatility process σ^2 becomes:

$$d\sigma_t^2 = -\lambda \sigma_t^2 dt dZ_{\lambda t}, \quad \sigma_0^2 > 0.$$

Notice the change in time of the background driving Lévy process and we get the solution

$$\sigma_t^2 = e^{-\lambda t}\sigma_0^2 + \int_0^t e^{-\lambda(t-s)} dZ_{\lambda s}.$$

By a change of variable, we get:

$$\sigma_t^2 = e^{-\lambda t}\sigma_0^2 + e^{-\lambda t} \int_0^{\lambda t} e^s dZ_s.$$

There is often a distribution $\mathcal{S}$, called the stationary distribution, that we can make σ^2 follow for all t if σ_0^2 is chosen according to $\mathcal{S}$. If $\mathcal{S}$ is one-dimensional, then we know that there exists an OU type process with $\mathcal{S}$ as its stationary distribution if and only if $\mathcal{S}$ is self-decomposable (Barndorff-Nielsen and Shephard, [7], p. 17).

Theorem 36. *(Stationarity, Wolfe, [41]) If X is self-decomposable, then there exists a stationary stochastic process σ_t^2 and a Lévy process Z_t, independent of σ_0^2, such that $\sigma_t^2 := X$ for all $t \geq 0$ and*

$$\sigma_t^2 = e^{-\lambda t}\sigma_0^2 + \int_0^t e^{-\lambda(t-s)} dZ_{\lambda s}, \quad \forall \lambda > 0.$$

Conversely, if σ_t^2 is a stationary stochastic process and Z_t is a Lévy process, independent of σ_0^2, such that σ_t^2 and Z_t satisfying the stochastic differential equation for squared volatility for all $\lambda > 0$, then σ_t^2 is self-decomposable.

We will now define some useful notation for the log-characteristic function and log-Laplace transform of a random variable X. The derivation of these equations can be found in [24], pages 43-45.

$$C_X(u) = log\Big(E[e^{iuX}]\Big) = log(\phi_X(u)) = \psi_X(u),$$
$$L_X(u) = log\Big(E[e^{-uX}]\Big) = log(\phi_X(iu)),$$

where ϕ denotes the characteristic function and ψ denotes the characteristic exponent of the random variable X. We can now relate Z and σ^2 through the following:

$$C_{\sigma_t^2}(u) = \textstyle\int_0^\infty C_{Z_1}(e^{-s}u)ds$$

and

$$C_{Z_1}(u) = u\tfrac{d}{du}C_{\sigma_t^2}(u) = u(C_{\sigma_t^2}(u))'.$$

Similarly,

$$L_{\sigma_t^2}(u) = \textstyle\int_0^\infty L_{Z_1}(e^{-s}u)ds$$

and

$$L_{Z_1}(u) = u\tfrac{d}{du}L_{\sigma_t^2}(u) = u(L_{\sigma_t^2}(u))'.$$

Theorem 37. *(Key Formula, Barndorff-Nielsen and Shephard [7], pg. 5) Let f denote a continuous function, Z a Lévy process and set $Y = \int_{R^+} f_t dZ_t$. Then*

$$C_Y(u) = \textstyle\int_{R^+} C_{Z_1}(uf(\tau))d\tau.$$

Proof.

$$\begin{aligned}\exp(C_Y(u)) &= E\Big[\exp(iu\textstyle\int_{R^+} f_\tau dZ_\tau)\Big]\\ &= E\Big(\exp\Big(iu\textstyle\int_{R^+} f_\tau dZ_\tau\Big)\Big)\\ &= E\Bigg(\textstyle\prod_{\tau\in R^+}\exp(iuf_\tau dZ_\tau)\Bigg)\\ &= \textstyle\prod_{\tau\in R^+} E(\exp(iuf_\tau dZ_\tau))\\ &= \textstyle\prod_{\tau\in R^+}(\exp(C_{dZ_\tau}(uf_\tau))\\ &= \textstyle\prod_{\tau\in R^+}(\exp(C_{Z_1}(uf_\tau)d\tau)\\ &= exp(\textstyle\int_{R^+} C_{Z_1}(uf(\tau))d\tau)\end{aligned}$$

and taking a logarithm of both sides will give us the desired result. □

6. THE BARNDORFF-NIELSEN AND SHEPARD (BN-S) MODEL AND HOW TO PRICE OPTIONS

6.1. The BN-S model

Looking back to the Black-Scholes-Merton equation, we are missing the volatility of the stock market. Volatility (or the environment) is what can cause the sudden jumps and dips that we can see if we look at real world data and changes stochastically over time. Therefore, we will build a model that not only takes into account the asset price process, but we must also include a suitable model for the volatility process.

In order to encapsulate what was discussed in the previous section, we will consider volatility models whose squared volatility process is a Lévy-driven Ornstein-Uhlenbeck process. This approach will allow our volatility to include jumps and gives the availability of closed form expressions for the integrals of the squared volatility process and returns. One advantage of a model with this setup is the ability to accomplish the statistical analysis of price series on scales from several minutes to several days.

Let the price process of the stock $S = \{S_t\}_{t\geq 0}$ be defined on a filtered probability space $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{0\leq t\leq T}, P)$. Let $W = \{W_t\}_{t\geq 0}$ be a standard Brownian motion and $Z = \{Z_{\lambda t}\}_{t\geq 0}$ be a positive and nondecreasing Lévy process independent of the Brownian motion. Then,

$$S_t = S_0 e^{X_t}, \tag{6.1}$$

and the dynamics for the logarithmic return satisfy:

$$dX_t = (\mu + \beta\sigma_t^2)dt + \sigma_t dW_t + \rho dZ_{\lambda t}, \tag{6.2}$$

with instantaneous variance process satisfying:

$$d\sigma_t^2 = -\lambda\sigma_t^2 dt + dZ_{\lambda t}. \tag{6.3}$$

The parameters μ, β, ρ, and λ are real constants with $\lambda > 0$ and $\rho \leq 0$. Note that the drift depends on the volatility. This is due to investors requiring a “risk premium” for holding stochastic assets.

This premium is due to the fact that they could just decide to keep their assets in a regular bank account which has no risk, but also has a very low interest rate. The term $\beta\sigma^2$ corresponds to this "risk premium" and increases the drift as the volatility increases. The term $\rho dZ_{\lambda t}$ links the upward jumps in volatility with the downward jumps in asset price. Using Nicolato and Venados [25], we require that Z satisfies the following assumptions:

1) Z has no deterministic drift ($\gamma = 0$) and its Lévy measure has density $\omega(x)$. Therefore, the cumulant transform $K_{Z_1}(u) = log\Big[E[e^{uZ_1}]\Big]$, when it exists takes the form:

$$K_{Z_1}(u) = \int_{R^+}(e^{ux} - 1)\omega(x)dx;$$

2) letting $\hat{u} = sup[u \in R : K_{Z_1}(u) < \infty]$, then $\hat{u} > 0$;

3) $lim_{u\uparrow\hat{u}} K_{Z_1}(u) = \infty$.

The model satisfying the previous three assumptions is the BN-S model. Recall from the previous section, volatility jumps when the driving process jumps and exponentially decays between consecutive jumps. When $\rho \neq 0$, each jump in the volatility is associated with a jump in the price process. The jump within the price process is proportional to the size of the jump in volatility and can be thought of as new (bad) information entering the market causing volatility to increase and thus stock prices fall. If $\rho = 0$, the volatility still has jumps, but the price process is continuous.

Using Itô's formula, the dynamics of the stock price process $S_t = e^{X_t}$ are given by:

$$\begin{aligned} dS_t &= S_{t-}dY_t \\ &= S_{t-}(b_t dt + \sigma_t dW_t + dM_t), \end{aligned}$$

where b_t is the appreciation rate and is given by:

$$b_t = \mu + \lambda K_{Z_1}(\rho) + (\beta + \tfrac{1}{2})\sigma_t^2,$$

and $M = \{M_t\}_{t\geq 0}$ is the martingale Lévy process

$$M_t = \textstyle\sum_{0\leq s\leq t}(e^{\rho \Delta Z_{\lambda s}} - 1) - \lambda K_{Z_1}(\rho)t.$$

We will adopt the notation of Jacob and Shiryaev [20] moving forward, thus:

$$\begin{aligned} M_t &= \textstyle\int_{[0,t]\times R}(e^{\rho x} - 1)(\mu_Z - \nu_Z)(ds, dx) \\ &= (e^{\rho x} - 1) \star (\mu_Z - \nu_Z)_t, \end{aligned}$$

where μ_Z is the random measure associated with the jumps of Z and

$$\nu_Z(\omega, dt, dx) = \lambda w(x) dx dt$$

is its compensator.

For the proof of this equation, see Jacob and Shiryaev [20].

6.2. Market completeness

As stated by Marshall [24], derivatives should be priced using a martingale measure. Let Q be an equivalent measure that transforms the discounted asset price $\widetilde{S} = \{\widetilde{S}_t = \exp(-rt)S_t\}_{t\geq 0}$ into a martingale. For Q to be an equivalent martingale measure (EMM), it must satisfy the following conditions:

1) Q is equivalent to the real world measure P, i.e. they have the same null sets;

2) the discounted stock price process $\widetilde{S}$ is a martingale under Q.

In order to be risk-neutral, e^{-rt}, $t \geq 0$ must be a martingale, thus:

$$E(e^{-rT} S_T) = S_0.$$

A risk-neutral world is impossible to reach, but in one, all individuals are indifferent to risk and thus require no compensation for risk. Also, the expected return for all assets is the risk-free rate which would be similar to just dumping your money into a savings account with a set interest rate.

Theorem 38. *(First Fundamental theorem of asset pricing, Applebaum [1], pg. 270) If the market is free of arbitrage opportunities, then there exists a probability measure Q, which is equivalent to P, with respect to which the discounted process $\widetilde{S}$ is a martingale.*

Definition. *(Completeness, Etheridge [13], pg 16) A market is said to be complete if every contingent claim can be exactly replicated by a self-financing portfolio, i.e. if every possible derivative can be perfectly hedged.*

Theorem 39. *(Second Fundamental theorem of asset pricing, Applebaum [1], pg. 271) An arbitrage-free market is complete if and only if there exists a unique probability measure Q, which is equivalent to P, with respect to which the discounted process $\widetilde{S}$ is a martingale.*

Therefore, if we can show that the martingale measure is unique, then we know that the market is complete. The uniqueness is linked with the predictable representation property (PRP) of a martingale stated below.

Definition. *(Predictable Representation Property, Schoutens [31], pg 46) Every square integral random variable* $F \in \mathcal{F}_T$ *has a representation of the form*

$$F = E[F] + \textstyle\sum_{i=1}^{\infty} \int_0^T a_s^{(i)} d(H_s^{(i)} - E[H_s^{(i)}]),$$

where $a^{(i)} = \{a_s^{(i)}\}_{0 \leq s \leq T}$ *is predictable and* $H^{(i)} = \{H_s^{(i)}\}_{0 \leq s \leq T}$ *is the power jump process of order* i*, i.e.,* $H_s^{(1)} = X_s$ *and*

$$H_s^{(i)} = \textstyle\sum_{0<u\leq s} (\Delta X_u)^i, \quad i = 2, 3, \ldots.$$

As a result of Brownian motion having continuous paths, there are no jumps and $H_s^{(i)} = 0$ for $i \geq 2$. Thus, the sum only has one term and we obtain the following PRP for Brownian motion:

$$F = E[F] + \int_0^T a_s dW_s, \tag{6.4}$$

where $a = a^{(1)}$ is predictable. Therefore, the Black-Scholes model is complete. Only the Poisson case can be simplified in a similar manner. This implies that even though most stochastic models used in option pricing are arbitrage free, most are not complete. The problem that this causes is that to more realistically model the real world market, we will get an incomplete model.

When we use a Lévy market model, there are many equivalent martingale measures we can choose leading to multiple prices for European options. Eberlein and Jacob [12] have shown that, for models based on infinite-variation Lévy processes, we can calculate the potential option prices such that we get the entire no-arbitrage interval. The boundary prices are set when there is a simple buy/sell-and-hold strategy that allows riskless arbitrage.

The question that arises is, how do we change from measure P to the risk-neutral measure Q?

Theorem 40. *(Girsanov Theorem for asset prices with jumps, Etheridge [13], pg. 178) Let* $\{W_t\}_{t\geq 0}$ *be a standard* P*-Brownian motion and* $\{N_t\}_{t\geq 0}$ *a (possibly time-inhomogeneous) Poisson process with intensity* $\{\lambda_t\}_{t\geq 0}$ *under* P*. We write* $\mathcal{F}_t$ *for the* σ*-field generated by* $\mathcal{F}_t^W \bigcup \mathcal{F}_t^N$*. Suppose that* $\{\theta_t\}_{t\geq 0}$ *and* $\{\phi_t\}_{t\geq 0}$ *are* $\{\mathcal{F}_t\}_{t\geq 0}$*-predictable processes with* ϕ_t *positive for each* t*, such that*

$$\textstyle\int_0^t ||\theta_s||^2 ds < \infty \quad \text{and} \quad \int_0^t \phi_s \lambda_s ds < \infty.$$

Then under the measure Q *whose Radon-Nikodym derivative with respect to* P *is given by*

$$\frac{dQ}{dP}\Big|_{\mathcal{F}_t} = L_t,$$

where $L_0 = 1$ *and*

$$\frac{dL_t}{L_{t-}} = \theta_t dW_t + (\phi_t - 1)dM_t,$$

the process $\{X_t\}_{t\geq 0}$ *defined by* $X_t = W_t - \int_0^t \theta_s ds$ *is a Brownian motion and* $\{N_t\}_{t\geq 0}$ *has intensity* $\{\phi_t \lambda_t\}_{t\geq 0}$.

6.3. The equivalent martingale measure

The BN-S model given earlier is arbitrage free, as proven by Barndorff-Nielsen and Shephard in [6], pg. 194-195, but incomplete. This leads to multiple equivalent martingale measures (EMM) existing. The structure for a general EMM for a BN-S model is covered in detail in Nicolato and Venardos [25]. We will be interested in the structure-preserving class of EMMs which have log returns that can also be described by a BN-S model, but with different parameters and potentially different stationary distributions. The following two theorems from Nicolato and Venardos [25] give a complete characterization of both classes.

Theorem 41. *(The set of EMMs for the BN-S model) Denote by* $\mathcal{M}$ *the set of EMMs for the BN-S model. Let* $Q \in \mathcal{M}$. *Then the density process* $L_t = \frac{dQ}{dP}|_{\mathcal{F}_t}$ *is given by the Doléans-Dade exponential process*

$$\begin{aligned} L_t &= \varepsilon(\psi \bullet W + (Y-1) \star (\mu_Z - \nu_Z))_t \\ &= \varepsilon\Big(\int_0^t \psi_s dW_s + \int_0^t \int_{R^+} (Y(s,x) - 1)(\mu_Z - \nu_Z)(dx, ds)\Big) \end{aligned}$$

where μ_Z, ν_Z *are from Jacob and Shiryaev [20],* $\psi = \{\psi_t\}$ *is a predictable process and* $Y = Y(\omega, t, x)$ *is a strictly positive predictable process such that*

$$\int_0^t ds \int_{R^+} (\sqrt{Y(s,x)} - 1)^2 w(x) dx < \infty \qquad P - a.s.$$

The function Y *and the process* ψ *are linked by*

$$\mu + (\beta + \tfrac{1}{2})\sigma_t^2 + \sigma_t \psi_t + \lambda \int_{R^+} Y(t,x)(e^{\rho x} - 1) w(x) dx - r = 0$$

$dP \otimes dt$ *almost surely.*

Proof. For a detailed derivation see Nicolato and Venardos [25] pg 8. □

We now move on to the subclass of EMMs which preserve the BN-S structure. We will call this class $\mathcal{M}'$, using the notation of Nicolato and Venardos [25]. This subclass has the property, for $Q \in \mathcal{M}'$, the log-price process and its volatility can be described by the second and third characteristics of a BN-S model, with different parameters and possibly different distribution increments of the Lévy process Z.

Theorem 42. *(The Structure-Preserving EMM) Let $y \in \mathcal{Y}'$ where*

$$\mathcal{Y}' := \Big\{ y : R^+ \to R^+ \Big| \int_{R^+} (\sqrt{y(x)} - 1)^2 w(x) dx < \infty \Big\}.$$

Then the process

$$\psi_t = \omega^{-1} \Big(r - \mu - (\beta + \tfrac{1}{2}) \sigma_t^2 - \lambda K^y_{Z_1}(\rho) \Big)$$

where

$$K^y_{Z_1}(u) = \int_{R^+} (e^{ux} - 1) w^y(x) dx \quad \textit{and} \quad w^y(x) = y(x) w(x),$$

for $Re(u) < 0$, is such that

$$P\Big(\int_o^T \psi_s^2 ds < \infty \Big) = 1,$$

and

$$L^y_t = \varepsilon(\psi \bullet W + (y - 1) \star (\mu_Z - \nu_Z))_t \quad 0 \leq t \leq T$$

is a density process. The probability measure Q^y defined by

$$\tfrac{dQ^y}{dP} = L^y_T$$

is an EMM and the dynamics under Q^y are given by

$$dX_t = (r - \lambda K^y_{Z_1}(\rho) - \tfrac{1}{2}\sigma_t^2) dt + \sigma_t dW^y_t + \rho dZ_{\lambda t},$$
$$d\sigma_t^2 = -\lambda \sigma_t^2 dt + dZ_{\lambda t},$$

where $W^y_t = W_t - \int_0^t \psi_s ds$ is a Q^y-Brownian motion, $Z_{\lambda t}$ is a Q^y-Lévy process. Z_1 has Lévy density $w^y(x)$ and cumulant transform $K^y_{Z_1}(u)$. In addition, the processes W^y and Z are independent under Q^y. Hence $Q^y \in \mathcal{M}'$.

6.4. Characteristic functions for the BN-S model

Our end goal has remained the same, compute the value of a derivative, now we are doing so under any given EMM. There are a myriad of methods used to do this including: analytical or numerical integration when the density function is known explicitly, Monte Carlo simulation estimation, and using finite difference methods when the price is described using partial differential equations. However, there are a lot of times we do not have an explicit expression for the density, but we do for its characteristic function. We can then use Fourier inversion techniques to rewrite the price of the derivative using the characteristic function of the terminal price of the underlying asset. This is the method we will focus on in this section.

6.4.1. The characteristic function

The log-asset price process, which appears in the BN-S model, has a characteristic function which we can derive a general expression for. The following theorems will accomplish this derivation for the characteristic function.

Theorem 43. *(Characteristic function for the BN-S model, Marshall, [24] pg. 79) In the case of the general OU-type model described by the BN-S model, the characteristic function* $\phi(u) = E[\exp(iuX_T)|\mathcal{F}_t]$ *is given by*

$$\phi(u) = \exp\Big(iu(X_t + \mu(T-t)) - \frac{1}{2}(u^2 - 2\beta iu)\epsilon(t,T)\sigma_t^2 + \lambda \int_t^T K_{Z_1}(f(s,u))\Big)ds \qquad (6.5)$$

where

$$\epsilon(s,T) = \lambda^{-1}(1 - e^{-\lambda(T-s)})$$
$$f(s,u) = \rho iu - \tfrac{1}{2}(u^2 - 2\beta iu)\epsilon(s,T).$$

Proof. See Marshall [24] pages 101-102. □

Whether we have an IG-OU or Γ-OU specification of the BN-S model, we can explicitly calculate the integral part of the previous theorem.

Theorem 44. *(Nicolato and Venardos [25], pg. 451) Set*

$$f_1 = \rho iu - \tfrac{1}{2}(u^2 - 2\beta iu)(1 - e^{-\lambda(T-t)}),$$
$$f_2 = \rho iu - \tfrac{1}{2}(u^2 - 2\beta iu),$$
$$f(s,u) = \rho iu - \tfrac{1}{2}(u^2 - 2\beta iu)\epsilon(s,T).$$

Then for a $\Gamma(a,b)$-OU process, the integral is:

$$\int_t^T K_{Z_1}(f(s,u))ds = \Big(blog\Big[\frac{b-f_1}{b-\rho iu}\Big] + f_2\lambda(T-t)\Big) \times \frac{a}{\lambda(b-f_2)}; \tag{6.6}$$

similarly, for an IG(a,b)-OU process, we get:

$$\int_t^T K_{Z_1}(f(s,u))ds = \frac{a}{\lambda}(\sqrt{b^2-2f_1} - \sqrt{b^2-2\rho iu}) + \frac{2af_2}{\lambda\sqrt{2f_2-b^2}}$$
$$\times\Big(arctan\Big(\sqrt{\tfrac{b^2-2\rho iu}{2f_2-b^2}}\Big) - arctan\Big(\sqrt{\tfrac{b^2-2f_1}{2f_2-b^2}}\Big)\Big). \tag{6.7}$$

6.5. The risk-neutral characteristic function

Using the structure-preserving EMM, we can determine the characteristic function of the terminal risk-neutral log asset price. We will let $\beta = -\frac{1}{2}$ and $\mu = r - \lambda K^y_{Z_1}(\rho)$ in the characteristic function model. The risk-neutral characteristic function will be denoted by $\phi^Q(u)$ and is given by Marshall [24] as:

$$\phi^Q(u) = \exp\Big(iu(X_t + [r - \lambda K^y_{Z_1}(\rho)](T-t)) - \frac{1}{2}(u^2+iu)\lambda^{-1}(1-e^{-\lambda(T-t)})\sigma_t^2\Big)$$
$$\times \exp\Big(\textstyle\int_t^T \lambda K_{Z_1}(f(s,u))ds\Big). \tag{6.8}$$

Therefore, we have related the risk-neutral characteristic function to the BN-S model with a $\mathcal{S}$-OU specification for the squared volatility process. When given a specific choice for the martingale distribution $\mathcal{S}$, the risk-neutral characteristic function for the log-asset price process can be explicitly stated.

Theorem 45. *(Risk-Neutral Characteristic Functions, Marshall [24], pg 81) Set*

$$f_1 = \rho iu - \tfrac{1}{2}(u^2 - 2\beta iu)(1-e^{-\lambda(T-t)})$$
$$f_2 = \rho iu - \tfrac{1}{2}(u^2 - 2\beta iu).$$

Then the characteristic function for a terminal risk-neutral log-asset price process under the BN-S model with squared volatility following a $\Gamma(a,b)$-OU process is

$$\phi^Q_\Gamma(u) = \exp\Big(iu(X_t + [r - a\lambda\rho(b-\rho)^{-1}](T-t)) - \frac{1}{2}(u^2+iu)\lambda^{-1}(1-e^{-\lambda(T-t)})\sigma_t^2\Big)$$
$$\times \exp\Big(\Big(blog\Big[\tfrac{b-f_1}{b-\rho iu}\Big] + f_2\lambda(T-t)\Big)\tfrac{a}{b-f_2}\Big). \tag{6.9}$$

Similarly with an IG(a,b)-OU process for the squared volatility,

$$\phi_{IG}^{Q}(u) = \exp\left(iu(X_t + [r - \lambda a\rho b^{-1}(1-2b^{-2}\rho)^{-\frac{1}{2}}](T-t)) - \frac{1}{2}(u^2+u)\lambda^{-1}(1-e^{-\lambda(T-t)})\sigma_t^2\right)$$

$$\times \exp\left(a(\sqrt{b^2-2f_1} - \sqrt{b^2-2\rho iu})\right)$$
$$\times \exp\left(\frac{2af_2}{\sqrt{2f_2-b^2}}\left[arctan\left(\sqrt{\frac{b^2-2\rho iu}{2f_2-b^2}}\right) - arctan\left(\sqrt{\frac{b^2-2f_1}{2f_2-b^2}}\right)\right]\right). \tag{6.10}$$

Proof. See Marshall [24] pages 102-104. □

6.6. Computational methods for the BN-S model

In this section, we will look at two methods used to compute the value of a European call option. Both of these methods utilize Fourier transform and Fourier inversion techniques. In the literature, (Heston [17], Raible [26], and Carr and Madan [9]), these techniques are referred to as transform-based methods.

6.6.1. Option pricing through the FFT

According to Casella and Berger [10], pg. 84, every cumulative distribution function has a unique characteristic function.

Theorem 46. *(Convergence of Characteristic Functions, Casella and Berger [10], pg. 84) Suppose X_k, $k = 1, 2, \ldots,$ is a sequence of random variables, each with characteristic function $\phi_{X_k}(u)$. Furthermore, suppose that*

$$lim_{k\uparrow\infty}\phi_{X_k}(u) = \phi_X(u),$$

for all u in a neighborhood of 0, and $\phi_X(u)$ is the characteristic function of a random variable X. Then, for all x where the cumulative distribution function $F_X(x)$ is continuous,

$$lim_{k\uparrow\infty}F_{X_k}(x) = F_X(x).$$

Let S_t be the asset price at time $t \geq 0$ under risk neutrality, F_{S_t} the cumulative distribution function of S_t and f_{S_t} the density function of S_t. We can then price a European call option which has a strike price K, T time to maturity, and a risk-free rate r by:

$$C_T(K) = e^{-rT}E(S_T - K)^+$$

$$= \mathrm{e}^{-rT}\int_R (s-K)^+ dF_{S_T}(s)$$

$$
\begin{aligned}
&= e^{-rT}\int_{s>K} sdF_{S_T}(s) - e^{-rT}\int_{s>K} KdF_{S_T}(s)\\
&= e^{-rt}E(S_T)\frac{\int_{s>K} sdF_{S_T}(s)}{E(S_T)} - Ke^{-rT}\int_{s>K} dF_{S_T}(s)\\
&= S_0\frac{\int_{s>K} sdF_{S_T}(s)}{E(S_T)} - Ke^{-rT}\int_R 1_{s>K}dF_{S_T}(s)\\
&= S_0\int_K^\infty \left(\frac{s}{E(S_T)}\right)f_{S_T}(s)ds - Ke^{-rT}\left(1-\int_0^K f_{S_T}(s)ds\right). \qquad (6.11)
\end{aligned}
$$

Letting $g(s) = (\frac{s}{E(S_T)})f_{S_T}(s)$, a density function, $\Pi_1 = \int_K^\infty g(s)ds$, and $\Pi_2 = 1 - \int_0^K f_{S_T}(s)ds$, we get:

$$C_T(K) = S_0\Pi_1 - Ke^{-rT}\Pi_2. \qquad (6.12)$$

As $K \to -\infty$, Π_1 tends to 1 and $\Pi_2 = 1$. When $K \to \infty$, Π_1 tends to 0 and $\Pi_2 = 0$.

Since Π_1 and Π_2 both involve density functions, if we know their corresponding characteristic functions, ϕ_{Π_1} and ϕ_{Π_2} respectively, then Π_1 and Π_2 can be uniquely determined. If the characteristic functions are known analytically, authors such as Bakshi and Madan [4] and Scott [32] have numerically determined the risk-neutral probability of finishing in the money as:

$$Pr(S_T > K) = \Pi_2 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left(\frac{\exp(-iuln(K))\phi(u)}{iu}\right)du.$$

For a derivation see Marshall [24] pg. 104-105. The delta of the option is:

$$\Pi_1 = \frac{1}{2} + \frac{1}{\pi}\int_0^\infty Re\left(\frac{\exp(-iuln(K))\phi(u-i)}{iu\phi_T(-i)}\right)du,$$

where $\phi(u)$ is the characteristic function of the random variable $logS_T$.

We will now look at the Fast Fourier Transform (FFT). Studies by Bakshi and Chen [3], Bates [8], Chen and Scott [11], and Heston [17] have applied Fourier Analysis to option price computation by decomposing the option price in a similar fashion as above. We need to overcome the singularity at $u = 0$ in order to apply the FFT in order to take advantage of its efficiency and computer implementation.

We will now discuss an approach introduced by Carr and Madan [9] which is based on the FFT. Since the density is often not known in closed form, but the characteristic function of the terminal log price is, we can utilize the FFT.

Let $C_T(k)$ denote the price of a European call option with maturity T and strike $K = \exp(k)$:

$$C_T(k) = \int_k^\infty e^{-rT}(e^s - e^k)q_T(s)ds, \qquad (6.13)$$

where $q_T(s)$ denotes a risk-neutral density of $s_T = logS_T$. As $k \to -\infty$, $C_T(k)$ converges to S_0 and therefore not square-integrable. We will therefore consider the modification:

$$c_T(k) = \exp(\alpha k) C_T(k), \tag{6.14}$$

which is square-integrable for a suitable $\alpha > 0$ which may depend on the model for S_t. The Fourier transform of c_T is:

$$\psi_T(v) = \int_{-\infty}^{\infty} e^{ivk} c_T(k) dk.$$

Interchanging the integrals gives:

$$\begin{aligned}
\psi_T(v) &= \int_{-\infty}^{\infty} e^{ivk} \int_{k}^{\infty} e^{\alpha k} e^{-rT}(e^s - e^k) q_T(s) ds dk \\
&= \int_{-\infty}^{\infty} e^{-rT} q_T(s) \int_{-\infty}^{s} (e^{\alpha k + s} - e^{(\alpha+1)k}) e^{ivk} dk ds \\
&= \int_{-\infty}^{\infty} e^{-rT} q_T(s) \left(\frac{e^{(\alpha+1+iv)s}}{\alpha + iv} - \frac{e^{(\alpha+1+iv)s}}{\alpha+1+iv} \right) ds \\
&= \frac{e^{-rT} \phi(v - (\alpha+1)i)}{\alpha^2 + \alpha - v^2 + i(2\alpha+1)v},
\end{aligned}$$

where $\phi(u)$ is the characteristic function of the risk-neutral log-asset price process. If $\alpha = 0$, then the denominator in the final equality vanishes when $v = 0$, which induces a singularity like we saw previously. Since the FFT is evaluated with $v = 0$, this is a requirement. In order for c_T to be square-integrable, it is sufficient for $\psi_T(0)$ to be finite which happens when $\phi(-(\alpha+1)i)$ is finite. Since,

$$\begin{aligned}
\phi(-(\alpha+1)i) &= E(e^{i[-(\alpha+1)i] logS_T}) \\
&= E(e^{(\alpha+1) logS_T}) \\
&= E(S_T^{\alpha+1}).
\end{aligned}$$

It was found by Schoutens et. al [30]. that $\alpha = .075$ leads to stable algorithms, i.e. the prices are well replicated for many model parameters and the conditions are satisfied. Now, using Fourier inversion,

$$C_T(k) = \frac{\exp(-\alpha k)}{\pi} \int_0^{\infty} e^{-ivk} \psi_T(v) dv. \tag{6.15}$$

From numerical computation, if $v_j = \eta j,\ j = 0, \ldots, N-1$ and $\eta > 0$ is the width of a rectangle making up the area of integration, then:

$$C_T(k) \approx \tfrac{\exp(-\alpha k)}{\pi} \textstyle\sum_{j=0}^{N-1} e^{-iv_j k}\psi_T(v_j)\eta.$$

Since the FFT is an efficient algorithm for computing sums of the type,

$$X(u) = \textstyle\sum_{j=0}^{N-1} e^{-i\frac{2\pi}{N}ju}x_j, \quad for \quad u = 0, \ldots, N-1,$$

then the FFT should be used to calculate the approximation of $C_T(k)$.

Generally, options are traded most frequently around the spot price. This means we want to consider log-strike prices around the log spot price s_0:

$$k_u = -\tfrac{1}{2}N\zeta + \zeta u + s_0, \quad for \quad u = 0, \ldots, N-1,$$

where $\zeta > 0$ is the distance between the log strikes. Therefore, our approximation for $C_T(k)$ becomes:

$$C_T(k_u) \approx \tfrac{\exp(-\alpha k)}{\pi} \textstyle\sum_{j=0}^{N-1} e^{i\zeta\eta ju}e^{i[(\frac{1}{2}N\zeta - s_0)v_j]}\psi(v_j)\eta.$$

Provided that

$$\zeta\eta = \tfrac{2\pi}{N},$$

and letting

$$x_j = e^{i[(\frac{1}{2}N\zeta - s_0)v_j]}\psi(v_j),$$

we can apply the FFT. Since N controls the computation time and is often determined by computational setup, we face the trade-off between accuracy and the number of strikes around the spot price. This is due to having to choose a larger η if ζ is small.

To improve the integration, we can incorporate Simpson's rule weightings into our summation. We then obtain:

$$C_T(k_u) \approx \frac{\exp(-\alpha k)}{\pi} \sum_{j=0}^{N-1} e^{-i\frac{2\pi}{N}ju}e^{ibv_j}\psi(v_j)\frac{\eta}{3}[3 + (-1)^j - \delta_j], \tag{6.16}$$

where $b = (\frac{1}{2}N\zeta - s_0)$ and δ_n is the Kronecker delta function.

6.6.2. Option pricing through the direct integration method

Heston [17] was the first to make use of the characteristic function of the terminal asset price in an analytical formula for option pricing (in stochastic volatility models). Bakshi and Madan [4] then extended Heston's approach and showed generally we can write the price of a European call at time t, with strike price K and maturity T as:

$$C_T(K) = S_t \Pi_1 - K e^{-r(T-t)} \Pi_2 \tag{6.17}$$

where Π_1 and Π_2 are as in the previous section.

These techniques where further improved by Attari [2] and Lewis [22] whose work reduced the work needed into a single numerical integration. We will adopt the naming of Kilin [21] and refer to this as the direct integration (DI) method.

Theorem 47. *(The Direct Integration (DI) Method, Attari [2], pg. 3) The value of a European call option at time $t \leq T$ is given by:*

$$C_T(K) = S_t - \tfrac{1}{2} e^{-r(T-t)} K - e^{-r(T-t)} K b$$

$$\textit{where}$$

$$b = \left(\frac{1}{\pi} \int_0^\infty \frac{\left(Re(\phi(\omega)) + \frac{Im(\phi(\omega))}{\omega}\right) cos(\omega l) + \left(Im(\phi(\omega)) - \frac{Re(\phi(\omega))}{\omega}\right) sin(\omega l)}{1+\omega^2} dw \right).$$

Here, S_t is the underlying price at time t, K is the strike price, T is the maturity of the option, r is the risk-free interest rate, $\phi(\omega)$ is the risk-neutral characteristic function of:

$$x = log\left(\frac{S_T}{S_t}\right) - r(T-t),$$

and

$$l = log\left(\frac{K e^{-r(T-t)}}{S_t}\right).$$

Proof. See Marshall [24] pages 105-109. □

The advantages of this formula are the single integral contained within it and the quadratic term in the denominator results in a faster rate of decay.

6.7. Some recent improvements on classical BN-S model

The classical BN-S model has some disadvantages and those lead to the exploration of modifications and refinements of that model. For example, the classical model has just one background driving Lévy process for both the log-return and volatility. This may not accurately reflect empirical data. There are some recent works that address this issue. For instance, in [33], the authors proposed a generalized model that incorporated the log-return and volatility in a correlated way. In effect, this incorporated the "lag-time" of volatility reaction to market fluctuations. It turns out that such models are effective in modeling various swap prices in the financial market. To that end, ([15] and [16]), developed an analysis of variance, volatility, and covariance swaps based on the BN-S model. The mathematical analysis for the BN-S model is further extended in ([18] and [19]), where the authors developed a concept of volatility/variance modulated price index and expressed transition probability density functions in terms of various special functions. This is further analyzed in [14], where stocks are modeled by a superposition of non-Gaussian OU-type processes.

In more recent works ([27] and [28]), it is shown that some sequential hypothesis testing-based improvements on the BN-S model are possible. In addition, there are two desirable properties of the variance process suggested by the empirical data: (i) long-term memory and (ii) jumps. In [29], the authors introduced and analyzed the fractional BN-S stochastic volatility model that incorporates both these properties. The model primarily depends on a Brownian motion, a Lévy subordinator, and a fractional Brownian motion. The BN-S-based models are also implemented to the improvement of the portfolio of financial assets. For example, [23], considered a BN-S model-based portfolio optimization problem in a financial market under a general utility function.

Even with the above generalizations, there remains one significant problem for the resulting BN-S models - the resulting models usually lack the long-range dependence property. In [34], the authors have shown that for some empirical data, there is a hidden deterministic component in the "'jumps", that can be analyzed through various data-science-based techniques. In effect, when this component is fed back to the BN-S model, the model incorporates a long-range dependence. However, there is no unique way to extract this deterministic component. In [36], the authors have shown a couple of different approaches of extracting the deterministic component (viz. the volatility approach and the duration approach). Research in this area is active and ongoing.

Finally, it is worth mentioning that in addition to the derivative markets (e.g. stock markets), the BN-S model is applicable to commodity markets. For instance, in [35], the authors applied a machine/deep learning-based refined BN-S model (as described in the last paragraph) and analyzed optimal hedging strategies in a commodity market - oil market. In addition to that, the "quantity" of a commodity is also known to be stochastic. In [40], the authors provided a BN-S model-based mathematical way of handling the quantity risk in connection to the BN-S model.

REFERENCES

[1] D. Applebaum. *Lévy Processes and Stochastic Calculus.* Cambridge University Press, Cambridge, 2004.

[2] M. Attari. Option pricing using fourier transforms: A numerically efficient simplification. *SSRN Electronic Journal*, 2004.

[3] G. Bakshi and Z. Chen. An alternative valuation for contingent claims. *The Journal of Financial Economics*, 44(1):123–165, 1997.

[4] G. Bakshi and D.B. Madan. Spanning and derivative security valuation. *Journal of financial economics*, 55(2):205–238, 2000.

[5] Bshouty D. Bar-Lev, S. and G. Letac. Natural exponential families and self-decomposability. *Statistics and Probability Letters*, 13:147–152, 1992.

[6] O.E. Barndorff-Nielsen and N. Shephard. Non-gaussian ornstein uhlenbeck based models and some of there uses in financial exonomics. *Journal of the Royal Statistical Society*, Series B(63):167–241, 2001a.

[7] O.E. Barndorff-Nielsen and N. Shephard. *Lévy process-Theory and Applications*, chapter Modelling by L evy processes for financial econometrics, pages 283–318. Boston: Birkhauser, 2001b.

[8] D. Bates. Jumps and stochastic volatility: Exchange rate processes implicit in deutschemark options. *Review of Financial Studies*, 9:69–108, 1996.

[9] P. Carr and D. Madan. Option valuation using the fast fourier transform. *Journal of Computational Finance*, 2:61–73, 1999.

[10] G. Casella and R. Berger. *Statistical Inference.* Duxbury Press, 2002.

[11] R. Chen and L. Scott. Pricing interest rate options in a two-factor cox-ingersoll-ross model of the term structure. *Review of Financial Studies*, 5:613–636, 1992.

[12] E. Eberlein and J. Jacob. On the range of options prices. *Finance and Stochastics*, 1:131–140, 1997.

[13] A. Etheridge. *A course in Financial Calculus.* Cambridge University Press, 2002.

[14] Ghebremichael M. Habtemicael, S. and I. SenGupta. Volatility and variance swap using superposition of the barndorff-nielsen and shephard type lévy processes. *Sankhya B: The Indian Journal of Statistics*, 81(1):75–92, 2019.

[15] S. Habtemicael and I. SenGupta. Pricing variance and volatility swaps for barndorff-nielsen and shephard process driven financial markets. *International Journal of Financial Engineering*, 3(4):1650027, 2016a.

[16] S. Habtemicael and I. SenGupta. Pricing covariance swaps for barndorff-nielsen and shephard process driven financial markets. *Annals of Financial Economics*, 11:1650012, 2016b.

[17] S. Heston. A closed-form solution for options with stochastic volatility with applications to bond and currency options. *Review of Financial Studies*, 6:327–343, 1993.

[18] A. Issaka and I. SenGupta. Feynman path integrals and asymptotic expansions for transition probability densities of some lévy driven financial markets. *Journal of Applied Mathematics and Computing*, 54(1):159–182, 2017a.

[19] A. Issaka and I. SenGupta. Analysis of variance based instruments for ornstein-uhlenbeck type models: swap and price index. *Annals of Finance*, 13(4):401–434, 2017b.

[20] J. Jacob and A.N. Shiryaev. *Limit theorems for stochastic processes.* Berlin: Springer, 1987.

[21] F. Kilin. Accelerating the calibration of stochastic volatility models. *University Library of Munich*, (MPRA 2975), 2007.

[22] A. Lewis. A simple option formula for general jump-diffusion and other exponential lévy processes. *SSRN Electronic Journal*, 2002.

[23] M. Lin and I. SenGupta. Analysis of optimal portfolio on finite and small time horizons for a stochastic volatility market model. *SIAM Journal on Financial Mathematics*, 12(4):1596–1624, 2021.

[24] J. Marshall. *Stochastic Volatility Modeling of the Ornstein Uhlenbeck Type: Pricing and Calibration*. Dissertation, University of Johannesburg, 2008.

[25] E. Nicolato and E. Venardos. Option pricing in stochastic volatility models of ornstein-uhlenbeck type. *Mathematical Finance*, 13:445–466, 2003.

[26] S. Raible. *Lévy processes in finance: Theory, Numerics, and Empirical Facts*. Dissertation, University of Freiburg, 2000.

[27] M. Roberts and I. SenGupta. Infinitesimal generators for two-dimensional lévy process-driven hypothesis testing. *Annals of Finance*, 16(1):121–139, 2020.

[28] M. Roberts and I. SenGupta. Sequential hypothesis testing in machine learning, and crude oil price jump size detection. *Applied Mathematical Finance*, 27(5):374–395, 2021.

[29] N. Salmon and I. SenGupta. Fractional barndorff-nielsen and shephard model: applications in variance and volatility swaps, and hedging. *Annals of Finance*, 17:529–558, 2021.

[30] Simons E. Schoutens, W. and J. Tistaert. *A Perfect Calibration! Now What?* New York: John Wiley and Sons, 2005.

[31] W. Schoutens. *Lévy processes in finance: pricing financial derivatives*. New York: John Wiley and Sons, 2003.

[32] L. Scott. Pricing stock options in a jump-diffusion model with stochastic volatility and interest rates: Application of fourier inversion methods. *Mathematical Finance*, 7:413–426, 1997.

[33] I. SenGupta. Generalized bn-s stochastic volatility model for option pricing. *International Journal of Theoretical and Applied Finance*, 19(02):1650014, 2016.

[34] Nganje W. SenGupta, I. and E. Hanson. Refinements of barndorff-nielsen and shephard model: an analysis of crude oil price with machine learning. *Annals of Data Science*, 8(1):39–55, 2021.

[35] Wilson W. SenGupta, I. and W. Nganje. Barndorff-nielsen and shephard model: oil hedging with variance swap and option. *Mathematics and Financial Economics*, 13(2):209–226, 2019.

[36] H. Shoshi and I. SenGupta. Hedging and machine learning driven crude oil data analysis using a refined barndorff-nielsen and shephard model. *International Journal of Financial Engineering*, 8(4):2150015, 2021.

[37] S.E. Shreve. *Stochastic Calculus for Finance II*. Springer, 2004.

[38] U. Wiersema. *Brownian Motion Calculus*. John Wiley and Sons, 2008.

[39] Howison S. Wilmott, P. and J. Dewynne. *The Mathematics of Financial Derivatives: A Student Introduction*. Cambridge University Press, 1995.

[40] Nganje W. Gebresilasie S. Wilson, W. and I. SenGupta. Barndorff-nielsen and shephard model for hedging energy with quantity risk. *High Frequency*, 2(3-4):202–214, 2019.

[41] S.J. Wolfe. On a continuous analogue of the stochastic difference equation $\rho x_{n+1}+b_n$. *Stochastic Processes and their Applications*, 12:301–312, 1982.