ROBUSTNESS OF THE EIGENVALUE TEST FOR COMMUNITY STRUCTURE

A Thesis
Submitted to the Graduate Faculty
of the
North Dakota State University
of Agriculture and Applied Science

By

Matthew Allan Rose

In Partial Fulfillment of the Requirements
for the Degree of
MASTER OF SCIENCE

Major Department:
Statistics

April 2021

Fargo, North Dakota

NORTH DAKOTA STATE UNIVERSITY

Graduate School

Title

ROBUSTNESS OF THE EIGENVALUE TEST FOR COMMUNITY STRUCTURE

By

Matthew Allan Rose

The supervisory committee certifies that this thesis complies with North Dakota State University's regulations and meets the accepted standards for the degree of

MASTER OF SCIENCE

SUPERVISORY COMMITTEE:

Mingao Yuan

Chair

Peter Oduor

Ronald Degges

Approved:

04/16/2021

Date

Rhonda Magel

Department Chair

ABSTRACT

Networks can take on many different forms, such as the people from the University you attend. Within these networks, community structure may exist. This "community structure" refers to the clustering of nodes by a common characteristic. There are many algorithms to extract communities within a network. These methods depend on the assumption that structure exists within the network. Statistical tests have been proposed to test this assumption. In practice, networks may have measurement errors. This usually comes in the form of missing data or other faults. As a result, networks may not tell the full story at surface level and network structure often suffer from some type of error, as there may be nodes or edges absent from the data or ones that should not exist within the network. We wish to observe the effectiveness of the largest eigenvalue test for community structure when error is introduced into the network.

TABLE OF CONTENTS

LIST OF FIGURES

LIST OF APPENDIX TABLES

LIST OF APPENDIX FIGURES

1. INTRODUCTION

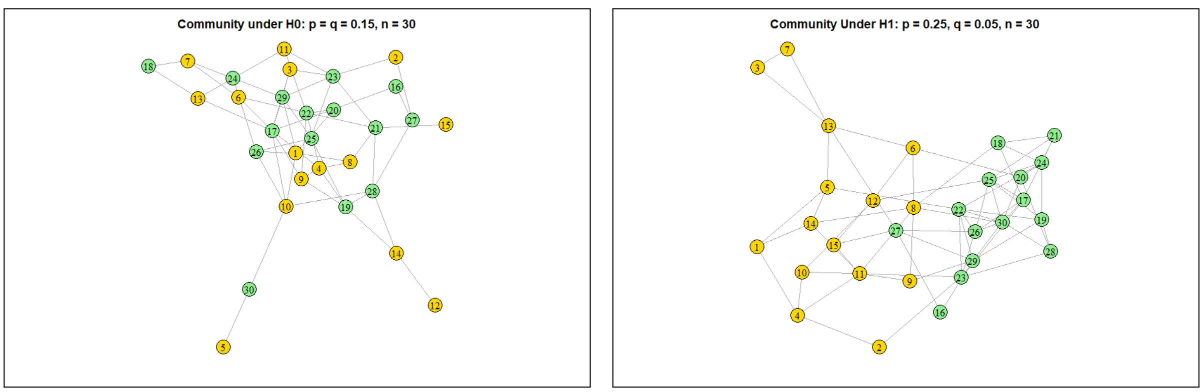
1.1. Background

Networks are a larger part of daily life than many of us realize. We exist within a network in nearly every part of society, such as the school you attend, the job you work at, or the neighborhood you live in. These networks do not stop at human beings either, they exist in many other facets, such as businesses and retail goods. With network analysis, we are able to observe such phenomena like: "what retail items are commonly bought together" and "whether someone knows another person because of mutual friends". When we look at graphs of these networks, we see a scattering of points connected by lines. Points are generally referred to as "nodes" and lines are referred to as "edges". In the interest of people, each "node" would represent a person, while the "edges" represent some connection between two people. These connections represent what we are trying to analyze. If we are looking at a graph of a group of people, the lines drawn may represent that those two people know each other or are friends. A simple method to represent this type of network is an Erdős-Rényi graph. An Erdős-Rényi graph can be represented by G(n,p_0), where n is the number of nodes in the graph and p_0 is the probability of having an edge between two nodes (Fortunato 2010). It can be seen that the probability of sharing an edge with another node is equal for every possible combination. The interest is to observe patterns or clusters within the data. We refer to a cluster in our graph as a "community". When clusters or separation are observed in the graph, we can say that a "community structure" may exist in the data. When referring to Erdős-Rényi graphs, we expect to see no community structure since the probability of an edge existing between any two nodes is equal (Fortunato 2010). A new parameter q can be defined in the model in this case that represents the probability of an edge existing between nodes of different communities, referred to as "between" probability. The parameter p can now be defined as the probability of an edge existing between two nodes of the same community, referred to as "within" probability. The new model $G(n, p, q)$, with the restriction $p \geq q$, is referred to as the stochastic block model, and can be used to model community structure (Mossel et al. 2018). The main purpose of community detection is to extract the communities. Whether there is community structure within a network is unknown in practice. To test the existence of community structure, the following hypothesis test can be constructed using the Erdős-Rényi model and the stochastic block model (Bickel and Sarkar 2016).

$$H_0 : \text{The network follows an Erdős-Rényi model: } G(n, p_0) \qquad \text{where } p = q = p_0 \tag{1.1}$$

$$H_1 : \text{The network follows a Stochastic Block Model: } G(n, p, q) \qquad \text{where } p > q \tag{1.2}$$

In the case where the network follows an Erdős-Rényi model, the probability of an edge existing within a group is the same as the probability of an edge existing between groups. As stated before, no community structure is expected in an Erdős-Rényi graph. In the case where the network follows a stochastic block model, we expect to see two well-defined communities (Mossel et al. 2018). An example of the two graphs can be seen below.



(a) A Network plot from an Erdős-Rényi model. (b) A Network plot from a Stochastic Block Model.

Figure 1.1. Network plots under null and alternative hypotheses.

As we can see from the two graphs, the Erdős-Rényi model shows no particular clusters as we see a similar number of edges between the two groups as we do within the groups. In the stochastic block model, we can observe a higher number of edges within the two groups than we can between them.

A network on vertex set $[n] = \{1, 2, \ldots, n\}$ can be represented by an adjacency matrix $\mathbf{A}$, a symmetric n x n matrix with all diagonal entries being equal to zero while all other entries are 0 or 1. A value of 1 indicates an edge between the i^{th} and j^{th} nodes while a value of 0 indicates no edge. To generate a graph from stochastic block model $G(n, p, q)$, we define an n x 1 vector $\mathbf{Z}$, where the elements of $\mathbf{Z}$ are randomly assigned a value 1 or -1 with probability 0.5. This defines our two groups of interest. Considering the within probability p and the between probability q, we can randomly generate the edges for two given nodes using $\mathbf{A}_{ij} \sim$ *Bernoulli(p)* when the i^{th} and the j^{th} element of $\mathbf{Z}$ are equal and $\mathbf{A}_{ij} \sim$ *Bernoulli(q)* when the i^{th} and the j^{th} element of $\mathbf{Z}$ differ. For the adjacency matrix we need only consider $i < j$.

$$\mathbf{A}_{ij} = \mathbf{A}_{ji} = \begin{cases} 0 & \text{if } i = j \\ Bernoulli(\text{p}) & \text{if } \mathbf{Z}_i = \mathbf{Z}_j \\ Bernoulli(\text{q}) & \text{if } \mathbf{Z}_i \neq \mathbf{Z}_j \end{cases} \qquad \text{for } i < j \tag{1.3}$$

The adjacency matrix help us accomplish one of the fundamental tasks: finding communities within a network. There are many algorithms used to detect community structure. The success of these algorithms depend on the assumption that community structure exists. The existence of community structure can be stated by the hypotheses (1.1) and (1.2), in which no community structure exists for (1.1) and community structure exists for (1.2). Several statistical tests are proposed to test this assumption, such as the subgraph-count tests in (Gao and Lafferty 2017, Jin et al. 2018, Yuan and Nan 2020, Yuan and Wen 2021) and the largest eigenvalue test in (Bickel and Sarkar 2016).

For simulated networks, data can usually be generated without the possibility of error. For most real-world networks, this is generally not the case. Surveys and other methods of data collection may give us a general idea of the structure of the network, but often imperfect (Newman 2018). This error may disturb the community structure of the network we are observing. This disruption is caused by misplaced nodes and edges. When a node or edge that should be accounted for in the network is missing, we have a false negative node/edge. When a node or edge that should not exist is present within the network, we have a false positive node/edge. In addition to this, there may exist duplications and merging of edges and nodes (Wang et al. 2012). In this thesis, we are interested in evaluating the effectiveness of the "largest eigenvalue" test when error is present. Here, we will consider error in the form of adding or deleting edges. We can observe the effect of false positive edges (adding an edge that was originally absent) and false negative edges (deleting an edge that was already present) and how the test's ability to detect communities is impacted.

1.2. The Largest Eigenvalue Test

For the largest eigenvalue test, we assume that the edge probabilities are constant. Again, the purpose of the test is to determine whether the network follows an Erdős-Rényi model (no community structure) or the network follows a stochastic block model (some community structure exists) (Bickel and Sarkar, 2016). We can refer to (1.1) and (1.2) for the respective null and alternative hypotheses. Under an Erdős-Rényi model, we define the statistic P as $P = E[\mathbf{A}]$:

$$P = np\vec{\boldsymbol{e}}\vec{\boldsymbol{e}}^T - p\mathbf{I}, \tag{1.4}$$

where $\vec{\boldsymbol{e}}$ is a vector of length n with all entries equal to $\frac{1}{\sqrt{n}}$ and $\mathbf{I}$ is an n x n identity matrix. We now use P to calculate the normalized matrix $\hat{\mathbf{A}}$ (Bickel and Sarkar, 2016):

$$\hat{\mathbf{A}} = \frac{\mathbf{A} - P}{\sqrt{(n-1)p(1-p)}}. \tag{1.5}$$

Now, we can derive the eigenvalues of the normalized matrix $\hat{\mathbf{A}}$. We will refer to the largest eigenvalue of $\hat{\mathbf{A}}$ as λ_1. In most cases, we do not know the actual value of p and q when sampling from networks. When this is the case, we must estimate p by $\hat{p}$. The estimate of $\hat{p}$ can be derived from the original adjacency matrix $\mathbf{A}$ and can be expressed as

$$\hat{p} = \frac{\sum_{j=1}^{n}\sum_{i=1}^{n}\mathbf{A}_{ij}}{n(n-1)} \tag{1.6}$$

Simply, we are taking the sum of all the edges in the graph and dividing it by the maximum number of edges possible in the graph. Since a node cannot share an edge with itself, the diagonal values of the matrix are not considered in the estimation of the edge probability. Given the estimate $\hat{p}$, the previous statistics can be redefined using the estimation. We define these new statistics as $\hat{P}$, $\hat{\mathbf{A}}^{'}$, and $\hat{\lambda_1}$, the largest eigenvalue of $\hat{\mathbf{A}}^{'}$ (Bickel and Sarkar 2016).

$$\hat{P} = n\hat{p}\vec{\boldsymbol{e}}\vec{\boldsymbol{e}}^T - \hat{p}\mathbf{I}. \tag{1.7}$$

$$\hat{\mathbf{A}}' = \frac{\mathbf{A} - \hat{P}}{\sqrt{(n-1)\hat{p}(1-\hat{p})}}. \tag{1.8}$$

With $\hat{\lambda_1}$ we can define our test statistic θ, which converges to a Tracy-Widom distribution with index 1 (TW_1) (Lee and Yin 2014, Tracy and Widom 1994).

$$\theta := n^{2/3}\{\hat{\lambda_1} - 2\} \to TW_1. \tag{1.9}$$

Using the test statistic θ, we can compute a p-value from the tables of the Tracy-Widom distribution for which P(X $\geq \theta$) equals said p-value. The p-value can then be compared to a chosen significance level and a decision on the hypothesis test can be made.

1.3. Network with Measurement Errors

We have discussed how the largest eigenvalue test can help determine whether community structure exists within a network, but now we wish to see how reliable the test is in the face of error. Human error tends to be a large culprit when it comes to incorrect data. In the case of networks, we may have scenarios where we say two nodes share an edge when they actually do not. Inversely, we may miss an edge between two nodes, especially when we are dealing with a large number of nodes. When measuring networks of social interaction, we often need to collect data by surveys. Many factors in these surveys can cause errors, such as incorrectly recording a response, a misleading or poorly-worded question, or non-response (Newman 2018). The factor of interest may cause response error as well, such as when observing a network where a "friendship" represents an edge, the conditions of what constitutes a "friendship" may be subjective (Wang et al. 2012).

This error can be simulated in our network, and a new adjacency matrix can be created that takes error probabilities into account. We can define two error probabilities: ϵ_a and ϵ_d, where ϵ_a is the probability of adding an edge and ϵ_d is the probability of deleting an edge. As mentioned previously, this simulates just one of many types of errors that may be present in a network. Using these error terms, the "error" adjacency matrix $\widetilde{\mathbf{A}}$ can be created. Given a membership vector $\mathbf{Z}$, the error adjacency matrix re-samples the outcomes from the adjacency matrix $\mathbf{A}$ from a Bernoulli distribution, this time considering the error probabilities. The error adjacency matrix $\widetilde{\mathbf{A}}$ is resampled as:

$$\widetilde{\mathbf{A}}_{ij} = \widetilde{\mathbf{A}}_{ji} = \begin{cases} Bernoulli(p\epsilon_a) & \text{if } \mathbf{Z}_i = \mathbf{Z}_j \text{ and } \mathbf{A}_{ij} = 0,\ i < j \\ Bernoulli(q\epsilon_a) & \text{if } \mathbf{Z}_i \neq \mathbf{Z}_j \text{ and } \mathbf{A}_{ij} = 0,\ i < j \\ Bernoulli(1 - p\epsilon_d) & \text{if } \mathbf{Z}_i = \mathbf{Z}_j \text{ and } \mathbf{A}_{ij} = 1,\ i < j \\ Bernoulli(1 - q\epsilon_d) & \text{if } \mathbf{Z}_i \neq \mathbf{Z}_j \text{ and } \mathbf{A}_{ij} = 1,\ i < j \end{cases} \quad \text{for } i \neq j \tag{1.10}$$

Logistically, when $\mathbf{A}_{ij} = 0$, we are dealing with the probability of adding an edge (ϵ_a), and when $\mathbf{A}_{ij} = 1$, we are dealing with the probability of deleting an edge (ϵ_d). For the first case, we have a $p\epsilon_a$ probability of adding an edge between two nodes that are within the same group. For case two, there is a $q\epsilon_a$ probability of adding an edge between two nodes that are in different groups. For the third case, the probability of deleting an edge shared by two nodes within the same group is 1 - $p\epsilon_d$. Finally, for the last case, the probability of deleting an edge shared by two nodes in different groups is 1 - $q\epsilon_d$. It should be noted that when $\epsilon_a = \epsilon_d = 0$, $\mathbf{A} = \widetilde{\mathbf{A}}$. When the error probabilities are zero, we have a Bernoulli with probability 0 when $\mathbf{A}_{ij} = 0$, meaning the entry will not change for $\widetilde{\mathbf{A}}_{ij}$. This also goes for cases three and four, as when the error probabilities are zero, we have a Bernoulli with probability 1 when $\mathbf{A}_{ij} = 1$, implying the entry will stay the same for $\widetilde{\mathbf{A}}_{ij}$. Now given the adjacency matrix $\mathbf{A}$ and the error adjacency matrix $\widetilde{\mathbf{A}}$, we can calculate the test statistic θ and determine the size and power of the test.

2. SIMULATION METHODOLOGY

2.1. Simulation Methods: Size

The simulations of size and power were computed using R software, where the adjacency matrices were simulated given the parameters n, p, q, ϵ_a, *and* ϵ_d along with the indicator vector $\mathbf{Z}$. For decisions regarding hypothesis testing, a significance level of $\alpha = 0.05$ was used. When estimating size and power, the results from 500 repetitions were used. To estimate size, we assume that the null hypothesis is true. This means that the network follows an Erdős-Rényi model $G(n, p_0)$ where $p_0 = p = q$. The adjacency matrix $\mathbf{A}$ will be generated with dimension n x n and the probability that the i^{th} and j^{th} nodes share an edge is p_0. The error adjacency matrix $\widetilde{\mathbf{A}}$ is then generated based on the entries of $\mathbf{A}$. Since it was shown that $\mathbf{A} = \widetilde{\mathbf{A}}$ when $\epsilon_a = \epsilon_d = 0$, it is sufficient to use $\widetilde{\mathbf{A}}$ to calculate $\hat{\mathbf{A}}'$ even when there is an absence of error. Although the value of p_0 is defined for the simulation and used to generate the data, we assume it is unknown when evaluating the hypothesis. The estimate $\hat{p}$ must be used instead to calculate $\hat{P}$ and $\hat{\mathbf{A}}'$. Once the statistic $\hat{P}$ and the normalized matrix $\hat{\mathbf{A}}'$ have been generated, we can extract the largest eigenvalue $\hat{\lambda_1}$ from $\hat{\mathbf{A}}'$ to calculate the test statistic θ. The p-value is calculated as $\mathrm{P}(\mathrm{X} \geq \theta)$ from TW_1 and is compared to the significance level $\alpha = 0.05$. We then record whether H_0 is rejected or not. If the null hypothesis is rejected, a type I error has been committed. This holds since the data was simulated under an Erdős-Rényi model, but the p-value of the test statistic warrants a rejection of H_0 which states that the network follows an Erdős-Rényi model. This process is repeated 500 times and the number of rejections of H_0 is recorded. The proportion of rejections out of the 500 repetitions is reported as the size. Simulations were done with both $p_0 = 0.15$ and $p_0 = 0.25$, with sample sizes of $n = 30$, 50, 70, 90, 120, 240, 360, and 480. The parameters ϵ_a and ϵ_d were assumed to be equal when used and took on values of 0, 0.05, 0.10, 0.15, 0.20, 0.25, 0.30, and 0.35.

2.2. Simulation Methods: Power

The simulation of power is very similar to size, with a few exceptions. When size was calculated, we assumed that H_0 was true and the network followed an Erdős-Rényi model. When simulating for power, we assume that the alternative hypothesis H_1 is true and that the network simulated follows a Stochastic Block Model $G(n,p,q)$ (Bickel and Sarkar 2016). The probabilities of

p and q differ in this case and $p > q$. The values of p and q also have the stipulation that $\frac{p+q}{2} = p_0$. This is due to using the estimate $\hat{p}$ to calculate the test statistic. The error adjacency matrix $\widetilde{\mathbf{A}}$ can be generated given n, p, q, ϵ_a, and ϵ_d. The test statistic θ can be calculated and a decision on the hypothesis can be rendered given the p-value from TW_1. Since the assumption was that H_1 was true and the data was generated from a Stochastic Block Model, a failure to reject H_0 would result in a type II error. For 500 repetitions, we generate a test statistic and make a decision on the hypothesis test. The number of times we fail to reject H_0 is recorded, and the proportion is the type II error β. The power can then be obtained by taking $1 - \beta$. When size was calculated, two different values of p_0 were used, $p_0 = 0.15$ and $p_0 = 0.25$. For the power simulations, the values of p and q can differ as long as $\frac{p+q}{2} = p_0$ is satisfied and $p > q$. Each value of p_0 was broken into three cases: a small difference between p and q, a moderate difference between p and q, and a large difference between p and q. This gives us estimates of power for six different combinations of p and q. The six combinations are as follows:

For $p_0 = 0.15$:

Small difference: $p = 0.17$, $q = 0.13$

Moderate difference: $p = 0.20$, $q = 0.10$

Large difference: $p = 0.25$, $q = 0.05$

For $p_0 = 0.25$:

Small difference: $p = 0.27$, $q = 0.23$

Moderate difference: $p = 0.30$, $q = 0.20$

Large difference: $p = 0.35$, $q = 0.15$

The condition $\frac{p+q}{2} = p_0$ is satisfied in all cases. The difference between p and q affects how quickly the power converges given the sample size n. For this reason, different sets of sample sizes are used for each case to display the convergence of power. When the probabilities $p$and q are closer, a large sample size is needed to achieve a high power. Primarily, we look to observe the changes in power when the error probabilities ϵ_a and ϵ_d are present. The same parameters are used for each error probability, with all combinations of ϵ_a and ϵ_d being observed. This leaves us with 64 combinations of values for ϵ_a and ϵ_d as each value ranges from 0 to 0.35 in intervals of 0.05. Power is then calculated for each selected sample size given the error probability combination. This process is repeated amongst the six sets of values for p and q.

3. SIMULATION RESULTS

3.1. Simulation Results: Size

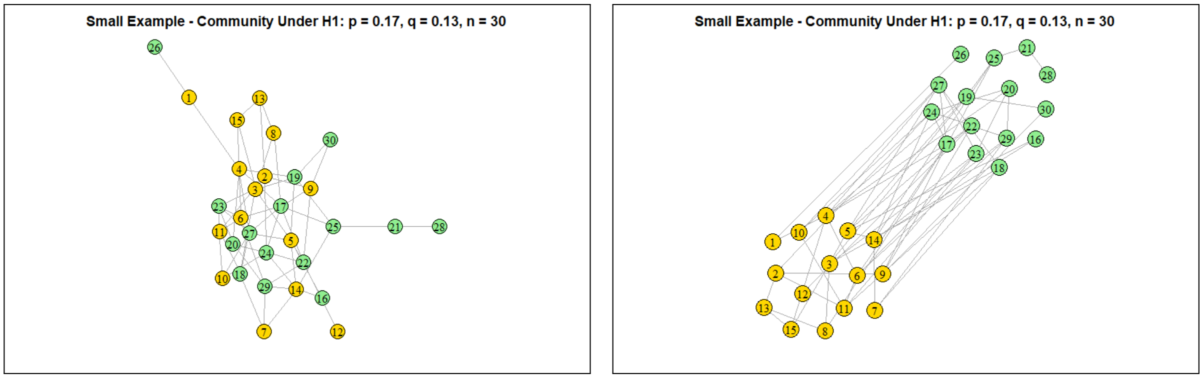
Before running the simulations for power, we first observe the size of the test for each value of p_0. The type I error (α) must be within an acceptable threshold, for these simulations, the acceptable range was $0.02 \leq \alpha \leq 0.06$. For $p_0 = 0.15$, the size was acceptable for all sample sizes in the absence of the error probabilities. When looking at how ϵ_a and ϵ_d affect the size, it appears the smaller sample sizes are impacted more by the error probabilities. When $n = 30$ and $\epsilon_a = \epsilon_d = 0$, we achieve a size of 0.024 whereas when $\epsilon_a = \epsilon_d = 0.35$, the size was 0.016. This slight decrease in size is noticeable up until $n = 120$ where the size begins to appear more constant in the presence of error. When p_0 is increased to 0.25, the size was much lower compared to $p_0 = 0.15$. The size is actually outside the acceptable range when the sample size is less than 90. The size appears to remain relatively constant across all sample sizes when the error probabilities increase.

3.2. Simulation Results: Power

The tests for power are simulated at each sample size for each of the six combinations of the within probability p and the between probability q. Different combinations of the error probabilities ϵ_a and ϵ_d are also used. By assuming the alternative H_1 is true and that the network follows a Stochastic Block Model, we estimate the proportion of times out of 500 that we commit a type II error (β) and calculate the power (1-β). We will observe the trends in power as well as the impact of the error probabilities for each of the six combinations of p and q. For this study, we seek to obtain an acceptable power of (1-β) ≥ 0.90.

3.2.1. Small Difference: $p = 0.17$ and $q = 0.13$

Observing the trends in power for when p and q are very similar, we notice that a much higher sample size is required to achieve an acceptable power. Three different sample sizes were used: 360, 480, and 600. The test crossed the acceptable 0.90 threshold between $n = 480$ (0.768) and $n = 600$ (0.994) in the absence of ϵ_a and ϵ_d. This should follow, as since p and q are very similar, we must have a sufficient number of nodes in order to distinguish the difference. The following community plot can help support this:



(a) Random network plot under alternative hypothesis.

(b) Clustered network plot under alternative hypothesis.

Figure 3.1. Network plots under alternative hypothesis $p = 0.17$ and $q = 0.13$.

The graph on the left is not clustered, although as n increases, the two groups will begin to appear more clustered. The graph on the right is clustered, with yellow representing the first $\frac{n}{2}$ nodes and light green representing nodes $\frac{n}{2} + 1$ to n. This graph allows us to have a better view of the edges between nodes. We can see the connections between nodes within each group as well as the connections between each group. As the probabilities of sharing an edge within groups (p) and between groups (q) are similar, we see a similar number of edges between and within at a small sample size. As sample size increases, this difference will become more apparent. Even when $n = 360$, the power of the test was low (0.266), so a sample size of $n = 30$ would result in an extremely low power. Taking the error probabilities into account, we can observe the change in power in several different ways such as the individual impact of ϵ_a and ϵ_d. When holding ϵ_a constant at 0, we can observe the change in power due to ϵ_d, the probability of deleting an edge. The plot of power by ϵ_d for each sample size is displayed below:

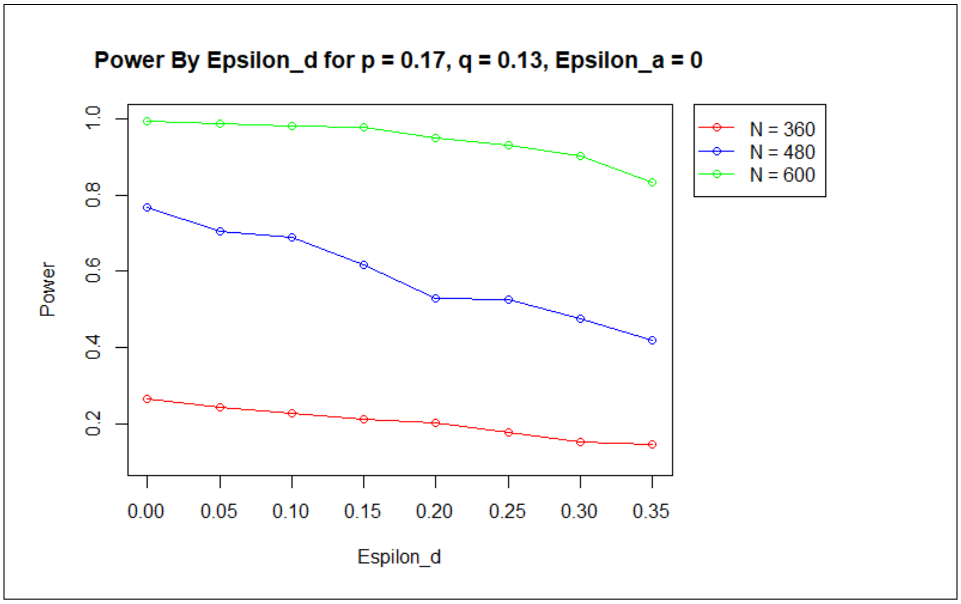


Figure 3.2. Plots of power under alternative hypothesis $p = 0.17$ and $q = 0.13$, $\epsilon_a = 0$.

As ϵ_d increases, there appears to be a decrease in power. This decrease is most apparent with the $n = 480$ sample size. In the absence of ϵ_d, the power of the test at a sample size of 480 was 0.768. When the probability of deleting an already existing edge was 0.35, the power shrunk to 0.42. The largest sample size of 600 was more resilient, especially with smaller values of ϵ_d. Next, we can observe power when both error probabilities are present. Holding the probability of adding an edge ϵ_a at 0.2, the obtained powers are displayed:

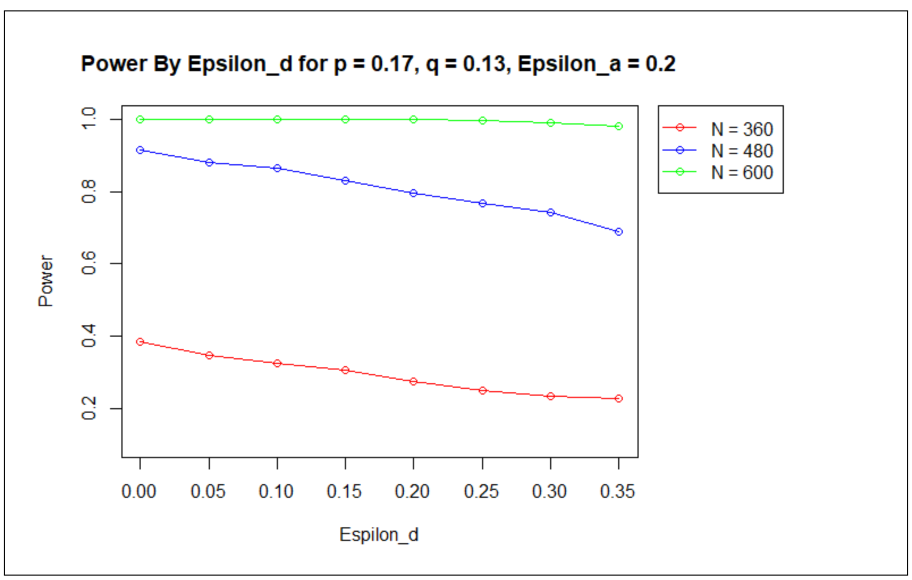


Figure 3.3. Plots of power under alternative hypothesis $p = 0.17$ and $q = 0.13$, $\epsilon_a = 0.20$.

It can be noted that the presence of ϵ_a causes an increase power. Meaning that when the probability of adding an edge exists, this can cause an overestimation of power. The plots follow a similar pattern as ϵ_d increases, with the plots being shifted upwards. Again, the middle sample size of 480 was affected the most, with a power of 0.918 when $\epsilon_d = 0$ and a power of 0.688 when $\epsilon_d = 0.35$. With a sample size of 600, the power stays near 1 even as ϵ_d approaches 0.35. Lastly, we can observe power when both errors approach their most extreme. Holding ϵ_a constant at 0.35, we obtain the following power plots:

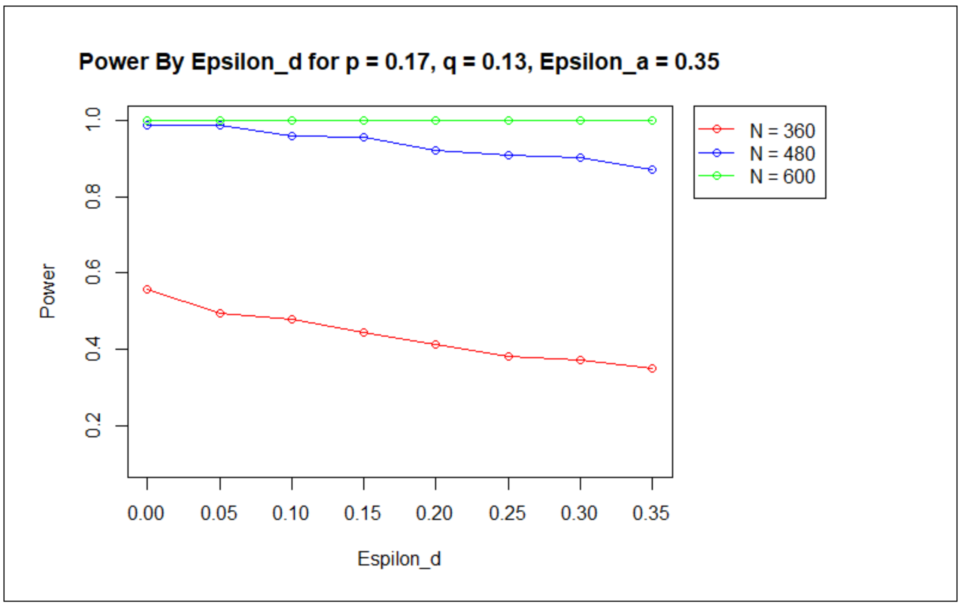
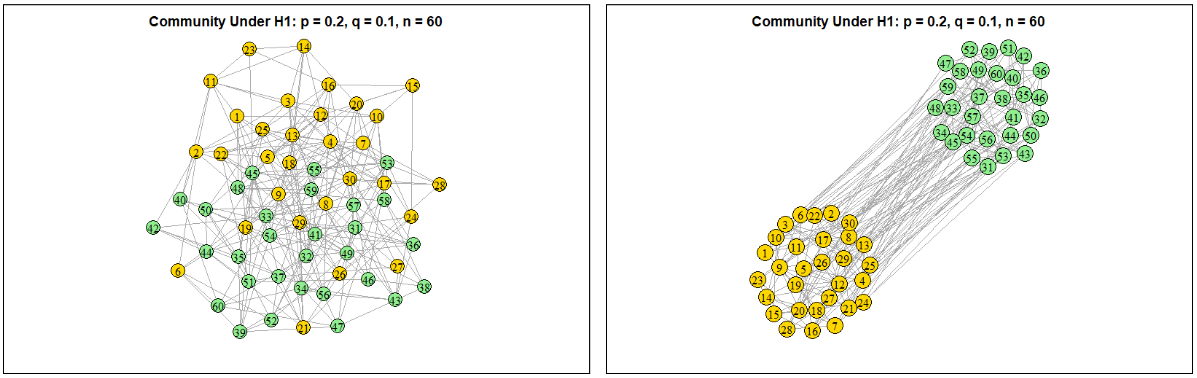


Figure 3.4. Plots of power under alternative hypothesis $p = 0.17$ and $q = 0.13$, $\epsilon_a = 0.35$.

With the probability of adding an edge being relatively high, this causes the power at a sample size of 480 to be nearly 1. With the initial power being overestimated due to ϵ_a, the decrease as ϵ_d increases are not quite as extreme. We can see that the absence of edges that should exist cause the power to decrease, while the addition of edges that should not be present leads to an overestimation of power.

3.2.2. Moderate difference: $p = 0.2$ and $q = 0.1$

Next, we will observe the power of the eigenvalue test and the impact of the error probabilities when p and q are more modestly distinct. Since the probabilities used to simulate this data are further apart, we should expect to have a higher power at lower sample sizes. For this set, sample sizes of 60, 90, 120, and 150 were used. When setting ϵ_a and ϵ_d equal to 0, we obtain a power of 0.162 at $n = 60$, a power of 0.53 at $n = 90$, a power of 0.91 at $n = 120$, and a power of 0.996 at $n = 150$. We can take a look at the network plot to get a visualization of the network. The following is a network with $n = 60$ nodes:



(a) Random network plot under alternative hypothesis.

(b) Clustered network plot under alternative hypothesis.

Figure 3.5. Network plots under alternative hypothesis $p = 0.20$ and $q = 0.10$

The plot on the left is the non-clustered community, however, we begin to see the clustering of each group at this stage. On the right, we can examine the edges between communities and see how there exists more edges within each group than between groups. Now, we can observe the change in power when holding ϵ_a constant at 0 with $p = 0.2$ and $q = 0.1$. The plot of power for each of the four sample sizes is displayed:

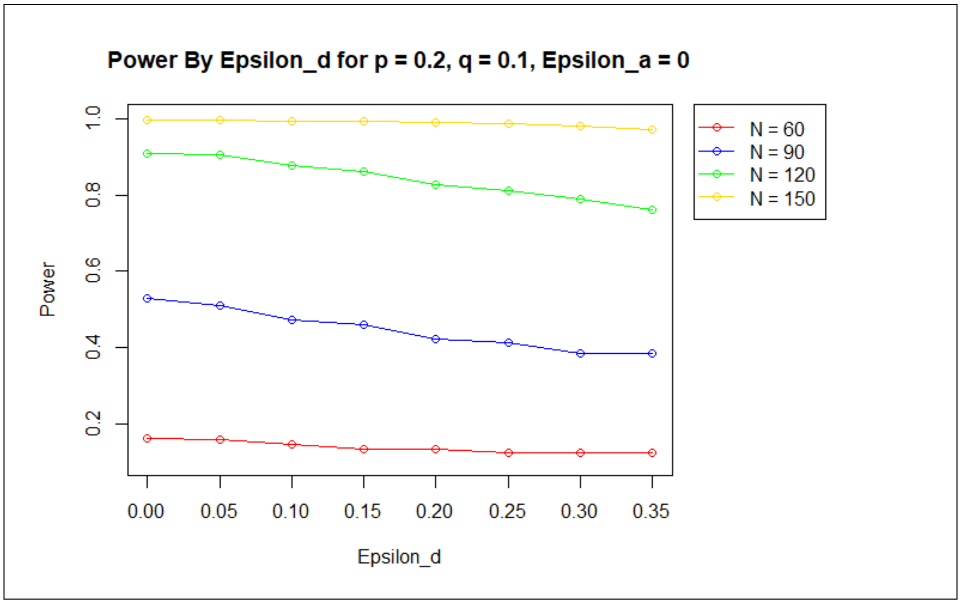


Figure 3.6. Plots of power under alternative hypothesis $p = 0.20$, $q = 0.10$, and $\epsilon_a = 0$.

We notice a similar pattern with the change in power that was observed with the small difference between p and q. At a sample size of 150, the test is able to withstand a large ϵ_d and maintain a high power. When the sample size was 120, the power was an acceptable 0.91. However, with a large error $\epsilon_d = 0.35$, the power shrinks to 0.76. When $n = 90$, there is also a noticeable decrease in power. Now, we can observe the change in power when we introduce ϵ_a into the test, setting it constant at 0.2. The following powers are plotted for each sample size:

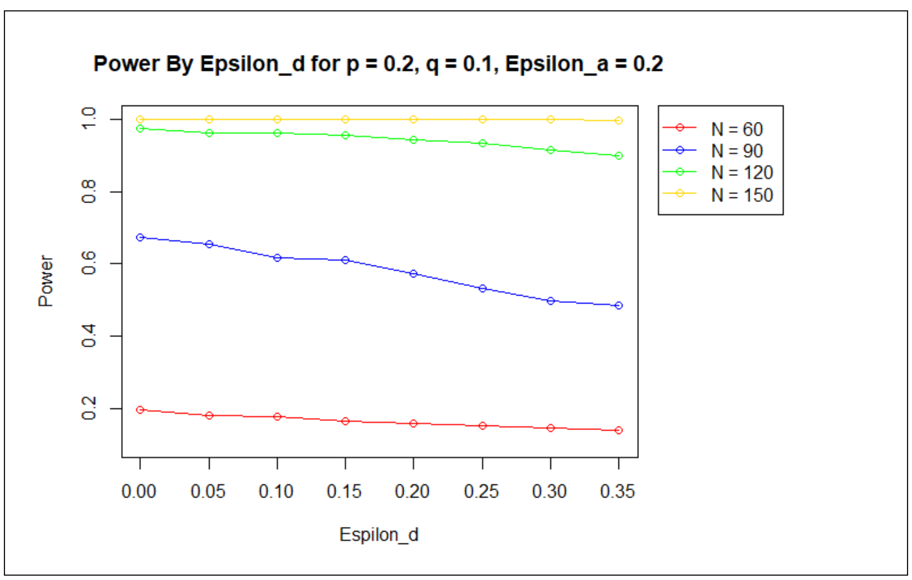


Figure 3.7. Plots of power under alternative hypothesis $p = 0.20$, $q = 0.10$, and $\epsilon_a = 0.20$.

Again, we observed a vertical shift upwards as ϵ_a went from 0 to 0.2. The addition of edges causes an overestimation in power, as it did with the small difference between p and q. At a sample size of 150, the sample size remains close to 1 even as ϵ_d increases. At $n = 120$, the power was a bit more resilient to the decrease that ϵ_d causes, with that decrease now being most apparent with $n = 90$. It appears that when the test has an initial range of power around 0.40 to 0.90, it is more influenced by ϵ_d than powers outside this range. Finally, we observed power holding ϵ_a constant at 0.35:

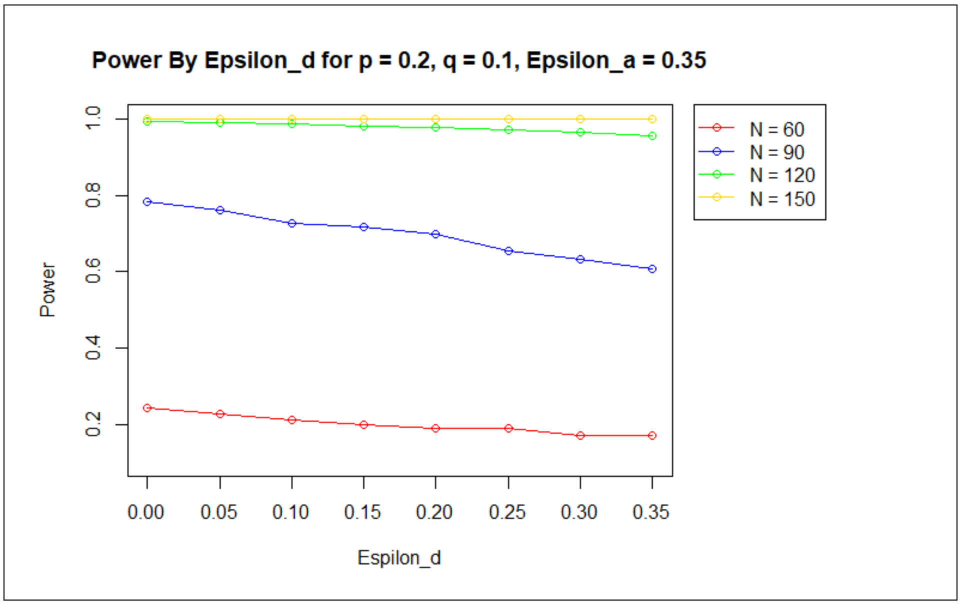
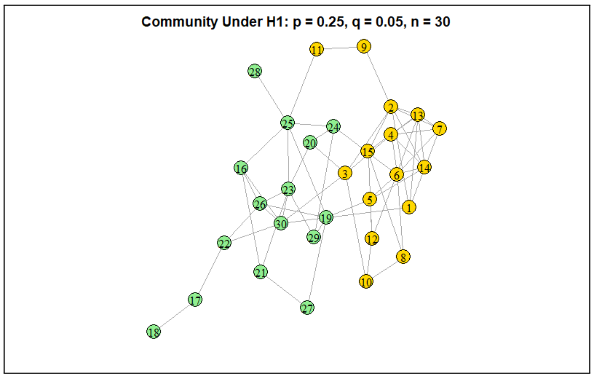


Figure 3.8. Plots of power under alternative hypothesis $p = 0.20$, $q = 0.10$, and $\epsilon_a = 0.35$.

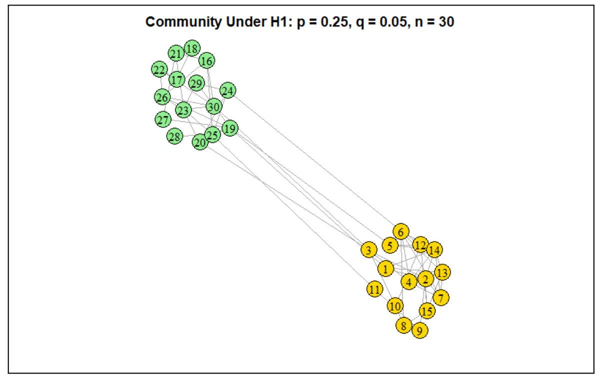
The same patterns can again be recognized. An upwards shift is again noted as ϵ_a increased from 0.2 to 0.35. The powers for $n = 120$ and $n = 150$ are very similar at $\epsilon_d = 0$. However, ϵ_d has a larger impact on the smaller sample size as it increases. At a sample size of 90, the initial power was 0.782, but as ϵ_d increased to 0.35, the power decreased to 0.608. The patterns observed in the medium difference of p and q were similar to that of the small difference.

3.2.3. Large difference: $p = 0.25$ and $q = 0.05$

For the final set when $p_0 = 0.15$, we wish to observe the power of the eigenvalue test when the probabilities of sharing an edge between groups and within groups are vastly different. Since the two values are distinct, we expect to only need a small sample size to achieve a high power. For these simulations, sample sizes of 30, 40, and 50 were used. With probabilities p and q being far apart, even at a small sample size $n = 30$, a power of 0.484 is achieved. The power quickly converges, reaching 0.994 at a sample size of 50. We can observe the network plot for this simulation to see the disparity between the two probabilities:



(a) Random network plot under alternative hypothesis.



(b) Clustered network plot under alternative hypothesis.

Figure 3.9. Network plots under alternative hypothesis $p = 0.25$ and $q = 0.05$.

On the left, even with an unorganized graph, we can see a pretty clear clustering of the two groups. On the right, the clustered graph shows us the minimal number of edges between the two groups, as most of the edges exist within each group. Next, we will analyze the power of the test at these parameters while setting ϵ_a equal to zero. The plot of power for each sample size is displayed as the value of ϵ_d changes:

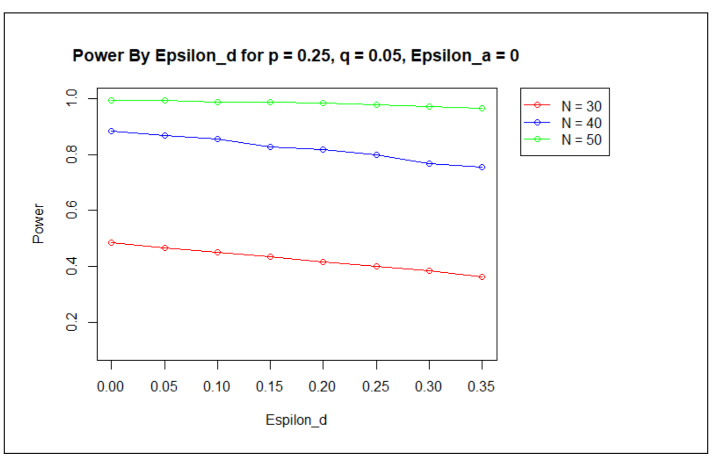


Figure 3.10. Plots of power under alternative hypothesis $p = 0.25$, $q = 0.05$, and $\epsilon_a = 0$.

The similarities of the previous plots are still present, as the largest sample size $n = 50$ keeps a high power even with a large value of ϵ_d. At a sample size of 40, the power ranged from 0.882 when $\epsilon_d = 0$ to 0.754 when $\epsilon_d = 0.35$. Next, the value of ϵ_a will be held constant at 0.2 and the power at each sample size will be plotted. The following plots are obtained for each sample size:

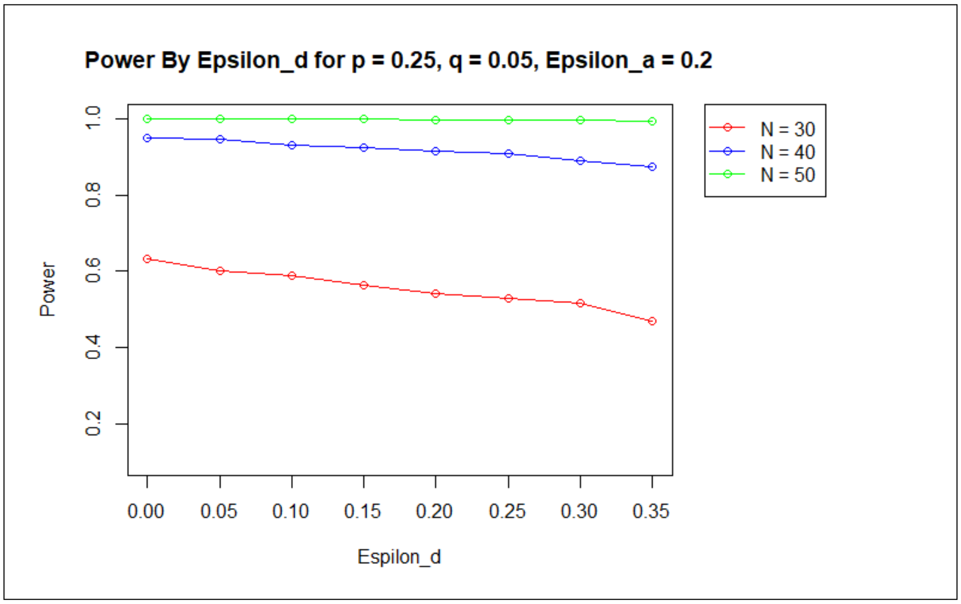


Figure 3.11. Plots of power under alternative hypothesis $p = 0.25$, $q = 0.05$, and $\epsilon_a = 0.20$.

The vertical shift upwards in power is once again present as ϵ_a is held constant at a higher value. At a sample size of 50, the power is maintained close to 1. At a sample size of 40, due to the overestimation of power, the decrease in power is minimized as ϵ_d increases. Finally, we hold ϵ_a constant at 0.35 and observe the change in power:

The power at $n = 40$ and $n = 50$ are very similar at $\epsilon_d = 0$, but the smaller sample size is affected more as the probability of deleting an edge increases. At a sample size of 30, the power is overestimated at 0.716 due to a large ϵ_a value. As ϵ_d increases, the power decreases to 0.578. The increasing error probability ϵ_d appears to have the largest impact on a test with power between 0.40 and 0.90.

3.3. Patterns for $p_0 = 0.25$

The patterns observed when $p_0 = 0.15$ seemed to carry on for $p_0 = 0.25$. These plots can be viewed in the appendix. The most notable observation is the small difference between p and q.

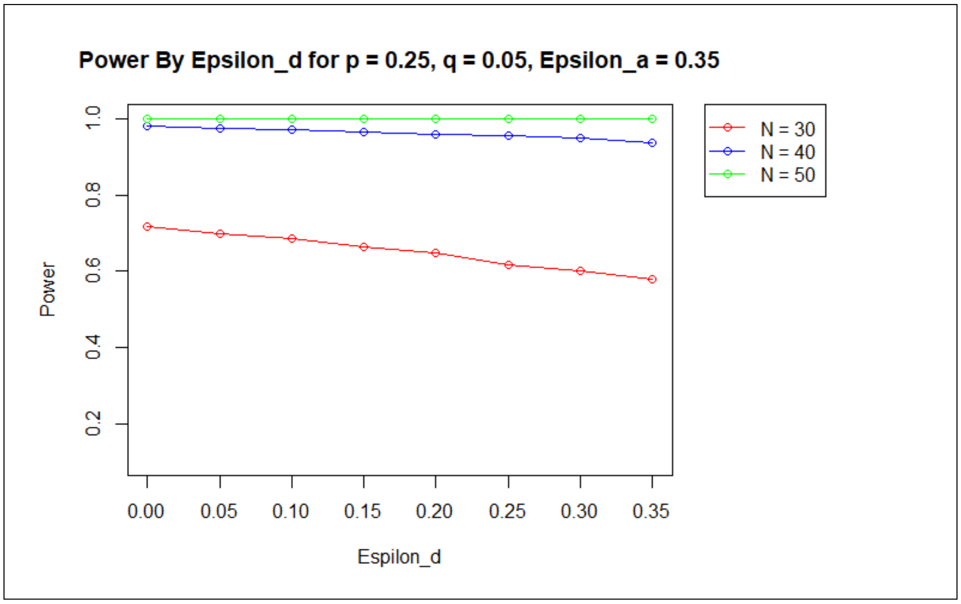


Figure 3.12. Plots of power under alternative hypothesis $p = 0.25$, $q = 0.05$, and $\epsilon_a = 0.35$.

In this case, $p = 0.27$ and $q = 0.23$. Even at a sample size of 720, a power of 0.832 is achieved when ϵ_a and ϵ_d are both zero. The decrease in power as ϵ_d increases appears to be more extreme in this case, as the power shrinks to 0.252 at $\epsilon_d = 0.35$. In general, the decrease in power due to ϵ_d seemed to be larger in the $p_0 = 0.25$ case. As the probabilities are higher for both p and q in the $p_0 = 0.25$ case, this means ϵ_a and ϵ_d will cause the addition and deletion of a higher number of edges, respectively.

3.4. Patterns for constant ϵ_d

Patterns for constant ϵ_d were also observed and can be viewed in the appendix. These plots also follow similar patterns. Rather than an upwards shift when ϵ_d is increased, we see a downwards shift in power. This was highlighted as the increase in ϵ_d caused a decrease in power. The presence of an increasing ϵ_a resulted in the overestimation of power, as we see the plots increasing as ϵ_a increased. Tests with power between 0.40 and 0.90 tended to be affected the most by ϵ_a, as this is when the increase seemed most apparent.

3.5. Behaviors for $\epsilon_a = \epsilon_d$

To help have a better idea of how the error probabilities interact with one another, we can observe the power of the tests when $\epsilon_a = \epsilon_d$ for all possible values. With these plots, we can observe the patterns for power and determine which error probability is having more influence on the power.

A total of six plots were created, one for each of the six combinations of p and q. For $p_0 = 0.15$, the following three plots are displayed:

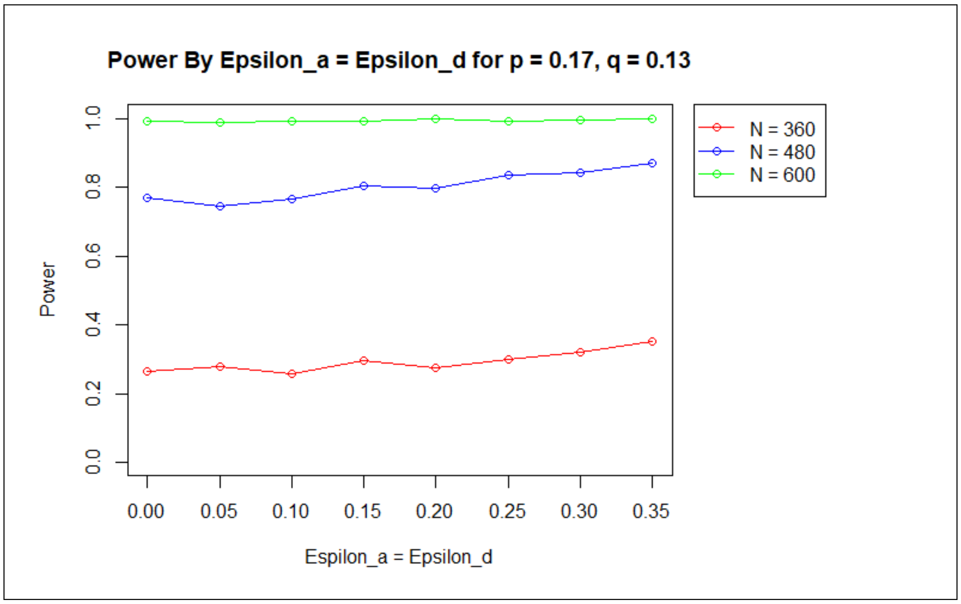


Figure 3.13. Plots of power under alternative hypothesis $p = 0.17$, $q = 0.13$, and $\epsilon_a = \epsilon_d$.

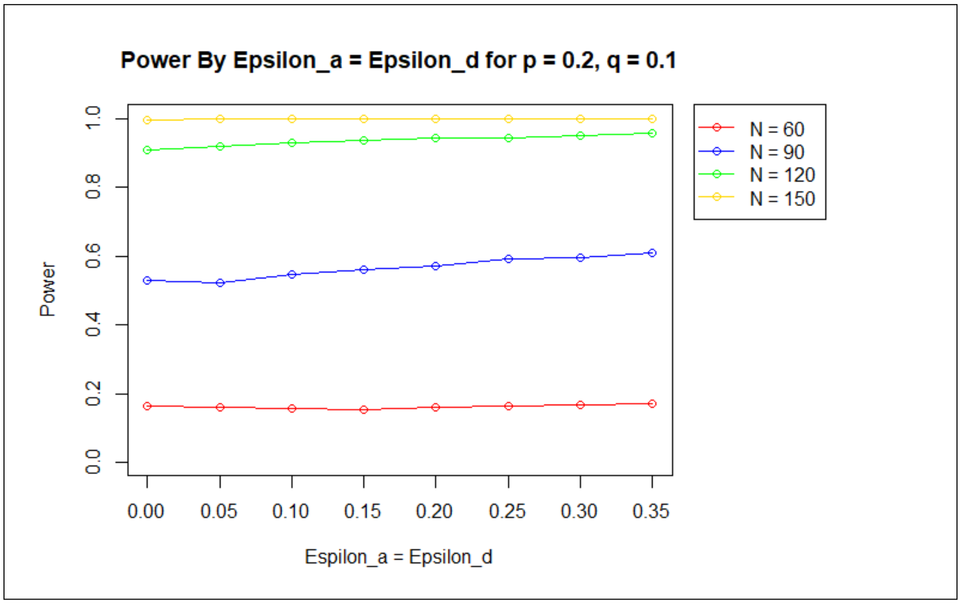


Figure 3.14. Plots of power under alternative hypothesis $p = 0.20$, $q = 0.10$, and $\epsilon_a = \epsilon_d$.

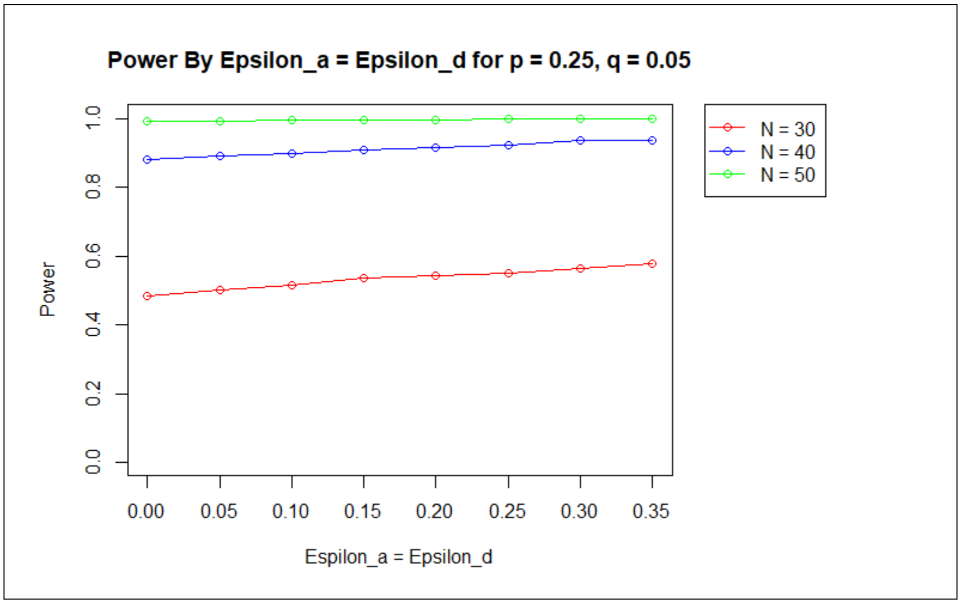


Figure 3.15. Plots of power under alternative hypothesis $p = 0.25$, $q = 0.05$, and $\epsilon_a = \epsilon_d$.

We can see that an increase in both of the error probabilities results in an increase in power overall. Earlier, it was stated that ϵ_a was correlated to an increase in power. We can make an argument that ϵ_a has more influence on power than ϵ_d in this case. Once again, tests with a somewhat weak to a somewhat strong power seem to be influenced the most by the error probabilities. We will also look at the other three combinations of p and q for when $p_0 = 0.25$. The plots for the following three combinations are shown:

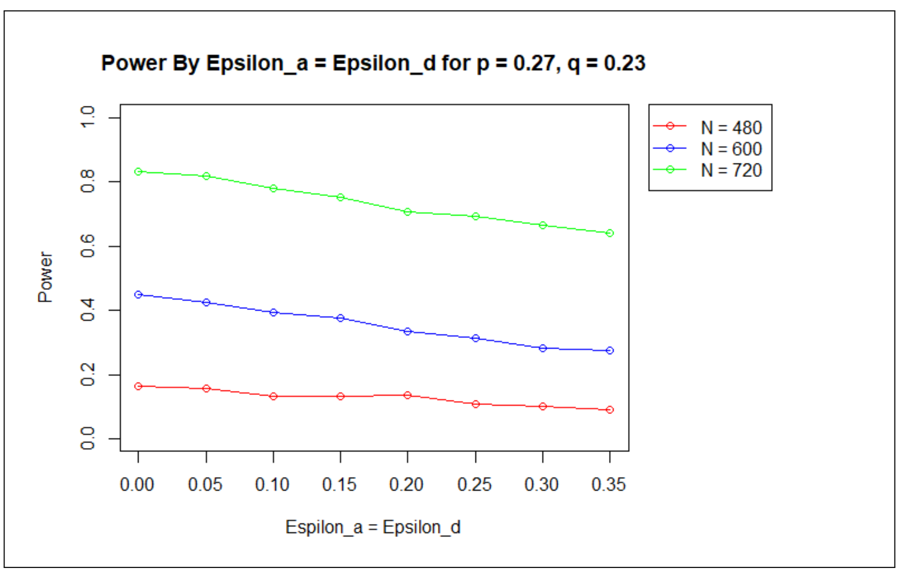


Figure 3.16. Plots of power under alternative hypothesis $p = 0.27$, $q = 0.23$, and $\epsilon_a = \epsilon_d$.

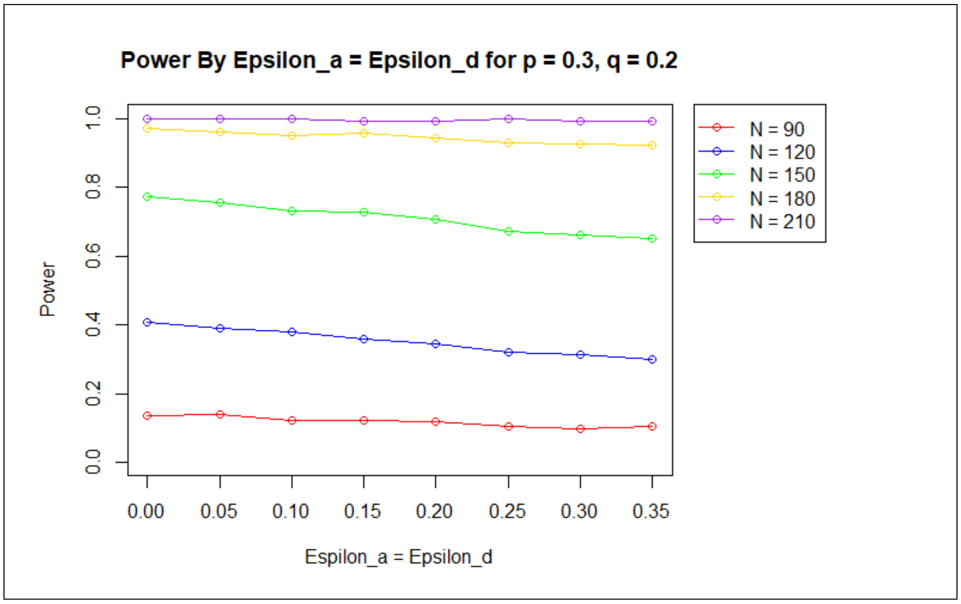


Figure 3.17. Plots of power under alternative hypothesis $p = 0.30$, $q = 0.20$, and $\epsilon_a = \epsilon_d$.

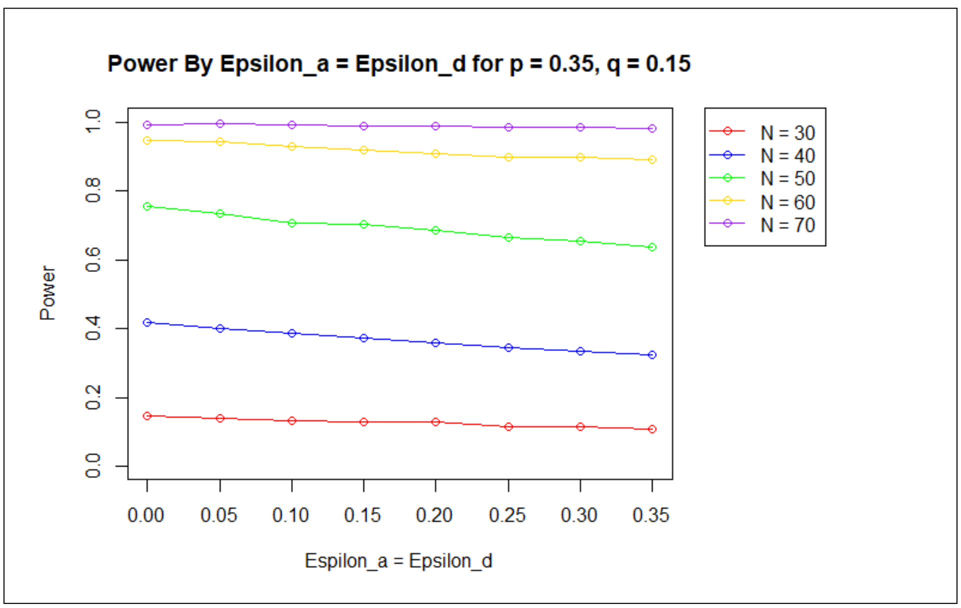


Figure 3.18. Plots of power under alternative hypothesis $p = 0.35$, $q = 0.15$, and $\epsilon_a = \epsilon_d$.

For the $p_0 = 0.25$ case, a decrease in power as ϵ_a and ϵ_d increase was observed. This is in contrast to the $p_0 = 0.15$ case. In this case, we can argue that ϵ_d has more influence on power. The decrease in power is most apparent when the baseline power ($\epsilon_a = \epsilon_d = 0$) is around the 0.40 to 0.90 range.

4. CONCLUSION

These simulations of the largest eigenvalue test allowed us to see how powerful the test can be, given the appropriate sample size. The results showed that even in the presence of error, a large sample size can stabilize the power. The most intriguing discovery was the change of influence with the error probabilities between $p_0 = 0.15$ and $p_0 = 0.25$, with ϵ_a having more influence on power for $p_0 = 0.15$, and ϵ_d having more influence on power when $p_0 = 0.25$. The probability of adding an edge ϵ_a, meaning edges that should not be present are added to the network, causes an inflation of the power. On the other hand, ϵ_d, the probability of deleting an edge, caused a decrease in power. When deleting an edge, it means we are not accounting for an edge that should be there. This seems more plausible in practice, as it is more likely that connections are glossed over and not recorded. The eigenvalue test performs very well in detecting community structure, given appropriate p and q probabilities as well as large sample sizes. We also observed that when the power of a test is initially in the range of 0.40 to 0.90 when the dependent error probability is 0, the power is affected more by the error. This region of "fragility" further supplements the need of a large sample size to attain the appropriate level of power.

REFERENCES

[Abbe, 2017] Abbe, E. (2017). Community detection and stochastic block models: recent developments. *The Journal of Machine Learning Research*, 18(1):6446–6531.

[Bickel and Sarkar, 2016] Bickel, P. J. and Sarkar, P. (2016). Hypothesis testing for automated community detection in networks. *Journal of the Royal Statistical Society: Series B: Statistical Methodology*, pages 253–273.

[Fortunato, 2010] Fortunato, S. (2010). Community detection in graphs. *Physics Reports*, 486(3-5):75–174.

[Gao and Lafferty, 2017] Gao, C. and Lafferty, J. (2017). Testing for global network structure using small subgraph statistics. *arXiv preprint arXiv:1710.00862*.

[Jin et al., 2018] Jin, J., Ke, Z., and Luo, S. (2018). Network global testing by counting graphlets. In *International Conference on Machine Learning*, pages 2333–2341. PMLR.

[Lee and Yin, 2014] Lee, J. O. and Yin, J. (2014). A necessary and sufficient condition for edge universality of wigner matrices. *Duke Mathematical Journal*, 163:117–173.

[Mossel et al., 2018] Mossel, E., Neeman, J., and Sly, A. (2018). A proof of the block model threshold conjecture. *Combinatorica*, 38:665–708.

[Newman, 2006] Newman, M. E. J. (2006). Modularity and community structure in networks. *Proceedings of the National Academy of Sciences*, 103(23):8577–8582.

[Newman, 2018] Newman, M. E. J. (2018). Estimating network structure from unreliable measurements. *Physical Review E*, 98(6).

[Tracy and Widom, 1994] Tracy, C. A. and Widom, H. (1994). Level-spacing distributions and the airy kernel. *Communications in Mathematical Physics*, 159(1):151–174.

[Wang et al., 2012] Wang, D. J., Shi, X., McFarland, D. A., and Leskovec, J. (2012). Measurement error in network data: A re-classification. *Soc. Networks*, 34:396–409.

[Yuan and Nan, 2020] Yuan, M. and Nan, Y. (2020). Test dense subgraphs in sparse uniform hypergraph. *Communications in Statistics - Theory and Methods*, 0(0):1–20.

[Yuan and Wen, 2021] Yuan, M. and Wen, Q. (2021). A practical test for a planted community in heterogeneous networks. *arXiv preprint arXiv:2101.05928*.

APPENDIX

Table A.1. These tables display power for when "p" and "q" are very close. Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.17, 0.13, 0, \epsilon_d)$.

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.17; 0.13; 0; 0)$	$(0.17; 0.13; 0; 0.05)$	$(0.17; 0.13; 0, 0.1)$	$(0.17; 0.13; 0, 0.15)$
360		0.266	0.244	0.228	0.214
480	$\lambda_1(\tilde{A})$	0.768	0.704	0.688	0.616
600		0.994	0.986	0.98	0.978
n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.17; 0.13; 0; 0.2)$	$(0.17; 0.13; 0; 0.25)$	$(0.17; 0.13; 0, 0.3)$	$(0.17; 0.13; 0, 0.35)$
360		0.204	0.178	0.154	0.146
480	$\lambda_1(\tilde{A})$	0.528	0.526	0.476	0.42
600		0.948	0.93	0.902	0.834

Table A.2. These tables display power for when "p" and "q" are very close. Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.17, 0.13, 0.05, \epsilon_d)$.

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.17; 0.13; 0.05; 0)$	$(0.17; 0.13; 0.05; 0.05)$	$(0.17; 0.13; 0.05, 0.1)$	$(0.17; 0.13; 0.05, 0.15)$
360		0.34	0.278	0.258	0.25
480	$\lambda_1(\tilde{A})$	0.796	0.744	0.714	0.654
600		0.996	0.99	0.99	0.976
n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.17; 0.13; 0.05; 0.2)$	$(0.17; 0.13; 0.05; 0.25)$	$(0.17; 0.13; 0.05, 0.3)$	$(0.17; 0.13; 0.05, 0.35)$
360		0.22	0.202	0.194	0.182
480	$\lambda_1(\tilde{A})$	0.618	0.588	0.516	0.482
600		0.982	0.968	0.928	0.90

Table A.3. These tables display power for when "p" and "q" are very close. Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.17, 0.13, 0.1, \epsilon_d)$.

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.17; 0.13; 0.1; 0)$	$(0.17; 0.13; 0.1; 0.05)$	$(0.17; 0.13; 0.1, 0.1)$	$(0.17; 0.13; 0.1, 0.15)$
360		0.328	0.312	0.258	0.256
480	$\lambda_1(\tilde{A})$	0.83	0.802	0.766	0.75
600		1	0.992	0.992	0.982

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.17; 0.13; 0.1; 0.2)$	$(0.17; 0.13; 0.1; 0.25)$	$(0.17; 0.13; 0.1, 0.3)$	$(0.17; 0.13; 0.1, 0.35)$
360		0.252	0.23	0.216	0.174
480	$\lambda_1(\tilde{A})$	0.698	0.646	0.608	0.558
600		0.98	0.97	0.974	0.932

Table A.4. These tables display power for when "p" and "q" are very close. Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.17, 0.13, 0.15, \epsilon_d)$.

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.17; 0.13; 0.15; 0)$	$(0.17; 0.13; 0.15; 0.05)$	$(0.17; 0.13; 0.15, 0.1)$	$(0.17; 0.13; 0.15, 0.15)$
360		0.362	0.33	0.314	0.296
480	$\lambda_1(\tilde{A})$	0.874	0.849	0.808	0.806
600		0.998	0.998	0.996	0.994

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.17; 0.13; 0.15; 0.2)$	$(0.17; 0.13; 0.15; 0.25)$	$(0.17; 0.13; 0.15, 0.3)$	$(0.17; 0.13; 0.15, 0.35)$
360		0.258	0.254	0.224	0.204
480	$\lambda_1(\tilde{A})$	0.766	0.708	0.662	0.616
600		0.986	0.982	0.972	0.97

Table A.5. These tables display power for when "p" and "q" are very close. Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.17, 0.13, 0.2, \epsilon_d)$.

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.17; 0.13; 0.2; 0)$	$(0.17; 0.13; 0.2; 0.05)$	$(0.17; 0.13; 0.2, 0.1)$	$(0.17; 0.13; 0.2, 0.15)$
360		0.384	0.348	0.326	0.308
480	$\lambda_1(\tilde{A})$	0.916	0.88	0.864	0.83
600		1	1	0.998	1

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.17; 0.13; 0.2; 0.2)$	$(0.17; 0.13; 0.2; 0.25)$	$(0.17; 0.13; 0.2, 0.3)$	$(0.17; 0.13; 0.2, 0.35)$
360		0.274	0.25	0.236	0.228
480	$\lambda_1(\tilde{A})$	0.796	0.768	0.742	0.688
600		0.998	0.996	0.99	0.982

Table A.6. These tables display power for when "p" and "q" are very close. Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.17, 0.13, 0.25, \epsilon_d)$.

n	Method	$(p, q, \epsilon_a, \epsilon_d)$ = $(0.17; 0.13; 0.25; 0)$	$(0.17; 0.13; 0.25; 0.05)$	$(0.17; 0.13; 0.25, 0.1)$	$(0.17; 0.13; 0.25, 0.15)$
360		0.448	0.408	0.402	0.362
480	$\lambda_1(\tilde{A})$	0.94	0.918	0.892	0.89
600		1	1	0.998	0.998
n	Method	$(p, q, \epsilon_a, \epsilon_d)$ = $(0.17; 0.13; 0.25; 0.2)$	$(0.17; 0.13; 0.25; 0.25)$	$(0.17; 0.13; 0.25, 0.3)$	$(0.17; 0.13; 0.25, 0.35)$
360		0.336	0.30	0.286	0.254
480	$\lambda_1(\tilde{A})$	0.846	0.834	0.808	0.742
600		0.998	0.994	0.992	0.986

Table A.7. These tables display power for when "p" and "q" are very close. Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.17, 0.13, 0.3, \epsilon_d)$.

n	Method	$(p, q, \epsilon_a, \epsilon_d)$ = $(0.17; 0.13; 0.3; 0)$	$(0.17; 0.13; 0.3; 0.05)$	$(0.17; 0.13; 0.3, 0.1)$	$(0.17; 0.13; 0.3, 0.15)$
360		0.508	0.47	0.442	0.42
480	$\lambda_1(\tilde{A})$	0.96	0.952	0.932	0.922
600		1	1	1	1
n	Method	$(p, q, \epsilon_a, \epsilon_d)$ = $(0.17; 0.13; 0.3; 0.2)$	$(0.17; 0.13; 0.3; 0.25)$	$(0.17; 0.13; 0.3, 0.3)$	$(0.17; 0.13; 0.3, 0.35)$
360		0.362	0.34	0.322	0.294
480	$\lambda_1(\tilde{A})$	0.902	0.878	0.844	0.78
600		1	0.998	0.996	0.992

Table A.8. These tables display power for when "p" and "q" are very close. Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.17, 0.13, 0.35, \epsilon_d)$.

n	Method	$(p, q, \epsilon_a, \epsilon_d)$ = $(0.17; 0.13; 0.35; 0)$	$(0.17; 0.13; 0.35; 0.05)$	$(0.17; 0.13; 0.35, 0.1)$	$(0.17; 0.13; 0.35, 0.15)$
360		0.558	0.494	0.48	0.446
480	$\lambda_1(\tilde{A})$	0.988	0.986	0.96	0.956
600		1	1	1	1
n	Method	$(p, q, \epsilon_a, \epsilon_d)$ = $(0.17; 0.13; 0.35; 0.2)$	$(0.17; 0.13; 0.35; 0.25)$	$(0.17; 0.13; 0.35, 0.3)$	$(0.17; 0.13; 0.35, 0.35)$
360		0.412	0.382	0.372	0.352
480	$\lambda_1(\tilde{A})$	0.922	0.908	0.902	0.872
600		1	1	0.998	0.998

Table A.9. These tables display power for when "p" and "q" are moderately distanced. Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.20, 0.10, 0, \epsilon_d)$.

n	Method	$(p, q, \epsilon_a, \epsilon_d)$ = $(0.2; 0.1; 0; 0)$	$(0.2; 0.1; 0, 0.05)$	$(0.2; 0.1; 0, 0.1)$	$(0.2; 0.1; 0, 0.15)$
60		0.162	0.16	0.148	0.136
90		0.53	0.51	0.472	0.46
120	$\lambda_1(\tilde{A})$	0.91	0.906	0.878	0.86
150		0.996	0.996	0.994	0.994

n	Method	$(p, q, \epsilon_a, \epsilon_d)$ = $(0.2; 0.1; 0; 0.2)$	$(0.2; 0.1; 0, 0.25)$	$(0.2; 0.1; 0, 0.3)$	$(0.2; 0.1; 0, 0.35)$
60		0.136	0.126	0.124	0.124
90		0.422	0.414	0.386	0.384
120	$\lambda_1(\tilde{A})$	0.828	0.81	0.79	0.76
150		0.99	0.988	0.982	0.972

Table A.10. These tables display power for when "p" and "q" are moderately distanced. Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.20, 0.10, 0.05, \epsilon_d)$.

n	Method	$(p, q, \epsilon_a, \epsilon_d)$ = $(0.2; 0.1; 0.05; 0)$	$(0.2; 0.1; 0.05, 0.05)$	$(0.2; 0.1; 0.05, 0.1)$	$(0.2; 0.1; 0.05, 0.15)$
60		0.162	0.16	0.156	0.144
90		0.554	0.524	0.506	0.48
120	$\lambda_1(\tilde{A})$	0.938	0.918	0.908	0.884
150		1	1	1	0.996

n	Method	$(p, q, \epsilon_a, \epsilon_d)$ = $(0.2; 0.1; 0.05; 0.2)$	$(0.2; 0.1; 0.05, 0.25)$	$(0.2; 0.1; 0.05, 0.3)$	$(0.2; 0.1; 0.05, 0.35)$
60		0.138	0.138	0.116	0.116
90		0.472	0.452	0.418	0.384
120	$\lambda_1(\tilde{A})$	0.866	0.854	0.816	0.786
150		0.994	0.988	0.988	0.98

Table A.11. These tables display power for when "p" and "q" are moderately distanced. Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.20, 0.10, 0.1, \epsilon_d)$.

n	Method	$(p, q, \epsilon_a, \epsilon_d)$ = $(0.2; 0.1; 0.1; 0)$	$(0.2; 0.1; 0.1, 0.05)$	$(0.2; 0.1; 0.1, 0.1)$	$(0.2; 0.1; 0.1, 0.15)$
60		0.174	0.168	0.158	0.15
90		0.608	0.572	0.548	0.522
120	$\lambda_1(\tilde{A})$	0.956	0.942	0.928	0.918
150		1	1	1	0.998

n	Method	$(p, q, \epsilon_a, \epsilon_d)$ = $(0.2; 0.1; 0.1; 0.2)$	$(0.2; 0.1; 0.1, 0.25)$	$(0.2; 0.1; 0.1, 0.3)$	$(0.2; 0.1; 0.1, 0.35)$
60		0.146	0.142	0.128	0.124
90		0.488	0.464	0.44	0.43
120	$\lambda_1(\tilde{A})$	0.902	0.886	0.866	0.836
150		0.996	0.996	0.994	0.994

Table A.12. These tables display power for when "p" and "q" are moderately distanced. Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.20, 0.10, 0.15, \epsilon_d)$.

n	Method	$(p, q, \epsilon_a, \epsilon_d)$ = $(0.2; 0.1; 0.15; 0)$	$(0.2; 0.1; 0.15, 0.05)$	$(0.2; 0.1; 0.15, 0.1)$	$(0.2; 0.1; 0.15, 0.15)$
60		0.186	0.178	0.17	0.154
90		0.648	0.614	0.60	0.56
120	$\lambda_1(\tilde{A})$	0.972	0.96	0.94	0.936
150		1	1	1	1
n	Method	$(p, q, \epsilon_a, \epsilon_d)$ = $(0.2; 0.1; 0.15; 0.2)$	$(0.2; 0.1; 0.15, 0.25)$	$(0.2; 0.1; 0.15, 0.3)$	$(0.2; 0.1; 0.15, 0.35)$
60		0.15	0.142	0.136	0.136
90		0.546	0.494	0.488	0.45
120	$\lambda_1(\tilde{A})$	0.926	0.908	0.882	0.856
150		1	0.998	0.996	0.996

Table A.13. These tables display power for when "p" and "q" are moderately distanced. Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.20, 0.10, 0.2, \epsilon_d)$.

n	Method	$(p, q, \epsilon_a, \epsilon_d)$ = $(0.2; 0.1; 0.2; 0)$	$(0.2; 0.1; 0.2, 0.05)$	$(0.2; 0.1; 0.2, 0.1)$	$(0.2; 0.1; 0.2, 0.15)$
60		0.196	0.182	0.178	0.166
90		0.674	0.656	0.618	0.61
120	$\lambda_1(\tilde{A})$	0.974	0.972	0.962	0.956
150		1	1	1	1
n	Method	$(p, q, \epsilon_a, \epsilon_d)$ = $(0.2; 0.1; 0.2; 0.2)$	$(0.2; 0.1; 0.2, 0.25)$	$(0.2; 0.1; 0.2, 0.3)$	$(0.2; 0.1; 0.2, 0.35)$
60		0.16	0.154	0.148	0.142
90		0.572	0.534	0.498	0.486
120	$\lambda_1(\tilde{A})$	0.944	0.934	0.914	0.90
150		1	1	0.998	0.996

Table A.14. These tables display power for when "p" and "q" are moderately distanced. Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.20, 0.10, 0.25, \epsilon_d)$.

n	Method	$(p, q, \epsilon_a, \epsilon_d)$ = $(0.2; 0.1; 0.25; 0)$	$(0.2; 0.1; 0.25, 0.05)$	$(0.2; 0.1; 0.25, 0.1)$	$(0.2; 0.1; 0.25, 0.15)$
60		0.202	0.20	0.19	0.17
90		0.704	0.686	0.658	0.636
120	$\lambda_1(\tilde{A})$	0.982	0.978	0.968	0.966
150		1	1	1	1
n	Method	$(p, q, \epsilon_a, \epsilon_d)$ = $(0.2; 0.1; 0.25; 0.2)$	$(0.2; 0.1; 0.25, 0.25)$	$(0.2; 0.1; 0.25, 0.3)$	$(0.2; 0.1; 0.25, 0.35)$
60		0.168	0.164	0.158	0.15
90		0.622	0.592	0.566	0.54
120	$\lambda_1(\tilde{A})$	0.96	0.944	0.938	0.922
150		1	1	1	0.998

Table A.15. These tables display power for when "p" and "q" are moderately distanced. Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.20, 0.10, 0.3, \epsilon_d)$.

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.2; 0.1; 0.3; 0)$	$(0.2; 0.1; 0.3, 0.05)$	$(0.2; 0.1; 0.3, 0.1)$	$(0.2; 0.1; 0.3, 0.15)$
60		0.218	0.208	0.206	0.188
90		0.738	0.722	0.70	0.67
120	$\lambda_1(\tilde{A})$	0.988	0.984	0.982	0.98
150		1	1	1	1
n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.2; 0.1; 0.3; 0.2)$	$(0.2; 0.1; 0.3, 0.25)$	$(0.2; 0.1; 0.3, 0.3)$	$(0.2; 0.1; 0.3, 0.35)$
60		0.186	0.174	0.166	0.156
90		0.642	0.634	0.596	0.594
120	$\lambda_1(\tilde{A})$	0.966	0.96	0.952	0.944
150		1	1	1	1

Table A.16. These tables display power for when "p" and "q" are moderately distanced. Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.20, 0.10, 0.35, \epsilon_d)$.

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.2; 0.1; 0.35; 0)$	$(0.2; 0.1; 0.35, 0.05)$	$(0.2; 0.1; 0.35, 0.1)$	$(0.2; 0.1; 0.35, 0.15)$
60		0.244	0.228	0.214	0.20
90		0.782	0.762	0.728	0.716
120	$\lambda_1(\tilde{A})$	0.992	0.99	0.986	0.982
150		1	1	1	1
n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.2; 0.1; 0.35; 0.2)$	$(0.2; 0.1; 0.35, 0.25)$	$(0.2; 0.1; 0.35, 0.3)$	$(0.2; 0.1; 0.35, 0.35)$
60		0.192	0.19	0.172	0.172
90		0.70	0.656	0.634	0.608
120	$\lambda_1(\tilde{A})$	0.976	0.972	0.966	0.956
150		1	1	1	1

Table A.17. These tables display power for when there is a large difference between "p" and "q". Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.25, 0.05, 0, \epsilon_d)$,

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.25; 0.05; 0; 0)$	$(0.25; 0.05; 0; 0.05)$	$(0.25; 0.05; 0, 0.1)$	$(0.25; 0.05; 0, 0.15)$
30		0.484	0.466	0.45	0.434
40	$\lambda_1(\tilde{A})$	0.882	0.868	0.854	0.828
50		0.994	0.994	0.988	0.986
n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.25; 0.05; 0; 0.2)$	$(0.25; 0.05; 0; 0.25)$	$(0.25; 0.05; 0, 0.3)$	$(0.25; 0.05; 0, 0.35)$
30		0.416	0.402	0.384	0.364
40	$\lambda_1(\tilde{A})$	0.818	0.80	0.768	0.754
50		0.984	0.976	0.972	0.964

Table A.18. These tables display power for when there is a large difference between "p" and "q". Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.25, 0.05, 0.05, \epsilon_d)$,

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.25; 0.05; 0.05; 0)$	$(0.25; 0.05; 0.05; 0.05)$	$(0.25; 0.05; 0.05, 0.1)$	$(0.25; 0.05; 0.05, 0.15)$
30		0.526	0.502	0.484	0.466
40	$\lambda_1(\tilde{A})$	0.90	0.892	0.876	0.866
50		0.996	0.994	0.994	0.99
n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.25; 0.05; 0.05; 0.2)$	$(0.25; 0.05; 0.05; 0.25)$	$(0.25; 0.05; 0.05, 0.3)$	$(0.25; 0.05; 0.05, 0.35)$
30		0.448	0.426	0.416	0.396
40	$\lambda_1(\tilde{A})$	0.836	0.828	0.812	0.784
50		0.99	0.984	0.982	0.974

Table A.19. These tables display power for when there is a large difference between "p" and "q". Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.25, 0.05, 0.1, \epsilon_d)$,

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.25; 0.05; 0.1; 0)$	$(0.25; 0.05; 0.1; 0.05)$	$(0.25; 0.05; 0.1, 0.1)$	$(0.25; 0.05; 0.1, 0.15)$
30		0.556	0.536	0.514	0.498
40	$\lambda_1(\tilde{A})$	0.922	0.91	0.898	0.884
50		0.998	0.998	0.996	0.994
n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.25; 0.05; 0.1; 0.2)$	$(0.25; 0.05; 0.1; 0.25)$	$(0.25; 0.05; 0.1, 0.3)$	$(0.25; 0.05; 0.1, 0.35)$
30		0.478	0.466	0.444	0.42
40	$\lambda_1(\tilde{A})$	0.874	0.854	0.838	0.822
50		0.99	0.988	0.986	0.98

Table A.20. These tables display power for when there is a large difference between "p" and "q". Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.25, 0.05, 0.15, \epsilon_d)$,

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.25; 0.05; 0.15; 0)$	$(0.25; 0.05; 0.15; 0.05)$	$(0.25; 0.05; 0.15, 0.1)$	$(0.25; 0.05; 0.15, 0.15)$
30		0.59	0.572	0.546	0.536
40	$\lambda_1(\tilde{A})$	0.94	0.928	0.918	0.91
50		0.998	0.998	0.996	0.996
n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.25; 0.05; 0.15; 0.2)$	$(0.25; 0.05; 0.15; 0.25)$	$(0.25; 0.05; 0.15, 0.3)$	$(0.25; 0.05; 0.15, 0.35)$
30		0.504	0.488	0.468	0.45
40	$\lambda_1(\tilde{A})$	0.898	0.884	0.876	0.848
50		0.994	0.99	0.992	0.986

Table A.21. These tables display power for when there is a large difference between "p" and "q". Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.25, 0.05, 0.2, \epsilon_d)$,

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.25; 0.05; 0.2; 0)$	$(0.25; 0.05; 0.2; 0.05)$	$(0.25; 0.05; 0.2, 0.1)$	$(0.25; 0.05; 0.2, 0.15)$
30		0.632	0.602	0.588	0.564
40	$\lambda_1(\tilde{A})$	0.95	0.946	0.93	0.924
50		0.998	0.998	0.998	0.998
n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.25; 0.05; 0.2; 0.2)$	$(0.25; 0.05; 0.2; 0.25)$	$(0.25; 0.05; 0.2, 0.3)$	$(0.25; 0.05; 0.2, 0.35)$
30		0.542	0.528	0.516	0.47
40	$\lambda_1(\tilde{A})$	0.916	0.908	0.89	0.874
50		0.996	0.996	0.996	0.994

Table A.22. These tables display power for when there is a large difference between "p" and "q". Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.25, 0.05, 0.25, \epsilon_d)$,

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.25; 0.05; 0.25; 0)$	$(0.25; 0.05; 0.25; 0.05)$	$(0.25; 0.05; 0.25, 0.1)$	$(0.25; 0.05; 0.25, 0.15)$
30		0.652	0.644	0.616	0.594
40	$\lambda_1(\tilde{A})$	0.966	0.958	0.95	0.942
50		1	1	1	1
n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.25; 0.05; 0.25; 0.2)$	$(0.25; 0.05; 0.25; 0.25)$	$(0.25; 0.05; 0.25, 0.3)$	$(0.25; 0.05; 0.25, 0.35)$
30		0.582	0.55	0.542	0.518
40	$\lambda_1(\tilde{A})$	0.938	0.922	0.92	0.902
50		0.998	0.998	0.996	0.996

Table A.23. These tables display power for when there is a large difference between "p" and "q". Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.25, 0.05, 0.3, \epsilon_d)$,

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.25; 0.05; 0.3; 0)$	$(0.25; 0.05; 0.3; 0.05)$	$(0.25; 0.05; 0.3, 0.1)$	$(0.25; 0.05; 0.3, 0.15)$
30		0.692	0.672	0.652	0.626
40	$\lambda_1(\tilde{A})$	0.972	0.964	0.962	0.954
50		1	1	1	1
n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.25; 0.05; 0.3; 0.2)$	$(0.25; 0.05; 0.3; 0.25)$	$(0.25; 0.05; 0.3, 0.3)$	$(0.25; 0.05; 0.3, 0.35)$
30		0.614	0.588	0.564	0.546
40	$\lambda_1(\tilde{A})$	0.948	0.938	0.936	0.918
50		1	1	0.998	0.998

Table A.24. These tables display power for when there is a large difference between "p" and "q". Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.25, 0.05, 0.35, \epsilon_d)$,

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.25; 0.05; 0.35; 0)$	$(0.25; 0.05; 0.35; 0.05)$	$(0.25; 0.05; 0.35, 0.1)$	$(0.25; 0.05; 0.35, 0.15)$
30		0.716	0.70	0.686	0.664
40	$\lambda_1(\tilde{A})$	0.98	0.974	0.972	0.966
50		1	1	1	1
n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.25; 0.05; 0.35; 0.2)$	$(0.25; 0.05; 0.35; 0.25)$	$(0.25; 0.05; 0.35, 0.3)$	$(0.25; 0.05; 0.35, 0.35)$
30		0.648	0.618	0.60	0.578
40	$\lambda_1(\tilde{A})$	0.96	0.954	0.948	0.938
50		1	1	1	0.998

Table A.25. These tables display power for when "p" and "q" are very close. Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.27, 0.23, 0, \epsilon_d)$.

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.27; 0.23; 0; 0)$	$(0.27; 0.23; 0, 0.05)$	$(0.27; 0.23; 0, 0.1)$	$(0.27; 0.23; 0, 0.15)$
480		0.162	0.132	0.11	0.11
600	$\lambda_1(\tilde{A})$	0.45	0.388	0.322	0.238
720		0.832	0.758	0.69	0.592
n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.27; 0.23; 0; 0.2)$	$(0.27; 0.23; 0, 0.25)$	$(0.27; 0.23; 0, 0.3)$	$(0.27; 0.23; 0, 0.35)$
480		0.064	0.062	0.06	0.06
600	$\lambda_1(\tilde{A})$	0.188	0.148	0.144	0.12
720		0.478	0.394	0.316	0.252

Table A.26. These tables display power for when "p" and "q" are very close. Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.27, 0.23, 0.05, \epsilon_d)$.

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.27; 0.23; 0.05; 0)$	$(0.27; 0.23; 0.05, 0.05)$	$(0.27; 0.23; 0.05, 0.1)$	$(0.27; 0.23; 0.05, 0.15)$
480		0.19	0.158	0.126	0.108
600	$\lambda_1(\tilde{A})$	0.464	0.426	0.364	0.276
720		0.862	0.818	0.706	0.626
n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.27; 0.23; 0.05; 0.2)$	$(0.27; 0.23; 0.05, 0.25)$	$(0.27; 0.23; 0.05, 0.3)$	$(0.27; 0.23; 0.05, 0.35)$
480		0.086	0.072	0.072	0.06
600	$\lambda_1(\tilde{A})$	0.232	0.176	0.152	0.15
720		0.564	0.462	0.372	0.30

Table A.27. These tables display power for when "p" and "q" are very close. Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.27, 0.23, 0.1, \epsilon_d)$.

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.27; 0.23; 0.1; 0)$	$(0.27; 0.23; 0.1, 0.05)$	$(0.27; 0.23; 0.1, 0.1)$	$(0.27; 0.23; 0.1, 0.15)$
480		0.204	0.168	0.132	0.116
600	$\lambda_1(\tilde{A})$	0.542	0.446	0.392	0.322
720		0.898	0.844	0.78	0.698
n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.27; 0.23; 0.1; 0.2)$	$(0.27; 0.23; 0.1, 0.25)$	$(0.27; 0.23; 0.1, 0.3)$	$(0.27; 0.23; 0.1, 0.35)$
480		0.10	0.074	0.068	0.068
600	$\lambda_1(\tilde{A})$	0.252	0.212	0.148	0.136
720		0.648	0.508	0.43	0.344

Table A.28. These tables display power for when "p" and "q" are very close. Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.27, 0.23, 0.15, \epsilon_d)$.

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.27; 0.23; 0.15; 0)$	$(0.27; 0.23; 0.15, 0.05)$	$(0.27; 0.23; 0.15, 0.1)$	$(0.27; 0.23; 0.15, 0.15)$
480		0.228	0.184	0.178	0.132
600	$\lambda_1(\tilde{A})$	0.572	0.48	0.40	0.376
720		0.936	0.89	0.814	0.752
n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.27; 0.23; 0.15; 0.2)$	$(0.27; 0.23; 0.15, 0.25)$	$(0.27; 0.23; 0.15, 0.3)$	$(0.27; 0.23; 0.15, 0.35)$
480		0.104	0.104	0.068	0.054
600	$\lambda_1(\tilde{A})$	0.276	0.25	0.186	0.158
720		0.656	0.574	0.458	0.362

Table A.29. These tables display power for when "p" and "q" are very close. Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.27, 0.23, 0.2, \epsilon_d)$.

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.27; 0.23; 0.2; 0)$	$(0.27; 0.23; 0.2, 0.05)$	$(0.27; 0.23; 0.2, 0.1)$	$(0.27; 0.23; 0.2, 0.15)$
480		0.248	0.196	0.168	0.14
600	$\lambda_1(\tilde{A})$	0.63	0.53	0.466	0.368
720		0.968	0.92	0.86	0.824
n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.27; 0.23; 0.2; 0.2)$	$(0.27; 0.23; 0.2, 0.25)$	$(0.27; 0.23; 0.2, 0.3)$	$(0.27; 0.23; 0.2, 0.35)$
480		0.136	0.122	0.084	0.084
600	$\lambda_1(\tilde{A})$	0.334	0.262	0.222	0.18
720		0.708	0.638	0.52	0.43

Table A.30. These tables display power for when "p" and "q" are very close. Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.27, 0.23, 0.25, \epsilon_d)$.

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.27; 0.23; 0.25; 0)$	$(0.27; 0.23; 0.25, 0.05)$	$(0.27; 0.23; 0.25, 0.1)$	$(0.27; 0.23; 0.25, 0.15)$
480		0.244	0.202	0.184	0.162
600	$\lambda_1(\tilde{A})$	0.70	0.58	0.56	0.478
720		0.972	0.924	0.878	0.844

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.27; 0.23; 0.25; 0.2)$	$(0.27; 0.23; 0.25, 0.25)$	$(0.27; 0.23; 0.25, 0.3)$	$(0.27; 0.23; 0.25, 0.35)$
480		0.138	0.108	0.098	0.082
600	$\lambda_1(\tilde{A})$	0.39	0.312	0.228	0.216
720		0.764	0.694	0.602	0.526

Table A.31. These tables display power for when "p" and "q" are very close. Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.27, 0.23, 0.3, \epsilon_d)$.

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.27; 0.23; 0.3; 0)$	$(0.27; 0.23; 0.3, 0.05)$	$(0.27; 0.23; 0.3, 0.1)$	$(0.27; 0.23; 0.3, 0.15)$
480		0.302	0.266	0.212	0.182
600	$\lambda_1(\tilde{A})$	0.738	0.66	0.58	0.518
720		0.974	0.944	0.936	0.874

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.27; 0.23; 0.3; 0.2)$	$(0.27; 0.23; 0.3, 0.25)$	$(0.27; 0.23; 0.3, 0.3)$	$(0.27; 0.23; 0.3, 0.35)$
480		0.144	0.134	0.102	0.094
600	$\lambda_1(\tilde{A})$	0.396	0.338	0.282	0.246
720		0.766	0.742	0.664	0.566

Table A.32. These tables display power for when "p" and "q" are very close. Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.27, 0.23, 0.35, \epsilon_d)$.

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.27; 0.23; 0.35; 0)$	$(0.27; 0.23; 0.35, 0.05)$	$(0.27; 0.23; 0.35, 0.1)$	$(0.27; 0.23; 0.35, 0.15)$
480		0.356	0.254	0.232	0.212
600	$\lambda_1(\tilde{A})$	0.792	0.72	0.658	0.568
720		0.988	0.966	0.948	0.924

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.27; 0.23; 0.35; 0.2)$	$(0.27; 0.23; 0.35, 0.25)$	$(0.27; 0.23; 0.35, 0.3)$	$(0.27; 0.23; 0.35, 0.35)$
480		0.142	0.14	0.102	0.09
600	$\lambda_1(\tilde{A})$	0.466	0.408	0.344	0.274
720		0.876	0.83	0.714	0.64

Table A.33. These tables display power for when "p" and "q" are moderately distanced. Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.3, 0.2, 0, \epsilon_d)$.

n	Method	$(p, q, \epsilon_a, \epsilon_d)$ = $(0.3; 0.2; 0; 0)$	$(0.3; 0.2; 0, 0.05)$	$(0.3; 0.2; 0, 0.1)$	$(0.3; 0.2; 0, 0.15)$
90		0.136	0.128	0.118	0.098
120		0.406	0.372	0.318	0.286
150	$\lambda_1(\tilde{A})$	0.774	0.742	0.668	0.61
180		0.972	0.954	0.93	0.888
210		0.998	0.996	0.996	0.99

n	Method	$(p, q, \epsilon_a, \epsilon_d)$ = $(0.3; 0.2; 0; 0.2)$	$(0.3; 0.2; 0, 0.25)$	$(0.3; 0.2; 0, 0.3)$	$(0.3; 0.2; 0, 0.35)$
90		0.094	0.076	0.076	0.066
120		0.248	0.216	0.188	0.158
150	$\lambda_1(\tilde{A})$	0.55	0.49	0.406	0.346
180		0.85	0.80	0.724	0.664
210		0.968	0.96	0.936	0.90

Table A.34. These tables display power for when "p" and "q" are moderately distanced. Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.3, 0.2, 0.05, \epsilon_d)$.

n	Method	$(p, q, \epsilon_a, \epsilon_d)$ = $(0.3; 0.2; 0.05; 0)$	$(0.3; 0.2; 0.05, 0.05)$	$(0.3; 0.2; 0.05, 0.1)$	$(0.3; 0.2; 0.05, 0.15)$
90		0.146	0.14	0.118	0.106
120		0.44	0.39	0.356	0.324
150	$\lambda_1(\tilde{A})$	0.816	0.756	0.716	0.652
180		0.976	0.962	0.944	0.914
210		0.998	0.998	0.998	0.994

n	Method	$(p, q, \epsilon_a, \epsilon_d)$ = $(0.3; 0.2; 0.05; 0.2)$	$(0.3; 0.2; 0.05, 0.25)$	$(0.3; 0.2; 0.05, 0.3)$	$(0.3; 0.2; 0.05, 0.35)$
90		0.09	0.086	0.074	0.068
120		0.276	0.232	0.186	0.174
150	$\lambda_1(\tilde{A})$	0.558	0.52	0.44	0.40
180		0.864	0.818	0.758	0.694
210		0.988	0.966	0.958	0.918

Table A.35. These tables display power for when "p" and "q" are moderately distanced. Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.3, 0.2, 0.1, \epsilon_d)$.

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.3; 0.2; 0.1; 0)$	$(0.3; 0.2; 0.1, 0.05)$	$(0.3; 0.2; 0.1, 0.1)$	$(0.3; 0.2; 0.1, 0.15)$
90		0.158	0.142	0.122	0.114
120		0.486	0.424	0.38	0.33
150	$\lambda_1(\tilde{A})$	0.826	0.794	0.73	0.678
180		0.982	0.97	0.95	0.936
210		1	0.998	0.998	0.992
n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.3; 0.2; 0.1; 0.2)$	$(0.3; 0.2; 0.1, 0.25)$	$(0.3; 0.2; 0.1, 0.3)$	$(0.3; 0.2; 0.1, 0.35)$
90		0.102	0.09	0.078	0.072
120		0.276	0.258	0.224	0.186
150	$\lambda_1(\tilde{A})$	0.626	0.56	0.502	0.436
180		0.892	0.848	0.818	0.742
210		0.99	0.984	0.968	0.946

Table A.36. These tables display power for when "p" and "q" are moderately distanced. Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.3, 0.2, 0.15, \epsilon_d)$.

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.3; 0.2; 0.15; 0)$	$(0.3; 0.2; 0.15, 0.05)$	$(0.3; 0.2; 0.15, 0.1)$	$(0.3; 0.2; 0.15, 0.15)$
90		0.168	0.154	0.144	0.122
120		0.516	0.464	0.418	0.36
150	$\lambda_1(\tilde{A})$	0.856	0.834	0.774	0.728
180		0.99	0.976	0.962	0.956
210		1	1	1	0.994
n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.3; 0.2; 0.15; 0.2)$	$(0.3; 0.2; 0.15, 0.25)$	$(0.3; 0.2; 0.15, 0.3)$	$(0.3; 0.2; 0.15, 0.35)$
90		0.104	0.094	0.08	0.076
120		0.316	0.26	0.24	0.196
150	$\lambda_1(\tilde{A})$	0.656	0.602	0.534	0.472
180		0.924	0.896	0.83	0.772
210		0.996	0.988	0.966	0.96

Table A.37. These tables display power for when "p" and "q" are moderately distanced. Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.3, 0.2, 0.2, \epsilon_d)$.

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.3; 0.2; 0.2; 0)$	$(0.3; 0.2; 0.2, 0.05)$	$(0.3; 0.2; 0.2, 0.1)$	$(0.3; 0.2; 0.2, 0.15)$
90		0.19	0.176	0.156	0.136
120		0.548	0.48	0.434	0.378
150	$\lambda_1(\tilde{A})$	0.89	0.856	0.808	0.74
180		0.992	0.986	0.98	0.968
210		1	1	1	1

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.3; 0.2; 0.2; 0.2)$	$(0.3; 0.2; 0.2, 0.25)$	$(0.3; 0.2; 0.2, 0.3)$	$(0.3; 0.2; 0.2, 0.35)$
90		0.118	0.10	0.088	0.078
120		0.344	0.302	0.262	0.226
150	$\lambda_1(\tilde{A})$	0.706	0.64	0.576	0.504
180		0.944	0.914	0.86	0.818
210		0.992	0.992	0.984	0.968

Table A.38. These tables display power for when "p" and "q" are moderately distanced. Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.3, 0.2, 0.25, \epsilon_d)$.

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.3; 0.2; 0.25; 0)$	$(0.3; 0.2; 0.25, 0.05)$	$(0.3; 0.2; 0.25, 0.1)$	$(0.3; 0.2; 0.25, 0.15)$
90		0.202	0.168	0.156	0.138
120		0.574	0.518	0.468	0.414
150	$\lambda_1(\tilde{A})$	0.91	0.88	0.836	0.786
180		0.994	0.992	0.982	0.978
210		1	1	1	1

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.3; 0.2; 0.25; 0.2)$	$(0.3; 0.2; 0.25, 0.25)$	$(0.3; 0.2; 0.25, 0.3)$	$(0.3; 0.2; 0.25, 0.35)$
90		0.118	0.104	0.092	0.084
120		0.376	0.32	0.274	0.248
150	$\lambda_1(\tilde{A})$	0.75	0.672	0.63	0.556
180		0.954	0.928	0.896	0.842
210		0.998	0.998	0.986	0.978

Table A.39. These tables display power for when "p" and "q" are moderately distanced. Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.3, 0.2, 0.3, \epsilon_d)$.

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.3; 0.2; 0.3; 0)$	$(0.3; 0.2; 0.3, 0.05)$	$(0.3; 0.2; 0.3, 0.1)$	$(0.3; 0.2; 0.3, 0.15)$
90		0.218	0.196	0.172	0.158
120		0.614	0.57	0.508	0.45
150	$\lambda_1(\tilde{A})$	0.934	0.91	0.856	0.812
180		0.994	0.992	0.99	0.98
210		1	1	1	1
n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.3; 0.2; 0.3; 0.2)$	$(0.3; 0.2; 0.3, 0.25)$	$(0.3; 0.2; 0.3, 0.3)$	$(0.3; 0.2; 0.3, 0.35)$
90		0.134	0.112	0.098	0.094
120		0.402	0.348	0.314	0.266
150	$\lambda_1(\tilde{A})$	0.76	0.71	0.66	0.61
180		0.966	0.952	0.926	0.89
210		1	0.998	0.994	0.994

Table A.40. These tables display power for when "p" and "q" are moderately distanced. Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.3, 0.2, 0.35, \epsilon_d)$.

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.3; 0.2; 0.35; 0)$	$(0.3; 0.2; 0.35, 0.05)$	$(0.3; 0.2; 0.35, 0.1)$	$(0.3; 0.2; 0.35, 0.15)$
90		0.232	0.214	0.18	0.16
120		0.624	0.59	0.528	0.50
150	$\lambda_1(\tilde{A})$	0.94	0.93	0.886	0.848
180		1	0.998	0.99	0.988
210		1	1	1	1
n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.3; 0.2; 0.35; 0.2)$	$(0.3; 0.2; 0.35, 0.25)$	$(0.3; 0.2; 0.35, 0.3)$	$(0.3; 0.2; 0.35, 0.35)$
90		0.144	0.124	0.112	0.104
120		0.434	0.396	0.356	0.298
150	$\lambda_1(\tilde{A})$	0.808	0.746	0.694	0.65
180		0.978	0.962	0.954	0.922
210		1	0.998	0.998	0.994

Table A.41. These tables display power for when there is a large difference between "p" and "q". Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.35, 0.15, 0, \epsilon_d)$.

n	Method	$(p, q, \epsilon_a, \epsilon_d)$ = $(0.35; 0.15; 0; 0)$	$(0.35; 0.15; 0, 0.05)$	$(0.35; 0.15; 0, 0.1)$	$(0.35; 0.15; 0, 0.15)$
30		0.148	0.13	0.116	0.106
40		0.418	0.376	0.342	0.304
50	$\lambda_1(\tilde{A})$	0.756	0.702	0.66	0.612
60		0.948	0.926	0.90	0.862
70		0.994	0.992	0.982	0.978
n	Method	$(p, q, \epsilon_a, \epsilon_d)$ = $(0.35; 0.15; 0; 0.2)$	$(0.35; 0.15; 0, 0.25)$	$(0.35; 0.15; 0, 0.3)$	$(0.35; 0.15; 0, 0.35)$
30		0.092	0.084	0.076	0.068
40		0.272	0.244	0.218	0.184
50	$\lambda_1(\tilde{A})$	0.566	0.498	0.46	0.414
60		0.838	0.782	0.734	0.67
70		0.958	0.94	0.912	0.882

Table A.42. These tables display power for when there is a large difference between "p" and "q". Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.35, 0.15, 0.05, \epsilon_d)$.

n	Method	$(p, q, \epsilon_a, \epsilon_d)$ = $(0.35; 0.15; 0.05; 0)$	$(0.35; 0.15; 0.05, 0.05)$	$(0.35; 0.15; 0.05, 0.1)$	$(0.35; 0.15; 0, 0.15)$
30		0.152	0.138	0.122	0.116
40		0.442	0.402	0.362	0.322
50	$\lambda_1(\tilde{A})$	0.766	0.736	0.688	0.648
60		0.96	0.942	0.912	0.89
70		0.996	0.996	0.992	0.984
n	Method	$(p, q, \epsilon_a, \epsilon_d)$ = $(0.35; 0.15; 0.05; 0.2)$	$(0.35; 0.15; 0.05, 0.25)$	$(0.35; 0.15; 0.05, 0.3)$	$(0.35; 0.15; 0.05, 0.35)$
30		0.104	0.088	0.08	0.068
40		0.29	0.256	0.232	0.208
50	$\lambda_1(\tilde{A})$	0.592	0.532	0.494	0.436
60		0.852	0.804	0.766	0.71
70		0.97	0.954	0.934	0.912

Table A.43. These tables display power for when there is a large difference between "p" and "q". Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.35, 0.15, 0.1, \epsilon_d)$.

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.35; 0.15; 0.1; 0)$	$(0.35; 0.15; 0.1, 0.05)$	$(0.35; 0.15; 0.1, 0.1)$	$(0.35; 0.15; 0.1, 0.15)$
30		0.162	0.15	0.132	0.12
40		0.456	0.43	0.386	0.348
50	$\lambda_1(\tilde{A})$	0.798	0.76	0.708	0.666
60		0.966	0.954	0.93	0.908
70		0.998	0.996	0.992	0.99

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.35; 0.15; 0.1; 0.2)$	$(0.35; 0.15; 0.1, 0.25)$	$(0.35; 0.15; 0.1, 0.3)$	$(0.35; 0.15; 0.1, 0.35)$
30		0.10	0.096	0.086	0.074
40		0.31	0.272	0.238	0.22
50	$\lambda_1(\tilde{A})$	0.616	0.57	0.52	0.472
60		0.868	0.836	0.792	0.748
70		0.976	0.966	0.95	0.918

Table A.44. These tables display power for when there is a large difference between "p" and "q". Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.35, 0.15, 0.15, \epsilon_d)$.

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.35; 0.15; 0.15; 0)$	$(0.35; 0.15; 0.15, 0.05)$	$(0.35; 0.15; 0.15, 0.1)$	$(0.35; 0.15; 0.15, 0.15)$
30		0.174	0.158	0.138	0.128
40		0.496	0.444	0.404	0.372
50	$\lambda_1(\tilde{A})$	0.828	0.776	0.738	0.704
60		0.972	0.958	0.946	0.92
70		0.998	0.996	0.994	0.99

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.35; 0.15; 0.15; 0.2)$	$(0.35; 0.15; 0.15, 0.25)$	$(0.35; 0.15; 0.15, 0.3)$	$(0.35; 0.15; 0.15, 0.35)$
30		0.114	0.10	0.092	0.08
40		0.34	0.294	0.266	0.23
50	$\lambda_1(\tilde{A})$	0.65	0.608	0.538	0.484
60		0.89	0.866	0.83	0.77
70		0.984	0.978	0.964	0.938

Table A.45. These tables display power for when there is a large difference between "p" and "q". Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.35, 0.15, 0.2, \epsilon_d)$.

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.35; 0.15; 0.2; 0)$	$(0.35; 0.15; 0.2, 0.05)$	$(0.35; 0.15; 0.2, 0.1)$	$(0.35; 0.15; 0.2, 0.15)$
30		0.184	0.168	0.154	0.138
40		0.512	0.48	0.442	0.394
50	$\lambda_1(\tilde{A})$	0.852	0.816	0.772	0.724
60		0.98	0.97	0.956	0.94
70		1	0.998	0.998	0.996
n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.35; 0.15; 0.2; 0.2)$	$(0.35; 0.15; 0.2, 0.25)$	$(0.35; 0.15; 0.2, 0.3)$	$(0.35; 0.15; 0.2, 0.35)$
30		0.128	0.106	0.096	0.084
40		0.358	0.314	0.284	0.25
50	$\lambda_1(\tilde{A})$	0.686	0.638	0.572	0.532
60		0.908	0.882	0.852	0.81
70		0.99	0.984	0.972	0.954

Table A.46. These tables display power for when there is a large difference between "p" and "q". Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.35, 0.15, 0.25, \epsilon_d)$.

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.35; 0.15; 0.25; 0)$	$(0.35; 0.15; 0.25, 0.05)$	$(0.35; 0.15; 0.25, 0.1)$	$(0.35; 0.15; 0.25, 0.15)$
30		0.202	0.18	0.16	0.146
40		0.544	0.50	0.454	0.416
50	$\lambda_1(\tilde{A})$	0.868	0.83	0.794	0.768
60		0.978	0.972	0.962	0.952
70		1	0.998	0.998	0.994
n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.35; 0.15; 0.25; 0.2)$	$(0.35; 0.15; 0.25, 0.25)$	$(0.35; 0.15; 0.25, 0.3)$	$(0.35; 0.15; 0.25, 0.35)$
30		0.132	0.116	0.102	0.096
40		0.382	0.346	0.31	0.268
50	$\lambda_1(\tilde{A})$	0.716	0.664	0.62	0.568
60		0.928	0.90	0.88	0.836
70		0.992	0.986	0.978	0.964

Table A.47. These tables display power for when there is a large difference between "p" and "q". Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.35, 0.15, 0.3, \epsilon_d)$.

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.35; 0.15; 0.3; 0)$	$(0.35; 0.15; 0.3, 0.05)$	$(0.35; 0.15; 0.3, 0.1)$	$(0.35; 0.15; 0.3, 0.15)$
30		0.22	0.192	0.172	0.158
40		0.57	0.536	0.488	0.452
50	$\lambda_1(\tilde{A})$	0.88	0.848	0.824	0.784
60		0.988	0.984	0.972	0.96
70		1	1	0.998	0.998
n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.35; 0.15; 0.3; 0.2)$	$(0.35; 0.15; 0.3, 0.25)$	$(0.35; 0.15; 0.3, 0.3)$	$(0.35; 0.15; 0.3, 0.35)$
30		0.148	0.12	0.116	0.10
40		0.408	0.37	0.334	0.294
50	$\lambda_1(\tilde{A})$	0.748	0.704	0.654	0.602
60		0.94	0.926	0.898	0.862
70		0.996	0.99	0.986	0.978

Table A.48. These tables display power for when there is a large difference between "p" and "q". Simulated Powers with measurement error: graphs generated from $\mathcal{G}(0.35, 0.15, 0.35, \epsilon_d)$.

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.35; 0.15; 0.35; 0)$	$(0.35; 0.15; 0.35, 0.05)$	$(0.35; 0.15; 0.35, 0.1)$	$(0.35; 0.15; 0.35, 0.15)$
30		0.232	0.21	0.188	0.166
40		0.59	0.562	0.52	0.468
50	$\lambda_1(\tilde{A})$	0.898	0.884	0.846	0.812
60		0.99	0.986	0.982	0.968
70		1	1	1	0.998
n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.35; 0.15; 0.35; 0.2)$	$(0.35; 0.15; 0.35, 0.25)$	$(0.35; 0.15; 0.35, 0.3)$	$(0.35; 0.15; 0.35, 0.35)$
30		0.15	0.132	0.116	0.108
40		0.442	0.398	0.358	0.324
50	$\lambda_1(\tilde{A})$	0.772	0.736	0.692	0.638
60		0.956	0.938	0.914	0.892
70		0.998	0.994	0.992	0.982

Table A.49. Simulated size with measurement error: graphs generated from $\mathcal{G}(0.15, 0.15, \epsilon_a, \epsilon_d)$. $p = q = p_0 = 0.15$.

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.15; 0.15; 0; 0)$	$(0.15, 0.15, 0.05, 0.05)$	$(0.15, 0.15, 0.1, 0.1)$	$(0.15, 0.15, 0.15, 0.15)$
30		0.024	0.024	0.022	0.02
50		0.034	0.032	0.028	0.028
70		0.036	0.036	0.032	0.03
90	$\lambda_1(\tilde{A})$	0.038	0.036	0.038	0.036
120		0.04	0.036	0.038	0.034
240		0.05	0.042	0.042	0.046
360		0.052	0.06	0.046	0.046
480		0.048	0.042	0.04	0.04
n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.15; 0.15; 0.2; 0.2)$	$(0.15, 0.15, 0.25, 0.25)$	$(0.15, 0.15, 0.3, 0.3)$	$(0.15, 0.15, 0.35, 0.35)$
30		0.018	0.016	0.014	0.016
50		0.024	0.024	0.022	0.022
70		0.028	0.024	0.024	0.024
90	$\lambda_1(\tilde{A})$	0.032	0.032	0.028	0.024
120		0.034	0.028	0.03	0.032
240		0.038	0.04	0.038	0.042
360		0.042	0.036	0.036	0.042
480		0.04	0.04	0.044	0.038

Table A.50. Simulated size with measurement error: graphs generated from $\mathcal{G}(0.25, 0.25, \epsilon_a, \epsilon_d)$. $p = q = p_0 = 0.25$.

n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.25; 0.25; 0; 0)$	$(0.25, 0.25, 0.05, 0.05)$	$(0.25, 0.25, 0.1, 0.1)$	$(0.25, 0.25, 0.15, 0.15)$
30		0.01	0.008	0.006	0.008
50		0.012	0.012	0.01	0.01
70		0.014	0.01	0.014	0.012
90	$\lambda_1(\tilde{A})$	0.02	0.016	0.016	0.016
120		0.02	0.018	0.018	0.016
240		0.022	0.022	0.024	0.024
360		0.024	0.032	0.028	0.03
480		0.026	0.03	0.03	0.032
n	Method	$(p, q, \epsilon_a, \epsilon_d) = (0.25; 0.25; 0.2; 0.2)$	$(0.25, 0.25, 0.25, 0.25)$	$(0.25, 0.25, 0.3, 0.3)$	$(0.25, 0.25, 0.35, 0.35)$
30		0.006	0.006	0.006	0.008
50		0.01	0.01	0.008	0.01
70		0.012	0.014	0.012	0.01
90	$\lambda_1(\tilde{A})$	0.014	0.014	0.016	0.012
120		0.016	0.016	0.016	0.018
240		0.022	0.022	0.02	0.02
360		0.024	0.024	0.03	0.03
480		0.038	0.032	0.032	0.03

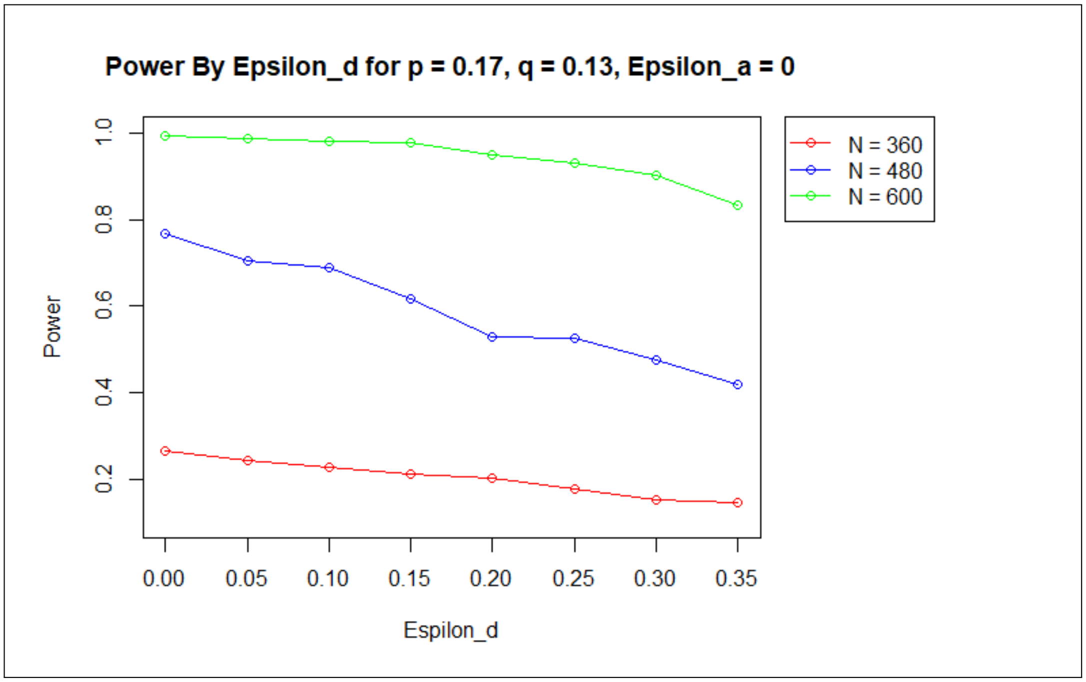


Figure A.1. Plot of Power for $p = 0.17$, $q = 0.13$, $\epsilon_a = 0$.

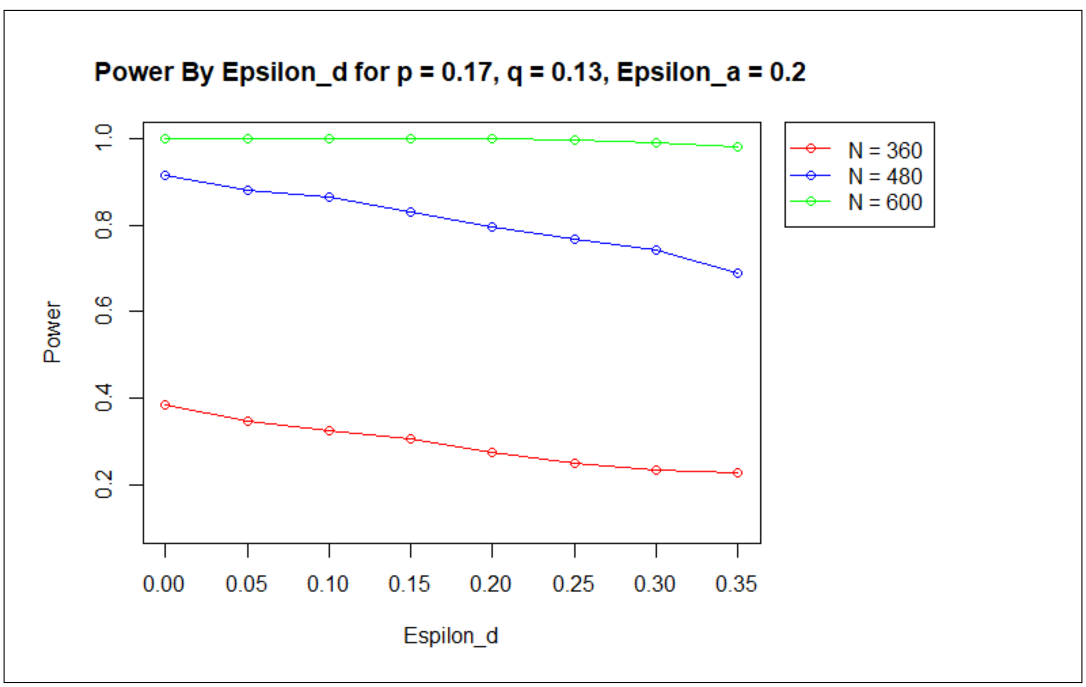


Figure A.2. Plot of Power for $p = 0.17$, $q = 0.13$, $\epsilon_a = 0.20$.

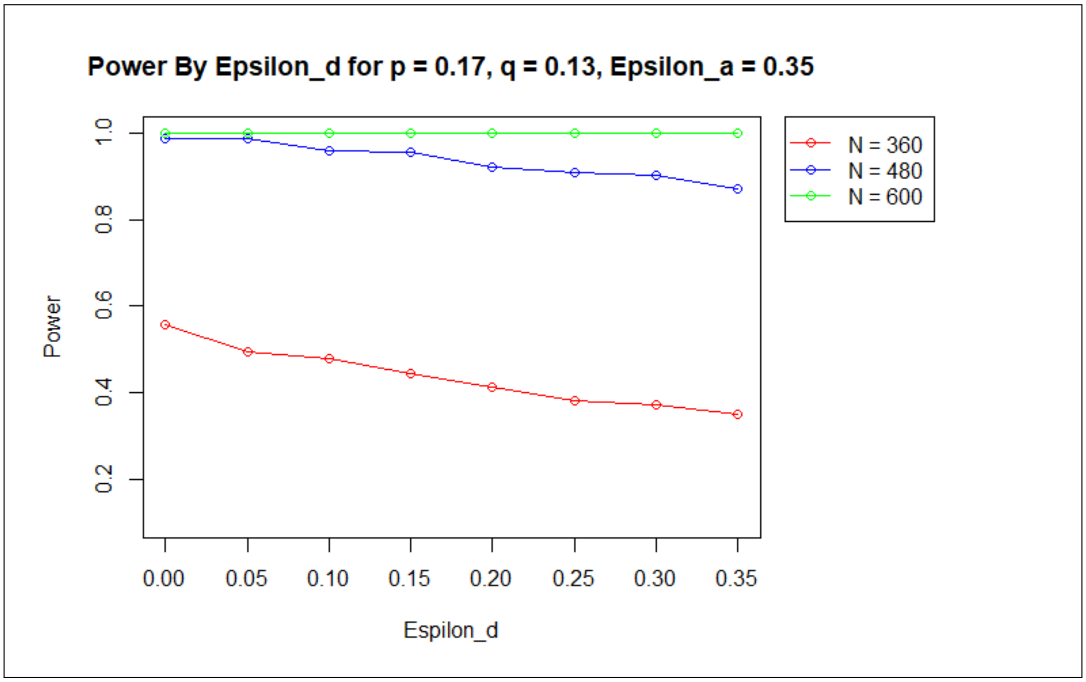


Figure A.3. Plot of Power for $p = 0.17$, $q = 0.13$, $\epsilon_a = 0.35$.

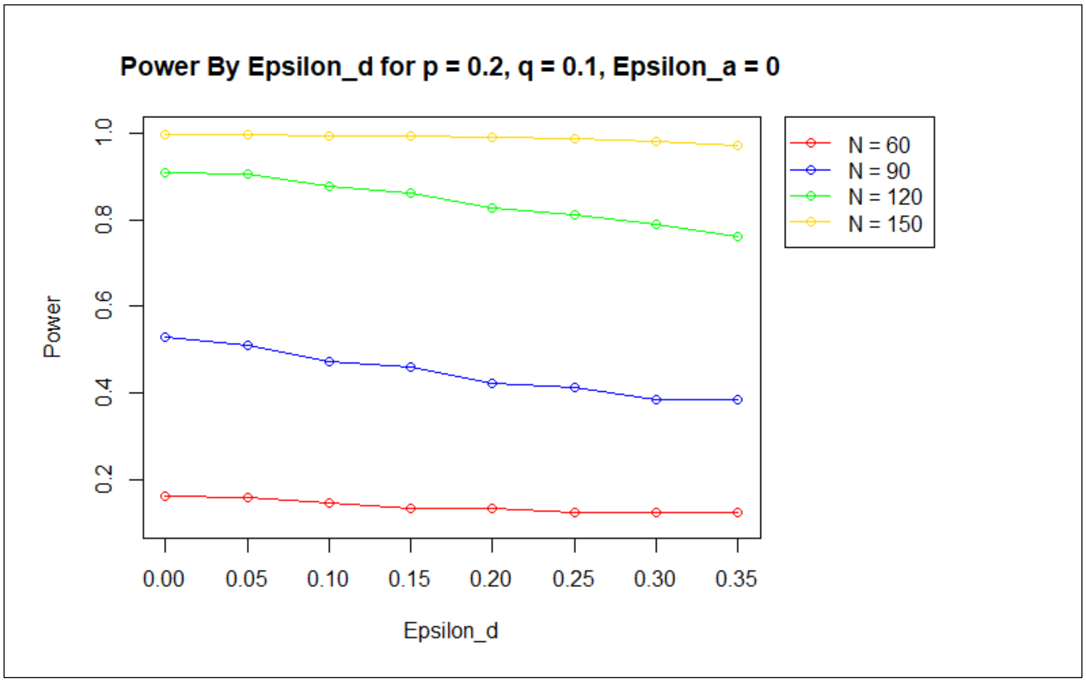


Figure A.4. Plot of Power for $p = 0.20$, $q = 0.10$, $\epsilon_a = 0$.

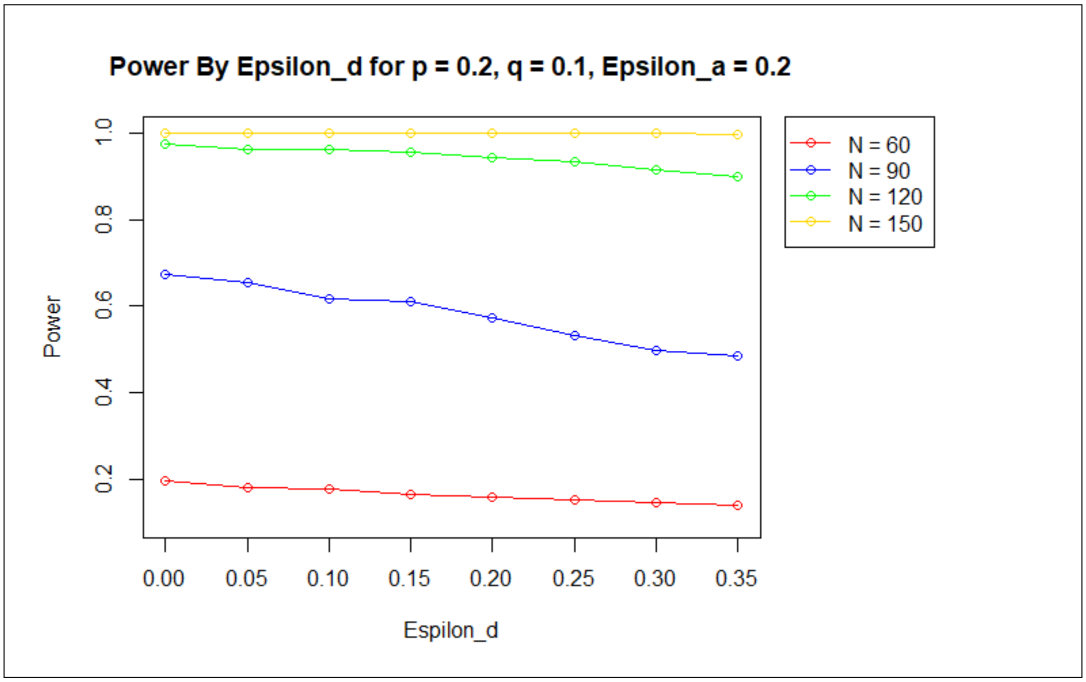


Figure A.5. Plot of Power for $p = 0.20$, $q = 0.10$, $\epsilon_a = 0.20$.

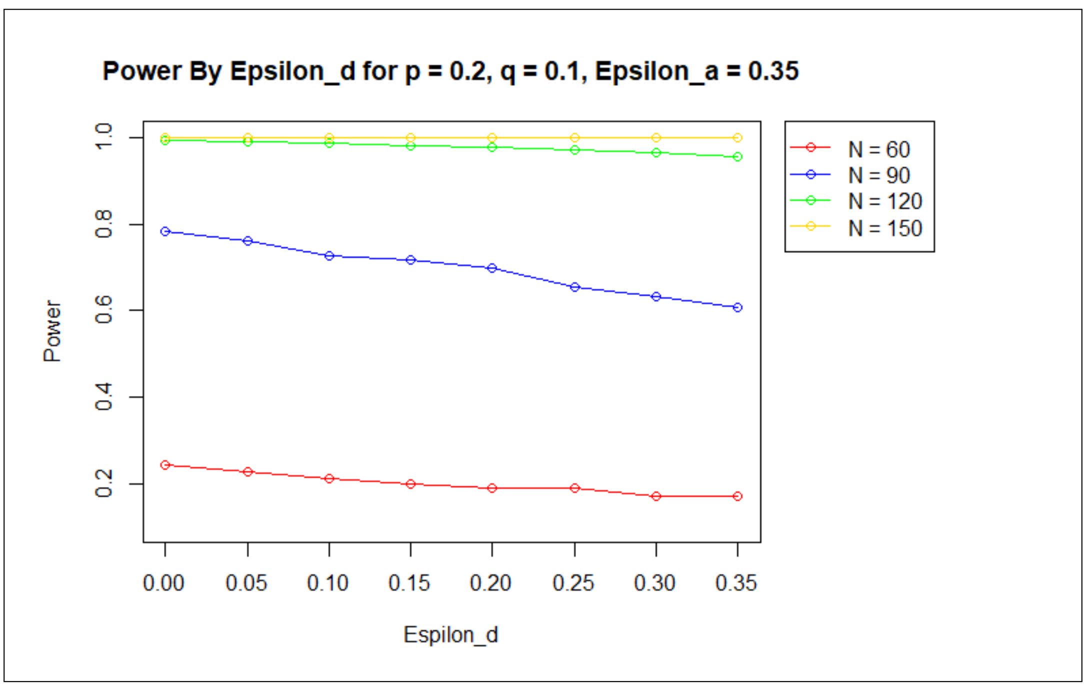


Figure A.6. Plot of Power for $p = 0.20$, $q = 0.10$, $\epsilon_a = 0.35$.

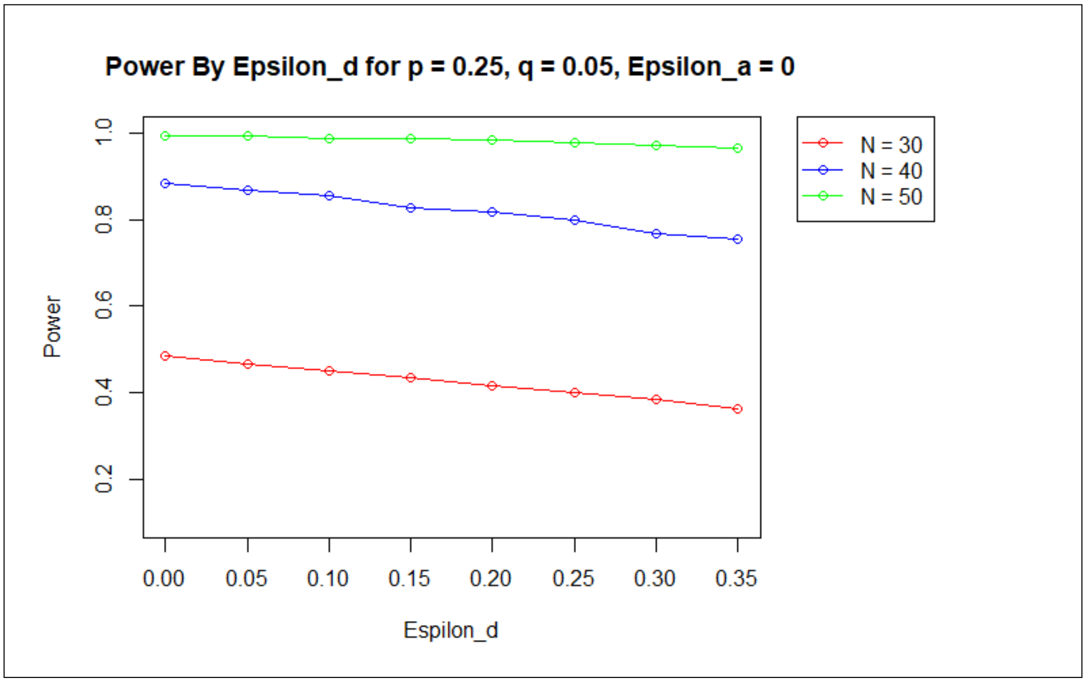


Figure A.7. Plot of Power for $p = 0.25$, $q = 0.05$, $\epsilon_a = 0$.

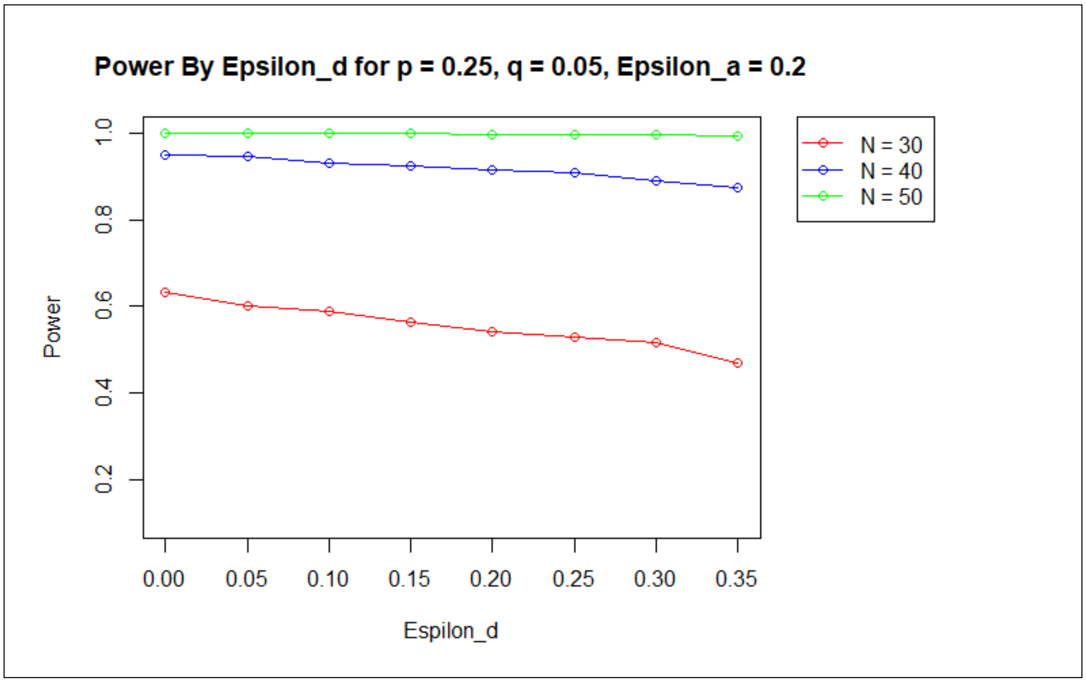


Figure A.8. Plot of Power for $p = 0.25$, $q = 0.05$, $\epsilon_a = 0.20$.

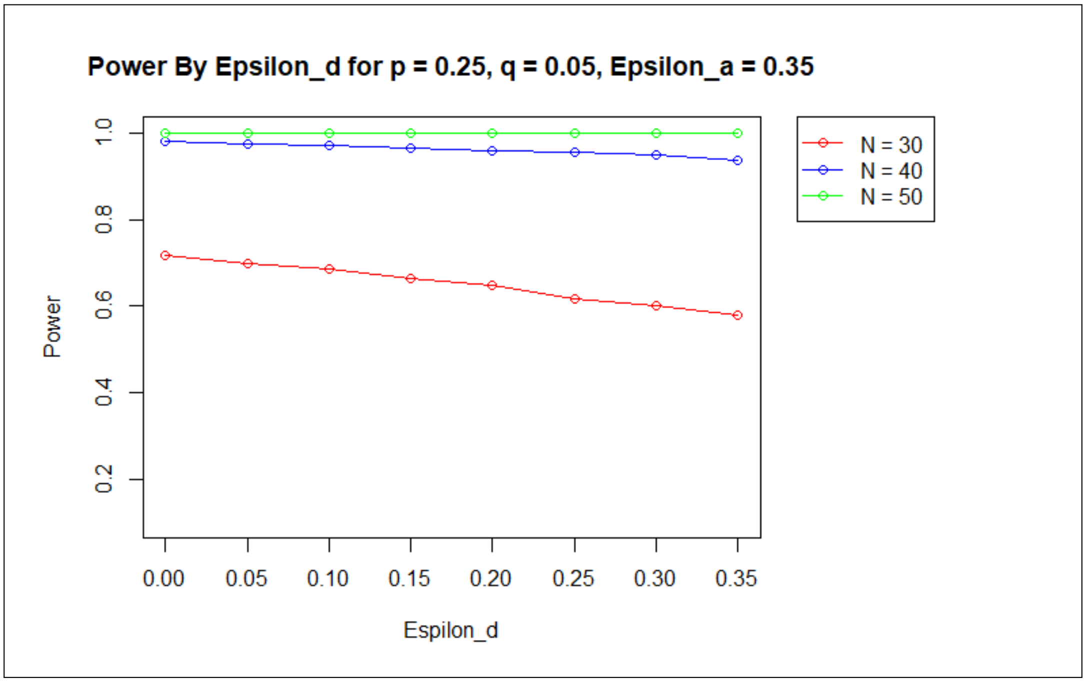


Figure A.9. Plot of Power for $p = 0.25$, $q = 0.05$, $\epsilon_a = 0.35$.

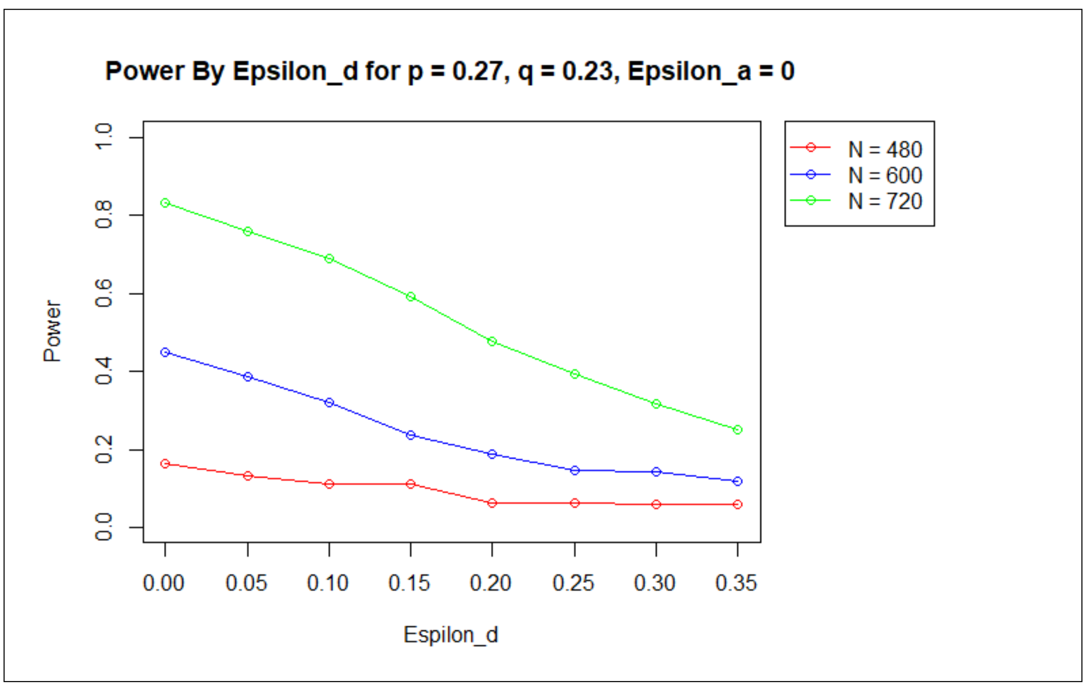


Figure A.10. Plot of Power for $p = 0.27$, $q = 0.23$, $\epsilon_a = 0$.

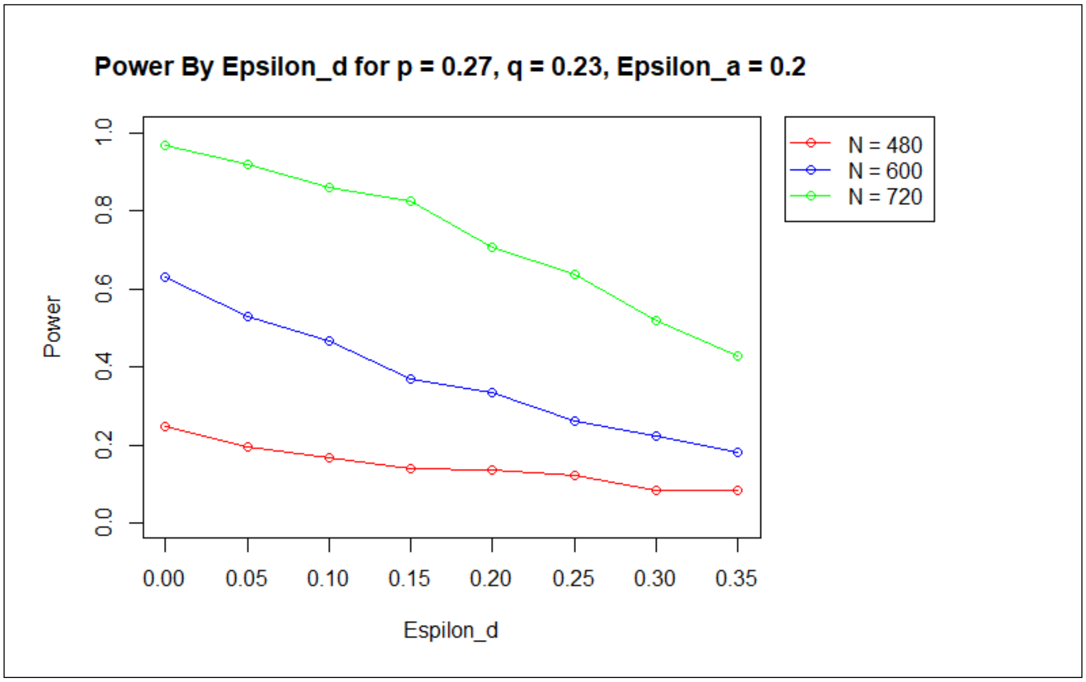


Figure A.11. Plot of Power for $p = 0.27$, $q = 0.23$, $\epsilon_a = 0.20$.

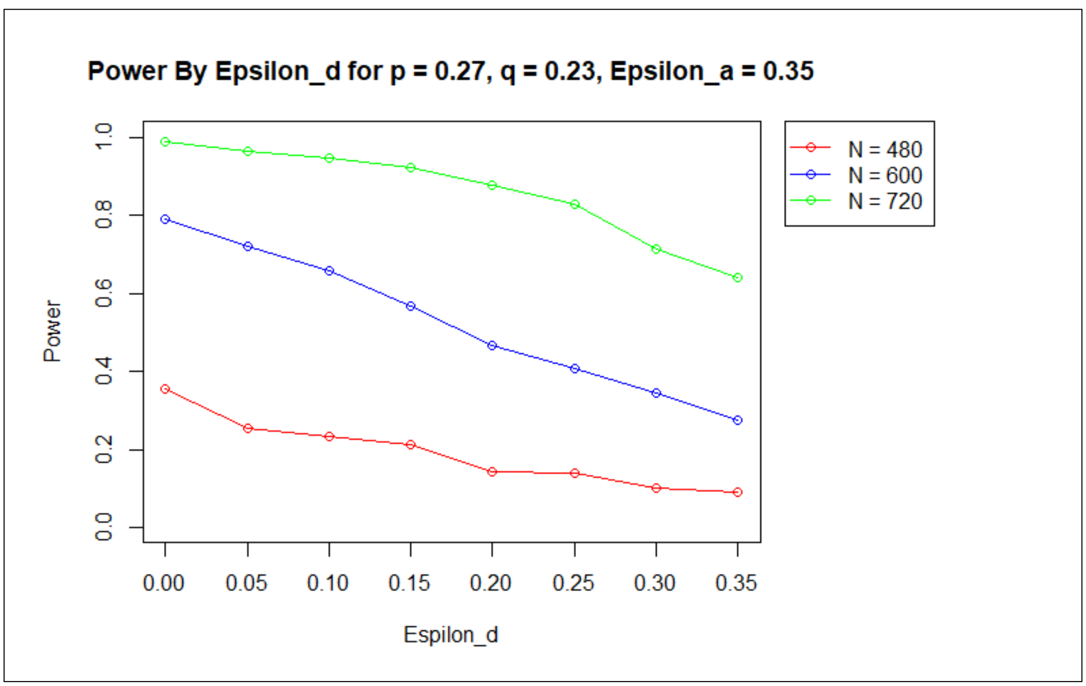


Figure A.12. Plot of Power for $p = 0.27$, $q = 0.23$, $\epsilon_a = 0.35$.

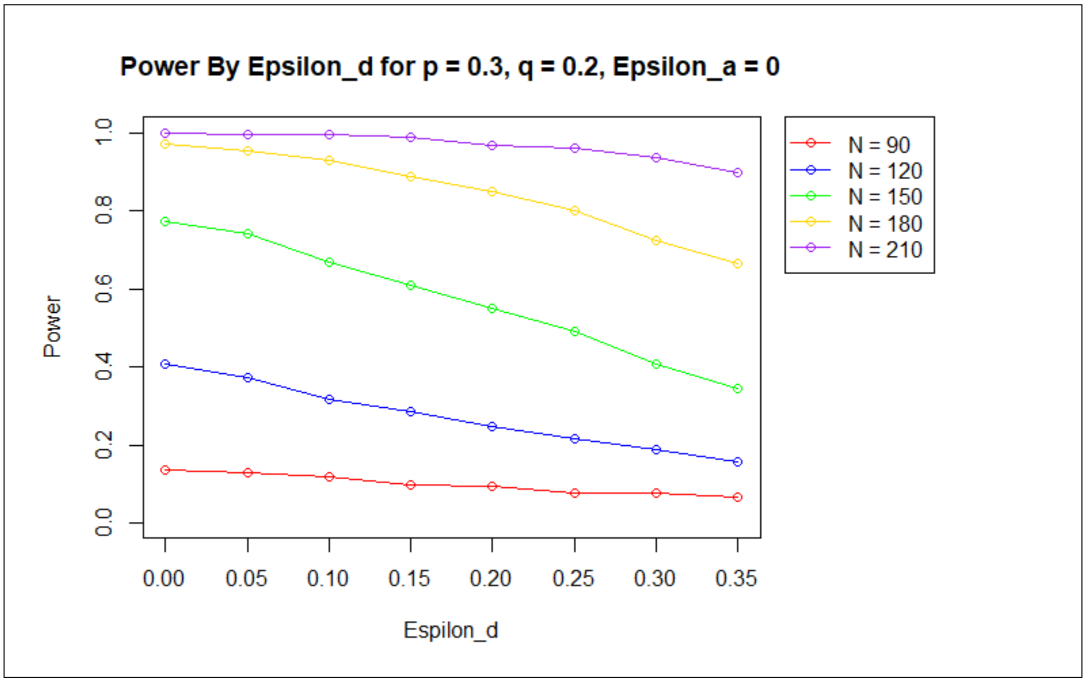


Figure A.13. Plot of Power for $p = 0.30$, $q = 0.20$, $\epsilon_a = 0$.

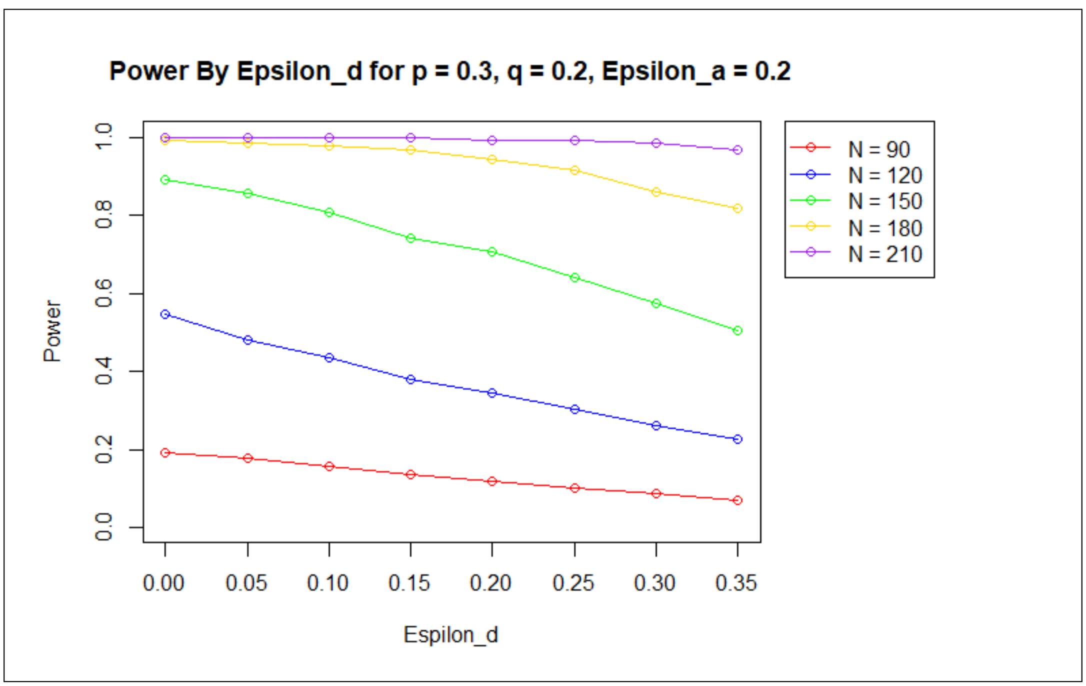


Figure A.14. Plot of Power for $p = 0.30$, $q = 0.20$, $\epsilon_a = 0.20$.

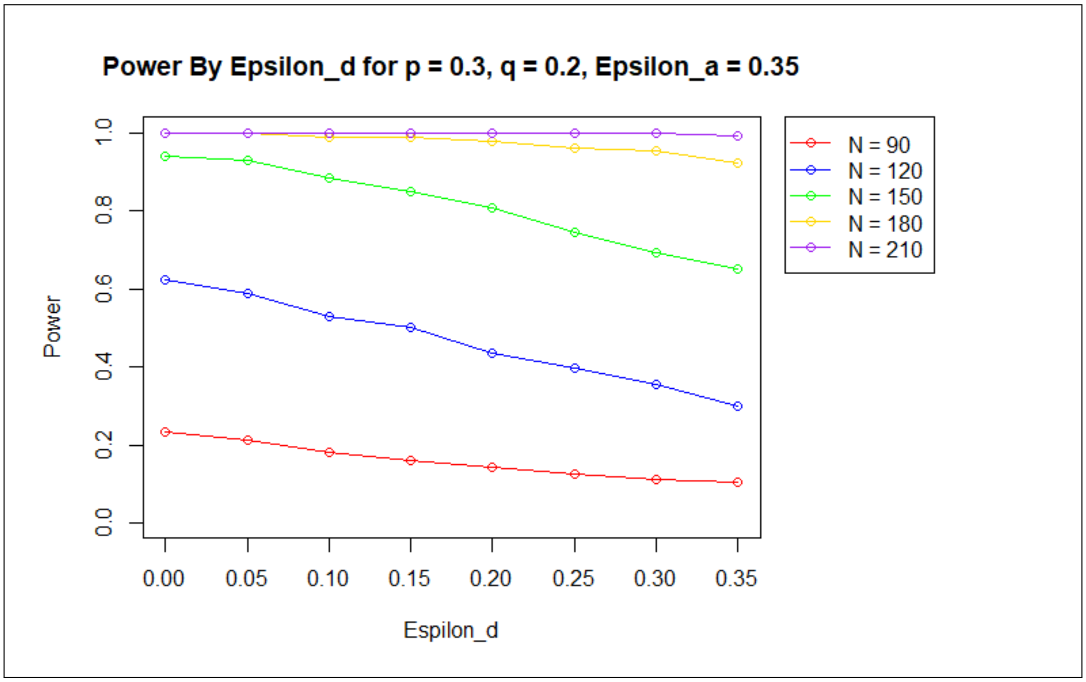


Figure A.15. Plot of Power for $p = 0.30$, $q = 0.20$, $\epsilon_a = 0.35$.

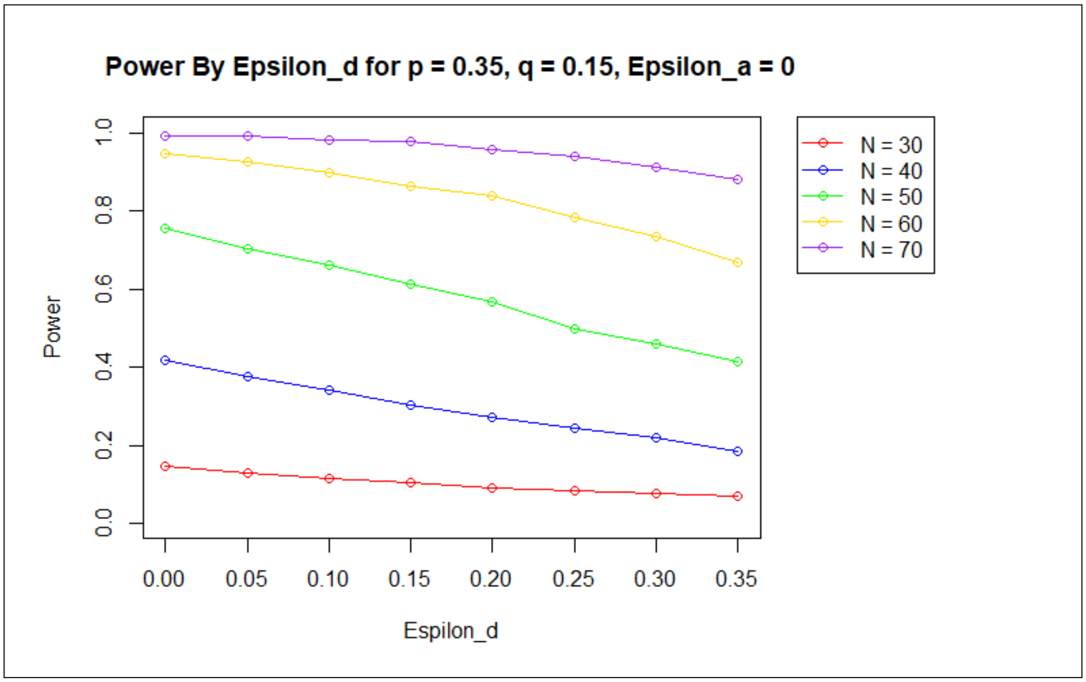


Figure A.16. Plot of Power for $p = 0.35$, $q = 0.15$, $\epsilon_a = 0$.

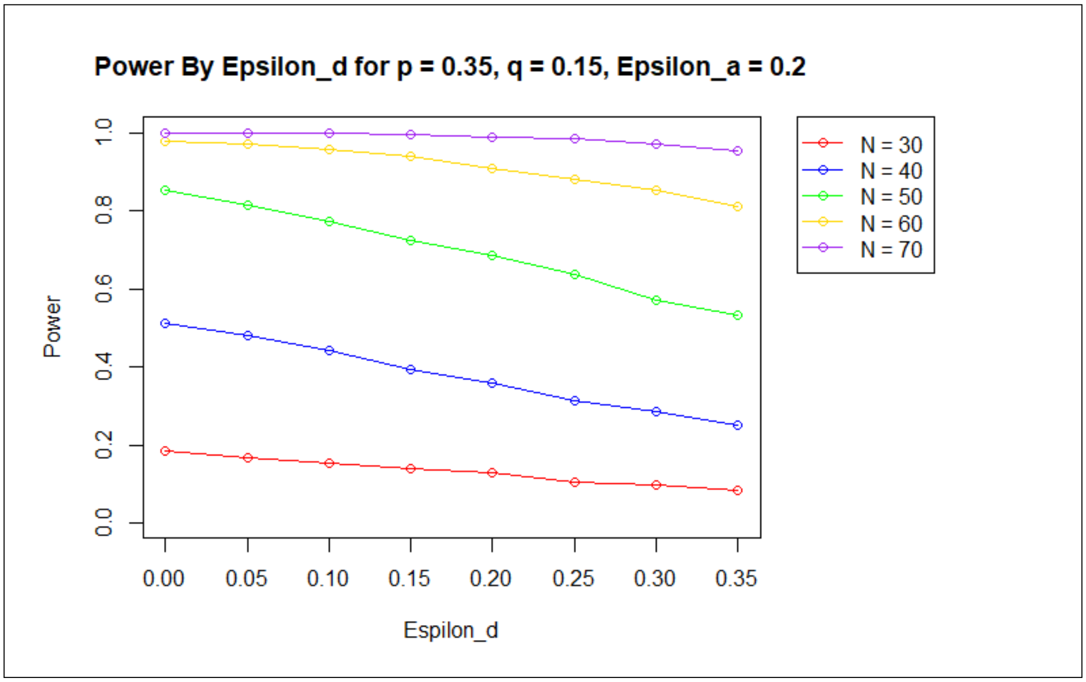


Figure A.17. Plot of Power for $p = 0.35$, $q = 0.15$, $\epsilon_a = 0.20$.

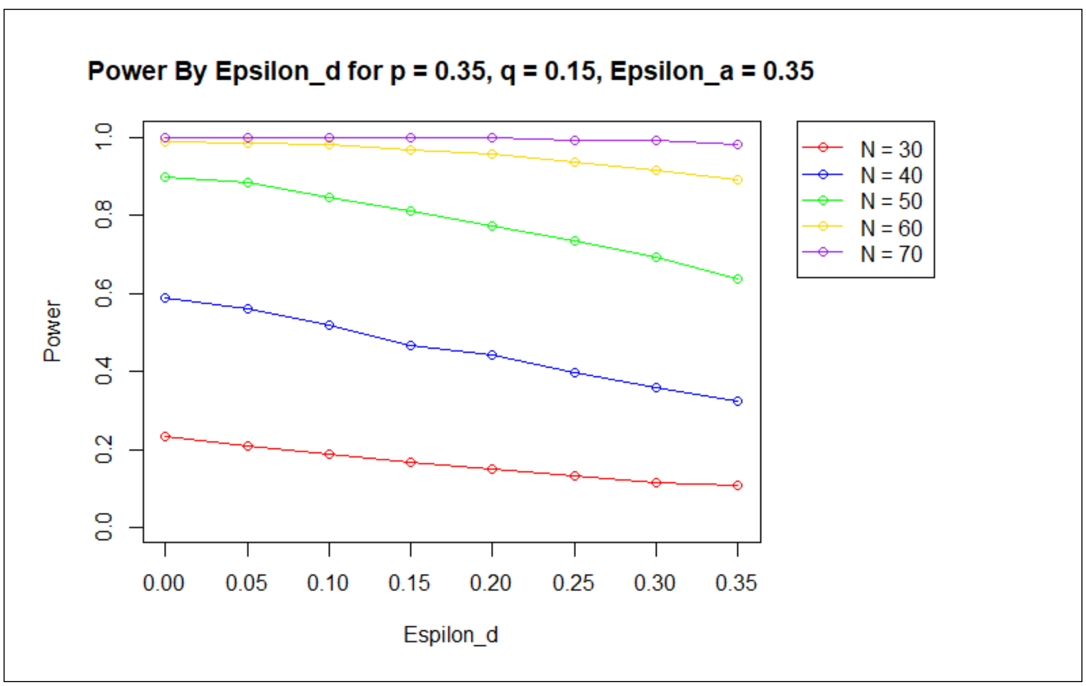


Figure A.18. Plot of Power for $p = 0.35$, $q = 0.15$, $\epsilon_a = 0.35$.

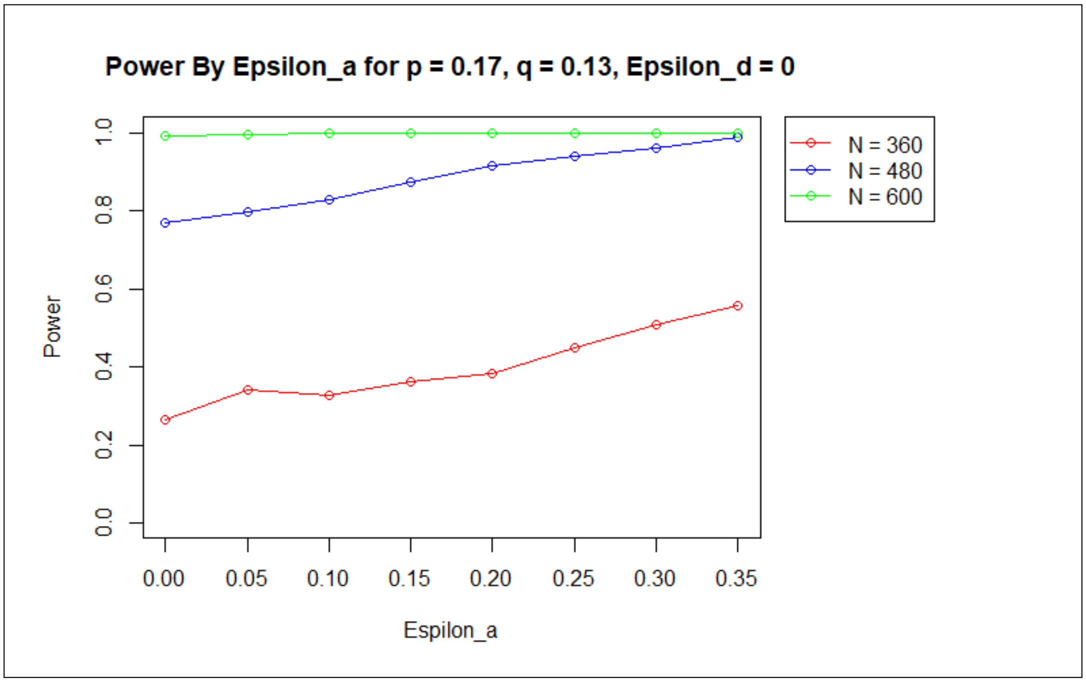


Figure A.19. Plot of Power for $p = 0.17$, $q = 0.13$, $\epsilon_d = 0$.

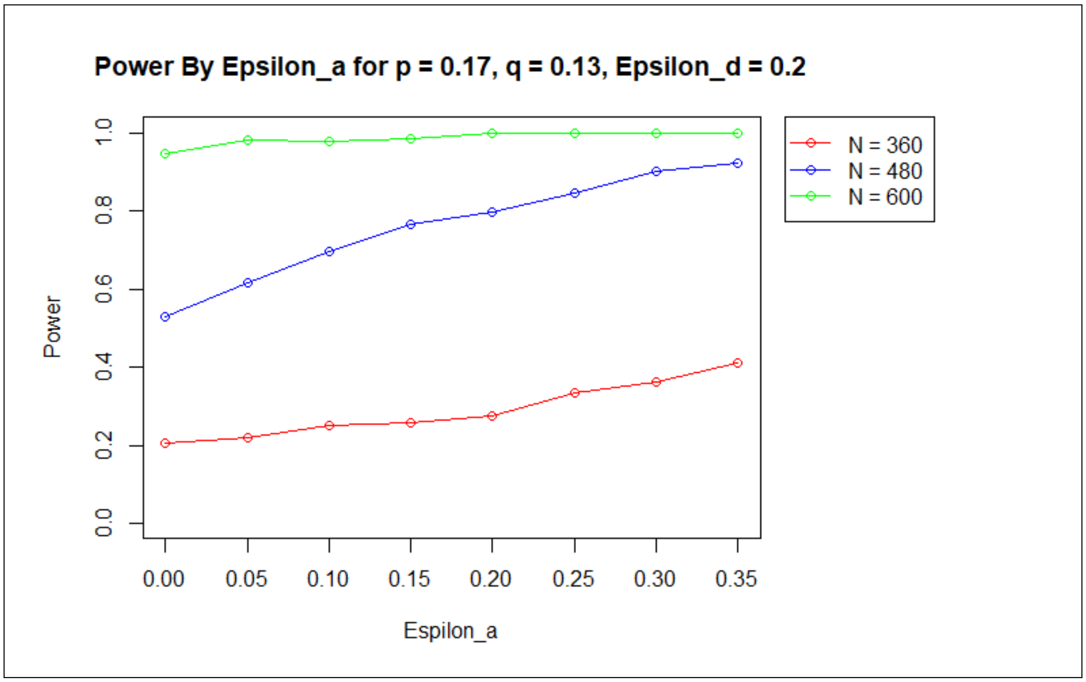


Figure A.20. Plot of Power for $p = 0.17$, $q = 0.13$, $\epsilon_d = 0.20$.

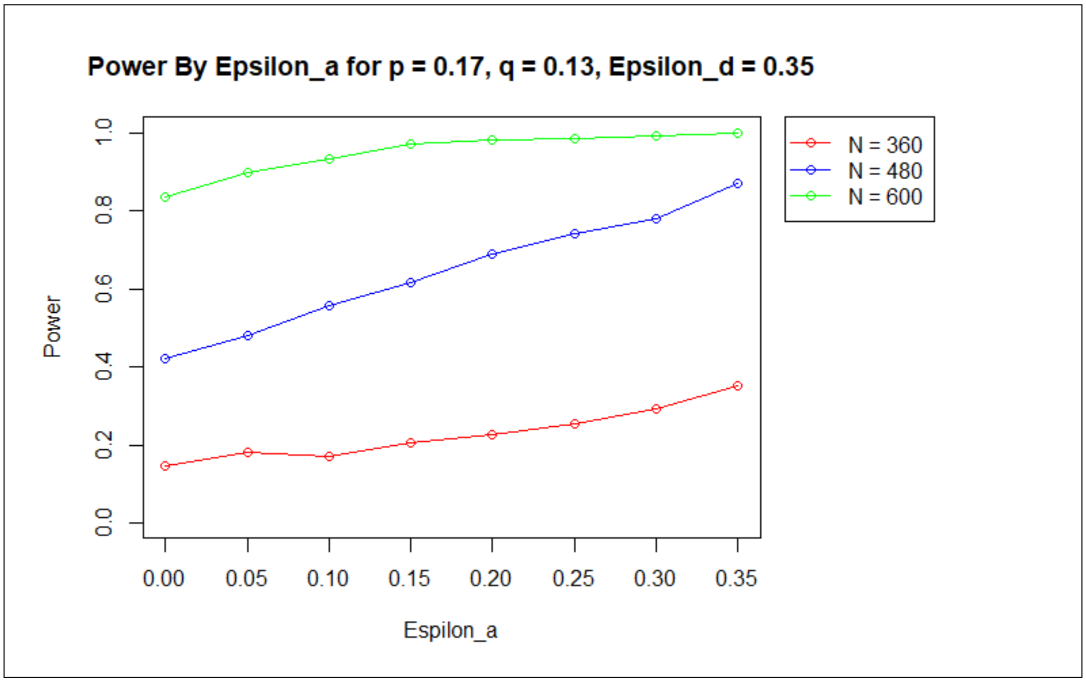


Figure A.21. Plot of Power for $p = 0.17$, $q = 0.13$, $\epsilon_d = 0.35$.

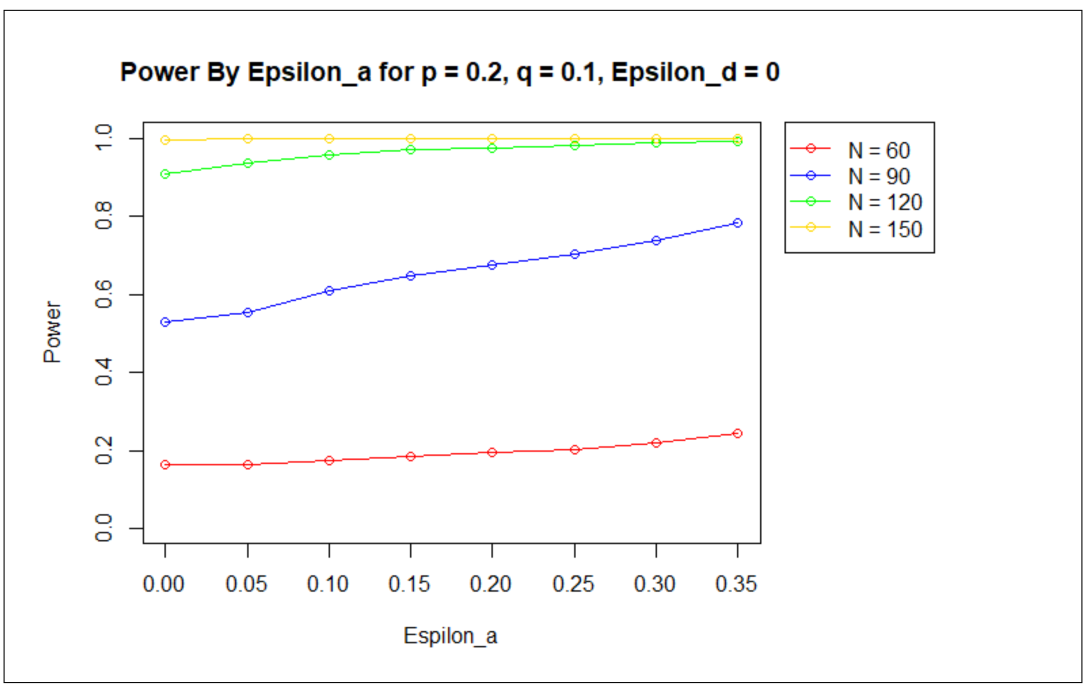


Figure A.22. Plot of Power for $p = 0.20$, $q = 0.10$, $\epsilon_d = 0$.

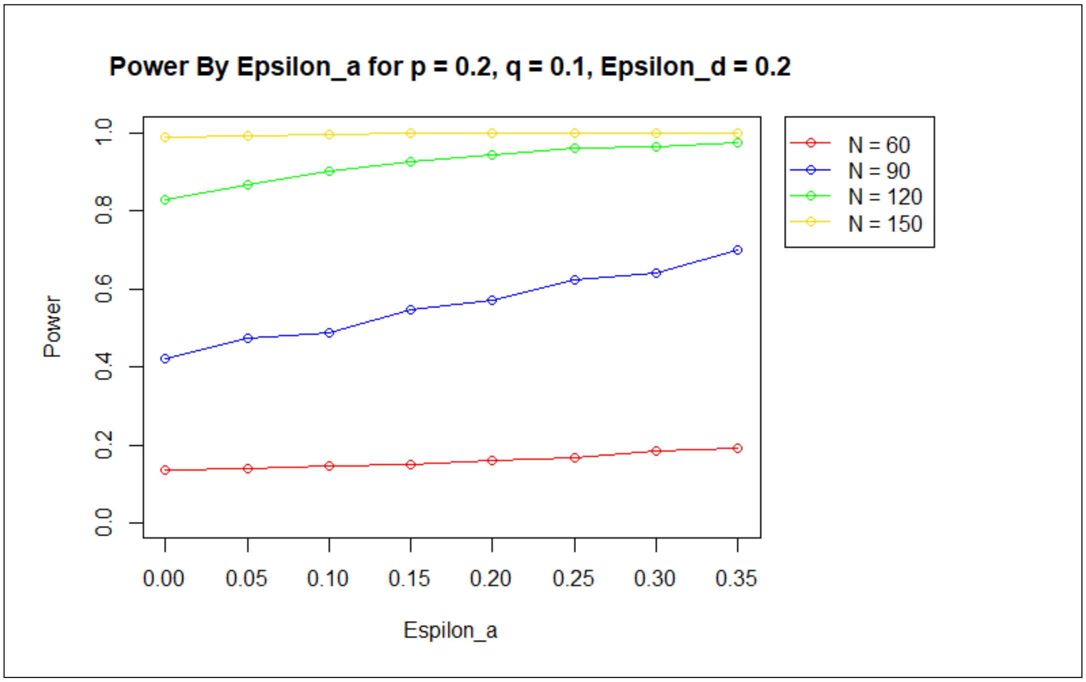


Figure A.23. Plot of Power for $p = 0.20$, $q = 0.10$, $\epsilon_d = 0.20$.

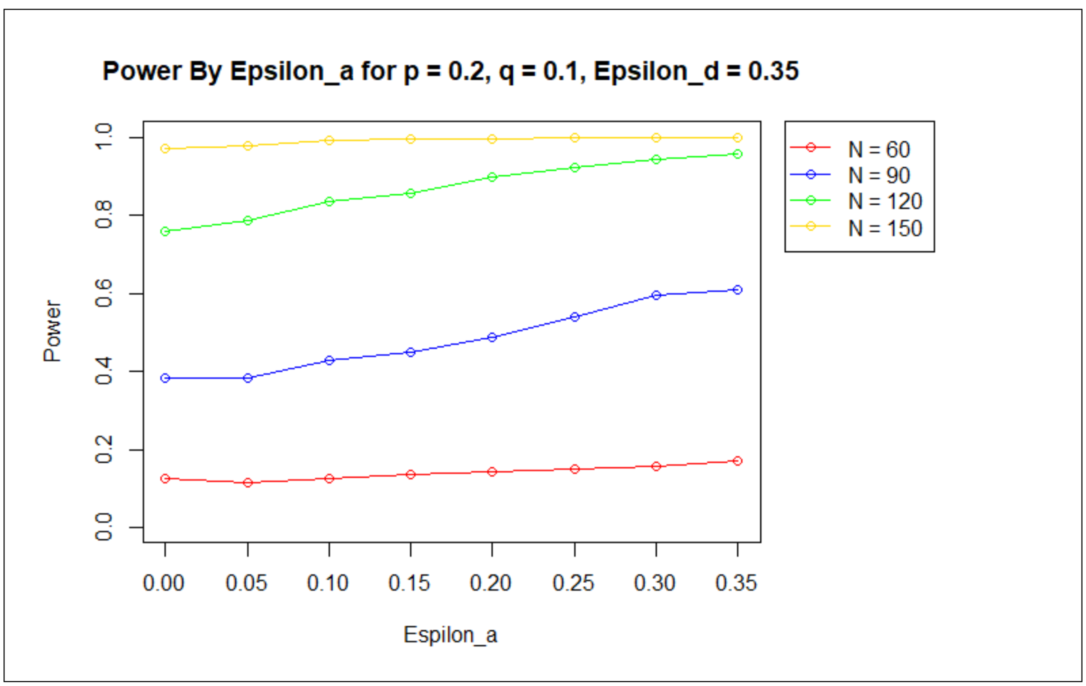


Figure A.24. Plot of Power for $p = 0.20$, $q = 0.10$, $\epsilon_d = 0.35$.

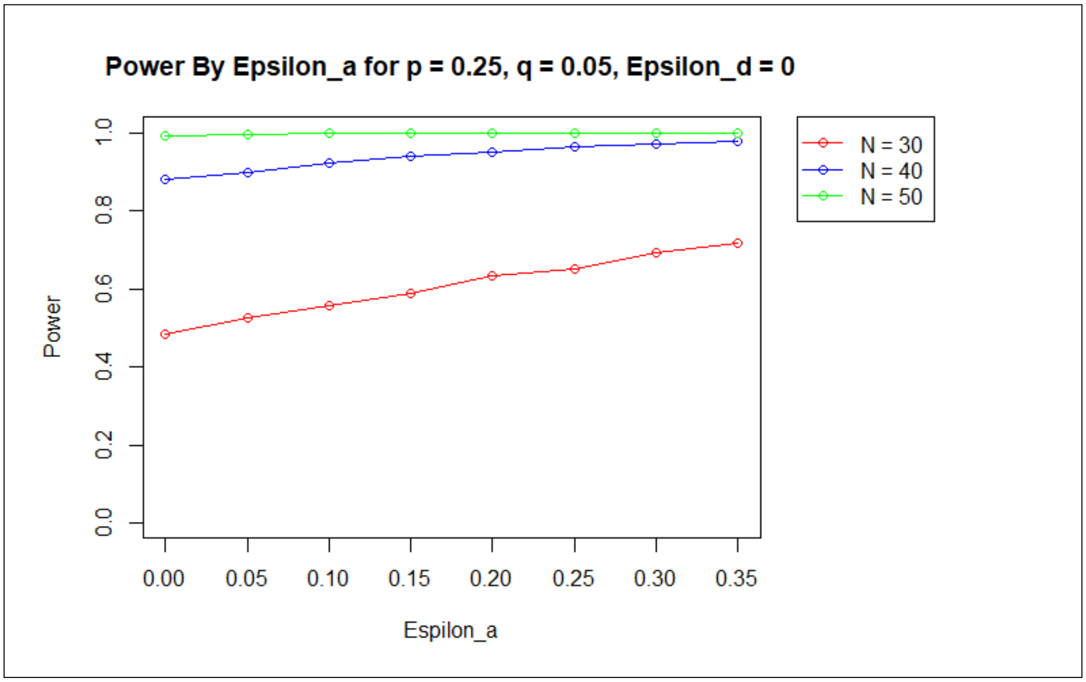


Figure A.25. Plot of Power for $p = 0.25$, $q = 0.05$, $\epsilon_d = 0$.

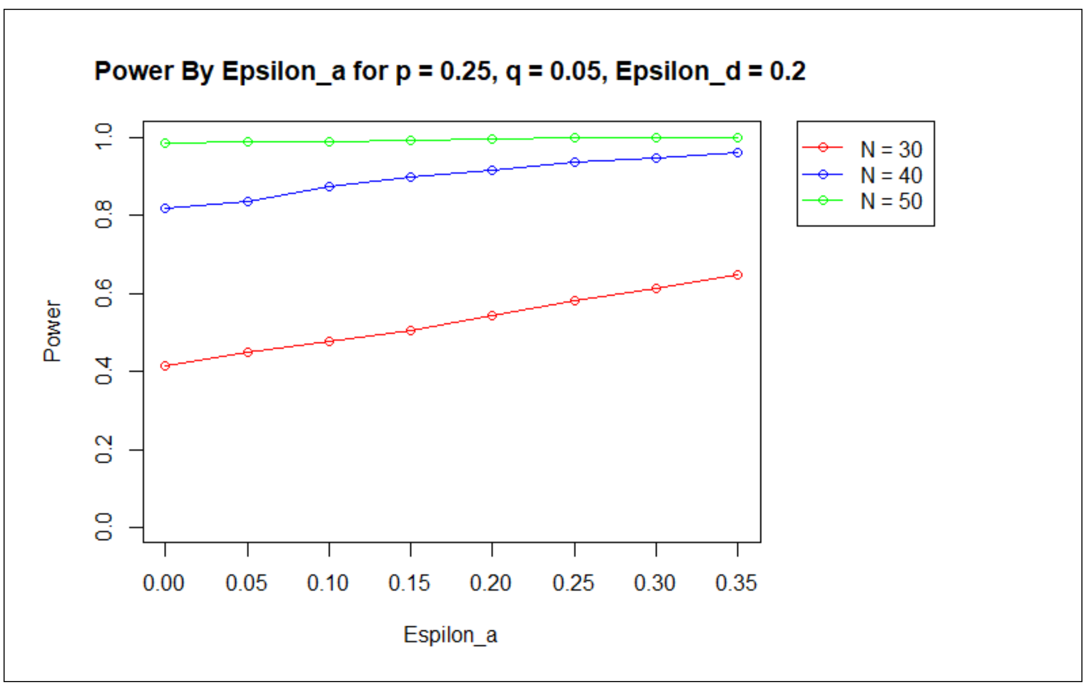


Figure A.26. Plot of Power for $p = 0.25$, $q = 0.05$, $\epsilon_d = 0.20$.

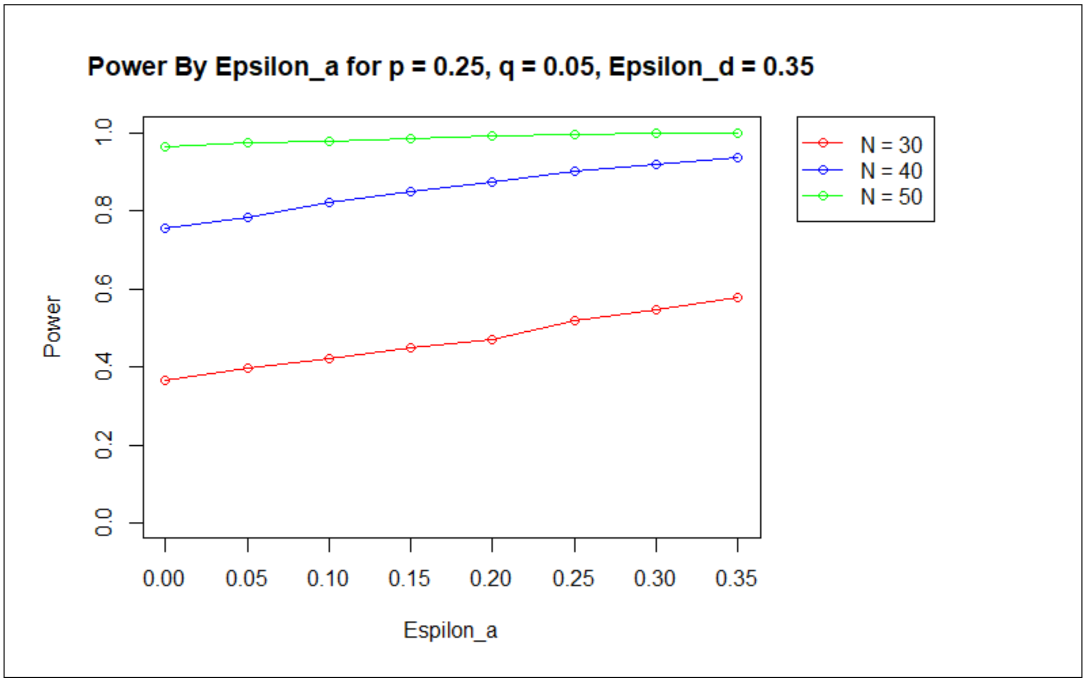


Figure A.27. Plot of Power for $p = 0.25$, $q = 0.05$, $\epsilon_d = 0.35$.

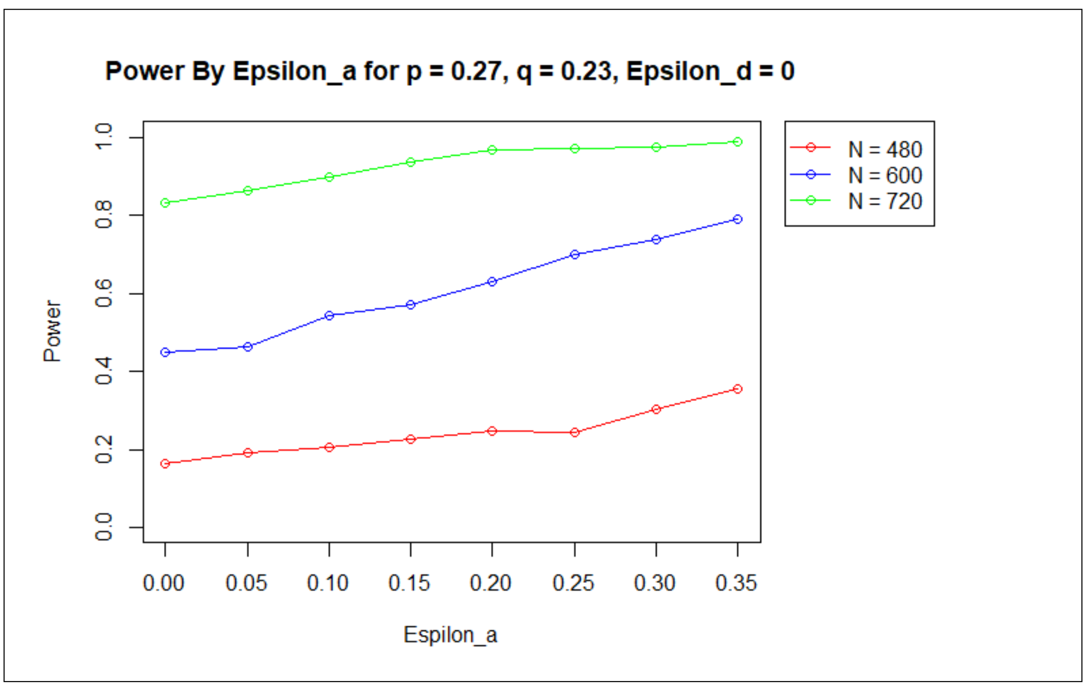


Figure A.28. Plot of Power for $p = 0.27$, $q = 0.23$, $\epsilon_d = 0$.

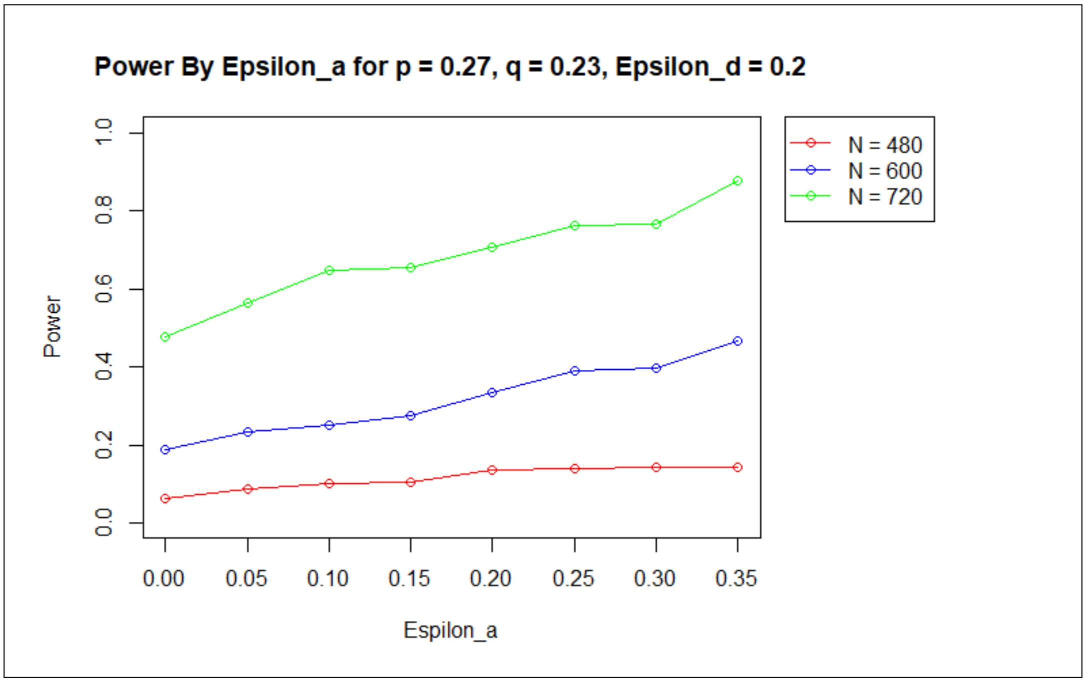


Figure A.29. Plot of Power for $p = 0.27$, $q = 0.23$, $\epsilon_d = 0.20$.

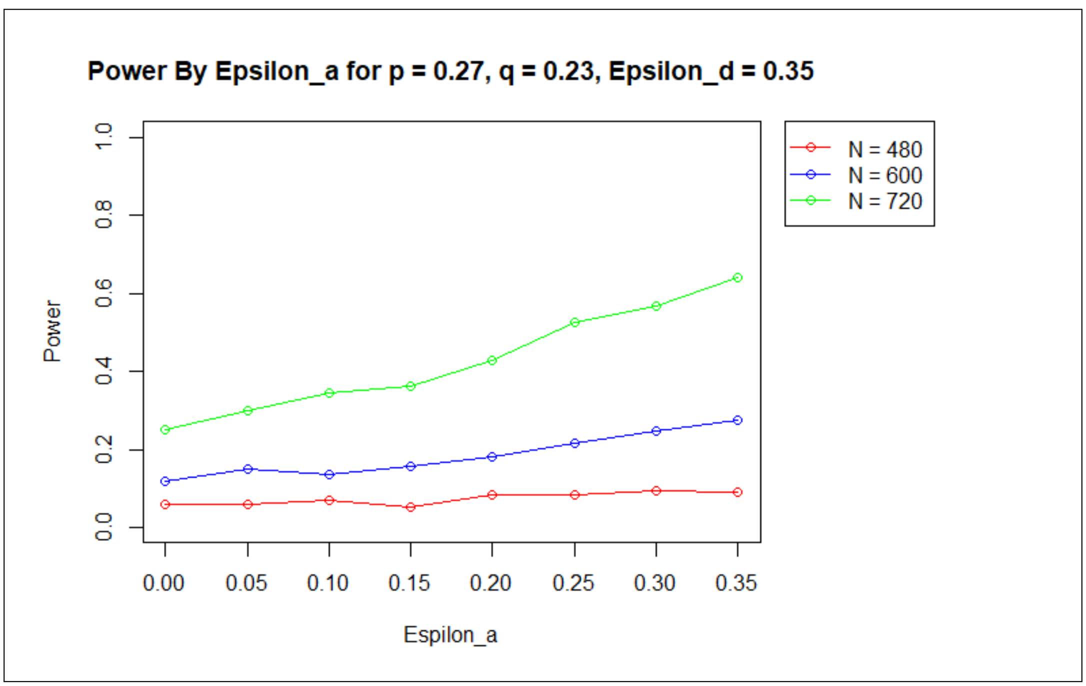


Figure A.30. Plot of Power for $p = 0.27$, $q = 0.23$, $\epsilon_d = 0.35$.

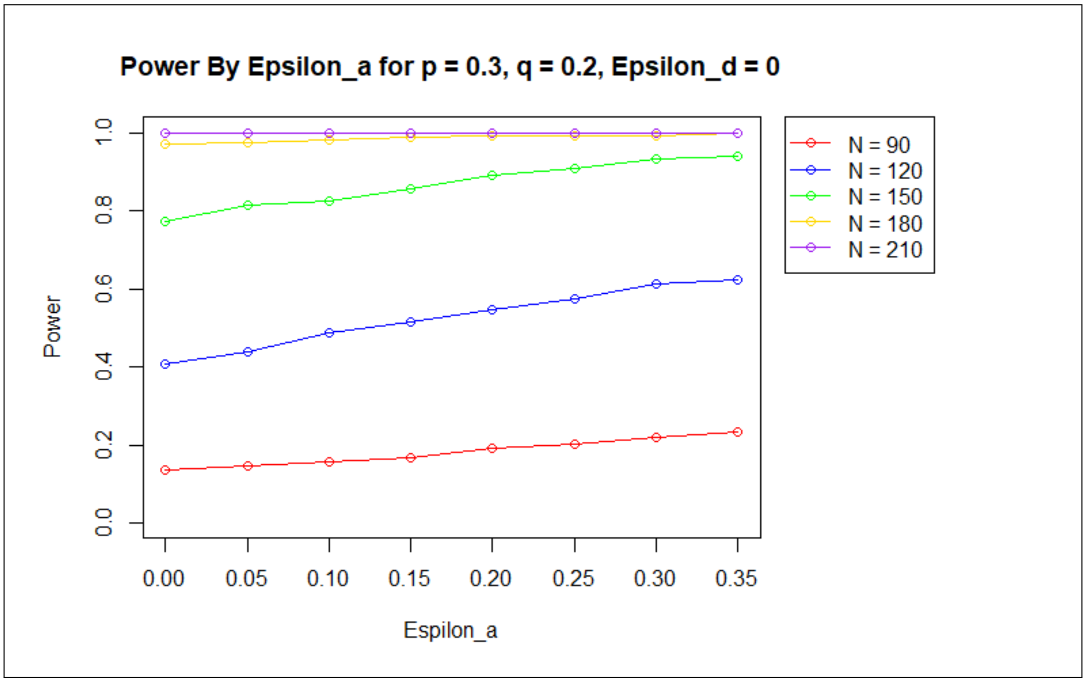


Figure A.31. Plot of Power for $p = 0.30$, $q = 0.20$, $\epsilon_d = 0$.

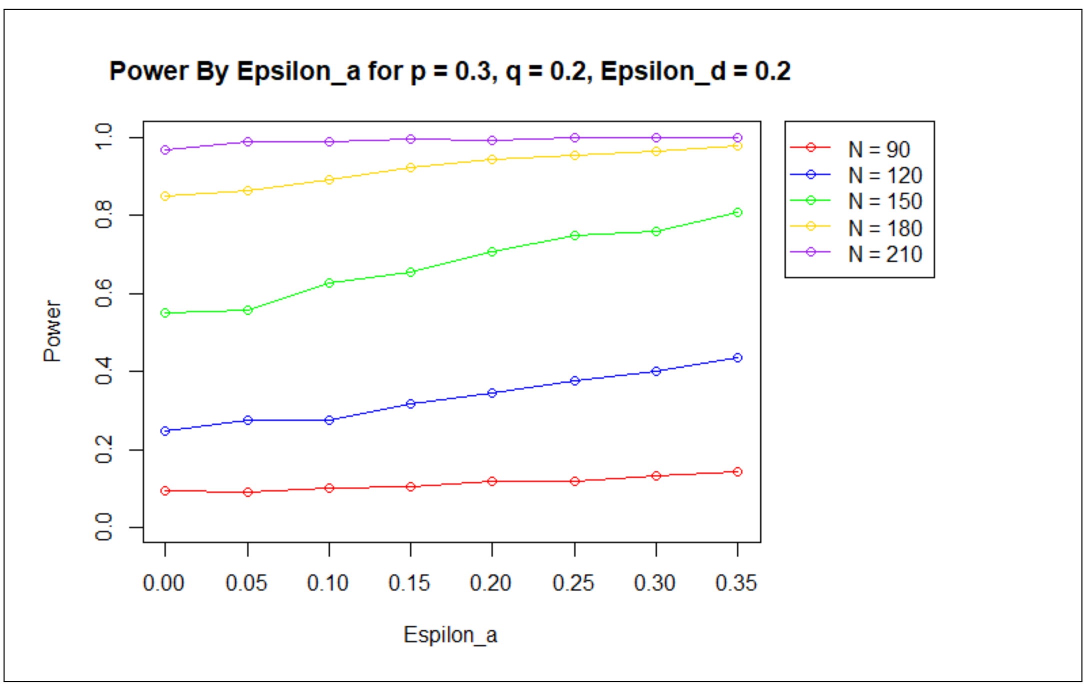


Figure A.32. Plot of Power for $p = 0.30$, $q = 0.20$, $\epsilon_d = 0.20$.

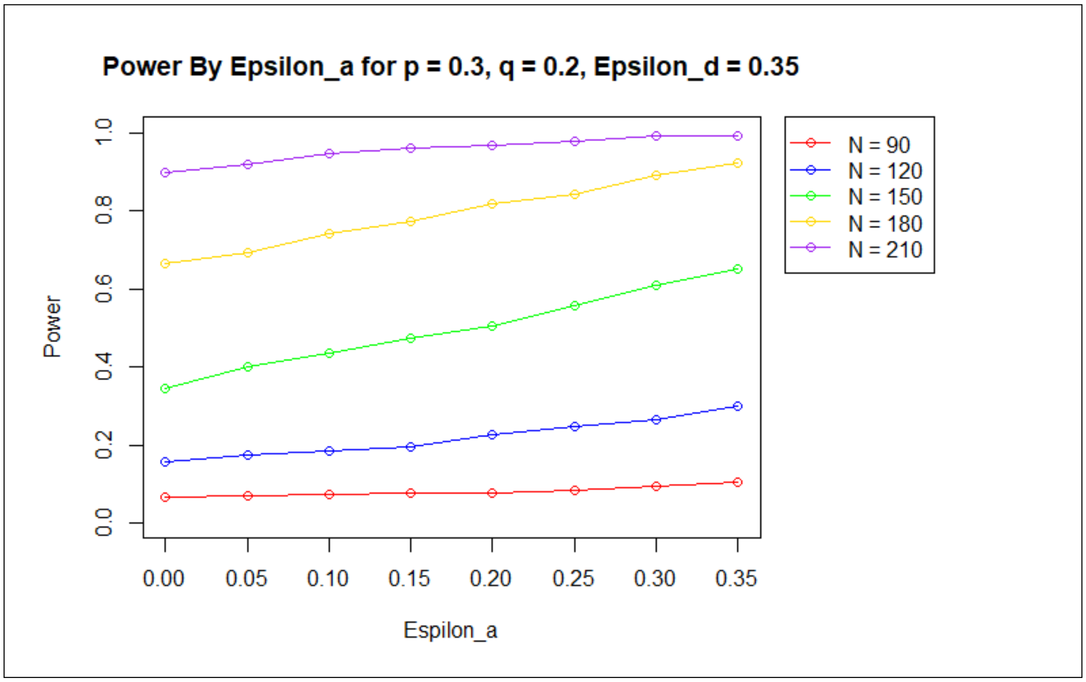


Figure A.33. Plot of Power for $p = 0.30$, $q = 0.20$, $\epsilon_d = 0.35$.

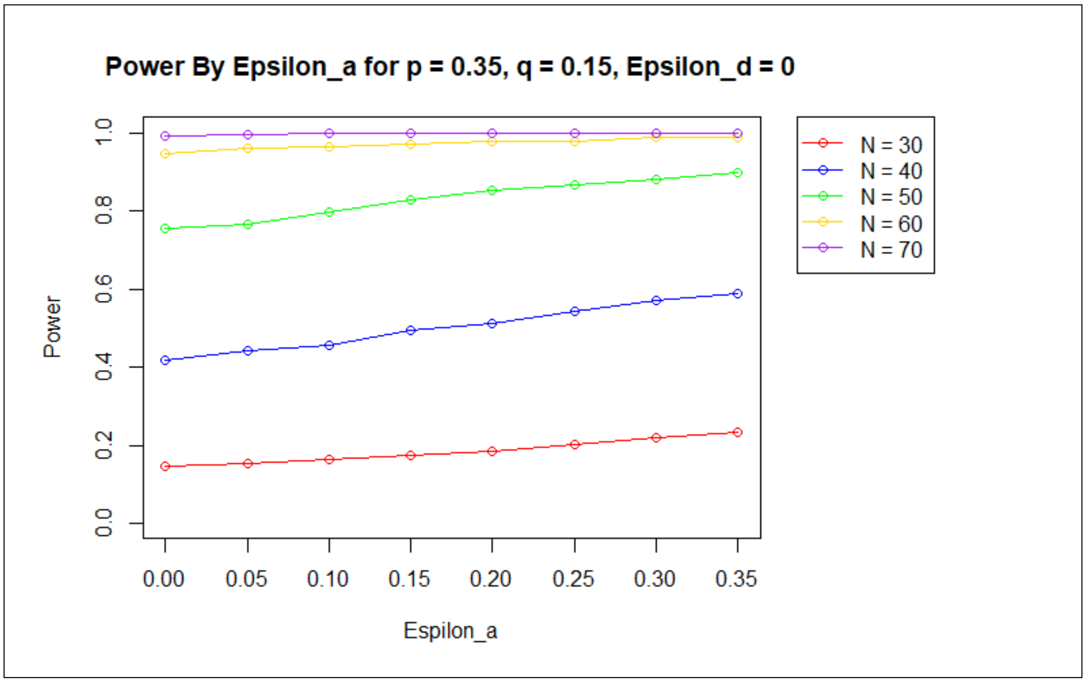


Figure A.34. Plot of Power for $p = 0.35$, $q = 0.15$, $\epsilon_d = 0$.

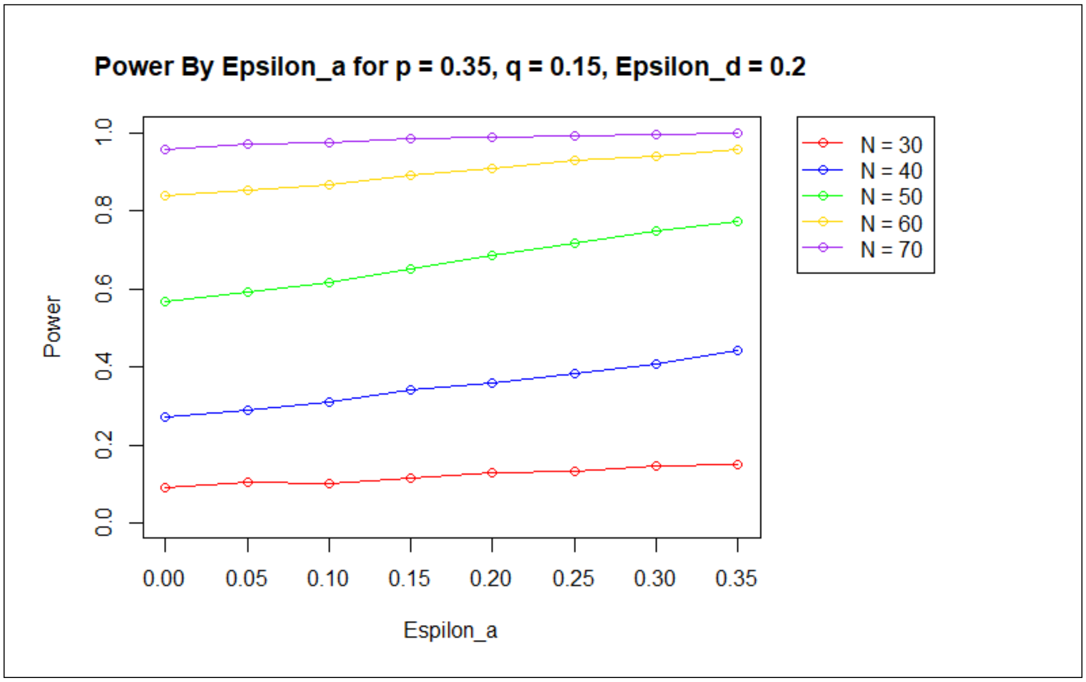


Figure A.35. Plot of Power for $p = 0.35$, $q = 0.15$, $\epsilon_d = 0.20$.

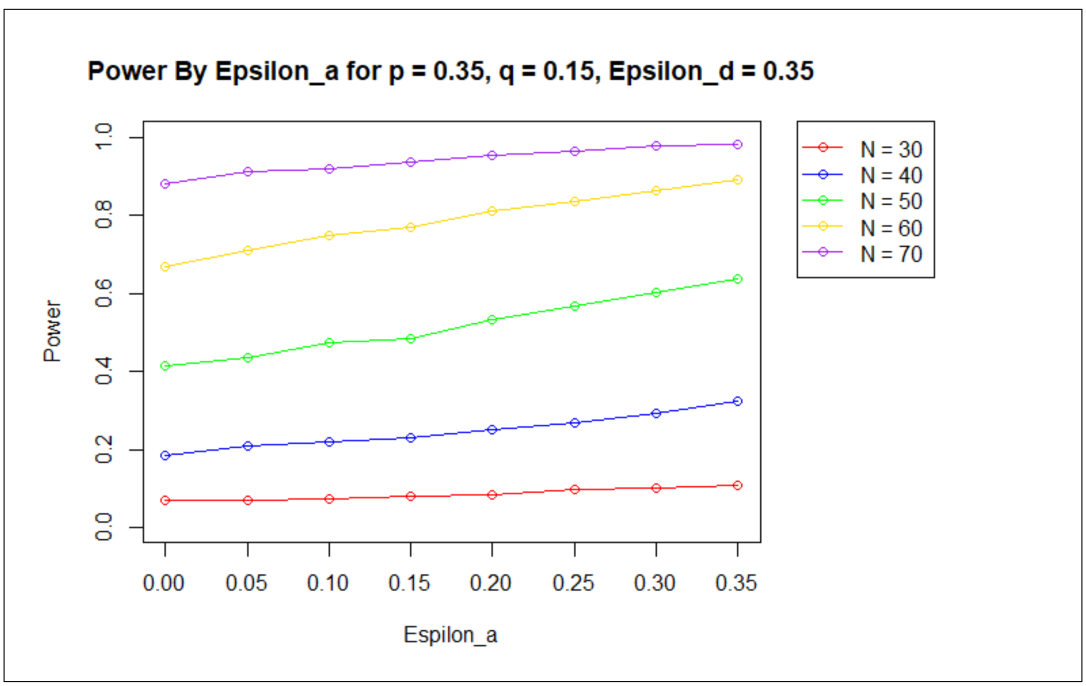


Figure A.36. Plot of Power for $p = 0.35$, $q = 0.15$, $\epsilon_d = 0.35$.

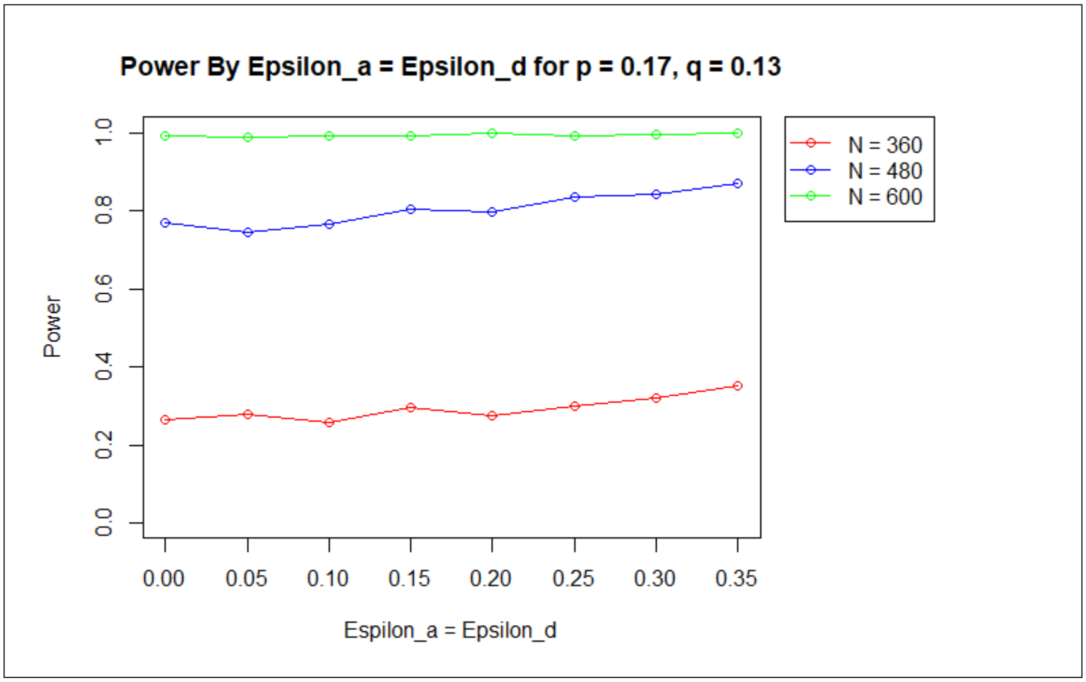


Figure A.37. Plot of Power for $p = 0.17$, $q = 0.13$, $\epsilon_a = \epsilon_d$.

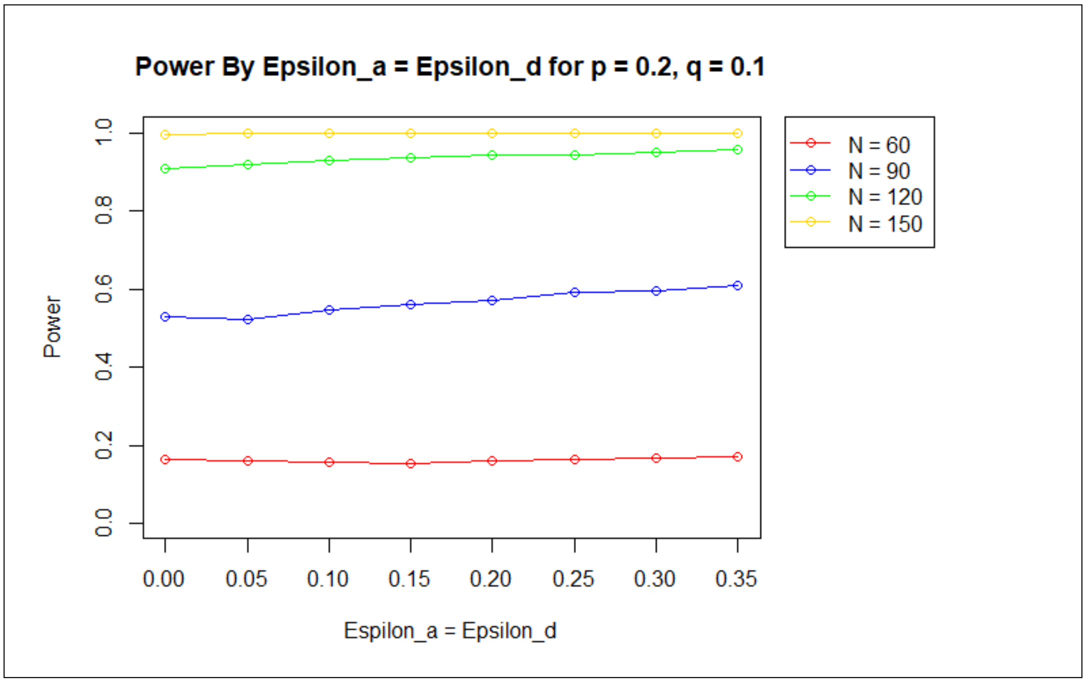


Figure A.38. Plot of Power for $p = 0.20$, $q = 0.10$, $\epsilon_a = \epsilon_d$.

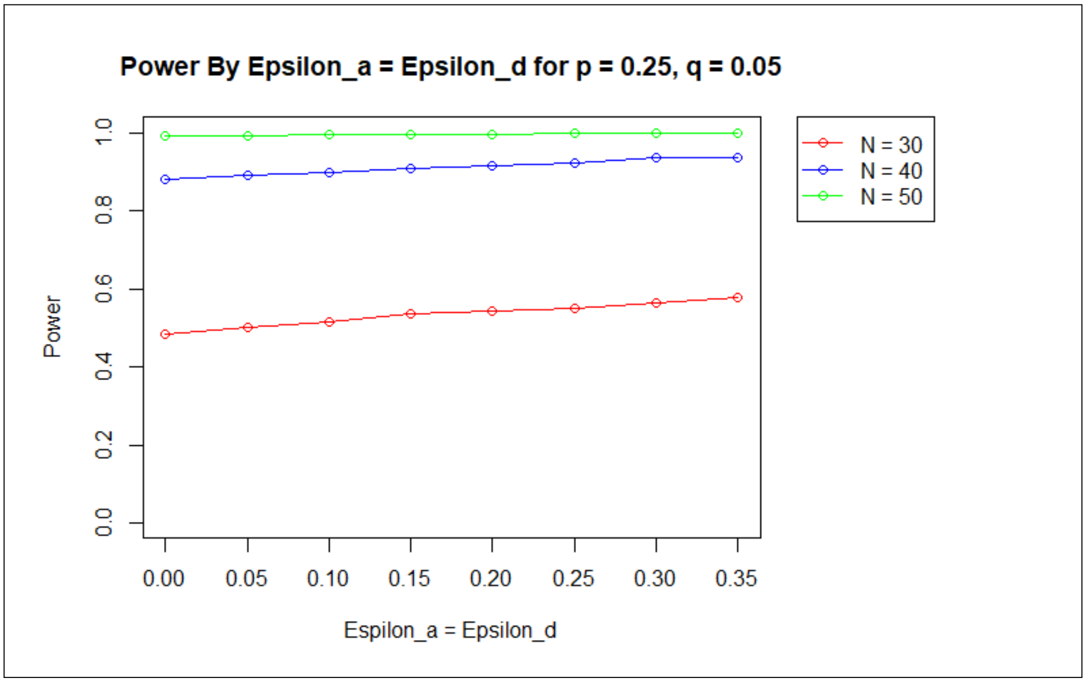


Figure A.39. Plot of Power for $p = 0.25$, $q = 0.05$, $\epsilon_a = \epsilon_d$.

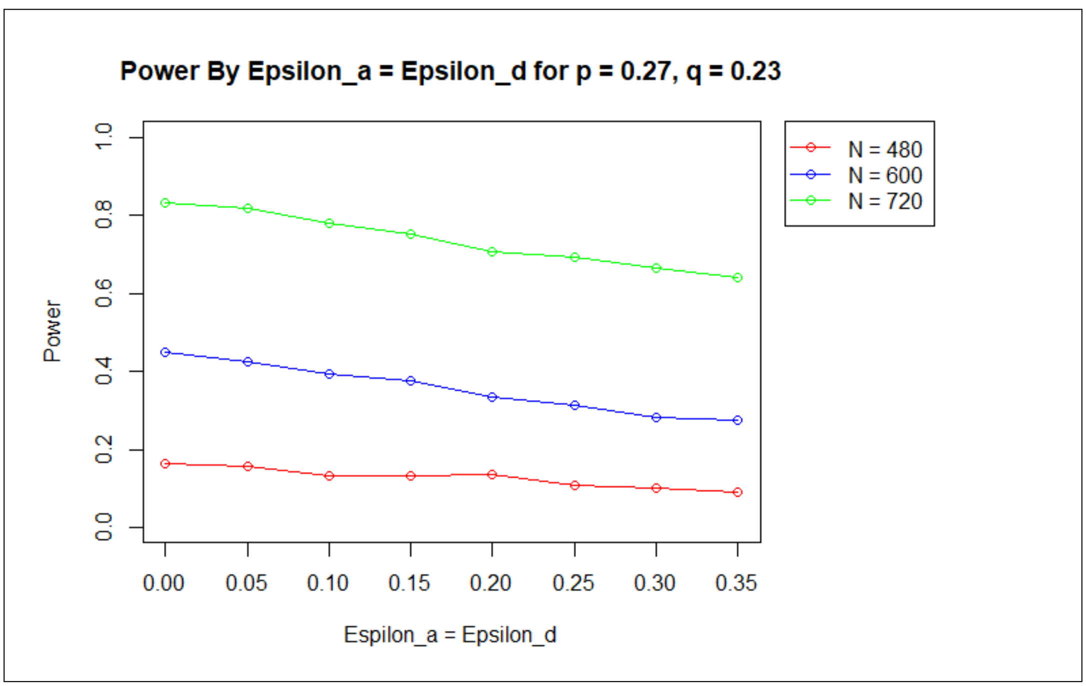


Figure A.40. Plot of Power for $p = 0.27$, $q = 0.23$, $\epsilon_a = \epsilon_d$.

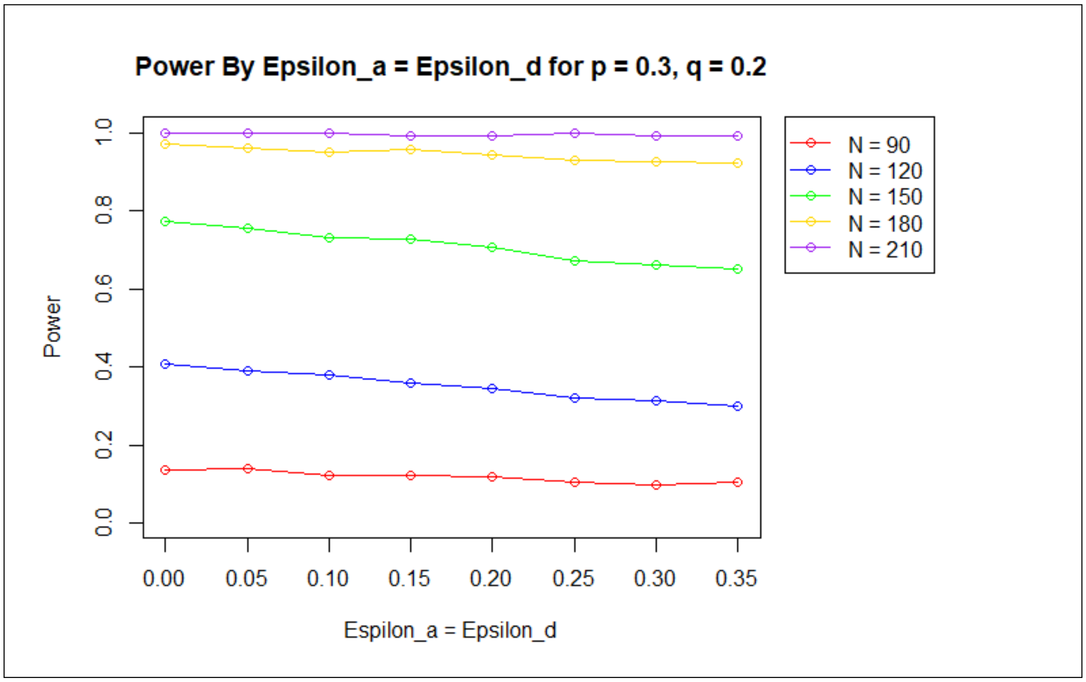


Figure A.41. Plot of Power for $p = 0.30$, $q = 0.20$, $\epsilon_a = \epsilon_d$.

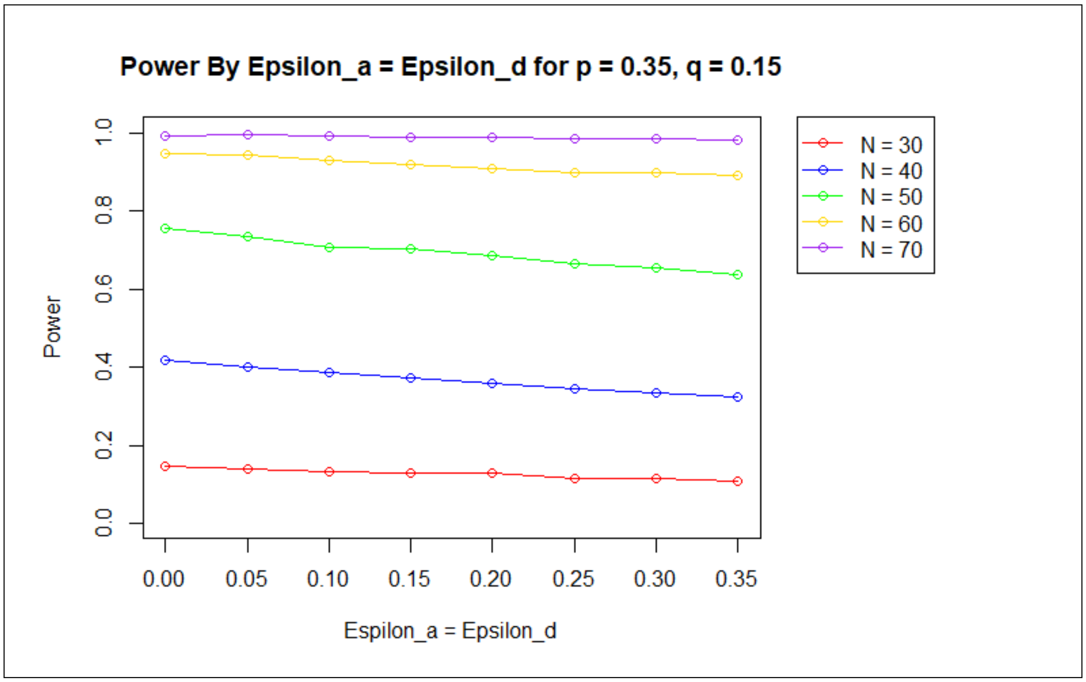


Figure A.42. Plot of Power for $p = 0.35$, $q = 0.15$, $\epsilon_a = \epsilon_d$.