SOME RESULTS ON SEMICROSSED PRODUCTS AND RELATED OPERATOR ALGEBRAS

A Dissertation
Submitted to the Graduate Faculty
of the
North Dakota State University
of Agriculture and Applied Science

By

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In Partial Fulfillment of the Requirements
for the Degree of
DOCTOR OF PHILOSOPHY

Major Department:
Mathematics

November 2021

Fargo, North Dakota

NORTH DAKOTA STATE UNIVERSITY

Graduate School

Title

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The supervisory committee certifies that this dissertation complies with North Dakota State University's regulations and meets the accepted standards for the degree of

DOCTOR OF PHILOSOPHY

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ABSTRACT

We investigate various properties of two classes of operator algebras: directed graph operator algebras and semicrossed products. First we consider analytic structure in the form of derivations and point derivations on these algebras. Our two main results describe the structure of derivations on graph operator algebras and point derivations on semicrossed product operator algebras. We then investigate multivariate semicrossed products and the maps on the associated, underlying compact Hausdorff space. We consider potential generalizations of classical 1-dimensional variants and look for which of our multivariate analogs have nice structure with a proposed invariant for multivariate dynamical systems. We close by developing a component-wise look at the maximal C^*-algebra of the $n \times n$ matrices, the simplest of the direct graph operator algebras. This is the first concrete example of a maximal C^*-algebra since the one example that accompanied the definition in the original paper about maximal C^*-envelopes.

ACKNOWLEDGEMENTS

I wish to thank my advisor, Benton Duncan, for his assistance and encouragement in completing this dissertation. It has been a fun math adventure.

I'd like to thank Dr. Dogan Comez for providing valuable feedback to improve the exposition of this dissertation. I also want to recognize the other committee members Nikita Barabanov and Jennifer Momsen.

Finally, I want to acknowledge the support of my family (both past and present) and friends over the years.

DEDICATION

Dedicated to my cat.

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1. INTRODUCTION AND PRELIMINARIES

In the study of operator algebras there are three main types of algebras: nonselfadjoint algebras, C^*-algebras, and von Neumann algebras. The latter two have an extensive literature, but nonselfadoint operator algebras are not as fully developed. One of the key factors in this lack of development is the dearth of classical examples to investigate. C^*-algebras and von Neumann algebras are connected to classically defined objects: continuous functions on compact Hausdorff spaces, and essentially bounded measurable functions on a finite measure space, respectively. The classical analogs of nonselfadjoint algebras do not necessarily lead directly to natural operator algebra analogues.

There have been two approaches, then, to nonselfadjoint operator algebras. On the one hand, looking at specific concrete examples and identifying commonalities between them drives much of the work. An alternative approach has been to, for a nonselfadjoint operator algebra, consider the C^*-algebra it generates and try to intuit properties on the nonselfadjoint operator algebra from the associated C^*-algebra. Both of these approaches present their own difficulties.

In this work we focus on two classes of nonselfadjoint operator algebras: semicrossed products and graph operator algebras. These two classes contain many of the examples studied in the literature and, in addition, have some overlap. In addition, there is enough structure in these examples to provide interesting avenues of study. After going through preliminary definitions, this work breaks out into three areas of focus which we outline now.

A classical nonselfadjoint operator algebra is the disk algebra. One property that the disk algebra has that is not true of classical C^*-algebras is an analytic structure. In fact, the derivative at a point in the interior of the disk is something that exists for the disk algebra but is not present for continuous functions on the unit disk. Hence, we use an abstraction of the derivative at a point, to consider analytic structure for semicrossed products. This extends work of [15], investigating analytic structure for one particular type of semicrossed product and work of [15] investigating point derivations of directed graph operator algebras. Here we investigate derivations on directed graph operator algebras, and then look at point derivations for certain representations of semicrossed products. We also spend time looking at the overlap, considering what types of graph operator algebras give rise to semicrossed products.

The next part of this disquisition focuses on a generalization of semicrossed products from [9]. The classical semicrossed product, defined in [20], begins with a single map acting on a compact Hausdorff space. Unlike the C^*-algebra context which typically requires that the map be a homeomorphism semicrossed products can be defined for more general continuous functions, see [9] for a survey of these algebras. The classical semicrossed product is generated by the continuous functions on the given compact Hausdorff space together with an isometry that implements the action of the continuous function. In the generalized context, Davidson and Katsoulis considered the algebra generated by more than one map. As their work was focused on the associated algebra, they did not spend much time investigating the dynamical-system type properties of multiple maps acting on a compact Hausdorff space.

We begin the process of filling this gap by considering three important properties in the classical case: periodicity, transitivity, and topological transitivity. It is not clear how to directly translate these properties into the multivariate setting so we consider, for each, multiple variants of the definitions. We consider the relationships between our different definitions and illustrate with examples the limits of our definitions. In the one variable case, two dynamical systems that are conjugate have the same properties. One of the advantages of the nonselfadjoint algebra is that topological conjugacy is an isomorphism invariant between the associated algebras. In [23], the author proposes that partition conjugacy is the right invariant for a multivariate dynamical system. Their intuition is driven by the desire to make an invariant that, mimicking the single variable case, is an isomorphism invariant. Here, taking partition conjugacy as the right invariant, we consider our notions of periodicity, transitivity, and topological transitivity with respect to this invariant.

In the last section, we return to the roots of nonselfadjoint operator algebras: trying to understand how a nonselfadjoint operator algebra sits inside a C^*-algebra. Given a single algebra, there are, in fact, many ways to embed the algebra into a C^*-algebra. Much of the classical approach has been to look for the smallest C^*-algebra that contains the nonselfadjoint operator algebra. While this smallest C^*-algebra has the benefit that it is often computable, it lacks any real connection with the underlying algebra. In [3], the author proposed an alternate enveloping C^*-algebra which encoded all of the representation of the underlying nonselfadjoint operator algebra. He called this the maximal C^*-algebra of an operator algebra, and in that paper he described the maximal C^*-algebra for the simplest nonselfadjoint operator algebra, the upper triangular 2×2 matrices.

While the maximal C^*-algebra has begun to be used in studying operator algebras [5], finding concrete representations of the maximal C^*-algebra for an operator algebra has not extended beyond the initial example. Recently, in as yet unpublished work [15] developed a framework for extending the Blecher example to upper triangular $n \times n$ matrices as well as one specific type of semicrossed products. Here we go through some of the calculations underlying that approach, elucidating this heretofore mysterious construction.

1.1. Introduction to Operator Algebras

In this section we provide some background definitions to put the results of this disquisition into context. We will be studying a particular type of operator algebra, called a semicrossed product. We start with some background material to provide context, and then we will develop some algebraic and analytic properties for semicrossed products.

Definition 1. A *vector space* X over a field $\mathbb{F}$ is a space with two operations, addition $+ : X \times X \to X$ and multiplication $\cdot : \mathbb{F} \times X \to X$, that satisfy the following:

1. $x + y = y + x$ for all $x, y \in X$
2. $(x + y) + z = x + (y + z)$ for all $x, y, z \in X$
3. there exists $0 \in X$ such that $x + 0 = x$ for all $x \in X$
4. for every $x \in X$, there exists $-x \in X$ such that $x + (-x) = 0$
5. there exists $1 \in \mathbb{F}$ such that $1 \cdot x = x$ for all $x \in X$
6. $a \cdot (x + y) = a \cdot x + a \cdot y$ for all $a \in \mathbb{F}$ and $x, y \in X$
7. $(a + b) \cdot x = a \cdot x + b \cdot x$ for all $a, b \in \mathbb{F}$ and $x \in X$
8. $a \cdot (b \cdot x) = (ab) \cdot x$ for every $a, b \in \mathbb{F}$ and $x \in X$.

Definition 2. A vector space X becomes a *normed vector space* when it is equipped with a function $\|\cdot\| : X \to [0, \infty)$, called a *norm*, that satisfies the following:

1. $\|x\| = 0$ if and only if $x = 0$
2. $\|x + y\| \leq \|x\| + \|y\|$ for all $x, y \in X$
3. $\|ax\| = |a|\|x\|$ for all $a \in \mathbb{F}$ and $x \in X$.

Definition 3. A *Banach space* is a normed vector space that is complete with respect to the norm.

Definition 4. A *Banach algebra* $\mathcal{A}$ is a Banach space equipped with multiplication from $\mathcal{A} \times \mathcal{A}$ into $\mathcal{A}$ satisfying

1. $a(bc) = (ab)c$ for all $a, b, c \in \mathcal{A}$
2. $(a + b)c = ac + bc$ for all $a, b, c \in \mathcal{A}$
3. $a(b + c) = ab + ac$ for all $a, b, c \in \mathcal{A}$
4. $\lambda(ab) = (\lambda a)b = a(\lambda b)$ for all $a, b \in \mathcal{A}$ and $\lambda \in \mathbb{F}$

such that the norm satisfies $\|ab\| \leq \|a\|\|b\|$ for all $a, b \in \mathcal{A}$.

Example 1. Let X be a compact Hausdorff space. The set $C(X)$ of all continuous, complex-valued functions on X is a Banach algebra. The algebra multiplication is defined pointwise, and the norm is the supremum norm $\|f\|_\infty = \sup_{x \in X} \{|f(x)|\}$.

Example 2. Let $B(\mathcal{H})$ denote the set of all bounded linear operators on a Hilbert space $\mathcal{H}$. Define addition pointwise and multiplication as composition ($fg = g \circ f$ for all $f, g \in B(\mathcal{H})$). Equipping $B(\mathcal{H})$ with norm $\|f\| = \sup_{\|h\|=1} \{\|f(h)\|\}$ makes $B(\mathcal{H})$ a Banach algebra.

When $\mathcal{H} = \mathbb{C}^n$, $B(\mathcal{H})$ is the set of $n \times n$ matrices with complex entries which we denote $M_n(\mathbb{C})$.

For more on these Example 1 and Example 2, we refer to Section 5.1 of [18].

Definition 5. A *concrete operator algebra* is a closed subalgebra $B(\mathcal{H})$ of some Hilbert space $\mathcal{H}$.

We refer to Chapter 2 of [3] for an overview of operator algebras.

Example 3. The subset of $M_n(\mathbb{C})$ of upper triangular matrices, denoted $T_n(\mathbb{C})$, is an operator algebra acting on $\mathbb{C}^n$.

There is an abstract characterization of an operator algebra. We start by considering a Banach algebra A. Given such an algebra there is a natural norm on the algebra of $n \times n$ matrices with entries from A. See [19] for more detail.

Definition 6. An *operator algebra* A is a Banach algebra A together with matricial norms $\|\cdot\|_n$ on $M_n(A)$ such that

$$\max\{\|X\|_n, \|Y\|_m\} \leq \left\| \begin{bmatrix} X & 0 \\ 0 & Y \end{bmatrix} \right\|_{m,n}$$

for all $X \in M_n(A)$ and $Y \in M_m(A)$ and so that $\|X \cdot Y\|_m \leq \|X\|_m \|Y\|_m$ for all $X, Y \in M_m(A)$.

When talking about operator algebras, one considers natural maps between operator algebras. If A, B are operator algebras, given any homomorphism $\pi : A \to B$, there is, for any n an induced homomorphism $\pi_n : M_n(A) \to M_n(B)$.

Definition 7. Given operator algebras A and B, we say that $\pi : A \to B$ is a *representation* if π is a homomorphism. We say it is faithful if the map is one-to-one. If π_n is an isometry for every n we say that π is a complete isometry and if π_n is a contraction for every n we say that π is completely contractive.

In practical terms, it is typically much easier to verify that something is an operator algebra by constructing a Hilbert space on which it acts, or by embedding it into a known algebra, rather than verifying facts about the matricial norms.

Example 4. If you consider the operator algebra $C(D)$ where D is the closed unit disk in the complex plane then the disk algebra is the subalgebra of $C(D)$ consisting of those functions which are analytic in the open unit disk. The notation for this algebra is $A(\mathbb{D})$ [10].

Definition 8. An *involution* on a Banach algebra A is a map on A, notated $a \mapsto a^*$, that satisfies

1. $(a^*)^* = a$

2. $(ab)^* = b^* a^*$

3. $(\lambda a + b)^* = \overline{\lambda} a^* + b^*$

for all $a, b \in A$ and $\lambda \in \mathbb{C}$.

Definition 9. A C^*-algebra is a Banach algebra with involution that satisfies the norm $\|a^*a\| = \|a\|^2$ for all $a \in A$. A homomorphism that preserves the adjoint is called a $*$-homomorphism.

The next two examples can be found in Section 5.1 of [18].

Example 5. Let X be a compact Hausdorff space. We used in Example 1 that $C(X)$ is a Banach algebra. $C(X)$ becomes a C^*-algebra when an involution is defined by $f^* = \overline{f}$.

Example 6. For any Hilbert space $\mathcal{H}$, $B(\mathcal{H})$ becomes a C^*-algebra when involution is defined by the usual adjoint operator.

A *concrete C^*-algebra* is a norm closed self-adjoint subalgebra of $B(\mathcal{H})$ for some Hilbert space $\mathcal{H}$. It is a well known result (Gelfand-Naimark Theorem, see Theorem I.9.12 of [6]) that every abstract C^*-algebra A is isometrically $*$-isomorphic to a concrete C^*-algebra of operators.

An important concept for nonselfadjoint operator algebras is that there may be many ways to embed the operator algebra into bounded operators on a Hilbert space, for details of the following see [3]. For example, by construction $A(\mathbb{D})$ embeds into $C(\mathbb{D})$. One can show that when you consider the unit circle $\mathbb{T}$ inside $\mathbb{D}$, there is an onto $*$-homomorphism $\pi : C(\mathbb{D}) \to C(\mathbb{T})$ given by restriction. Interestingly, $\pi|_{A(\mathbb{D})}$ is a completely isometric isomorphism (because of the maximum modulus theorem for analytic functions), so it embeds in a smaller C^*-algebra. Similarly, given any contractive operator T in a Hilbert space, there is a completely contractive homomorphism $\pi : A(\mathbb{D}) \to C^*(T)$, and hence $A(\mathbb{D})$ can sit inside larger, non-commutative operator algebras. Understanding how an operator algebra sits inside a C^*-algebra becomes an important question.

Definition 10. A *C^*-cover* of an operator algebra A is a C^*-algebra C and a complete isometry $\pi : A \to C$ such that $i(A)$ generates C as a C^*-algebra. The smallest such cover is called the *C^*-envelope*, denoted $C_e^*(A)$, and the largest is called the *maximal C^*-algebra for A*, denoted $C_{max}^*(A)$.

The existence of the C^*-envelope [9] and the maximal C^*-algebra [2] are important results in the theory of nonselfadjoint operator algebras.

An important fact is that there is one known computed example of the maximal C^*-algebra of an operator algebra. In [3] it was established that for the upper triangular 2×2 matrices, T_2,

$$C^*_{max}(T_2) \quad = \quad \{f \in M_2\left(C([0,1])\right) : f(0) \text{ is a diagonal scalar matrix}\}$$

We now provide the background for one of the important algebras we are going to study.

Definition 11. A *topological dynamical system* is a pair (X, σ) consisting of a topological space X and a continuous map σ acting on X.

In what follows we will always be dealing with topological dynamical systems, so we will drop the adjective topological. In addition, unless stated otherwise, we will assume that X is a compact Hausdorff space.

Definition 12. A C^**-dynamical system* is a triple $(A, \mathcal{G}, \sigma)$ consisting of a C^*-algebra A, a locally compact group $\mathcal{G}$, and homomorphism $\sigma : \mathcal{G} \to Aut(A)$.

Definition 13. A *covariant representation* of a C^*-dynamical system $(A, \mathcal{G}, \sigma)$ is a pair of representations (π, ρ) on a Hilbert space $\mathcal{H}$ consisting of a *-representation $\pi : A \to \mathcal{H}$ and unitary $\rho : \mathcal{G} \to \mathcal{H}$ that satisfies

$$\rho(g)\pi(a)\rho(g)^* = \pi\left(\alpha(g)(a)\right)$$

for all $a \in A$ and $g \in \mathcal{G}$.

Definition 14. Let (A, σ) be a dynamical system, and assume that $\mathcal{G}$ is a countable discrete group. Equip the polynomial algebra of continuously compactly supported A-valued functions on $\mathcal{G}$

$$\mathcal{Q}(A,t) \quad = \quad \left\{ \sum_{\substack{1 \le i \le n \\ g_i \in \mathcal{G}}} a_{g_i} t_{g_i} : a_{g_i} \in A \right\}$$

with multiplication $t_g a t_g^{-1} = \alpha(t_g)(a)$, and define the norm

$$\left\| \sum_{\substack{1 \leq i \leq n \\ g_i \in \mathcal{G}}} a_{g_i} t_{g_i} \right\| = \sup_{(\pi,\rho) \text{ covariant}} \left\| \sum_{\substack{1 \leq i \leq n \\ g_i \in \mathcal{G}}} \pi\left(a_{g_i}\right) \rho\left(t_{g_i}\right) \right\|$$

The crossed product $A \rtimes_\sigma \mathcal{G}$ is the completion of $\mathcal{Q}(A, t)$ with respect to the norm.

Remark 1. Definitions 12, 13, and 14 are found in [6]. Refer to Chapter VIII of [6] for more in depth information about this algebra.

2. FINITE DIRECTED GRAPHS

In this chapter we explore derivations on and representations of finite directed graphs.

2.1. Regular Representations of Semigroups

Here, we introduce the directed graph operator algebras and investigate point derivations on these algebras. We then describe the form of a derivation for a directed graph operator algebra, establishing that there exists a nontrivial derivation on a graph algebra, a question left open in [12].

We first recall some facts on semigroups.

Definition 15. A *semigroup* S is a non-empty set with binary operation $*$ that is associative, i.e., $a * (b * c) = (a * b) * c$ for all $a, b, c \in S$.

Definition 16. Let X be a set. A *transformation* of X is a single-valued mapping from X into itself. The *full transformation semigroup* of X, denoted $\mathcal{T}_X$, is the set of all transformations of X.

Remark 2. Section 1.1 of [4] gives an overview of semigroups and transformations. In notation, for $x \in X$ and $\alpha \in \mathcal{T}_X$, we denote α applied to x as αx. $\mathcal{T}_X$ is a semigroup with composition given by $(\beta * \alpha)x = \beta(\alpha x)$. The associativity is a simple check of moving symbols:

$$(\gamma * (\beta * \alpha))\, x = \gamma\left((\beta * \alpha)\, x\right) = \gamma\left(\beta\left(\alpha x\right)\right) = (\gamma * \beta)\left(\alpha x\right) = ((\gamma * \beta) * \alpha)\, x$$

Example 7. Let $X = \{a, b, c\}$. $\mathcal{T}_X$ has 27 elements: in mapping, each element has three possible destinations, and combining the three possible mappings of each element yields 27 possibilities. Denote these transformations as α_i, $i = 1, \cdots, 27$. The full list is seen in Figure 2.1.

Definition 17. The *right regular representation* of a semigroup $\mathcal{S}$ is a mapping that takes each $s \in \mathcal{S}$ to $\rho_s \in \mathcal{T}_{\mathcal{S}}$ given by $\rho_s(t) = ts$. Similarly, the *left regular representation* of a semigroup $\mathcal{S}$ is a mapping that takes each $s \in \mathcal{S}$ to $\lambda_s \in \mathcal{T}_{\mathcal{S}}$ given by $\lambda_s(t) = st$.

Remark 3. Section 1.3 of [4] gives an overview of these important transformations. We just point out a few of the facts. We drop the binary operation notation $*$ when talking of $\mathcal{S}$. Of note is $\rho_{s_1 s_2} = \rho_{s_2}\rho_{s_1}$ and $\lambda_{s_1 s_2} = \lambda_{s_2}\lambda_{s_1}$ for every $s_1, s_2 \in \mathcal{S}$ via the associative property of semigroups.

	a	b	c
α_1	a	a	a
α_2	a	a	b
α_3	a	a	c
α_4	a	b	a
α_5	a	b	b
α_6	a	b	c
α_7	a	c	a
α_8	a	c	b
α_9	a	c	c

	a	b	c
α_{10}	b	a	a
α_{11}	b	a	b
α_{12}	b	a	c
α_{13}	b	b	a
α_{14}	b	b	b
α_{15}	b	b	c
α_{16}	b	c	a
α_{17}	b	c	b
α_{18}	b	c	c

	a	b	c
α_{19}	c	a	a
α_{20}	c	a	b
α_{21}	c	a	c
α_{22}	c	b	a
α_{23}	c	b	b
α_{24}	c	b	c
α_{25}	c	c	a
α_{26}	c	c	b
α_{27}	c	c	c

Figure 2.1. All 27 elements of $\mathcal{T}_X$, where $X = \{a, b, c\}$, is shown.

Additionally, observe that the right regular representation is faithful if and only if $\mathcal{S}$ is left reductive (i.e., for every $s, t, x \in \mathcal{S}$, if $xs = xt$, then $s = t$). Similarly, the left regular representation is faithful if and only if $\mathcal{S}$ is right reductive.

Let X be a set, and let $E = \{e\}$ be a group consisting only of the identity. Let E^0 be E with an added 0 element, and let $M_X(E^0)$ be the set of $X \times X$ matrices with elements in E^0. We consider $V_X \subseteq M_X\left(E^0\right)$ as the set of matrices in which each row has exactly one element as e.

Definition 18. The *natural isomorphism* will be the mapping of $\alpha \in \mathcal{T}_X$ to $V(\alpha) = [v_{xy}(\alpha)] \in V_X$, which e in its $x - y$ entry if $\alpha(x) = y$ and 0 otherwise.

This definition leads to the following example (as found in Section 3.5 of [4]).

Example 8. Take a semigroup $\mathcal{S}$. For $s \in \mathcal{S}$, apply the natural isomorphism to the regular representations ρ_s, λ_s to obtain matrices $R(s) = [r_{xy}(s)]$ and $L(s) = [\ell_{xy}(s)]$ in $M_{\mathcal{S}}\left(E^0\right)$ given by

$$r_{xy}(s) \quad = \quad \begin{cases} e & \text{if } xs = y \\ 0 & \text{otherwise} \end{cases}$$

and

$$\ell_{xy}(s) \quad = \quad \begin{cases} e & \text{if } sx = y \\ 0 & \text{otherwise.} \end{cases}$$

We refer to $R(s)$ and $L(s)$ as the regular representations on the space.

We let $P = \prod_{a \in \mathcal{S}} \{0, e\}$ with projection maps p_a. For $t \in \mathcal{S}$, define $\xi_t \in P$ with $p_a(\xi_t) = e$ if $a = t$ and $p_a(\xi_t) = 0$ otherwise. We can think of $\{\xi_t\}_{t \in \mathcal{S}}$ as vectors on which $R(s)$ and $L(s)$ can be applied. Take any $y, s \in \mathcal{S}$. We have

$$\begin{aligned} R(s)\xi_y &= \left[r_{ab}(s)\right]_{a,b \in \mathcal{S}} \left[p_c(\xi_y)\right]_{c \in \mathcal{S}} \\ &= \left[\sum_{c \in \mathcal{S}} r_{ac}(s) p_c(\xi_y)\right]_{a \in \mathcal{S}} \\ &= \left[r_{ay}(s)\right]_{a \in \mathcal{S}}. \end{aligned}$$

Hence, $R(s)\xi_y = 0$ unless there is $x \in \mathcal{S}$ such that $xs = y$. If $xs = y$, then $r_{ay}(x) = e$ only when $a = x$, so $R(s)\xi_y = \xi_x$.

Similarly, if there is $x \in \mathcal{S}$ with $sx = y$, then $L(s)\xi_y = \xi_x$. $L(s)\xi_y = 0$ if no such x exists.

Definition 19. Suppose $\mathcal{G}$ is a group with identity e and X is a set. If there is a map $H : \mathcal{G} \times X \to X : (g, x) \mapsto g \cdot x$ so that $H(e, x) = x$ and $H(g_1, H(g_2, x)) = H(g_1 g_2, x)$ for every $g_1, g_2 \in \mathcal{G}$ and $x \in X$, we say that $\mathcal{G}$ *acts* on the left of X.

Remark 4. Suppose $\mathcal{G}$ is a topological group and X is a topological space, and assume H from Definition 19 is continuous. The pair (G, X) is called a *transformation group*. We refer to Section 2.1 of [25].

2.2. Directed Graphs

We first remind the reader of some basic facts about directed graphs.

Definition 20. A *graph* G is a set of vertices $V(G)$ and a set of edges $E(G)$ which represent pairs of vertices that they connect. The graph becomes *directed* if it is equipped with a range map r and a source map s that maps $E(G)$ into $V(G)$ that gives direction to each edge. That is, for every $e \in E(G)$, $s(e) \in V(G)$ is vertex where the edge begins, and $r(e) \in V(G)$ is where the edge terminates. We use the notation $G = (V(G), E(G), r, s)$ to describe the graph.

One can extend the range and source maps to apply to vertices by setting $r(x) = x = s(x)$ for any vertex, which is helpful in notation.

Definition 21. A graph $G = (V(G), E(G), r, s)$ is *finite* if $V(G)$ and $E(G)$ are finite sets.

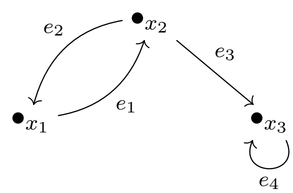
Definition 22. A *path* in G is either a vertex x or an object of the form $e_{i_m}e_{i_{m-1}}\cdots e_{i_1}$ with $e_i \in E(G)$ and $r(e_{i_p}) = s(e_{i_{p+1}})$.

Remark 5. For $e \in E(G)$, it is a common technique to write the path $e = r(e)es(e)$ to use in calculations, as seen below.

Definition 23. We let $\mathbb{F}^+(G)$ denote the set of finite paths in G. We make $\mathbb{F}^+(G)$ a semigroup by equipping it with the concatenation operation, noting if $w_1, w_2 \in \mathbb{F}^+(G)$, then $w_2w_1 \in \mathbb{F}^+(G)$ if and only if $r(w_1) = s(w_2)$. In the language of [17], $\mathbb{F}^+(G)$ is called the *path space of* G.

Chapter 1 of [22] is a standard introduction to these objects.

Example 9. Consider the following finite directed graph:



Here, for example, $s(e_2) = x_2$ and $r(e_2) = x_1$. We also have $e_3e_1 \in \mathbb{F}^+(G)$, but $e_1e_3 \notin \mathbb{F}^+(G)$.

We want to explore the algebra generated by these graphs. We let $\{\xi_w : w \in \mathbb{F}^+(G)\}$ be as defined in Example 8, and we consider the Hilbert space $\mathcal{H}_G = \ell^2(\mathbb{F}^+(G))$ of square summable sequences generated by $\{\xi_w : w \in \mathbb{F}^+(G)\}$. On this space, the creation operator $\lambda_G : \mathbb{F}^+(G) \to \mathcal{B}(\mathcal{H}_G)$ is given by

$$\lambda_G(w)\xi_v = \begin{cases} \xi_{wv}, & \text{if } r(v) = s(w) \\ 0, & \text{otherwise.} \end{cases}$$

and its adjoint is given by

$$\lambda_G(w)^*\xi_v = \begin{cases} \xi_u, & \text{if } v = wu' \text{ for some } u \in \mathbb{F}^+(G) \\ 0, & \text{otherwise.} \end{cases}$$

It is clear that for $w_2w_1 \in \mathbb{F}^+(G)$ we can see that $\lambda_G(w_2w_1) = \lambda_G(w_2)\lambda_G(w_1)$ and $\lambda_G(w_2w_1)^* = \lambda_G(w_2)^*\lambda_G(w_1)^*$. In the spirit of Example 8, we say that λ_G is the left regular representation on $\mathbb{F}^+(G)$.

We will use $L_x = \lambda_G(x)$ for $x \in V(G)$ and $L_e = \lambda_G(e)$ for $e \in E(G)$ to denote these operators. Notice that $L_{r(e)}L_e = L_e = L_eL_{s(e)}$ for any e.

Definition 24. The *free semigroupoid algebra generated by* G, denoted $\mathfrak{L}(G)$, is the closure under the WOT-topology of the operators L_e and L_x.

$\mathfrak{L}(G)$ is studied in depth in [17] and [16].

From Remark 4.3 in [17], we note that every $a \in \mathfrak{L}(G)$ has unique Fourier expansion

$$a \sim \sum_{w\in\mathbb{F}^+(G)} a_w\lambda_G(w).$$

We will be working with the norm closed algebra but the result about Fourier expansion works in this context as well, as long as we consider the Fourier sequence as converging in the strong operator topology to the element of the algebra. In particular the norm-closed algebra $A(G) \subset \mathfrak{L}(G)$. In our notation we will write this similarity as an equality to minimize confusion.

For our purposes in later calculations, we need the following notation

$$A_{i,j} = \left\{w \in \mathbb{F}^+(G) : s(w) = x_j,\ r(w) = x_i\right\}$$

in order to subdivide a Fourier series.

2.3. Derivations

We introduce a type of function that we will explore using the algebras we're studying.

Definition 25. A *derivation* on an algebra $\mathcal{A}$ is a linear function D on $\mathcal{A}$ so that for every $a, b \in \mathcal{A}$,

$$D(ab) = aD(b) + D(a)b.$$

Example 10. The usual derivative is a derivation on $C^\infty(\mathbb{R})$.

Remark 6. If $\mathcal{A}$ is an algebra with generators $\{g_\lambda\}$ and D is a continuous derivation on the set of generators, then D extends linearly to a derivation on $\mathcal{A}$. That is, if taking any generators g_{λ_1} and g_{λ_2} gives $D\left(g_{\lambda_1} g_{\lambda_2}\right) = g_{\lambda_1} D\left(g_{\lambda_2}\right) + D\left(g_{\lambda_1}\right) g_{\lambda_2}$, then D is a derivation on $\mathcal{A}$.

Definition 26. Suppose π is a representation of an algebra $\mathcal{A}$ acting as an operator on a Hilbert space $\mathcal{H}$. A continuous linear function $D : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ is a *point derivation at* π if for every $a, b \in \mathcal{A}$,

$$D(ab) = \pi(a)D(b) + D(a)\pi(b).$$

We will see examples of point derivations involving directed graphs and semicrossed products in later sections.

Proposition 1. *If $\mathcal{A}$ has an identity* 1*, then $D(1) = 0$ for any derivation D.*

Proof. As $1 = 1 \cdot 1$, $D(1) = D(1) + D(1)$ from the definition of a derivation. □

Example 11. Take any $u \in \mathcal{A}$ for an algebra $\mathcal{A}$. Then $D_u(a) = ua - au$ is a derivation. To see this, take any $a, b \in \mathcal{A}$ and observe that

$$\begin{aligned} D_u(ab) &= uab - abu \\ &= aub - aub + uab - abu \\ &= aub - abu + uab - aub \\ &= a(ub - bu) + (ua - au)b \\ &= aD_u(b) + D_u(a)b \end{aligned}$$

Definition 27. A derivation D on an algebra $\mathcal{A}$ is called an *inner derivation* if there exists $u \in \mathcal{A}$ such that $D(a) = ua - au$ for all $a \in \mathcal{A}$.

Proposition 2. *If an algebra $\mathcal{A}$ is commutative, then there are no nontrivial inner derivations.*

Proof. Suppose D is an inner derivation on $\mathcal{A}$, and take $u \in \mathcal{A}$ so that $D(a) = ua - au$ for all $a \in \mathcal{A}$. Then $D(a) = au - au = 0$. □

Example 12. Let π be a representation of and algebra $\mathcal{A}$ on a Hilbert space $\mathcal{H}$, and fix $U \in \mathcal{B}(\mathcal{H})$. The function $D : \mathcal{A} \to \mathcal{B}(\mathcal{H})$ given by $D(a) = U\pi(a) - \pi(a)U$ is a point derivation at π as

$$\begin{aligned} D(ab) &= U\pi(a)\pi(b) - \pi(a)\pi(b)U \\ &= \pi(a)U\pi(b) - \pi(a)\pi(b)U + U\pi(a)\pi(b) - \pi(a)U\pi(b) \\ &= \pi(a)(U\pi(b) - \pi(b)U) + (U\pi(a) - \pi(a)U)\pi(b) \\ &= \pi(a)D(b) + D(a)b \end{aligned}$$

for every $a, b \in \mathcal{A}$.

Definition 28. A point derivation D at π is said to be *inner* if there is $U \in \mathcal{B}(\mathcal{H})$ such that $D(a) = U\pi(a) - \pi(a)U$ for every $a \in \mathcal{A}$.

Proposition 3. *Suppose an algebra $\mathcal{A}$ has generators $\{g_\lambda\}_{\lambda\in\Lambda}$ and representation π. If a point derivation D at π is inner on the generators, then D is inner.*

Proof. Suppose there is $U \in \mathcal{B}(\mathcal{H})$ such that $D(g_\lambda) = U\pi(g_\lambda) - \pi(g_\lambda)U$ for every $\lambda \in \Lambda$. Suppose for some $n \geq 1$,

$$D(g_{\lambda_n} \cdots g_{\lambda_1}) = U\pi(g_{\lambda_n} \cdots g_{\lambda_1}) - \pi(g_{\lambda_n} \cdots g_{\lambda_1})U$$

Then

$$\begin{aligned} D(g_{\lambda_{n+1}} \cdots g_{\lambda_1}) &= \pi(g_{\lambda_{n+1}})D(g_{\lambda_n} \cdots g_{\lambda_1}) + D(g_{\lambda_{n+1}})\pi(g_{\lambda_n} \cdots g_{\lambda_1}) \\ &= \pi(g_{\lambda_{n+1}})U\pi(g_{\lambda_n} \cdots g_{\lambda_1}) - \pi(g_{\lambda_{n+1}})\pi(g_{\lambda_n} \cdots g_{\lambda_1})U \\ &\quad + U\pi(g_{\lambda_{n+1}})\pi(g_{\lambda_n} \cdots g_{\lambda_1}) - \pi(g_{\lambda_{n+1}})U\pi(g_{\lambda_n} \cdots g_{\lambda_1}) \\ &= U\pi(g_{\lambda_{n+1}} \cdots g_{\lambda_1}) - \pi(g_{\lambda_{n+1}} \cdots g_{\lambda_1})U. \end{aligned}$$

Calculations for showing sums of monomials with variables from $\{g_\lambda\}_{\lambda\in\Lambda}$ will be inner when D is applied are similar. □

Example 13. Let $\mathcal{A}$ be an algebra generated by two elements a and b. Define D on $\mathcal{A}$ with $D(a) = ab - ba$ and $D(b) = ba - ab$, extending linearly. Using Proposition 3, it is a quick check to show that D is a derivation on $\mathcal{A}$.

Let $U = -a - b$. As $D(a) = Ua - aU$ and $D(b) = Ub - bU$, we apply Proposition 3 to show that D is inner.

2.4. Derivations on Finite Directed Graphs

Take any directed graph $G = (V(G), E(G), r, s)$; say that $V(G) = \{x_k\}_{k \in K}$ and $E(G) = \{e_\ell\}_{\ell \in L}$, and let D be an arbitrary derivation on $A(G)$. Our goal is to say what form D must take for arbitrary elements of $A(G)$.

Proposition 4. *For $x_k \in V(G)$ there exist a_w^k such that,*

$$D\left(L_{x_k}\right) = \sum_{\substack{j \in K \\ j \neq k}} \sum_{w \in A_{k,j}} a_w^k \lambda_G(w) + \sum_{\substack{i \in K \\ i \neq k}} \sum_{w \in A_{i,k}} a_w^k \lambda_G(w).$$

Moreover, for $i, j \in K$ with $i \neq j$, $a_w^j = -a_w^i$ for $w \in A_{i,j}$.

Proof. Since $D(L_x)$ is in $A(G)$, we let $\{a_w^k\}_{w \in \mathbb{F}^+(G)}$ be the Fourier coefficients of L_{x_k}, where $x_k \in V(G)$. Then for $k \in K$,

$$D\left(L_{x_k}\right) = \sum_{i,j \in K} \sum_{w \in A_{i,j}} a_w^k \lambda_G(w).$$

Take $k_1, k_2 \in K$; observe that

$$\begin{aligned} D\left(L_{x_{k_1}} L_{x_{k_2}}\right) &= L_{x_{k_1}} D\left(L_{x_{k_2}}\right) + D\left(L_{x_{k_1}}\right) L_{x_{k_2}} \\ &= \sum_{j \in K} \sum_{w \in A_{k_1,j}} a_w^{k_2} \lambda_G(w) + \sum_{i \in K} \sum_{w \in A_{i,k_2}} a_w^{k_1} \lambda_G(w) \end{aligned} \tag{2.1}$$

If $k_1 = k_2 =: k$, equation 2.1 and the fact that L_{x_k} is idempotent force $a_w^k = 0$ if $w \in A_{k,k}$ and

$$D\left(L_{x_k}\right) = \sum_{\substack{j \in K \\ j \neq k}} \sum_{w \in A_{k_j}} a_w^k \lambda_G(w) + \sum_{\substack{i \in K \\ i \neq k}} \sum_{w \in A_{i,k}} a_w^k \lambda_G(w).$$

In the case where $k_1 \neq k_2$,

$$0 = \sum_{w \in A_{k_1,k_2}} \left(a_w^{k_2} + a_w^{k_1} \right) \lambda_G(w)$$

since $L_{x_{k_1}} L_{x_{k_2}} = 0$. $\square$

Proposition 5. *There is $U \in A(G)$ such that for every $x_k \in V(G)$,*

$$D\left(L_{x_k}\right) = U L_{x_k} - L_{x_k} U.$$

Proof. Define $U = \sum_{w \in \mathbb{F}^+(G)} u_w \lambda_G(w)$ by $u_w = -a_w^i$ for $w \in A_{i,j}$ with $i \neq j$. Choose $u_w = 0$ for $w \in A_{i,i}$.

Building from equation 2.1 and Proposition 4 itself, we obtain

$$\begin{aligned}
D\left(L_{x_k}\right) &= \sum_{\substack{j \in K \\ j \neq k}} \sum_{w \in A_{k,j}} a_w^k \lambda_G(w) + \sum_{\substack{i \in K \\ i \neq k}} \sum_{w \in A_{i,k}} a_w^k \lambda_G(w) \\
&= \sum_{\substack{j \in K \\ j \neq k}} \sum_{w \in A_{k,j}} a_w^k \lambda_G(w) + \sum_{\substack{i \in K \\ i \neq k}} \sum_{w \in A_{i,k}} \left(-a_w^i\right) \lambda_G(w) \\
&= \sum_{\substack{i \in K \\ i \neq k}} \sum_{w \in A_{i,k}} \left(-a_w^i\right) \lambda_G(w) - \sum_{\substack{j \in K \\ j \neq k}} \sum_{w \in A_{k,j}} \left(-a_w^k\right) \lambda_G(w) \\
&= \sum_{i \in K} \sum_{w \in A_{i,k}} u_w \lambda_G(w) - \sum_{j \in K} \sum_{w \in A_{k,j}} u_w \lambda_G(w) \\
&= U L_{x_k} - L_{x_k} U,
\end{aligned}$$

as desired. $\square$

Proposition 6. *For $e \in E(G)$, $D(L_e) = D_1(L_e) + D_2(L_e)$, where $D_1(L_e) = UL_e - L_e U$ and $U \in A(G)$ is as in Proposition 5, and D_2 is a (not necessarily inner) derivation.*

Proof. For $e \in E(G)$, say that $s(e) = x_\ell$ and $r(e) = x_k$. Let $\{b_w^e\}$ be the Fourier coefficients of $D(L_e)$. Then Proposition 4 yields

$$\begin{aligned} D(L_e) &= D\left(L_{x_k}\right) L_e + L_{x_k} D(L_e) L_{x_\ell} + L_e D(L_{x_\ell}) \\ &= \sum_{\substack{i \in K \\ i \neq k}} \sum_{w \in A_{i,k}} a_w^k \lambda_G(we) + \sum_{w \in A_{k,\ell}} b_w^e \lambda_G(w) + \sum_{\substack{j \in K \\ j \neq \ell}} \sum_{w \in A_{\ell,j}} a_w^\ell \lambda_G(ew). \end{aligned}$$

Set

$$\begin{aligned} D_1(L_e) &= \sum_{\substack{i \in K \\ i \neq k}} \sum_{w \in A_{i,k}} a_w^k \lambda_G(we) + \sum_{\substack{j \in K \\ j \neq \ell}} \sum_{w \in A_{\ell,j}} a_w^\ell \lambda_G(ew) \\ D_2(L_e) &= \sum_{w \in A_{k,\ell}} b_w^e \lambda_G(w). \end{aligned}$$

Notice that

$$\begin{aligned} UL_e - L_e U &= \sum_{i \in K} \sum_{w \in A_{i,k}} u_w \lambda_G(we) - \sum_{j \in K} \sum_{w \in A_{\ell,j}} u_w \lambda_G(ew) \\ &= \sum_{\substack{i \in K \\ i \neq k}} \sum_{w \in A_{i,k}} \left(-a_w^i\right) \lambda_G(we) - \sum_{\substack{j \in K \\ j \neq \ell}} \sum_{w \in A_{\ell,j}} \left(-a_w^\ell\right) \lambda_G(ew) \\ &= \sum_{\substack{i \in K \\ i \neq k}} \sum_{w \in A_{i,k}} a_w^k \lambda_G(we) + \sum_{\substack{j \in K \\ j \neq \ell}} \sum_{w \in A_{\ell,j}} a_w^\ell \lambda_G(ew) \\ &= D_1\left(L_e\right), \end{aligned}$$

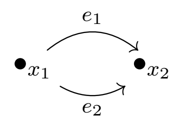
which is the desired result. □

Proposition 7. *If D is continuous, $D = D_1 + D_2$, where D_1 and D_2 are derivations and D_1 is inner.*

Proof. For $x \in V(G)$, we set $D_1(L_{x_k}) = D(L_{x_k})$, where D_1 is as in Proposition 4, and we set $D_2(L_{x_k}) = 0$. For $e \in E(G)$, we use D_1 and D_2 as they are in Proposition 6. Applying Proposition 3 gives the result. □

It is not true in general that derivations on a directed graph are inner, as the next example shows.

Example 14. Consider the following graph G:



Define D on $\mathfrak{L}(G)$ as follows:

$$\begin{aligned} D\left(L_{x_1}\right) &= L_{e_2} - L_{e_1} \\ D\left(L_{x_2}\right) &= L_{e_1} - L_{e_2} \\ D\left(L_{e_1}\right) &= L_{e_1} + L_{e_2} \\ D\left(L_{e_2}\right) &= L_{e_1} + L_{e_2} \end{aligned}$$

Then Propositions 4 and 6 are satisfied. Since $\mathbb{F}^+(G)$ is finite, D can be extended to a continuous function on $\mathfrak{L}(G)$, which by Proposition 7 is a derivation on the algebra.

Take $U = u_{x_1}L_{x_1} + u_{x_2}L_{x_2} - L_{e_1} + L_{e_2}$. Such a U satisfies Proposition 5. However, notice that

$$\begin{aligned} UL_{e_1} - L_{e_1}U &= (u_{x_2} - u_{x_1})L_{e_1} \\ UL_{e_2} - L_{e_2}U &= (u_{x_2} - u_{x_1})L_{e_2}. \end{aligned}$$

Hence, D is not inner.

3. SEMICROSSED PRODUCTS

3.1. Semicrossed Products

In this chapter we introduce the semicrossed products and relate them to the graph operator algebras of the previous section. We look at derivations in general, and then we consider point derivations for a specific representation of a semicrossed product. Parts of the content of this chapter appeared in [11].

The crossed product is a much studied object. We refer to Chapter VIII of [6], Chapter 2 of [25], and II.10.3 of [1] for the full construction. Semicrossed products are constructed similary. In a crossed product, the *covariant system* is a triple $(A, \mathcal{G}, \alpha)$, where $\mathcal{G}$ is a locally compact group acting on an operator algebra A by a homomorphism $\alpha : \mathcal{G} \to Aut(A)$. A *covariant representation* of a covariant system $(\mathcal{A}, \mathcal{G}, \alpha)$ is a pair of representations (π, U) that satisfies the covariance relation $U(g)\pi(a)U(g)^* = \pi(\alpha(g)(a))$, where $\pi : A \to B(\mathcal{H})$ is a non-degenerate representation, $U : \mathcal{G} \to B(\mathcal{H})$ is a unitary operator, and $\mathcal{H}$ is some Hilbert space.

In a semicrossed product, we utilize semigroups instead of groups. Semigroups were first studied in [20], and we remind the reader of this setting. A *semigroup dynamical system* is a triple $(A, \mathcal{S}, \alpha)$, where A is an operator algebra, $\mathcal{S}$ is a semigroup, and α is a completely contractive endomorphism of A. A *semigroup dynamical system* is called a *C^*-dynamical system* if A is a C^*-algebra. A pair (π, V) is called a *isometric covariant representation*, where $\pi : A \to B(\mathcal{H})$, $V : A \to B(\mathcal{H})$ is an isometry, and $\mathcal{H}$ is some Hilbert space. The *covariant relation* can be given by $V(s)\pi(a) = \pi(\alpha(g)(a))V(s)$ or $\pi(a)V(s) = V(s)\pi(\alpha(g)(a))$. Since the first choice forces $\ker\alpha \subseteq \ker\pi$, the relation used in [20] and other sources is given by $\pi(a)V(s) = V(s)\pi(\alpha(g)(a))$.

From (π, V), we have a representation of $(A, \mathcal{S}, \alpha)$ into $\ell^1(\mathbb{Z}^+, A, \alpha)$. The seminorm of $\ell^1(\mathbb{Z}^+, A, \alpha)$ is obtained by taking the supremum of all pairs (π, V), denoted $\mathcal{F}$. The *semicrossed product of $\mathcal{A}$ by $\mathcal{S}$ with respect to $\mathcal{F}$*, denoted $\mathcal{A} \times_\alpha^{\mathcal{F}} \mathcal{S}$, is the completion of $\mathcal{P}(\mathcal{A}, T)/N$. For the computation of this seminorm and discussion of the completion, we refer to Section 2 of [8].

When discussing semigroups, we use standardized notation as seen in [21]. We consider a dynamical system (X, σ), where X is a compact Hausdorff space and σ is continuous. The semicrossed product is the completion of the algebra generated by $C(X)$ and symbol S, where $fS = S(f \circ \sigma)$ for all $f \in C(X)$. Elements of this algebra are thought of as series $\sum S^n f_n$.

3.2. Finite Graphs Related to Semicrossed Products

Before we investigate derivations on semicrossed products we look at some relations between semicrossed product algebras and directed graph operator algebras.

Let $G = (V(G), E(G), r, s)$ be a finite directed graph with the property that each vertex is the source of exactly m ($m \in \mathbb{N}$) edges. Let $n := |V(G)|$, noting that $|E(G)| = nm$. We let $e_{i,j} \in E(G)$ ($j = 1, \cdots, m$) be such that $s(e_{i,j}) = v_i$, i.e., $\{e_{i,j}\}_{j=1}^m$ is the set of edges with source v_i.

Let $X = \{x_i\}_{i=1}^n$ be endowed with the discrete topology, so X is a compact Hausdorff space. Define $\rho : V(G) \to \{1, \cdots, n\}$ by $\rho(v_i) = i$, and for $j \in \{1, \cdots, m\}$, define $\sigma_j : X \to X$ by $\sigma_j(x_i) = x_{\rho(r(e_{i,j}))}$.

Define π as follows:

$$\begin{aligned} \pi\left(L_{v_i}\right) &= \chi_{\{x_i\}} \\ \pi\left(\sum_{i=1}^{n} L_{e_{i,j}}\right) &= S_j \end{aligned}$$

where χ is the characteristic function on X. Extend π linearly over $\mathfrak{L}(G)$.

Notice also that every $f \in C(X)$ can be written as

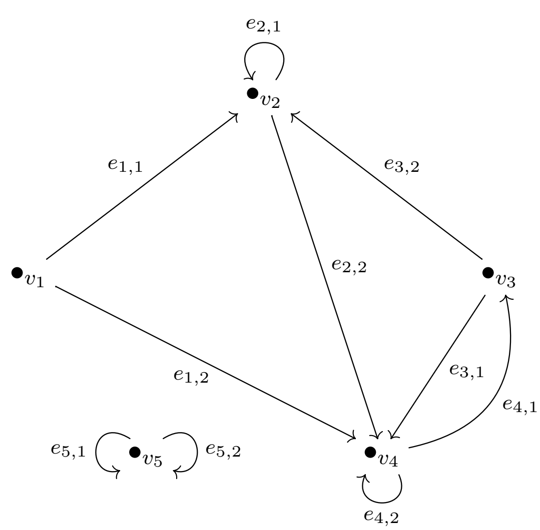
$$\begin{aligned} f &= \sum_{i=1}^{n} f\left(x_i\right) \chi_{\{x_i\}} \\ &= \sum_{i=1}^{n} f\left(x_i\right) \pi\left(L_{v_i}\right), \end{aligned}$$

and the covariance relation is satisfied:

$$\begin{aligned}
fS_j &= \left(\sum_{i=1}^{n} f\left(x_i\right)\pi\left(L_{v_i}\right)\right)\left(\sum_{k=1}^{n}\pi\left(L_{e_{k,j}}\right)\right)\\
&= \sum_{i=1}^{n} f\left(x_{\rho(r(e_{i,j}))}\right)\pi\left(L_{r(e_{i,j})}L_{e_{i,j}}\right)\\
&= \sum_{i=1}^{n} f\left(\sigma_j(x_i)\right)\pi\left(L_{e_{i,j}}\right)\\
&= \sum_{i=1}^{n} \pi\left(L_{e_{i,j}}\right) f\left(\sigma_j(x_i)\right)\pi\left(L_{v_i}\right)\\
&= \left(\sum_{k=1}^{n}\pi\left(L_{e_{k,j}}\right)\right)\left(\sum_{i=1}^{n} f\left(\sigma_j(x_i)\right)\pi\left(L_{v_i}\right)\right)\\
&= S_j(f\circ\sigma_j).
\end{aligned}$$

Hence, we can represent the given graph as a semicrossed product.

Example 15. Consider the following graph G:



Here, $m = 2$ and $n = 5$. We have $X = \{x_1, \cdots, x_5\}$, as well as $\pi(L_{v_i}) = \chi_{x_i}$ and $S_j = \sum_{i=1}^{5} L_{e_{i,j}}$. Renumbering the edges out of a vertex or changing the range of an edge will give us a different semicrossed product to work with.

3.3. Derivations on Semicrossed Products

The semicrossed product does not have a structure that is conducive to studying derivations on it directly since continuity makes the question too complex. Our goal is to develop a representation of a semicrossed product that has certain conditions and to study the resulting point derivation.

In the rest of this chapter, we let X be a compact Hausdorff space with maps $\sigma = \{\sigma_\ell\}_{\ell=1}^m$ acting on it, and $S_1, \cdots, S_m$ are contractions on a Hilbert space that satisfy the covariance relation $fS_\ell = S_\ell(f \circ \sigma_\ell)$ for every $\ell \in \{1, \cdots, m\} =: M$. Our semigroup is the free semigroup generated by M. Let $\mathcal{A}$ denote the semicrossed product generated by $C(X)$ and $S_1, \cdots, S_m$, and let $\mathcal{W}$ be the set of finite words from the alphabet M.

The following example is of a derivation on a semicrossed product. We introduce this example to demonstrate that derivations on semicrossed products are not necessarily inner. As seen later on, the representation that we construct will have point derivations that may be inner.

Example 16. Let $m = 2$, and let σ_ℓ, S_ℓ, and $\mathcal{A}$ be as above. We take some $\alpha_1, \alpha_2 \in C(X)$, and define D on $\mathcal{A}$ by

$$\begin{aligned} D(f) &= S_1(f - f \circ \sigma_1) + S_2(f - f \circ \sigma_2) \\ D(S_1) &= S_1\alpha_1 - S_1S_2 + S_2S_1 \\ D(S_2) &= S_2\alpha_2 + S_1S_2 - S_2S_1. \end{aligned}$$

It is routine to check that for any $f, g \in C(X)$,

$$D(fg) = fD(g) + D(f)g,$$

as well as

$$\begin{aligned} D(fS_1) &= D(S_1(f \circ \sigma_1)) \\ D(fS_2) &= D\left(S_2(f \circ \sigma_2)\right). \end{aligned}$$

Thus, if D is continuous, D is a derivation on $\mathcal{A}$, as noted by Remark 6.

We introduce X and maps to show that this derivation is not necessarily inner. Let $X = \{x_1, x_2, x_3\}$. Define $\sigma_1, \sigma_2, \alpha_1, \alpha_2$ as follows:

	σ_1	σ_2	α_1	α_2
x_1	x_3	x_2	1	0
x_2	x_2	x_3	0	1
x_3	x_2	x_2	0	1

Suppose for a contradiction that D is inner, so we have

$$u = u_\emptyset + S_1 u_1 + S_2 u_2 + S_1 S_1 u_{11} + S_2 S_1 u_{21} + S_1 S_2 u_{12} + S_2 S_2 u_{22} + \cdots \in \mathcal{A}$$

such that $D(a) = ua - au$ for every $a \in \mathcal{A}$. In particular, $D(S_\ell) = S_\ell \left(u_\emptyset \circ \sigma_\ell - u_\emptyset + \cdots\right)$. Equate this with the known form of $D(S_\ell)$ in this example. Using $\ell = 1, 2$, we obtain a system of six equations to determine $u_\emptyset$. It is routine to check that this system has no solution. Therefore, no such u exists, and D is not inner.

Example 17. Modifying the setting of Example 16, we instead let α_1 and α_2 be the zero functions. Then D is inner with $u = S_1 + S_2$, provided D is continuous.

We will no longer work with derivations on $\mathcal{A}$ itself. Rather, we want to study point derivations. As we will see in the remainder of the chapter, a point derivation on $\mathcal{A}$ will have a nice form.

Remark 7. In [11], this was studied with $m = 1$. We will build the multivariate version of what was studied in that paper. Theorem 2 of the paper concluded that point derivations at the constructed representation were all inner derivations. In what follows, we will show that this conclusion has an analogue in the multivariate case.

3.4. Notation and Setting

For our problem, we need to introduce some matricial notation. As usual, $M_n(\mathbb{C})$ denotes the set of $n \times n$ matrices with complex entries. We let $0_n \in M_n(\mathbb{C})$ be the matrix with all zero entries and I_n be the identity in $M_n(\mathbb{C})$. We denote $[a_{i,j}] \in M_n(\mathbb{C})$ as the matrix with $i - j$ entry $a_{i,j}$. If there is a subscript $P(i, j)$ on $[a_{i,j}]_{P(i,j)}$, we set $a_{i,j} = 0$ if $P(i, j)$ is false and $a_{i,j}$ as stated if

$P(i,j)$ is true. In $P(i,j)$, we use "*" on indices to denote that we use all indices in the "*" position that make $P(i,j)$ true. Note that this does not mean $a_{i,} \neq 0$ if $P(i,j)$ is true.

For example, in the following matrix

$$[a_{i,j}]_{i<j} = \begin{pmatrix} 0 & a_{1,2} & a_{1,3} & \cdots & a_{1,n-1} & a_{1,n} \\ 0 & 0 & a_{2,3} & \cdots & a_{2,n-1} & a_{2,n} \\ \vdots & \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 0 & a_{n-1,n} \\ 0 & 0 & 0 & \cdots & 0 & 0 \end{pmatrix}$$

the notation $i < j$ means the matrix is populated with entries where the $i-j$ entry has $i < j$. Entries where $i \geq j$ are equal to zero.

For another example of this notation, suppose $\{w_1, \cdots, w_5\} = \{1, \emptyset, 2, 12, 21\}$, noting that $w_1 = 1w_2$ and $w_4 = 1w_3$. In the matrix,

$$[a_{i,j}]_{\substack{w_i = 1w_* \\ * \neq j}} = \begin{pmatrix} a_{1,1} & 0 & a_{1,3} & a_{1,4} & a_{1,5} \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \\ a_{4,1} & a_{4,2} & 0 & a_{4,4} & a_{4,5} \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix},$$

the nonzero entries are in the $i-j$ entries where w_i is equal to $1w_*$ for some $* \in \{1,2\}$ and $* \neq j$. Since $w_1 = 1w_2$, we populate row 1, excluding column 2. Since $w_4 = 1w_3$, we populate row 4, excluding column 3.

For $w = i_p \cdots i_2 i_1 \in \mathcal{W}$, we use the notation

$$\begin{aligned} \sigma_w &= \sigma_{i_p} \circ \cdots \circ \sigma_{i_2} \circ \sigma_{i_1} \\ S_w &= S_{i_p} \cdots S_{i_1}. \end{aligned}$$

If w is the empty word, σ_w and S_w are the identities on their respective spaces.

In our problem, we fix $x \in X$ and $n \in \mathbb{N}$, and choose a set $\{w_1, \cdots, w_n\} \subset \mathcal{W}$ such that $\sigma_{w_i}(x) = \sigma_{w_j}(x)$ if and only if $i = j$. Let $N = \{1, \cdots, n\}$, and to save space, we will let x_i denote $\sigma_{w_i}(x)$. Let $N_x = \{x_i\}_{i=1}^n$.

Remark 8. For the rest of this chapter, it will only be relevant that each x_i of N_x is distinct. We will drop the notation $\sigma_{w_i}(x)$ and use only x_i.

3.5. A Representation π on $\mathcal{A}$

In Section 4 of [15], they developed a representation for a semigroup $\mathcal{S}$ given by

$$\begin{aligned} \pi_{x,\gamma}(f)\xi_s &= f(\sigma_s(x))\,\xi_s \\ \pi_{x,\gamma}(S_t)\xi_s &= \gamma(t)\xi_{ts}, \end{aligned}$$

where $\{\xi_s\}$ is as in Example 8, $x \in X$ is fixed, and γ is in the dual group of $\mathcal{G} = \mathcal{S} - \mathcal{S}$

We slightly modify this representation so that we work with finite matrices. We define $\pi : \mathcal{A} \to M_n(\mathbb{C})$ on $C(X)$ by

$$\pi(f) = [f(x_i)]_{i=j}$$

for all $f \in C(X)$, where $\{x_i\}_{i=1}^n$ is as above. Once we know where each S_ℓ is sent, we can extend π to all of $\mathcal{A}$.

Lemma 1. *For any $\ell \in M$,*

$$\pi(S_\ell) = \left[b_{i,j}^\ell\right]_{x_i = \sigma_\ell(x_j)}$$

for some fixed $\{b_{i,j}^\ell\}_{x_i=\sigma_\ell(x_j)} \subset \mathbb{C}$.

Proof. We know that $\pi(S_\ell) = \left[b_{i,j}^\ell\right]$ is fixed. For any $f \in C(X)$,

$$\pi(fS_\ell) = \left[f(x_i) \cdot b_{i,j}^\ell\right]$$

and

$$\pi\left(S_\ell(f\circ\sigma_\ell)\right) \quad = \quad \left[f\left(\sigma_\ell(x_j)\right)\cdot b_{i,j}^\ell\right]$$

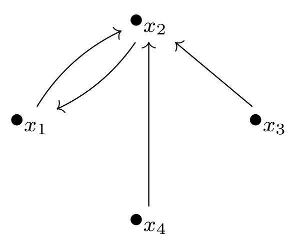
Using the covariance relation, we have for every i,j,

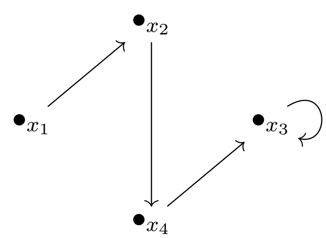
$$0 \quad = \quad \left(f(x_i) - f\left(\sigma_\ell(x_j)\right)\right)\cdot b_{i,j}^\ell.$$

As X is compact Hausdorff, we require $x_i = \sigma_\ell(x_j)$ in order for $b_{i,j}^\ell$ to be non-zero. □

Remark 9. Take $\ell \in M$. If we construct a directed graph with N_x as the vertices and the edges determined by σ_ℓ in the obvious way, then $\pi\left(S_\ell\right)$ is a generalized adjacency matrix of the directed graph.

Example 18. Let $X = \{x_i\}_{i=1}^4$, and consider σ_1 (left) and σ_2 (right) on X as follows:





We let S_ℓ ($\ell = 1,2$) be contractions that satisfy the covariance relation, and let $\mathcal{A}$ be the semicrossed product as usual. By choice, $\pi : \mathcal{A} \to M_3(\mathbb{C})$ is given for $f \in C(X)$ by

$$\pi(f) \quad = \quad \begin{pmatrix} f(x_1) & 0 & 0 \\ 0 & f(x_2) & 0 \\ 0 & 0 & f(x_3) \end{pmatrix}$$

For notation, $\pi\left(S_\ell\right) = \left[b_{i,j}^\ell\right]$. By Lemma 1, we know that

$$\pi\left(S_1\right) \quad = \quad \begin{pmatrix} 0 & b_{1,2}^1 & 0 \\ b_{2,1}^1 & 0 & b_{2,3}^1 \\ 0 & 0 & 0 \end{pmatrix}$$

This is as expected since within $\{x_1, x_2, x_3\}$, we have one step paths from x_1 to x_2, x_2 to x_1, and x_3 to x_2, which gives the 2-1, 1-2, and 2-3 entries, respectively, when we think of this in the perspective of an adjacency matrix.

Similarly, since under σ_2, we have one step paths x_1 to x_2 and x_3 to itself, and hence, as expected, Lemma 1 yields

$$\pi(S_2) = \begin{pmatrix} 0 & 0 & 0 \\ b^2_{2,1} & 0 & 0 \\ 0 & 0 & b^2_{3,3} \end{pmatrix}$$

It should be noted that π is not onto as the 3-1 and 3-2 entries will always be zero when multipling different combinations of $\pi(S_1)$ and $\pi(S_2)$. Thinking in terms of adjacency matrices, this means that there is no path via σ_1 and σ_2 from x_1 or x_2 to x_3 when we look at paths that stay strictly within the set $\{x_1, x_2, x_3\}$.

In practice, we may want π to be surjective.

Proposition 8. *The representation π is onto only if every element of N_x has a path to every other element in N_x using only the set itself and the maps $\{\sigma_\ell\}$.*

Proof. Take $w = \ell_p \cdots \ell_1 \in \mathcal{W}$. Observe that

$$\begin{aligned} \pi(S_{\ell_2})\pi(S_{\ell_1}) &= \left[b^{\ell_2}_{i,j}\right]_{x_i=\sigma_{\ell_2}(x_j)} \left[b^{\ell_1}_{i,j}\right]_{x_i=\sigma_{\ell_1}(x_j)} \\ &= \left[\sum_{\substack{1\le k\le n \\ x_i=\sigma_{\ell_2}(x_k) \\ x_k=\sigma_{\ell_1}(x_j)}}^{n} b^{\ell_2}_{i,k} b^{\ell_1}_{k,j} \right] \\ &= \left[b^{\ell_2}_{i,*_1} b^{\ell_1}_{*_1,j}\right]_{\substack{x_i=\sigma_{\ell_2}(x_{*_1}) \\ x_{*_1}=\sigma_{\ell_1}(x_j)}} \end{aligned}$$

In our notation, this means that for the $i-j$ position to be nonzero, $x_i = \sigma_{\ell_2\ell_1}(x_j)$ and $x_{*_1} = \sigma_{\ell_1}(x_j) \in N_x$.

Continue multiplying on the left by $\pi(S_{\ell_k})$ to obtain

$$\pi(S_w) = \left[b^{\ell_k}_{i,*_{k-1}} b^{\ell_{k-1}}_{*_{k-1},*_{k-2}} \cdots b^{\ell_2}_{*_2,*_1} b^{\ell_1}_{*_1,j}\right]_{\substack{x_i=\sigma_{\ell_p}(x_{*_{p-1}}) \\ x_{*_{p-1}}=\sigma_{\ell_{p-1}}(x_{*_{p-2}}) \\ \vdots \\ x_{*_2}=\sigma_{\ell_2}(x_{*_1}) \\ x_{*_1}=\sigma_{\ell_1}(x_j)}}.$$

In the $i-j$ position, there is a non-zero entry only if there is a path from x_j to x_i using elements of N_x via maps $\{\sigma_\ell\}$. □

3.6. Point Derivation at π

Let D be any point derivation at π. The goal is to show what form D must take.

Lemma 2. *There is fixed* $\{c_{i,j}\}_{\substack{1\leq i,j\leq n \\ i\neq j}} \subset \mathbb{C}$ *so that for every* $f \in C(X)$,

$$D(f) = [c_{i,j}(f(x_i) - f(x_j))]_{i\neq j}.$$

Proof. Observe that the entries of $D|_{C(X)}$ are bounded and linear, so by the Riesz representation theorem, for every $f \in C(X)$,

$$D(f) = \left[\int f \, d\mu_{i,j}\right]$$

for some unique (not necessarily positive) measures $\mu_{i,j}$. In particular, as D is a derivation, for every $f, g \in C(X)$,

$$\begin{aligned} D(fg) &= \pi(f)D(g) + D(f)\pi(g) \\ &= [f(x_i)]_{i=j}\left[\int g \, d\mu_{i,j}\right] + \left[\int f \, d\mu_{i,j}\right][g(x_i)]_{i=j}; \end{aligned}$$

hence,

$$\left[\int fg \, d\mu_{i,j}\right] = \left[f(x_i)\int g \, d\mu_{i,j} + g(x_j)\int f \, d\mu_{i,j}\right]. \tag{3.1}$$

Let $f = g$ be the function that sends every $x \in X$ to 1. By equation (3.1), $\mu_{i,j}(X) = 0$ for every i, j. Take measurable closed sets E_1 and E_2 in X, and let f and g be their respective characteristic functions. Via Urysohn's Lemma, we have continuous functions $\{f_n : X \to [0,1]\}_{n\in\mathbb{N}}$ and $\{g_n : X \to [0,1]\}_{n\in\mathbb{N}}$ that converge to f and g, respectively. Using the Dominated Convergence Theorem and equation (3.1), we see that

$$\mu_{i,j}\left(E_1 \cap E_2\right) \quad = \quad f\left(x_i\right) \cdot \mu_{i,j}\left(E_2\right) + g\left(x_j\right) \cdot \mu_{i,j}\left(E_1\right). \tag{3.2}$$

For any measurable $E \subseteq X$, equation 3.2 forces

$$\mu_{i,j}(E) \quad = \quad \left(f(x_i) + f(x_j)\right) \cdot \mu_{i,j}(E).$$

In order for E to have nonzero measure, either (i) $x_i \in E$ and $x_j \notin E$, or (ii) $x_j \in E$ and $x_i \notin E$.

Take sets E_1, E_2 that both satisfy (i). Using (3.2) again, we see that

$$\mu_{i,j}(E_1) = \mu_{i,j}(E_1 \cap E_2) = \mu_{i,j}(E_2),$$

so sets that satisfy (i) have the same measure. A similar result holds for two sets that both satisfy (ii). Considering $E_1 = \{x_i\}$ and $E_2 = \{x_j\}$, equation 3.2 shows that sets that satisfy (i) have measures of same magnitude but opposite signs as those that satisfy (ii).

If $i \neq j$, set $c_{i,j} := \mu_{i,j}(E)$, where E satisfies (i); this gives $\mu_{i,j} = c_{i,j}\left(\delta_{\{x_i\}} - \delta_{\{x_j\}}\right)$, where δ is the Dirac measure. □

Lemma 3. *For $\ell \in M$,*

$$\begin{aligned} D(S_\ell) \quad = \quad & \left[b^\ell_{i,*_1} c_{*_1,j} - c_{i,*_2} b^\ell_{*_2,j}\right]_{\substack{x_i=\sigma_\ell(x_{*_1}) \\ x_{*_2}=\sigma_\ell(x_j) \\ x_i\neq\sigma_\ell(x_j)}} + \left[b^\ell_{i,*} c_{\ell,j}\right]_{\substack{x_i=\sigma_\ell(x_*) \\ \sigma_\ell(x_j)\notin N_x}} \\ & - \left[c_{i,*} b^\ell_{*,j}\right]_{\substack{x_*=\sigma_\ell(x_j) \\ x_i\notin\sigma_\ell(N_x)}} + \left[a^\ell_{i,j}\right]_{x_i=\sigma_\ell(x_j)}, \end{aligned}$$

where $\left\{b^\ell_{i,j}\right\}$ is as in Lemma 1, $\{c_{i,j}\}$ is as in Lemma 2, and $\{a_{i,j}\} \subset \mathbb{C}$.

Proof. As a placeholder, we denote $D(S_\ell) = \left[a^\ell_{i,j}\right]$. We will use the following sets for $\ell \in M$:

$$\begin{aligned}
E^\ell_1 &= \{(i,j) \in N \times N : x_i = \sigma_\ell(x_j)\} \\
E^\ell_2 &= \{(i,j) \in N \times N : \sigma_\ell(x_j) \in N_x \text{ and } x_i \neq \sigma_\ell(x_j)\} \\
E^\ell_3 &= \{(i,j) \in N \times N : x_i \in \sigma_\ell(N_x) \text{ and } x_i \neq \sigma_\ell(x_j)\} \\
E^\ell_4 &= \{(i,j) \in N \times N : x_i \notin \sigma_\ell(N_x) \text{ and } \sigma_\ell(x_j) \notin N_x\}.
\end{aligned}$$

Observe that $\{E^\ell_1, E^\ell_2 \backslash E^\ell_3, E^\ell_3 \backslash E^\ell_2, E^\ell_2 \cap E^\ell_3, E^\ell_4\}$ gives a partition of $N \times N$ for every $\ell \in M$ and that the equation in the statement of the lemma can be written as

$$\begin{aligned}
D(S_\ell) &= \left[b^\ell_{i,*_1} c_{*_1,j} - c_{i,*_2} b^\ell_{*_2,j}\right]_{(i,j) \in E^\ell_2 \cap E^\ell_3} + \left[b^\ell_{i,*} c_{*,j}\right]_{(i,j) \in E^\ell_3 \backslash E^\ell_2} \\
&\quad - \left[c_{i,*} b^\ell_{*,j}\right]_{(i,j) \in E^\ell_2 \backslash E^\ell_3} + \left[a^\ell_{i,j}\right]_{(i,j) \in E^\ell_1}.
\end{aligned}$$

Take $f \in C(X)$ and $\ell \in M$. Notice that

$$\begin{aligned}
D(S_\ell f) &= \pi(S_\ell) D(f) + D(S_\ell) \pi(f) \\
&= \left[b^\ell_{i,j}\right]_{x_i = \sigma_\ell(x_j)} \left[c_{i,j} \left(f(x_i) - f(x_j)\right)\right]_{i \neq j} + \left[a^\ell_{i,j}\right] \left[f(x_i)\right]_{i=j} \\
&= \left[b^\ell_{i,*} c_{*,j} \left(f(x_*) - f(x_j)\right)\right]_{\substack{x_i = \sigma_\ell(x_*) \\ * \neq j}} + \left[a^\ell_{i,j} f(x_j)\right],
\end{aligned}$$

and in particular,

$$D(S_\ell(f \circ \sigma_\ell)) = \left[b^\ell_{i,*} c_{*,j} \left(f(\sigma_\ell(x_*)) - f(\sigma_\ell(x_j))\right)\right]_{\substack{x_i = \sigma_\ell(x_*) \\ * \neq j}} + \left[a^\ell_{i,j} f(\sigma_\ell(x_j))\right].$$

We also have

$$\begin{aligned}
D(f S_\ell) &= \pi(f) D(S_\ell) + D(f) \pi(S_\ell) \\
&= \left[f(x_i)\right]_{i=j} \left[a^\ell_{i,j}\right] + \left[c_{i,j} \left(f(x_i) - f(x_j)\right)\right]_{i \neq j} \left[b^\ell_{i,j}\right]_{x_i = \sigma_\ell(x_j)} \\
&= \left[a^\ell_{i,j} f(x_i)\right] + \left[c_{i,*} b^\ell_{*,j} \left(f(x_i) - f(x_*)\right)\right]_{\substack{x_* = \sigma_\ell(x_j) \\ i \neq *}}.
\end{aligned}$$

As $fS_\ell = S_\ell(f \circ \sigma_\ell)$, we have

$$
\begin{aligned}
0_n \;=\;& \left[b^\ell_{i,*}c_{*,j}\left(f(\sigma_\ell(x_*)) - f(\sigma_\ell(x_j))\right)\right]_{\substack{x_i=\sigma_\ell(x_*)\\ *\neq j}} \\
&- \left[c_{i,*}b^\ell_{*,j}\left(f(x_i) - f(x_*)\right)\right]_{\substack{x_*=\sigma_\ell(x_j)\\ i\neq *}} + \left[a^\ell_{i,j}\left(f\left(\sigma_\ell(x_j)\right) - f(x_i)\right)\right]
\end{aligned} \tag{3.3}
$$

This equation will be useful in determining most of the values of $\{a^\ell_{i,j}\}$.

If we take $(i,j) \in E^\ell_4$, equation (3.3) yields

$$0 \;=\; a^\ell_{i,j}\left(f\left(\sigma_\ell(x_j)\right) - f(x_i)\right).$$

Since $\sigma_\ell(x_j) \neq x_i$ and X is compact Hausdorff, we must have $a^\ell_{i,j} = 0$.

Now choose $(i,j) \in E^\ell_2 \cap E^\ell_3$. Again utilizing (3.3), we obtain $i_0, j_0 \in N$ with $i_0 \neq i$, $j_0 \neq j$, $w_{i_0} = \ell w_j$, and $w_i = \ell w_{j_0}$ so that

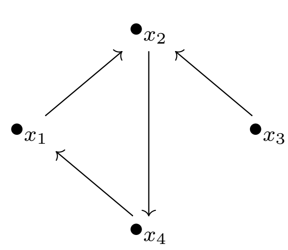
$$
\begin{aligned}
0 \;&=\; b^\ell_{i,j_0}c_{j_0,j}\left(f(x_i) - f(x_{i_0})\right) - c_{i,i_0}b^\ell_{i_0,j}\left(f(x_i) - f(x_{i_0})\right) + a^\ell_{i,j}\left(f(x_{i_0}) - f(x_i)\right) \\
&=\; \left(b^\ell_{i,j_0}c_{j_0,j} - c_{i,i_0}b^\ell_{i_0,j} - a^\ell_{i,j}\right)\left(f(x_i) - f(x_{i_0})\right).
\end{aligned}
$$

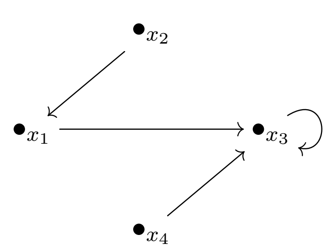
As $x_i \neq x_{i_0}$, we have $a^\ell_{i,j} = b^\ell_{i,j_0}c_{j_0,j} - c_{i,i_0}b^\ell_{i_0,j}$.

Similarly, $a^\ell_{i,j} = -c_{i,i_0}b^\ell_{i_0,j}$ for $(i,j) \in E^\ell_2 \backslash E^\ell_3$, and if $(i,j) \in E^\ell_3 \backslash E^\ell_2$, we obtain $a^\ell_{i,j} = b^\ell_{i,j_0}c_{j_0,j}$.

Finally, take $(i,j) \in E^\ell_1$; from (3.3), we get $0 = 0$, which gives us no known relation for $a^\ell_{i,j}$, so we leave these entries as unknown values $a^\ell_{i,j}$ in our form for $D(S_\ell)$. □

Example 19. Let $X = \{x_i\}_{i=1}^4$, and let σ_1 (left) and σ_2 (right) on X be as follows:





Let S_i $(i = 1, 2)$ and $\mathcal{A}$ be as usual.

Set $\pi : \mathcal{A} \to M_3(\mathbb{C})$ by

$$\pi(f) = \begin{pmatrix} f(x_1) & 0 & 0 \\ 0 & f(x_2) & 0 \\ 0 & 0 & f(x_3) \end{pmatrix}.$$

Lemma 3 tells us

$$D(S_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & b^1_{2,1}c_{1,2} + b^1_{2,3}c_{3,2} & 0 \\ 0 & 0 & 0 \end{pmatrix} - \begin{pmatrix} c_{1,2}b^1_{2,1} & 0 & c_{1,2}b^1_{2,3} \\ 0 & 0 & 0 \\ c_{3,2}b^1_{2,1} & 0 & c_{3,2}b^1_{2,3} \end{pmatrix} + \begin{pmatrix} 0 & 0 & 0 \\ a^1_{2,1} & 0 & a^1_{2,3} \\ 0 & 0 & 0 \end{pmatrix}$$

since $x_2 = \sigma_1(x_1)$ and $x_2 = \sigma_1(x_3)$, and

$$D(S_2) = \begin{pmatrix} b^2_{1,2}c_{2,1} & 0 & b^2_{1,2}c_{2,3} \\ 0 & 0 & 0 \\ 0 & b^2_{3,1}c_{1,2} + b^2_{3,3}c_{3,2} & 0 \end{pmatrix} - \begin{pmatrix} c_{1,3}b^2_{3,1} & 0 & c_{1,3}b^2_{3,3} \\ c_{2,3}b^2_{3,1} & c_{2,1}b^2_{1,2} & c_{2,3}b^2_{3,3} \\ 0 & c_{3,1}b^2_{1,2} & 0 \end{pmatrix}$$
$$+ \begin{pmatrix} 0 & a^2_{1,2} & 0 \\ 0 & 0 & 0 \\ a^2_{3,1} & 0 & a^2_{3,3} \end{pmatrix}.$$

since $x_1 = \sigma_2(x_2)$, $x_3 = \sigma_2(x_1)$, and $x_3 = \sigma_3(x_3)$.

Proposition 9. *There is a $U \in M_n(\mathbb{C})$ so that $D(f) = U\pi(f) - \pi(f)U$ for every $f \in C(X)$.*

Proof. Set $U = [u_{i,j}] \in M_n(\mathbb{C})$ by $u_{i,j} = -c_{i,j}$ if $i \neq j$. A simple calculation shows that

$$\begin{aligned} U\pi(f) - \pi(f)U &= [u_{i,j}][f(x_i)]_{i=j} - [f(x_i)]_{i=j}[u_{i,j}] \\ &= [u_{i,j}(f(x_j) - f(x_i))]_{i\neq j} \\ &= [c_{i,j}(f(x_i) - f(x_j))]_{i\neq j} \\ &= D(f), \end{aligned}$$

as desired. □

Proposition 10. *D is inner if and only if*

$$\left\{\left[a_{i,j}^{\ell}\right]_{x_i=\sigma_\ell(x_j)}\right\}_{\ell=1}^{m} = \left\{\left[b_{i,j}^{\ell}\left(u_{i,i}-u_{j,j}\right)\right]_{x_i=\sigma_\ell(x_j)}\right\}_{\ell=1}^{m}$$

has a solution for $\{u_{i,i}\}_{i=1}^{n}$.

Proof. Let U be as in Proposition 9. Observe that

$$\begin{aligned} U\pi(S_\ell)-\pi(S_\ell)U &= [u_{i,j}]\left[b_{i,j}^{\ell}\right]_{x_i=\sigma_\ell(x_j)} - \left[b_{i,j}^{\ell}\right]_{x_i=\sigma_\ell(x_j)}[u_{i,j}] \\ &= \left[u_{i,*}b_{*,j}^{\ell}\right]_{x_*=\sigma_\ell(x_j)} - \left[b_{i,*}^{\ell}u_{*,j}\right]_{x_i=\sigma_\ell(x_*)} \\ &= \left[u_{i,*}b_{*,j}^{\ell}\right]_{(i,j)\in E_2^\ell} + \left[u_{i,*}b_{*,j}^{\ell}\right]_{(i,j)\in E_1^\ell} - \left[b_{i,*}^{\ell}u_{*,j}\right]_{(i,j)\in E_3^\ell} - \left[b_{i,*}^{\ell}u_{*,j}\right]_{(i,j)\in E_4^\ell} \\ &= \left[b_{i,*}^{\ell}c_{*,j}\right]_{(i,j)\in E_3^\ell} - \left[c_{i,*}b_{*,j}^{\ell}\right]_{(i,j)\in E_2^\ell} + \left[b_{i,j}^{\ell}\left(u_{i,i}-u_{j,j}\right)\right]_{(i,j)\in E_1^\ell} \\ &= D(S_\ell) - \left[a_{i,j}^{\ell}\right]_{(i,j)\in E_1^\ell} + \left[b_{i,j}^{\ell}\left(u_{i,i}-u_{j,j}\right)\right]_{(i,j)\in E_1^\ell}. \end{aligned}$$

We need to find $\{u_{i,i}\}_{i=1}^{n}$ that works for every $\ell \in M$, so we require $a_{i,j}^{\ell} = b_{i,j}^{\ell}\left(u_{i,i}-u_{j,j}\right)$ for $(i,j) \in E_1^\ell$. □

Corollary 1. *If $m = 1$ in Proposition 10, then D is inner.*

4. MULTIVARIATE DYNAMICAL SYSTEMS

In this chapter we introduce multivariate dynamical systems, in analogue to the regular notion of dynamical system and we introduce the three important notions we wish to consider generalizations of: periodicity, transitivity, and topological transitivity. We then consider different variations of these in the multivariate context, looking at relations between our definitions and how they are impacted by the partition conjugacy invariant.

In Section 4.1, we will give classical definitions for the situation of a single map acting on a compact Hausdorff space. We are interested in dynamical systems (X, σ), where the map σ is a family of maps $\sigma = \{\sigma_i\}_{i \in I}$ acting on X. As further notation, $\mathbb{F}^+(I)$ denotes the set of finite words using elements of I as letters, and as before, when we have $w = i_k \cdots i_1 \in \mathbb{F}^+(I)$, the notation

$$\sigma_w = \sigma_{i_k} \circ \cdots \circ \sigma_{i_1}$$

is used. For $w \in \mathbb{F}^+(I)$ and $n \in \mathbb{N}$, we let u^n denote the word with n copies of u concatenated, and we use the convention that u^0 is the empty word.

We want to extend the definitions from Section 4.1 to the multivariate case. There is no universal agreement on how these definitions would carry over to a multivariate dynamical system. We will begin our discussion by looking at some possible definitions.

4.1. Dynamical Systems

In this section, the map σ in the dynamical system (X, σ) is a single map.

Definition 29. A point $x \in X$ is a *periodic point* for the system (X, σ) if there is $n \in \mathbb{N}$ so that $\sigma^n(x) = x$. The smallest such n that gives this is the *period* of x.

Example 20. Every $x \in \mathbb{R}/\mathbb{Z}$ is a periodic point for the rational shift σ on $\mathbb{R}/\mathbb{Z}$ given by $\sigma(x) = x + q$, where $q \in \mathbb{Q}$. If $q = \frac{a}{b}$ $(a, b \in \mathbb{Z})$ is in reduced form, then each x has period b since $\sigma^b(x) = x + bq = x + a$ and $a \in \mathbb{Z}$.

Furthermore, a map on $\mathbb{R}/\mathbb{Z}$ in the form $\sigma(x) = x + q$ has periodic points if and only if $q \in \mathbb{Q}$ since $\sigma^n(x) = x + nq \equiv x$ $(n \in \mathbb{N})$ if and only if $x + nq = x + k$ for some $k \in \mathbb{Z}$, i.e., if and only if $q = \frac{k}{n}$.

The periodic points of (X, σ) are *dense* if every nonempty, open $U \subseteq X$ contains a periodic point of (X, σ).

Example 21. As every rational point is periodic, the periodic points of the rational shift of Example 20 are dense.

Example 22. The map σ on $\mathbb{R}/\mathbb{Z}$ given by $\sigma(x) = 2x$ is called the doubling map. Notice that $\frac{1}{6} \mapsto \frac{1}{3} \mapsto \frac{2}{3} \mapsto \frac{1}{3}$, so $\frac{1}{6}$ is not a periodic point in $(\mathbb{R}/\mathbb{Z}, \sigma)$.

For $k \in \mathbb{N}$, a periodic point $x \in \mathbb{R}/\mathbb{Z}$ satisfies $\sigma^k(x) = 2^k x \equiv x$, so $x + n = 2^k x$ for some $n \in \mathbb{Z}$, i.e., $x = \frac{n}{2^k-1}$. Notice that for any $n \in \{1, \cdots, 2^k - 1\}$, $\sigma^k\left(\frac{n}{2^k-1}\right) = \frac{2^k n}{2^k-1} = \frac{n}{2^k-1} + n$, so $\frac{n}{2^k-1}$ is periodic. Observe that $\left\{\frac{1}{2^k-1}, \frac{2}{2^k-1}, \cdots, \frac{2^k-2}{2^k-1}, \frac{2^k-1}{2^k-1}\right\}$ are distinct members of $(0, 1]$. Furthermore, $\left\{\left(\frac{m}{2^k-1}, \frac{m+1}{2^k-1}\right]\right\}_{m=1}^{2^k-2}$ is a partition of $(0, 1]$ and each interval is of length $\frac{1}{2^k-1}$.

Take any $(a, b) \subset (0, 1]$, and choose $k \in \mathbb{N}$ large enough so that one element of the partition $\left\{\left(\frac{m}{2^k-1}, \frac{m+1}{2^k-1}\right]\right\}_{m=1}^{2^k-2}$ is contained entirely in (a, b). Call this element $\left(\frac{m_0}{2^k-1}, \frac{m_0+1}{2^k-1}\right]$ and note that we have the periodic point $\frac{m_0+1}{2^k-1} \in (a, b)$. Hence, the periodic points of this system are dense, but not all points are periodic.

Definition 30. The *orbit* of a point x in a dynamical system (X, σ) is the set $\left\{\sigma^k(x)\right\}_{k \in \mathbb{N}}$.

Definition 31. A point $x \in X$ is a *transitive point* for the system (X, σ) if its orbit under σ is dense in X.

Example 23. This is a modification of the proof in Example 11.3.1 of [7]. Every $x \in \mathbb{R}/\mathbb{Z}$ is a transitive point for the irrational shift σ on $\mathbb{R}/\mathbb{Z}$ given by $\sigma(x) = x + p$, where $p \in \mathbb{R}\backslash\mathbb{Q}$.

To see this, fix some $x \in \mathbb{R}/\mathbb{Z}$ and $p \in \mathbb{R}\backslash\mathbb{Q}$. Take any $y \in \mathbb{R}/\mathbb{Z}$ and $\epsilon \in [0, 1)$. We use the metric $d(x, y) = \min\{|x - y|, 1 - |x - y|\}$.

Choose $N \in \mathbb{N}$ so that $\frac{1}{N} < \epsilon$. For $1 \leq m \leq N$, set $I_m = \left[\frac{m-1}{N}, \frac{m}{N}\right)$. Observe that $\{I_m\}_{m=1}^N$ is a partition of $[0, 1)$ and $|I_m| = \frac{1}{N}$ for all m. For $0 \leq j \leq N$, set $x_j = pj \in \mathbb{R}/\mathbb{Z}$. Since p is irrational, we know from Example 20 that each x_j is distinct as this set is simply the first N interations of the orbit of 0 under σ. Since there are $N + 1$ x_j's and N I_m's, we can apply the Pigeonhole Principle to obtain $1 \leq m_0 \leq N$ and distinct $0 \leq j_1, j_2 \leq N$ such that $x_{j_1}, x_{j_2} \in I_{m_0}$. Observe that $d(x_{j_1}, x_{j_2}) < \frac{1}{N} < \epsilon$ and that $x_{j_2} - x_{j_1} \in \left[0, \frac{1}{N}\right) \cup \left[\frac{N-1}{N}, 1\right)$. It follows that if $x_{j_2} - x_{j_1} \in \left[0, \frac{1}{N}\right)$, then $d(x_{j_1}, x_{j_2}) = |x_{j_2} - x_{j_1}|$, and if $x_{j_2} - x_{j_1} \in \left[\frac{N-1}{N}, 1\right)$, then $d(x_{j_1}, x_{j_2}) = 1 - |x_{j_2} - x_{j_1}|$.

Without loss of generality, $x_{j_2} > x_{j_1}$, which gives $d(x_{j_1}, x_{j_2}) = |x_{j_2} - x_{j_1}|$. Let $\ell = j_2 - j_1$. Set $x_\ell = x_{j_2} - x_{j_1} = p\ell$, and note that for any $i \in \mathbb{N}$, $|[ix_\ell, (i+1)x_\ell)| < \epsilon$. Notice that the set $\{ix_\ell\}_{i\in\mathbb{N}}$ moves across $\mathbb{R}/\mathbb{Z}$ in steps of length less than ϵ and that with enough iterations, the intervals $[ix_\ell, (i+1)x_\ell)$ will cover $\mathbb{R}/\mathbb{Z}$. Choose $i_\ell \in \mathbb{N}$ such that $y - x \in [i_\ell x_\ell, (i_\ell + 1)x_\ell)$ In particular, $d(y - x, i_\ell x_\ell) < \epsilon$.

Observe that

$$\begin{aligned} d\left(y, \sigma^{i_\ell \ell}(x)\right) &= \min\left\{|y - \sigma^{i_\ell \ell}(x)|, 1 - |y - \sigma^{i_\ell \ell}(x)|\right\} \\ &= \min\left\{|y - x - pi_\ell \ell|, 1 - |y - x - pi_\ell \ell|\right\} \\ &= \min\left\{|(y - x) - i_\ell x_\ell|, 1 - |(y - x) - i_\ell x_\ell|\right\} \\ &= d\left(y - x, i_\ell x_\ell\right) \\ &< \epsilon. \end{aligned}$$

Hence, the orbit of x comes arbitrarily close to y. Since y was arbitrary, x is a transitive point.

Definition 32. A dynamical system (X, σ) is *topologically transitive* if for every pair of nonempty, open sets U and V in X, there is $n \in \mathbb{N} \cup \{0\}$ such that $\sigma^n(U) \cap V \neq \emptyset$.

Example 24. The irrational shift of Example 23 is topologically transitive.

Remark 10. In general, a dynamical system being topologically transitive and a dynamical system having a transitive point are not equivalent conditions.

4.2. Semicrossed Product and Dynamics

Let X be a compact Hausdorff space.

Theorem 1. *The periodic points of $\sigma = \{\sigma_i\}_{i\in I}$ are dense in X if and only if for every $f \in C(X)$ and $\epsilon > 0$, there is $n \in \mathbb{N}$ and a surjective $\pi : \mathcal{A} \to M_n(\mathbb{C})$ with $\|\pi(f)\| > \|f\| - \epsilon$.*

Proof. Suppose the periodic points of (X, σ) are dense. Take arbitrary $f \in C(X)$ and $\epsilon > 0$. Let $y \in X$ be where f attains its maximum modulus, and let V be the open ball centered on $f(y)$ of radius ϵ. As f is continuous, we may choose open $U \ni y$ in X so that $f(U) \subseteq V$. Take periodic $x \in V$. Let $n \in \mathbb{N}$ be the period of x, and let $w = i_n \cdots i_1 \in \mathbb{F}^+(I)$ be such that $\sigma_w(x) = x$. Set $N_x = \{x_i\}_{i=1}^n$ by $x_1 = x$, $x_2 = \sigma_{i_1}(x)$, $\cdots$, and $x_n = \sigma_{i_{n-1}} \circ \cdots \circ \sigma_{i_1}(x)$. Define $\pi : \mathcal{A} \to M_n(\mathbb{C})$ as in Section 3.5, and by Proposition 8, we know that π is surjective. Then

$$\begin{aligned} 0 &\leq \|f\| - \|\pi(f)\| \\ &\leq |f(y)| - |f(x)| \\ &= ||f(y)| - |f(x)|| \\ &\leq |f(y) - f(x)| \\ &< \epsilon. \end{aligned}$$

Hence, $\|f\| - \epsilon < \|\pi(f)\|$.

Suppose the periodic points are not dense. Choose $y \in X$ and ϵ so that there are no periodic points in the ball U centered on y with radius ϵ. Let $f \in C(X)$ be so that f is supported on U. Notice that $\|\pi(f)\| = 0$ for every periodic x, and hence, π is not surjective. □

Let $\mathcal{A} = C(X) \times_\sigma \mathbb{Z}^+$. Let J be an ideal of $C(X)$, and let $K \subset X$ be the closed set such that $J = \{f \in C(X) : f|_K = 0\}$. Let I be the ideal in $\mathcal{A}$ generated by J. Define $\pi : \mathcal{A} \to M_k(\mathcal{A})$ by

$$\begin{aligned} \pi(f) &= [f|_K]_{i=j} \\ \pi(S_\ell) &= [S_\ell]_{i=j+1 \bmod k} \end{aligned}$$

Let $\mathcal{A} = C(X) \times_\sigma \mathbb{F}^+$. Let I be the ideal in $\mathcal{A}$ generated by $\left\{S_{m_1}^{k_1}, \cdots, S_{m_p}^{k_p}\right\}$. Set $k = \max\{k_i\}$ and $M = \{m_i\}$. Define $\pi : \mathcal{A} \to M_k(\mathcal{A})$ by

$$\begin{aligned} \pi(f) &= [f]_{i=j} \\ \pi(S_\ell) &= \begin{cases} [S_\ell]_{i=j+1 \bmod k} & \text{if } \ell \notin M \\ [S_\ell]_{\substack{i=j+1 \\ j \leq \ell_k}} & \ell \in M \end{cases} m \end{aligned}$$

Theorem 2. *$\sigma : X \to X$ is topologically transitive if and only if for every pair of ideals J_1, J_2 of $C(X)$, there is $k \in \mathbb{N}$ so that $\left(\pi_x(S^k)\pi_x(J_1)\pi_x(S^k)\right) \cap (\pi_x(J_2)) \neq \{0\}$.*

Proof. Suppose σ is topologically transitive. Take any pair of ideals J_1, J_2 of $C(X)$, and let U_1, U_2 denote the respective open sets on which their elements are supported. Let k denote the smallest integer for which $\sigma^k(U_1)$ intersects U_2. Fix $x \in U_1$, and note that for any $f_2 \in U_2$, $f_2(\sigma^m(x)) = 0$, $m < k$. The $k-k$th entries of $\pi_x(S^k)\pi_x(f_1)\pi_x(S^k)$ and $\pi_x(f_2)$ are $f_1(x)$ and $f_2(\sigma^k(x))$, respectively, where $f_i \in J_i$. Choose f_1 and f_2 to give the entries the same non-zero value.

If σ is not topologically transitive then there are open sets U and V such that $U \cap \sigma^k(V) = \emptyset$ for all k. Then the ideals $\{f : f|_{X\setminus U} = 0\}$ and $\{g : G|_{X\setminus V} = 0\}$ will satisfy $\big(\pi_x(S^k)\pi_x(U)\pi_x(S^k)\big) \cap (\pi_x(V)) = \{0\}$ for all k. $\square$

The following example shows why the above theorem cannot be strengthened to use the whole matrix.

Example 25. Consider the irrational shift σ on $X = \mathbb{R}/\mathbb{Z}$ given by $\sigma(x) = x + \frac{1}{\sqrt{5}}$. Let $U_1 = (1/10, 1/5)$ and $U_2 = (1/2, 3, 5)$, and let $J_i = \big\{f \in C(X) : f|_{X-U_i} = 0\big\}$. Fix $x \in U_1$, and suppose $\pi_x(S)\pi_x(f_1)\pi_x(S)^* = \pi_x(f_2)$ for some $f_i \in J_i$. It follows that $f_1(\sigma^m(x)) = f_2(\sigma^{m+1}(x))$, $m \geq 0$. In particular, $f_1(\sigma^m(x)) \neq 0$ implies $\sigma^m(x) \in U_1$ and $\sigma^{m+1}(x) \in U_2$, i.e., $\sigma^m(x) \in (1/10, 3/5 - 1/\sqrt{5}) =: W$.

As x is transitive, the orbit of x is dense in W. However, f_1 vanishes off of W and off of the orbit x. Hence, f_1 must be the zero function since it is continuous, which is a contradiction. Therefore, $\pi_x(S)\pi_x(J_1)\pi_x(S) \cap \pi_x(J_2) = \{0\}$.

4.3. Beginning Multivariate Definitions

In this section, the map σ in the dynamical system (X, σ) is now a family of maps.

4.3.1. Periodicity

Definition 33. For a dynamical system (X, σ), we consider six possible definitions of what it means for a point $x \in X$ to be *P(i) periodic* for the system. In a dynamical system (X, σ), a point $x \in X$ is

1. *P1-periodic* if there is $v \in \mathbb{F}^+(I)$ so that $\sigma_v(x) = x$

2. *P2-periodic* if x is periodic for (X, σ_i) for every $i \in I$

3. *P3-periodic* if there is $v \in \mathbb{F}^+(I)$ so that every letter of I is used in v and $\sigma_v(x) = x$

4. *P4-periodic* if there is $v \in \mathbb{F}^+(I)$ so that every letter of I is used in v nontrivially and $\sigma_v(x) = x$

5. *P5-periodic* if for every $u \in \mathbb{F}^+(I)$, there is $a \in \mathbb{F}^+(I)$ so that $\sigma_{au}(x) = x$

6. *P6-periodic* if for every $w \in \mathbb{F}^+(I)$, there is $k \in \mathbb{N}$ so that $\sigma_{w^k}(x) = x$

Remark 11. We will drop the word *periodic* and just refer to $P(i)$ below.

Remark 12. When we discuss P3 or P4, it will be assumed that I is finite.

Remark 13. When we discuss P5, we assume that x is not a fixed point for at least one of the maps.

We will first look at relations between these different possibilities before observing certain properties.

Proposition 11. *We have the following relations between the choices of Definition 33.*

1. *If x is P2 for (X, σ), then x is P1 for (X, σ).*

2. *If x is P3 for (X, σ), then x is P1 for (X, σ).*

3. *If x is P4 for (X, σ), then x is P3 for (X, σ).*

4. *If x is P5 for (X, σ), then x is P4 for (X, σ).*

5. *If x is P6 for (X, σ), then x is P5 for (X, σ).*

6. *If x is P2 for (X, σ), then x is P3 for (X, σ).*

Proof. Points 1, 2, and 3 are immediate.

For 4, take $v \in \mathbb{F}^+(I)$ so that every letter of I is used exactly once. We may assume $\sigma_v(x) \neq x$ and choose $a \in \mathbb{F}^+(I)$ so that $\sigma_{av}(x) = x$. Since $av \in \mathbb{F}^+(I)$ uses each letter of I nontrivially, x is P4.

For 5, take any $u \in \mathbb{F}^+(I)$, and choose $k \in \mathbb{N}$ so that $\sigma_{w^k}(x) = x$. As $w^{k-1} \in \mathbb{F}^+(I)$, x is P5.

For 6, for every $i \in I$, choose $n_i \in \mathbb{N}$ so that $\sigma_{i^{n_i}}(x) = x$. The concatenation v of each i^{n_i} is in $\mathbb{F}^+(I)$ and $\sigma_v(x) = x$. As v contains each letter of I, x is P3. □

The following examples show that the converses of Proposition 11 are not necessarily true.

Example 26. Let $X = \mathbb{R}/\mathbb{Z}$, and let σ_1, σ_2 on X be given by $\sigma_1(x) = x + 2/5$ and $\sigma_2(x) = x/2$. Notice that $\sigma_{21}(2/5) = 1/5$, so $x = 1/5$ is P1. However, $x = 2/5$ is not periodic for σ_2, so $x = 2/5$ is not P2.

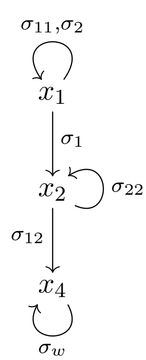
Example 27. Let $X = \{x_1, x_2, x_3\}$, and let σ_1 (left) and σ_2 (right) on X be as follows:

Since $\sigma_{11}(x_1) = x_1$, x_1 is P1. However, if σ_2 is ever introduced to the orbit of x_1, the orbit stays at x_2. Hence, x_1 is not P3.

Example 28. Let $X = \{x_1, x_2, x_3, x_4\}$, and let σ_1 (left) and σ_2 (right) on X be as follows:

As $\sigma_{1221}(x_1) = x_1$, x_1 is P3. Note that this path contains σ_2 trivially since the application of σ_{22} on $\sigma_1(x_1)$ and the application is simply the identity.

Observe that for a nontrivial path for x_1, we exclude loops. Since σ_2 and σ_{11} immediately lead x_1 back to itself, a nontrivial path for x_1 back to itself must begin with σ_{21}. Introducing σ_1 at this step leads to x_4, but x_4 is fixed under both σ_1 and σ_2. Hence, to get back to x_1, σ_2 must be applied. However, σ_{22} loops $\sigma_1(x_1)$ back to itself, so σ_2 is in the path trivially. From x_2, the only way to return to x_1 is through σ_1. The possible paths are shown below:



We are not able to add nontrivial paths from x_1 back to itself. Hence, x_1 is not P4.

Example 29. Let $X = \mathbb{R}/\mathbb{Z}$, and let σ_1, σ_2 on X be given by $\sigma_1(x) = x/3$ and $\sigma_2(x) = x + 1/2$.

First notice that $\sigma_{21}(3/4) = 3/4$, so $x = 3/4$ is P4.

Now observe that σ_{22} is the identity on X, so for any nontrivial $w \in \mathbb{F}^+(I)$, σ_w reduces to σ_{1^n}, where $n \in \mathbb{N} \cup \{0\}$, or $\sigma_{1^{i_n}21^{i_{n-1}}2\cdots 21^{i_1}}$, where $i_1, i_n \in \mathbb{N} \cup \{0\}$ and $i_2, \cdots, i_{n-1} \in \mathbb{N}$. We have $\sigma_{11}(3/4) = 1/12$, so for any $i \in \mathbb{N}$, $\sigma_{1^i 11}(3/4) \in (0, 1/12)$ and $\sigma_{21^i 11}(3/4) \in (1/2, 19/36) \subset (0, 3/4)$. Continuing in this manner, we see that $\sigma_{w111}(3/4) \in (0, 3/4)$ for every $w \in \mathbb{F}^+(I)$. Hence, $x = 3/4$ is not P5.

Example 30. We use again the space $X = \mathbb{R}/\mathbb{Z}$. Let σ_1, σ_2 on X be given by $\sigma_1(x) = x + 2/5$ and $\sigma_2(x) = 1 - |2x - 1|$.

For any $w \in \mathbb{F}^+(I)$, $\sigma_w(1/5) \in \{0, 1/5, 2/5, 3/5, 4/5\}$. Applying σ_1 an appropriate number of times to $\sigma_w(1/5)$ returns $1/5$, so $x = 1/5$ is P5.

Now observe that $x = 1/5$ is not P6 as $\sigma_{(12)^k}(1/5) = 2/5$ for every $k \in \mathbb{N}$.

For completion in examples of periodicity types, the following is an example of a P6 point.

Example 31. Let $X = \{1, 2, 3, 4, 5, 6\}$, and let σ_a (left), σ_b (center), and σ_c (right) be given by

$1 \to 3$	$1 \to 3$	$1 \to 4$
$2 \to 6$	$2 \to 2$	$2 \to 3$
$3 \to 5$	$3 \to 6$	$3 \to 5$
$4 \to 5$	$4 \to 6$	$4 \to 5$
$5 \to 1$	$5 \to 1$	$5 \to 1$
$6 \to 1$	$6 \to 1$	$6 \to 1$

For every $w \in \mathbb{F}^+(\{a, b, c\})$ with $|w| = 3$, $\sigma_w(1) = 1$. Hence, for any $w \in \mathbb{F}^+(\{a, b, c\})$, $\sigma_{w^3}(1) = 1$, so 1 is P6 for (X, σ).

Remark 14. In summary, we have chains P2 $\implies$ P1 and P6 $\implies$ P5 $\implies$ P4 $\implies$ P3 $\implies$ P1, and in these implications, we are not guaranteed the converse. As will be seen, P2 $\implies$ P3, but it is not true that P2 $\implies$ P4 or P3 $\implies$ P2.

Example 32. Let $X = \{x_1, x_2, x_3\}$, and let σ_1 (left) and σ_2 (right) on X be as follows:

Here, x_1 is P2 but not P4.

Example 33. Let $X = \{x_1, x_2\}$, and let σ_1 (left) and σ_2 (right) on X be as follows:

In this case, x_1 is P3 but not P2.

Remark 15. We use the standard definition of periodic points being *dense* that was stated in Section 4.1. In the multivariate case, we need only specifiy which version of periodicity is being used when discussing the periodic points being dense.

Proposition 12. *If the periodic points are dense for at least one σ_i of $\sigma = \{\sigma_i\}$, then P1 points are dense for σ.*

Proof. Suppose the periodic points of $\sigma_{i_0} \in \sigma$ are dense in X, and take $U \subseteq X$. Choose $x_0 \in U$ and $n \in \mathbb{N}$ so that $\sigma_{i_0^n}(x_0) = x_0$. Note that x_0 is P1 using $v = i_0^n$. □

4.3.2. Transitivity

Now we consider what it could mean for a point to be transitive.

Definition 34. For a dynamical system (X, σ), we consider four possible definitions of what it means for a point $x \in X$ to be *Tr(i) transitive* for the system. We add the notation of $\mathbb{F}^\infty(I)$ being the sets of all finite or infinite words of I. In a dynamical system (X, σ), a point $x \in X$ is

1. *Tr1-transitive* if for any $y \in X$ and open $U \ni y$ in X, there is $w \in \mathbb{F}^+(I)$ so that $\sigma_w(x) \in U$.
2. *Tr2-transitive* if x is transitive for (X, σ_i) for every $i \in I$.
3. *Tr3-transitive* if for every $y \in X$, open $U \ni y$ in X, and $u \in \mathbb{F}^+(I)$, there is $b \in \mathbb{F}^+(I)$ such that $\sigma_{bu}(x) \in U$.

4. *Tr4-transitive* if there is $v \in \mathbb{F}^\infty(I)$ so that for every $y \in X$ and open $U \ni y$ in X, there is $a \in \mathbb{F}^+(I)$ with $v = ba$ ($b \in \mathbb{F}^\infty(I)$) and $\sigma_a(x) \in U$.

Remark 16. As with periodicity, we drop the word *transitive* when talking about a space that is Tr(i) transitive.

We will first look at relations between these different possibilities before observing certain properties.

Proposition 13. *We have the following relations between the choices of Definition 34.*

1. *If x is Tr2, Tr3, or Tr4 for (X, σ), then x is Tr1 for (X, σ).*
2. *Suppose X is compact. If x is Tr3, then it is Tr4.*
3. *Suppose X is countable. If x is Tr3, then it is Tr4.*

Proof. 1 is immediate.

For 2, for every $y \in X$, choose a neighbourhood U_y. Take $\{y_i\}_{i=1}^n \subset X$ so that $X = \cup_{i=1}^n U_{y_i}$. Take any $u \in \mathbb{F}^+(I)$, and we may inductively choose $b_i \in \mathbb{F}^+(I)$ $(i = 1, \cdots, n)$ so that $\sigma_{b_i \cdots b_1 u}(x) \in U_{y_i}$. Let $v = b_n \cdots b_1 u$, and the definition is satisfied. □

The following examples show that the converses of Proposition 11 are not necessarily true.

Example 34. Consider $X = \{x_i\}_{i=1}^3$ with the discrete topology, and let σ_1 (left) and σ_2 (right) on X be as follows:

Observe that x_1 is Tr1, but not Tr2, Tr3, or Tr4 as $\sigma_{w1}(x_1) = x_2$ and $\sigma_{w2}(x_1) = x_3$ for every $w \in \mathbb{F}^\infty(I)$.

Example 35. Consider $X = \{x_i\}_{i=1}^4$ with the discrete topology, and define σ_1 (left) and σ_2 (right) on X as follows:

Observe that x_1 is Tr4 using $v = 211$. However, x_1 is not Tr3 since $\sigma_{w21}(x_1) = x_4$ for any $w \in \mathbb{F}^\infty(I)$.

Proposition 14. *If the transitive points are dense for at least one σ_i of $\sigma = \{\sigma_i\}_{i\in I}$, then Tr1 points are dense for σ.*

Proof. Immediate. □

The following gives an example of a Tr3 point.

Example 36. Define $\sigma = \{\sigma_i\}_{i=1}^2$ on $[0,1]$ by $\sigma_1(z) = z^3$ and $\sigma_2(z) = z^{1/2}$. Observe that σ_1 and σ_2 are commutative and that $(\sigma_{1^\alpha} \circ \sigma_{2^\beta})(z) = z^{3^\alpha/2^\beta}$ for $\alpha, \beta \in \mathbb{N} \cup \{0\}$.

Take any $x \in (0,1)$. Take any $y \in (0,1)$ and consider any open $J \subseteq [0,1]$ containing y. We may assume without loss of generality that J is an interval (a_1, a_2) in $(0,1)$. Then $J = x^I$, where $I = \left(\frac{\ln a_1}{\ln x}, \frac{\ln a_2}{\ln x}\right) \subseteq \mathbb{R}^+$. We want to find $\alpha, \beta \in \mathbb{N} \cup \{0\}$ so that $x^{3^\alpha/2^\beta} \in J$, i.e., so that $3^\alpha/2^\beta \in I$.

It is well known that we have a dyadic rational $\frac{p}{2^n} \in I$, where $p \in \mathbb{N}$ and $n \in \mathbb{N} \cup \{0\}$. Let $m \in \mathbb{N} \cup \{0\}$ be such that $3^m \leq p < 3^{m+1}$. Then at least one of the following three conditions are satisfied: $\frac{3^m}{2^n} \in I$, $\frac{3^{m+1}}{2^n} \in I$, or $I \subset \left(\frac{3^m}{2^n}, \frac{3^{m+1}}{2^n}\right)$. We will only do more work if only the third condition is met. In this case, note that $\frac{2^n}{3^m} I \subset (1,3) = 3^{(0,1)}$.

For $y \in \mathbb{R}$, use the notation $\lfloor y \rfloor = \max\{\alpha \in \mathbb{Z} : \alpha \leq y\}$ and $\{y\} = y - \lfloor y \rfloor$. Observe that for any $y, z \in \mathbb{R}$, $y - z = \lfloor y - z \rfloor + \{y - z\}$ and $y - z = (\lfloor y \rfloor + \{y\}) - (\lfloor z \rfloor + \{z\})$. Notice that $\{y\} - \{z\} \in (-1,1)$, and that $\{y - z\} = \{y\} - \{z\}$ if $\{y\} - \{z\} \in [0,1)$ and $\{y - z\} = \{y\} - \{z\} + 1$ otherwise. In particular,

$$\begin{aligned} \lfloor y - z \rfloor &= \lfloor y \rfloor - \lfloor z \rfloor + \{y\} - \{z\} - \{y - z\} \\ &= \begin{cases} \lfloor y \rfloor - \lfloor z \rfloor & \text{if } \{y\} - \{z\} \in [0,1) \\ \lfloor y \rfloor - \lfloor z \rfloor + 1 & \text{otherwise.} \end{cases} \end{aligned} \tag{4.1}$$

From basic inequality and logarithm properties, we have for any $k, \ell \in \mathbb{N}$, $\frac{3^{m+\ell}}{2^{n+k}} \in \left(\frac{3^m}{2^n}, \frac{3^{m+1}}{2^n}\right)$ if and only if $0 < \ell - k \cdot \frac{\ln 2}{\ln 3} < 1$. This implies $\ell - k \cdot \frac{\ln 2}{\ln 3} = \left\{\ell - k \cdot \frac{\ln 2}{\ln 3}\right\}$, which gives $\lfloor \ell - k \cdot \frac{\ln 2}{\ln 3} \rfloor = 0$. Notice that $\{\ell\} = 0$ and $\left\{k \cdot \frac{\ln 2}{\ln 3}\right\} \in (0,1)$. Applying Equation 4.1 yields $\ell = 1 + \left\lfloor k \cdot \frac{\ln 2}{\ln 3} \right\rfloor$.

Define $B, a : \mathbb{N} \to \mathbb{R}$ by $B(b) = b \cdot \frac{\ln 2}{\ln 3}$ and $a(b) = 1 + \lfloor B(b) \rfloor$. Observe that for any $b \in \mathbb{N}$,

$$\frac{3^{a(b)}}{2^b} = \frac{3^{1+B(b)-\{B(b)\}}}{2^b} = 3^{1-\{B(b)\}} \in (1,3) = 3^{(0,1)}.$$

Notice also that $\{B(b)\}_{b\in\mathbb{N}}$ is an irrational rotation of 0 on $\mathbb{R}/\mathbb{Z}$, so its orbit is dense in $[0,1]$ according to Example 23. Choose $b \in \mathbb{N}$ so that $3^{1-\{B(b)\}} \in \frac{2^n}{3^m} I$. Observe that $\frac{3^{m+a(b)}}{2^{n+b}} \in I$; hence, $x^{3^{m+a(b)}/2^{n+b}} \in J$. Hence, x is Tr1, and as x was arbitrary, any element of $(0,1)$ is Tr1 for (X, σ). Furthermore, this implies every $x \in X$ is Tr3.

4.3.3. Topological Transitivity

Definition 35. For a dynamical system (X, σ), where $\sigma = \{\sigma_i\}_{i\in I}$, we consider three possibilities for what it means for the system to be *TTr(i) topologically transitive.* A dynamical system (X, σ) is

1. *TTr1-topologically transitive* if given open, nonempty $U, V \subseteq X$, there is $w \in \mathbb{F}^+(I)$ such that $\sigma_w(U) \cap V \neq \emptyset$

2. *TTr2-topologically transitive* (X, σ_i) is topologically transitive for every $i \in I$

3. *TTr3-topologically transitive* if given open, nonempty $U, V \subseteq X$ and $u \in \mathbb{F}^+(I)$, there is $b \in \mathbb{F}^+(I)$ such that $\sigma_{bu}(U) \cap V \neq \emptyset$

4.4. Partition Conjugacy

4.4.1. Definition

In [9], Davidson and Katsoulis introduced a concept called piecewise conjugacy when discussing multivariate systems, and a few properties that pass between piecewise conjugate systems were explored. In [23], Ramsey refined the definition further. Definition 36 is sourced from [23].

Definition 36. We say that two dynamical systems (X, σ) and (Y, τ), where $\sigma = \{\sigma_i\}_{i\in I}$ and $\tau = \{\tau_i\}_{i\in I}$, are *partition conjugate* if there are clopen sets $\{V_{i,j}\}_{i,j\in I}$ in X and a homeomorphism $\gamma : X \to Y$ such that

1. for any fixed j, $\cup_{i \in I} V_{i,j} = X$ and $V_{i,j} \cap V_{i',j} = \emptyset$ when $i \neq i'$

2. for any fixed i, $\cup_{j \in I} V_{i,j} = X$ and $V_{i,j} \cap V_{i,j'} = \emptyset$ when $j \neq j'$

3. for any i, j, $\sigma_i|_{V_{i,j}} = \gamma^{-1} \circ \tau_j \circ \gamma|_{V_{i,j}}$

4. for any i, j, $\sigma_i^{-1}(\sigma_i(V_{i,j})) = V_{i,j} = \gamma^{-1}\left(\tau_j^{-1}(\tau_j \circ \gamma(V_{i,j}))\right)$

Remark 17. If X and Y are connected, this is the standard definition of topological conjugacy, i.e., the indexing set I has a single member.

Remark 18. As noted in [23], point (4) of Definition 36 is equivalent to saying for any $i, j \in I$, $\sigma_i^{-1}(\sigma_i(V_{i,j})) = V_{i,j} = \gamma^{-1}\left(\tau_j^{-1}(\tau_j \circ \gamma(V_{i,j}))\right)$ if and only if $\sigma_i(V_{i,j}) \cap \sigma_i(V_{i,j'}) = \emptyset$ when $j \neq j'$.

Example 37. Let $X = (0,1) \cup (1,2) \cup (2,3)$, and let $V_{1,1} = V_{2,2} = V_{3,3} = (0,1)$ and $V_{1,2} = V_{2,3} = V_{3,1} = (1,2)$ and $V_{1,3} = V_{2,1} = V_{3,2} = (2,3)$. Clearly, these sets satisfy (1) and (2) of Definition 36.

Consider $Y = (-4,-3) \cup (-3,-2) \cup (-2,-1)$, and let $\gamma : X \to Y$ be given by

$$\gamma(x) = \begin{cases} x - 2 & x \in (0,1) \\ -x - 2 & x \in (1,2) \\ (x-2)^2 - 3 & x \in (2,3), \end{cases}$$

noting that $\gamma(0,1) = (-2,-1)$, $\gamma(1,2) = (-4,-3)$, and $\gamma(2,3) = (-3,-2)$. Observe also that

$$\gamma^{-1}(y) = \begin{cases} -y - 2 & y \in (-4,-3) \\ 2 + \sqrt{y+3} & y \in (-3,-2) \\ y + 2 & y \in (-2,-1). \end{cases}$$

Set $\sigma_1, \sigma_2, \sigma_3$ on X by

$$\sigma_1(x) = \begin{cases} \sqrt{x}+2 & x \in (0,1) = V_{1,3} \\ 2-(x-1)^2 & x \in (1,2) = V_{1,2} \\ x-2 & x \in (2,3) = V_{1,3} \end{cases}$$

$$\sigma_2(x) = \begin{cases} 3-x^2 & x \in (0,1) = V_{2,2} \\ (x-1)^2 & x \in (1,2) = V_{2,3} \\ 4-x & x \in (2,3) = V_{2,1} \end{cases}$$

$$\sigma_3(x) = \begin{cases} 2-x & x \in (0,1) = V_{3,3} \\ \sqrt{x-1}+2 & x \in (1,2) = V_{3,1} \\ \sqrt{3-x} & x \in (2,3) = V_{3,2}. \end{cases}$$

The respective graphs of $\sigma_1, \sigma_2, \sigma_3$ are shown in Figures 4.1 and 4.2. It is clear that the ranges of $V_{i,j}$ and $V_{i,j'}$ with $j \neq j'$ are separate under σ_i, so (4) of Definition 36 is satisfied (via Remark 18).

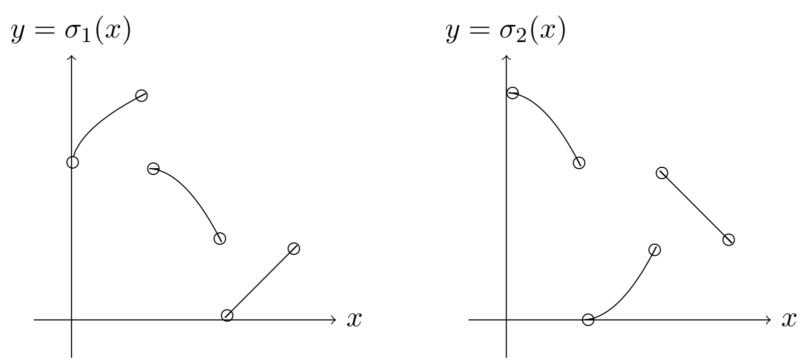


Figure 4.1. The graphs of σ_1 (left) and σ_2 (right) of Example 37 are shown.

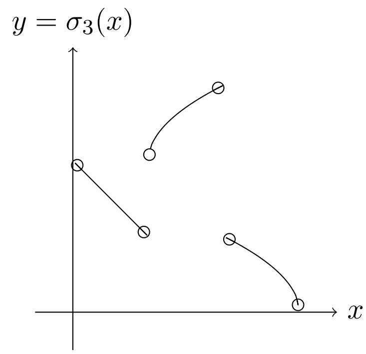


Figure 4.2. The graph of σ_3 of Example 37 is shown.

Set τ_1, τ_2, τ_3 on Y by

$$\tau_1(y) = \begin{cases} -y-6 & y \in (-4,-3) = \gamma(V_{3,1}) \\ \sqrt{y+3}-4 & y \in (-3,-2) = \gamma(V_{2,1}) \\ y-1 & y \in (-2,-1) = \gamma(V_{1,1}) \end{cases}$$

$$\tau_2(y) = \begin{cases} (y+3)^2-4 & y \in (-4,-3) = \gamma(V_{1,2}) \\ \sqrt{1-\sqrt{y+3}}-2 & y \in (-3,-2) = \gamma(V_{3,2}) \\ \left(1-(y+2)^2\right)^2-3 & y \in (-2,-1) = \gamma(V_{2,2}) \end{cases}$$

$$\tau_3(y) = \begin{cases} (y+3)^2-2 & y \in (-4,-3) = \gamma(V_{2,3}) \\ \sqrt{y+3}-2 & y \in (-3,-2) = \gamma(V_{1,3}) \\ y-2 & y \in (-2,-1) = \gamma(V_{3,3}). \end{cases}$$

It is an easy (albeit tedious) check to verify that (3) of Definition 36 is satisfied.

Thus, (X,σ) and (Y,τ) are partition conjugate.

Proposition 15. *Two dynamical systems (X,σ) and (Y,τ), where $\sigma = \{\sigma_i\}_{i\in I}$ and $\tau = \{\tau_i\}_{i\in I}$, are partition conjugate if there are clopen sets $\{U_{i,j}\}_{i,j\in I}$ in Y and a homeomorphism $\delta : Y \to X$ such that*

1. *for any fixed j, $\cup_{i\in I} U_{i,j} = Y$ and $U_{i,j} \cap U_{i',j} = \emptyset$ when $i \neq i'$.*
2. *for any fixed i, $\cup_{j\in I} U_{i,j} = Y$ and $U_{i,j} \cap U_{i,j'} = \emptyset$ when $j \neq j'$.*

3. *for any* i, j, $\tau_j|_{U_{i,j}} = \delta^{-1} \circ \sigma_i \circ \delta|_{U_{i,j}}$.

4. *for any* i, j, $\tau_j(U_{i,j}) \cap \tau_{j'}(U_{i,j'}) = \emptyset$ *when* $j \neq j'$.

Proof. Let $\delta = \gamma^{-1}$ and for each $i, j \in I$, $U_{i,j} = \gamma(V_{i,j})$, where γ and $V_{i,j}$ are as in Definition 36.

Take any $j \in I$. As $V_{i,j} \cap V_{i',j} = \emptyset$ when $i \neq i'$ and γ is injective, $U_{i,j} \cap U_{i',j} = \emptyset$ for $i \neq i'$. Since $\cup_{i \in I} V_{i,j} = X$ and γ is surjective, $\cup_{i \in I} U_{i,j} = Y$. (2) is seen similarly.

Take $i, j \in I$ and $y \in U_{i,j}$. Then $y = \gamma(x)$ for unique $x \in V_{i,j}$; hence, $\tau_j(y) = \tau_j \circ \gamma(x)$. By Definition 36, $\tau_j(y) = \gamma \circ \sigma_i(x) = \gamma \circ \sigma_i \circ \gamma^{-1}(x)$.

In order to see (4), we utilize Remark 18. Since $\sigma_i(V_{i,j}) \cap \sigma_i(V_{i,j'}) = \emptyset$ and γ is injective, $\gamma \circ \sigma(V_{i,j}) \cap \gamma \circ \sigma_i(V_{i,j'}) = \emptyset$, and hence, $\tau_j(V_{i,j}) \cap \tau(V_{i,j'}) = \emptyset$. □

The following is proved similarly as Remark 18, but (4) of Proposition 15 is easier to work with.

Proposition 16. *For any* $i, j \in I$, $\tau_j^{-1}(\tau_j(U_{i,j})) = U_{i,j} = \gamma_0^{-1}\left(\sigma_i^{-1}(\sigma_i \circ \gamma_0(U_{i,j}))\right)$ *if and only if* $\tau_j(U_{i,j}) \cap \tau_{j'}(V_{i,j'}) = \emptyset$ *when* $j \neq j'$.

Example 38. In the setting of Example 37, $U_{1,1} = U_{2,2} = U_{3,3} = (-2, -1)$ and $U_{1,2} = U_{2,3} = U_{3,1} = (-4, -3)$ and $U_{1,3} = U_{2,1} = U_{3,2} = (-3, -2)$.

We check just (4) of Proposition 15.

The respective graphs of τ_1, τ_2, τ_3 are shown in Figures 4.3 and 4.4.

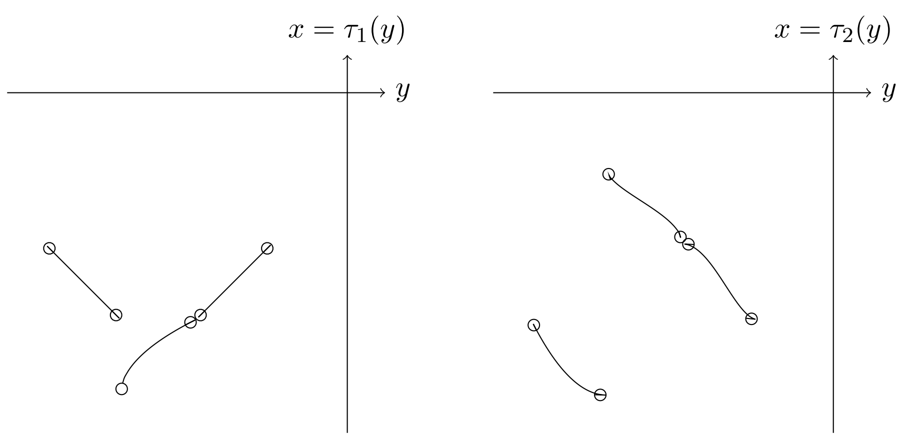


Figure 4.3. The graphs of τ_1 (left) and τ_2 (right) of Example 38 are shown.

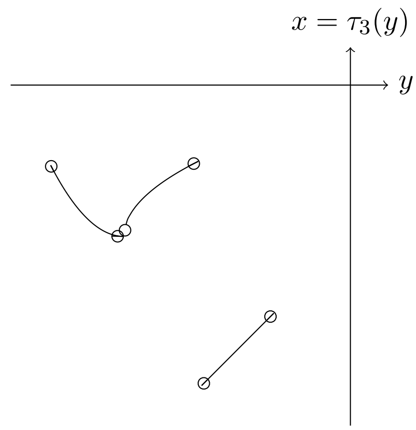


Figure 4.4. The graph of τ_3 of Example 38 is shown.

It is not as easy visually to see (4) of Proposition 15 as it is in Proposition 18, so consider Figure 4.5 for $\tau_j(U_{i,j})$.

$j\backslash i$	1	2	3
1	$(-3,-2)$	$(-4,-3)$	$(-3,-2)$
2	$(-4,-3)$	$(-3,-2)$	$(-2,-1)$
3	$(-2,-1)$	$(-2,-1)$	$(-4,-3)$

Figure 4.5. $\tau_j(U_{i,j})$ is shown.

In each column, the entries are disjoint, so (4) is satisfied.

4.4.2. Partition Conjugacy and Periodicity

Certain dynamics properties are preserved in the multivariate case under partition conjugacy as they are in one map systems under conjugacy.

Proposition 17. *Suppose (X, σ) and (Y, τ) are partition conjugate, and let γ and $V_{i,j}$ be as in Definition 36. If x is P1 for (X, σ), then $\gamma(x)$ is P1 for (Y, τ).*

Proof. Let $w = i_m \cdots i_1 \in \mathbb{F}^+(I)$ be such that $\sigma_w(x) = x$. For every $k = 1, \cdots, m$, we let $j_k \in I$ be the unique index so that $\sigma_{i_{k-1}\cdots i_1}(x) \in V_{i_k,j_k}$ (using $\sigma_{i_0}(x) = x$). Set $v = j_m \cdots j_1 \in \mathbb{F}^+(I)$. We apply (3) of Definition 36 multiple times to obtain

$$\begin{aligned}\sigma_w(x) &= \gamma^{-1} \circ \tau_{j_m} \circ \gamma \circ \gamma^{-1} \circ \tau_{j_{m-1}} \circ \gamma \circ \cdots \circ \gamma^{-1} \circ \tau_{i_2} \circ \gamma \circ \gamma^{-1} \circ \tau_{i_1} \circ \gamma(x) \\ &= \gamma^{-1} \circ \tau_v \circ \gamma(x).\end{aligned}$$

Thus, $\gamma(x) = \tau_v(\gamma(x))$. □

Example 39. In the setting of Example 37, $x = 1/2$ and $\gamma(x) = -\frac{3}{2}$ are P1 for (X, σ) for (Y, τ), respectively, since we have the relations in Figure 4.6.

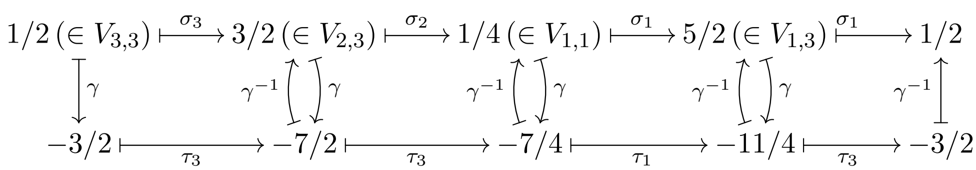


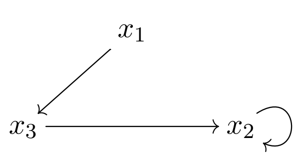
Figure 4.6. Orbit of 1/2 under partition conjugacy

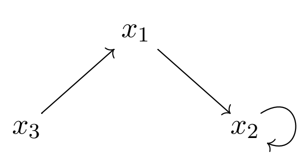
In the construction of Proposition 17, $w = 1123$ and $v = 3133$. Observe that Proposition 17 does not extend to P3 or P4. $x = 1/2$ is P3 and P4, but the construction used did not demonstrate that $\gamma(x)$ is P3 or P4. It is true, though, that $\gamma(x)$ is P3 and P4 since

$$-\frac{3}{2} \overset{\tau_2}{\mapsto} \frac{9}{16} - 3 \overset{\tau_3}{\mapsto} -\frac{5}{4} \overset{\tau_3}{\mapsto} -\frac{13}{4} \overset{\tau_1}{\mapsto} -\frac{11}{4} \overset{\tau_3}{\mapsto} -\frac{3}{2}.$$

However, the next example shows that there is not an equivalent Proposition 17 for P3 or P4.

Example 40. Let $X = \{x_1, x_2, x_3\}$. Let σ_α (left) and σ_β (right) on X be as follows:





Since $\sigma_{\beta\alpha}(x_1) = x_1$, x_1 is P4 (and hence P3) for (X, σ).

Let $V_{\alpha,\alpha} = \{x_1, x_2\} = V_{\beta,\beta}$ and $V_{\alpha,\beta} = \{x_3\} = V_{\beta,\alpha}$. Consider $Y = \{a, b, c\}$, and set $\gamma : X \to Y$ by $x_1 \mapsto b$, $x_2 \mapsto c$, and $x_3 \mapsto a$.

Let τ_α (left) and τ_β (right) on Y be as follows:

It is a routine check to see that (X,σ) and (Y,τ) are partition conjugate. It is also clear that $\gamma(x_1) = b$ is not P3 (and hence, not P4) for (Y,τ).

There is not an equivalent Proposition 17 for P2 or P6, as we will discuss in the next example.

Example 41. Let $X = \{x_1, x_2\}$, and let σ_a (left) and σ_b (right) on X be as follows:

As σ_b is the identity on X, every σ_w, $w \in \mathbb{F}^+(I)$ reduces to σ_{b^k} for some $k \in \mathbb{N} \cup \{0\}$. Hence, $\sigma_{w^2}(1) = 1$, so 1 is P6 for (X,σ). It is also clear that 1 is P2.

Let $V_{a,a} = \{1\} = V_{b,b}$ and $V_{a,b} = \{b\} = V_{b,a}$, and let τ_a (left) and τ_b (right) on X be as follows:

Using $\gamma = \sigma_a$, it is an easy check to show that (X,σ) and (X,τ) are partition conjugate.

However, $\gamma(1)$ is not P6 since for every $k \in \mathbb{N}$, $\tau_{1^k}(\gamma(1)) = 1 \neq \gamma(1)$. Notice also that since τ_a and τ_b are constant maps, $\gamma(1)$ is not P2.

However, we do have an equivalent statement of Proposition 17 for P5.

Proposition 18. *Suppose (X,σ) and (Y,τ) are partition conjugate. If x is P5 for (X,σ), then $\gamma(x)$ is P5 for (Y,τ).*

Proof. Let γ and $U_{i,j}$ be as in Definition 36. Take any $u = j_m \cdots j_1 \in \mathbb{F}^+(I)$. For every $k = 1, \cdots, m$, we let $i_k \in I$ be the unique index so that $\tau_{i_{k-1}\cdots i_1}(\gamma(x)) \in U_{i_k,j_k}$, using $\tau_{i_0}(\gamma(x)) = \gamma(x)$, and let $w = i_m \cdots i_1 \in \mathbb{F}^+(I)$. From (3) of Proposition 15, we get

$$\tau_u(\gamma(x)) \quad = \quad \gamma \circ \sigma_w(x).$$

Choose $b = i_p \cdots i_{m+1} \in \mathbb{F}^+(I)$ so that $\sigma_{bw}(x) = x$. For every $k = m+1, \cdots, p$, we let $j_k \in I$ be the unique index so that $\tau_{i_{k-1} \cdots i_{m+1}}(\gamma(x)) \in U_{i_k, j_k}$, and set $a = j_p \cdots j_{m+1}$. As before, we have

$$\tau_{au}(\gamma(x)) \quad = \quad \gamma \circ \sigma_{bw}(x),$$

so $\tau_{au}(\gamma(x)) = \gamma(x)$. □

4.4.3. Partition Conjugacy and Transitivity

We now explore some properties of transitivity that are preserved under partition conjugacy.

Proposition 19. *Suppose (X, σ) and (Y, τ) are partition conjugate. If x is Tr1 for (X, σ), then $\gamma(x)$ is Tr1 for (Y, τ).*

Proof. Let γ and $V_{i,j}$ be as in Definition 36. Take any $U \subseteq Y$ nonempty and open, and note that $V := \gamma^{-1}(U)$ is nonempty and open in X. Choose $v = i_m \cdots i_1 \in \mathbb{F}^+(I)$ so that $\sigma_v(x) \in V$. There is unique $j_1 \in I$ so that $x \in V_{i_1, j_1}$, and there is unique $j_2 \in I$ so that $\sigma_{i_1}(x) \in V_{i_2, j_2}$, and so on until we have unique $j_m \in I$ so that $\sigma_{i_{m-1} \cdots i_1}(x) \in V_{i_m, j_m}$. Set $u = j_m \cdots j_1$. As usual, $\sigma_v(x) = \gamma^{-1} \circ \tau_u \circ \gamma(x)$, so $\tau_u \circ \gamma(x) \in \gamma(V)$. Hence, $\tau_u(\gamma(x)) \in U$, and $\gamma(x)$ is Tr1 for (Y, τ). □

Remark 19. Example 41 shows that there is not an equivalent Proposition 19 for Tr2.

Proposition 20. *Suppose (X, σ) and (Y, τ) are partition conjugate. If x is Tr3 for (X, σ), then $\gamma(x)$ is Tr3 for (Y, τ).*

Proof. Let γ and $V_{i,j}$ be as in Definition 36. Take any $u = j_m \cdots j_1 \in \mathbb{F}^+(I)$ and nonempty, open $U \subseteq Y$. Again, for every $k = 1, \cdots, m$, we let $i_k \in I$ be the unique index such that $\sigma_{i_{k-1} \cdots i_1}(x) \in V_{i_k, j_k}$ (using $\sigma_{i_0}(x) = x$). Let $w = i_m \cdots i_1 \in \mathbb{F}^+(I)$. As usual, $\sigma_w(x) = \gamma^{-1} \circ \tau_u(\gamma(x))$.

As x is Tr3, choose $a = i_p \cdots i_{m+1} \in \mathbb{F}^+(I)$ so that $\sigma_{aw}(wx) \in \gamma^{-1}(U)$. For $k = m+1, \cdots, p$, we let $j_k \in I$ be the unique index such that $\sigma_{i_{k-1}, \cdots, i_1} \in V_{i_k, j_k}$, and again, as usual, $\sigma_{aw}(x) = \gamma^{-1} \circ \tau_{bu}(\gamma(x))$. Thus, $\tau_{bu}(\gamma(x)) \in U$, so $\gamma(x)$ is Tr1. □

4.4.4. Partition Conjugacy and Topological Transitivity

We now explore some properties of topological transitivity that are preserved under partition conjugacy.

Proposition 21. *Suppose (X, σ) and (Y, τ) are partition conjugate. If (X, σ) is TTr1, then (Y, τ) is TTr1.*

Proof. Let γ and $V_{i,j}$ be as in Definition 36. Take $A, B \subseteq Y$ nonempty and open. Then $A_0 := \gamma^{-1}(A)$ and $B_0 := \gamma^{-1}$ are nonempty and open in X. Choose $w = i_m \cdots i_1 \in \mathbb{F}^+(I)$ so that $\sigma_w(A_0) \cap B_0 \neq \emptyset$. Take any $z_m \in \sigma_w(A_0) \cap B_0$.

First observe that $\gamma(z_m) \in \gamma(B_0) = B$.

Now, $z_m = \sigma_{i_m}(z_{m-1})$ for some $z_{m-1} \in \sigma_{i_{m-1} \cdots i_1}(A_0)$, and $z_{m-1} = \sigma_{i_{m-1}}(z_{m-2})$ for some $z_{m-2} \in \sigma_{i_{m-2} \cdots i_1}(A_0)$. Continue in this manner until we have $z_2 = \sigma_{i_2}(z_1)$ for some $z_1 \in \sigma_{i_1}(A_0)$ and $z_1 = \sigma_{i_1}(z_0)$ for some $z_0 \in A_0$. For $k = 1, \cdots, m$, we let $j_k \in I$ be the unique index so that $z_{k-1} \in V_{i_k, j_k}$, and set $v = j_m \cdots j_1 \in \mathbb{F}^+(I)$. Then

$$\begin{aligned} z_m &= \sigma_{i_m}(z_{m-1}) \\ &= \gamma^{-1} \circ \tau_{j_m} \circ \gamma(z_{m-1}) \\ &= \cdots \\ &= \gamma^{-1} \circ \tau_v \circ \gamma(z_0). \end{aligned}$$

Since $\gamma(z_0) \in A$, $\gamma(z_m) \in \tau_v(A)$, so $\tau_v(A) \cap B \neq \emptyset$. □

Proposition 22. *Suppose (X, σ) and (Y, τ) are partition conjugate. If (X, σ) is TTr3, then (Y, τ) is TTr3.*

Proof. Let γ and $V_{i,j}$ be as in Definition 36. Take any $A, B \subseteq Y$ nonempty and open, and take any $v = j_m \cdots j_1 \in \mathbb{F}^+(I)$. Notice that $A_0 := \gamma^{-1}(A)$ and $B_0 := \gamma^{-1}(B)$ are nonempty and open in X. There is unique $i_1 \in I$ such that $A_0 \cap V_{i_1, j_1} \neq \emptyset$. Fix $z_1 \in A_0 \cap V_{i_1, j_1}$.

Set $z_2 = \sigma_{i_1}(z_1)$. There is unique $i_2 \in I$ such that $z_2 \in V_{i_2, j_2}$. Since σ_{i_1} is continuous, there is open A_1 with $z_1 \in A_1$ such that $\sigma_{i_1}(A_1) \subseteq V_{i_2, j_2}$. Set $z_3 = \sigma_{i_2}(z_2)$. There is unique $i_3 \in I$ such that $z_3 \in V_{i_3,j_3}$. Since σ_{i_2} is continuous, there is open $A_2 \ni z_2$ such that $\sigma_{i_2}(A_2) \subseteq V_{i_3,j_3}$. Notice that $z_1 \in \sigma_{i_1}^{-1}(A_2)$.

Continue inductively in this manner. For $3 \leq k \leq m$, set $z_k = \sigma_{i_{k-1}}(z_{k-1})$. There is unique $i_k \in I$ such that $z_k \in V_{i_k,j_k}$. As $\sigma_{i_{k-1}}$ is continuous, we may choose $A_{k-1} \ni z_{k-1}$ so that $\sigma_{i_{k-1}}(A_{k-1}) \subseteq V_{i_k,j_k}$, noting that $z_1 \in \sigma_{i_{k-2}\cdots i_1}^{-1}(A_{k-1})$.

Set $M = \sigma_{i_{m-2}\cdots i_1}^{-1}(A_{m-1}) \cap \sigma_{i_{m-3}\cdots i_1}^{-1}(A_{m-2}) \cap \cdots \cap \sigma_{i_1}^{-1}(A_2) \cap A_1 \cap A_0 \cap V_{i_1,j_1}$. Since $z_1 \in M$ and each σ_i is continuous, M is nonempty and open. Set $u = i_m \cdots i_1 \in \mathbb{F}^+(I)$. As (X, σ) is TTr3, choose $a = i_p \cdots i_{m+1} \in \mathbb{F}^+(I)$ so that $\sigma_{au}(M) \cap B_0 \neq \emptyset$.

Take $z \in \sigma_{au}(M) \cap B_0$. Note that $\gamma(z) \in B$. Choose $y \in M$ so that $\sigma_{av}(y) = z$. Since $y \in V_{i_1,j_1}$, we have $\sigma_{i_1}(y) = \gamma^{-1} \circ \tau_{j_1} \circ \gamma(y)$. Since $y \in A_1$, we have $\sigma_{i_1}(y) \in V_{i_2,j_2}$, so $\sigma_{i_2 i_1}(y) = \gamma^{-1} \circ \tau_{j_2 j_1} \circ \gamma(y)$. Since $y \in \sigma_{i_1}^{-1}(A_2)$, we see $\sigma_{i_2 i_1}(y) \in V_{i_3,j_3}$, so $\sigma_{i_3 i_2 i_1}(y) = \gamma^{-1} \circ \tau_{j_3 j_2 j_1} \circ \gamma(y)$. Continuing in this manner, we get $\sigma_u(y) = \gamma^{-1} \circ \tau_v \circ \gamma(y)$.

There is unique $j_{m+1} \in I$ so that $\sigma_u(y) \in V_{i_{m+1},j_{m+1}}$, noting $\sigma_{i_{m+1}u}(y) = \gamma^{-1} \circ \tau_{j_{m+1}v} \circ \gamma(y)$. Continue this process until we have unique $j_p \in I$ so that $\sigma_{i_{p-1}\cdots i_{m+1}u}(y) \in V_{i_p,j_p}$, noting $\sigma_{au}(y) = \gamma^{-1} \circ \tau_{j_p \circ j_{m+1} v} \circ \gamma(y)$. Set $b = j_p \cdots j_{m+1} \in \mathbb{F}^+(I)$. Thus, $\gamma(z) = \tau_{bv}(\gamma(y))$. Since $y \in A_0$, $\gamma(y) \in A$. Thus, $\gamma(z) \in \tau_{bv}(A)$. Hence, $\tau_{bv}(A) \cap B \neq \emptyset$, so (Y, τ) is TTr3. □

4.5. The surjective extension $\widetilde{X}$ of a single map σ

In Section 5 of [15], they developed a surjection extension $\widetilde{X}$ of a dynamical system. We remind of the notation in that paper as we will use it throughout this section. We start with a dynamical system $(X, \sigma, \mathcal{S})$, where σ is a family of surjective maps on a compact Hausdorff space X indexed by an abelian semigroup $\mathcal{S}$. A surjective map p from $(X, \sigma, \mathcal{S})$ to a similarly defined dynamical system $(Y, \beta, \mathcal{S})$ is called an *extension* map from $(X, \sigma, \mathcal{S})$ into $(Y, \beta, \mathcal{S})$ if $p \circ \beta_s = \sigma_s \circ p$ for every $s \in \mathcal{S}$. We let $\mathcal{G}$ be the group $\mathcal{S} - \mathcal{S}$. A partial order on $\mathcal{G}$ is given by $h < g$ if $g - h \in \mathcal{S}$. Set $\{X_g\}_{g \in \mathcal{G}}$ so that $X_g = X\ \forall g$, and for $g, h \in \mathcal{G}$ with $h < g$, we set σ_u with $u = g - h$ as $X_g \to X_h$. We then define

$$\widetilde{X} \quad = \quad \left\{ \sum_{g \in G} x_g \xi_g \in \Pi X_g : x_h = \sigma_u(x_g)\ \forall h < g \in G,\ u = g - h \right\}$$

We set $\widetilde{\sigma} = \{\sigma_t\}_{t\in\mathcal{S}}$ on $\widetilde{X}$ by $\widetilde{\sigma}_t\left((x_g)_{g\in\mathcal{G}}\right) = \left(\sigma_t(x_g)\right)_{g\in\mathcal{G}}$. Set $p : \widetilde{X} \to X$ by $p\left((x_g)_{g\in\mathcal{G}}\right) = x_0$, where 0 is the identity in $\mathcal{G}$. From Proposition 3 of [15], we know that p gives us a homeomorphism extension form $\left(\widetilde{X}, \widetilde{\sigma}, \mathcal{S}\right)$ into $(X, \sigma, \mathcal{S})$, and by Lemma 6 of [15], it is minimal.

In [21] the one variable case was studied. It was shown that certain properties such as transitivity are preserved in this extension. We now want to show that this can be applied to the multivariate definitions that we are using. We observe that if $\widetilde{x} \in p^{-1}\{x\}$ is P1 for $(\widetilde{X}, \widetilde{\sigma})$, then x is P1 for (X, σ) since $w \in \mathbb{F}^+(I)$ so that $\widetilde{\sigma}_w(\widetilde{x}) = \widetilde{x}$. Then $p\left(\widetilde{\sigma}_w(\widetilde{x})\right) = p\left(\widetilde{x}\right)$, so $\sigma_w(x) = x$. From this we can conclude the following

Corollary 2. *If $\widetilde{x} \in p^{-1}\{x\}$ is P2/P3/P4/P5/P6 for $(\widetilde{X}, \widetilde{\sigma})$, then x is P2/P3/P4/P5/P6 for (X, σ).*

As the next example shows, the converse of our observation about $P1$ and $\tilde{X}$ is not true.

Example 42. Let $X = \mathbb{R}/\mathbb{Z}$, and consider $x = \frac{1}{5}$. Set σ_a and σ_b on X by $\sigma_a(x) = 2x$ and $\sigma_b(x) = 3x$. Notice that x is P1 for (X, σ) since $\sigma_{a+b}(x) = x$.

Choose $\widetilde{x} \in p^{-1}\{x\}$ so that $x_{a+b} = \frac{1}{30}$, noting $x_a = \frac{1}{10}$ and $x_b = \frac{1}{15}$. Observe that in order for $\sigma_w(x_a) = x_a$ or $\sigma_w(x_b) = x_b$, we must have, respectively, the forms $w = 3^k$ or $w = 2^k$, $k \in \{0\} \cup \mathbb{N}$. Since no w will work for both x_a and x_b, $\widetilde{x}$ is not P1 for $(\widetilde{X}, \widetilde{\sigma})$.

Proposition 23. *x is Tr1 for (X, σ) if and only if every $\widetilde{x} \in p^{-1}\{x\}$ is Tr1 for $\left(\widetilde{X}, \widetilde{\sigma}\right)$.*

Proof. Start with the forward implication. Take arbitrary $\widetilde{y} = \sum_{g\in\mathcal{G}} y_g\xi_g \in \widetilde{X}$ and any open $\widetilde{U} \subseteq \widetilde{X}$ containing $\widetilde{y}$. We may assume that $\widetilde{U}$ is a basic set; hence $\widetilde{U} = \widetilde{X} \cap \prod_{g\in\mathcal{G}} U_g$, where each U_g is open in X and $U_g = X$ for every $g \in \mathcal{G}$ except for some finite set $\{g_i\}_{i=1}^n$. For every $i = 1, \cdots, n$, fix $s_i, t_i \in \mathcal{S}$ so that $g_i = s_i - t_i$, and set $s \in \mathcal{S}$ by $s = \sum_{i=1}^n s_i$. Notice we have the relations for $\widetilde{y}$ seen in Figure 4.7.

We look at $U = \bigcap_{i=1}^n \sigma_{s-g_i}^{-1}\left(U_{g_i}\right)$, which is nonempty since y_s lives in it. As x is Tr1, we may choose $a \in \mathbb{F}^+(\mathcal{S})$ so that $\sigma_a(x) \in U$, noting $\sigma_{a+s-g_i}(x) \in U_{g_i}$ for every i. Take any $\widetilde{x} = \sum_{g\in\mathcal{G}} x_g\xi_g \in p^{-1}\{x\}$, and observe that for every $i = 1, \cdots, n$, we obtain Figure 4.8.

From this, we see that $\widetilde{\sigma}_{s+a}\left(\widetilde{x}\right) = \sum_{g\in\mathcal{G}} x_{g-s-a}\xi_g \in \widetilde{U}$.

For the converse, choose any $y \in X$ and neighbourhood U of y. Set $\widetilde{U} = p^{-1}(U)$, and choose $w \in \mathbb{F}^+(\mathcal{S})$ so that $\widetilde{\sigma}_w\left(\widetilde{x}\right) \in \widetilde{U}$. Then $p\left(\widetilde{\sigma}_w\left(\widetilde{x}\right)\right) = \sigma_w(x)$ lives in $p\left(\widetilde{U}\right) = U$. $\square$

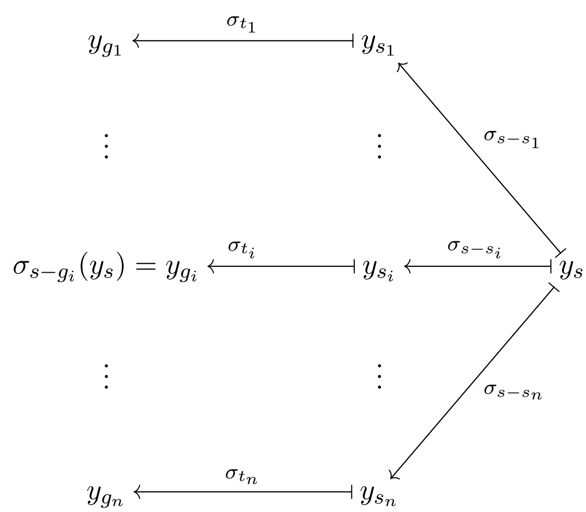


Figure 4.7. The relation between terms in $\widetilde{y}$

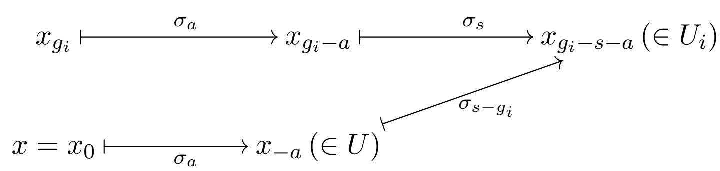


Figure 4.8. The relation between terms of $\widetilde{x}$

Proposition 24. *x is Tr3 for (X, σ) if and only if any $\widetilde{x} \in p^{-1}\{x\}$ is Tr3 for $\left(\widetilde{X}, \widetilde{\sigma}\right)$.*

Proof. Start with the forward direction. Take any $\widetilde{y} = \sum_{g \in \mathcal{G}} y_g \xi_g \in \widetilde{X}$ and open $\widetilde{U} \subseteq \widetilde{X}$ containing $\widetilde{y}$. Now take any $\widetilde{x} = \sum_{g \in \mathcal{G}} x_g \xi_g \in p^{-1}\{x\}$ and $u \in \mathbb{F}^+(\mathcal{S})$. We may assume $\widetilde{U}$ has form $\widetilde{U} = \widetilde{X} \cap \Pi_{g \in \mathcal{G}} U_g$, where each U_g is open in X and $U_g = X$ except for g in some finite set $\{g_i\}_{i=1}^n$. For every $i = 1, \cdots, n$, choose $s_i, t_i \in \mathcal{S}$ so that $g_i = s_i - t_i$, and set $s = \sum_{i=1}^n s_i \in \mathcal{S}$.

Observe that $\sigma_{s-g_i}(y_s) = y_{g_i}$, so $y_s \in \sigma_{s-g_i}(U_{g_i})$, $i = 1, \cdots, n$. Set $U = \cap_{i=1}^n \sigma_{s-g_i}(U_{g_i})$, which is open and nonempty. As x is Tr3, we may choose $a \in \mathbb{F}^+(\mathcal{S})$ so that $\sigma_{u+a}(x) \in U$; notice that $\sigma_{u+a+s-g_i}(x) = \sigma_{u+a+s}(x_{g_i}) \in U_{g_i}$, $i = 1, \cdots, n$. Hence, $\widetilde{\sigma}_{u+a+s}\left(\widetilde{x}\right) = \sum_{g \in \mathcal{G}} \sigma_{u+a+s}(x_g)\xi_g$. Thus, $\widetilde{x} \in \widetilde{\sigma}_{u+a+s}\left(\widetilde{U}\right)$. Since $a + s \in \mathbb{F}^+(\mathcal{S})$, $\widetilde{x}$ is Tr3 for $\left(\widetilde{X}, \widetilde{\sigma}\right)$.

Now the converse. Take any $y \in X$ and open $U \subseteq X$ containing y, and take any $u \in \mathbb{F}^+(\mathcal{S})$. Set $\widetilde{U} = p^{-1}(U)$, and choose $a \in \mathbb{F}^+(\mathcal{S})$ so that $\widetilde{\sigma}_{u+a}\left(\widetilde{x}\right) \in \widetilde{U}$. It follows that $p\left(\widetilde{\sigma}_{u+a}\left(\widetilde{x}\right)\right) = \sigma_{u+a}(x)$ is in U. $\square$

Remark 20. A space having a transitive point and a space being topologically transitive are not equivalent conditions in general. However, a proposition for topological transitivity analogous to the one above for transitive points can be proved similarly.

Proposition 25. *(X, σ) is TTr1 if and only if $\left(\widetilde{X}, \widetilde{\sigma}\right)$ is TTr1.*

Proof. Suppose (X, σ) is topologically transitive. Take any open, nonempty $\widetilde{U}, \widetilde{V} \subset \widetilde{X}$; we can assume that these are basic sets of the form $\widetilde{U} = \widetilde{X} \cap \prod_{g \in \mathcal{G}} U_g$ and $\widetilde{V} = \widetilde{X} \cap \prod_{g \in \mathcal{G}} V_g$, where U_g and V_g are X except finitely often. Let $\{g_i\}_{i=1}^n \subseteq \mathcal{G}$ be where $U_{g_i} \neq X$ or $V_{g_i} \neq X$. For every i, choose $s_i, t_i \in \mathcal{S}$ so that $g_i = s_i - t_i$, and set $s = \sum_{i=1}^n s_i$. Consider $U := p\left(\widetilde{U}\right)$ and $V := \bigcap_{i=1}^n \sigma_{s-g_i}^{-1}(V_{g_i})$. Since (X, σ) is TTr1, we have $a \in \mathbb{F}^+(\mathcal{S})$ so that $\sigma_a(U) \cap V \neq \emptyset$.

Take $x \in \sigma_a(U) \cap V$, and choose $\widetilde{x} = \sum_{g \in \mathcal{G}} x_g \xi_g \in \widetilde{X}$ so that $x_s = x$. Observe that $\sigma_{s-g_i}(x_s) = x_{g_i} \in V_{g_i}$ for every i, so $\widetilde{x} \in \widetilde{V}$. Now, since $x \in (p \circ \widetilde{\sigma}_a)\left(\widetilde{U}\right)$, we have $\widetilde{y} = \sum_{g \in \mathcal{G}} y_g \xi_g \in \widetilde{U}$ so that $x = y_{-a}$. In particular, $\widetilde{\sigma}_{s+a}(\widetilde{y}) = \widetilde{x}$, so $\widetilde{\sigma}_{s+a}\left(\widetilde{U}\right) \cap \widetilde{V} \neq \emptyset$.

Now for the converse. Take open, nonempty $U, V \subset X$. Let $\widetilde{U} = p^{-1}(U)$ and $\widetilde{V} = p^{-1}(V)$. Choose $w \in \mathbb{F}^+$ so that $\widetilde{\sigma}_w\left(\widetilde{U}\right) \cap \widetilde{V} \neq \emptyset$; take $\widetilde{x}$ in this intersection. Notice that $p(\widetilde{x}) \in (p \circ \widetilde{\sigma}_w)\left(\widetilde{U}\right) = (\sigma_w \circ p)\left(\widetilde{U}\right)$, so $p(\widetilde{x}) \in \sigma_w(U)$. □

5. MAXIMAL ENVELOPE EXAMPLE

In this section we consider from [15] that the maximal C^*-algebra for the upper triangular $n \times n$ matrices is contained in the matrices over the free product of copies of $C[0,1]$. We investigate how the free product multiplication interacts with the matrix multiplication to describe what component-wise entries in the maximal C^*-algebra will look like.

In Section 1.1, we saw that there are many way to embed an operator algebra into a C^*-algebra. In this example, we focus on the maximal C^*-algebra $C^*_{max}(A)$. Recall Definition 10: for an operator algebra A, there exists a largest C^*-cover $C^*_{\max}(A)$. Formally, there is a completely isometric $j : A \to C^*_{\max}(A)$ with the following universal property: if π is a completely contractive homomorphism from A into a C^*-algebra D, then there is a unique *-homomorphism $\widetilde{\pi} : C^*_{\max}(A) \to D$ such that $\widetilde{\pi} \circ j = \pi$.

We will connect this to another C^*-algebra, the amalgamated free product.

Definition 37. Suppose A, B, D are operator-algebras and $\alpha_A : D \to A$ and $\alpha_B : D \to B$ are completely isometric invlusions. The *amalgamated free product* $A *_D B$ is the unique operator algebra generated by $f_A(A) \cup f_B(B)$, where $f_A : A \to A *_D B$ and $f_B : B \to A *_D B$ are injective completely isometric representations that satisfy $f_A \circ \alpha_A = f_B \circ \alpha_B$ and have the universal property that given an operator algebra H and a completely contractive homomorphism $h_A : A \to H$ and $h_B : B \to H$ satisfying $h_A \circ \alpha_A = h_B \circ \alpha_B$, there is $\widetilde{h} : A *_D B \to H$ with $\widetilde{h} \circ f_A = h_A$ and $\widetilde{h} \circ f_B = h_B$.

In general, the algebra $A *_D B$ is spanned by elements of the form $a_1 * b_1 * \cdots * a_n * b_n$ which is a formal multiplication with the only way for elements to "pass" across $*$ is for them to be in d. For more information about free products see [24]. In [2], it is shown that these two constructions are related in the following sense.

Proposition 26. *[2] Let A, B, and D be operator algebras with D a C^*-algebra. Then $C^*_{max}(A *_D B) \cong C^*_{max}(A) *_D C^*_{max}(B)$.*

(Sketch of proof)

We denote by $j_A : A \to C^*_{max}(A)$ and $j_B : B \to C^*_{max}(B)$ the completely isometric inclusions. Then by the way the free product is constructed $j_A|_D = j_A|_D$ and hence there is a a completely contractive representation $j_A * j_B : A *_D B \to C^*_{max}(A) *_D C^*_{max}(B)$ and by the universal property for C^*_{max} there is a completely contractive homomorphism $\pi : C^*_{max}(A *_D B) \to C^*_{max}(A) *_D C^*_{max}(B)$.

On the other hand, there are completely isometric inclusions $i_A : A \to A *_D B$ and $i_B : B \to A *_D B$. By the universal property for C^*_{max} these lift to $*$-homomorphisms $I_A : C^*_{max}(A) \to C^*_{max}(A *_D B)$ and $I_B : C^*_{max}(B) \to C^*_{max}(A *_D B)$. Verifying that $I_A|_D = I_B|_D$, yields a map $I_A * I_B : C^*_{max}(A) *_D C^*_{max}(B) \to C^*_{max}(A *_D B)$. Composing π and $I_A *_D I_B$ induces the identity map on the subalgebra generated by A and B and hence the two maps must be $*$-isomorphisms, proving the proposition.

How this helps us understand C^*-max for graph operator algebras is due to two facts about graph algebras. The first is that T_n the algebra of upper triangular $n \times n$ matrices is the graph algebra for the graph with n-vertices

$$\bullet \longrightarrow \bullet \longrightarrow \cdots \longrightarrow \bullet\,.$$

The second requires some notation. We let D_n denote the algebra of $n \times n$ diagonal matrices inside M_n.

The result is a direct application of Theorem 4 from [13] which in our context tells us that

Proposition 27. *The algebra T_n is completely isometrically isomorphic to*

$$(T_2 \oplus C^{n-2}) *_{D_n} (\mathbb{C} \oplus T_2 \oplus \mathbb{C}^{n-3}) *_{D_n} \cdots *_{D_n} (\mathbb{C}^{n-2} \oplus T_2).$$

From there we can conclude the following about $C^*_{Max}(T_n)$.

Proposition 28. *For all n*

$$C^*_{max}(T_n) \cong (C^*_{max}(T_2) \oplus C^{n-2}) *_{D_n} (\mathbb{C} \oplus C^*_{max}(T_2) \oplus \mathbb{C}^{n-3}) *_{D_n} \cdots *_{D_n} (\mathbb{C}^{n-2} \oplus C^*_{max}(T_2)).$$

When we recall that

$$C^*_{max}(T_2) = \left\{ \begin{bmatrix} f_1 & f_2 \\ f_3 & f_4 \end{bmatrix} : f_i \in C[0,1] \text{ and } f_2(0) = 0 = f_3(0) \right\}$$

we can think about $C^*_{max}(T_n)$ as free products of matrices over functions algebras. There are many technicalities to prove but in, as yet unpublished work, Duncan [14] showed that

$$C^*_{max}(T_n) \subset M_n(C([0,1]) *_{\mathbb{C}} C([0,1]) *_{\mathbb{C}} \cdots *_{\mathbb{C}} C([0,1])).$$

In what follows we will look at what elements of C^*_{max} look like inside $M_n(C([0,1]) *_{\mathbb{C}} C([0,1]) *_{\mathbb{C}} \cdots *_{\mathbb{C}} C([0,1]))$.

We build sets as follows where $F = C^*_{max}(T_2)$.

$$H_{j,k} = \overbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}^{j-1} \oplus F \oplus \overbrace{\mathbb{C} \oplus \cdots \oplus \mathbb{C}}^{k-1-j}$$

Thinking of those as matrices in M_k, we have

$$\begin{aligned} H_{j,k} = & \Big\{ \lambda^j_{1,1} e_{1,1} + \cdots + \lambda^j_{j-1,j-1} e_{j-1,j-1} + f^j_{j,j} e_{j,j} + f^j_{j+1,j+1} e_{j+1,j+1} \\ & + \lambda^j_{j+2,j+2} e_{j+2,j+2} + \cdots + \lambda^j_{k,k} e_{k,k} + g^j_{j+1,j} e_{j+1,j} + g^j_{j,j+1} e_{j,j+1} \Big\} \end{aligned}$$

In these sets, the non zero diagonal off entries are in these positions:

$$\begin{aligned} \{a_{i,j} e_{i,j} : i - j = 1\} &= \left\{ g^i_{i+1,i} e_{i+1,i} \right\} \\ \{a_{i,j} e_{i,j} : i - j = -1\} &= \left\{ g^i_{i,i+1} e_{i,i+1} \right\} \end{aligned}$$

Each $H_{j,k}$ is allowed non zero entries along each diagonal entry.

If we look at any free product, we only consider generating elements $h^k_{i,j} e_{i,j}$ rather than full matrices. We introduce a notational convention for looking at non zero entries, noting these only occur in $i - i$, $(i+1) - i$, or $i - (i+1)$ positions. For

$$h^{k_1}_{i_1,j_1} e_{i_1,j_1} *_D \cdots *_D h^{k_m}_{i_m,j_m} e_{i_m,j_m} = h^{k_1}_{i_1,j_1} *_D \cdots *_D h^{k_m}_{i_m,j_m} e_{i_1,i_{m+1}},$$

where $j_\ell = i_{\ell+1}$ for $\ell = 1, \cdots, m-1$ and $i_{m+1} = j_m$. In this, $i_\ell - i_{\ell+1} \in \{-1, 0, 1\}$.

If $i_\ell = i_{\ell+1}$, then $h^{k_\ell}_{i_\ell, i_{\ell+1}} \in \left\{\mathbb{C}, \left\{f^\ell_{\ell,\ell}\right\}, \left\{f^{\ell-1}_{\ell,\ell}\right\}\right\}$

If $i_\ell = i_{\ell+1} + 1$, then $h^{k_\ell}_{i_\ell, i_{\ell+1}} \in \left\{g^\ell_{\ell+1,\ell}\right\}$

If $i_\ell = i_{\ell+1} - 1$, then $h^{k_\ell}_{i_\ell, i_{\ell+1}} \in \left\{g^\ell_{\ell,\ell+1}\right\}$

Observe that

$$\begin{aligned} i_1 - i_{m+1} &= (i_1 - i_2) + (i_2 - i_3) + \cdots + (i_m - i_{m+1}) \\ &= |\{\ell : i_\ell - i_{\ell+1} = 1\}| - |\{\ell : i_\ell - i_{\ell+1} = -1\}| \end{aligned}$$

Take any $e_{i,j}$. If we multiply by a matrix unit on the right and obtain a nonzero result, that matrix is of the form $e_{j,*}$, if we require $j - * \in \{-1, 0, 1\}$. That is, we have $e_{j,j-1}$, $e_{j,j}$, or $e_{j,j+1}$. Following this pattern, we can construct a diagram, as seen in Figure 5.1, where three arrows point from the unit matrix to each of the possibilities of unit matrices that can give us this result.

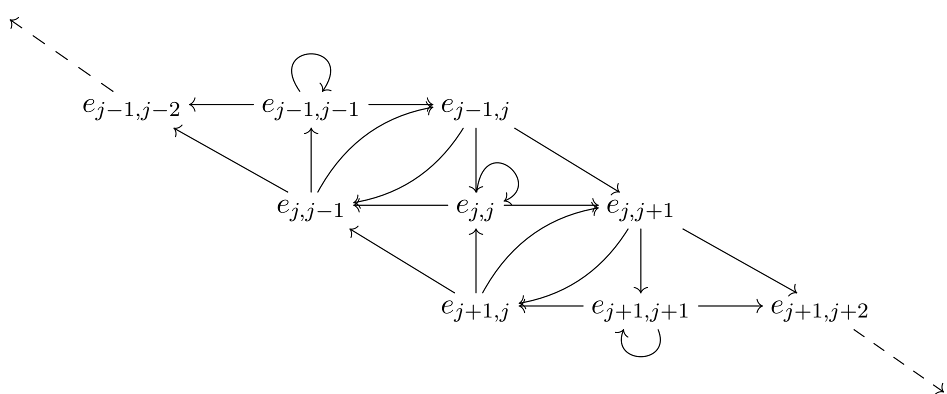


Figure 5.1. The possible paths are shown.

To get from $e_{j,j-1}$ to an element of the form $e_{j-1,*}$, we must pass through $e_{j-1,j-2}$. Adjusting the indices above as appropriate, for $n > 0$, we see that to get from $e_{j-n,j-(n+1)}$ to something of the form $e_{j-(n+1),*}$, we must pass through $e_{j-(n+1),j-(n+2)}$.

To get from $e_{j,j+1}$ to something of the form $e_{j+1,*}$, we must pass through $e_{j+1,j+2}$. Adjusting the indices above as appropriate, for $n > 0$, we see that to get from $e_{j+n,j+(n+1)}$ to something of the form $e_{j+(n+1),*}$, we must pass through $e_{j+(n+1),j+(n+2)}$.

We look again at

$$e_{i_1,i_{m+1}} = e_{i_1,i_2}e_{i_2,i_3}\cdots e_{i_{m-1},i_m}e_{i_m,i_{m+1}}$$

Let $M = i_{m+1} - i_1$. Without loss of generality, assume $M > 0$. We let $N_1 = i_1 + 1$, $N_2 = N_1 + 1 = i_1 + 2$, $\cdots$, and $N_{M-1} = N_{M-1} + 1 = i_1 + M - 1$. Let $N_0 = i_1$ and $N_M = i_{m+1}$.

In the sequence of indices $\{i_1, \cdots, i_{m+1}\}$, we have a subsequence $\{j_\ell\}_{\ell=1}^{M}$ with j_ℓ being the first instance of N_ℓ that gives us $e_{N_{\ell-1},N_\ell}$.

While this allows us to consider what the entries in the matrices might look like, it does not give us a general method that extends outside this context. This method does not practically translate into understanding C^*_{max} for even other finite graphs, even when using the free product construction.

REFERENCES

[1] Blackadar, B. *Operator Algebras and Non-Commutative Geometry. 3 : Theory of C*-Algebras and von Neumann Algebras (Encyclopedia of Mathematics)* Springer, 2005.

[2] Blecher, D. Modules over operator algebras, and the maximal C^*-dilation. *J. Funct. Anal.* **169** (1999), 251-288.

[3] Blecher, D. and C. Le Merdy. *Operator Algebras and Their Modules : An Operator Space Approach* Oxford, Clarendon, 2010.

[4] Clifford, A. and G. Preston. *The Algebraic Theory of Semigroups. 2nd ed., vol. 1* American Mathematical Society, 1961.

[5] Clouâtre, R. and C. Ramsey. Residually finite-dimensional operator algebras. *J. Funct. Anal.* **277** (2019), 2572-2616.

[6] Davidson, K. *C*-Algebras by Example* American Mathematical Society, 1997.

[7] Davidson, K. and A. Donsig. *Real analysis with real applications* Prentice Hall, 2002.

[8] Davidson, K., Fuller, A. and E. Kakariadis. Semicrossed Products of Operator Algebras by Semigroups, *Mem. Amer. Math. Soc.* **247** (2014), no. 1168.

[9] Davidson, K. and E. Katsoulis. Operator algebras for multivariable dynamics *Mem. Amer. Math. Soc.* **209** (2011), no. 982.

[10] Douglas, R. *Banach Algebra Techniques in Operator Theory (Graduate Texts in Mathematics, 179) (2nd ed.)* Springer, 1998.

[11] Duchsherer, M. and B. Duncan. Some point derivations for semicrossed products *Linear Algebra Appl.* **459** (2014), 237-247.

[12] Duncan, B. Finite dimensional point derivations for graph algebras *Illinois J. Math.* **52** (2008), 419-435.

[13] Duncan, B. Certain free products of graph operator algebras *J. Math. Anal. Appl.* **364** (2010), 534-543.

[14] Duncan, B. Identification of maximal C^*-covers of some operator algebras, in preparation, 2021.

[15] Duncan, B. and J. Peters. Operator algebras and representations from commuting semigroup actions *J. Oper. Theory* **74** (2010), 23-43.

[16] Katsoulis, E. Local maps and the representation theory of operator algebras *Trans. Amer. Math. Soc.* **368** (2015), 5377-5397.

[17] Kribs, D. and S. Power. Free semigroupoid algebras *J. Ramanujan Math. Soc.* **19** (2004), 117-159

[18] MacCluer, B. *Elementary Functional Analysis* Springer New York, 2009.

[19] Paulsen, V. *Completely Bounded Maps and Operator Algebras* Cambridge University Press, 2002.

[20] Peters, J. Semi-crossed products of C^*-algebras *J. Func. Anal.* **59** (1984), 498-534.

[21] Peters, J. The C^*-envelope of a semicrossed product and nest representations *Contemp. Math.* **503** (2009) 197-215.

[22] Raeburn Iain. Graph algebras *CBMS Regional Conference Series in Mathematics* **103** American Mathematical Society, 2005.

[23] Ramsey, C. Automorphisms of free products and their application to multivariable dynamics *Indiana Univ. Math. J.* **65** (2016), 899-924.

[24] Voiculescu, D., Dykema, K. and A. Nica. Free random variables. A noncommutative probability approach to free products with applications to random matrices, operator algebras and harmonic analysis on free groups. *CRM Monograph Series, 1* American Mathematical Society, 1992.

[25] Williams, D. *Crossed Products of C^*- Algebras* American Mathematical Society, 2007.