ON PARTIAL PERMUTATIONS AND ALTERNATING SIGN MATRICES: BIJECTIONS AND POLYTOPES

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The supervisory committee certifies that this dissertation complies with North Dakota State University's regulations and meets the accepted standards for the degree of

DOCTOR OF PHILOSOPHY

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ABSTRACT

Motivated by the study of chained permutations and alternating sign matrices, we investigate partial permutations and alternating sign matrices. We give a length generating function for partial permutations and show bijections relating certain subsets to decorated permutations and set partitions. We prove bijections among partial alternating sign matrices and several other combinatorial objects as well as results related to their dynamics, analogous to those in the usual alternating sign matrix setting. We also study families of polytopes which are the convex hulls of these matrices. We determine inequality descriptions, facet enumerations, and face lattice descriptions. Finally, we study partial permutohedra which arise naturally as projections of these polytopes, revealing connections to graph associahedra.

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Next, I would like to thank the rest of the NDSU mathematics department. In particular, those professors who agreed to be on my committee, Jason Boynton, Cătălin Ciupercă, and Torin Greenwood - whether they realize it or not, they have each played a key part in my journey through graduate school. A special thanks goes to Maxx Kureczko for his friendship and mentorship throughout my time at NDSU as well. I would be remiss if I did not also mention Tina Exner. Her smile and positive attitude truly helped make even the most stressful days more bearable. And of course, thank you to Mark Strand for being on my committee on short notice.

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DEDICATION

For Oma and Opa.

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1. INTRODUCTION

A classical enumeration problem in combinatorics is to find the number of ways to place m non-attacking rooks on an $n \times n$ chess board. Many variations of this problem have been studied as well: one can change the shape of the board, the types of pieces, introduce restricted positions, and more. One recent generalization of the non-attacking rook problem was inspired by 3-person chess [25]. The motivating problem for this study was to figure out the number of ways to place m non-attacking rooks on the three-person chessboard in Figure 1.1. This was solved and further generalized to new families of boards, namely k boards of size $n \times n$ chained together in circular or linear configurations. The original case corresponding to 3-person chess is, in this context, 6 chained together 4×4 boards in the circular configuration. An example of the original board and how to interpret it in the chained case is given in Figure 1.1.

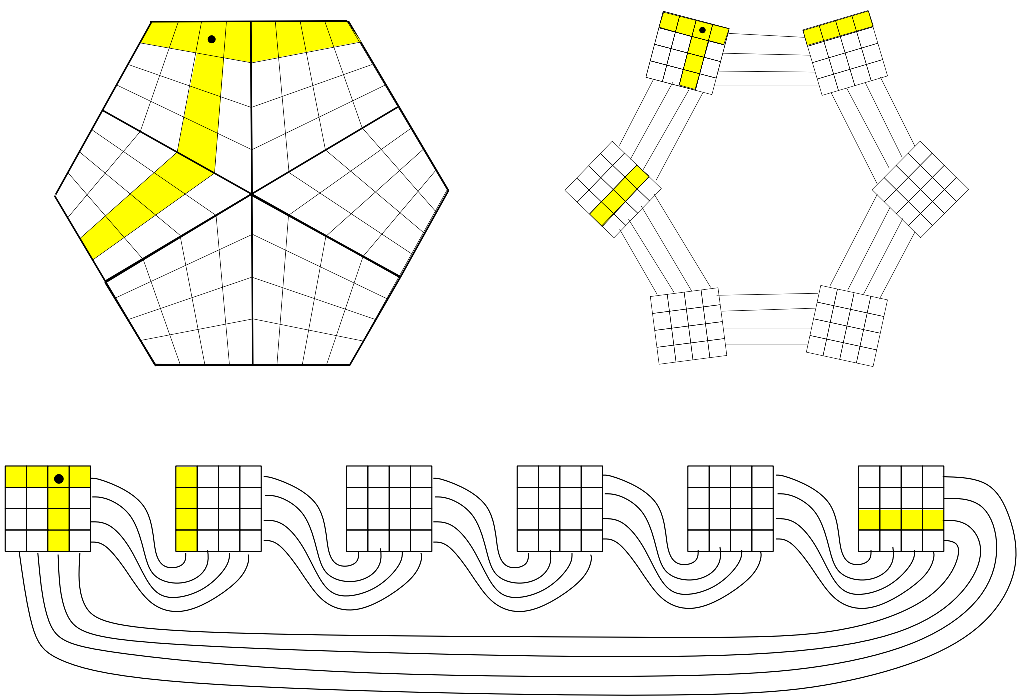


Figure 1.1. Top left: the original 3-person chess board, with a dot representing a rook, and highlights representing the positions where that rook can attack. Top right: the board "pulled apart" into six 4×4 boards. Bottom: the board in a "standard position," helping interpret rows and columns.

In the classical setting, placements of n non-attacking rooks on an $n \times n$ board corresponds naturally with *permutation matrices*, which have an interesting generalization called *alternating sign matrices*. Relevant background information on permutations and alternating sign matrices can be found in Chapter 2. In the chained setting, there is a similar correspondence between rook placements on chained boards and *chained permutations*, which can be then generalized to *chained alternating sign matrices*. Examples of these are shown found in Figure 1.2, with definitions and explanations in [25].

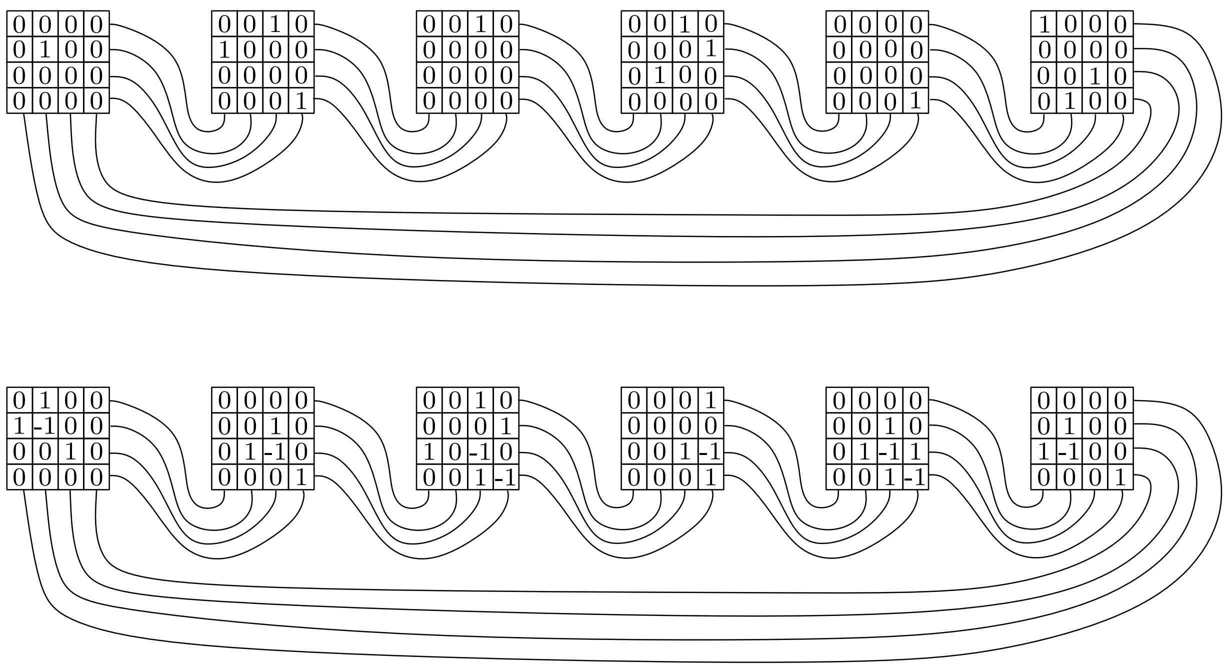


Figure 1.2. An example of a chained permutation (top) and chained alternating sign matrix (bottom).

In [25], we enumerated chained permutations (maximum rook placements) in both the linear and circular case. The enumeration depends on the *composition* of a chained permutation, which is a tuple recording the number of ones on each $n \times n$ *component*. For example, in Figure 1.2, the chained permutation has six 4×4 components and composition $(1, 3, 1, 3, 1, 3)$. We also found bijections to other combinatorial objects on both chained permutations and alternating sign matrices.

These chained generalizations of permutations and alternating sign matrices inspired further study. In particular, we consider a *partial order* on chained permutations analogous to the *strong Bruhat order* in the usual permutation setting. We hope to show that this partial order extends chained alternating sign matrices in such a way that the *poset* of chained alternating sign matrices is the *MacNeille completion* of the poset of chained permutations, which would be an interesting analogue to a result of Lascoux and Schützenberger in the classic setting [27]. See Chapter 5 for a short summary of progress on this problem. In order to aid in this endeavor, we sought to better understand the properties and structure of the individual components of chained permutations and alternating sign matrices, which led us to the main subjects of this thesis: *partial permutations and alternating sign matrices.*

We begin in Chapter 2 by providing a summary of well-known combinatorial definitions and results which may be useful to help understand the main results, found in Chapters 3 and 4. We split this into four main topics: permutations (Section 2.1), alternating sign matrices (Section 2.2), dynamics (Section 2.3), and polytopes (Section 2.4).

Then in Chapter 3, we study rectangular analogues of permutations and alternating sign matrices. We begin in Section 3.1, by focusing on $P_{m,n}$, the set of $m \times n$ partial permutations matrices. Our main results therein are comprised of the generating function result below and two interesting bijections. These bijections related certain subsets of partial permutations to *decorated permutations* and *set partitions.*

Theorem 3.1.6. *The length generating function of $m \times n$ partial permutations is given by:*

$$\sum_{w \in P_{m,n}} q^{\ell(w)} = \sum_{r=0}^{m} \left(\begin{bmatrix} m \\ r \end{bmatrix}_q ([n]_q)_r \cdot q^{\frac{r(r+1)}{2}} \right).$$

In Section 3.2, turn to our main topic: $m \times n$ partial alternating sign matrices. We explore several bijections, which in most cases, are analogues of those studied in the usual alternating sign matrix setting (see [37, 47]). These bijections culminate in our main result of this chapter, which is the following theorem. See Figure 1.3 for an example of each of the objects in the statement of the theorem.

Theorem 3.2.6. *There are explicit bijections among $m \times n$ partial alternating sign matrices, (m,n)-partial monotone triangles, (m,n)-partial height function matrices, (m,n)-partial fully-packed loop configurations, (m,n)-rectangular ice configurations, the set of order ideals of $\boldsymbol{P}_{m,n}$, and (m,n)-nests of osculating paths.*

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 \end{pmatrix} \qquad \begin{array}{ccccccc} & & & 1 & & & \\ & & 1 & & 3 & & \\ & 0 & & 2 & & 3 & \\ 0 & & 1 & & 2 & & 4 \end{array} \qquad \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 3 \\ 2 & 3 & 2 & 3 & 2 \\ 3 & 4 & 3 & 2 & 3 \\ 4 & 3 & 4 & 3 & 2 \end{pmatrix}$$

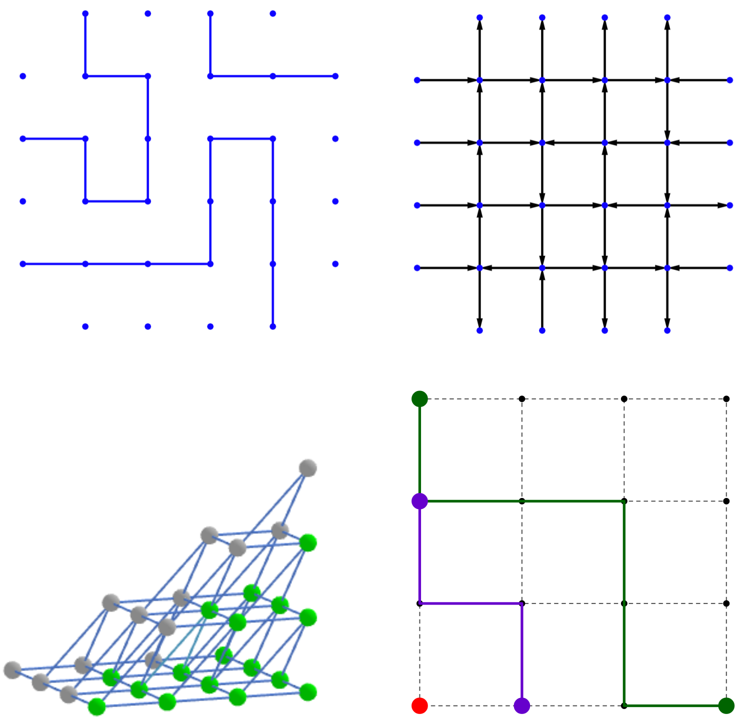


Figure 1.3. From left to right, top to bottom: a 4×4 partial alternating sign matrix, along with its corresponding partial monotone triangle, partial height function matrix, partial fully-packed loop configuration, rectangular ice configuration, order ideal, and nest of osculating paths.

Such bijections among combinatorial objects provide new angles from which to study them. For example, after these bijections, in Section 3.3, we apply an analogue of Wieland's gyration action [51] on fully-packed loop configurations, and following [47], we relate this to the study of toggles and order ideals. We obtain the following theorem, showing that rowmotion on order ideals of a certain poset and gyration on partial fully-packed loop configurations have the same orbit structure.

Theorem 3.3.6. *$J(\boldsymbol{P}_{m,n})$ under Row and (m,n)-partial fully-packed loop configurations under gyration are in equivariant bijection.*

In Chapter 4, we study polytopes which are built from the matrices defined in Sections 3.1 and 3.2. We continue the study of analogous results to those in the usual permutation and alternating sign matrix cases, now in the realm of polytopes, and reveal new connections to graph associahedra. In Section 4.1, we briefly discuss the matrices of the previous chapter, giving equivalent definitions that are more consistent with existing polytope literature. In Section 4.2, we summarize some known results on *partial permutation polytopes*. In Section 4.3, we define a new family of polytopes, *partial alternating sign matrix polytopes*, denoted $\mathrm{PASM}(m,n)$. We determine their inequality descriptions, facet enumerations and face lattice description in terms of labeled graphs, as stated in the theorems below.

Theorem 4.3.6. *$\mathrm{PASM}(m,n)$ consists of all $m \times n$ real matrices $X = (X_{ij})$ such that:*

$$0 \leq \sum_{i'=1}^{i} X_{i'j} \leq 1, \qquad \text{for all } 1 \leq i \leq m, 1 \leq j \leq n,$$

$$0 \leq \sum_{j'=1}^{j} X_{ij'} \leq 1, \qquad \text{for all } 1 \leq i \leq m, 1 \leq j \leq n.$$

Theorem 4.3.8. *The number of facets of $\mathrm{PASM}(m,n)$ equals $4mn - 3m - 3n + 5$.*

Theorem 4.3.18. *Let F be a face of $\mathrm{PASM}(m,n)$ and $\mathcal{M}(F)$ be equal to the set of partial alternating sign matrices that are vertices of F. The map $\psi : F \mapsto g(\mathcal{M}(F))$ induces an isomorphism between the face lattice of $\mathrm{PASM}(m,n)$ and the set of sum-labelings of $\Gamma_{(m,n)}$ ordered by containment. Moreover, the dimension of F equals the number of regions of $\psi(F)$.*

In Section 4.4, we discuss *partial permutohedra*, denoted $\mathcal{P}(m,n)$. We then determine inequality descriptions and facet enumerations and characterize their face lattice using chains in the Boolean lattice. The following three theorems from Section 4.4 comprise our second set of main results of Chapter 4.

Theorem 4.4.9. *$\mathcal{P}(m,n)$ consists of all vectors $u \in \mathbb{R}^m$ such that:*

$$\sum_{i \in S} u_i \leq \binom{n+1}{2} - \binom{n-k+1}{2}, \qquad \text{where } S \subseteq [m], |S| = k \neq 0, \text{ and}$$

$$u_i \geq 0, \qquad \text{for all } 1 \leq i \leq m.$$

Theorem 4.4.10. *The number of facets of $\mathcal{P}(m,n)$ equals $m + 2^m - 1 - \sum_{r=1}^{m-n} \binom{m}{m-r}$.*

We reinterpret the result [29, Prop. 56] that $\mathcal{P}(m,m)$ is a graph associahedron called the *stellohedron* to prove an alternate characterization of its face lattice in terms of chains in the Boolean lattice. This connection is helpful conceptually, as it relates the face structure of these polytopes to familiar combinatorial objects.

Theorem 4.4.23. *The face lattice of $\mathcal{P}(m,m)$ is isomorphic to the lattice of chains in $\mathcal{B}_m$, where $C < C'$ if C' can be obtained from C by iterations of (1) and/or (2) from Lemma 4.4.20. A face of $\mathcal{P}(m,m)$ is of dimension k if and only if the corresponding chain has k missing ranks.*

We conjecture a similar face lattice characterization for $\mathcal{P}(m,n)$ in the case $m \neq n$.

We furthermore connect these polytopes by showing PASM(m,n) projects to $\mathcal{P}(m,n)$, by a similar technique used to show that alternating sign matrix polytopes project to permutohedra [46]. Our last main result is as follows; here ϕ_z is the map that multiplies a matrix by z on the right and $\mathcal{P}_z$ is a generalized partial permutohedron determined by z.

Theorem 4.4.27. *Let z be a strictly decreasing vector in $\mathbb{R}^m$. Then $\phi_z(\text{PASM}(m,n)) = \mathcal{P}_z(m,n)$.*

This projection gives us a way to connect matrix polytopes to graph associahedra.

Finally, we have computed the volume and Ehrhart polynomials of the polytopes studied in this paper. We note that the Ehrhart polynomials we were able to compute have positive coefficients, and have found the following result and conjecture regarding the volume of the partial permutohedron.

Theorem 4.4.28. *$\mathcal{P}(2, n)$ has normalized volume equal to $2n^2 - 1$.*

Conjecture 4.4.29. *$\mathcal{P}(m, 2)$ has normalized volume equal to $3^m - m$.*

2. PRELIMINARIES

In this chapter, we give preliminary definitions, results, and references that the reader may find helpful in understanding the relevance and content of our main results, which appear in Chapters 3 and 4.

2.1. Permutations

Permutations are foundational combinatorial objects. They are well-studied and lie at the core of many enumeration problems, such as counting non-attacking rook placements. Permutations are also widely used in mathematics and other disciplines; they make appearances in group theory, computer science, biology, and even quantum physics. Simply put, a permutation of length n is a linear arrangement of the elements of the set $[n] := \{1, 2, \ldots, n\}$. For example, 462513 is a permutation of length 6. We will denote the set of all permutations of length n as S_n. A *fixed point* of a permutation $w = w_1 \ldots w_n$ is an element i such that $w_i = i$. An *r-permutation* of $[n]$ is a linear arrangement of r elements of $[n]$. For example, 6413 is a 4-permutation of [6]. The representation of the above permutation is called *one-line notation*; we will also view permutations as *permutation matrices*.

Definition 2.1.1. An $n \times n$ *permutation matrix* is a matrix whose entries are in $\{0, 1\}$ and whose rows and columns sum to 1. In other words, there is exactly one 1 in each row and column, and zeros elsewhere.

Permutation matrices can be viewed as maximal non-attacking rook placements on square boards. Given a permutation in one-line notation, $w = w_1 w_2 \cdots w_n$, one can construct its associated permutation matrix by putting a 1 in position (i, w_i) and zeros everywhere else. For example, the permutation 462513 has the permutation matrix given in Figure 2.1. The enumeration of permutations is a well-known, simple counting problem: there are $n!$ permutations of length n.

To help study the structure of combinatorial objects such as permutations, *combinatorial statistics* are often used. A *combinatorial statistic* is an integer given to each element of a set

$$\begin{pmatrix} 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}$$

Figure 2.1. The matrix representation of the permutation 462513.

of combinatorial objects. A statistic can be used to refine the enumeration of the set and help understand its structure. One such statistic on permutations is *inversion number.*

Definition 2.1.2. Let $w = w_1w_2 \dots w_n$ be a word whose elements are nonnegative integers. An *inversion of* w is a pair (i, j) such that $i < j$ but $w_i > w_j$. The *inversion number of* w, denoted $\mathrm{inv}(w)$, is the number of inversions of w. That is, $\mathrm{inv}(w) = |\{(i, j) \mid i < j,\ w_i > w_j\}|$.

In particular, we can compute the inversion number of a permutation. For our running example of $w = 462513$, we have $\mathrm{inv}(w) = 10$. The inversion number can be used to help quantify how "far away" a given permutation is from the *identity permutation*, $12 \cdots n$. It can also be used to refine the enumeration of permutations. A *q-analogue* of an expression is an extension or generalization of it involving the parameter q such that when you take the limit as q approaches 1, the original expression is returned. Define the q-analogue of a positive integer n to be $[n]_q := \left(1 + q + q^2 + \cdots + q^{n-1}\right)$. Let $[n]_q! := [n]_q[n-1]_q \cdots [2]_q[1]_q$ be the q-analogue of $n!$. We can similarly define the q-analogue of binomial coefficients: $\begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{[n]_q!}{[k]_q![n-k]_q!}$. It can be shown that $\sum_{w \in S_n} q^{\mathrm{inv}(w)} = [n]_q!$. This gives us a way to find how many inversions of a given length have inversion number of k by inspecting the coefficient of q^k in this polynomial. For example, when $n = 3$, we have $\sum_{w \in S_3} q^{\mathrm{inv}(w)} = [3]_q! = (1 + q + q^2)(1 + q) = 1 + 2q + 2q^2 + q^3$. This tells us that in S_3, there is one permutation with inversion number 0, two permutations with inversion number 1, two with inversion number 2, and one with inversion number 3. There is a similar well-known q-analogue result on *binary words*, that is, words whose digits are in $\{0, 1\}$. Denote by $B_{n,k}$ the set of binary words with n total digits, k of which are zeros. There are $\binom{n}{k}$ such binary words. Further, we have that $\displaystyle\sum_{w \in B_{n,k}} q^{\mathrm{inv}(w)} = \begin{bmatrix} n \\ k \end{bmatrix}_q$, giving a refined enumeration of binary numbers [44].

It is also of interest to study how certain actions affect words, and in particular permutations. An *adjacent transposition*, denoted s_i (where $1 \leq i \leq n-1$), is an action which swaps w_i and w_{i+1} in the permutation w written in one-line notation. This action swaps rows i and $i+1$ in the corresponding permutation matrix. Applying an adjacent transposition to a permutation produces another permutation whose inversion number differs by exactly one. For example, with $w = 462513$, we have $s_2 w = u = 426513$. Notice that $\text{inv}(w) = 10$ and $\text{inv}(u) = 9$. Another way to view the inversion number of a permutation is as the minimum number of adjacent transpositions that would need to be applied to get to the identity permutation. For example, $\text{inv}(4213) = 4$ and we can apply, in order, s_1, s_2, s_3, and s_1 to get from 4213 to 1234.

We can use the inversion number to construct a *partially ordered set* on S_n.

Definition 2.1.3. A *partially ordered set*, or *poset*, is a set P with a binary relation $\leq$ which is reflexive, antisymmetric, and transitive. We say y *covers* x in P (or x is covered by y) if $x < y$ and no element z exists in P such that $x < z < y$. A poset is often represented by its *Hasse diagram*, a graph whose vertices are the elements of P and whose edges are the covering relations; if $x < y$, then y is drawn above x in the Hasse diagram.

For more information about partially ordered sets, and for additional relevant terminology, see Chapter 3 of [44].

Two common partial orders on S_n are the *weak and strong Bruhat orders.* The weak order is defined as follows: v is covered by w if and only if $w = s_i v$ and $\text{inv}(w) = \text{inv}(v) + 1$. The strong Bruhat order allows any transposition (swapping w_i and w_j for any $i \neq j$) instead of just adjacent ones. Figure 2.2 shows the Hasse diagrams for both the weak and strong Bruhat orders on S_3.

2.2. Alternating Sign Matrices

A well-studied generalization of permutation matrices is *alternating sign matrices.* Alternating sign matrices have a very interesting history, enumeration, and connection to other areas of mathematics and science [9]. There exist beautiful bijections between alternating sign matrices and many other combinatorial objects, including *monotone triangles*, *height function matrices*, *fully-packed loop configurations*, *square ice configurations* and *order ideals* of a particular poset. In particular, the bijection with square ice configurations revealed connections to physics which

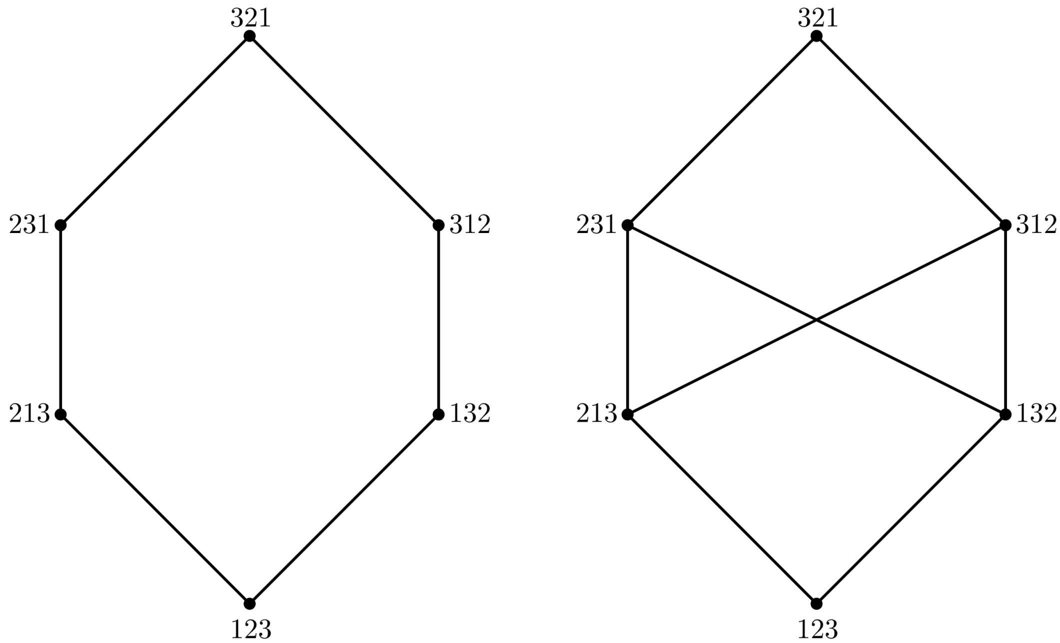


Figure 2.2. The weak Bruhat order (left) and strong Bruhat order (right) on S_3.

proved useful in one of the proofs of the enumeration of alternating sign matrices [26]. We will include a description of the bijection with monotone triangles in this section. See [37] and [47] for details on the other bijections. Generalizations of these bijections are given in Section 3.2. We begin with the definition of an alternating sign matrix.

Definition 2.2.1. An *alternating sign matrix of order* n is an $n \times n$ matrix with entries in $\{-1, 0, 1\}$ whose rows and columns sum to 1 and whose entries alternate in sign across each row and column. We will denote the set of all alternating sign matrices of order n by A_n.

Note that permutation matrices are the alternating sign matrices with no -1 entries. The only alternating sign matrix of order 3 which is not a permutation matrix is $\begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \\ 0 & 1 & 0 \end{pmatrix}$. Alternating sign matrices have a beautiful counting formula:

$$|A_n| = \prod_{j=0}^{n-1} \frac{(3j+1)!}{(n+j)!}.$$

This formula was conjectured in 1983 [32], and not proven until several years later [52, 26].

We now give the definition of a monotone triangle, and describe the bijection between monotone triangles and alternating sign matrices.

Definition 2.2.2. A *monotone triangle of order n* is a triangular array of positive integers $(a_{i,j})_{1 \leq i \leq j \leq n}$ taken from the set $[n]$ whose rows are strictly increasing and whose southwest-to-northeast and northwest-to-southeast diagonals are weakly increasing.

Given an alternating sign matrix, first construct its matrix of partial column sums. Then create the corresponding monotone triangle whose ith row consists of the values j for which entry (i, j) of this partial sum matrix is 1. An example is given in Figure 2.3. A natural partial order to put on monotone triangles is component-wise comparison. That is, $(a_{ij}) \leq (b_{ij})$ if $a_{ij} \leq b_{ij}$ for all i, j. The poset of monotone triangles with this partial order is closely related to the strong Bruhat order on S_n. It is known to be the *MacNeille completion* of the strong Bruhat order; in other words it is the smallest *lattice* which contains the strong Bruhat order [27]. See Figure 2.4 for an example of this poset with $n = 3$.

$$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & -1 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \longleftrightarrow \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 1 \\ 1 & 0 & 1 & 1 \\ 1 & 1 & 1 & 1 \end{pmatrix} \longleftrightarrow \begin{array}{ccccccc} & & & 2 & & & \\ & & 1 & & 4 & & \\ & 1 & & 3 & & 4 & \\ 1 & & 2 & & 3 & & 4 \end{array}$$

Figure 2.3. A 4×4 alternating sign matrix along with its matrix of partial column sums and its corresponding monotone triangle.

Another of the objects alternating sign matrices are in bijection with are fully-packed loop configurations (or just fully-packed loops), which we define below. In order to do so, we must first define a certain graph.

Definition 2.2.3. Define the graph $G_{m,n}$ as follows. The vertex set is:

$$V_{m,n} := \{v_{i,j} \ : \ 0 \leq i \leq m+1, \ 0 \leq j \leq n+1\} - \{v_{0,0}, v_{0,n+1}, v_{m+1,0}, v_{m+1,n+1}\}.$$

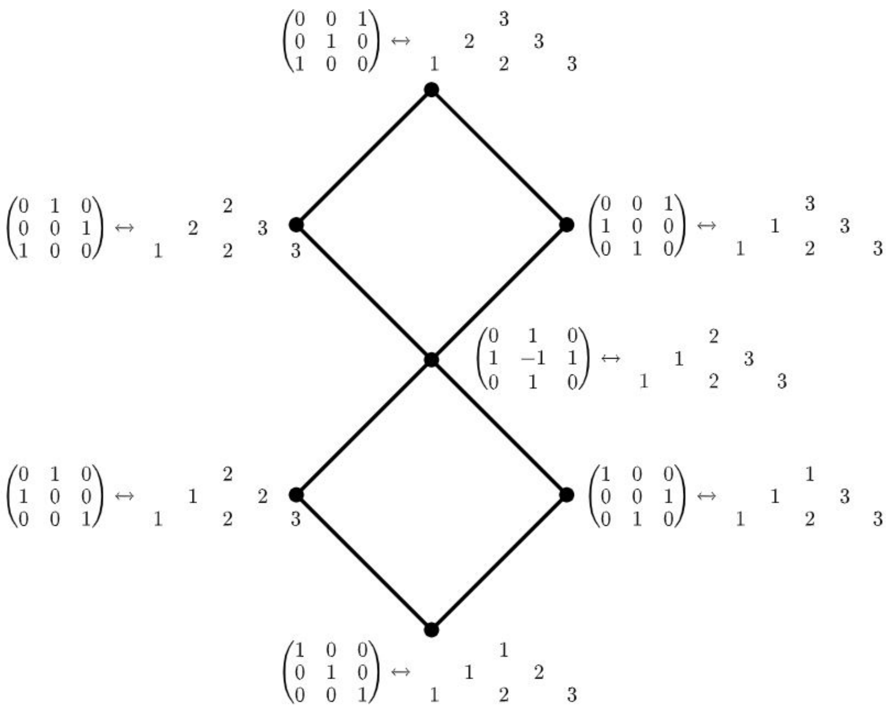


Figure 2.4. The poset of monotone triangles of order 3, ordered by component-wise comparison.

We say the *internal vertices* are $\{v_{i,j} : 1 \leq i \leq m,\ 1 \leq j \leq n\}$, and the remaining vertices are *boundary vertices.* We also say a vertex $v_{i,j}$ is *even* (resp. *odd*) if $i + j$ is even (resp. odd). The edge set is:

$$E_{m,n} := \begin{cases} v_{i,j}v_{i+1,j} & 0 \leq i \leq m, 1 \leq j \leq n \\ v_{i,j}v_{i,j+1} & 1 \leq i \leq m, 0 \leq j \leq n. \end{cases}$$

See Figure 2.5 for a visual example of $G_{m,n}$.

Definition 2.2.4. A *fully-packed loop configuration* of order n is a subgraph of $G_{n,n}$ such that each interior vertex has exactly two incident edges and the following boundary conditions are met. When n is odd, edges include $v_{0,j}v_{1,j}$ and $v_{n,j}v_{n+1,j}$ for j odd, as well as $v_{i,0}v_{i,1}$ and $v_{i,n}v_{i,n+1}$ for i even. When n is even, edges include $v_{0,j}v_{1,j}$ for j odd, $v_{n,j}v_{n+1,j}$ for j even, $v_{i,0}v_{i,1}$ for i even, and $v_{i,n}v_{i,n+1}$ for i odd.

See Figure 2.6 for examples of fully-packed loops.

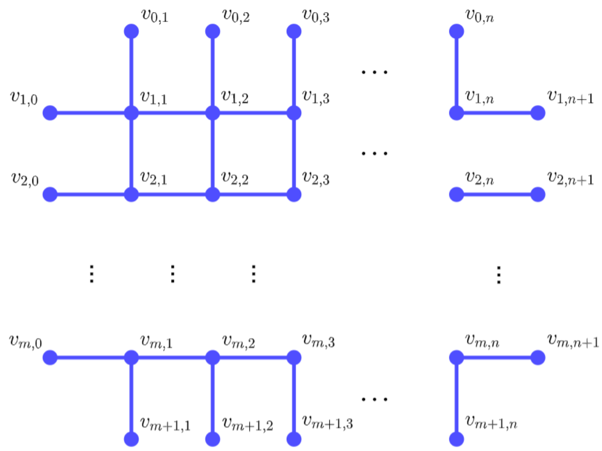


Figure 2.5. The graph $G_{m,n}$.

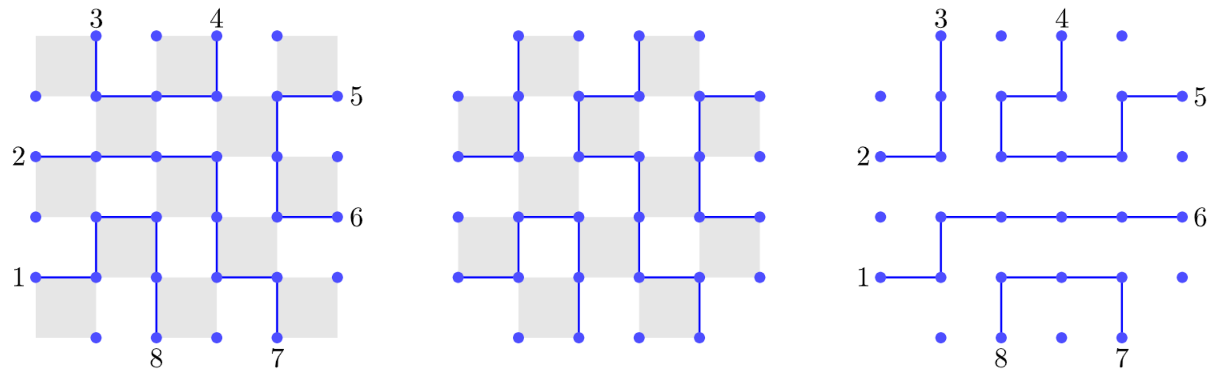


Figure 2.6. The gyration action on a fully-packed loop configuration, first performing the local action on even squares (shaded on the left) and then odd squares (shaded in the middle). The initial and final fully-packed loops have labeled boundary edges for constructing their link patterns.

We now introduce an action on fully-packed loops which has some very nice properties [51]. First, define a local action on a square of a fully-packed loop as follows. If its only edges are a pair of parallel lines, swap them to a pair of parallel lines in the other orientation. Otherwise, do nothing.

Also assign a parity to each square, starting with even in the top left corner, and alternating between even and odd for adjacent squares. Then define an action on a fully-packed loop as applying this local move first to all of the even squares, and then all of the odd squares. This produces another fully-packed loop since the local action does not change the degree of any interior vertex and does not affect the boundary conditions. Call this action *gyration*. An example of this action can be seen in Figure 2.6.

One can label along the boundary of a fully-packed loop where the edges "exit" the graph with the numbers $\{1, \ldots, 2n\}$. Then each number is connected to one other by a path, and each fully-packed loop can be reduced to a *non-crossing matching*, meaning that if the numbers are arranged in a circle and connected with arcs, none of the arcs cross. This is called the *link pattern* of a fully-packed loop. Wieland showed the following theorem [51].

Theorem 2.2.5 ([51])**.** *Gyration on a fully-packed loop rotates the corresponding link pattern. Specifically, if i and j are connected in a link pattern, then in the corresponding link pattern after gyration is applied, $i-1$ and $j-1$ (mod $2n$) will be connected.*

See Figure 2.7 for an example of this rotation.

Figure 2.7. The link patterns for the fully-packed loops from Figure 2.6.

2.3. Dynamics

Some combinatorial objects behave particularly nicely with respect to various actions - notably those which break up the set of objects into *orbits*, such as when the action is bijective and the set is finite. Given a set of objects and such an action, it is especially noteworthy when the *order* of the action is predictable and when the action exhibits special properties, such as the *cyclic sieving phenomenon* [39] or *homomesy phenomenon* [38]. For more information on the background and an overview of some results in this area called *dynamical algebraic combinatorics*, see [48]. For more detailed information on results related to homomesy, see, for example [40].

While much could be said about this topic, we now provide a few relevant definitions which will help the reader understand the results of Section 3.3.

Definition 2.3.1. Let P be a poset. An *order ideal* of P is a subset $X \subseteq P$ such that if $y \in X$ and $z \leq y$, then $z \in X$. The set of order ideals of P is denoted $J(P)$.

$J(P)$, ordered by inclusion, itself is a poset. In fact it is a *distributive lattice* (see Chapter 3 of [44]).

We now define an action on order ideals of poset, introduced by Cameron and Fon-Der-Flaass [12] and studied later by Striker and Williams [49].

Definition 2.3.2. Let P be a poset and let $X \in J(P)$. For any $q \in P$, the *toggle* $t_q : J(P) \to J(P)$ is defined as follows:

$$t_q(X) = \begin{cases} X \cup \{q\} & \text{if } q \notin X \text{ and } X \cup \{q\} \in J(P) \\ X - \{q\} & \text{if } q \in X \text{ and } X - \{q\} \in J(P) \\ X & \text{otherwise.} \end{cases}$$

The *toggle group* of P is the group generated by the t_q for all $q \in P$, with operation composition.

In other words, if an element is not in the order ideal, and adding it in would result in another order ideal, then toggling the element adds it in. If an element is in the order ideal, and removing it would result in another order ideal, then toggling the element removes it. Otherwise, toggling the element does nothing. Notice that toggles are involutions, and that toggles commute whenever there are no covering relations between the elements [12].

Definition 2.3.3. Let P be a poset, and let $X \in J(P)$. Then *rowmotion*, $\mathrm{Row}(X)$, is the order ideal generated by the minimal elements of P not in X.

In [12], it was shown that rowmotion on X is the order ideal obtained from X by toggling the elements of P from top to bottom (in other words, toggling by rows). The order of rowmotion on order ideals of particular posets, such as *root posets* and *minuscule posets* exhibits very nice properties, such as having predictable order, cyclic sieving, and homomesy. In some cases, rowmotion corresponds to another action, called promotion, on *tableaux* or tableaux-like objects [18, 19, 49]. Interpreting actions as toggle group actions on order ideals of certain posets can be advantageous. There may be results or properties of the original action that reveal something new about the posets, or conversely, known results of posets may be used to prove something about the original action.

2.4. Polytopes

Polytopes are objects which lie at the intersection of combinatorics and geometry. See [54] for an excellent introduction and summary of known polytope results and further definitions.

Definition 2.4.1. A *(convex) polytope* P can be defined in two ways:

1. as the convex hull of a finite set of points in $\mathbb{R}^n$. That is, given $\vec{x} = \{x_1, x_2, \dots, x_k\} \subset \mathbb{R}^n$, then $P = \left\{\lambda_1 x_1 + \cdots + \lambda_k x_k \ : \ \lambda_i \geq 0, \sum_{i=1}^{k} \lambda_i = 1\right\}$.

2. as the bounded intersection of finitely many closed halfspaces in $\mathbb{R}^n$.

In 2 dimensions, polytopes are just convex polygons. Informally speaking, one can think of a 2-dimensional polytope as taking a rubber band and stretching around a set of points in the plane. In 3 dimensions, one can think of tightly shrink-wrapping a set of points in space.

Of particular interest may be the *faces* of polytopes. Faces include *vertices* (extreme points), *edges*, and higher dimensional analogues of these. Maximal proper faces (that is, the highest-dimensional faces which are not the polytope itself) are called *facets*. One can study the structure of the faces of a polytope by constructing its *face lattice*, which is the poset of its faces ordered by inclusion. An example is given in Figure 2.8. Different interpretations of face lattices can be useful conceptually or for computations involving large-dimensional polytopes.

Many examples of polytopes are either *simple* (meaning that every vertex is contained in the minimal number of facets) or *simplicial* (meaning that every proper face is a *simplex*). Some important examples of non-simple and non-simplicial polytopes related to objects we have already discussed include the nth *Birkhoff polytope* for $n > 3$ (the convex hull of $n \times n$ permutation matrices) [8, 50] and the nth alternating sign matrix polytope for $n \geq 3$ (the convex hull of $n \times n$ alternating sign matrices) [6, 46]. The nth Birkhoff polytope is $(n-1)^2$-dimensional, has vertices which are exactly the $n \times n$ permutation matrices, has n^2 facets, and has a characterization of its face lattice in terms of *elementary bipartite graphs* [7]. The nth alternating sign matrix polytope is $(n-1)^2$-

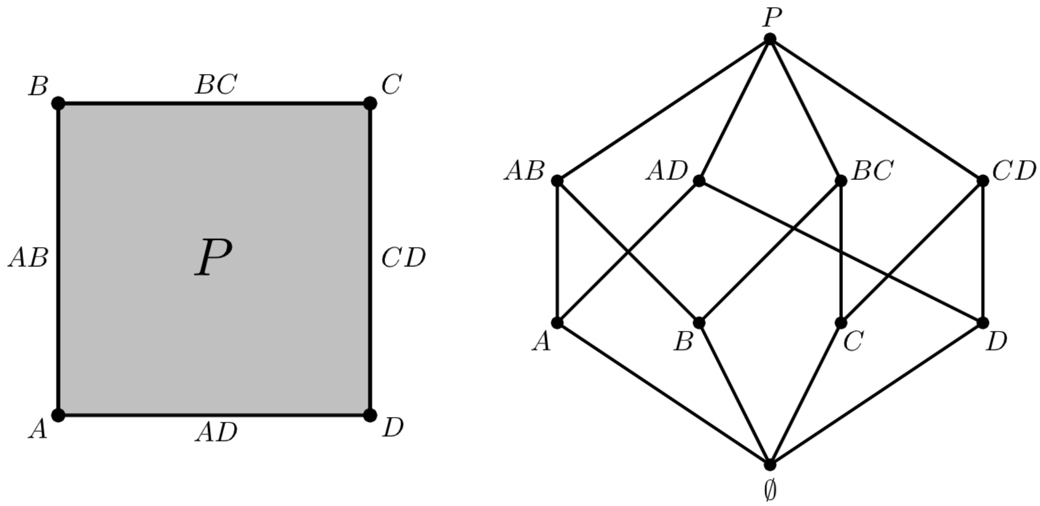


Figure 2.8. An example of a polytope and its face lattice.

dimensional, has vertices which are exactly the $n \times n$ alternating sign matrices, has $4[(n-2)^2+1]$ facets, and has a characterization of its face lattice in terms of *elementary flow grids* [6, 46].

Both Birkhoff and alternating sign matrix polytopes project [46] to another well-studied family of polytopes called *permutohedra.* The nth permutohedron is $(n-1)$-dimensional, has vertices which are exactly the one-line representation of permutations, has $k!\left\{{n \atop k}\right\}$ faces of size k (where $\left\{{n \atop k}\right\}$ are *Stirling numbers of the second kind*) and in particular has 2^n-2 facets. It is also a *zonotope*, meaning that it is the *Minkowski sum* of line segments that connect pairs of standard basis vectors, and has a characterization of its face lattice in terms of *ordered set partitions* [7, 42, 54].

Another topic of interest in the study of polytopes involves their *Ehrhart polynomials* and *normalized volume.* The *t-th dilation* of a polytope P is $tP = \{t\vec{x} \ : \ \vec{x} \in P\}$. Denote by $L_P(t)$ the number of *lattice points* in tP, that is, points with integer coordinates. $L_P(t)$ was shown by Ehrhart [20] to be a polynomial when P is a *lattice polytope*, a polytope whose vertices are lattice points. This polynomial is known as the Ehrhart polynomial of a polytope. The *normalized volume* of a polytope of dimension d is the volume multiplied by $d!$, and it is known that this is the leading coefficient of the Ehrhart polynomial. Of special interest is when the coefficients of the Ehrhart polynomial of P are positive. In this case, P is called *Ehrhart positive.* Some classes of polytopes have been shown to be (or to not be) Ehrhart positive, but in general, this is a very difficult problem with no standard approach. For background, further information, and references regarding these topics, see for example [4] or [28].

3. BIJECTIONS

In this chapter, we study analogues of permutations and alternating sign matrices. We allow the matrices to be rectangular, and in doing so relax the condition that each row and column must sum to 1.

3.1. Partial Permutations

In this section, we define *partial permutations* and give straightforward enumerative results. We then state and prove a q-analogue result, providing a refined way of counting partial permutations based on their *length*. We also show a connection to *decorated permutations*. Finally, we define a subfamily of partial permutations which has an interesting relation to *set partitions* and a previously studied polytope.

Definition 3.1.1. An $m \times n$ *partial permutation matrix* is an $m \times n$ matrix $M = (M_{ij})$ with entries in $\{0, 1\}$ such that the entries in each row and column sum to either 0 or 1. We denote the set of all $m \times n$ partial permutation matrices as $P_{m,n}$. We may also refer to partial permutation matrices simply as partial permutations.

Remark 3.1.2. Partial permutations matrices are sometimes called *subpermutation matrices*, see, for example, [11]. We choose the terminology partial permutation since we consider our matrices as objects in their own right, rather than as submatrices of larger (square) permutation matrices. The use of the term partial permutation is consistent with literature on square partial permutation matrices, such as [13]. Rectangular partial permutation matrices are mentioned in [34].

3.1.1. Enumeration

We now enumerate partial permutations, using standard counting arguments. See Table 3.1 for the number of $m \times n$ partial permutations for $m, n \leq 6$.

Proposition 3.1.3. *$P_{m,n}$ and $P_{n,m}$ are each enumerated by*

$$\sum_{k=0}^{m} \binom{m}{k} (n)_k,$$

where $(n)_k$ denotes the falling factorial $(n)_k := n(n-1)(n-2)\cdots(n-k+1)$.

Proof. Let $m \leq n$. For any partial permutation matrix in $P_{m,n}$, there can be at most one 1 in each row and column, and all other entries must be 0. We begin by choosing which k rows will have a 1. There are $\binom{m}{k}$ ways to do this. Label these rows $r_{i_1}, r_{i_2}, \ldots, r_{i_k}$. Now consider which columns will contain these ones. There are n possible columns for the 1 in row r_{i_1}. This leaves $n-1$ possible columns for the 1 in row r_{n_2}. Continuing in this manner, there will be $n-t+1$ columns available for the 1 in row r_{i_t} for $1 \leq t \leq k$. So there are $(n)_k$ possibilities of columns for each of the $\binom{m}{k}$ choices of rows. Since $m \leq n$, there can be at most m ones in the matrix, so summing k over these values gives us the total number of matrices in $P_{m,n}$. $P_{n,m}$ has the same cardinality as $P_{m,n}$, since there is a bijection given by transposing the matrix, so the result follows. $\square$

Table 3.1. The number of $m \times n$ partial permutation matrices.

m \ n	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	7	13	21	31	43
3	4	13	34	73	136	229
4	5	21	73	209	501	1045
5	6	31	136	501	1546	4051
6	7	43	229	1045	4051	13327

Example 3.1.4. The 13 elements of $P_{2,3}$ are:

$$M_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad M_4 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix} \qquad M_7 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_2 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad M_5 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad M_8 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$M_3 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \end{pmatrix} \qquad M_6 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} \qquad M_9 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$M_{10} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix} \qquad M_{12} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$$

$$M_{11} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \qquad M_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}$$

A one-line representation of a partial permutation matrix can be obtained in the following way: for each row, record the column of the 1, unless there is no 1 in that row, in which case record a 0. This will be a word with m digits in letters $\{0, 1, \ldots, n\}$, where each nonzero number may be used at most once. An example of a partial permutation matrix along with its one-line notation can be found in Figure 3.1.

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \longleftrightarrow 1302$$

Figure 3.1. A 4×4 partial permutation matrix along with its one-line notation.

Definition 3.1.5. The *length* of an $m \times n$ partial permutation w, denoted $\ell(w)$, is its inversion number plus the sum of its entries.

In [23], a partial order is put on the set of partial injective functions of order n, which are equivalent to $n \times n$ partial permutation matrices. This partial order is a generalization of Bruhat order on S_n. The length of a partial permutation corresponds to its rank in this poset, and this is easily extended to the $m \times n$ case.

We now give some notation and a q-analogue result on partial permutation matrices which is analogous to that of usual permutations (as mentioned in Section 2.1). For $w = w_1 w_2 \cdots w_m \in P_{m,n}$, let $\mathrm{inv}(w)$ be the usual inversion number. Denote by $\mathrm{inv}_0(w)$ the number of inversions involving a 0 and by $\mathrm{inv}_1(w)$ the number of inversions of w not involving a 0, so that $\mathrm{inv}(w) = \mathrm{inv}_0(w) + \mathrm{inv}_1(w)$. Let $\mathrm{inc}(w) := \sum_{i=1}^{m} w_i - \frac{r(r+1)}{2}$, where r is the number of nonzero entries of w. Then since

$\mathrm{inv}(w) = \mathrm{inv}_0(w) + \mathrm{inv}_1(w)$ and the sum of entries of w is $\mathrm{inc}(w) + \frac{r(r+1)}{2}$, we have that $\ell(w) = \mathrm{inv}_0(w) + \mathrm{inv}_1(w) + \mathrm{inc}(w) + \frac{r(r+1)}{2}$. This deconstruction of $\ell(w)$ will be useful in our proof of Theorem 3.1.6. Finally, let $([n]_q)_k := [n]_q[n-1]_q \cdots [n-k+1]_q$. That is $([n]_q)_k$ is the q-analogue of the falling factorial.

Theorem 3.1.6. *The length generating function of $m \times n$ partial permutations is given by:*

$$\sum_{w \in P_{m,n}} q^{\ell(w)} = \sum_{r=0}^{m} \left(\begin{bmatrix} m \\ r \end{bmatrix}_q ([n]_q)_r \, q^{\frac{r(r+1)}{2}} \right).$$

Proof. First note that $([n]_q)_k = [r]_q! \begin{bmatrix} n \\ r \end{bmatrix}_q$, so the summand on the right hand side can be expanded to $\begin{bmatrix} m \\ r \end{bmatrix}_q [r]_q! \begin{bmatrix} n \\ r \end{bmatrix}_q q^{\frac{r(r+1)}{2}}$. We will begin by keeping track of what is contributed by each disjoint substatistic: $\mathrm{inv}_0(w)$, $\mathrm{inv}_1(w)$, and $\mathrm{inc}(w)$.

Let $w = w_1 w_2 \cdots w_m \in P_{m,n}$. Let $W = \{w_{i_1}, w_{i_2}, \ldots, w_{i_r}\}$ be the ordered set of nonzero entries of w, that is $0 < w_{i_1} < w_{i_2} < \cdots < w_{i_r} \leq n$. Let $\{j_1, j_2, \ldots, j_{m-r}\}$ be the positions of the zeros in w. Then $\mathrm{inv}_0(w) = \mathrm{inv}(\tilde{w})$, where $\tilde{w} \in B_{m,m-r}$ is the corresponding binary word with zeros in positions $\{j_1, j_2, \ldots, j_{m-r}\}$. Since $\displaystyle\sum_{w \in B_{m,m-r}} q^{\mathrm{inv}(w)} = \begin{bmatrix} m \\ r \end{bmatrix}_q$, we have the portion of our length generating function corresponding to inv_0.

Now, consider the word u obtained by removing all of the zeros in w. u is an r-permutation of $[n]$, and in particular is a permutation of W as defined above. Associate to u its *normalized permutation* $\hat{u} \in S_r$ by replacing w_{i_k} with k. Then $\mathrm{inv}_1(w) = \mathrm{inv}(u) = \mathrm{inv}(\hat{u})$. Since $\displaystyle\sum_{\hat{u} \in S_r} q^{\mathrm{inv}(\hat{u})} = [r]_q!$, we have the portion of our length generating function corresponding to inv_1.

Consider once again the set W. There is a bijection between r-subsets of $[n]$ and $B_{n,r}$, where the r zeros are in positions $i_1, i_2, \ldots, i_r$. Call the image of W under this bijection v. Now, counting $\mathrm{inc}(w)$ is the same as finding the difference $w_{i_k} - k$ for each k and summing the results. But this is just $\mathrm{inv}(v)$, because the difference $w_{i_k} - k$ is precisely how many ones are to the left of the 0 in the position i_k. So we have that $\mathrm{inc}(w) = \mathrm{inv}(v)$. Again, we can use the fact that $\displaystyle\sum_{v \in B_{n,r}} q^{\mathrm{inv}(v)} = \begin{bmatrix} n \\ r \end{bmatrix}_q$, and so we have the portion of our length generating function corresponding to $\mathrm{inc}(w)$. We now simply need to multiply by $q^{\frac{r(r+1)}{2}}$ to account for what was subtracted from the total sum of entries.

Finally, since each of these substatistics are disjoint, combining their contributions and summing from $r = 0$ to m gives us the final result. $\square$

The following example shows the deconstruction described above for a particular partial permutation.

Example 3.1.7. Let $w = 250704 \in P_{6,7}$. Then $\mathrm{inv}(w) = 7$ and the sum of the entries of w is 18, so $\ell(w) = 25$. Following the proof of Theorem 3.1.6, we have the following: $r = 4$ (the number of nonzero entries of w), $W = \{2, 4, 5, 7\}$ (the ordered set of nonzero entries of w), $\{j_1, j_2\} = \{3, 5\}$ (the positions of the zeros in w), $\tilde{w} = 110101$ (the binary word with 6 digits whose zeros are in the same as the zeros of w), $u = 2574$ (the 4-permutation of [7] obtained by deleting the zeros of w), $\hat{u} = 1342$ (the normalized permutation of u), and $v = 1010010$ (the binary word with 7 digits whose zeros are in the positions of the elements of W). We see that $\mathrm{inv}_0(w) = \mathrm{inv}(\tilde{w}) = 5$, $\mathrm{inv}_1 w = \mathrm{inv}(u) = \mathrm{inv}(\hat{u}) = 2$, $\mathrm{inc}(w) = \mathrm{inv}(v) = 8$, $\frac{4(4+1)}{2} = 10$, and adding these together we get $5 + 2 + 8 + 10 = 25 = \ell(w)$.

3.1.2. Relation to Decorated Permutations

We now state a connection between certain partial permutations and decorated permutations, which are defined below. Decorated permutations are in bijection with various other combinatorial objects, such as *Grassmannian necklaces* and *Le-diagrams*, and arise in the study of *positroids* (see, for example [35]).

Definition 3.1.8. A *decorated permutation of length* n is a permutation $\pi = \pi_1 \dots \pi_n$ where fixed points $\pi_i = i$ are replaced by either a positive or negative decoration, $\bar{i}$ or $\underline{i}$ respectively. We denote the set of decorated permutations of size n as D_n.

Remark 3.1.9. Another way to write decorated permutations is to instead write $\pi = (\pi_1, \dots, \pi_n)$ and replace each positively decorated fixed point with a "+" and each negatively decorated fixed point with a "−." This representation is equivalent because only fixed points are decorated. For example, the $\pi = \bar{1}5\underline{3}\bar{4}2$ would be written as $(+, 5, -, +, 2)$. This is the convention we will use in the upcoming examples and theorem.

Example 3.1.10. The following are the 16 elements of D_3:

$$(+,+,+), \quad (-,+,+), \quad (+,-,+), \quad (+,+,-), \quad (-,-,+), \quad (-,+,-), \quad (+,-,-), \quad (-,-,-),$$

$$(+,3,2), \quad (-,3,2), \quad (2,1,+), \quad (2,1,-), \quad (2,3,1), \quad (3,1,2), \quad (3,+,1), \quad \text{and} \quad (3,-,1).$$

Theorem 3.1.11. *Decorated permutations of length n are in bijection with the partial permutations written in one-line notation whose nonzero entries are maximized.*

Proof. Let $\pi \in D_n$. First, strip π of any positive decorations, that is replace any "+" with the position that "+" occupies. Then replace any negative decorations with 0. Note that if there are no negative decorations, the result is simply a permutation of n. Let the number of negative decorations of π (i.e. the number of zeros in the result) be z. The final step is to replace the remaining nonzero entries with the $n - z$ largest entries from $[n]$ (that is, $\{z+1, z+2, \ldots, n\}$), while keeping the same relative order. By construction, the end result is a unique element of $P_{n,n}$ whose nonzero entries are maximized.

Furthermore, since each of the steps is easily reversible without loss of information, this map is a bijection. For the reverse map, consider $v = (v_1, \ldots, v_n)$ of $\mathcal{P}(n,n)$ whose nonzero entries are maximized. Let $Z_v = \{i : v_i = 0\}$, that is Z_v is the set of positions whose entries are 0 in v. Let $I_v = [n] - Z_v$. Replace the nonzero entries in v with those from I_v while keeping the same relative order. Then replace the zeros with negative decorations. Finally, decorate any remaining fixed points with "+". □

See Example 3.1.12 for an example of this bijection.

Example 3.1.12. Let $\pi = (+,4,-,6,-,2) \in D_6$. We first strip π of any positive decorations: $(1,4,-,6,-,2)$. We then replace any negative decorations with 0: $(1,4,0,6,0,2)$. Finally, we replace the remaining nonzero entries with the numbers $\{3,4,5,6\}$ while keeping the same relative order: $(3,5,0,6,0,4) = v$, an element of $\mathcal{P}(6,6)$ whose nonzero entries are maximized.

For the reverse direction, start with $v = (3,5,0,6,0,4)$. Then $Z_v = 3,5$ and so $I_v = 1,2,4,6$. We replace the nonzero entries of v with those from I_v while keeping the same relative order:

$(1, 4, 0, 6, 0, 2)$, then positively decorate any fixed points: $(+, 4, 0, 6, 0, 2)$. Finally, replace the zeros with negative decorations: $(+, 4, -, 6, -, 2) = \pi$.

The partial permutations which are in bijection with decorated permutations are exactly the vertices of the *partial permutohedra* studied in Section 4.4.

3.1.3. Relation to Set Partitions

In [15], the authors investigated certain families of permutation matrices whose entries below the second diagonal are fixed to be zero. The polytopes formed by these matrices, called *CRY polytopes*, are certain faces of the Birkhoff polytope. They were conjectured [15] and later shown [53] to have very nice volume formulas given by the product of *Catalan numbers*. Motivated by this, the authors of [31] studied the alternating sign matrix analogue of these polytopes. They showed, among other things, that these polytopes are *order polytopes*, allowing for explicit formulas for their volumes and Ehrhart polynomials.

We investigated "partial analogues" of these polytopes but were unable to find analogous results. However, we found an interesting bijection in the case of partial permutations which allows us to enumerate such families in terms of *Bell numbers*, so we include it here.

Definition 3.1.13. A *k-CRY partial permutation matrix* is a square partial permutation matrix $M \in P_{n,n}$ such that $M_{ij} = 0$ for $i + j > n + k$. In other words, there is an additional restriction requiring that the entries below an antidiagonal depending on the parameter k are equal to 0. We denote the set of $n \times n$ k-CRY partial permutation matrices as $P^k_{n,n}$.

Example 3.1.14. The 15 matrices in $P^1_{3,3}$ are below. The entries which must follow the additional restriction from Definition 3.1.13 are bolded and blue.

$$M_1 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathbf{0} \\ 0 & \mathbf{0} & \mathbf{0} \end{pmatrix} \qquad M_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \mathbf{0} \\ 1 & \mathbf{0} & \mathbf{0} \end{pmatrix} \qquad M_3 = \begin{pmatrix} 0 & 0 & 0 \\ 1 & 0 & \mathbf{0} \\ 0 & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

$$M_4 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & \mathbf{0} \\ 0 & \mathbf{0} & \mathbf{0} \end{pmatrix} \qquad M_8 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & \mathbf{0} \\ 1 & \mathbf{0} & \mathbf{0} \end{pmatrix} \qquad M_{12} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & \mathbf{0} \\ 1 & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

$$M_5 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \mathbf{0} \\ 0 & \mathbf{0} & \mathbf{0} \end{pmatrix} \qquad M_9 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & \mathbf{0} \\ 0 & \mathbf{0} & \mathbf{0} \end{pmatrix} \qquad M_{13} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & \mathbf{0} \\ 0 & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

$$M_6 = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \mathbf{0} \\ 0 & \mathbf{0} & \mathbf{0} \end{pmatrix} \qquad M_{10} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & \mathbf{0} \\ 1 & \mathbf{0} & \mathbf{0} \end{pmatrix} \qquad M_{14} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & \mathbf{0} \\ 0 & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

$$M_7 = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & \mathbf{0} \\ 0 & \mathbf{0} & \mathbf{0} \end{pmatrix} \qquad M_{11} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & \mathbf{0} \\ 0 & \mathbf{0} & \mathbf{0} \end{pmatrix} \qquad M_{15} = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & \mathbf{0} \\ 1 & \mathbf{0} & \mathbf{0} \end{pmatrix}$$

In the next theorem, we relate $P^k_{n,n}$ to *set partitions*, which we now define.

Definition 3.1.15. A *partition of a set* A, or *set partition* is a grouping of its elements into nonempty subsets, called *blocks*.

Theorem 3.1.16. *$P^k_{n,n}$ is bijection with set partitions of $\{1, 2, \dots, n+k\}$ such that none of $1, \dots, k$ may be in the same block and none of $n+1, \dots, n+k$ may be in the same block.*

Proof. We begin with the case $k = 1$. Let $M \in P^1_{n,n}$, so every entry of M below the main antidiagonal is 0. We construct a bijection to set partitions of $\{1, 2, \dots, n+1\}$. Construct the corresponding set partition by iterating the following: if $M_{ij} = 1$, then i must be in the same block as $n - j + 2$. If one of these numbers has already been put in a block, merge the blocks.

To reverse this, start with a set partition of $\{1, 2, \dots, n+1\}$. Construct a matrix $(M_{i,j})$ by the following rule: If there is a group of more than 2, say $\{a_1, a_2, \dots, a_m\}$ such that $a_1 < a_2 < \cdots < a_m$, then put a 1 in row a_i, column $n - a_{i+1} + 2$ for $1 \leq i \leq m - 1$. Set the rest of the entries equal to zero. Note in particular that the entries for which $i + j > n + 1$ will be zero, resulting in a matrix in $P^1_{n,n}$. So we have a bijection between $P^1_{n,n}$ and set partitions of $n + 1$.

Now consider any $1 < k \leq n$. We apply a similar bijection. In this case, we construct a set partition by iterating the following: if $M_{ij} = 1$, then i must be in the same block as $n - j + k + 1$. Since $M_{ij} = 0$ for $i + j > n + k$, this results in a set partition of $n + k$ such that none of $1, \ldots, k$ may be in the same block and none of $n + 1, \ldots, n + k$ may be in the same block. We follow the same rule as above to reverse this. Note that in particular, starting with a set partition of $n + k$ such that none of $1, \ldots, k$ are in the same block and none of $n + 1, \ldots, n + k$ are in the same block guarantees that the resulting matrix will have zeros in entries for which $i + j > n + k$. $\square$

An example of this bijection is given in Example 3.1.17.

Example 3.1.17. Let $M = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \in P^1_{4,4}$. A straightforward way to see the bijection is to label the rows of M with the numbers $1, 2, \ldots n$ and the columns with the numbers $n + k, n + k - 1, \ldots, k + 1$. Then a 1 in a position labeled (i, j) corresponds to i being in the same block of j in the set partition. Our labeling of M in this example is the following:

$$\begin{array}{c c} & \begin{matrix} 5 & 4 & 3 & 2 \end{matrix} \\ \begin{matrix} 1 \\ 2 \\ 3 \\ 4 \end{matrix} & \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \end{array}$$

There are ones in positions with labels $(1, 4)$, $(2, 3)$, and $(4, 5)$, so the blocks are $\{1, 4, 5\}$ and $\{2, 3\}$. That is, the set partition of 5 corresponding to M is $\{\{1, 4, 5\}, \{2, 3\}\}$.

Theorem 3.1.16 relates k-CRY partial permutations to set partitions. Set partitions are counted by Bell numbers. This allows us to enumerate k-CRY permutations for small k completely in terms of Bell numbers.

Definition 3.1.18. The *nth Bell number*, denoted B_n, is defined to be the number of ways that a set of size n can be partitioned into nonempty subsets. Bell numbers are calculated as $B_n = \sum_{i=0}^{n} \left\{ {n \atop i} \right\}$, where $\left\{ {n \atop i} \right\}$ are the *Stirling numbers of the second kind*, the number of ways to partition a set of cardinality n into exactly i nonempty subsets.

Definition 3.1.19. Given a sequence $A = \{a_1, a_2, a_3, \ldots\}$, the *first difference sequence* of A is $\Delta A = \{a_2 - a_1, a_3 - a_2, \ldots\}$. Denote the nth term of ΔA as ΔA_n. The *second difference sequence* of A, $\Delta^2 A$, is the first difference sequence of ΔA. Likewise the *kth difference sequence* of A, $\Delta^k A$, is the first difference sequence of $\Delta^{k-1} A$ for $k \in \mathbb{N}$.

If B is the sequence of Bell numbers, ΔB_n is the number of set partitions of n where n is not a singleton. This is because we are taking the number of set partitions of n and subtracting the number of set partitions of $n-1$, which we can count by removing all set partitions where n is a singleton (since the rest of the set partition can be interpreted as a set partition of $n-1$). Similarly, $\Delta^2 B_n$ is the number of set partitions of n where neither n nor $n-1$ are singletons. Equivalently, $\Delta^2 B_n$ is the number of set partitions of n where neither i nor j are singletons for $i \neq j$. So in particular, $\Delta^2 B_n$ can be thought of as the number of set partitions where neither 1 nor n are singletons, which is the interpretation we use in the following lemma.

Proposition 3.1.20. *Let S be the set of set partitions of n where neither 1 nor n are singletons, so that $|S| = \Delta^2 B_n$, and T be the set of set partitions of n where 1 is not in the same block as 2 and $n-1$ is not in the same block as n. There exists an explicit bijection between S and T.*

Proof. Let $x \in S$. Then x is a set partition of n where neither 1 nor n are singletons. Map x to $y \in T$ as follows. If 1 is in the same block as 2, remove 1 from that block, making it a singleton. Likewise, if n is in the same block as $n-1$, remove n from that block, making it a singleton. Then y is necessarily a set partition such that 1 is not in the same block as 2 and $n-1$ is not in the same block as n. This is easily reversible; the reverse map is the following. Given $y \in T$, map it to $x \in S$ as follows. If 1 is a singleton, move it to the block containing 2. Likewise, if n is a singleton, move it to the block containing $n-1$. Thus we have a bijection. $\square$

Corollary 3.1.21. $|P^1_{n,n}| = B_{n+1}$. *Moreover, $P^2_{n,n}$ is enumerated by the second difference of Bell numbers. That is, $|P^2_{n,n}| = (B_{n+2} - B_{n+1}) - (B_{n+1} - B_n) = B_{n+2} - 2B_{n+1} + B_n$.*

Proof. This follows directly from Theorem 3.1.16 and Lemma 3.1.20. □

3.2. Partial Alternating Sign Matrix Bijections

In this section, we first define an analogue of alternating sign matrices, which we call *partial alternating sign matrices*. We then describe several bijections akin to those in the usual alternating sign matrix setting (see, for example [37, 47]). Our initial motivation for the study of partial alternating sign matrices was to better understand the structure of *chained alternating sign matrices*, as each component of a chained alternating sign matrix is a partial alternating sign matrix (see [25] as well as Chapter 5 for more information). While this is still in progress, the study of partial alternating sign matrix bijections themselves is robust and presented here.

Definition 3.2.1. An $m \times n$ *partial alternating sign matrix* is an $m \times n$ matrix $M = (M_{ij})$ with entries in $\{-1, 0, 1\}$ such that:

- the entries in each row and column sum to either 0 or 1,
- the nonzero entries in each row and column alternate in sign, and
- the first nonzero entry (if any) in each column and last nonzero entry (if any) in each row are 1.

We denote the set of all $m \times n$ partial permutation matrices as $\mathrm{PASM}_{m,n}$.

Remark 3.2.2. It may be more natural to require that the first nonzero entry (if any) in each row is one rather than the last nonzero entry. This would give us an equivalent set of objects, since the matrices would just be reflections of the ones we have defined. In fact, in Chapter 4, we will use the opposite convention. However, for the sake of defining the upcoming bijections, it is more convenient to use the requirement in the definition above.

Remark 3.2.3. The set of matrices in $\mathrm{PASM}_{m,n}$ with no -1 entries is the set $P_{m,n}$ of partial permutation matrices, since the second condition is vacuous.

Example 3.2.4. $\mathrm{PASM}_{2,3}$ consists of the 13 matrices from Example 3.1.4 plus the following four additional matrices:

$$M_{14} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & -1 & 1 \end{pmatrix} \qquad M_{15} = \begin{pmatrix} 0 & 1 & 0 \\ 1 & -1 & 1 \end{pmatrix}$$

$$M_{16} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 0 & 1 \end{pmatrix} \qquad M_{17} = \begin{pmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \end{pmatrix}$$

The cardinality of $\mathrm{PASM}_{n,n}$ is given by sequence A202751 in the Online Encyclopedia of Integer Sequences [1]. It is unlikely that there exists a product formula for $|\mathrm{PASM}_{m,n}|$, since, for example, $|\mathrm{PASM}_{6,6}| = 1442764 = 2^2 \cdot 373 \cdot 967$. See Table 3.2 for the number of $m \times n$ partial alternating sign matrices for $m, n \leq 6$, calculated with SageMath [45].

Table 3.2. The number of $m \times n$ partial alternating sign matrices.

m \ n	1	2	3	4	5	6
1	2	3	4	5	6	7
2	3	8	17	31	51	78
3	4	17	62	184	462	1022
4	5	31	184	924	3809	13197
5	6	51	462	3809	26394	150777
6	7	78	1022	13197	150777	1442764

Remark 3.2.5. $n \times n$ partial permutation and alternating sign matrices were studied in a different context by Fortin [23]. He showed that, with a poset structure analogous to the strong Bruhat order, the lattice of partial alternating sign matrices is the *MacNeille completion* of the poset of partial permutations. That is, the lattice of partial alternating sign matrices is the smallest lattice containing the poset of partial permutations. This is analogous to the result of Lascoux and Schützenberger [27] that the lattice of $n \times n$ alternating sign matrices is the MacNeille completion of the strong Bruhat order on S_n.

We now state our main result of this section, which is the goal of the remainder of this section.

Theorem 3.2.6. *There are explicit bijections among $m \times n$ partial alternating sign matrices, (m, n)-partial monotone triangles, (m, n)-partial height function matrices, (m, n)-partial fully-packed loop configurations, (m, n)-rectangular ice configurations, the set of order ideals of $\boldsymbol{P}_{m,n}$, and (m, n)-nests of osculating paths.*

We prove this theorem via a series of lemmas and define the needed objects along the way. We begin with partial monotone triangles.

Definition 3.2.7. An (m, n)-*partial monotone triangle* is a triangular array of non-negative integers $(a_{i,j})_{1 \leq j \leq i \leq m}$ with entries in $\{0, 1, \ldots, n\}$ and with the following properties:

- rows are weakly increasing: $a_{i,j} \leq a_{i,j+1}$,
- nonzero entries in rows are strictly increasing, and
- diagonals are weakly increasing: $a_{i,j} \leq a_{i-1,j}$ and $a_{i,j} \leq a_{i+1,j+1}$.

An example of the triangular array with indexing as described above is given in Figure 3.2, and an example of a partial monotone triangle can be found in Figure 3.3

Remark 3.2.8. Objects similar to this definition have appeared in the literature before. In particular, a certain transformation of an (n, n)-partial monotone triangle is what is referred to as a "generalized key" in [23].

$$\begin{array}{ccccccc} & & & a_{11} & & & \\ & & a_{12} & & a_{22} & & \\ & \iddots & & \cdots & & \ddots & \\ a_{1m} & & a_{2m} & & \cdots & & a_{mm} \end{array}$$

Figure 3.2. A triangular array of numbers with the indexing as described in Definition 3.2.7.

$$
\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 1 & 0 & 0 \\ 1 & 0 & -1 & 1 \end{pmatrix}
\qquad
\begin{array}{ccccccc}
 & & & 1 & & & \\
 & & 1 & & 3 & & \\
 & 0 & & 2 & & 3 & \\
0 & & 1 & & 2 & & 4
\end{array}
\qquad
\begin{pmatrix} 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 0 \\ 2 & 2 & 1 & 0 & 0 \\ 3 & 2 & 1 & 1 & 0 \end{pmatrix}
\qquad
\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 3 \\ 2 & 3 & 2 & 3 & 2 \\ 3 & 4 & 3 & 2 & 3 \\ 4 & 3 & 4 & 3 & 2 \end{pmatrix}
$$

Figure 3.3. A 4×4 partial alternating sign matrix along with its corresponding partial monotone triangle, corner-sum matrix, and partial height function matrix.

Lemma 3.2.9. *There is an explicit bijection between $m \times n$ partial alternating sign matrices and (m, n)-partial monotone triangles.*

Proof. Given an $m \times n$ partial alternating sign matrix M, construct $(C_{i,j})_{1 \le i \le m, 1 \le j \le n}$, its $m \times n$ matrix of partial column sums, by setting $C_{ij} = \sum_{k=1}^{i} M_{kj}$. By the alternating condition on partial alternating sign matrices, this will be a $\{0, 1\}$-matrix. Construct an array of numbers from $(C_{i,j})$ as follows: in the ith row, record in increasing order the values j for which $C_{ij} = 1$. If there are less than i such values, fill in zeros from the left until there are i values in row i. By construction, the nonzero entries in each row are strictly increasing. The restrictions on partial alternating sign matrices, namely the rows and columns summing to 0 or 1, the alternating condition, and the fact that the first (last) nonzero entry in each column (row) must be 1 guarantee that the diagonals are weakly increasing from southwest to northeast and northwest to southeast. Thus, the result is an (m, n)-partial monotone triangle. Since any matrix can be uniquely determined by its partial column sums, this is a one-to-one map.

Furthermore, this map is easily reversible. Given an (m, n)-partial monotone triangle $(a_{i,j})$, first build an $m \times n$ $\{0, 1\}$-matrix $(C_{i,j})$ by setting $C_{i,k} = 1$ whenever $a_{i,j} = k > 0$ and filling the rest of the matrix with zeros. This matrix $(C_{i,j})$ records the partial column sums of a unique partial alternating sign matrix, M, where $M_{1j} = C_{1j}$ and $M_{ij} = C_{ij} - C_{i-1,j}$ for $2 \le i \le m$. $\square$

Putting the natural partial order (component-wise comparison) on partial monotone triangles forms a lattice, which is the MacNeille completion of the poset of partial permutations as mentioned in Remark 3.2.5. This is proven for $m = n$ in [23], and can be extended to $m \neq n$. A similar poset construction using all sign matrices was studied in [10].

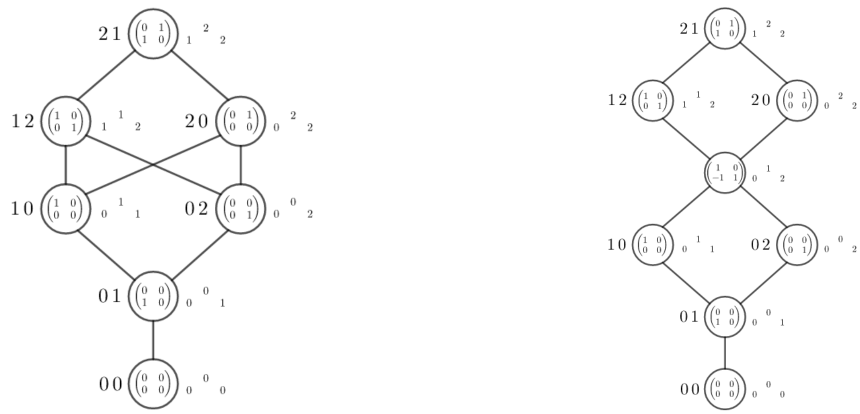


Figure 3.4. The poset of partial permutations (left) and the poset of partial alternating sign matrices (right), for $m = n = 2$.

We now move on to *corner-sum matrices*, which are useful for showing the correspondence with *partial height function matrices*.

Definition 3.2.10. Given any $m \times n$ matrix $(M_{i,j})_{1 \leq i \leq m, 1 \leq j \leq n}$, we can define its (north-east) *corner-sum matrix* $(c_{i,j})_{0 \leq i \leq m, 0 \leq j \leq n}$ by setting $c_{i,j} = \sum_{i' \geq i, j' \leq j} M_{i',j'}$.

Lemma 3.2.11. *There is an explicit bijection between $m \times n$ partial alternating sign matrices and $(m+1) \times (n+1)$ matrices whose first row and last column are all zeros, and whose entries are increasing by from top to bottom and right to left and whose entries increase by at most one.*

Proof. We claim the set of matrices described above is exactly the set of corner-sum matrices of $m \times n$ partial alternating sign matrices. The first row and last column of any such corner-sum matrix are all zeros by definition. The fact that an $m \times n$ partial alternating matrix has entries in $\{-1, 0, 1\}$, along with the alternating condition guarantees that adjacent entries in the corresponding corner-sum matrix differ by at most one. Note that given any corner-sum matrix c, one can recover the original matrix M by $M_{ij} = c_{ij} - c_{i-1,j} - c_{i,j+1} + c_{i-1,j+1}$. $\square$

See Figure 3.3 for an example of a partial alternating sign matrix along with its corresponding corner-sum matrix.

Definition 3.2.12. An *(m,n)-partial height function matrix* is a matrix $(h_{i,j})_{0\leq i\leq m,1\leq j\leq n}$ with $h_{0,k} = k$ for $0 \leq k \leq n$, and $h_{\ell,0} = \ell$ for $0 \leq \ell \leq m$, and such that adjacent entries in any row or column are non-negative and differ by exactly 1.

Lemma 3.2.13. *There is an explicit bijection between $m \times n$ partial alternating sign matrices and (m, n)-partial height function matrices.*

Proof. The bijection is given by first mapping a partial alternating sign matrix $(M_{i,j})_{1\leq i\leq m,1\leq j,\leq n}$ to its corner-sum matrix $(c_{i,j})$. By Lemma 3.2.11, this is an $(m+1)\times(n+1)$ matrices whose first row and last column are all zeros, and whose entries increase by at most one from top to bottom and right to left. Then define the matrix $(h_{i,j})_{0\leq i\leq m,0\leq j,\leq n}$ as $h_{i,j} = i + j - 2c_{i,n-j}$. $(h_{i,j})$ satisfies Definition 3.2.12 because of the following. The condition that the first row and last column of the corner-sum matrix are 0 guarantees that the first row and first column start at 0 and increase by 1. The condition that the entries of the corner-sum matrix increase by at most one from top to bottom and right to left is equivalent to the condition that adjacent entries in $(h_{i,j})$ differ by exactly one. In particular, when moving along a row (right-to-left) or column (top-to-bottom), if the entry stays the same in the corner-sum matrix, then the corresponding entry in $(h_{i,j})$ goes down by one, and if the entry increases by one in the corner-sum matrix, then the corresponding entry in $(h_{i,j})$ goes up by one (this can be seen using the formula $h_{i,j} = i + j - 2c_{i,n-j}$). Thus $(h_{i,j})$ is a partial height function matrix.

Given a partial height function matrix $(h_{i,j})$, we can recover $(c_{i,j})$ by setting $c_{i,n-j} = \frac{i+j-h_{i,j}}{2}$, from which we can recover the partial alternating sign matrix as mentioned in the proof of Lemma 3.2.11. $\square$

An example of a partial alternating sign matrix and its corresponding partial height function matrix is given in Figure 3.3.

The next several bijections make use of the graph $G_{m,n}$ of Definition 2.2.3. These bijections will give us more interesting angles from which to study partial alternating sign matrices, with visualizable objects and connections to physics and dynamics.

Definition 3.2.14. An (m, n)-partial fully-packed loop configuration is a subgraph of $G_{m,n}$ whose edges include $v_{0,j}v_{1,j}$ for j odd and $v_{i,0}v_{i,1}$ for i even, and such that each interior vertex has exactly two incident edges.

See Figure 3.5, left, for an example.

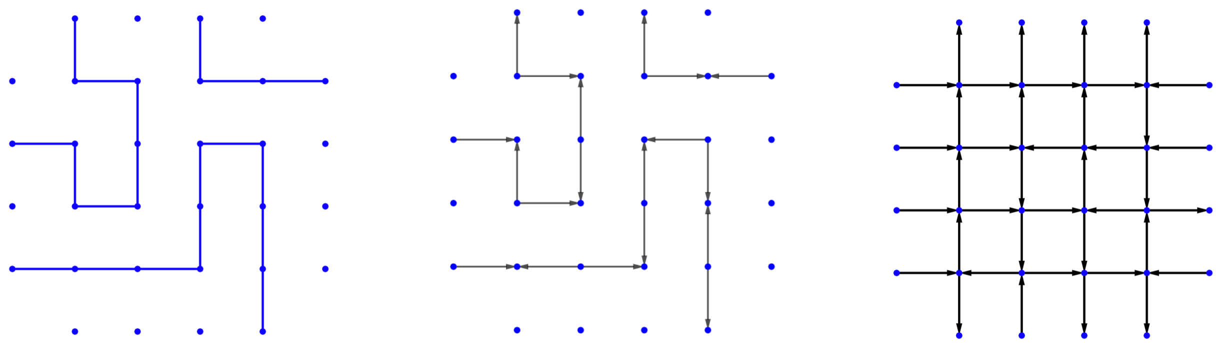


Figure 3.5. An example of the $(4, 4)$-partial fully-packed loop configuration, this partial fully-packed loop with its edges directed, and the associated $(4, 4)$-rectangular ice configuration corresponding to the partial alternating sign matrix from Figure 3.3.

Lemma 3.2.15. *There is an explicit bijection between (m, n)-partial fully-packed loop configurations and (m, n)-partial height function matrices.*

Proof. Starting with an (m, n)-partial height function matrix $(h_{i,j})$, overlay $G_{m,n}$ so that each interior number in the partial height function matrix has four surrounding vertices. Separate two horizontally adjacent numbers by an edge if the numbers are $2k$ and $2k + 1$ (in either order) for any integer k. Separate two vertically adjacent numbers by an edge if the numbers are $2k - 1$ and $2k$ (in either order) for any integer k. Since $h_{0,k} = k$ for $0 \leq k \leq n$, and $h_{\ell,0} = \ell$ for $0 \leq \ell \leq m$, we will have the required boundary conditions for an (m, n)-partial fully-packed loop. Also, each partial height function matrix entry differs by exactly one, so this yields the condition that interior vertices have exactly two incident edges. For the reverse map, start with the partial fully-packed loop configuration and fill in the boundary conditions for the partial height function matrix. Then, simply use the rule described above in reverse, using the condition that each adjacent entry in the partial height function matrix must be exactly one apart. $\square$

See Figure 3.6 for an example.

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 3 \\ 2 & 3 & 2 & 3 & 2 \\ 3 & 4 & 3 & 2 & 3 \\ 4 & 3 & 4 & 3 & 2 \end{pmatrix}$$

Figure 3.6. The partial height function matrix from Figure 3.3 overlaid with its corresponding partial fully-packed loop configuration.

Definition 3.2.16. An (m,n)-rectangular ice configuration is a directed graph whose underlying graph is $G_{m,n}$ such that the directed edges along the left point inward, the directed edges on the top point outward, and each interior vertex has both in-degree and out-degree equal to two.

In general when a directed graph whose underlying graph is $G_{m,n}$ has all directed edges pointing inward along the left and right and outward along the top and bottom, that graph is said to have *domain wall boundary conditions.* Here, we have a partial analogue of those conditions. An example of a rectangular ice configuration is given in Figure 3.5, right.

Lemma 3.2.17. *There is an explicit bijection between* (m,n)*-rectangular ice configurations and* (m,n)*-partial fully-packed loop configurations.*

Proof. Given an (m,n)-rectangular ice configuration, keep only those edges which start at an even vertex and end at an odd vertex. Note that there is no loss of information here, as each interior vertex had in-degree and out-degree equal to two. Then make those edges undirected. Again, we lose no information here because we know that each of these edges was originally directed towards an odd vertex. Each vertex will now have degree equal to two, resulting in an (m,n)-partial fully-packed loop configuration by definition. Note that since we started with edges along the left pointing inward and edges along the top pointing outward, keeping edges that were originally directed towards an odd vertex give us the fixed boundary conditions along the top and left edges for a partial fully-packed loop.

For the reverse direction, we simply start with an (m, n)-partial fully-packed loop configuration "undo" each of the steps: direct the edges so that they go from even vertices to odd vertices. Then, fill in the remaining directed edges so that each interior vertex has in- and out-degree equal to 2. The boundary condition on rectangular ice that the directed edges along the left point inward and the directed edges on the top point outward is satisfied because we started with a partial fully-packed loop, which by definition has edges $v_{0,j}v_{1,j}$ for j odd and $v_{i,0}v_{i,1}$ for i even. $\square$

We now relate partial alternating sign matrices to order ideals of a certain poset (recall Definitions 2.1.3 and 2.3.1). This, along with the connection to partial fully-packed loop configurations lays the groundwork for the study of related dynamics in Section 3.3.

Definition 3.2.18. Define the poset elements $\mathbf{P}_{m,n}$ as the coordinates (i, j, k) in $\mathbb{Z}^3$ such that $0 \leq k \leq m-1$, $k \leq j \leq n-1$, and $k \leq i \leq m-1$. Define the partial order via the following covering relations: (i, j, k) covers $(i+1, j, k)$, $(i, j+1, k)$, $(i-1, j, k-1)$, and $(i, j-1, k-1)$ whenever these coordinates are poset elements.

See Figure 3.7 for an example of $\mathbf{P}_{m,n}$. The construction of this poset directly gives us the following proposition and corollary.

Proposition 3.2.19. *$\boldsymbol{P}_{m,n}$ is a ranked poset in which the rank of (i, j, k) in $\boldsymbol{P}_{m,n}$ equals $m+n-2-i-j+2k$. It has a unique minimal element $(m-1, n-1, 0)$ which has rank 0. The maximal elements are of the form (i, i, i) and have rank $m+n-2$.*

Corollary 3.2.20. *$J(\boldsymbol{P}_{m,n})$ is a distribute lattice of rank $\frac{-m^3+3m^2n+3m+m}{6}$.*

Proof. Since we have an explicit construction of $\mathbf{P}_{m,n}$, we can use this to determine the rank of $J(\mathbf{P}_{m,n})$ (viewed as the lattice of order ideals by inclusion). The maximum element of $J(\mathbf{P}_{m,n})$ has rank equal to the number of elements in $P_{m,n}$. At each k value, we have $(m-k) \times (n-k)$ elements, and so we have $\displaystyle\sum_{k=0}^{m-1}(m-k)(n-k) = \frac{-m^3+3m^2n+3m+m}{6}$ total elements. $\square$

We now show that the order ideals of this poset are in correspondence with the other objects we have been studying.

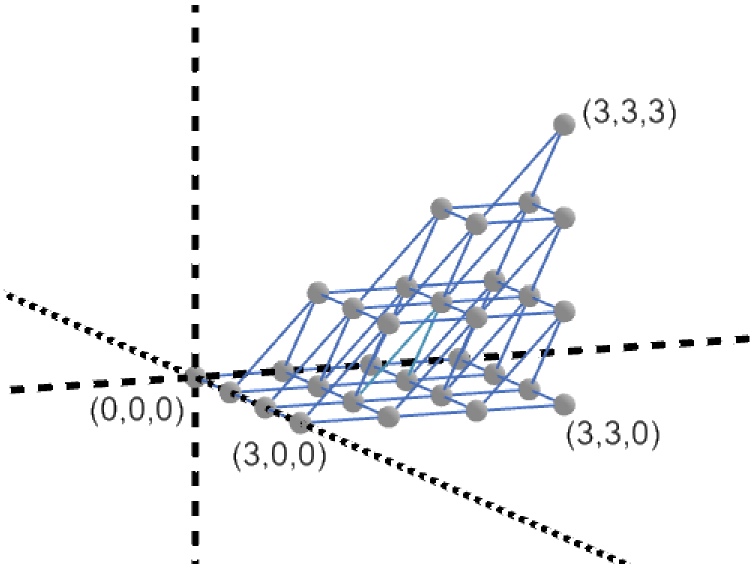


Figure 3.7. An example of $\mathbf{P}_{4,4}$ plotted in $\mathbb{R}^3$.

Lemma 3.2.21. *There is an explicit bijection between $J(\boldsymbol{P}_{m,n})$ and (m,n)-partial height function matrices.*

Proof. Let $\mathcal{O} \in J(\mathbf{P}_{m,n})$. For $0 \le i \le m-1$, $0 \le j \le n-1$, define the subset $S_{i,j} := \{(i,j,t) \mid (i,j,t) \in \mathbf{P}_{m,n}\}$. That is, $S_{i,j}$ is the intersection of the elements of $\mathbf{P}_{m,n}$ from the explicit construction in Definition 3.2.18 with the line $(x,y,z) = (i,j,t)$. Construct an $(m+1) \times (n+1)$ matrix $(h_{i,j})$ as follows. First, set $h_{0,k} = k$ for $0 \le k \le n$ and $h_{\ell,0} = \ell$ for $0 \le \ell \le m$. Then, entry $h_{i,j}$ for $1 \le i \le m$, $1 \le j \le n$ is determined by the cardinality of the intersection between $\mathcal{O}$ and $S_{i,j}$. If this cardinality equals k, then, $h_{i,j} = i + j - 2k$. The fact that $\mathcal{O}$ is an order ideal, along with covering relations of $\mathbf{P}_{m,n}$ guarantee that adjacent entries in $(h_{i,j})$ differ by exactly one, thus the result is an (m,n)-partial height function matrix. Notice that if the cardinality of the intersection is k, then $h_{i,j}$ will be $2k$ less than the maximum possible value of that partial height function entry. $\square$

An example of this bijection is given in Figure 3.8.

Finally, we relate partial alternating sign matrices to certain sets of nested lattice paths. This allows us to explicitly count the number of partial alternating sign matrices with total sum 1, which gives some hope of finding a refined enumeration of partial alternating sign matrices.

$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 3 \\ 2 & 3 & 2 & 3 & 2 \\ 3 & 4 & 3 & 2 & 3 \\ 4 & 3 & 4 & 3 & 2 \end{pmatrix} \longleftrightarrow$$

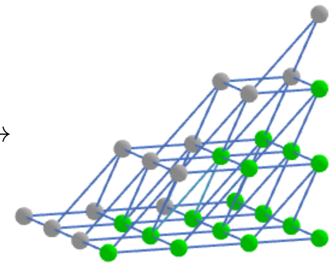


Figure 3.8. A $(4,4)$-partial height function matrix and its corresponding order ideal in $\mathbf{P}_{4,4}$.

Definition 3.2.22. An (m,n)-*nest of osculating lattice paths* is any set of osculating paths (non-crossing paths which can only touch at corners) in the $m \times n$ grid with south and east steps, whose starting points are on the left side, and whose end points are on the bottom of the grid.

See Figure 3.9 for an example. The next lemma follows from known results of osculating paths (see [5] for more information further references) and so we present it without proof, though we do give a description of the bijection.

Lemma 3.2.23. *There is an explicit bijection between $m \times n$ partial alternating sign matrices with total sum t and the set of (m,n)-nests of osculating lattice paths with t paths.*

The map from partial alternating sign matrices to nests of osculating paths can be described as follows. Given an $m \times n$ partial alternating sign matrix, first construct its (m,n)-rectangular ice configuration, and reflect it horizontally. Then, starting at the topmost inward-facing directed edge along the left, create a path by following the directed edges to the right when possible and down otherwise. Do this similarly for each other inward-facing directed edge along the left. Finally, delete the first and last edge of each path. An example is given in Figure 3.9. This correspondence gives us the following enumeration for partial alternating sum matrices with total sum 1.

Corollary 3.2.24. *The number of $m \times n$ partial alternating sign matrices with sum 1 is $\binom{m+n}{m} - 1$.*

Proof. By Lemma 3.2.23, the set of $m \times n$ partial alternating sign matrices with sum 1 is in bijection with the set of (m,n)-nests of osculating paths with 1 path. That is, the set of paths in an $m \times n$ grid with starting point on the left and ending point on the bottom of the grid. Each of these paths can be uniquely extended to a lattice path in an the $(m+1) \times (n+1)$ grid (where the additional column has been added on the left and additional row on the bottom) which starts in the upper left corner and ends in the lower right corner. To do so, the start of the path gets extended one step to the left, and then up to the upper left corner, and the end of the path gets extended one step down and then right to the lower right corner. This will account for all such paths in the $(m+1)\times(n+1)$ grid, except for the one which is all south steps followed by all east steps. (This path is excluded because no path starting on the left edge of the $m\times n$ grid can have its starting point as the lower left corner of the $(m+1)\times(n+1)$ grid.) Since those paths are counted by $\binom{m+n}{m}$, the total number of $m\times n$ partial alternating sign matrices with sum 1 is $\binom{m+n}{m}-1$. $\square$

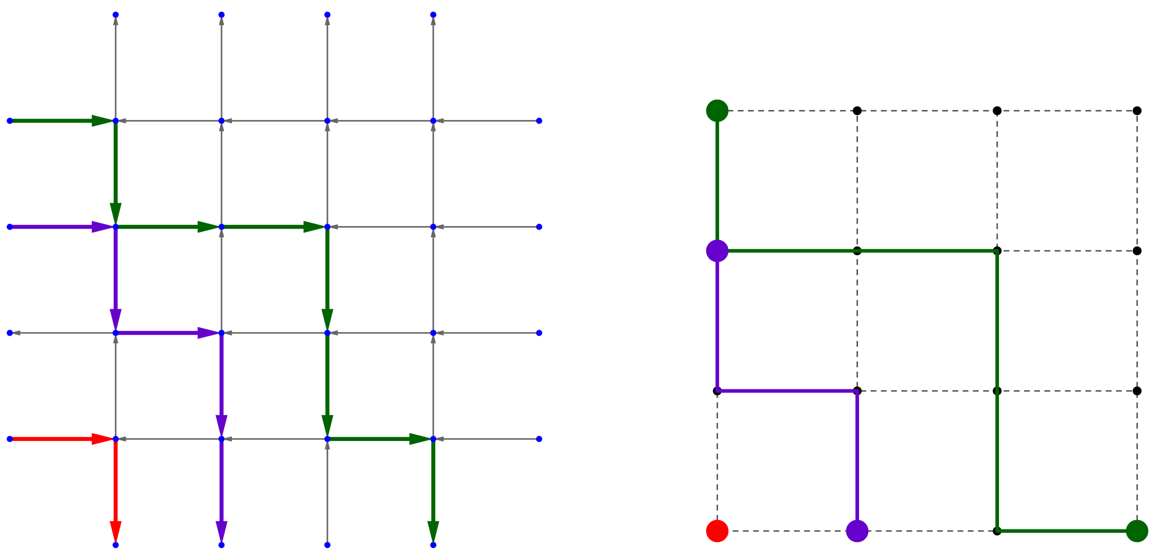


Figure 3.9. An example of how to map a $(4,4)$-rectangular ice to its corresponding nest of osculating paths.

We leave it as an open question to enumerate $\{M \in \mathrm{PASM}_{m,n} \mid \mathrm{sum}(M)=t\}$ for $t>1$. Table 3.3 gives the number of $n\times n$ partial alternating sign matrices with total sum t, for $n,t\le 6$ calculated using SageMath. Note the value for $n=6$, $t=2$ is prime, so there will not be a product formula in general, though this does not preclude the existence of a sum formula. It would be interesting to find a formula for the cardinality of the set $\{M \in \mathrm{PASM}_{m,n} \mid \mathrm{sum}(M)=t\}$, since for $t=m=n$, this is the set of $n\times n$ alternating sign matrices, enumerated by $\displaystyle\prod_{j=0}^{n-1}\frac{(3j+1)!}{(n+j)!}$ [52, 26].

Table 3.3. The number of $n \times n$ partial alternating sign matrices with total sum t.

n \ t	0	1	2	3	4	5	6
1	1	1					
2	1	5	2				
3	1	19	35	7			
4	1	69	425	387	42		
5	1	251	4845	13861	7007	429	
6	1	923	55897	458263	709242	210912	7436

Proof of Theorem 3.2.6. Explicit bijections among the objects are given in Lemmas 3.2.9, 3.2.13, 3.2.15, 3.2.17, 3.2.21, and 3.2.23. $\square$

3.3. Partial Alternating Sign Matrix Dynamics

In this section, we explore dynamics related to partial alternating sign matrices, inspired by [47]. We first describe a local move on partial fully-packed loop configurations, which leads to the definition of an action called *gyration* on these configurations. This is analogous to Wieland's gyration of fully-packed loops [51], as discussed in Section 2.2. We then show how gyration acts on the corresponding height function matrices and order ideals. Finally, we use Theorem 2.2.5 to prove a rotation-like result on *partial link patterns*.

Given an (m, n)-partial fully-packed loop configuration, call a square *even* (resp. *odd*) if the vertex in its upper-left corner (or lower-right corner) is even (resp. odd). Call a square *interior* if all of its surrounding vertices are interior, and *exterior* otherwise. Additionally, we say a square is a *boundary square* if it is in the first row or column.

Define a *local action* on a square as follows. For an interior square, if its only edges are a pair of parallel lines, swap it to a pair of parallel lines in the other orientation. For an exterior square, we define the local action based on location. If the square is on the far right (and not a corner), we swap only the left edge with the pair of horizontal parallel edges (and vice versa). If the square is on the bottom (and not a corner), we swap only the top edge with the pair of vertical parallel edges (and vice versa). If the square is the bottom-right corner, we swap only the left edge with only the top edge (and vice versa). In all other cases, we do nothing. Note that with this definition, the action will never act on a boundary square. See Figure 3.10 for an example of each of the local actions described above.

Definition 3.3.1. Define the action *gyration* on an (m, n)-partial fully-packed loop configuration by first performing the local action on all even squares, then on all odd squares.

An example of this action is given in Figure 3.11. Note that by construction, since this action never changes the total degree of any interior vertex and does not affect the fixed boundary, it always produces another (m, n)-partial fully-packed loop configuration.

Figure 3.10. From left to right: the local moves on interior squares, exterior squares along the right side, exterior squares along the bottom side, and the exterior square in the bottom-right corner.

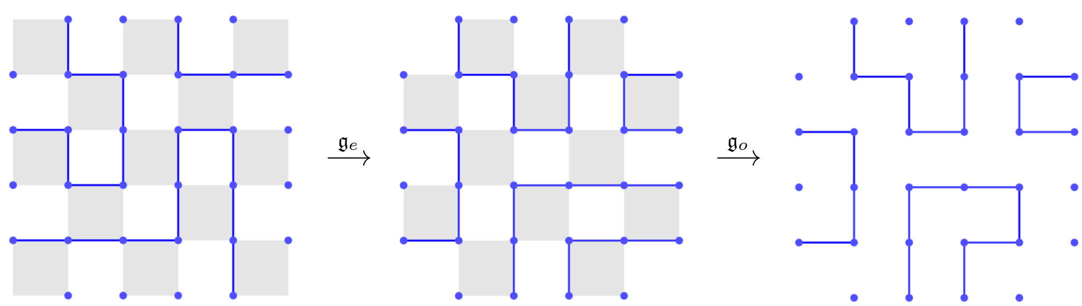


Figure 3.11. The gyration action on a partial fully-packed loop configuration. $\mathfrak{g}_e$ denotes performing the local action on the even squares (shaded in the first diagram), and $\mathfrak{g}_o$ denotes performing the local action on the odd squares (shaded in the second diagram).

Given the results of the previous section, we can examine how the local action described above affects the different objects in bijection with partial fully-packed loop configurations. In particular, we can describe exactly what happens on the corresponding partial height function matrices and order ideals.

Lemma 3.3.2. *Let $(h_{i,j})$ be an (m,n)-partial height function matrix, $\mathcal{O} \in J(\boldsymbol{P}_{m,n})$ the corresponding order ideal, and F the corresponding (m,n)-partial fully-packed loop, via the bijections of Lemmas 3.2.15 and 3.2.21. Then the following are equivalent:*

(1) The local action applied to F at the square in row i and column j, where the rows (columns) are numbered from the top (left) starting at 1.

(2) Incrementing or decrementing the partial height function matrix entry $h_{i,j}$ by 2, if possible.

(3) Toggling $S_{i,j}$ (from the proof of Lemma 3.2.21).

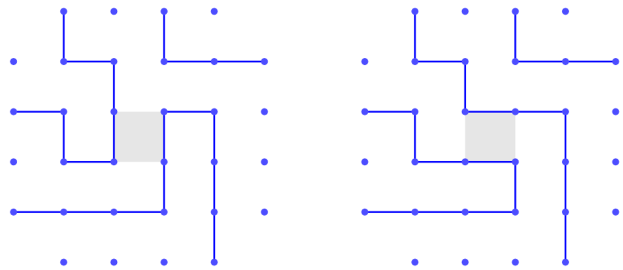
Proof. To see the equivalence between (1) and (2), we follow the bijection in Lemma 3.2.15. We see that when the local action would change edges on F, this corresponds to $h_{i,j}$ being surrounded by the same number (each entry above, below, to the right, and to the left is either $h_{i,j} - 1$ or $h_{i,j} + 1$). Making the local action change in edges corresponds exactly to changing the value $h_{i,j}$ to the only other possible height function value: if the surrounding values are all $h_{i,j} - 1$, then $h_{i,j}$ gets changed to $h_{i,j} - 2$, and if the surrounding values are all $h_{i,j} + 1$, then $h_{i,j}$ gets changed to $h_{i,j} + 2$. In the case where the local action would do nothing, this corresponds to $h_{i,j}$ having surrounding values which are not all the same.

To see the equivalence between (2) and (3), we follow the bijection in Lemma 3.2.21. We see that incrementing $h_{i,j}$ by 2 (and having it stay a partial height function matrix) is only possible when removing an element from $S_{i,j}$ (in $\mathcal{O}$) results in another order ideal. Likewise, decrementing $h_{i,j}$ by 2 is only possible when adding an element of $S_{i,j}$ results in another order ideal. When it is not possible to change $h_{i,j}$ and have the result be a partial height function matrix, then adding or removing an element from $S_{i,j}$ results in a subset that is not an order ideal. $\square$

See Figure 3.12 for an example of this correspondence.

We now define a toggle group action and show that this action corresponds to gyration on partial fully-packed loops.

Definition 3.3.3. For any finite ranked poset P, define $\mathrm{Gyr} : J(P) \to J(P)$ as the toggle group action which toggles the elements in even ranks first, then odd ranks. Its inverse, Gyr^{-1}, toggles the elements in odd ranks first, then even ranks.



$$\begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 3 \\ 2 & 3 & {\color{blue}2} & 3 & 2 \\ 3 & 4 & 3 & 2 & 3 \\ 4 & 3 & 4 & 3 & 2 \end{pmatrix} \qquad \begin{pmatrix} 0 & 1 & 2 & 3 & 4 \\ 1 & 2 & 3 & 4 & 3 \\ 2 & 3 & {\color{blue}4} & 3 & 2 \\ 3 & 4 & 3 & 2 & 3 \\ 4 & 3 & 4 & 3 & 2 \end{pmatrix}$$

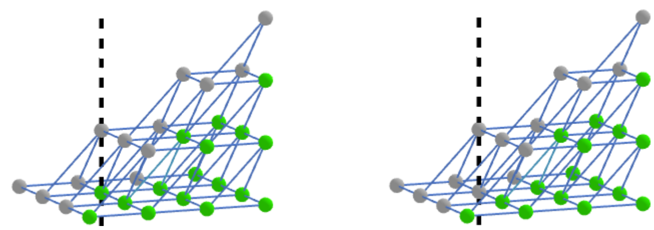


Figure 3.12. Performing the local action on a single square of a partial fully-packed loop (top), the result on corresponding partial height function matrices (middle), and the result on the corresponding order ideals (bottom).

Note that this this action is well-defined because elements in ranks of the same parity do not have covering relations between them, so the corresponding toggles commute.

Proposition 3.3.4 (Proposition 6.4, [47])**.** *For any finite ranked poset P, there is an equivariant bijection between $J(P)$ under Row and under Gyr, that is, Row and Gyr are conjugate elements in the toggle group $T(P)$.*

In the case where $P = \mathbf{P}_{m,n}$, we have the following proposition, which follows from Lemma 3.2.21 and Lemma 3.3.2.

Proposition 3.3.5. *Gyration on (m, n)-partial fully-packed loop configurations is equivalent to Gyr acting on $J(\boldsymbol{P}_{m,n})$ when $m + n$ is even, and Gyr^{-1} acting on $J(\boldsymbol{P}_{m,n})$ when $m + n$ is odd.*

Proof. Notice that the parity of a square in an (m, n)-partial fully-packed loop configuration corresponds to the parity of rank in $\mathbf{P}_{m,n}$. When $m + n$ is even, even squares correspond to even ranks, and when $m + n$ is odd, even squares correspond to odd ranks. Thus, when we perform gyration on an (m, n)-partial fully-packed loop (which applies the local move on all of the even squares and then all of the odd squares), by Lemma 3.3.2, this corresponds to toggling even ranks followed by odd ranks when $m + n$ is even, which is exactly Gyr from Definition 3.3.3. When $m + n$ is odd, performing gyration on an (m, n)-partial fully-packed loop corresponds to toggling odd ranks followed by even ranks, which is Gyr^{-1}. □

The previous two propositions give the following theorem, which is an analogue of Theorem 8.13 in [49].

Theorem 3.3.6. *$J(\boldsymbol{P}_{m,n})$ under Row and (m, n)-partial fully-packed loops (or (m, n)-partial height function matrices) under gyration are in equivariant bijection.*

Recall Theorem 2.7, which says that in the usual $n \times n$ alternating sign matrix setting, gyration acting on a fully-packed loops rotates the link pattern. We close this section with a result on partial fully-packed loops that follows from this, and for completeness provide a table of rowmotion (or equivalently gyration) orbit sizes in the partial setting (see Table 3.4).

Definition 3.3.7. Given an (m, n)-partial fully-packed loop configuration, F, label the places where the paths exit the graph along the left and top (the fixed boundary conditions) with the numbers $\left\{1, \ldots, \lfloor \frac{m}{2} \rfloor + \lceil \frac{n}{2} \rceil\right\}$, starting with 1 in the lower left. Arrange these numbers in a circular arc, and connect any numbers that are connected by a path in F with an arc. Call this the *partial link pattern* for F.

See Figure 3.13 for an example of partial link patterns.

Corollary 3.3.8. *Gyration on (m, n)-partial fully-packed loop configurations exhibits a partial rotation on the corresponding partial link patterns. Specifically, if i and j are connected in a partial link pattern, then in the corresponding partial link pattern after gyration is applied, $i - 1$ and $j - 1$ will either be connected to each other or not at all.*

Proof. This follows from Theorem 2.2.5. Any (m, n)-partial alternating sign matrix can be interpreted as the upper-left corner of a larger alternating sign matrix; for any rows or columns that have a sum of 0, we can systematically add zeros and ones to the left and below until we have a larger matrix with each row and column summing to 1. So, the corresponding (m,n)-partial fully-packed loop is really the upper-left corner of a larger fully-packed loop. Let F_1 be a partial fully-packed loop and F_2 be the result of applying gyration on F_1. Let F_1', be a larger fully-packed loop for which F_1 is the corner, and F_2' be the result after gyration is applied to F_1' so that F_2 is the corner of F_2'. Let i and j be connected in the partial link pattern for F_1. Then they will also be connected in the link pattern for F_1'. When gyration is applied on F_1', the link pattern is rotated, $i-1$ and $j-1$ will be connected in the link pattern for F_2'. If the path connecting them stays entirely within the corner that is F_2, then $i-1$ and $j-1$ will be connected in the partial link pattern for F_2. However, the path connecting them in F_2' may leave that corner, in which case they will not be connected to anything in the partial link pattern for F_2. $\square$

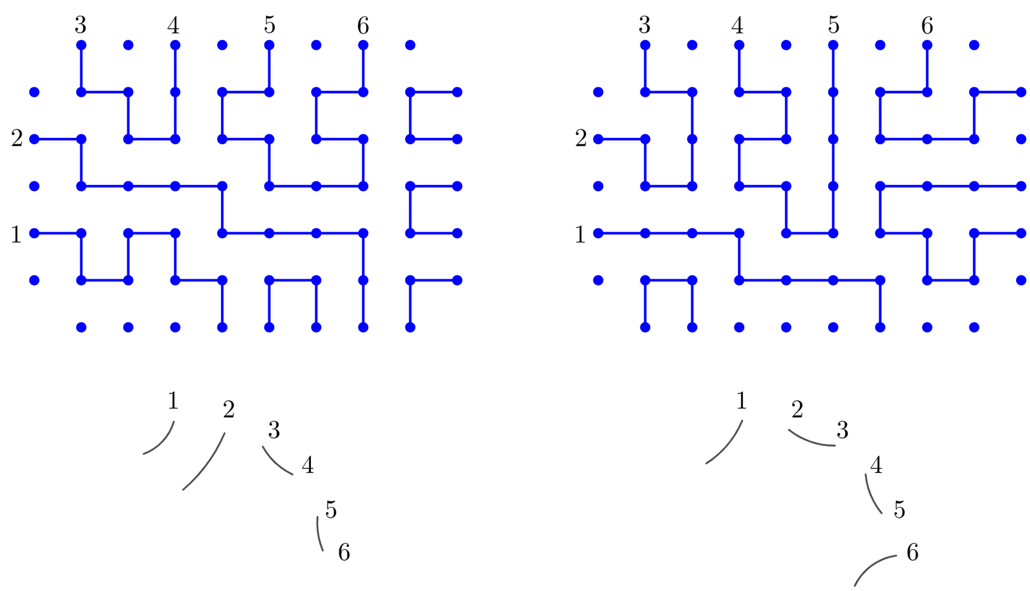


Figure 3.13. A $(5,7)$-partial fully-packed loop configuration along with is partial link pattern (left), and the result of each after gyration is applied (right). For any numbers that do not have a connection, we choose to draw a line that terminates so that it does not cross any other lines or connect to any other numbers, to more closely mimic the feeling of being a piece of a larger link pattern.

Table 3.4. A table of rowmotion orbit sizes for $\mathbf{P}_{m,n}$ for small values of m and n.

	Orbit Size	Number of Orbits
$m = 1$, $n = 1$	2	1
$m = 1$, $n = 2$	3	1
$m = 1$, $n = 3$	4	1
$m = 1$, $n = 4$	5	1
$m = 1$, $n = 5$	6	1
$m = 2$, $n = 2$	2	2
	4	1
$m = 2$, $n = 3$	2	1
	5	3
$m = 2$, $n = 4$	3	2
	6	3
	7	1
$m = 2$, $n = 5$	7	5
	8	2
$m = 3$, $n = 3$	2	4
	6	7
	12	1
$m = 3$, $n = 4$	2	2
	7	16
	26	1
	42	1
$m = 3$, $n = 5$	2	2
	8	27
	9	2
	14	1
	18	1
	36	1
	46	1
	52	1
	58	1
$m = 4$, $n = 4$	2	7
	4	10
	8	55
	10	1
	11	2
	14	1
	16	14
	18	1
	24	2
	30	2
	34	1

4. POLYTOPES[1]

In this chapter, we study polytopes which are built from partial permutations and alternating sign matrices.

4.1. Matrices

In this section, we briefly discuss the matrices from the previous chapter. We give equivalent definitions that are more useful in the context of polytopes and consistent with existing polytope literature.

Definition 4.1.1. An $m \times n$ *partial permutation matrix* is an $m \times n$ matrix $M = (M_{ij})$ with entries in $\{0, 1\}$ such that:

$$\sum_{i'=1}^{m} M_{i'j} \in \{0,1\}, \qquad \text{for all } 1 \leq j \leq n. \tag{4.1}$$

$$\sum_{j'=1}^{n} M_{ij'} \in \{0,1\}, \qquad \text{for all } 1 \leq i \leq m. \tag{4.2}$$

We denote the set of all $m \times n$ partial permutation matrices as $P_{m,n}$.

Remark 4.1.2. This definition is exactly Definition 3.1.1, since requiring row and column sums of a $\{0, 1\}$ matrix to be in $\{0, 1\}$ is equivalent to there being at most one 1 in each row and column.

Definition 4.1.3. An $m \times n$ *partial alternating sign matrix* is an $m \times n$ matrix $M = (M_{ij})$ with entries in $\{-1, 0, 1\}$ such that:

$$\sum_{i'=1}^{i} M_{i'j} \in \{0,1\}, \qquad \text{for all } 1 \leq i \leq m, 1 \leq j \leq n. \tag{4.3}$$

$$\sum_{j'=1}^{j} M_{ij'} \in \{0,1\}, \qquad \text{for all } 1 \leq i \leq m, 1 \leq j \leq n. \tag{4.4}$$

[1]The material in this chapter was coauthored by Dylan Heuer and Dr. Jessica Striker. Heuer had primary responsibility for the calculations and finding connections to existing research. Heuer and Striker worked collaboratively on the proofs of most of the theorems. Heuer was the primary drafter of this chapter; both Heuer and Striker revised and proofread this chapter.

We denote the set of all $m \times n$ partial alternating sign matrices as $\text{PASM}_{m,n}$.

Remark 4.1.4. Note that this is equivalent to Definition 3.2.1, but with the opposite convention, as mentioned in Remark 3.2.2. Here, we require that the partial row and column sums must be equal to 0 or 1. This means that the first nonzero entry in each row or column (if any) must be 1 (if it were -1, a partial row or column sum would then be -1). Also, the nonzero entries must alternate in sign (if not, a partial row or column sum could be greater than 1 or less than 0). Finally, the total row and column sums must be equal to 0 or 1 since the total sums are the last partial sums. An example of a 4×4 partial alternating sign matrix with this convention is given in Figure 4.1.

$$\begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & -1 \\ 1 & -1 & 0 & 1 \end{pmatrix}$$

Figure 4.1. A 4×4 partial alternating sign matrix.

Remark 4.1.5. Partial alternating sign matrices are a subset of *sign matrices*, which differ from Definition 4.1.3 in that each row partial sum is not restricted to $\{0,1\}$ as in (4.4), but may equal any non-negative integer. See [43] for information about polytopes whose vertices are sign matrices and Lemma 4.3.10 for the relationship between these polytopes.

4.2. Partial Permutation Polytopes

In this section, we give the definition of partial permutation polytopes, review their inequality descriptions, and provide the enumeration of their vertices and facets. All of these results are known (see Remark 4.2.3) or easily deduced, but we include them for completeness and for comparison to the polytopes of the next section. We also compute data for the volume of these polytopes and conjecture a formula for $m = 2$.

Definition 4.2.1. Let $\text{PPerm}(m,n)$ be the polytope defined as the convex hull, as vectors in $\mathbb{R}^{mn}$, of all the matrices in $P_{m,n}$. Call this the *(m,n)-partial permutation polytope.*

Remark 4.2.2. The dimension of $\text{PPerm}(m,n)$ is mn. To see this, let $U_{i,j}$ be the $m \times n$ matrix with (i,j) entry equal to 1 and zeros elsewhere. Note that $U_{i,j} \in \text{PPerm}(m,n)$ for all $1 \le i \le m$,

$1 \leq j \leq n$. Since $\mathrm{PPerm}(m,n)$ contains each of these mn unit vectors, its dimension equals the ambient dimension mn.

Remark 4.2.3. The polytopes $\mathrm{PPerm}(m,n)$ have been previously studied in different contexts. Since any partial permutation matrix can be reinterpreted as an incidence vector of some matching, $\mathrm{PPerm}(m,n)$ is a *matching polytope*. In [3, 16], adjacency conditions of vertices of matching polytopes were studied. For a nice summary and proof of these results, see [41, Chapter 25]. Also, Mirsky showed that the set of $n \times n$ doubly substochastic matrices is the convex hull of all $n \times n$ partial permutation matrices [33]. This is easily extendable to the $m \times n$ case, which is stated below in Proposition 4.2.4. In [2], partial permutations were viewed as rook placements, and edges and faces of $\mathrm{PPerm}(m,n)$ were enumerated.

Proposition 4.2.4. $\mathrm{PPerm}(m,n)$ *consists of all $m \times n$ real matrices $X = (X_{ij})$ such that:*

$$X_{ij} \geq 0, \qquad \text{for all } 1 \leq i \leq m, 1 \leq j \leq n, \tag{4.5}$$

$$\sum_{j'=1}^{n} X_{ij'} \leq 1, \qquad \text{for all } 1 \leq i \leq m, \tag{4.6}$$

$$\sum_{i'=1}^{n} X_{i'j} \leq 1, \qquad \text{for all } 1 \leq j \leq n. \tag{4.7}$$

Since these inequalities are irredundant, we obtain the following corollary by counting the inequalities.

Corollary 4.2.5. *The number of facets of* $\mathrm{PPerm}(m,n)$ *equals $mn + m + n$.*

Proposition 4.2.6. *The vertices of* $\mathrm{PPerm}(m,n)$ *are exactly the matrices in $P_{m,n}$, so* $\mathrm{PPerm}(m,n)$ *has* $\sum_{k=0}^{m} \binom{m}{k} (n)_k$ *vertices.*

Proof. Let $B \in P_{m,n}$. In order to show that B is a vertex of $\mathrm{PPerm}(m,n)$, we will find a hyperplane in $\mathbb{R}^{mn}$ with B on one side and all other matrices in $P_{m,n}$ on the other. Then since $\mathrm{PPerm}(m,n)$ is the convex hull of $P_{m,n}$, B must be a vertex.

Let $H_B(X) := \sum_{1 \leq i \leq m, 1 \leq j \leq n} a_{ij} X_{ij}$, where $a_{ij} = \begin{cases} +1, & \text{if } B_{ij} = 1 \\ -1, & \text{if } B_{ij} = 0 \end{cases}$. Then define a hyperplane in $\mathbb{R}^{mn}$ by the equation $H_B(X) = \mathrm{sum}(B) - \frac{1}{2}$, where $\mathrm{sum}(B)$ is the sum of all the entries of B.

Given a hyperplane formed this way, we can easily recover the matrix from which it was formed, and so H_B is unique for each B. Let $B' \neq B$ be any other matrix in $P_{m,n}$. Then there is some (i,j) such that either $B_{ij} = 0$ and $B'_{ij} = 1$ or $B_{ij} = 1$ and $B'_{ij} = 0$. Thus by construction, $H_B(B) = \text{sum}(B) > \text{sum}(B) - \frac{1}{2}$, and $H_B(B') < \text{sum}(B) - \frac{1}{2}$. Therefore the vertices of $\text{PPerm}(m,n)$ are the $m \times n$ partial permutation matrices, and Proposition 3.1.3 gives the enumeration. □

Example 4.2.7. The 7 matrices in $P_{2,2}$ are:

$$A_1 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix},\ A_2 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix},\ A_3 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix},\ A_4 = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix},\ A_5 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix},$$
$$A_6 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \text{ and } A_7 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}.$$

Then we have $H_{A_2}(X) = X_{11} - X_{12} - X_{21} - X_{22}$, and the hyperplane is defined by $H_{A_2}(X) = 0.5$. When we substitute the entries of each matrix in $P_{2,2}$ into the equation for H_{A_2}, we see that A_2 is on one side of this hyperplane, while all the other matrices are on the other side.

$$A_1 : X_{11} = 0, X_{12} = 0, X_{21} = 0, X_{22} = 0 \to H_{A_2}(A_1) = 0 - 0 - 0 - 0 = 0 < 0.5;$$
$$A_2 : X_{11} = 1, X_{12} = 0, X_{21} = 0, X_{22} = 0 \to H_{A_2}(A_2) = 1 - 0 - 0 - 0 = 1 > 0.5;$$
$$A_3 : X_{11} = 0, X_{12} = 1, X_{21} = 0, X_{22} = 0 \to H_{A_2}(A_3) = 0 - 1 - 0 - 0 = -1 < 0.5;$$
$$A_4 : X_{11} = 0, X_{12} = 0, X_{21} = 1, X_{22} = 0 \to H_{A_2}(A_4) = 0 - 0 - 1 - 0 = -1 < 0.5;$$
$$A_5 : X_{11} = 0, X_{12} = 0, X_{21} = 0, X_{22} = 1 \to H_{A_2}(A_5) = 0 - 0 - 0 - 1 = 0 < 0.5;$$
$$A_6 : X_{11} = 1, X_{12} = 0, X_{21} = 0, X_{22} = 1 \to H_{A_2}(A_6) = 1 - 0 - 0 - 1 = 0 < 0.5;$$
$$A_7 : X_{11} = 0, X_{12} = 1, X_{21} = 1, X_{22} = 0 \to H_{A_2}(A_7) = 0 - 1 - 1 - 0 = -2 < 0.5.$$

Remark 4.2.8. The normalized volume of $\text{PPerm}(m,n)$ for small values of m and n is given in Table 4.1, computed using SageMath [45]. Due to the large size of the polytopes, further computations are not easily obtained. Note that there does not appear to be a nice general formula for these volumes. However, when one parameter is set equal to two, we have the following conjecture.

Table 4.1. The normalized volume of $\mathrm{PPerm}(m,n)$ for small values of m and n.

$m \backslash n$	1	2	3	4	5
1	1	1	1	1	1
2	1	4	17	66	247
3	1	17	642	22148	622791
4	1	66	22148	12065248	5089403019
5	1	247	622791	5089403019	53480547965190

Conjecture 4.2.9. *The normalized volume of* $\mathrm{PPerm}(n,2)$ *(or equivalently* $\mathrm{PPerm}(2,n)$*) is equal to* $\binom{2n}{n} - n$.

We have confirmed this conjecture for $m \leq 14$ using SageMath.

Remark 4.2.10. We have used SageMath to compute the Ehrhart polynomials for $\mathrm{PPerm}(m,n)$ for $m,n \leq 5$ and note that in all of these cases their coefficients are positive.

4.3. Partial Alternating Sign Matrix Polytopes

In this section, we define partial alternating sign matrix polytopes. We give an inequality description and facet enumeration in Subsection 4.3.1. In Subsection 4.3.2, we determine the face lattice. We also compute the volume for small values of m and n in Subsection 4.3.3.

4.3.1. Vertices, Facets, Inequality Description

In this subsection, we give the definition of partial alternating sign matrix polytopes. In Proposition 4.3.3, we determine the vertices. We prove an inequality description in Theorem 4.3.6. Then in Theorem 4.3.8, we enumerate the facets. Finally, we relate these polytopes to $\mathrm{PPerm}(m,n)$ in Lemma 4.3.10.

Definition 4.3.1. Let $\mathrm{PASM}(m,n)$ be the polytope defined as the convex hull, as vectors in $\mathbb{R}^{mn}$, of all the matrices in $\mathrm{PASM}_{m,n}$. Call this the (m,n)-*partial alternating sign matrix polytope.*

Remark 4.3.2. $\mathrm{PASM}(m,n)$ contains $\mathrm{PPerm}(m,n)$, since, as noted in Remark 3.2.3, the set of partial alternating sign matrices $\mathrm{PASM}_{m,n}$ contains all the partial permutation matrices $P_{n,m}$. So the dimension of $\mathrm{PASM}(m,n)$ is the ambient dimension mn, since by Remark 4.2.2, this is the dimension of $\mathrm{PPerm}(m,n)$.

Proposition 4.3.3. *The vertices of* $\mathrm{PASM}(m,n)$ *are exactly the matrices in* $\mathrm{PASM}_{m,n}$.

Proof. In [43, Theorem 4.3], a hyperplane is constructed that separates a given $m \times n$ sign matrix from all other $m \times n$ sign matrices. Since $m \times n$ partial alternating sign matrices are a subset of $m \times n$ sign matrices, this hyperplane must separate a given $m \times n$ partial alternating sign matrix from all others. The hyperplane construction is as follows. Let M be an $m \times n$ partial alternating sign matrix and $C_M = \{(i,j) \ : \ \sum_{i'=1}^{i} M_{i'j} = 1\}$. Then the hyperplane in $\mathbb{R}^{mn}$ that separates M from the other elements of $\mathrm{PASM}_{m,n}$ is $\sum_{(i,j)\in C_M} \sum_{i'=1}^{i} X_{i'j} - \sum_{(i,j)\notin C_M} \sum_{i'=1}^{i} X_{i'j} = |C_M| - \frac{1}{2}$. Thus the vertices of $\mathrm{PASM}(m,n)$ are the $m \times n$ partial alternating sign matrices. □

We now give the following definitions from [43], which we will use in the proof of Theorem 4.3.6.

Definition 4.3.4 ([43, Definition 3.3])**.** We define the $m \times n$ *grid graph* $\Gamma_{(m,n)}$ as follows. The vertex set is $V(m,n) := \{(i,j) : 1 \le i \le m+1,\ 1 \le j \le n+1\} - \{(m+1,n+1)\}$. We separate the vertices into two categories. We say the *internal vertices* are $\{(i,j) : 1 \le i \le m,\ 1 \le j \le n\}$ and the *boundary vertices* are $\{(m+1,j) \text{ and } (i,n+1) : 1 \le i \le m,\ 1 \le j \le n\}$. The edge set is:

$$E(m,n) := \begin{cases} (i,j) \text{ to } (i+1,j) & 1 \le i \le m, 1 \le j \le n \\ (i,j) \text{ to } (i,j+1) & 1 \le i \le m, 1 \le j \le n. \end{cases}$$

Edges between internal vertices are called *internal edges* and any edge between an internal and boundary vertex is called a *boundary edge*. We draw the graph with i increasing to the right and j increasing down, to correspond with matrix indexing.

Definition 4.3.5 ([43, Definition 3.4])**.** Given an $m \times n$ matrix X, we define a labeled graph, $\hat{X}$, which is a labeling of the vertices and edges of $\Gamma_{(m,n)}$ from Definition 4.3.4. The internal vertices (i,j), $1 \le i \le m$, $1 \le j \le n$, are each labeled with the corresponding entry of X: $\hat{X}_{ij} = X_{ij}$. The horizontal edges from (i,j) to $(i,j+1)$ are each labeled by the corresponding row partial sum $r_{ij} = \displaystyle\sum_{j'=1}^{j} X_{ij'}$ $(1 \le i \le m, 1 \le j \le n)$. Likewise, the vertical edges from (i,j) to $(i+1,j)$ are each labeled by the corresponding column partial sum $c_{ij} = \displaystyle\sum_{i'=1}^{i} X_{i'j}$ $(1 \le i \le m, 1 \le j \le n)$.

The following theorem gives an inequality description of PASM(m,n). The proof uses a combination of ideas from [43, 46].

Theorem 4.3.6. PASM(m,n) *consists of all* $m \times n$ *real matrices* $X = (X_{ij})$ *such that:*

$$0 \le \sum_{i'=1}^{i} X_{i'j} \le 1, \qquad \text{for all } 1 \le i \le m, 1 \le j \le n, \tag{4.8}$$

$$0 \le \sum_{j'=1}^{j} X_{ij'} \le 1, \qquad \text{for all } 1 \le i \le m, 1 \le j \le n. \tag{4.9}$$

Proof. Let $X \in$ PASM(m,n). First we need to show that X satisfies (4.8) and (4.9). Now $X = \sum_\gamma \mu_\gamma M_\gamma$ where $\sum_\gamma \mu_\gamma = 1$ and the $M_\gamma \in \text{PASM}_{m,n}$. Since we have a convex combination of partial alternating sign matrices, by Definition 4.1.3, we obtain (4.8) and (4.9) immediately. Thus PASM(m,n) fits the inequality description.

Let X be a real-valued $m \times n$ matrix satisfying (4.8) and (4.9). We wish to show that X can be written as a convex combination of partial alternating sign matrices in $\text{PASM}_{m,n}$, so that X is in PASM(m,n).

Consider the corresponding labeled graph $\hat{X}$ of Definition 4.3.5. We will construct a trail in $\hat{X}$ all of whose edges are labeled by inner numbers and show it is a simple path or cycle. (A number α is *inner* if $0 < \alpha < 1$.) Let $r_{i0} = 0 = c_{0j}$ for all i, j. Then for all $1 \le i \le \lambda_1, 1 \le j \le n$, we have $\hat{X}_{ij} = r_{ij} - r_{i,j-1} = c_{ij} - c_{i-1,j}$. Thus,

$$r_{ij} + c_{i-1,j} = c_{ij} + r_{i,j-1}. \tag{4.10}$$

Note that if there are no inner edge labels, then X is already a partial alternating sign matrix. If there exists i or j such that $\hat{X}_{i,n+1}$ or $\hat{X}_{m+1,j}$ is inner, begin constructing the trail at the adjacent boundary edge. If no such i or j exist, start the trail on any vertex, say $\hat{X}_{ij}$ adjacent to an edge with inner label. By (4.10), at least one of $c_{(i\pm1,j)}, r_{(i,j\pm1)}$ is also inner, so we may begin forming a trail by moving through edges with inner labels. From the starting point, construct the trail as follows. Go along a row or column from the starting point along edges with inner labels. Continue in this manner until either (1) you reach a vertex adjacent to an edge that was previously in the trail, or (2) you reach a new boundary edge. If (1), then the part of the trail constructed between the first and second time you reached that vertex will be a simple cycle. That is, we cut off any part that was constructed before the first time that vertex was reached. If (2), then the starting point for the trail must have been a boundary edge, since there exist at least one boundary vertex with inner label. Thus our trail is actually a path.

Label the corners of the path or cycle (not the boundary vertices) alternately $(+)$ and $(-)$. Set ℓ^+ equal to the largest number that we could subtract from the $(-)$ entries and add to the $(+)$ entries while still satisfying (4.8) and (4.9). Construct a matrix X^+ by subtracting and adding in this way. X^+ is a matrix which stills satisfies (4.8) and (4.9) and which has least one more non-inner edge label than X.

Now give opposite labels to the corners and set ℓ^- equal to the largest number we could subtract from $(-)$ entries and add to $(+)$ entries while still satisfying (4.8) and (4.9). Add and subtract in a similar way to create X^-, another matrix satisfying (4.8) and (4.9) and which has at least one more non-inner edge label than X.

Both X^+ and X^- satisfy (4.8) and (4.9) by construction. Also by construction,

$$X = \frac{\ell^-}{\ell^+ + \ell^-} X^+ + \frac{\ell^+}{\ell^+ + \ell^-} X^-$$

and $\frac{\ell^-}{\ell^+ + \ell^-} + \frac{\ell^+}{\ell^+ + \ell^-} = 1$. So X is a convex combination of the two matrices X^+ and X^- that still satisfy the inequalities and are each at least one step closer to being partial alternating sign matrices, since that have at least one more partial sum attaining its maximum or minimum bound. By repeatedly applying this procedure, X can be written as a convex combination of partial alternating sign matrices.

See Figure 4.2 and Example 4.3.7 for an example of this construction. $\square$

Example 4.3.7. Let $X = \begin{pmatrix} 0.2 & 0.4 & 0.3 \\ 0.7 & -0.3 & -0.1 \\ 0 & 0.5 & -0.2 \end{pmatrix}$. Then by the construction described in the proof of Theorem 4.3.6 and shown in Figure 4.2, X can be decomposed as $X = \frac{0.3}{0.1+0.3} X^+ + \frac{0.1}{0.1+0.3} X^-$, where $X^+ = \begin{pmatrix} 0.2 & 0.5 & 0.2 \\ 0.8 & -0.4 & 0 \\ 9 & 0.5 & -0.2 \end{pmatrix}$ and $X^- = \begin{pmatrix} 0.2 & 0.1 & 0.6 \\ 0.4 & 0 & -0.4 \\ 0 & 0.5 & -0.2 \end{pmatrix}$. In this step of decomposing, $\ell^+ = 0.1$ and $\ell^- = 0.3$. Continuing the process of decomposition, one could write X as a convex combination of partial alternating sign matrices.

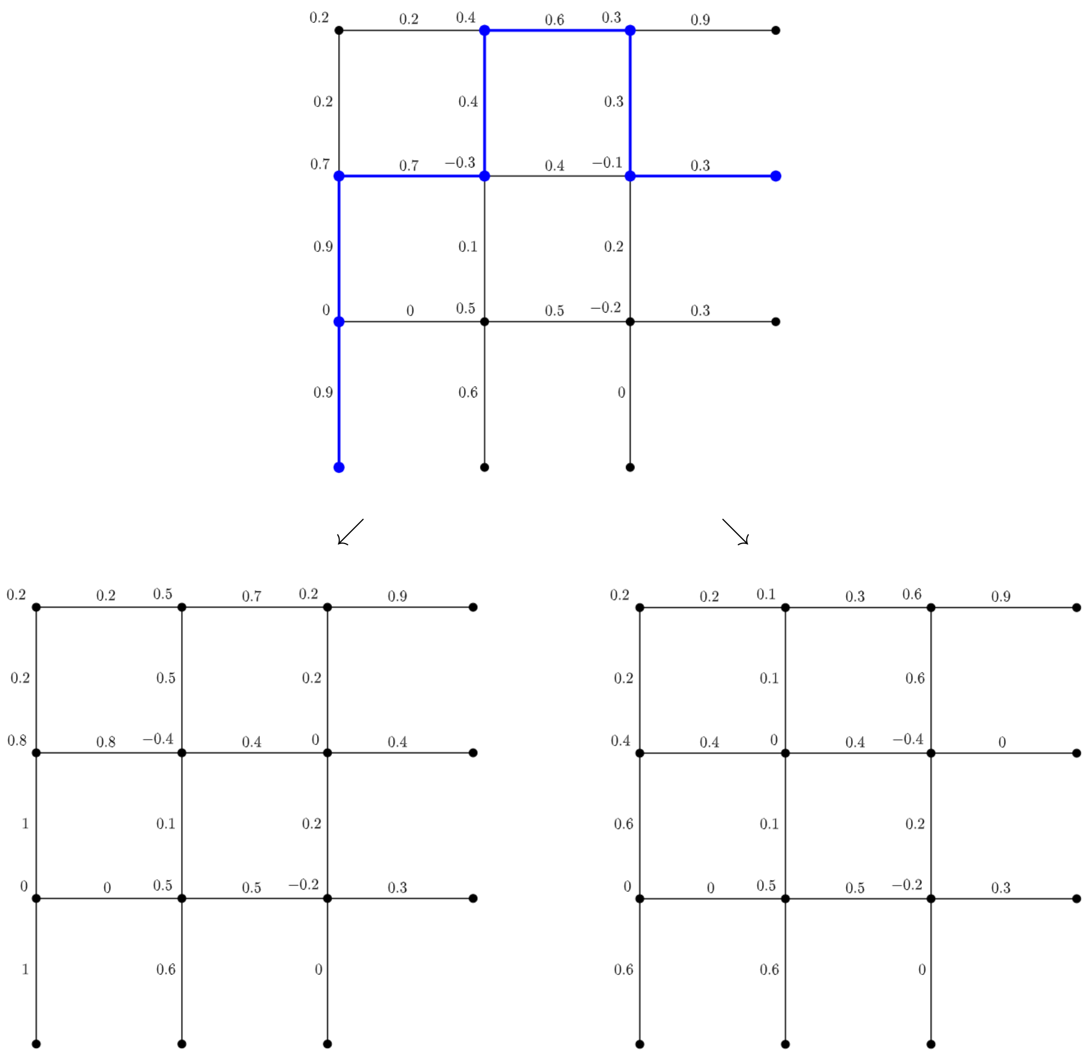


Figure 4.2. An example of the path construction described in the proof of Theorem 4.3.6. The bold blue edges and vertices are those included in the path.

Theorem 4.3.6 gives a simple inequality description, but it is not a *minimal* inequality description. That is, some of the inequalities in (4.8) and (4.9) are redundant. In the following theorem, we determine these redundancies to count the inequalities that determine facets.

Theorem 4.3.8. *The number of facets of* $\mathrm{PASM}(m,n)$ *equals* $4mn - 3m - 3n + 5$.

Proof. There are $4mn$ total defining inequalities given in (4.8) and (4.9). Therefore there are at most $4mn$ facets, each made by turning one of the inequalities into an equality. We begin by showing how $3m + 3n - 5$ of these are redundant.

First, we have that $0 \le X_{1j}$ from the column partial sums, so $0 \le \sum_{j'=1}^{j} X_{1j'}$ for $1 \le j \le n$ are all unnecessary. This is n total redundant inequalities.

From the column partial sums we already have that $0 \le X_{11}$ and from row partial sums we have $0 \le X_{21}$. Together these imply $0 \le X_{11} + X_{21}$. Similarly, all of the partial column sums $0 \le \sum_{i'=1}^{i} X_{i'1}$ for $2 \le i \le m$ are implied by the partial row sums $0 \le X_{i'1}$. This gives another $m - 1$ redundancies.

Now note that $\sum_{i'=1}^{m} X_{i'1} \le 1$, and that $0 \le X_{m1}$. This implies that $\sum_{i'=1}^{m-1} X_{i'1} \le 1 - X_{m1} \le 1$. Similarly, all of the $m-1$ inequalities of the form $\sum_{i'=1}^{i} X_{i'1} \le 1$ for $1 \le i < m$ are all implied by the partial row sums $0 \le X_{i'1}$. This gives us another $m-1$ redundant inequalities. By a similar argument, we will also have that the $n-1$ inequalities of the form $\sum_{j'=1}^{j} X_{1j'} \le 1$ for $1 \le j < n$ are implied by the partial column sums $0 \le X_{1j'}$. This give us another $n-1$ redundant inequalities.

Finally, note that $\sum_{i'=1}^{i-1} X_{i'1} \ge 0$ and $\sum_{i'=1}^{i} X_{i'1} \le 1$. This implies that $X_{i1} \le 1 - \sum_{i'=1}^{i-1} X_{i'1} \le 1$ for $2 \le i \le m$. This gives us another $m-1$ redundancies. By a similar argument we also have that $\sum_{j'=1}^{j-1} X_{1j'} \ge 0$ and $\sum_{j'=1}^{j} X_{1j'} \le 1$ combine to imply that $X_{1j} \le 1$ for $2 \le j \le n$. This gives $n-1$ additional redundancies.

Overall, this means that the number of facets is at most $4mn - 3m - 3n + 5$. We claim that this upper bound is the facet count. That is, a facet can be defined as all $X \in \mathrm{PASM}(m,n)$ which satisfy exactly one of the following:

$$r_{ij} = \sum_{j'=1}^{j} X_{ij} = 0, \qquad \text{for all } 2 \le i \le m \text{ and } 1 \le j \le n \tag{4.11}$$

$$r_{ij} = \sum_{j'=1}^{j} X_{ij} = 1, \qquad \text{for all } 2 \le i \le m \text{ and } 2 \le j \le n \tag{4.12}$$

$$c_{ij} = \sum_{i'=1}^{i} X_{ij} = 0, \qquad \text{for all } 1 \le i \le m \text{ and } 2 \le j \le n \tag{4.13}$$

$$c_{ij} = \sum_{i'=1}^{i} X_{ij} = 1, \qquad \text{for all } 2 \le i \le m \text{ and } 2 \le j \le n \tag{4.14}$$

$$c_{m1} = \sum_{i=1}^{m} X_{i1} = 1 \tag{4.15}$$

$$r_{1m} = \sum_{j=1}^{n} X_{1j} = 1 \tag{4.16}$$

$$r_{11} = c_{11} = X_{11} = 0 \tag{4.17}$$

To show this, let two generic equalities of the form (4.11)-(4.17) be denoted as $\alpha_{ij} = \gamma$ and $\beta_{k\ell} = \delta$ for $\alpha, \beta \in \{r, c\}$ and $\gamma, \delta \in \{0, 1\}$, where the choice of r or c for each of α and β indicates whether the equality involves partial row sum r_{ij} or partial column sum c_{ij}, and the indices (i, j) and (k, ℓ) must be in the corresponding ranges indicated by (4.11)–(4.17). In the cases below, we will construct an $m \times n$ partial alternating sign matrix M, such that M satisfies $\alpha_{ij} = \gamma$ and not $\beta_{k\ell} = \delta$.

Case 1: $\alpha_{ij} = 0$ and $\beta_{k\ell} = 1$. We set M equal to the zero matrix.

In each of the following, we will specify the nonzero entries of M, and assume all other entries are zero.

Case 2: $\alpha_{ij} = 0$ and $\beta_{k\ell} = 0$.

- If $i \neq k$ and $j \neq \ell$ let $M_{k\ell} = 1$.
- Suppose $\alpha = \beta = c$. If $j \neq \ell$, let $M_{k\ell} = 1$. If $j = \ell$ and $i < k$, let $M_{k\ell} = 1$. If $j = \ell$ and $i > k$, let $M_{k\ell} = M_{k+1,\ell-1} = 1$ and $M_{k+1,\ell} = -1$.

- Suppose $\alpha = \beta = r$. If $i \neq k$, let $M_{k\ell} = 1$. If $i = k$ and $j < \ell$, let $M_{k\ell} = 1$. If $i = k$ and $j > \ell$, let $M_{k\ell} = M_{k-1,\ell+1} = 1$ and $M_{k+1,\ell} = -1$.

- If $\alpha = r$ and $\beta = c$, let $M_{1\ell} = 1$.

- If $\alpha = c$ and $\beta = r$, let $M_{k1} = 1$.

Case 3: $\alpha_{ij} = 1$ and $\beta_{k\ell} = 1$.

- If $i \neq k$ and $j \neq \ell$, let $M_{ij} = 1$.

- Suppose $\alpha = \beta = c$. If $j \neq \ell$, let $M_{ij} = 1$. If $j = \ell$ and $i < k$, let $M_{ij} = M_{i+1,j-1} = 1$ and $M_{i+1,j} = -1$ If $j = k$ and $i > \ell$, let $M_{ij} = 0$.

- Suppose $\alpha = \beta = r$. If $i \neq k$, let $M_{ij} = 1$. If $i = k$ and $j < \ell$, let $M_{ij} = M_{i-1,j+1} = 1$ and $M_{i,j+1} = -1$. If $i = k$ and $j > \ell$, let $M_{ij} = 1$.

- If $\alpha = r$ and $\beta = c$, let $M_{1j} = 1$.

- If $\alpha = c$ and $\beta = r$, let $M_{i1} = 1$.

Case 4: $\alpha_{ij} = 1$ and $\beta_{k\ell} = 0$.

- If $i \neq k$ and $j \neq \ell$, let $M_{ij} = M_{k\ell} = 1$.

- Suppose $\alpha = \beta = c$. If $i = k$ and $j \neq \ell$, let $M_{ij} = M_{1\ell} = 1$. If $j = \ell$, let $M_{1j} = 1$.

- Suppose $\alpha = \beta = r$. If $i = k$, let $M_{i1} = 1$. If $j = \ell$ and $i \neq k$, let $M_{ij} = M_{k1} = 1$.

- Suppose $\alpha = r$ and $\beta = c$. If $i = k$ and $j < \ell$, let $M_{ij} = M_{1\ell} = 1$. If $i = k$ and $j > \ell$, let $M_{k\ell} = 1$. If $j = \ell$ and $i \leq k$, let $M_{ij} = 1$. If $j = \ell$ and $i < k$, let $M_{k\ell} = M_{i1} = 1$.

- Suppose $\alpha = c$ and $\beta = r$. If $i = k$ and $j \leq \ell$, let $M_{ij} = 1$. If $i = k$ and $j > \ell$, let $M_{k\ell} = M_{1j} = 0$. If $j = \ell$ and $i < k$, let $M_{ij} = M_{k1} = 1$. If $j = \ell$ and $i > k$, let $M_{k\ell} = 1$.

In each of these cases, M is constructed so that it satisfies $\alpha_{ij} = \gamma$ but not $\beta_{ij} = \delta$, so each of the equalities in (4.11)–(4.17) gives rise to a unique facet. Thus there are $4mn - 3m - 3n + 5$ facets of $\mathrm{PASM}(m, n)$. □

Remark 4.3.9. The above inequality description may make one wonder whether the matrix defining $\mathrm{PASM}(m,n)$ is *totally unimodular*. Consider the case when $m = n = 2$. Then there are 3×3 submatrices with determinant 2 and -2, so the matrix is not totally unimodular.

Recall from Remark 4.1.5 that partial alternating sign matrices are a subset of sign matrices [43]. It was shown in [43, Theorem 5.3] that the convex hull of $m \times n$ sign matrices, denoted $P(m,n)$, has inequality description as in Theorem 4.3.6, except in (4.9) the ≤ 1 is not present. More specifically, we have the following relation.

Lemma 4.3.10. $\mathrm{PASM}(m,n)$ *is the intersection of* $P(m,n)$ *with the subspace of* $m \times n$ *real matrices* $X = (X_{ij})$ *such that:*

$$\sum_{i'=1}^{i} X_{i'j} \leq 1, \text{ for all } 1 \leq i \leq m, 1 \leq j \leq n.$$

4.3.2. Face Lattice

In this subsection, we characterize the face lattice of $\mathrm{PASM}(m,n)$ in Theorem 4.3.18, using *sum-labelings* of the graph $\Gamma(m,n)$ (see Definition 4.3.4).

We first state some definitions and a lemma that will help prove Theorem 4.3.18 describing the face lattice of $\mathrm{PASM}(m,n)$. This theorem is analogous to [43, Theorems 7.15 and 7.16] which describe the face lattice of $P(m,n)$. The proof is also similar.

Recall $\hat{M}$ from Definition 4.3.5.

Definition 4.3.11. A *basic sum-labeling* of $\Gamma_{(m,n)}$ is a labeling of the edges of $\Gamma_{(m,n)}$ with 0 or 1 such that the edge labels equal the corresponding edge labels of $\hat{M}$ for some $M \in \mathrm{PASM}_{m,n}$.

Remark 4.3.12. Recall we can recover any matrix from its column partial sums, thus basic sum-labelings of $\Gamma_{(m,n)}$ are in bijection with partial alternating sign matrices $\mathrm{PASM}_{m,n}$.

Definition 4.3.13. Let δ and δ' be labelings of the edges of $\Gamma_{(m,n)}$ with 0, 1, or $\{0,1\}$ (where by 0 we mean $\{0\}$ and similarly for 1). Define the *union* $\delta \cup \delta'$ as the labeling of $\Gamma_{(m,n)}$ such that each edge is labeled by the union of the corresponding labels on δ and δ'. Define *intersection* $\delta \cap \delta'$ and *containment* $\delta \subseteq \delta'$ similarly.

Definition 4.3.14. A *sum-labeling* δ of $\Gamma_{(m,n)}$ is either the empty labeling of $\Gamma_{(m,n)}$ (denoted $\emptyset$) or a labeling of the edges of $\Gamma_{(m,n)}$ with 0, 1, or $\{0,1\}$ such that there exists a set S of basic sum-labelings of $\Gamma_{(m,n)}$ so that $\delta = \bigcup_{\delta' \in S} \delta'$.

Definition 4.3.15. Given $M \in \text{PASM}_{m,n}$, let $g(M)$ denote the sum-labeling of $\Gamma_{(m,n)}$ associated to M. Given a collection of partial alternating sign matrices $\mathcal{M} = \{M_1, M_2, \ldots, M_r\} \subseteq \text{PASM}_{m,n}$, define the map $g(\mathcal{M}) = \bigcup_{i=1}^{r} g(M_i)$.

Definition 4.3.16. Given a sum-labeling δ, consider the planar graph G composed of the edges of δ labeled by the two-element set $\{0,1\}$ (and all incident vertices), where we regard any external edges on the right and bottom as meeting at a point in the exterior. We say a *region* of δ is defined as a planar region of G, excluding the exterior region. Let $\mathcal{R}(\delta)$ denote the number of regions of δ. (For consistency we set $\mathcal{R}(\emptyset) = -1$.)

See Figure 4.3 for an example of a sum-labeling of $\Gamma_{(2,3)}$ with 4 regions.

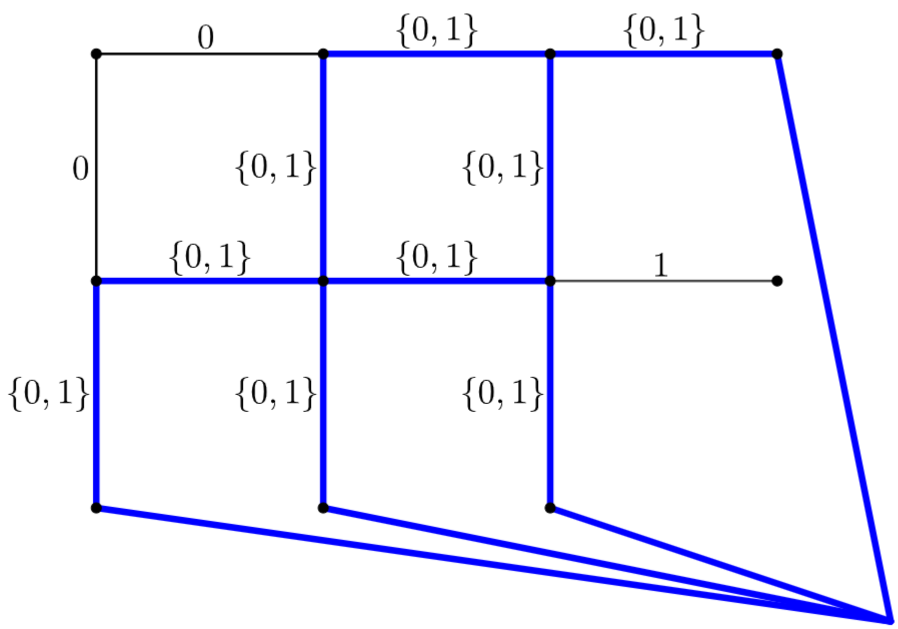


Figure 4.3. The sum-labeling of $\Gamma_{(2,3)}$ which is $g(M_3) \cup g(M_{13}) \cup g(M_{15})$, where M_3, M_{13}, and M_{15} are as in Examples 3.1.4 and 3.2.4. Edges labeled $\{0,1\}$ are bolded and blue to accentuate the regions.

Lemma 4.3.17. *Suppose a sum-labeling δ has $\mathcal{R}(\delta) = \omega$. If $\delta \subset \delta'$ then $\mathcal{R}(\delta') > \omega$.*

Proof. By convention, the empty labeling has $\mathcal{R}(\emptyset) = -1$. If δ is a basic sum-labeling, $\mathcal{R}(\delta) = 0$, as there are no edges labeled $\{0, 1\}$ in a basic sum-labeling. Suppose a sum-labeling δ has $\mathcal{R}(\delta) = \omega > 0$. We wish to show if $\delta \subset \delta'$ then $\mathcal{R}(\delta') > \omega$. $\delta \subset \delta'$ implies that the labels of each edge of δ are subsets of the labels of each edge of δ', where at least one of these containments is strict. So there is an edge in δ' labeled $\{0, 1\}$ that was labeled 0 or 1 in δ. So δ' contains a basic sum labeling β' that differs from all the basic sum labelings in δ at edge e. Let β denote a basic sum labeling such that $\beta \subseteq \delta$. By Equation (4.10), at least one edge label of β' adjacent to e must also differ from the corresponding edge label of β. By iterating this (as in the proof of Theorem 4.3.6), β' differs from β by at least one simple path (connecting boundary vertices) or cycle of differing partial sums. This path or cycle appears as edges labeled by $\{0, 1\}$ in δ', and at least one of these edges was not labeled by $\{0, 1\}$ in δ. So δ' has at least one new region. Therefore, $\mathcal{R}(\delta') > \omega$. $\square$

We are now ready to state and prove the main theorem of this subsection.

Theorem 4.3.18. *Let F be a face of* PASM(m, n) *and $\mathcal{M}(F)$ be equal to the set of partial alternating sign matrices that are vertices of F. The map $\psi : F \mapsto g(\mathcal{M}(F))$ induces an isomorphism between the face lattice of* PASM(m, n) *and the set of sum-labelings of $\Gamma_{(m,n)}$ ordered by containment. Moreover, dim $F = \mathcal{R}(\psi(F))$.*

Proof. Let F be a face of PASM(m, n). Then $g(\mathcal{M}(F))$ is a sum-labeling of $\Gamma_{(m,n)}$ since $g(\mathcal{M}(F)) = \bigcup_{i=1}^{r} g(M_i)$ is a union of basic sum-labelings. We now construct the inverse of ψ, call it φ. Given a sum-labeling ν of $\Gamma_{(m,n)}$, let $\varphi(\nu)$ be the face that results as the intersection of the facets corresponding to the edges of ν with label 0 or 1.

We wish to show $\psi(\varphi(\nu)) = \nu$. First, we show $\nu \subseteq \psi(\varphi(\nu))$. Let $M \in$ PASM(m, n) such that $g(M) \subset \nu$ is a basic sum-labeling. M is in the intersection of the facets that yields $\varphi(\nu)$, since otherwise $g(M)$ would not be a basic sum-labeling such that $g(M) \subset \nu$. Thus $g(M) \subseteq \psi(\varphi(\nu))$ as well. So $\nu \subseteq \psi(\varphi(\nu))$.

Next, we show $\psi(\varphi(\nu)) \subseteq \nu$. Suppose not. Then there exists some edge e of $\Gamma_{(m,n)}$ whose label in $\psi(\varphi(\nu))$ strictly contains the label of e in ν. The label of e in ν is 0 or 1 and the label of e in

$\psi(\varphi(\nu))$ is $\{0, 1\}$. Let γ denote the label of e in ν. As in the previous case, the facet corresponding to the label γ on e would have been one of the facets intersected to get $\varphi(\nu)$. Therefore the matrix partial column sum corresponding to edge e would be fixed as γ in each partial alternating sign matrix in $\varphi(\nu)$. So in the union $\psi(\varphi(\nu))$, that edge label would be the union of the edge labels of all the partial alternating sign matrices in $\varphi(\nu)$, and this union would be γ. This is a contradiction. Thus $\nu = \psi(\varphi(\nu))$.

Let F_1 and F_2 be faces of PASM(m, n) such that $F_1 \subset F_2$. Then F_1 is an intersection of F_2 and some facet hyperplanes. In other words, F_1 is obtained from F_2 by setting at least one of the inequalities in Theorem 4.3.6 to an equality. We have that $\psi(F_1)$ is obtained from $\psi(F_2)$ by changing at least edge label of $\{0, 1\}$ to a label of 0 or 1. Therefore we have $\psi(F_1) \subset \psi(F_2)$.

Conversely, suppose that $\psi(F_1) \subset \psi(F_2)$. Recall the inverse of ψ is φ, where for any sum-labeling ν of $\Gamma_{(m,n)}$, $\varphi(\nu)$ is the face of PASM(m, n) that results as the intersection of the facets corresponding to the edges of ν with labels 0 or 1. Now if $\psi(F_1) \subset \psi(F_2)$, the edges of $\psi(F_1)$ with label $\{0, 1\}$ are a subset of such edges of $\psi(F_2)$, so the edges of $\psi(F_2)$ with labels of either 0 or 1 are a subset of such edges of $\psi(F_1)$. So $\varphi(\psi(F_1))$ is an intersection of the facets intersected in $\varphi(\psi(F_2))$ and one or more additional facets. Thus $F_1 = \varphi(\psi(F_1)) \subset \varphi(\psi(F_2)) = F_2$.

Now, we prove the dimension claim. Recall from Remark 4.3.2 that $\dim(\text{PASM}(m, n)) = mn$. Since ψ is a poset isomorphism, ψ maps a maximal chain of faces $F_0 \subset F_1 \subset \cdots \subset F_{mn}$ to the maximal chain $\psi(F_0) \subset \psi(F_1) \subset \cdots \subset \psi(F_{mn})$ in the sum-labelings of $\Gamma_{(m,n)}$. The sum-labeling whose labels are all equal to $\{0, 1\}$ contains all other sum-labelings, and this sum-labeling has mn regions. Thus the result follows by Lemma 4.3.17. □

4.3.3. Volume

The normalized volume of PASM(m, n) for small values of m and n is given in Table 4.2 (computed in SageMath). Due to the large size of the polytopes, further computations are not easily obtained. Note that there does not appear to be a nice formula for the volume.

Remark 4.3.19. We have used SageMath to compute the Ehrhart polynomials for PASM(m, n) for $m, n \leq 4$ and note that in all of these cases their coefficients are positive.

Table 4.2. The normalized volume of PASM(m, n) for small values of m and n.

$m \backslash n$	1	2	3	4
1	1	1	1	1
2	1	6	43	308
3	1	43	5036	696658
4	1	308	696658	3106156252

4.4. Partial Permutohedron

In this section, we study partial permutohedra that arise naturally as projections of PPerm(m, n) and PASM(m, n). After giving the definition, we count vertices and facets and find an inequality description in Subsection 4.4.1. Then in Subsection 4.4.2, we note the relation between the partial permutohedron and the stellohedron and give a new combinatorial description of its face lattice. We show in Subsection 4.4.3 that partial permutation and partial alternating sign matrix polytopes project to partial permutohedra. Finally, in Subsection 4.4.4, we give a result and conjecture on volume.

4.4.1. Vertices, Facets, Inequality Description

In this subsection, we first give the definition of partial permutohedra. We enumerate the vertices in Proposition 4.4.6 and the facets in Theorem 4.4.10 and prove an inequality description in Theorem 4.4.9.

Definition 4.4.1. Given a partial permutation matrix $M \in P_{m,n}$, its *one-line notation* $w(M)$ is a word $w_1 w_2 \dots w_m$ where $w_i = j$ if there exists j such that $M_{ij} = 1$ and 0 otherwise.

Example 4.4.2. Let $M = \begin{pmatrix} 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \end{pmatrix}$. Then $w(M) = 3502$.

Proposition 4.4.3. *$w(P_{m,n})$ can be characterized as the set of all words of length m whose entries are in $\{0, 1, \dots, n\}$ and whose nonzero entries are distinct.*

Proof. By definition, any matrix in $P_{m,n}$ has m rows and n columns with at most one 1 in any given row or column. Thus its image under w will be a word of length m with entries in $\{0, 1, \dots, n\}$

such that the nonzero entries are all distinct. It follows from the definition of w that this map is bijective. $\square$

Definition 4.4.4. Let $\mathcal{P}(m,n)$ be the polytope defined as the convex hull, as vectors in $\mathbb{R}^m$, of the words in $w(P_{m,n})$. Call this the (m,n)-*partial permutohedron.*

Definition 4.4.5. Let $z \in \mathbb{R}^n$ be a vector with distinct nonzero entries. Define $\phi_z : \mathbb{R}^{m\times n} \to \mathbb{R}^m$ as $\phi_z(X) = Xz$. Also define $w_z(P_{m,n})$ as the set of all words of length m whose entries are in $\{0, z_1, z_2, \ldots, z_n\}$ and whose nonzero entries are distinct. Then $\mathcal{P}_z(m,n)$ is the polytope defined as the convex hull, as vectors in $\mathbb{R}^m$, of the words in $w_z(P_{m,n})$.

Note that we will not use Definition 4.4.5 until later in this section, but the upcoming results about the structure of partial permutohedra can also be extended to $\mathcal{P}_z$ polytopes.

Proposition 4.4.6. *The number of vertices of* $\mathcal{P}(m,n)$ *equals*

$$\sum_{k=\max(m-n,0)}^{m} \frac{m!}{k!}.$$

Proof. The extreme points of $\mathcal{P}(m,n)$ are those whose nonzero entries are maximized. That is, if k is the number of zeros, the $(m-k)$ nonzero entries must be precisely $\{n, n-1, \ldots, n-(m-k)+1\}$. Now, since there are m total entries and k zeros, there are $\frac{m!}{k!}$ distinct vectors whose $m-k$ nonzero elements are maximized. $\square$

For the proof of the next theorem, and for that of Theorem 4.4.27, we need the concept of (weak) majorization [30].

Definition 4.4.7 ([30, Definition A.2])**.** Let u and v be vectors of length N. Then $u \prec_w v$ (that is, u is *weakly majorized* by v) if

$$\sum_{i=1}^{k} u_{[i]} \leq \sum_{i=1}^{k} v_{[i]}, \text{ for all } 1 \leq k \leq N$$

where the vector $\left(u_{[1]}, u_{[2]}, \ldots, u_{[N]}\right)$ is obtained from u by rearranging its components so that they are in decreasing order (and similarly for v).

Proposition 4.4.8 ([30, Proposition 4.C.2])**.** *For vectors u and v of length n, $u \prec_w v$ if and only if u lies in the convex hull of the set of all vectors z which have the form $z = \big(\varepsilon_1 v_{\pi(1)}, \dots, \varepsilon_n v_{\pi(n)}\big)$, where π is a permutation and each ε_i is either 0 or 1.*

Theorem 4.4.9. *$\mathcal{P}(m,n)$ consists of all vectors $u \in \mathbb{R}^m$ such that:*

$$\sum_{i\in S} u_i \leq \binom{n+1}{2} - \binom{n-k+1}{2}, \qquad \text{where } S \subseteq [m], |S| = k \neq 0, \text{ and} \tag{4.18}$$

$$u_i \geq 0, \qquad \text{for all } 1 \leq i \leq m. \tag{4.19}$$

Proof. First, note that if $P \in P_{m,n}$, then $w(P)$ satisfies (4.18) and (4.19). This is because the largest values that may appear are the m largest non-negative integers less than or equal to n, and the nonzero integers must be distinct. Since $w(P)$ satisfies the inequalities for any P, so must any convex combination.

Now, suppose $x \in \mathbb{R}^m$ satisfies (4.18) and (4.19). We will proceed by using Proposition 4.4.8. Fix n and let $v = (n, n-1, n-2, \dots, 1, 0, \dots, 0)$ be the decreasing vector in $\mathbb{R}^m$ whose largest entry is n, and whose subsequent nonzero entries decrease by 1 and for which all other entries are 0. Note that if $n \geq m$, then v will have no 0 entries: it will be $(n, n-1, \dots, n-m+1)$. Since x satisfies (4.18) and (4.19), it is by definition weakly majorized by v; note in particular that (4.18) requires that the sum of the k largest entries is never more than the k largest integers less than or equal to n. But now the convex hull described in Proposition 4.4.8 is actually $\mathcal{P}(m,n)$, thus $x \in \mathcal{P}(m,n)$. □

Theorem 4.4.10. *The number of facets of $\mathcal{P}(m,n)$ equals $m + 2^m - 1 - \sum_{r=1}^{m-n} \binom{m}{m-r}$.*

Proof. There are $2^m - 1$ total inequalities given in (4.18), and m inequalities given in (4.19). Note that $\binom{n-k+1}{2} = 0$ whenever $k \geq n$. When $m > n$, there are $m-n$ values of k such that $\binom{n-k+1}{2} = 0$, creating redundancies. For each r between 1 and $m-n$, we have redundant inequalities for the subsets of $[m]$ of size $m-r$. These are counted by $\binom{m}{m-r}$.

When $m \leq n$, none of the inequalities in (4.18) are redundant, since $\binom{n-k+1}{2} = 0$ may only be satisfied by $k = n$. □

Remark 4.4.11. When $m \geq n$, the number of facets of $\mathcal{P}(m,n)$ can also be written as:

$$m + \sum_{r=m-n+1}^{m} \binom{m}{m-r}.$$

4.4.2. Face Lattice

In this subsection, we give a combinatorial description of the face lattice of $\mathcal{P}(m,m)$ in Theorem 4.4.23 involving chains in the Boolean lattice. We furthermore state Conjecture 4.4.24, which extends this characterization to $m \neq n$.

We begin by relating $\mathcal{P}(m,m)$ to a specific graph associahedron, the stellohedron. But first, we need the following definitions.

Definition 4.4.12 ([14, Definition 2.2])**.** Let G be a connected graph. A *tube* is a proper nonempty set of vertices of G whose induced graph is a proper, connected subgraph of G. There are three ways that two tubes t_1 and t_2 may interact on the graph:

1. Tubes are *nested* if $t_1 \subset t_2$.

2. Tubes *intersect* if $t_1 \cap t_2 \neq \emptyset$ and $t_1 \not\subset t_2$ and $t_2 \not\subset t_1$.

3. Tubes are *adjacent* if $t_1 \cap t_2 = \emptyset$ and $t_1 \cup t_2$ is a tube in G.

Tubes are *compatible* if they do not intersect and they are not adjacent. A *tubing* T of G is a set of tubes of G such that every pair of tubes is compatible. A *k–tubing* is a tubing with k tubes.

Definition 4.4.13 ([17, Definition 2])**.** For a graph G, the *graph associahedron* $\mathrm{Assoc}(G)$ is a simple, convex polytope whose face poset is isomorphic to the set of tubings of G, ordered such that $T < T'$ if T obtained from T' by adding tubes.

Of particular interest to us is the graph associahedron of the star graph, $\mathrm{Assoc}(K_{1,m})$, also called the *stellohedron*.

Definition 4.4.14. The *star graph* (with $m+1$ vertices) is the complete bipartite graph $K_{1,m}$. We label the lone vertex $*$, and call it the *inner vertex*. We label the other m vertices $x_1, x_2, \ldots, x_m$, and call them *outer vertices*.

Remark 4.4.15. Note that if G has n nodes, vertices of $\mathrm{Assoc}(G)$ correspond to maximal tubings of G (*i.e.* $(n-1)$–tubings), and in general, faces of dimension k correspond to $(n-k-1)$-tubings of G. Thus for the star graph $K_{1,m}$, which has $m+1$ nodes, vertices of $\mathrm{Assoc}(K_{1,m})$ correspond to m-tubings, and in general, faces of dimension k correspond to $(m-k)$–tubings.

We examine the polytope $\mathrm{Assoc}(K_{1,m})$ through the lens of partial permutations, which allows us to understand it in a different way. Lemmas 4.4.19 and 4.4.20 and Corollary 4.4.22, which culminate in Theorem 4.4.23, shed light on a way to view these tubings, and thus the faces of the stellohedron, as certain chains in the Boolean lattice. Furthermore, in Conjecture 4.4.24 we describe what we think happens for $\mathcal{P}(m,n)$, where $m \neq n$. But first, we review the following result that relates $\mathcal{P}(m,m)$ to the stellohedron; this can be found, in other language, in [29]. See also [24], which gives connections to representation theory.

Theorem 4.4.16 ([29, Proposition 56])**.** *$\mathcal{P}(m,m)$ is a realization of Assoc$(K_{1,m})$.*

We describe the explicit map for vertices in the remark below.

Remark 4.4.17. The map which sends maximal tubings of $K_{1,m}$ to the vertices of $\mathcal{P}(m,m)$ is as follows. Let T be a maximal tubing of $K_{1,m}$, and for each outer vertex x_i, let t_i be the smallest tube containing x_i. Then the coordinate in $\mathbb{R}^m$ corresponding to T is $(|t_1|-1, |t_2|-1, \ldots, |t_m|-1)$. Note that two tubes of the star graph are compatible only if they each contain a single outer vertex and do not contain $*$, or one is contained in the other. So a maximal tubing will have r tubes which are singleton outer vertices and tubes of each size from $r+1$ to $m+1$. Moreover, the tube of size $r+1$ must contain each of the r singleton outer vertices along with the inner vertex. Thus such a tubing gets mapped to a coordinate in $\mathbb{R}^m$ with r zeros and whose nonzero entries are $\{m, m-1, \ldots, r+1\}$, which is a vertex of $\mathcal{P}(m,m)$.

One can view a tubing instead as its corresponding spine, defined below. This will help in our goal of describing a bijection between tubings of the star graph and chains in the Boolean lattice.

Definition 4.4.18. Let T be a tubing of the star graph. The *spine* of T is the poset of tubes of T ordered by inclusion, whose elements are labeled not by the tubes themselves but by the set of new vertices in each tube. For simplicity, we will use the label i in place of x_i.

Spines are defined (in more generality) in [29, Remark 10] and are called B-trees in [36, Definition 7.7]. See Figure 4.4 for examples of tubings with their corresponding spines, as well as their corresponding chains from the bijection in the following lemma. The *Boolean lattice* B_m is the poset of all subsets of $[m]$, ordered by inclusion.

Lemma 4.4.19. *Tubings of $K_{1,m}$ are in bijection with chains in the Boolean lattice $\mathcal{B}_m$.*

Proof. Given a spine S of a tubing T of $K_{1,m}$, we can construct the corresponding chain in the Boolean lattice as follows. The bottom element of the chain is the subset including anything that is grouped with $*$ in S. Each subsequent subset is made by adding in the elements in the next level of S, until we reach the top level. Any elements not used in the subsets of the chain will be those that appear below the $*$ in S.

Starting with a chain $C \in \mathcal{B}_m$, we can recover the corresponding spine S (and thus the tubing) by reversing this process. Any elements not in the maximal subset of C will be in the bottom level of S as singletons. Any elements in the minimal chain of C will appear with $*$ in S. The new elements that appear in each subsequent subset in C appear together as a new level in S. Once we have S, we can, of course, recover T. $\square$

Lemma 4.4.20. *Let T be a k-tubing and T' be a $(k+j)$-tubing of $K_{1,m}$, and let C and C' be their corresponding chains in $\mathcal{B}_m$ via the bijection in Lemma 4.4.19. $T \subset T'$ if and only if C' can be obtained from C by j iterations of the following:*

1. *adding a non-maximal subset, or*
2. *removing the same element from every subset.*

Proof. Consider $T \subset T'$, *i.e.* T' is obtained from T by adding tubes. Suppose T and T' differ by adding a single tube, that is, $T = \{t_1, t_2, \ldots, t_k\}$ and $T' = \{t_1, t_2, \ldots, t_k, t'\}$. Let S and S' be their corresponding spines, and let C and C' be their corresponding chains. First note that by the

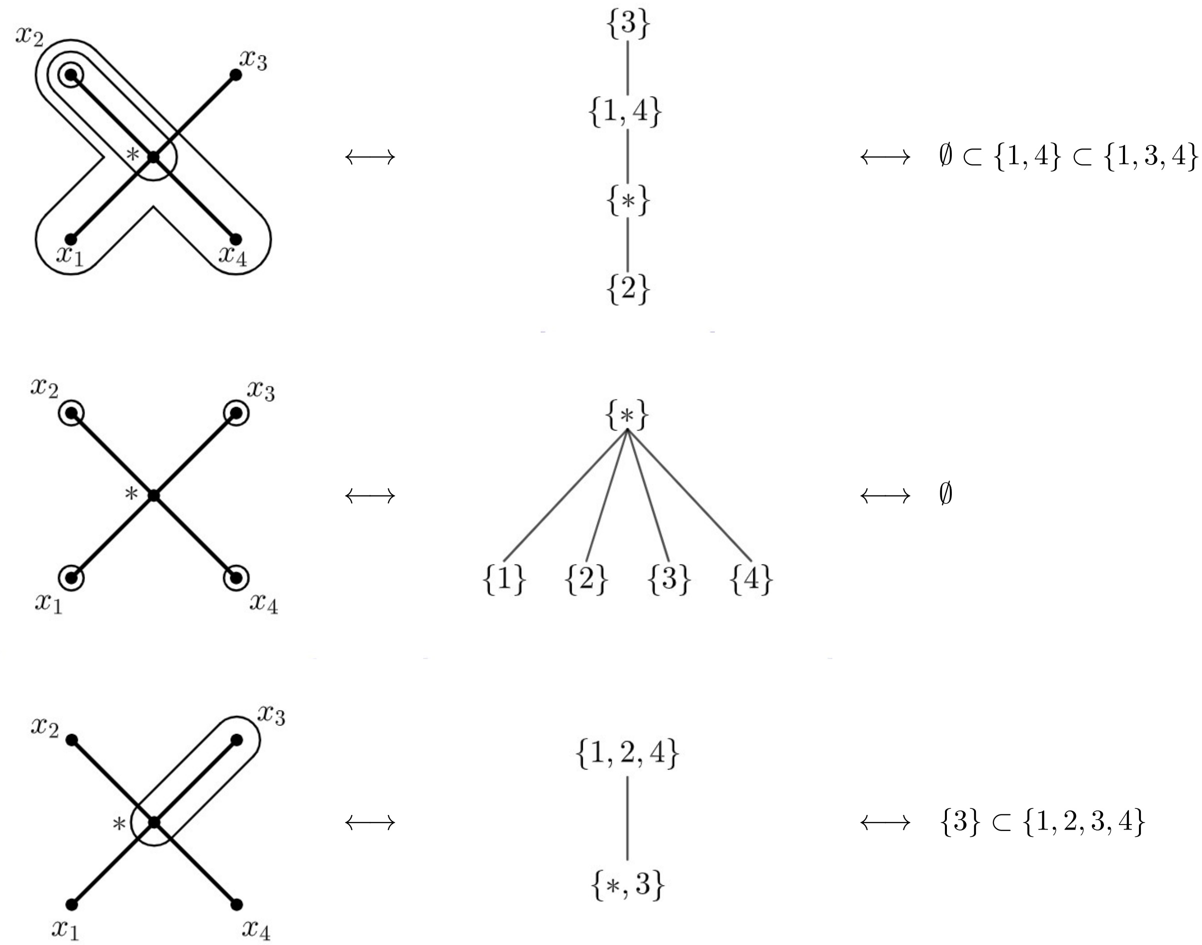


Figure 4.4. Examples of tubings of $K_{1,4}$ along with their corresponding spines (see Definition 4.4.18) and chains in $\mathcal{B}_4$ (via the bijection in Lemma 4.4.19)

nature of the star graph, a tube either is a singleton outer vertex, x_i, or contains the inner vertex, $*$. Note that a singleton x_i and the singleton $*$ cannot coexist as tubes in a tubing since they are not compatible (they are adjacent).

First consider the case that t' was a singleton outer vertex, x_i. This means that in S, i was grouped with $*$, while in S', $\{i\}$ now appears below $*$. On the level of chains, this means that i is removed from all of the subsets in C to obtain C'.

Now consider the case that t' was not a singleton outer vertex. Then it necessarily contains $*$. In this case, S' has a new level which was not present in S. In particular, this level contains $*$ (and possibly other labels). A new level containing $*$ corresponds to a non-maximal subset being added on the level of chains. In other words, C' is obtained from C by adding a non-maximal subset.

Now suppose T and T' differ by more than one tube, say T is a k-tubing and T' is a $(k+j)$-tubing for some j. Then T' is obtained from T by adding one tube at a time, j times, and thus C' is obtained from C by j iterations of (1) and/or (2) above. $\square$

We now give a description of the dimension of a face in terms of its corresponding chain. This description involves *missing ranks*, which we define below.

Definition 4.4.21. Given a chain $C \in \mathcal{B}_m$, we say a rank j is *missing* from C if there is no subset of size j in C and there is a subset of size greater than j in C.

Corollary 4.4.22. *A face of $\mathcal{P}(m, m)$ is of dimension k if and only if the corresponding chain has k missing ranks.*

Proof. We know that adding a tube reduces the dimension of the corresponding face by one. Also, by Lemma 4.4.20, we know that adding a tube corresponds to either adding a non-maximal subset or removing an element from every subset in the corresponding chain. In either case, this reduces the number of missing ranks in the chain by one. So, having k missing ranks in the chain corresponds to having $m-k$ tubes, which by definition of the graph associahedron corresponds to a face being of dimension k. $\square$

The theorem below follows directly from the above lemmas and corollary.

Theorem 4.4.23. *The face lattice of $\mathcal{P}(m, m)$ is isomorphic to the lattice of chains in $\mathcal{B}_m$, where $C < C'$ if C' can be obtained from C by iterations of (1) and/or (2) from Lemma 4.4.20. A face of $\mathcal{P}(m, m)$ is of dimension k if and only if the corresponding chain has k missing ranks.*

As chains in the Boolean lattice are generally more familiar objects than tubings of graphs, presenting results in terms of these chains is conceptually helpful. In fact, because of the description of the faces of $\mathcal{P}(m, m)$ in terms of chains, we are able to form the following conjecture for $\mathcal{P}(m, n)$.

Conjecture 4.4.24. *Faces of $\mathcal{P}(m,n)$ are in bijection with chains in $\mathcal{B}_m$ whose difference between largest and smallest nonempty subsets is at most $n-1$. A face of $\mathcal{P}(m,n)$ is of dimension k if and only if the corresponding chain has k missing ranks*

Remark 4.4.25. This conjecture has been tested and verified for $m, n \leq 4$ using SageMath.

4.4.3. Projection from Partial Alternating Sign Matrix Polytopes

In this subsection, we show that the partial permutohedron is a projection of both PPerm(m,n) (in Theorem 4.4.26) and PASM(m,n) (in Theorem 4.4.27). Recall ϕ_z from Definition 4.4.5.

Theorem 4.4.26. $\phi_z(\mathrm{PPerm}(m,n)) = \mathcal{P}_z(m,n)$.

Proof. First we need to show $\mathcal{P}_z(m,n) \subseteq \phi_z(\mathrm{PPerm}(m,n))$. Suppose $v \in \mathcal{P}_z(m,n)$. We wish to show $v \in \phi_z(\mathrm{PPerm}(m,n))$. By definition, $v = \sum \lambda_i w_i$ where the sum is over all length m words w_i whose entries are in $\{0, z_1, z_2, \ldots, z_n\}$ and whose nonzero entries are distinct. But $w_i = X_i z$ where $X_i \in P_{m,n}$. So $v = \sum \lambda_i X_i z = (\sum \lambda_i X_i) z$, which proves our claim.

Then we need to show that $\phi_z(\mathrm{PPerm}(m,n)) \subseteq \mathcal{P}_z(m,n)$. Define $\hat{z}$ as z with $m-n$ zeros appended if $m \geq n$ and as the largest $n-m$ components of z if $m < n$. Let $X = \{x_{ij}\}$ be an $m \times n$ partial permutation matrix. Then, by Proposition 4.4.8, the proof will be completed by showing $Xz \prec_w \hat{z}$ since the convex hull described will then be $\mathcal{P}_z(m,n)$. So, by Definition 4.4.7, we need to show:

$$\sum_{i=1}^{k} (Xz)_{[i]} \leq \sum_{i=1}^{k} \hat{z}_{[i]}, \text{ for } 1 \leq k \leq m.$$

This is true, since each component of the vector Xz is either 0 or z_j for some $1 \leq j \leq n$, because each column of X has at most one nonzero entry. □

Theorem 4.4.27. *Let z be a strictly decreasing vector in $\mathbb{R}^n$. Then $\phi_z(\mathrm{PASM}(m,n)) = \mathcal{P}_z(m,n)$.*

Proof. Let z be a strictly decreasing vector in $\mathbb{R}^n$. It follows from Theorem 4.4.26 and PPerm$(m,n) \subseteq$ PASM(m,n) that $\mathcal{P}_z(m,n) \subseteq \phi_z(\mathrm{PASM}(m,n))$. Thus it only remains to be shown that $\phi_z(\mathrm{PASM}(m,n)) \subseteq \mathcal{P}_z(m,n)$.

As in the previous theorem, define $\hat{z}$ as z with $m-n$ zeros appended if $m \geq n$ and as the largest $n-m$ components of z if $m < n$. Let $X = \{x_{ij}\}$ be an $m \times n$ partial alternating sign matrix. Then, by Proposition 4.4.8, the proof will be completed by showing $Xz \prec_w \hat{z}$ since the convex hull described will then be $\mathcal{P}_z(m,n)$. So, by Definition 4.4.7, we need to show:

$$\sum_{j=1}^{k} (Xz)_{[i]} \leq \sum_{i=1}^{k} \hat{z}_i, \text{ for } 1 \leq k \leq m.$$

To prove this, we will show that $\sum_{i \in I} (Xz)_i \leq \sum_{i=1}^{|I|} \hat{z}_i$ given any $I \subseteq \{1, \ldots, m\}$, so that, in particular, $\sum_{i=1}^{|I|} (Xz)_{[i]} \leq \sum_{i=1}^{|I|} \hat{z}_i$.

We will need to verify the following:

$$\sum_{j=1}^{\ell} \sum_{i \in I} x_{ij} \leq \min(\ell, |I|), \text{ for } 1 \leq \ell \leq n \tag{4.20}$$

To prove this, note that

$$\sum_{j=1}^{\ell} \sum_{i \in I} x_{ij} = \sum_{i \in I} \sum_{j=1}^{\ell} x_{ij} \leq |I|$$

since $\displaystyle\sum_{j=1}^{\ell} x_{ij} \leq 1$. But since $\displaystyle\sum_{j=1}^{\ell} x_{ij} \geq 0$ and $\displaystyle\sum_{i=1}^{m} x_{ij} \in \{0,1\}$, we also have that:

$$\sum_{i \in I} \sum_{j=1}^{\ell} x_{ij} \leq \sum_{i=1}^{m} \sum_{j=1}^{\ell} x_{ij} = \sum_{j=1}^{\ell} \sum_{i=1}^{m} x_{ij} \leq \ell.$$

Now using (4.20) we see that

$$\begin{aligned}
\sum_{i\in I}(Xz)_i &= \sum_{i\in I}\sum_{j=1}^{\ell} x_{ij}z_i = \sum_{j=1}^{n} z_i \sum_{i\in I} x_{ij} = \sum_{\ell=1}^{n-1}(z_\ell - z_{\ell+1})\sum_{i=1}^{\ell}\sum_{j\in J} x_{ij} + z_n \sum_{i=1}^{m}\sum_{j\in J} x_{ij} \\
&\le \sum_{\ell=1}^{m-1}(z_\ell - z_{\ell+1})\sum_{j=1}^{\ell}\sum_{i\in I} x_{ij} + z_n \min(n,|I|) \qquad\qquad \text{by (4.20)} \\
&= \sum_{\ell=1}^{\min(n,|I|)-1}(z_\ell - z_{\ell+1})\sum_{j=1}^{\ell}\sum_{i\in I} x_{ij} + \sum_{\ell=\min(n,|I|)}^{n-1}(z_\ell - z_{\ell+1})\sum_{j=1}^{\ell}\sum_{i\in I} x_{ij} + z_n \min(n,|I|) \\
&\le \sum_{\ell=1}^{\min(n,|J|)-1}(z_\ell - z_{\ell+1})\ell + \sum_{\ell=\min(n,|I|)}^{n-1}(z_\ell - z_{\ell+1})|I| + z_n \min(n,|I|)
\end{aligned}$$

by (4.20) and since $z_\ell \ge z_{\ell+1}$. Furthermore, this equals

$$\begin{aligned}
&\sum_{\ell=1}^{\min(n,|I|)} z_\ell \qquad \text{by telescoping sums,} \\
&= \sum_{\ell=1}^{|I|} \hat{z}_\ell, \qquad \text{since the last } m-n \text{ entries of } \hat{z} \text{ are zero in the case } n<m.
\end{aligned}$$

Thus $Xz \prec_w \hat{z}$ and so Xz is contained in the convex hull of the partial permutations of z. Therefore $\phi_z(\mathrm{PASM}(m,n)) = \mathcal{P}_z(m,n)$. □

4.4.4. Volume

Regarding the volume of $\mathcal{P}(m,n)$, we have the following theorem for $m=2$ and conjecture for $n=2$. We also give normalized volume computations for $m,n\le 7$ in Table 4.3.

Theorem 4.4.28. *$\mathcal{P}(2,n)$ has normalized volume equal to $2n^2-1$.*

Proof. $\mathcal{P}(2,n)$ is a 2-dimensional polytope whose extreme points consist of exactly $(0,0)$, $(n,0)$, $(0,n)$, $(n,n-1)$, and $(n-1,n)$. This forms an $n\times n$ square with one corner "cut off" by the line segment connecting $(n,n-1)$ to $(n-1,n)$. We can explicitly calculate the area of this region to be $n^2-\frac{1}{2}$. To obtain the normalized volume we multiply by $\dim(\mathcal{P}(2,n))! = 2!$ giving us $2n^2-1$. Refer to Figure 4.5 for the case $m=n=2$. □

Conjecture 4.4.29. *$\mathcal{P}(m,2)$ has normalized volume equal to $3^m - m$.*

Using SageMath, we have confirmed this conjecture for $m \leq 50$.

Table 4.3. Some normalized volume computations for $\mathcal{P}(m,n)$.

$m \backslash n$	1	2	3	4	5	6	7
1	1	2	3	4	5	6	7
2	1	7	17	31	49	71	97
3	1	24	129	342	699	1236	1989
4	1	77	954	4554	12666	27882	53370
5	1	238	6521	59040	262410	751380	1741950
6	1	723	42207	707669	5295150	22406130	65379150
7	1	2180	264501	7975502	99170254	651354480	2657217150

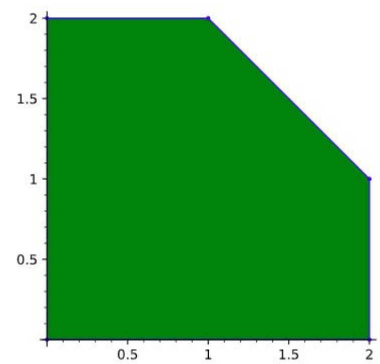


Figure 4.5. A plot of $\mathcal{P}(2,2)$.

Remark 4.4.30. We have used SageMath to compute the Ehrhart polynomials for $\mathcal{P}(m,n)$ for $m,n \leq 7$ and note that in all of these cases their coefficients are positive.

5. FUTURE WORK

While the study of partial permutations and alternating sign matrices, as well as their corresponding polytopes, has proved to be a rich and interesting area of study, it is by no means finished. One obvious direction for future work would be to prove Conjectures 4.2.9, 4.4.24, and 4.4.29. Along with this, we noted that in the cases which we could compute, the Ehrhart polynomials of the polytopes we studied had positive coefficients. This would be interesting to prove in general if it were true.

In the usual alternating sign matrix setting, there are intriguing connections to *domino tilings* of the *Aztec diamond* (see for example [21, 22]). In particular, there is a way to interleave posets related to alternating sign matrices such that order ideals in the resulting poset correspond to tilings of the Aztec diamond. This correspondence can also be seen by using *compatible pairs* of alternating sign matrices (pairs of alternating sign matrices, one of order $n+1$ and one of order n, along with certain compatibility conditions). Such pairs are in bijective correspondence with domino tilings of the Aztec diamond of order n. We have begun to explore an analogue of this in the partial setting. We can interleave posets in a similar way, and use the same notion of compatible pairs to produce "partial tilings" (see Figure 5.1 for an example).

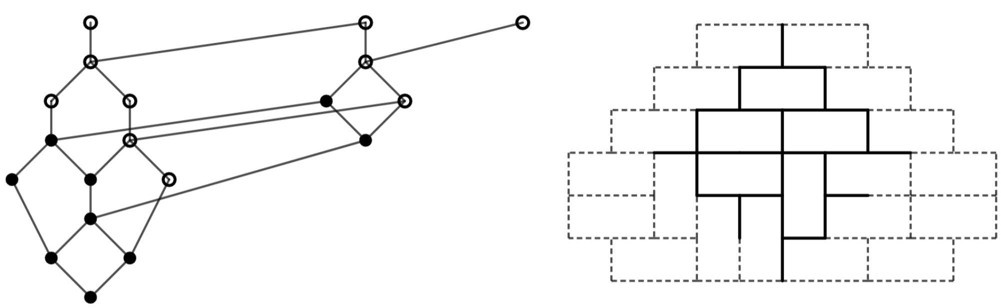


Figure 5.1. An order ideal of a poset which corresponds to a "partial tiling" of an Aztec diamond.

There is also the subject of the structure of chained permutations and alternating sign matrices, which was our original motivation for studying partial permutations and alternating sign matrices. It would be very interesting to find a partial order on chained alternating sign matrices which is the MacNeille completion of the corresponding poset of chained permutations. We have put a partial order on chained permutations, using analogues of adjacent transpositions, and with SageMath have been able to look at these posets. This order was built using chained permutations with fixed composition corresponding to maximum rook placements on the corresponding boards.

We have few results thus far in this direction, but have made some observations relating to the structure of the posets. For example, in the circular case, when there are k components of size 2×2, there is a connection to hypercube posets. In particular, when k is odd, the poset is exactly the hypercube poset of dimension k. When k is even, the poset is either the hypercube poset of dimension k or $\frac{k}{2}$, depending on the composition. Additionally (also in the circular case), when the size of each component is fixed, it appears that increasing the number of components from one even number to the next corresponds to taking a Cartesian product of posets. That is, if $k = 2\ell$ for a positive integer ℓ, and we let $C_{m,2\ell} = (n-m, m, n-m, m, \cdots, n-m, m)$ for $1 \leq m \leq n$ be the allowable compositions (see [25] for more information on allowable compositions) and $Q_{m,2\ell}$ be the corresponding posets, then we conjecture that $Q_{m,2\ell} = Q_{m,2} \times Q_{m,2} \times \cdots \times Q_{m,2}$ (the cartesian product of $Q_{m,2}$ with itself ℓ times). Finally, while this is not easy to describe, it may be worth noting that it does appear that certain "pieces" of these posets seem to appear often within many different larger posets. See Figure 5.2 for some examples.

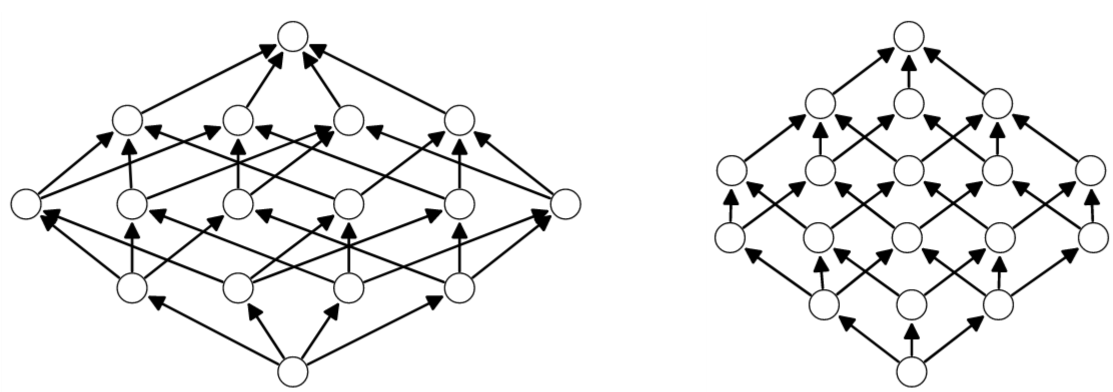


Figure 5.2. Left: the poset of chained permutations with 4 components of size 2×2 and with composition $(1, 1, 1, 1)$. This poset is isomorphic to the hypercube poset of dimension 4. Right: the poset of chained permutations with 2 components of size 3×3 and composition $(2, 1)$. This is an example of a poset which also appears as a piece of larger posets of chained permutations.

We were unable to extend this poset structure naturally to the case of chained alternating sign matrices. In the future, we would like to study a similar poset structure that includes all "partial chained permutations" which fit into a maximum composition. This would more closely mimic the poset of partial permutations, and may then extend nicely to the chained alternating sign matrix case. However, looking at it this way also complicates matters, as there are many more objects to consider, making the posets much harder to compute, produce, and study.

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