THE GAME OF NIM ON GRAPHS

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The Supervisory Committee certifies that this **disquisition** complies with North Dakota State University's regulations and meets the accepted standards for the degree of

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ABSTRACT

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The ordinary game of Nim has a long history and is well-known in the area of combinatorial game theory. The solution to the ordinary game of Nim has been known for many years and lends itself to numerous other solutions to combinatorial games. Nim was extended to graphs by taking a fixed graph with a playing piece on a given vertex and assigning positive integer weight to the edges that correspond to a pile of stones in the ordinary game of Nim. Players move alternately from the playing piece across incident edges, removing weight from edges as they move. Few results in this area have been found, leading to its appeal.

This dissertation examines broad classes of graphs in relation to the game of Nim to find winning strategies and to solve the problem of finding the winner of a game with both unit weighting assignments and with arbitrary weighting assignments. Such classes of graphs include the complete graph, the Petersen graph, hypercubes, and bipartite graphs. We also include the winning strategy for even cycles.

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CHAPTER 1. BACKGROUND

We will first look at some necessary background information in game theory, the general game of Nim, the game of Nim on Graphs, and some preliminary definitions that will be used throughout before diving into the results. Within the context of game theory, we will explain decision problems, define games and players, and explain two player combinatorial game theory. Next we will demonstrate how to play the general game of Nim, Nim addition, and how to use Nim addition to win the game. Following that we will explain our version of Nim on graphs and give a brief survey of previous results in this area.

1.1. Game theory

The study of game theory takes various forms across many disciplines. In economics, game theory is studied in the context of optimization and equilibrium points. The focus of this dissertation is strictly deterministic and will not rely on probabilities or optimal value solutions. All forms of game theory depend on the idea of decision problems and a few fundamental features common to all games.

Definition 1.1. *A **decision problem** is a problem of choosing among a set of alternatives [9].*

There are two conditions of decision problems. The first is that the decision maker must know the consequences of the decision. The second is that the decision maker must have a preference amongst the set of alternatives since the choice is only meaningful if the chooser has preferences. This means that we will always assume a player with a winning strategy would choose to use the winning strategy.

All of game theory is concerned with situations which have the following features:

1. There must be at least two players in a game.

2. Any game begins by one or more of the players making a choice among a number of specified alternatives.

3. After the choice associated with the first move is made, a specified situation results.

4. The choices made by the players may or may not become known depending on the rules of the game.

5. If a game is described in terms of successive choices, called moves, there is a termination move.

6. Every play of a game ends in a situation [9].

All games also require players, which necessitates the following definition:

Definition 1.2. *A **player** of a game must both make choices and receive payoffs [9].*

The concept of making choices is relatively self-explanatory. However, the idea of payoffs requires a small explanation. It may or may not be the case that a player of a game actually receives a monetary payoff. The satisfaction of winning is enough of a payoff to fulfill the definition for our purposes. For example, two people playing a card game or checkers with no money on the game constitute players. However, there are a few instances where the definition is highly necessary to determine the status of a game in terms of players. For example, when playing a game of solitaire, one would suspect that the luck of the draw through the shuffle of cards qualifies as a player. The cards do make decisions that affect the outcome of the game, but they do not receive payoffs, and henceforth are not players. Similarly, when playing Blackjack in a casino, one assumes that they are playing against the house. The house

certainly does receive payoffs, but the house does not make decisions since its moves are predetermined by a set of well-defined rules.

A further non-example of a player is surprising. Slot machines are a common game found in casinos. To play a slot machine, one puts in a coin, pulls a lever or presses a button, and awaits an outcome. Interestingly enough, "playing" a slot machine is a misnomer. The slot machine both makes decisions and receives payoffs. However, the "player" of the slot machine does not make decisions on the outcome of the game. Hence the slot machine is a player, but the person using the machine is not.

In situations defined as games in game theory, there are no unforeseen developments. Everything that can possibly occur in a game or during a move is known.

Definition 1.3. *A **strategy** is a plan which provides for every possible choice on the part of the other player.*

Game theory always follows the assumption that if a player has an optimal strategy, the player will use it. This means that no player would intentionally let the other player win, nor would the player make a move that is not advantageous. This follows from the definition of a decision problem since a strategy is simply a sequence of decisions.

Definition 1.4. *A game in which each player may know the entire history of the moves in a game is said to be a game of **perfect information** [8].*

Games of perfect information include chess, checkers, backgammon, and even tic-tac-toe. Games in which players do not have perfect information include all card games where players hold cards in their hands away from other players, such as poker, blackjack, and go-fish. Oftentimes, games not of perfect information require statistical evaluation to study in depth. It is precisely the chance involved in rolling

dice or shuffling cards that make statistical evaluation necessary in such games. These types of games, more often studied in economic game theory, only allow for optimal moves based on optimization functions. Games of this nature are set apart from combinatorial games which never rely on chance and always have perfect information.

Two-player combinatorial games are defined by the following features.

1. There are two players. For the purposes of this dissertation, we will denote these players by P_1 and P_2.

2. Moves are defined in terms of successive positions and typically a fixed starting position.

3. There are clearly defined rules which specify the moves either player is able to make from a position to the different options of the position.

4. The two players move alternately.

5. In typical game play, the player unable to move loses. There is also a form of game play known as the misére form in which the player who is unable to move wins.

6. Every game has an ending condition which ensures that every game comes to completion since some player is unable to move.

7. Both players have perfect information.

8. There are no chance moves [1].

The sixth feature ensures that there are no ties or draws in games. Thus tic-tac-toe would not qualify as a two-player combinatorial game. The last feature scraps any card or dice games from qualifying for our study. All two-player combinatorial games are games of perfect information, but it is not true that all games of perfect

information are combinatorial games. As mentioned earlier, backgammon is a game of perfect information since it satisfies the criteria, but it is not a combinatorial game since it involves dice which is an element of chance. Since go-fish is not a game of perfect information, it is also not a combinatorial game. Checkers, chess, go, and Nim are all two-player combinatorial games. Complete information and exempting chance moves take any probabilistic aspect out of the study of combinatorial of games.

1.2. The general game of Nim

The general game of Nim is a two-person combinatorial game consisting of at least three piles of stones where players alternate turns, selecting first a pile from which stones will be removed, and then a strictly positive number of stones to remove from that pile. The game terminates when there are no more stones on the playing surface, and the winner is the player who takes the last stone or stones. Players must always remove at least one stone and can only remove stones from a single pile during their turn. It is common for players to agree at the onset of the game that no two piles will have the same number of stones [2]. The reason for this is that in a game with three piles, the first player can always win if two piles have the same number of stones.

It is speculated that the game of Nim originated in China. The name Nim may have originated in Germany where *nimm* means "to take." The first known study of this game was done by Charles Bouton at the turn of the twentieth century, and his paper, *Nim, a game with complete mathematical theory,* [2] is the first mathematical record of the game. In this paper, Bouton not only describes the game, but also identifies what he calls "safe combinations" for the three-piles. The paper also describes Nim addition, p-positions, and 0-positions, long before modern interpretations came about. Bouton solved the normal form of the game completely and alluded to the generalization to n piles leaving the proof of his addition scheme to

the reader The paper ends with a description of the misére form of the game which remains unsolved to this day

To win the general game of Nim, one first writes the the number of stones in each pile as a binary digit Next, the place values of the binary digits are added modulo 2 in each column There is no carrying of any digits to the next place value This sum is called a *Nim sum* If the sum is not zero, the three-pile combination is not a safe combination Shortly, we will define a non-safe combination as a *p*-position because it has a positive Nim sum If the sum is zero, the three piles form a safe combination, which we will define as a 0-position since it has a Nim sum of zero The goal of a player is to remove stones from a single pile in such a way that the resulting three-piles are a safe combination, or a 0-position For an illustration of the Nim addition, see Figure 1

Definition 1.5. *From a particular position, if the first player to move can win for any of the second player's moves, we call the first player's position a p-**position** If the second player to move from this position can win for any of the first player's moves, we call this a 0-**position** [5].*

Another way to think about the positions is as follows if at a given position, the current player has a winning strategy, the position is a *p*-position, if at that same given position, the current player does not have a winning strategy, the position is a 0-position Additionally, one can consider 0-positions as winning positions for the player who produces them [10]

The terms *p*-position and 0-position come from the *positive* and *zero* Grundy number of that particular position [1] The term Grundy number, or *g*-number, is in honor of Patrick Michael Grundy, who in 1939 developed a function to analyze the large class of Nim-type games The Grundy function is constructed recursively by defining the terminal position to have a *g*-number of 0, and then obtaining the *g*-

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Figure 1. Here is an example of a game situation with three piles of stones. We write the 6, 4, and 3 stones in the piles in binary form and then add them modulo 2 column wise. The Nim sum in this example is 1.

$$
\begin{array}{c|ccc}
6 & 1 & 1 & 0 \\
4 & 1 & 0 & 0 \\
3 & & 1 & 1 \\
\hline
& 0 & 0 & 1
\end{array}
$$

number for any other position by considering the set of positions that can be reached by a move and their g-numbers. The g-number of the position is the smallest non-negative integer not used in the set of the g-numbers of the follower or previous positions [10]. In Figure 1, the Nim sum was positive. Thus the player about to move is in a p-position and the Grundy number of the position is positive.

Grundy numbers are used heavily in many areas of two-person combinatorial game theory. Two properties especially important to keep in mind are that when a player is on a 0-position all moves are to p-positions, and that when a player is on a p-position there is always at least one move to a 0-position. Essentially, this means that a player with an advantage at the beginning can keep it with a winning strategy.

To win using g-numbers and Nim addition, a player first finds the Nim sum according to the binary addition described above. Recall that the goal of a move for a player in a p-position is to put their opponent into a 0-position since any player with an advantage can keep it with a winning strategy. Thus it is the goal of a player starting in a p-position to continually put the opponent in a 0-position until the terminal 0-position ends the game. With the Nim sum in mind, the player next deduces how to put the opponent into the 0-position. Certainly the 0-position has a Nim sum of 0, hence the player looks to see which column sum will benefit from a reduction of the found Nim sum. If we consider Figure 1, then we see in this example that the player about to move wants to remove exactly one stone from one of the

piles. Further examination tells us that the pile from which the player should remove the stone is the pile with three stones in it currently. This would leave the piles at two, four, and six. Converting this back to binary and adding, we see that indeed this is a 0-position (Figure 2).

Figure 2. After finding the Nim sum we can determine the move to make. We see that the player should remove one stone from the pile with 3 stones to put the next player into a 0-position.

$$
\begin{array}{c|ccc}
6 & 1 & 1 & 0 \\
4 & 1 & 0 & 0 \\
2 & & 1 & 0 \\
\hline
 & 0 & 0 & 0
\end{array}
$$

In *On numbers and games* [4], Conway describes Nim sums and Nim addition using an alternate notation. This is also described in great detail in [1]. We use only a simplified version of the notation for our purposes here. All of these calculations rely on the ideas of Sprague and Grundy, which they describe separately. Interestingly, the Sprague-Grundy theory reduces all impartial games to the game of Nim [7].

In addition to the general game of Nim, many more forms spun off from this game. All forms of Nim rely on the ideas of Nim addition and positional values, and are lumped under the general classification of subtraction or take-away games. Such spin-off forms of Nim include Binary Nim, Fibonacci Nim, Schwenk's Nim, Northcott's Nim, the Silver Dollar Game, Welter's Game, and Kayles, amongst numerous others.

Binary Nim is played with a single pile and a subtraction set $S(s_1, s_2, \ldots, s_n), s_1 < s_2 < \ldots < s_n$ specifying how many stones a player may remove from the pile on a move. Furthermore, no player starts play by removing the whole pile and no player may remove more stones than his opponent [10]. The last player to move wins. This game was invented by Schwenk in 1970.

Fibonacci Nim is a form created by Whinihan in 1963. It is played with a single

pile according to the following rules: The first player may not remove the entire pile on the first move and neither player may remove more than twice the number of stones removed on the previous move. The last player who can move wins.

Schwenk's Nim is similar to the previous two forms with the change that the number of stones each player may remove may not exceed some given function that is dependent on the number of stones the last player removed.

Northcott's Nim and the Silver Dollar Game are described in much detail in [10] and [1] amongst various other forms of Nim.

CHAPTER 2. THE GAME OF NIM ON GRAPHS

To play Nim on graphs, two players first agree on a finite, undirected, integrally weighted graph and a fixed starting position. The position of the game is indicated by a positional piece which we will denote by Δ. The game starts with P_1 choosing an edge incident with Δ to move across. As a player moves across an edge, the player must lower the weight of the edge by an integer amount. The positional piece Δ moves with the move of the player so that when a player comes to rest on the other vertex incident with that edge, the next player must start with that vertex and move across edges incident with the new position of Δ. If either player lowers the weight of an edge to zero, the edge is no longer playable. For ease of notation, we will delete the edge from the picture of the graph if the weight is decreased to zero (see Figure 3). Play continues in this back-and-forth fashion until a player can no longer move since there are no edges incident with Δ.

Figure 3. An example of the first two moves in a game of Nim on graphs.

| Player 1's Turn | Player 2's Turn | Player 1's Turn |
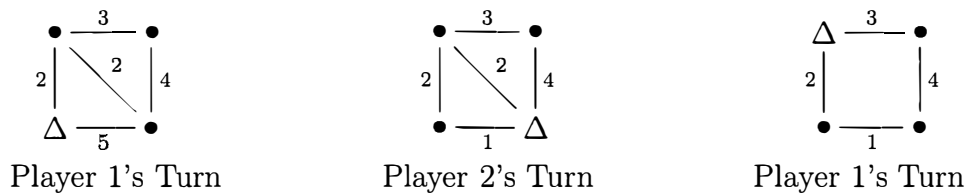
2.1. Preliminary definitions

Definition 2.1. *A **graph** G is a finite nonempty set of objects called **vertices** together with a possibly empty set of unordered pairs of distinct vertices of G called **edges**. The vertex set of G is denoted by $V(G)$, and the edge set is denoted by $E(G)* [3].*

The graphs we will consider are finite and undirected with no multiple edges or loops We will often want to label the vertices and edges When we do, the edge between vertex v_i and v_j will be denoted e_{ij} Additional graph theory terminology, including path, vertex degree, and graph isomorphism, will be assumed as found in [3] Paths are defined to be odd or even as the number of edges is odd or even respectively

Definition 2.2. *A graph G is **complete** if for every $u, v \in V(G)$ there exists $e_{uv} \in E(G)$ We denote the complete graph on n vertices by K_n*

A graph G is *k-partite* for $k \geq 1$ if it is possible to partition $V(G)$ into k subsets $V_1, V_2, \quad , V_k$, called partite sets, such that every element of $E(G)$ joins a vertex of V_i to a vertex of V_j for $i \neq j$ For $k = 2$ such graphs are called *bipartite*

Definition 2.3. *A **complete bipartite graph** with partite sets V_1 and V_2, where $|V_1| = r$ and $|V_2| = s$ is denoted by $K_{r\,s}$ and is a bipartite graph with the additional property that if $u \in V_1$ and $v \in V_2$ then $uv \in E(G)$*

Let u and v be two vertices of a graph, not necessarily distinct

Definition 2.4. *A u-v **walk** is a finite, alternating sequence $u = u_0, e_1, u_1, e_2, \quad , u_{k-1}, e_k u_k = v$ of vertices and edges, beginning with vertex u and ending with vertex v A u-v **trail** is a u-v walk in which no edge is repeated, while a u-v **path** is a $u - v$ trail in which no vertex is repeated [3]*

Definition 2.5. *A **cycle** is a u-v path in which $u = v$*

In other words, a cycle is a path that starts and ends at the same place Since it is a path, no vertex or edge is repeated within the cycle

For the game of Nim on graphs, we will not have physical piles of stones as in the general game of Nim Instead of this, we have "piles" of stones in the form of weight on the edges of a graph

Definition 2.6. *Given a graph G with edge set $E(G)$ and vertex set $V(G)$ we will call the non-negative integer value assigned to each $e \in E(G)$ the **weight** of the edge and denote the weight of edge e_{ij} by $\omega(e_{ij})$.*

When we say a graph has *unit weight*, we precisely mean that $\omega(e) = 1$ for all $e \in E(G)$.

For any graph G we assume $\omega(e_{ij}) \neq 0$ for all $e_{ij} \in E(G)$ at the start of a game. When an edge is such that $\omega(e) = 0$ we will delete it from the graph entirely, since it is no longer a playable edge. Given a game graph G with weight assignment $\omega_G(e)$, denote by P_1 the first player to move from the starting vertex, and denote by P_2 the player to move after P_1. The indicator piece Δ denotes the vertex from which a player is to move. We will always enumerate vertices in such a way that Δ is on v_1 at the start of a game.

Definition 2.7. *For either player and from a given position Δ on vertex v_j, we define the set of vertices to which a player may legally move to from Δ to be the **option** of the player. The set of options of player i at vertex v_j will be denoted by $O(P_i, v_j)$.*

Certainly for a vertex to exist in the set of options the incident edge must be adjacent to Δ. Thus $O(P_i, v_j) = \{v_k \in V(G) : \Delta = v_j;\ e_{jk} \in E(G);\ \omega(e_{jk}) \neq 0\}$. We will omit v_j when the position of Δ is apparent.

Definition 2.8. *For either player and from a given position Δ on vertex v_j, we call the decision of how much weight to remove from an edge e_{ji} the **choice** of the player.*

Thus for any given option, the player has a choice of whether or not to remove all weight, or exactly how much weight to remove.

Definition 2.9. *We will say that a pair of P_i's options are **isomorphic** if given two options, $v_j, v_k \in O(P_i, v_i)$, there exists a graph isomorphism between v_j and its*

*neighbors and v_k and its neighbors. We will say that two options are **identical** if in addition to being isomorphic, the subgraph induced by v_i and each $v_j \in O(P_i, v_i)$ have the same weight assignment.*

We will use the word *option* exclusively when we are referring to the vertex a player will move to, and the word *choice* to refer to the amount of weight across the option's edge to be removed during play. Hence during any given move, a player will have the option of which vertex to move to, and the choice of how much weight to remove.

Notice that the definition of isomorphic requires that the vertices in the set of options have the same degree, and that there is a bijection between the options of the vertices within the set of isomorphic options (see Figure 4). In other words, if for all $v_j, v_k \in O(P_i, v_i)$ we have that $O(P_i, v_j) \cong O(P_i, v_k)$ then the options of v_i are isomorphic. We will also talk about graphs being isomorphic within the context of Nim on graphs. This will be necessary to cut down on cases to consider within games.

Figure 4. The options at Δ are isomorphic but not identical.
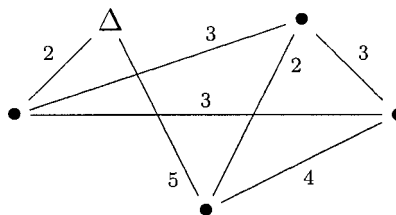
2.2. Nim on graphs basics

When we refer to a player winning a graph, we precisely mean that a player can win a game played on that graph under the specified weight assignment and starting at a specified vertex. If no starting vertex is specified, it is said that a player can

win that graph provided that the player can win starting at any vertex. When we say that a player is on an odd path or that a player has an odd path option, we mean that there is an odd path in terms of the number of edges from Δ to a vertex of degree one. However, when we refer to an even path, we precisely mean that all options from Δ to a vertex of degree one are even paths. Notice that if G is an odd path itself, there is an odd path option at any vertex. Hence for an odd path there is no loss of generality in not specifying the initial position of Δ. Such is not the case if G is an even path, since it is possible to position Δ on vertices of G in which both options are indeed odd paths.

Assume that the weight of the game graph is arbitrary for the time being. Any position on an odd path is a p-position for P_1, as is any position on an odd cycle. Starting at either vertex of degree one on an even path is a 0-position for P_1, and starting at any vertex that leaves two even paths from Δ is also a 0-position for P_1. However, as noted above, if Δ started on a vertex that leaves two odd paths on this even path, the position is a 0-position for the second player.

The Grundy number of a position in ordinary Nim not only told which player has an advantage at any given position, it also told the advantaged player what move to make according to the Grundy number. This is not the case with Nim on graphs. Knowing that you can win with Grundy number calculations does not tell you what strategies should be employed to defeat your opponent. The calculations of the Grundy numbers for trees, paths, cycles, and certain bipartite graphs can be found in [5, 6].

Just as there are spin-off versions of Nim, there are also other Nim-type games that are played on graphs. One of the most common is quite similar to our version, with the exception that the piles are placed on the vertices instead of the edges. Thus the weight function is defined for the vertex set instead of the edge set of a given

graph. Along the same lines are vertex take-away games where there is no playing piece, but players instead move back and forth removing a vertex and all incident edges. The player who is unable to move loses in normal game play. Other games played on graphs include Hackendot, Round Table, Marguerite, Sprouts, Jocasta, Sim, and Cram [10].

2.3. Previous results

Masahiko Fukuyama, in a paper entitled *A Nim game played on graphs*, introduced the form of Nim on graphs that we study in this dissertation. This paper used Grundy numbers to find a few results that we outline below. In *A Nim game played on graphs II*, Fukuyama expanded the previous results to include some theorems we use to find new results in this dissertation.

2.3.1. Nim on Graphs I

This paper follows a few blanket assumptions that play a crucial role in the types of graphs studied.

1. All graphs are finite, bipartite, and without multiple edges.

2. The degree of any odd vertex of the graph is two.

Note that in this context an odd vertex is one that takes an odd number of steps to reach from the starting vertex. Since the graphs studied in this paper are bipartite, that means that all of the vertices of one partite set have degree two at the start of the game. A further observation notices that any of the vertices P_2 moves from have degree at most two. Oftentimes we will see that P_2's vertices have degree one since P_1 removed all weight from the previous edge.

Under these assumptions, Fukuyama finds the first two results:

Proposition 2.10. *A position which has at least one odd path is a p-position.*

Lemma 2.11. *Let G be a graph and C be a cycle. Suppose that C is a subgraph of G.*

1. *The superposition of a p-position with the piece at an even vertex v and a 1-cycle is a p-position.*

2. *The superposition of a 0-position with the piece at odd vertex v and a 1-cycle is a 0-position.*

Here, what Fukuyama means by "superposition" is essentially sticking together two graphs. The superposition of a p-position and a 1-cycle results in a cycle where each edge has the weighting assignment of the p-position plus one.

One should note that, in general, the proposition is not true. It relies heavily on the assumption that the degrees of the odd vertices are only two. This assumption assures P_1 that P_2 cannot move onto a branch of a tree, for instance, that contains an even path for P_1. This is the prototypical example where there will be only one option for P_2 since P_1 will remove all weight from the previous edge.

Proposition 3.1 used in our result concerning the strategy for even cycles uses these results. By a 1-cycle, it is meant that there is a cycle where each edge has weight one. Since the graphs here are bipartite, we know that they only refer to even 1-cycles. An even cycle arbitrarily weighted could have multiple 1-cycles. When these 1-cycles are stripped away, we find only a path. This path gives the indication as to which player will win. The lemma lends itself to the following:

Lemma 2.12. *A position with the piece at even vertex is a p-position if and only if this position can be regarded as the superposition of a trivial p-position and 1-cycles.*

The paper then takes time to define thin edges and thick edges as those adjacent edges having less and more weight respectively. The paper uses Menger's Theorem in the proofs of the following two main results of the paper.

Lemma 2.13. *Take a position on a weighted graph G with the starting piece at even vertex v. Let G_ω^u denote the weighted graph which results by cutting off G_ω at odd vertex u of G. Then G can be regarded as the superposition of 1-cycles and a trivial p-position with an odd path which terminates at u if and only if the following three conditions are satisfied:*

1. *The weights of the two edges incident with u are different from each other.*

2. *The minimum capacity of cuts separating the two sections of G_ω^u is equal to the weight of the thin edge of u.*

3. *Even if any minimum cut separating the two sections is removed from the weighted graph G_ω^u, the vertex v is always connected with the thick section.*

Theorem 2.14. *Let $G_{\omega,v}$ be a weighted graph with starting piece at even vertex v. Then $G_{\omega,v}$ is a p-position if and only if $G_{\omega,v}$ has an odd vertex u satisfying all three conditions of the previous lemma.*

In what follows of the paper, Fukuyama looks at Nim on graphs with multiple edges where it is not required that the graphs are bipartite. The paper mentions extended Nim on graphs which allows entire new games to be assigned to particular edges. Meaning once a player decides to move across such an edge, the original game is suspended, and a new game is started just within the edge. It is determined that within the context of such games, the Grundy number of this extended game can be reduced to that for a normal game of Nim on graphs. For Nim on graphs with multiple edges, it is determined that the finding the Grundy number for multiple edged graphs is no different from finding the Grundy number for graphs without multiple edges.

2.3.2. Nim on Graphs II

In *Nim on Graphs II*, Nim on graphs with maximum matchings are considered, along with Nim on trees and Nim on cycles. Again, we need to consider the blanket

assumptions of all graphs being bipartite and all odd vertices having degree two. A *matching* here is defined as a set E_0 of edges of G such that no two edges of E_0 are adjacent. An E_0-*alternating trail* is a trail whose edges are alternately in E_0 and $G \setminus E_0$. Matchings are called *perfect* when all vertices of E_0 are matched.

Fukuyama defines M to be a matching without alternating cycles, $O(E_0)$ to be the set of vertices which can be connected by an E_0-alternating trail with even length starting from an unmatched vertex in E_0, and $I(E_0)$ to be the set of vertices which can be connected by an E_0-alternating trail with odd length starting from an unmatched vertex. Using a theorem of Berge that states a matching is maximum if and only if there are no alternating paths between any two distinct unmatched vertices, Fukuyama is able to prove the following:

Lemma 2.15. *Let E_0 be a maximum matching of G. Consider the graph H determined by $V(H) = V(G)$ and $E(H) = M \triangle E_0$. Each connected component of H is one of the following types:*

1. *an isolated vertex.*

2. *an even path whose edges are alternately in M and E_0. One vertex of this path is unmatched in E_0 and the other end vertex is unmatched in M.*

Lemma 2.16. *Let E_0 be a maximum matching of G. The following assertions hold:*

1. *For each vertex $u \in O(E_0)$, any vertex v incident with u belongs to $I(E_0)$ and the edge uv is contained in the subgraph induced by all the edges contained in E_0-alternating trails starting from unmatched vertices.*

2. *For each vertex $u \in I(M)$, there exists the vertex v incident with u such that $vu \in M$. This vertex v belongs to $O(E_0)$ and the edge vu is contained in the subgraph induced by all the edges contained in M-alternating trails starting from unmatched vertices.*

3 The subgraph induced by all the edges contained in E_0-alternating trails is a subgraph of the subgraph induced by all the edges contained in M-alternating trails

4 $O(M) \cap I(M) = \emptyset$ and the subgraph induced by all the edges contained in M-alternating trails is bipartite

Eventually, the lemmas and theorems solve the problem of finding which player has the winning strategy depending on the starting position It also solves the problem of finding the grundy number of a given position in a graph with a maximum matching without alternating cycles It is rather restrictive to consider only graphs with maximum matchings without alternating cycles, since this does not even cover the entire class of complete bipartite graphs, let alone bipartite graphs in general

Following these solutions, the paper goes on to explain the solution for finding the grundy number of Nim on cycles It should be noted that since we know that the first player can always win Nim on odd cycles, the actual value of the position should have no bearing on the game play This is not the case with even cycles, however, and Proposition 6 2 from the paper helps solve the strategy problem for Nim on even cycles

Proposition 2.17. *Let $G_{\omega,v}$ be a position of Nim on even cycles Denote the two perfect matchings of G by E_0 and E_1 Then the Grundy number of $G_{\omega\ v}$ is given by the formula*

$$g(G_{\omega,v}) = g_k(m,n),$$

where $m = \min\{\omega(e) | e \in E_0\}$, $n = \min\{\omega(e) | e \in E_1\}$, and $k = g(G_{\omega - m1_{E_0} - n1_{E_1},\ v})$ Here $\omega - m1_{E_0} - n1_{E_1}, v \quad E(G) \to \mathbb{N}$ is the weight mapping defined by $\omega(e) - m$ if $e \in E_0$ and $\omega(e) - n$ if $e \in E_1$

This proposition essentially tells us that we can subtract off 1-cycles to find the

Grundy number of a given position of Nim on even cycles. Subtracting off the 1-cycles on an even cycle leaves just a forest. Since this reduces Nim on even cycles to Nim on trees, it also solves the problem of finding the Grundy number for Nim on trees in the process. It should be strongly noted here that the trees under consideration must have degree two at all odd vertices. The result is not true in general for any tree.

CHAPTER 3. STRATEGY FOR NIM ON EVEN CYCLES

In the previous works of Fukuyama, great detail went into describing how to calculate the g-numbers of positions on particular graphs, including even cycles. However, despite the lengthy descriptions, there is no mention on what a player should actually do, given the information provided by the g-numbers. For this reason, we first look at the strategy of either player on even cycles. Here, cycles are denoted C_n, and we will denote even and odd cycles by C_{2n} and C_{2n-1} respectively.

The g-number is 0 at any starting position of Δ when $\omega(e) = k$ for all edges in an even cycle and for any $k \geq 0$; hence, the second player to start has the advantage [6]. Consider first the strategy for P_2 when $\omega(e) = 2$ for all edges on an arbitrary even cycle.

From any starting position and for either edge, P_1 only has the choice of reducing that edge to a weight of 1 or 0. Notice that P_1 would not want to make an odd path for P_2 by reducing to 0. Thus assume without loss of generality that P_1 moves to v_2 and reduces e_{12} to $\omega(e_{12}) = 1$. Then P_2's next 0-position option is to move to v_3 leaving $\omega(e_{23}) = 1$, since moving back to v_1 requires that P_2 create an odd path for P_1. Continuing on in this way P_1 and P_2 will move to v_{2j} and $v_{2j+1}, (1 \leq j \leq n - 1)$ respectively until P_1 is back at v_1 and there is only an even cycle with $\omega(e) = 1$ for all $e \in E(C_{2n})$, which as mentioned is a P_2 victory.

In the above case, P_1 was immediately forced to reduce the weight of an edge beyond the minimum weight of any edge. Now assume that the weights of the edges on an even cycle are arbitrary. It will still be the case that neither player wants to break the even cycle, and that the first player forced to decrease a weight below the minimum will lose. This means we can look at even cycles with arbitrary weighting assignments in the following way:

Proposition 3.1. *Assume $G = C_{2n}$ and that ω_G is some arbitrary weight assignment*

for G. Assume $\min_{e \in E(G)}(\omega_G(e)) = m$. Let G' be the graph formed from G under $\omega_{G'}(e) = \omega_G(e) - m$ with the same starting vertex. Then the p-positions of G are the p-positions of G' with the winning strategies for P_1 and P_2 on G following from those on G'.

Proof. Note that G' is no longer an even cycle since at least one edge of G is deleted under $\omega_{G'}(e)$. By Proposition 6.2 in [6] which gives a calculation of the Grundy number of even cycles, the Grundy number of G is determined in part by the Grundy number of G'. As the Grundy number of an odd path is positive and an even path is zero, the first player wins G if there is at least one odd path starting from Δ in G', and the second player wins G if all paths starting from Δ in G' are even.

To see that the strategy for playing G follows from that for G', first consider a graph with a positive Grundy number. On an odd path, we know that P_1 removes all weight on the incident edge. Since the Grundy number of G is positive, so is the Grundy number of G'. Hence G' contains an odd path. The previous paragraph implies that P_1 will move in the direction of the odd path in G' decreasing the weight of e_{12} to zero. In G, this corresponds to a move from v_1 to v_2 and a decrease of $\omega_G(e_{12})$ by $\omega_{G'}(e_{12})$ to m since $\omega_{G'}(e) = \omega_G(e) - m$ for all $e \in E(G)$.

First assume that P_2 moves back to v_1. Following this move, $\omega_G(e_{12}) = m' < m$ and we can now compare the strategy for P_1 to the strategy for some graph G'' formed from G with $\omega_{G''}(e) = \omega_G(e) - m'$ where G has been played two moves (see Figure 5). Since G'' is an odd path of length $2n - 1$ the first player has a winning strategy.

Now assume that P_2 moves to v_3 and sets $\omega(e_{23})$ to k. If $k > m$ then we know from G' that $P1$ moves back to v_2 setting $\omega(e_{23}) = m$. Since P_2 is on an even path in G', the first player will win. If $k = m$, then P_1 still has an odd path in G' and thus will win G. Finally if $k < m$ then k is the new minimum weight and there exists G'' with $\omega_{G''}(e) = \omega_G - k$ that is an odd path of length $2n - 1$ for P_1. In any case, the

Figure 5. Graphs of G and G' at the start of a game on even cycles. In this game $m = 2$, and since an odd path exists in G' we have a winning strategy for P_1.
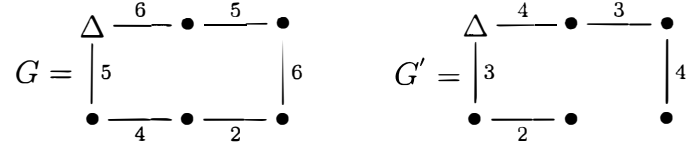
In the case that P_2 goes back to v_1 lowering the weight of the edge to less than two, we have the following graphs G and G''. Notice that in G'' there is an odd path of length $2n - 1$ since $\min_{e \in E(G)}(\omega(e)) = 1$ after the first two moves.
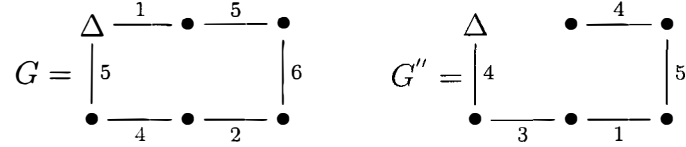
strategy for P_1 follows that for a graph with the lowest weight removed from every edge.

When the Grundy number of G' is zero, P_2 mimics the strategy of P_1 above.

To establish the uniqueness of this strategy, we must show that any move except one to reduce the edge weight to m on the odd path option results in a loss for the player who began on a p-position. In fact, the strategy holds at every stage of game play.

Assume P_1 begins the game on a p-position and let m be the minimum weight of any edge of $G = C_{2n}$ and $\Delta = v_1$ as before. There exists an odd path option in G', the graph formed from G under $\omega_{G'}(e) = \omega_G(e) - m$ for all $e \in E(G)$. Since taking an even path option results in a loss for P_1 by the above arguments, we assume that P_1 takes an odd path option.

Suppose that P_1 does not reduce the weight of e_{12} to m. We consider first the case when $\omega(e_{12}) = m'$ for $0 \leq m' < m$ following P_1's move. With $\Delta = v_2$ and P_2's turn, we can look at a graph G'' formed from G under $\omega_{G''}(e) = \omega_G(e) - m'$. Since $m' < m$ we have that G'' is a path of length $2n - 1$. Now P_2 may move along G in

the direction of the odd path in G'' reducing the edges to m' as play progresses for the win. Thus P_1 reducing any edge below m results in a loss of advantage and a P_2 win.

Now suppose that P_1 reduces e_{12} to m'' for $m < m''$ if possible, and that only one odd path option exists. If $\omega(e_{12}) = m + 1$ then we have nothing to show at this step, and if there are two odd path options, we will simply repeat this following argument a second time. With $\Delta = v_2$ and P_2's turn we will let P_2 move back to v_1 reducing the weight of e_{12} from m'' to m. In doing this, P_2 has left P_1 on an even or trivial path in the graph G' formed under the weight assignment $\omega_{G'}(e) = \omega_G(e) - m$ after two moves on G. Since this is a 0-position for P_1, we have that P_2 now holds the winning strategy. Thus using any other strategy on even cycles shifts the advantage to the player who originally started in a 0-position. $\qquad\square$

Once the minimum weight is removed from each of the edges, it becomes clear that P_1 will win if there is an odd path option from the starting vertex. In the same way we know that P_2 will win if all first player options from the starting vertex are even paths in G' (Figure 5).

CHAPTER 4. NIM ON THE COMPLETE GRAPH

4.1. A structure theorem

Theorem 4.1. *Let $G = K_{2,j}$ for $j \geq 1$ and $\omega(e) = 1$ for each $e \in K_{2,j}$. Assume that Δ is on a vertex in the partite set of size 2. Then P_2 will always win the $K_{2,j}$.*

Proof. We proceed by induction on j. Enumerate the vertices in the following way: Let $\Delta = v_1$ and v_2 be the other vertex in the partite set of size 2. Enumerate the vertices in the partition of size j by $v_3, v_4, \ldots, v_{j+2}$.

For $j = 1$ we have an even path. By previous work, this is a win for P_2. Similarly, for $j = 2$ we have an even cycle in which each edge has $\omega(e) = 1$ which we have also seen to be a win for P_2. Now assume that this is true for all complete $K_{2,i}$ for $i \leq j$. Consider the $K_{2,j+1}$ with Δ on v_1 in the partition of size 2. Notice that all of P_1's moves are identical since $O(P_1, \Delta = v_1) = \{v_3, v_4, \ldots, v_{j+3}\}$, all incident edges have weight 1, and $d(v_i) = 2$ for $3 \leq i \leq j + 3$.

Without loss of generality, assume that P_1 moves to v_3. Since e_{13} is now gone, as $\omega(e_{13}) = 1$ at the start, P_2 only has one move, namely to v_2. Now with Δ on v_2 and both players unable to move to v_3, we have P_1 on a $K_{2,j}$ (Figure 6).

Figure 6. $G = K_{2,j+1}$ after the first two moves. This isolates a vertex leaving a $K_{2,j}$.
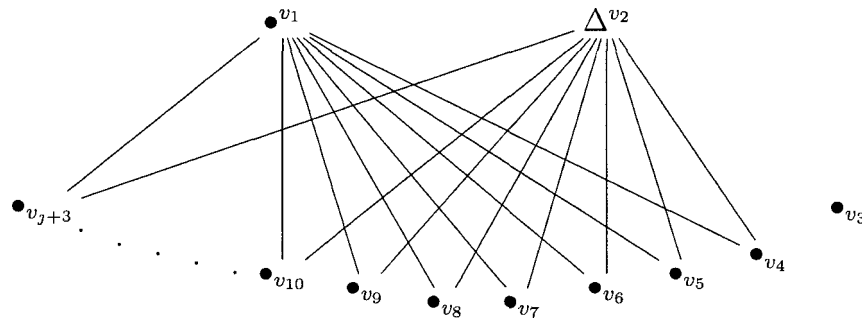
By our inductive assumption, the second player will win the $K_{2,j}$. Hence P_2 wins the $K_{2,j}$ for all $j \geq 1$ and Δ on a vertex in the partition of size 2. $\square$

Now consider the $K_{2,j} + e_{12}$ with Δ still on a vertex in the partite set of size 2 and the same vertex enumeration as above.

We will call this the SSB_j graph of order j (Figure 7). When the order of the graph is understood or insignificant, we will simply write SSB. Removing e_{12} on the first move yields a $K_{2,j}$ with Δ on v_2. This lends itself to the following corollary:

Corollary 4.2. *The first player will win the SSB_j for any j when $\omega(e) = 1$ for all $e \in E(SSB_j)$ and Δ is on v_1 or v_2.*

Proof. The first player removes e_{12} and lets P_2 start on the $K_{2,j}$ with Δ on a vertex in the partite set of size two, guaranteeing P_1 the win by the previous theorem. $\square$

Figure 7. An example of the SSB_j.
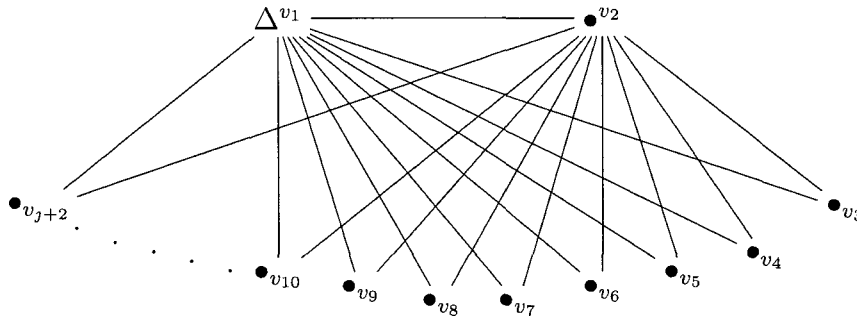
It is not the case that P_1 will always win the SSB if $\omega(e) \neq 1$ for every edge. The winner can be determined by arguments similar to those for even cycles.

4.2. The complete graph with unit weight

In Corollary 4.2, the first player has no option but to move back to either v_1 or v_2 since all other vertices only have degree 2 following the first move. Suppose now that P_1 had more options so that the move is not forced back to v_1 or v_2 in the SSB. We continue to assume $\omega(e) = 1$ but give P_1 more options by adding edges between the vertices in the partition of size j in the SSB. We show next that additional edges

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do not affect a player's strategy to play the SSB when such a structure exists as a subgraph.

Lemma 4.3. *Assume that $G = K_n$ and that $\omega(e) = 1$ for all $e \in E(G)$. Then P_1 can force P_2 to move within the confines of an SSB_{n-2} contained in K_n.*

Proof. Assume $G = K_n$ with $\Delta = v_1$ and $\omega(e) = 1$ for all $e \in G$. Then all of P_1's moves are identical. Without loss of generality, assume that P_1 moves from v_1 to v_2.

Then we have $O(P_2, v_2) = \{v_3, v_4, \ldots, v_n\}$ and each option is identical. So assume without loss of generality that P_2 moves to v_3. With P_1 on $\Delta = v_3$ there are two non-isomorphic moves for P_1. One of these is to move to v_1 and the other is to move to one of the $v_4, v_5, \ldots, v_n$. Since we want to show that P_1 can move along the SSB, he would naturally choose the v_1 option.

Now $O(P_2, v_1) = \{v_4, v_5, \ldots, v_n\}$ and all of these moves are identical. Assume that P_2 moves to v_4. Then since $v_2 \in O(P_1, v_4)$ we know P_1, in keeping with the strategy to move along the SSB, will choose to move to v_2.

Continuing on in this manner we will have that $v_1 \notin O(P_2, v_2)$, $v_2 \notin O(P_2, v_1)$ since e_{12} was the first edge removed. In general, every option at every move is identical for P_2. Since $v_1 \in O(P_1, \Delta = v_i)$ for all $v_i \in O(P_2, v_2)$ and $v_2 \in O(P_1, v_j)$ for all $v_j \in O(P_2, v_1)$, P_1 is able to choose to move along the SSB.

Keeping up game play in this fashion, i.e., P_1 choosing to move to whichever of the v_1 or v_2 options exist in $O(P_1, \Delta)$ and P_2's moves identical, we will exhaust the edges incident with v_1 and v_2 leaving P_2 on an isolated vertex. Precisely, if n is even, P_2 will be stuck on v_2, and if n is odd, P_2 will be stuck on v_1. $\qquad\square$

Notice that since P_1 never opted to use any edges outside of the SSB, the existence of those edges did not affect the strategy of P_1. We will call the technique of P_1 continually choosing to move to v_1 or v_2 from Δ the *SSB strategy* and employ this strategy in Theorem 4.6 below.

Before diving into the main theorem of Nim on the complete graph with unit weight, we need two additional definitions.

Definition 4.4. *We say two distinct vertices are **mutually adjacent** if they have the same set of neighbors and are neighbors themselves.*

Definition 4.5. *If two adjacent vertices of degree $k + 1$ have k common neighbors, we will call them **k-mutually adjacent**.*

Thus saying a graph contains two k-mutually adjacent vertices implies that the graph contains an SSB subgraph of order k. We will also speak of vertices that are k-mutually adjacent without being adjacent to each other. Notice that this implies the graph contains a $K_{2,k}$ subgraph.

Theorem 4.6. *Let G be a graph with $\omega(e) = 1$ for all $e \in E(G)$. If there exists at least two mutually adjacent vertices in G with Δ at one such vertex, then P_1 will win G.*

Proof. Assume that G is a graph of order n with $\omega(e) = 1$ for all $e \in E(G)$. Assume further that v_1 and v_2 are mutually adjacent. We proceed by induction on the k-mutual adjacency.

If v_1 and v_2 are 1-mutually adjacent and $\Delta = v_1$ then $d(v_1) = d(v_2) = 2$ and both are adjacent to some other vertex, say v_3. When P_1 moves to v_2 we have $O(P_2, v_2) = \{v_3\}$ forcing P_2's move. Then P_1 moves to v_1 for the win. Notice that this is consistent with the SSB strategy.

Assume that for all $k \leq j$ the first player to move on a graph G with at least two k-mutually adjacent vertices v_1 and v_2 and $\Delta \in \{v_1, v_2\}$ wins G by moving from v_1 to v_2 on the first move and continually choosing the v_1 or v_2 option. This implies that the second player to move from $G - e_{12}$ and $\Delta \in \{v_1, v_2\}$ wins by employing the SSB strategy.

Assume G is a graph of order n with $(j+1)$-mutually adjacent vertices v_1 and v_2 for $1 < j < n-2$ and $\Delta = v_1$. Enumerate the vertices of G in such a way that $O(P_1, v_1) = \{v_2, \ldots, v_{j+3}\}$. Suppose that P_1 moves to v_2. Then $O(P_2, v_2) = \{v_3, \ldots, v_{j+3}\}$. Without loss of generality, assume that P_2 moves to v_3. Since $v_1 \in O(P_1, v_3)$, let P_1 move to v_1. Now $O(P_2, v_1) = \{v_4, \ldots, v_{j+3}\}$. Thus we have P_2 on a j-mutually adjacent graph minus e_{12}. This means P_2 is on a complete bipartite subgraph of order j contained in G controlled by P_1. By Theorem 3.1, the second player to start from a bipartite graph will win, and by Lemma 4.1, since P_1 can force P_2 to move within the confines of this structure, P_1 will win this graph. Thus P_1 wins every graph G with at least two $(j+1)$-mutually adjacent vertices and Δ on a mutually adjacent vertex. $\qquad\square$

Corollary 4.7. *Assume that $G = K_n$ and that $\omega(e) = 1$ for all $e \in K_n$. Then P_1 can win the K_n for all $n > 1$.*

Proof. When $n = 2$ or 3 we have graphs that have been reduced to trivial wins for P_1. Any two vertices in the K_n are $(n-2)$-mutually adjacent. Thus for Δ at any vertex, P_1 will win the complete graph. $\qquad\square$

We have now successfully solved the problem of complete graphs when each edge has weight one. As shown, the existence of the SSB structure and appropriate starting position solves a large class of graphs. A quick check will show that the SSB strategy will not work for the complete graph and arbitrary weight assignments. However, we can show that for $n \leq 7$ the first player can win the complete graph with any weight assignment. To do this, we modify the SSB strategy slightly to account for the additional options given to the second player.

4.3. The complete graph with arbitrary weight

Currently, the complete graph with arbitrary weight is not solved for all values of n. However, we pose the following conjecture:

Conjecture 4.8. *Let $G = K_n$ with arbitrary weighting assignment and Δ at any vertex. Then P_1 can win the K_n for any $n \geq 2$.*

This conjecture is known to be true for $2 \leq n \leq 7$. The proof is via case-by-case basis for each of P_2's possible moves. We demonstrate the proof in Appendix A also contained on the accompanying CD. Although the computation is quite tedious, there are many graph isomorphisms that can cut down the number of cases significantly We do not include the isomorphisms in the proofs of the complete graphs for readability. Also, we have identified a strategy that works through the case K_7. We call this strategy *the complete graph strategy*. According to the complete graph strategy, P_1 should always move to the vertex of lowest degree and remove all weight from the corresponding edge.

Conjecture 4.9. *Let $G = K_n$ with arbitrary weighting assignment and Δ at any vertex. Then P_2 is forced to move to v_n where v_n is the last vertex of degree $n - 1$ during normal game play.*

Conjecture 4.10. *Let $G = K_n$ with arbitrary weighting assignment and Δ at any vertex. P_1 can stay within the confines of a K_{n-1} on the K_n so that eventually P_2 is forced to move to v_n.*

Conjecture 4.11. *Let $G = K_n$ with arbitrary weighting assignment and Δ at any vertex. Assume P_1 uses the complete graph strategy. Then with the exception of P_1's first move, P_1 always has an option to a vertex of degree strictly less than $n - 1$.*

Conjecture 4.12. *Let $G = K_n$ with arbitrary weighting assignment and Δ at any vertex. If P_1 can win the K_n, then P_1 can win the K_n with an additional pendant edge and end vertex at any vertex on the K_n.*

If either Conjecture 4.10 or Conjecture 4.11 are true, then Conjecture 4.12 holds. This lends itself to the following theorem:

**Theorem 4.13.** *If Conjecture* 4.10 *holds, then Conjecture* 4.12 *holds.*

Proof. To see this, suppose that P_1 can win the K_n. Call the pendant edge $e_{n,n+1}$ and the end vertex v_{n+1}. If $e_{n,n+1}$ is incident with Δ at the start of the game, then P_1 will take $e_{n,n+1}$ to v_{n+1} for the immediate win. If not, suppose without loss of generality that $e_{n,n+1}$ is incident with v_n. Recall that we can always reorder the vertices in such a way that v_n is the last vertex used by either player in a game on the K_n. By Conjecture 4.10, P_2 is the first player forced to move to v_n. Thus P_1 moves across $e_{n,n+1}$ to v_{n+1} for the win. $\square$

CHAPTER 5. NIM ON THE PETERSEN GRAPH

The Petersen graph is well-known in graph theory as providing counterexamples to many ideas once conjectured. For this reason, we shift our attention in this direction to first solve the unit edge case of Nim on the Petersen graph. and then explore the arbitrary weight case. Although we will only need it for the three-path Lemma (Lemma 5.2), we will adopt the following definition of a Petersen graph

Definition 5.1. *The Petersen graph is the simple graph whose vertices are the 2-element subsets of a 5-element set and whose edges are the pairs of disjoint 2-element subsets [11] (see Figure 8).*

Figure 8. The Petersen graph with the 2-element labels.
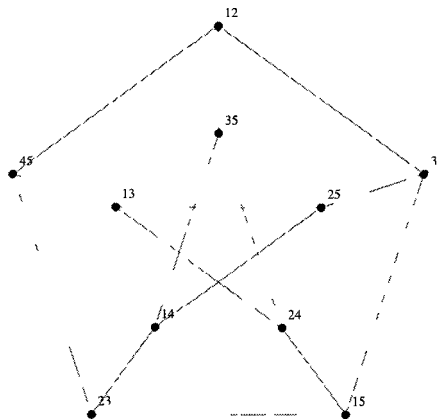
5.1. The isomorphic three-path lemma

To help us with our examination of the Petersen graph, we use the following lemma concerning three-paths in the Petersen graph. Because of the high vertex and edge transitivity of the Petersen graph, we will see that all paths of length three are isomorphic. This will diminish numerous cases from our consideration in both the unit weight case as well as the arbitrary weight case of Nim on the Petersen graph.

Lemma 5.2. *All paths of length three in the Petersen graph are isomorphic.*

Proof. Let $\{1, 2, 3, 4, 5\} = \{i, j, k, l, m\}$. Consider a path in the Petersen graph consisting of four vertices and three edges. Without loss of generality, let the first vertex be labeled $\{i, j\}$, the second vertex be labeled $\{k, l\}$, and the third vertex be labeled $\{i, m\}$. Note that the labeling of the second vertex could not use i or j and that the third vertex must use m and one of i or j but neither of k or l. Then for the fourth vertex, we may not use i or m and are allowed any two of $j, k,$ or l. Let $\{j, k\}$ be the fourth vertex. Also note that since there are no cycles of length three in the Petersen graph, we do not have to omit vertex j from the possible choices of labeling. Thus our path is $\{i, j\}, \{k, l\}, \{i, m\}, \{j, k\}$.

Let θ be a map from $\{i, j, k, l, m\}$ to $\{1, 2, 3, 4, 5\}$ such that $\theta(i) = 1, \theta(j) = 2, \theta(k) = 3, \theta(l) = 4,$ and $\theta(m) = 5$. Notice θ induces an automorphism of the Petersen graph that takes path $\{i, j\}, \{k, l\}, \{i, m\}, \{j, k\}$ to path $\{1, 2\}, \{3, 4\}, \{1, 5\}, \{2, 3\}$. Thus the automorphism group of the Petersen graph acts transitively on paths of length three. Hence all three-path in the Petersen graph are isomorphic.

$\square$

5.2. The Petersen graph with unit weight

Using the lemma of the previous section, we are now in a good place to solve the Petersen graph with unit weight. This is done on a case by case basis, and the reader is encouraged to examine the figures for further explanation of the proof.

Theorem 5.3. P_2 *can win the Petersen graph when each edge has unit weight.*

Proof. By Lemma 5.2 we know that the first three moves are isomorphic no matter how they are made. Recall that since each edge has unit weight, as a player moves across an edge it is deleted. Thus the first three moves create a three-path. Without
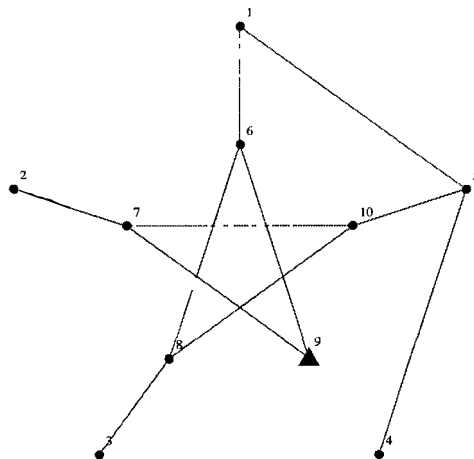
loss of generality, we may label the vertices according to Figure 9 and assume that the edges e_{12}, e_{23}, and e_{34} have been removed.

Figure 9. The Petersen graph after the first three-path is played out.
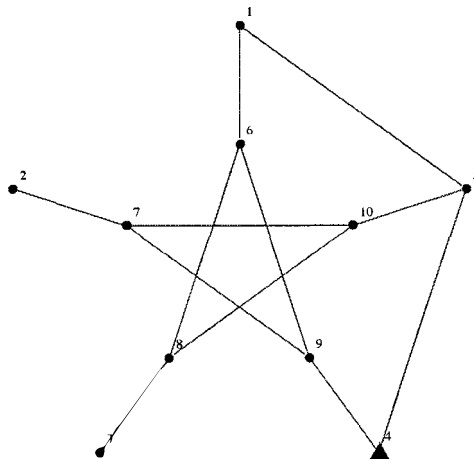
We have Figure 9 with P_2's turn and Δ on v_4. Here P_2 has two non-isomorphic options, one to v_9 and the other to v_5. Consider first P_2's move to v_9 (see Figure 10).
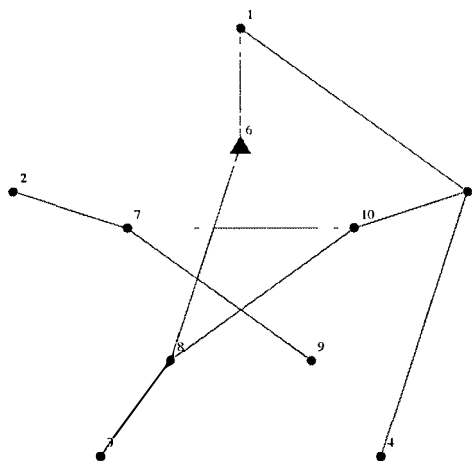
Figure 10. P_2 moves to v_9.

P_1 has two non-isomorphic options at this move, one to v_7 and the other to v_6. After brief study, one can see that P_1 would not choose to move to v_7 as doing so

results in a loss for P_1. To see this, note that if P_1 moves to v_7, P_2 will move to v_2 for the win. Thus we will discard this as an option for P_1 and focus on P_1's move to v_6.

Now we have P_2's move with Δ on v_6 (see Figure 11). Again, P_2 has two non-isomorphic options, one to v_1 and the other to v_8. For the same reasons as above, P_2 would not choose to move to v_8 since then P_1 would move to v_3 for the win. Thus we will discard the v_8 option and assume that P_2 moves to v_1.

Figure 11. P_1 moves to v_6.

At this point, P_1 has only one move to v_5. We can see (in Figure 12) that from v_5, P_2 moves to v_4 for the win.

Figure 12. P_2 moves to v_1.
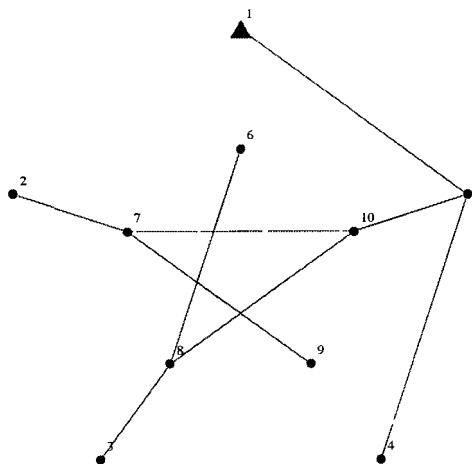
Since P_2 had the choice between v_9 and v_5 at the start of the game and P_2 can win playing the v_9 option for any of P_1's moves, P_2 can win the Petersen graph with unit weight. □

## 5.3. The Petersen graph with arbitrary weight

The Petersen graph has but 10 vertices and 15 edges. Compare this to the K_7, which has 7 vertices and 21 edges. For this reason, it would seem that the Petersen graph should be relatively easy to solve for the arbitrary weight case. Unfortunately, such is not the case with the arbitrary weight case of the Petersen graph. What has been determined for the Petersen graph with arbitrary weight is that *the complete graph strategy* does not work for P_1. In fact, if P_1 uses this strategy, P_2 can win the arbitrary weight case. Also, despite the high vertex and edge transitivity of the Petersen graph, much of the usefulness of this is lost as soon as edges are removed from the game. Almost always, it is necessary to consider every vertex as a separate option for a player.

At this point, we can show that the weighting assignment matters for the Petersen graph. A strategy exists for either player, but it is dependent on the weight of the edges. Also, we will provide propositions on which player will win based on the distance from Δ to the minimum weighted edge when only one minimum weighted edge exists. This will appear quite similar to the theorem for even cycles. For ease of notation, we will use the vertex enumeration from Figure 13.

Theorem 5.4. *The weight of the edges in relation to the position of Δ determines the winner of the Petersen graph.*

Proof. Suppose the weight assignment of the Petersen graph is arbitrary. We will break this proof up into two cases. The first case is when the game starts with

Figure 13. Vertex enumeration of the Petersen graph.
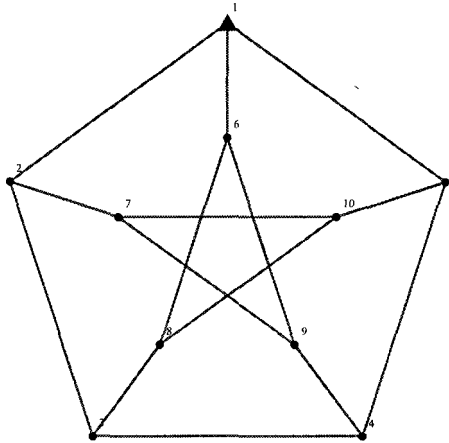
a three-path removed. The second case is when the game starts with a two-path removed. We will show that in both cases, Player 2 can win the Petersen graph.

Case 1: Suppose we start with a 3-path removed from the Petersen graph as in the proof of the arbitrary weight case (see Figure 14). Since both players are removing all weight in this case, we can assume that the three-path starts at v_1 and follows v_1, v_2, v_3, v_4.

Figure 14. Petersen graph with a 3-path removed.
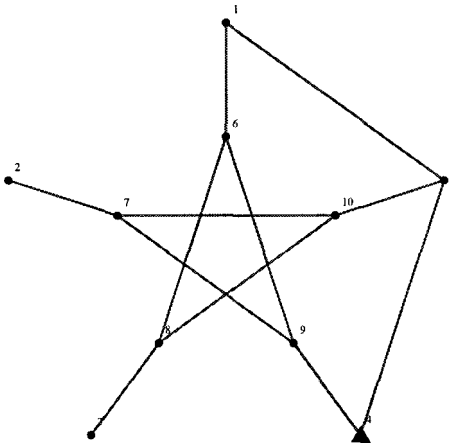
From this position, P_2 has two nonisomorphic options, each having two choices. Consider first the case that P_2 moves to v_9 and removes all weight. From here, if P_1

moves to v_7 then P_2 can move to v_2 for the win. So we may assume that P_1 moves to v_6 (Figure 15). Note that in this move, if P_1 failed to remove all weight, P_2 would move back across e_{69} removing all weight, thereby forcing P_1 to move to v_7.

Figure 15. Petersen graph after the first five moves.
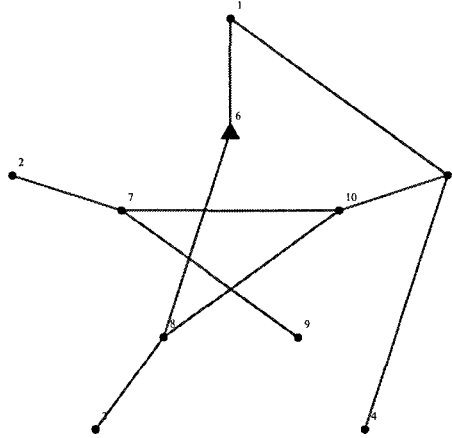
Now P_2 would not move to v_8 since doing so allows P_1 to move to v_3 for the win. Thus P_2 will move to v_1. Again here, P_2 will remove all weight on e_{16}. If not P_1 can move back across e_{16} to v_6, thereby forcing P_2 to move to v_8. What we have at this point in the game is an entire 6-cycle removed from the original Petersen graph arbitrarily weighted (Figure 16).

Figure 16. Petersen graph with a 6-cycle removed.
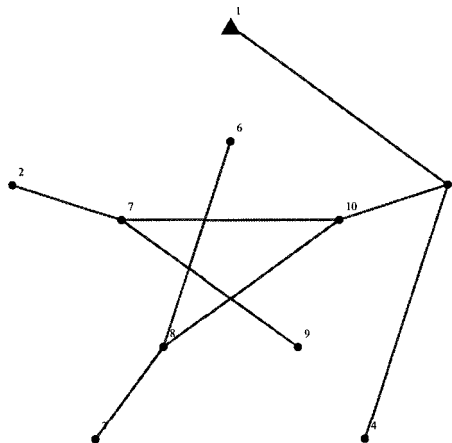
From here, P_1's move is forced to v_5. Clearly it really does not matter whether P_1 removes all weight or not, since from v_5, P_2 will move to v_4 for the win. Therefore, P_2 can win the arbitrarily weighted Petersen graph when the beginning of the game consists of removing a three-path. We should note here that it was necessary for P_2 to move to v_9 and not to v_5 on the first move following the removal of the three-path. Since P_2 had the option at this point, our conclusion still holds.

Case 2: Suppose that e_{12} is removed on P_1's first move, and that e_{23} is removed on P_2's subsequent move (see Figure 17). Now suppose that P_1 does not remove e_{34} as with the previous case. Now along with moving to v_5 or v_9, P_2 can move back to v_3. We will assume that P_2 moves back to v_3 instead of v_9 since we can see that the end game will not play out the same as in the previous case since e_{34} is still intact. Also, since v_5 did not produce a P_2 win in the previous case, we will not take this as our first assumption either.

Figure 17. Petersen graph at the start of the second case.
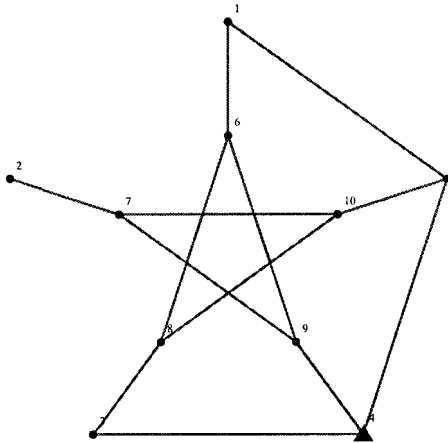
Suppose P_2 moves to v_3 and removes all weight from e_{34}. From here P_1 has only one move to v_8. Clearly P_1 must remove all weight from e_{38} since failing to do so would allow P_2 to move back to v_3 for the win. Now $O(P_2, v_8) = \{v_6, v_{10}\}$ and seemingly there is no danger for P_2 in either move (see Figure 18). Notice that if P_2

moves to v_{10} and removes all weight, then P_1 can only move to v_5 since moving to v_7 will result in a loss when P_2 moves to v_2. We consider that option first.

Figure 18. P_2 here forces P_1 to move to v_5.
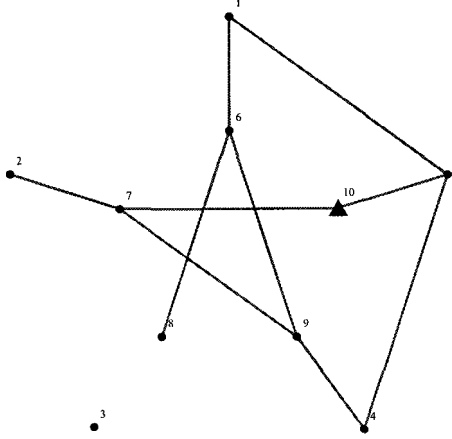
With Δ on v_{10} and P_1 to move, we will suppose that P_1 moves to v_5. Again, P_1 should remove all weight else P_2 moves back across $e_{5,10}$ removing all weight and forcing P_1 to move to v_7. Now if P_2 moves to v_1 and removes all weight, it forces P_1 to move to v_6 (see Figure 19).

Figure 19. P_1 is now forced to move to v_6.
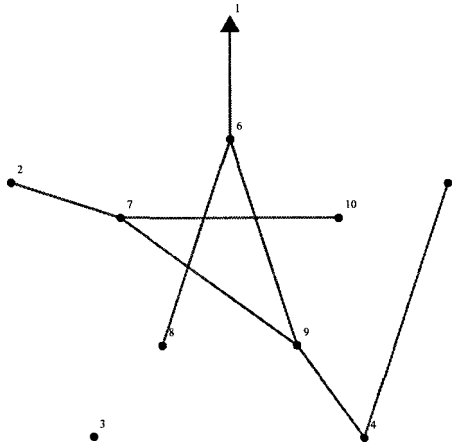
Here, with P_1 now forced to move to v_6, it does not matter whether or not all weight is removed from e_{16} since P_2 will be able to move to v_8 for the win. Therefore,

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P_2 can win the Petersen graph with arbitrary weight for any of P_1's second moves assuming that P_1 and P_2 both remove all weight on their first moves. Since P_2 can win for any of P_1's second moves, we are justified in assuming that P_2 removes all weight on P_2's first move.

What is more, if P_1 did not remove all weight on the first move, P_2 would also not want to remove all weight on the P_2's first move. Doing so would simply reverse the roles of P_1 and P_2 in our above argument. Hence the weight determines the winner a game of Nim played on the Petersen graph. $\qquad\square$

This shows that the weight of the Petersen graph plays a central role in determining the winner of the game. Just as in the even cycle case, first consider the situation that each edge has weight two. Since neither player wants to entirely remove an edge, the first player would have to lower the weight to 1 on the first move. P_2 would also choose to only lower the weight to one on each subsequent move. We have seen in the unit weight case that P_2 can win. This means P_2 should keep P_1 on the unit weight path. Hence, P_2 can win if each edge has weight two.

The same line of logic proving that P_2 can win will hold true f●r any amount of weight equally placed around the edges. That is to say, if there is uniform weight, P_2 will win Nim on the Petersen graph. Now just as with Nim on even cycles, we can think of the position of the minimally weighted edges. Let us denote the edge containing $\min_{e \in E(G)} \omega(e)$ by e_m. Certainly if we assume there is only one e_m and that this edge is incident with Δ at the start of the game, then P_1 can win. To see this, consider that if e_m is incident with Δ, it would be as if P_2 lowered the weight of the graph to the lowest weight right before P_1's turn.

Still assuming that there is a single e_m, we may also show by exhaustion that if e_m is distance 1 from Δ at the beginning of the game, then P_2 will win. Similarly, P_1 will win if e_m is distance 2 from Δ. Since the diameter of the Petersen graph is 2

the maximum distance from Δ to e_m is 2. Thus assuming that there is a single e_m, we can show which player will win the Petersen graph with arbitrary weight.

For the following propositions, we will refer to the vertex enumeration of Figure 13.

Proposition 5.5. *Assume the Petersen graph is weighted arbitrarily with the position of Δ fixed. If there is only one edge of minimum weight and it is incident with Δ at the start of the game, then P_1 can win.*

Proof. First note that neither player desires to lower the weight any edge below the current minimum. We have seen that if all edges are weighted uniformly, that the second player can win. The proof of this proposition is immediate when we consider that if Δ is incident with e_m then the roles of P_1 and P_2 have been switched from the uniform case, and that P_1 would play as P_2 as did in the unit weight case. $\qquad\square$

Proposition 5.6. *Assume the Petersen graph is weighted arbitrarily with the position of Δ fixed. If there is only one edge of minimum weight and it is distance one from Δ at the start of the game, then P_2 can win.*

Proof. Without loss of generality, we may assume that Δ is at v_1 and that $e_m = e_{23}$. Then $O(P_1, v_1) = \{v_2, v_6, v_5\}$. Since we only need to find a winning strategy for P_2 in each case, we show by exhaustion that P_2 can win for each of P_1's three options. Also, just as in the proof of the strategy for even cycles, a player can win more quickly if either player chooses not to lower the weight to the current lowest weight. Thus we can assume that both players will always choose to lower the weight to $\omega(e_m)$.

Case 1: P_1 moves to v_2. In this case, P_2 may only move to v_7. Now, $O(P_1, v_7) = \{v_{10}, v_9\}$.

First, if P_1 moves to v_{10}, then P_2 will opt to move to v_8. Then $O(P_1, v_8) = \{v_3, v_6\}$.

If P_1 moves to v_3, then P_2 has only one option to v_4. From here, $O(P_1, v_4) = \{v_9, v_5\}$. If P_1 moves to v_9 then P_2 will force P_1 to reduce the lowest weight at v_7. If P_1 moves to v_5 then P_2 will force P_1 to reduce the lowest weight at v_{10}.

If P_1 moves to v_6, then P_2 has only one option to v_1 since moving to v_9 results in a loss when P_1 moves to v_6. From here, P_1's only move it to v_5. Then P_2 moves to v_{10} where P_1 is forced to reduce the lowest weight. Thus P_2 can force reduction of the minimum weight for any of P_1's options at v_8.

Second, if P_1 moves to v_7, then P_2 will opt to move to v_6. From here, $O(P_1, v_6) = \{v_1, v_8\}$.

If P_1 moves to v_1, then P_2 only has one move to v_5. From v_5, if P_1 moves to v_{10} then P_2 will move to v_7 at which point P_1 is forced to reduce the lowest weight beyond the current minimum. If P_1 instead moves to v_4 from v_5, then P_2 will move to v_9 at which point P_1 is forced to reduce the lowest weight beyond the current minimum.

If P_1 moves to v_8, then P_2 will opt to move to v_3 since a move to v_{10} will result in a loss. Then from v_3, P_1 can only move to v_4. From v_4, P_2 will move to v_9 at which point P_1 is forced to reduce the lowest weight beyond the current minimum.

Thus for any of P_1's moves following a move to v_2, P_2 can force P_1 to reduce the lowest weight beyond the current minimum.

Case 2: P_1 moves to v_6. In this case P_2 will opt to move to v_8. Then $O(P_1, v_8) = \{v_3, v_{10}\}$.

First, if P_1 moves to v_3, then P_2 can only move to v_4. P_1 would not opt to move to v_9 from v_4 since doing so would allow P_2 to move to v_6 for a faster win. Thus P_1 will move to v_5. Here again, P_2 would not choose to move to v_{10} since that would allow P_1 to move to v_8 for a win. Thus P_2 will move to v_1. From here, P_1's only option is to v_2, and then P_2's only option is to v_7. With $O(P_1, v_7)$, if P_1 moves to v_{10} then P_2 will move to v_5 forcing P_1 to reduce the weight beyond the current minimum.

Likewise if P_1 moves to v_9 then P_2 will move to v_4 forcing P_1 to reduce the weight beyond the current minimum.

Second, if P_1 moves to v_{10}, then P_2 will move to v_7. At v_7, P_1 would not opt to move to v_9 since doing so will result in a faster win for P_2. Thus P_1 moves to v_2. Here, P_2 only has one move to v_1. Similarly, P_2 has only one move to v_5. Then P_2 can move from v_5 to v_{10} forcing P_1 to reduce the weight beyond the current minimum.

Thus for any of P_1's moves following a move to v_6, P_2 can force P_1 to reduce the lowest weight beyond the current minimum.

Case 3: P_1 moves to v_5. In this case P_2 will opt to move to v_4. Then $O(P_1, v_4) = \{v_3, v_9\}$.

First, if P_1 moves to v_3, then P_2 has only one option to move to v_8. From here, P_1 must opt to move to v_6 since a move to v_{10} ensures a faster win for P_2 who would then move to v_5 for the win. For P_2 at v_6, a move to v_9 would allow P_1 to move to v_4 for a win. Thus P_2 must opt to move to v_1. This only leaves P_1 able to move to v_2. With P_2 at v_2 the only option is to move to v_7. With $O(P_1, v_7) = \{v_{10}, v_9\}$ either option leaves a P_2 win. If P_1 moves to v_{10}, P_2 will move to v_5 forcing P_1 to reduce the weight beyond the current minimum, and if P_1 move to v_9, P_2 will move to v_4 forcing P_1 to reduce the weight beyond the current minimum.

Second, if P_1 moves to v_9, P_2 will opt to move to v_7. From here, P_1 will not opt to move to v_{10} since doing so would allow P_2 to move to v_5 for a faster win. Thus P_1 opts to move to v_2. Then P_2 has only one move to v_1. P_1 also has only one move from v_1 to v_6. At v_6, P_2 may move to v_9 forcing P_1 to reduce the weight beyond the current minimum.

Thus for any of P_1's moves following a move to v_5, P_2 can force P_1 to reduce the lowest weight beyond the current minimum.

Therefore, if there is only one edge of minimum weight and it is distance one

from Δ at the start of the game, then P_2 can win.

$\square$

Proposition 5.7. *Assume the Petersen graph is weighted arbitrarily with the position of Δ fixed. If there is only one edge of minimum weight and it is distance two from Δ at the start of the game, then P_1 can win.*

Proof. Since we only need to find a winning strategy for P_1, we show by exhaustion that P_1 can win for each of P_2's following moves. Also, just as in the proof of the strategy for even cycles, a player can win more quickly if either other player chooses not to lower the weight to the current lowest weight. Thus we can assume that both players will always choose to lower the weight to ωe_m.

Without loss of generality, we may assume that Δ is at v_1 and that $e_m = e_{34}$. Then $O(P_1, v_1) = \{v_2, v_6\}$ and note that v_2 and v_5 are isomorphic options for P_1. P_1 has the first choice of vertex to move to, so assume that P_1 moves to v_6. Then $O(P_2, v_6) = \{v_8\}$ since v_8 and v_9 are isomorphic options for P_2.

At v_8, P_1 will opt to move to v_3. From here, P_2 has only one option to move to v_2. At v_2, P_1 will opt to move to v_1, and again P_2 has only one move to v_5. At v_5, P_1 would not opt to move to v_{10} since doing so would allow P_2 to move to v_8 for a win. Thus P_1 moves to v_4, and P_2 has only one option to move to v_9. At v_9, P_1 may move to v_6 forcing P_2 to reduce the weight beyond the current minimum.

Thus for any of P_2's moves, P_1 can force P_2 to reduce the weight of an edge beyond $\omega(e_m)$ if there is only one edge of minimum weight and it is distance two from Δ at the start of the game allowing P_1 to win the Petersen graph. $\square$

The last proposition had play for P_2 which was forced at each move. This would not have been the case if P_1 had moved to v_2 or v_5 instead of v_6. In fact, P_2 would have had a winning strategy for either of these options by P_1.

Notice that these propositions suppose that there is but one edge of minimum weight in the arbitrarily weighted Petersen graph. Such would not be the case for multiple minimum weight edges. For example, if there were two adjacent edges, both with $\omega(e_m)$, and one incident with Δ at the start of the game, then P_2 would have a winning strategy. To see this, we can simply reverse the roles of P_1 and P_2 in Proposition 5.5. Likewise, if there were a minimally weighted 3-path with one edge incident with Δ at the start of the game, then P_1 would have a winning strategy, and so forth. It is not yet known how two or more minimally weighted non-adjacent edges affect the outcome of the game.

CHAPTER 6. NIM ON THE HYPERCUBE WITH UNIT WEIGHT

Definition 6.1. *The n-dimensional hypercube, or the n-cube, Q_n is the graph K_2 if $n = 1$, while for $n \geq 2$, Q_n is defined recursively as $Q_{n-1} \times K_2$ [3].*

We can also think of the n-cube as the graph whose vertices are labeled by the binary n-tuples $(a_1, a_2, \ldots, a_n)$ where each a_i is either 0 or 1 for $1 \leq i \leq n$ and such that two vertices are adjacent if and only if their corresponding n-tuples differ at precisely one coordinate. This is the view of hypercubes that we will adopt in what follows, along with the following alternate labeling. Label each vertex $a = (a_1, a_2, \ldots, a_n)$ of the hypercube Q_n by the corresponding set $X_a = \{i : a_i = 1\}$ [11]. Then we will draw the Q_n in the plane so that the vertical coordinates of the vertices are in order by the size of the sets labeling them (see Figure 20). We will call this the level labeling scheme and use it throughout the hypercube section.

Figure 20. Here is the Q_3 with the level labeling scheme.
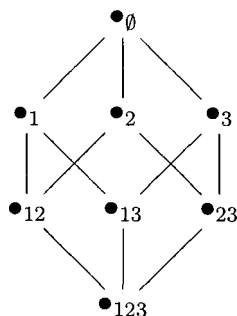
Definition 6.2. *The parity of a vertex in Q_n is the parity of the number of 1's in its name, even or odd [11].*

This implies that each edge of the Q_n has an even vertex and an odd vertex as endpoints (see Figure 20). This means that the even vertices form an independent set, as do the odd vertices. Hence Q_n is bipartite for any n [11].

Since we typically start with Δ on the lowest numerically denoted vertex. Here we will start with Δ on vertex $\emptyset$. With this level labeling scheme, we can think of the vertices at different levels corresponding to the number of digits in the vertex labels. Thus in the example of the Q_3 we have levels $\emptyset, 1, 2$, and 3.

Throughout this section we will assume that the weight of each edge of the hypercube has unit weight.

Lemma 6.3. *P_1 can keep game play on the Q_{2n+1} within the confines of levels $\emptyset, 1$, and 2.*

Proof. Let Q_{2n+1} have unit weight and label each vertex by the X_a scheme described above so that $\emptyset$ is at vertex $(0, 0, \ldots, 0)$. Give the Q_{2n+1} the level labeling scheme. Assume that Δ starts at vertex $\emptyset$. Note that since the Q_{2n+1} is regular of order $2n+1$ any choice of starting vertex is isomorphic.

Every hypercube is bipartite. Thus we can observe that P_1's vertices all have even parity, and P_2's vertices have odd parity according to the labeling scheme.

Suppose P_1 is at vertex ij in level 2. Since we want to show that P_1 can opt not to move down to level 3, we will show that there is always an option in level 1 for any ij in level 2. Since P_1 is playing from vertex ij, either P_2 moved from i or from j in level 1. Without loss of generality, assume P_2 moved from i so that $e_{i,ij}$ is no longer an option for P_1.

By way of contradiction, suppose that P_1 cannot move to j from ij. This implies that $e_{j,ij}$ has been used already. This can only occur in one of two ways: the first case is that P_1 moved to j via $e_{j,ij}$ on a previous move, and the second case is that P_2 moved from j to ij via $e_{j,ij}$ on a previous move.

In the first case, if P_1 moved from ij to j then it must be the case that P_2 was on level three and moved from some ijk to ij. This is because we are assuming that just now P_2 moved from i to ij and thus could not have made that move previously.

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(Recall that since we have unit weight, once an edge has been moved across once it is no longer a playable edge.) This contradicts the fact that P_1 would not make such a move unless forced to. Clearly P_1 was not forced to previously move down to level 3 since it is only now that a move to vertex i is no longer possible.

In the second case, if P_2 moved from j to ij but P_1 did not move from ij to i since it remains, then P_1 moved down to some ijk, again a contradiction.

Thus P_1 always has a level 1 option and hence can keep P_2 within levels $\emptyset, 1$, and 2. $\square$

Theorem 6.4. *Assume $\omega(e) = 1$ for all $e \in Q_{2n+1}$. Then P_1 can win the Q_{2n+1} for any $n \geq 1$.*

Proof. Assume $\omega(e) = 1$ for all $e \in E(Q_{2n+1})$, that $n \in \mathbb{Z}$, and $n \geq 1$. Label the digits according to X_a and the level labeling scheme. Start with Δ on $\emptyset$.

Since all hypercubes are bipartite, we know that P_1's vertices have even parity, and P_2 vertices have odd parity. By Lemma 6.3, P_1 can keep P_2 within the confines of levels $\emptyset, 1$, and 2. Because of this, consider only these three levels. In essence, "chop off" levels 3 through $2n + 1$.

With P_1 at $\emptyset$ at the start, notice that the vertices in level $\emptyset$ and 1 are all odd degree. Since we are considering the graph without levels 3 through $2n + 1$, the vertices in level 2 are all of degree 2. Also, since we are assuming each edge has unit weight, when a player moves across an edge, it is deleted. Thus P_1 starts on an odd degree vertex and P_2 starts on an even degree vertex at each of their respective moves. This implies that P_1 always has an edge to move away from at any vertex (since odd degree implies at least degree 1). However, since Q_{2n+1} is finite, eventually P_2 will come to a vertex of degree 0 and not be able to move.

Thus P_1 always wins the Q_{2n+1} for any positive integer value of n. $\square$

Theorem 6.5. *Assume that $\omega(e) = 1$. Then P_2 wins the Q_{2n} for all $n \geq 1$.*

Proof. Assume that $n \in \mathbb{Z}$, $n \geq 1$, and $\omega(e) = 1$ for all $e \in E(Q_{2n})$. Label the vertices according to X_a and the level labeling scheme. Start with Δ on $\emptyset$.

Note that Q_{2n} is regular of degree 2n, and Q_{2n} is bipartite. Thus P_1 moves from vertices with even parity, and P_2 moves from vertices with odd parity. Also notice that P_1 starts from a vertex of even degree, and each time P_1 moves from $\emptyset$ it is of even degree. Each other vertex is of odd degree when either player moves from it. This is because the degree lowers by one each time a player arrives at the vertex. Thus on the first move, P_1 moves from an even degree vertex to what was an even degree vertex. Since the process of moving to a vertex lowers the degree by one each time because of unit weight of the edges, P_2 starts from a vertex that has odd degree. This is true for each player at each vertex except for P_1 at vertex $\emptyset$.

If a vertex has odd degree when moving from it, a player is guaranteed to be able to move away from the vertex, since an odd degree vertex implies that the degree is at least 1. Thus the only vertex that a player could possibly get stuck at is the $\emptyset$ vertex. Since P_1 is the only player to move from $\emptyset$ by virtue of Q_{2n} being bipartite, P_1 is the only player who is able to lose.

Therefore, since there are only a finite number of moves, P_2 wins the Q_{2n} for any positive integer value of n. $\square$

With the previous two theorems, we can formulate the following two corollaries.

Corollary 6.6. *P_1 wins the unit weight hypercube if and only if n is odd.*

Corollary 6.7. *P_2 wins the unit weight hypercube if and only if n is even.*

6.1. A note about the hypercube with arbitrary weight

The unit weight hypercube had a nice parity argument to show the winner. Unfortunately, the hypercube weighted arbitrarily is not so easy to solve. We know very quickly that weight matters with the arbitrarily weighted hypercube. Take for a

simple example, $Q_2 = C_4$. By previous work in the even cycle section, we know that the winner of the game is decided by the distances to the lowest weight edge. Hence we can tell at least for the even values of n that the weight of the Q_n will matter in determining the winner of the game.

CHAPTER 7. BIPARTITE GRAPHS

In this section, we will consider only complete bipartite graphs, since non-complete bipartite graphs will require complete specification on the edge set. We provide a complete solution when the edges have unit weight, and then consider the arbitrary weight case.

7.1. Bipartite graphs with unit weight

As alluded to in the title of this section, we will assume that each edge of the complete bipartite graph has unit weight. We will first extend the result in the complete graph section to include the other partite set when considering $K_{2,j}$. Then we shall use this result to prove that the second player can win any complete bipartite graph.

Theorem 7.1. *Assume $\omega(e) = 1$ for all edges. Then P_2 wins the $K_{2,j}$ for all $j \geq 2$ and with Δ in either partite set.*

Proof. Let $\omega(e) = 1$ for all $e \in K_{2,j}$. If Δ is in the partite set of size 2, we know the theorem is true by Lemma 4.1.

If Δ is in the partite set of size j, then on P_1's first move, the choices of the two vertices are isomorphic. Similarly on P_2's move, the choices of the $j - 1$ vertices are isomorphic. Now on P_1's next turn, there is only one move to the other vertex in the partite set of size 2. From this position, P_2 can move to the starting vertex for the win. $\square$

Theorem 7.2. *Assume $\omega(e) = 1$ for all edges. Then P_2 wins the $K_{m,n}$ for all $m, n \geq 2$ and starting position in either partite set.*

Proof. Let $\omega(e) = 1$ for all $e \in K_{m,n}$, and let $m, n \geq 2$. Enumerate the vertices in such a way that Δ is on v_1, and then any other vertex in the same partite set has

label v_2. Starting at v_1, notice that there exists a $K_{2,n}$ or $K_{2,m}$ depending on which partite set Δ starts in. As long as P_2 continually chooses to move to v_1 or v_2 on his turn, P_2 can keep P_1 within the confines of the $K_{2,n}$ or $K_{2,m}$. By the previous theorem, we know P_2 wins the $K_{2,j}$ for all $j \geq 1$.

Thus P_2 wins any complete bipartite graph. $\qquad\square$

7.2. Bipartite graphs with arbitrary weight

The general solution to bipartite graphs with arbitrary weight is a much more difficult question to answer. At the moment, the solution for Nim on bipartite graphs with unit weight is not known. If we consider the $K_{2,2}$ we can quickly see that weight will matter for complete bipartite graphs. Since the $K_{2,2} = C_4$, our previous results show that the solution is determined by the distance from the starting piece to the lowest weighted edge.

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APPENDIX A. K_N WITH ARBITRARY WEIGHT FOR $2 \leq N \leq 7$

Due to the amount of information required to demonstrate the case-by-case basis of the K_n, we have included the play analysis for $3 \leq n \leq 7$ in the accompanying compact disc. We assume throughout the analysis that P_1 uses the Complete Graph Strategy. Furthermore, P_1 always has isomorphic options as the lowest degree option. Hence there is no reason to label P_1's moves. However, it is necessary to not only tell where P_2 moves in some cases, we must also show both the cases when P_2 removes all weight and when there is some amount of weight left by P_2. For this reason, we will denote the removal of all weight on an edge during P_2's turn by 0, and the choice of leaving weight behind by +. Furthermore, when P_2's options are not isomorphic, we will denote the choice of vertex to which P_2 moves by subscripts.

Note that the case of the K_2 is a trivial P_1 win since it is an odd path.