IMPEDANCES OF A CUBICAL QUAD ANTENNA

A Thesis

Submitted to the Graduate Faculty

of the

North Dakota State University

of Agriculture and Applied Science

by

Tsi-lung Choong

In Partial Fulfillment of the Requirements

For the Degree of

Master of Science

April, 1965

Fargo, North Dakota

This Thesis is Approved by:

ACKNOWLEDGMENT

The author wishes to express his sincere gratitude to Professor E. G. Anderson of the Electrical Engineering Department, North Dakota State University, for his helpful guidance and supervision without which this thesis would not have been possible.

TABLE OF CONTENTS

INTRODUCTION

Rectangular loop antennas and short electric dipoles are two of the oldest antennas in existance. In 1888, twenty years after Maxwell invented his famous Maxwell's equations, Hertz used these two antennas to prove that high frequency electric energy sources could radiate electromagnetic waves.

The "Cubical Quad"[1] or, simply, "Quad" antenna is a development of the rectangular loop antenna. It consists of a pair of square loops, one-quarter wavelength on a side or one wavelength around the periphery; one loop being driven and the other used as a parasitic reflector. The separation between the two is usually of the order of 0.15 to 0.2 wavelength, with the planes of the loops parallel.

While studying the properties of this antenna, it was discovered that little had been done to develop it from a theoretical aspect. The purpose of this thesis is to obtain values of the self and mutual impedances existing in such an antenna array. The values are obtained from mathematical analysis and experimental measurements and may be used in field pattern and gain calculations.

---

[1]The Radio Amateur's Handbook, American Radio Relay League, 39th Edition, 1962.

CHAPTER I

CALCULATION OF MUTUAL IMPEDANCE

Before doing any mathematical analysis some assumptions, that cannot be realized in the practical system, must be described. They are:

(1) The antenna is located at a place which is completely free from obstructions in all directions.

(2) The gap between the two input terminals is infinitely small.

(3) The ohmic losses along the antenna are negligible.

The following analysis is based on ideal situations.

In the derivation of the mutual impedance between two loops it is necessary, first of all, to derive an expression for the current distribution along one loop. Then, the induced electric field intensity at any point P along the second loop, which is produced by the retarded charges and currents on the first loop, can be determined. The power required to produce current against the opposition of the induced emf on the first loop is computed for each infinitely-small element. The total power is obtained by integrating over the whole length of the first loop. This gives total power, real and reactive, required to establish the current against the induced emf and from this the mutual impedance may be calculated. This method is well known as the "induced emf method".

1-1 Current Distribution on the Radiating Loop of the Cubical Quad Antenna

The square radiating loop is one-quarter wavelength on a side or one wavelength around the periphery. If it is fed by a balanced two wire line, the potential of one wire must be equal and opposite to that of the other with respect to the ground and equal out-of-phase currents must flow at the feed point.[1] Assuming the conductivity of the loop is infinite, it can be viewed as a lossless transmission line short-circuited at the point "e" (see fig. 1-1). Moreover, if the bâlanced two wire line transmits a sinusoidal wave to the input terminals of the loop the current of the incident wave may be expressed as

$$I_i e^{j(wt+\beta D)}$$

where I_i=maximum incident r.m.s. current.

$=2\pi/\lambda$ phase constant

D =reference distance. Taken as zero at the short circuit point.

The expression for current of the reflected wave will be

$$I_r e^{j(wt-\beta D)} \quad (1\text{-}2)$$

where I_r=maximum reflected r.m.s. current.

---

[1] Skilling, Electric Transmission Lines, McGraw-Hill Book Company, Inc., p. 93, 1961.

Krause, Antennas, McGraw-Hill Book Company, Inc., p. 415, 1950.

At the short circuited point "e", $I_i = I_r$ hence, the total current will be

$$I_t(\ell) = I_i e^{j(wt+\beta D)} + I_r e^{j(wt-\beta D)}$$

$$= 2I_i \mathrm{Cos}(\beta D) e^{jwt} \qquad (1-3)$$

Equation (1-3) shows that the incident and reflected waves combine to produce a standing wave which does not progress. The current distribution curves are shown in fig. 1-1 and fig. 1-2.

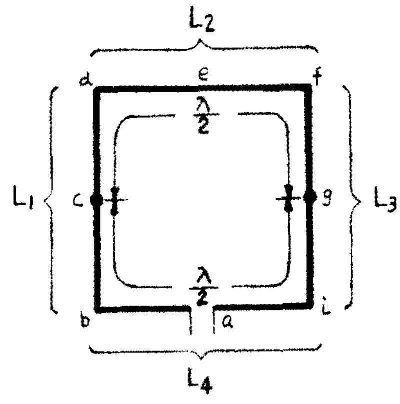

Fig. 1-1

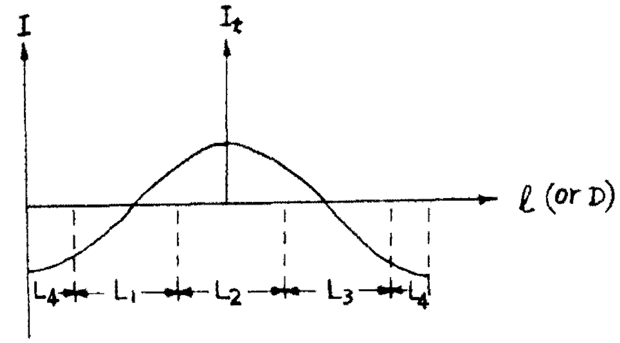

Fig. 1-2

In fig. 1-1, the four sides of the loop are marked L_1, L_2, L_3, L_4 respectively. The arrows indicate the instantaneous current directions and the dots indicate the locations of the current minima. For convenience, it is better to shift the D=0 point from "e" to "a" such that:

$$D = \ell - \frac{\lambda}{2}$$

$$I(\ell) = 2I_i \mathrm{Cos}\left[\beta\left(\ell - \frac{\lambda}{2}\right)\right] e^{jwt} \qquad (1-4)$$

$$= -2I_i \mathrm{Cos}(\beta\ell) e^{jwt}$$

Where ℓ is the distance along the radiating loop measured from point "a", defined as follows:

$$\ell = \frac{n\lambda}{8} \pm h$$

$$n = 0,\ 1,\ 3,\ 5,\ 7,\ 9\ ----$$

Referring to fig. 1-1 and fig. 1-2, n is equal to 1, 3, 5, 7 for L_1, L_2, L_3, L_4 respectively. The distance h, in wavelengths is measured from the point b, d, f or i in the clockwise direction.

1-2 Retarded Scalar Potential

The electric scalar potential due to a point charge is a linear function of the value of its charge. It follows that the potentials of more than one point charge are linearly superposable by scalar addition. In static electric fields, the potential at P (x, y, z) due to distribution charges along a line is

$$V = \frac{1}{4\pi\epsilon_0}\int \frac{\rho_L}{r}\, dh \qquad (1-6)$$

where ρ_L = linear charge density (coulomb/meter)

ϵ_0 = permittivity (dielectric constant for vacuum)

$= \frac{1}{36\pi 10^9}$ (farad/meter)

$r = \sqrt{x^2 + y^2 + (z-h)^2}$ (meter)

dh = element of length of line in meters

The integration is carried out wherever ρ_L has value.

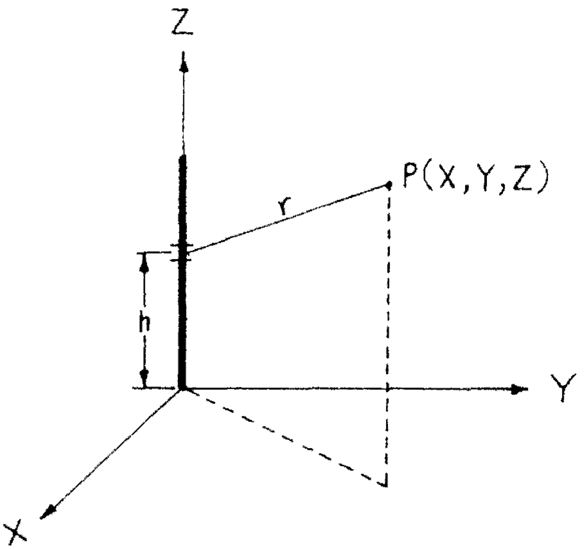

Fig. 1-3

In time-changing fields, ρ_L is changing with time. Its expression can be deduced from the continuity relation between current and charge density. The continuity of current states that a net flow of current out of a volume (positive current flow) must be equal to the negative rate of change of charge with respect to time.

$$\int_S \vec{J} \cdot d\vec{s} = - \frac{\partial \rho}{\partial t} \Delta v \qquad (1\text{-}7)$$

or

$$\nabla \cdot \vec{J} = - \frac{\partial \rho}{\partial t}$$

Now I is everywhere in the h direction (in fig. 1-3, z and h are in the same direction). The above expression becomes

$$\nabla \cdot I = \frac{\partial I_h}{\partial h} = - \frac{\partial \rho_L}{\partial t} \qquad (1\text{-}8)$$

or $$\frac{\partial I_h}{\partial h} = - \frac{\partial \rho_L}{\partial t} \tag{1-9}$$

$$\rho_L = - \int \frac{\partial I_h}{\partial h} \, dt \tag{1-10}$$

where I_h = current in the wire (amps)

ρ_L = linear charge density along the antenna (coulomb/meter)

Substituting equation (1-4) into equation (1-10)

$$\rho_L = 2I_i \int \frac{\partial}{\partial h} \left\{ \cos\left[\beta\left(\frac{n\lambda}{8} + h\right)\right] e^{jwt} \right\} dt$$

$$= -2I_i \int \sin\left[\beta\left(\frac{n\lambda}{8} + h\right)\right] e^{jwt} \, dt$$

$$= \frac{2jI_i \sin\left[\beta\left(\frac{n}{8}\lambda + h\right)\right]}{w} e^{jwt} + C$$

The constant of integration C indicates a linear charge density independent of t could be present. Since such a charge distribution, if it does exist, will not contribute to radiation its existance will be ignored.

Hence $$\rho_L = \frac{2I_i \sin\left[\beta\left(\frac{n\lambda}{8} + h\right)\right]}{w} e^{j(wt + \frac{\pi}{2})} \tag{1-11}$$

The space charge distribution curve is shown in fig. 1-4.

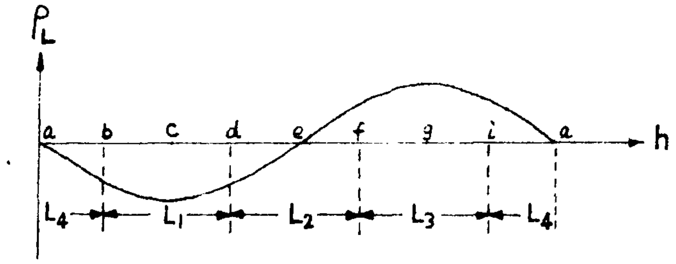

Fig. 1-4

In time-changing fields the effect of charge is not felt instantaneously at the point P, but only after an interval equal to the time required for the disturbance to propagate over the distance r; this time interval is

$$\frac{r}{c} \text{ seconds}$$

where c = velocity of light (= $3\text{x}10^8$ meters/sec.)

We can introduce this time of propagation, called the time of retardation, and write

$$\left[\rho_L\right] = -\,2I_i\beta\int \sin\left[\beta\left(\frac{n\lambda}{8}+h\right)\right] e^{jw\left(t-\frac{r}{c}\right)}\,dt$$

$$= \frac{2jI_i\beta\,\sin\left[\beta\left(\frac{n\lambda}{8}+h\right)\right] e^{jw\left(t-\frac{r}{c}\right)}}{w} \qquad (1\text{-}12)$$

$\left[\rho_L\right]$ is called the retarded charge density. Substituting it into equation (1-6) gives

$$\left[V\right] = \frac{jI_i e^{jwt}}{2\pi\epsilon_0 W}\int\left[\frac{\sin\left[\beta\left(\frac{n}{8}\lambda+h\right)\right]}{r}\,e^{-j\beta r}\right]dh$$

$$= \frac{jI_i e^{jwt}}{2\pi\epsilon_0 c}\int\frac{\sin\left[\beta\left(\frac{n}{8}\lambda+h\right)\right]\,e^{-j\beta r}}{r}\,dh \qquad (1\text{-}13)$$

$\left[V\right]$ is called the retarded scalar potential.

1-3 Retarded Vector Magnetic Potential

In static magnetic fields, the vector potential can be expressed in the form[2]

---

[2] Jordan derives $\vec{A}$ from the magnetic intensity $\vec{H}$, hence the expression for $\vec{A}$ does not involve μ_0; while (1-14) is derived from $\vec{B}$.

$$\vec{A} = \frac{\mu_0}{4\pi}\iiint \frac{\vec{J}}{r}\, dv \qquad \text{(Webers/meter)} \qquad (1\text{-}14)$$

Where $\vec{A}$ = vector magnetic potential at point P

μ_0 = permeability of vacuum (henrys/meter)

$\vec{J}$ = current density at volume element (amp/meter^2)

dv = volume element (meter^3)

r = distance from each volume element to the point P (meters).

If $\vec{J}$ is confined in a thin wire as stated in § 1-2, $\vec{J}$ is everywhere in some particular direction h and also is uniform. Thus

$$\vec{J} = \vec{a}_h J_h$$

Then

$$\iiint \vec{J} dv = \vec{a}_h \iiint J_h \, ds\, dh = \vec{a}_h \int I\, dh \qquad (1\text{-}15)$$

Where $\vec{a}_h$ = unit vector in h direction

ds = area element

dh = length element

$I = J_h a$ = current in wire

Substituting (1-15) into (1-14) gives

$$\vec{A} = \frac{\vec{a}_h \mu_0}{4\pi}\int \frac{I}{r}\, dh \qquad (1\text{-}16)$$

As stated in § 1-2, in time-changing fields, the effect of current changes on the antenna are not felt instantaneously at the point P, but only after an interval equal to the time required for the radiated wave to reach a distance r from the radiating element. This time internal is

$$\frac{r}{c} \text{ seconds}$$

Hence, equation (1-16) must be modified by a time factor.

$$[\vec{A}] = \frac{\vec{a}_h \mu_0}{4\pi} \int \frac{I\, e^{-jw(\frac{r}{c})}}{r}\, dh \qquad (1\text{-}17)$$

$[\vec{A}]$ is called the retarded vector magnetic potential. Substituting equation (1-4), the current in the antenna, into equation (1-17) gives

$$[\vec{A}] = \frac{-I_i \vec{a}_h \mu_0}{2\pi} \int \frac{\cos\left[\beta\left(\frac{n\lambda}{8} + h\right)\right] e^{jw(t-\frac{r}{c})}}{r}\, dh$$

$$= \frac{-I_i \vec{a}_h \mu_0}{2\pi} \int \frac{\cos\left[\beta\left(\frac{n}{8}\lambda + h\right)\right] e^{j(wt-\beta r)}}{r}\, dh \qquad (1\text{-}18)$$

1-4 The Induced EMF on the Reflecting Loop

Set the cubical antenna in rectangular coordinates with the two identical loops parallel and with their centers on the same axis, as shown in fig. 1-5. The four sides of the radiating loop are marked L_1, L_2, L_3, L_4 respectively, while the four sides of the reflecting loop are marked L_I, L_{II}, L_{III}, L_{IV}.

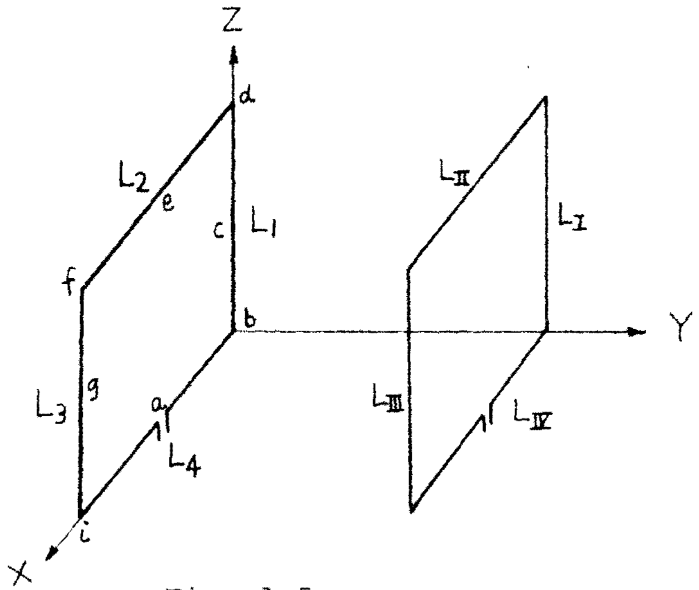

Fig. 1-5

The current and charge distribution on the radiating loop have been shown in fig. 1-1 and fig. 1-4. The points a,b,c,d,e,f,g and i on the loop are the same as those of fig. 1-1 and fig. 1-4.

Knowing the current and charge distribution, the retarded vector potential $\vec{A}$ and the retarded scalar potential V may be obtained by equations (1-13) and (1-18). Knowing the retarded scalar potential and retarded vector potential, the electric field is everywhere obtainable from the relation

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} \qquad (1-19)$$

where $\nabla = \vec{a}_x \frac{\partial}{\partial x} + \vec{a}_y \frac{\partial}{\partial y} + \vec{a}_z \frac{\partial}{\partial z}$

in rectangular coordinates.

Since the field intensities are superposable by vector addition, the four sides of the radiating loop can be treated as four radiating elements. Each element induces an emf at a point on the reflecting loop. The vector sum of the four emfs will be the total emf due to the radiating loop. The following sections deal with this kind of derivation.

1-5 The Induced EMF in the z Direction Due to the Current Element in the z Direction

In the following derivation let the current element be coincident with the z-axis. A point on the current element is designated h. A point in space is given in rectangular coordinates by P(x,y,z). The electric field intensity at P(x,y,z) is

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t} \qquad (1\text{-}19)$$

where ∇V = The gradient of retarded scalar potential at point P(x,y,z)

$$\nabla V = \vec{a}_x \frac{\partial V}{\partial x} + \vec{a}_y \frac{\partial V}{\partial y} + \vec{a}_z \frac{\partial V}{\partial z}$$

$\vec{A}$ = The retarded vector potential at point P(x,y,z)

When only the z component of the electric field is required equation (1-19) reduces to

$$E_z = -\frac{\partial V}{\partial z} - \frac{\partial A_z}{\partial t} \qquad (1\text{-}20)$$

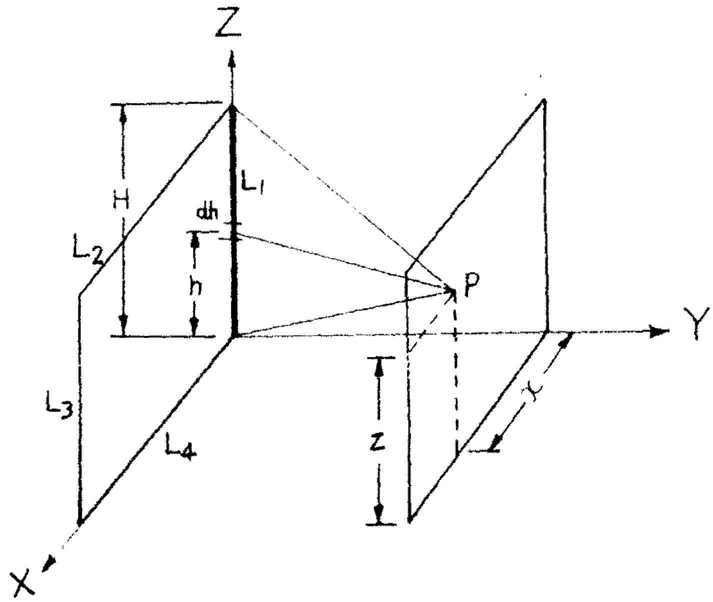

Fig. 1-6

Introducing the retarded scalar potential V and the retarded vector potential $\vec{A}$ into equation (1-20)

$$-\frac{\partial V}{\partial z} = -\frac{jI_i e^{jwt}}{2\pi\epsilon_0 c}\int \frac{\partial}{\partial z}\left\{\frac{\sin\left[\beta\left(\frac{n}{8}\lambda+h\right)\right]}{r}e^{-j\beta r}\right\}dh$$

$$-\frac{\partial A_z}{\partial t} = \frac{jwI_i\mu_0}{2\pi}\int\frac{\cos\left[\beta\left(\frac{n\lambda}{8}+h\right)\right]e^{j(wt-\beta r)}}{r}dh$$

$$\because \quad \sin\left[\beta\left(\frac{n}{8}\lambda+h\right)\right] = \frac{1}{2j}\left[e^{j\beta\left(\frac{n\lambda}{8}+h\right)} - e^{-j\beta\left(\frac{n}{8}\lambda+h\right)}\right]$$

$$\cos\left[\beta\left(\frac{n}{8}\lambda+h\right)\right] = \frac{1}{2}\left[e^{j\beta\left(\frac{n\lambda}{8}+h\right)} + e^{-j\beta\left(\frac{n}{8}\lambda+h\right)}\right]$$

$$\therefore \quad -\frac{\partial V}{\partial z} = \frac{-I_i e^{jwt}}{4\pi\epsilon_0 c}\int\frac{\partial}{\partial z}\left[\frac{e^{j\beta(\ell-r)}-e^{-j\beta(\ell+r)}}{r}\right]dh \qquad (1-21)$$

$$-\frac{\partial A_z}{\partial t} = \frac{jwI_i\mu_0}{4\pi}e^{jwt}\int\frac{e^{j\beta(\ell-r)}+e^{-j\beta(\ell+r)}}{r}dh \qquad (1-22)$$

Substituting equations (1-21) and (1-22) into equation (1-20) yields

$$E_z = -\frac{\partial V}{\partial z} - \frac{\partial A_z}{\partial t}$$

$$E_z = -\frac{I_i e^{jwt}}{4\pi\epsilon_0 C}\int \frac{\partial}{\partial z}\left[\frac{e^{j\beta(\ell-r)} - e^{-j\beta(\ell+r)}}{r}\right] dh$$

$$+ \frac{jwI_i\mu_0 e^{jwt}}{4\pi}\int \frac{e^{j\beta(\ell-r)} + e^{-j\beta(\ell+r)}}{r}\, dh \qquad (1\text{-}23)$$

where $\ell = \frac{n\lambda}{8} + h$ it is defined in equation (1-5)

$$w\mu_0 = 2\pi f \frac{1}{C^2\epsilon_0} = \frac{2\pi}{\lambda C\epsilon_0} = \frac{\beta}{\epsilon_0 C} \qquad (1\text{-}24)$$

Substituting equation (1-24) into equation (1-23) yields

$$E_z = -\frac{I_i e^{jwt}}{4\pi\epsilon_0 C}\int \frac{\partial}{\partial z}\, \frac{e^{j\beta(\ell-r)} - e^{-j\beta(\ell+r)}}{r}\, dh$$

$$+ \frac{jI_i e^{jwt}}{4\pi\epsilon_0 C}\int \frac{e^{j\beta(\ell-r)} + e^{-j\beta(\ell+r)}}{r}\, dh \qquad (1\text{-}25)$$

Equation (1-25) represents the field intensity at P due to the retarded charges and current. The integration of equation (1-25) is carried out everywhere along the Z axis. The total field intensity due to all the retarded charges and current distributed on the element of length H will be

$$E_z = +\frac{I_i e^{jwt}}{4\pi\epsilon_0 C}\left\{\int_0^H \frac{\partial}{\partial z}\left[\frac{e^{-j\beta(\ell+r)} - e^{+j\beta(\ell-r)}}{r}\right] dh + j\beta\int_0^H \frac{e^{-j\beta(\ell+r)} + e^{j\beta(\ell-r)}}{r}\, dh\right\} \qquad (1\text{-}26)$$

One can prove[3]

$$E_z = \frac{I_i e^{jwt}}{4\pi\epsilon_0 C}\left[\frac{e^{j\beta(\ell-r)}}{r} - \frac{e^{-j\beta(\ell+r)}}{r}\right]_0^H \qquad (1\text{-}27)$$

r is a function of x, y, z, and h; ℓ is a function of n and h.

$$r = r(x,y,z,h)$$

$$\ell = \ell(n,h) = \frac{n\lambda}{8} \pm h$$

When h = H, set $\quad r = r(x,y,z,H) = r_H$

[3] Appendix I

$$
\left.
\begin{aligned}
&\text{and} \qquad \ell = \ell(n,H) = \ell_H \\
&\text{when } h = 0, \text{ set} \\
&\qquad r = r(x,y,z,o) = r_o \\
&\qquad \ell = \ell(n,o) = \ell_o
\end{aligned}
\right\} \quad (1\text{-}28)
$$

Expanding equation (1-27) yields

$$E_z = \frac{I_i e^{jwt}}{4\pi\epsilon_0 C}\left[\frac{e^{j\beta(\ell_H - r_H)}}{r_H} - \frac{e^{-j\beta(\ell_H + r_H)}}{r_H} - \frac{e^{j\beta(\ell_o - r)}}{r_o} + \frac{e^{-j\beta(\ell_o + r_o)}}{r_o}\right]$$

$$= \frac{I_i e^{jwt}}{4\pi\epsilon_0 C}\left[\frac{e^{-j\beta r_H}}{r_H}(e^{j\beta\ell_H} - e^{-j\beta\ell_H}) + \frac{e^{-j\beta r_o}}{r_o}(e^{-j\beta\ell_o} - e^{j\beta\ell_o})\right]$$

or

$$
\left.
\begin{aligned}
&E_{zL_1} = \frac{jI_i e^{jwt}}{2\pi\epsilon_0 C}\left[\frac{e^{-j\beta r_H}}{r_H}\sin(\beta\ell_H) - \frac{e^{-j\beta r_o}}{r_o}\sin(\beta\ell_o)\right] \\
&r = \sqrt{x^2 + y^2 + (z-h)^2} \\
&\ell_H = \frac{\lambda}{8} + H \quad , \ell_o = \frac{\lambda}{8}
\end{aligned}
\right\} \quad (1\text{-}29)
$$

E_{zL_1} is the induced emf in z direction due to the current and charges on L_1. Now if the point P is brought to the surface of L_I or L_{III}, equation (1-29) represents the tangential field intensity at P due to time-changing current distributed on L_1.

For the field intensity due to current element L_3, it is necessary to consider the field due to the charges and the field due to the current separately.

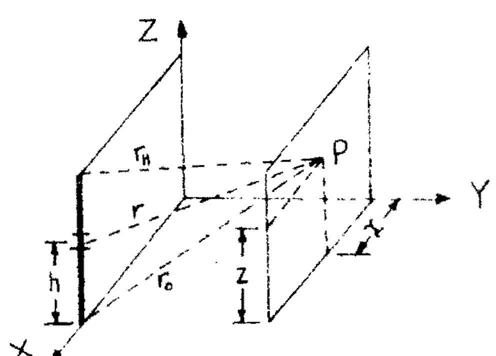

Fig. 1-7

Looking back to the equation (1-26), it is understood that the first integral was the result of the retarded charges, while the second integral was the result of the retarded current. Charges on the side L_3 were given by the equation (1-11) which is

$$\rho_L = \frac{2I_i\beta \sin\left[\beta\left(\frac{5\lambda}{8} + h\right)\right] e^{j\left(wt+\frac{\pi}{2}\right)}}{w}$$

where the positive h direction is in the negative z direction when the antenna is located as shown in fig. 1-7. Now if the positive h direction is changed to the positive z direction, the charge distribution on L_3 can be expressed as

$$\rho_L = \frac{2I_i\beta \sin\left[\beta\left(\frac{7\lambda}{8} - h\right)\right] e^{j\left(wt+\frac{\pi}{2}\right)}}{w} \qquad (1\text{-}30)$$

Hence, if $\ell = \frac{7\lambda}{8} - h$ the first integral of the equation (1-26) represents the field intensity due to the charges on the side L_3.

In Fig. 1-2, the current on the side i-g-f is flowing in the same direction as in b-c-d. However, in fig. 1-1, the current in i-g-f is flowing in the reverse direction to that in b-c-d. Hence equation (1-4) requires a sign change when it is used in conjunction with L_3.

$$I(\ell) = 2I_i \cos\left[\beta\left(\frac{5\lambda}{8} + h\right)\right] e^{jwt}$$

where the positive "h" direction is in the negative z direction. Now if the positive "h" direction is changed to the positive z direction, the current expression on L_3 can be expressed as

$$I(\ell) = 2I_i \cos\left[\beta\left(\frac{7\lambda}{8} - h\right)\right] e^{jwt} \tag{1-31}$$

For the z direction field intensity due to the current element L_3, equation (1-26) can be used if

$$\ell = \frac{7\lambda}{8} - h$$

and the sign of the second integral is changed

$$E_z = \frac{I_i e^{jwt}}{4\pi\epsilon_0 C}\left[\int_0^H \frac{\partial}{\partial z}\, \frac{e^{-j\beta(\ell+r)} - e^{j\beta(\ell-r)}}{r}\, dh - j\beta\int_0^H \frac{e^{-j\beta(\ell+r)} + e^{j\beta(\ell-r)}}{r}\, dh\right] \tag{1-32}$$

It can be proved that [4]

$$E_z = \frac{I_i e^{jwt}}{4\pi\epsilon_0 C}\left[\frac{e^{j\beta(\ell-r)}}{r} - \frac{e^{-j\beta(\ell+r)}}{r}\right]_0^H$$

$$= \frac{I_i e^{jwt}}{4\pi\epsilon_0 C}\left\{\left[\frac{e^{j\beta(\ell_H - r_H)}}{r_H} - \frac{e^{-j\beta(\ell_H + r_H)}}{r_H}\right] - \left[\frac{e^{j\beta(\ell_o - r_o)}}{r_o} - \frac{e^{-j\beta(\ell_o + r_o)}}{r_o}\right]\right\}$$

$$= \frac{jI_i e^{jwt}}{2\pi\epsilon_0 C}\left[\frac{e^{-j\beta r_H}}{r_H}\sin(\beta\ell_H) - \frac{e^{-j\beta r_o}}{r_o}\sin(\beta\ell_o)\right]$$

or

$$E_{zL_3} = \frac{jI_i e^{jwt}}{2\pi\epsilon_0 C}\left[\frac{e^{-j\beta r_H}}{r_H}\sin(\beta\ell_H) - \frac{e^{-j\beta r_o}}{r_o}\sin(\beta\ell_o)\right]$$

$$\text{where } r = \sqrt{(x-H)^2 + y^2 + (z-h)^2} \tag{1-33}$$

$$\ell = \frac{7\lambda}{8} - h$$

E_{zL_3} is the induced emf in the z direction due to the current and charges on L_3.

If point P is brought to the surface of L_I or L_{III}, equation (1-33) represents the induced emf on L_I or L_{III} due to the charges and current on L_3.

---

[4] Appendix I

1-6 The Induced EMF in X Direction Due to the Current Element in X Direction.

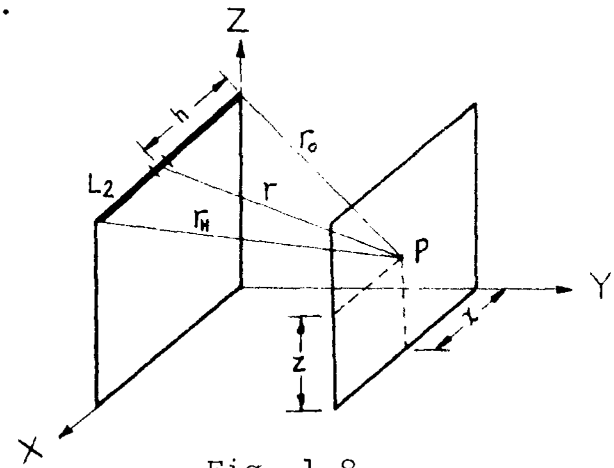

Fig. 1-8

The induced emf at point P has been given by

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$$

Particularly, if only the x component of the electric field is required

$$E_x = -\frac{\partial V}{\partial x} - \frac{\partial A_x}{\partial t} \qquad (1\text{-}34)$$

Exactly following the derivation of the last section, equation (1-34) can be expressed in the following form

$$E_x = \frac{I_i e^{jwt}}{4\pi\epsilon_0 C}\left\{\int_0^H \frac{\partial}{\partial x}\left[\frac{e^{-j\beta(\ell+r)} - e^{j\beta(\ell-r)}}{r}\right] dh + j\beta\int_0^H \frac{e^{-j\beta(\ell-r)} + e^{+j\beta(\ell-r)}}{r}\, dh\right\} \qquad (1\text{-}35)$$

Where the first integral is the result of the retarded scalar potential and the second integral is the result of the retarded vector potential. As was shown in § 1-5, equation (1-35) becomes:

$$\left.\begin{aligned} E_{xL_2} &= \frac{jI_i e^{jwt}}{2\pi\epsilon_o C}\left[\frac{e^{-j\beta r_H}}{r_H}\sin(\beta\ell_H) - \frac{e^{-j\beta r_o}}{r_o}\sin(\beta\ell_o)\right] \\ \text{where } r &= \sqrt{(x-h)^2 + y^2 + (z-H)^2} \\ \ell_H &= \frac{3}{8}\lambda + H \;;\; \ell_o = \frac{3}{8}\lambda \end{aligned}\right\} \quad (1\text{-}36)$$

If the point P is brought to the surface of L_{II} or L_{IV}, equation (1-36) represents the tangential induced emf on L_{II} or L_{IV} due to the current and charges on L_2.

For the field intensity due to the current element L_4 equation (1-35) can be used, but requires some changes. Charges on the side L_4 were given by equation (1-11) which is

$$\rho_L = \frac{2I_i\beta\sin\left[\beta\left(\frac{7\lambda}{8} + h\right)\right]e^{j(wt+\frac{\pi}{2})}}{w}$$

Where the positive h direction is in the negative x direction. If the positive h direction is changed to the positive x direction, the charge distribution on L_4 can be expressed as

$$\rho_L = \frac{2jI_i\beta\sin\left[\beta\left(\frac{\lambda}{8} - h\right)\right]e^{jwt}}{w}$$

Hence, if

$$\ell = \frac{\lambda}{8} - h$$

the first integral of equation (1-35) represents the x direction field intensity due to the charges on L_4.

In fig. 1-2 it was shown that the current on sides d-e-f and b-a-i flow in the opposite directions; while in fig. 1-1 the currents flow in the same direction. Hence, the expression for current on L_4 must be changed in sign.

$$I(\ell) = 2I_i \cos\left[\beta\left(\frac{7\lambda}{8} + h\right)\right] e^{jwt}$$

where the positive h direction is in the negative x direction. If the positive h direction is changed to the positive x direction, the current expression on L_4 can be expressed as

$$I(\ell) = 2I_i \cos\left[\beta\left(\frac{\lambda}{8} - h\right)\right] e^{jwt}$$

If the sign of the second integral of equation (1-35) is changed and ℓ is specified as

$$\ell = \frac{\lambda}{8} - h$$

The second integral of equation (1-35) represents the x direction field intensity due to the current on L_4. The x direction field intensity due to the current element L_4 will be

$$E_{xL_4} = \frac{Iie^{jwt}}{4\pi\epsilon_0 C}\left\{\int_0^H \frac{\partial}{\partial x}\left[\frac{e^{-j\beta(\ell+r)} - e^{j\beta(\ell-r)}}{r}\right] dh - j\beta \int_0^H \frac{e^{-j\beta(\ell+r)} + e^{j\beta(\ell-r)}}{r} dh\right\} \quad (1-37)$$

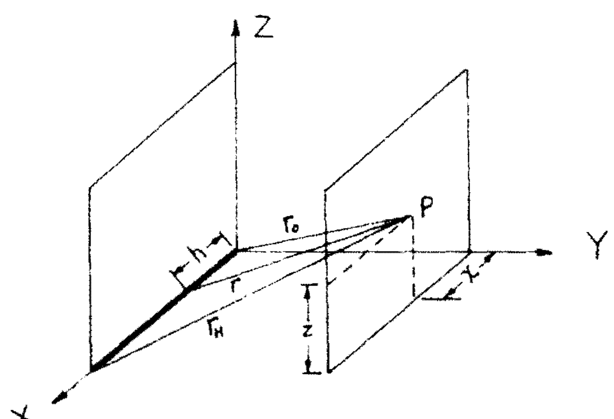

Fig. 1-9

Observing that equations (1-32) and (1-37) are of the same form. E_{xL_4} can be written:

$$E_{xL4} = \frac{jI_ie^{jwt}}{2\pi\epsilon_0 C}\left[\frac{e^{-j\beta r_H}}{r_H}\sin(\beta\ell_H) - \frac{e^{-j\beta r_o}}{r_o}\sin(\beta\ell_o)\right]$$

$$\text{where } r = \sqrt{(x-h)^2 + y^2 + z^2} \qquad (1-38)$$

$$\text{and} \quad \ell = \frac{\lambda}{8} - h$$

If the point P is brought to the surface of L_{II} or L_{IV} the equation (1-38) represents the tangential induced emf on L_{II} or L_{IV}.

1-7 The Induced EMF in the x Direction at Point P Due to Charges Distributed along the z direction.

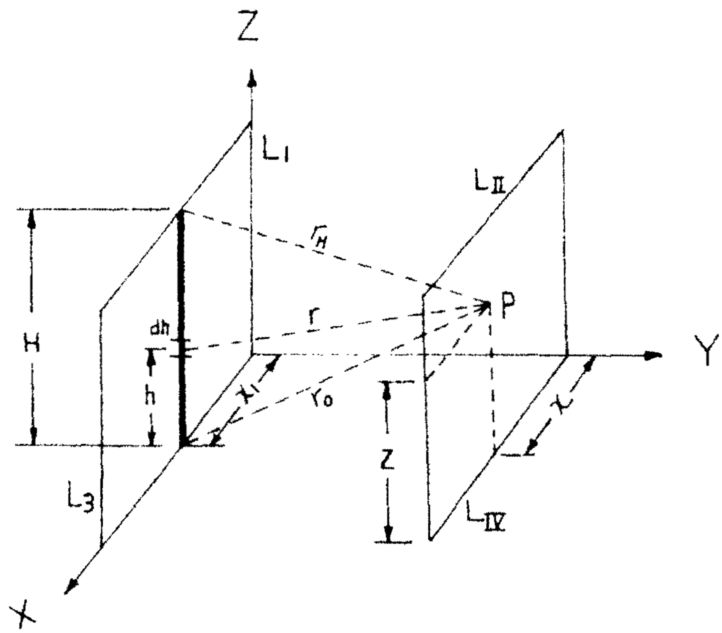

Fig. 1-10

The induced emf at a point P is given by equation (1-19). In rectangular coordinates:

$$\vec{E} = -(\vec{a}_x \frac{\partial v}{\partial x} + \vec{a}_y \frac{\partial v}{\partial y} + \vec{a}_z \frac{\partial v}{\partial z}) - \frac{\partial}{\partial t}(\vec{a}_x A_x + \vec{a}_y A_y + \vec{a}_z A_z)$$

If only the x direction field intensity is required,

$$E_x = -\frac{\partial v}{\partial x} - \frac{\partial A_x}{\partial t}$$

$A_x = 0$, since the current element is in the z direction.

Hence $$E_x = -\frac{\partial v}{\partial x} \tag{1-39}$$

Equation (1-39) shows that the x direction field intensity at P is a function of only the charges on the current element. Introducing equation (1-13) into equation (1-39) with the limits of the integration from h=o to h=H yields.

$$E_x = -\frac{j I_i e^{jwt}}{2\pi \epsilon_0 C} \int_0^H \frac{\partial}{\partial x}\left[\frac{\sin\left[\beta(\frac{n\lambda}{8} + h)\right]}{r} e^{-j\beta r}\right] dh$$

$$= \frac{-j I_i e^{jwt}}{2\pi \epsilon_0 C} \int_0^H \sin\left[\beta(\frac{n\lambda}{8} + h)\right] \frac{\partial}{\partial x}\left(\frac{e^{-j\beta r}}{r}\right) dh \tag{1-40}$$

where $r = \sqrt{(x-x_1)^2 + y^2 + (z-h)^2}$

$$\frac{\partial}{\partial x}\left(\frac{e^{-j\beta r}}{r}\right) = -\frac{j\beta(x-x_1) e^{-j\beta r}}{r^2} - \frac{(x-x_1) e^{-j\beta r}}{r^3} \tag{1-41}$$

$$\sin\left[\beta(\frac{n\lambda}{8} + h)\right] = \sin(\beta \ell)$$

$$= \frac{e^{j\beta\ell} - e^{-j\beta\ell}}{2j} \tag{1-42}$$

Substituting equations (1-41) and (1-42) into equation (1-40)

$$E_x = \frac{I_i e^{jwt}}{4\pi \epsilon_0 C} \int_0^H (e^{j\beta\ell} - e^{-j\beta\ell}) \left[\frac{j\beta(x-x_1)}{r^2} + \frac{(x-x_1)}{r^3}\right] e^{-j\beta r}\, dh$$

$$E_x = \frac{I_i e^{jwt}}{4\pi\epsilon_0 C}(x-x_1)\left[\int_0^H e^{j\beta(\ell-r)}\left(\frac{j\beta}{r^2}+\frac{1}{r^3}\right)dh - \int_0^H e^{-j\beta(\ell+r)}\left(\frac{j\beta}{4^2}+\frac{1}{r^3}\right)dh\right]$$

$$= \frac{I_i e^{jwt}}{4\pi\epsilon_0 C}(x-x_1)\left[e^{j\beta\frac{n\lambda}{8}}\int_0^H\left(\frac{j\beta}{r^2}+\frac{1}{r^3}\right)e^{j\beta(h-r)}\,dh\right.$$

$$\left.-e^{-j\beta\frac{n\lambda}{8}}\int_0^H\left(\frac{j\beta}{r^2}+\frac{1}{r^3}\right)e^{-j\beta(h+r)}\,dh\right] \qquad (1\text{-}43)$$

Equation (1-43) represents the x direction field intensity at P due to time-changing charges distributed on the current element of length H set in the z direction.

The first integrand of equation (1-43) turns out to be a perfect differential of the form[5]:

$$\frac{d}{dh}\,\frac{e^{j\beta(h-r)}}{r(r-h+z)} \qquad (1\text{-}44)$$

Also the second integrand of equation (1-43) turns out to be a perfect differential of the form[6]:

$$\frac{d}{dh}\left[-\frac{e^{-j\beta(h+r)}}{r(r+h-z)}\right] \qquad (1\text{-}45)$$

Thus equation (1-43) becomes

$$E_x = \frac{I_i e^{jwt}}{4\pi\epsilon_0 C}(x-x_1)\left\{e^{j\beta\frac{n\lambda}{8}}\left[\frac{e^{j\beta(H-r_H)}}{r_H(r_H-H+z)} - \frac{e^{j\beta r_o}}{r_o(r_o+z)}\right]\right.$$

$$\left.+\,e^{-j\beta\frac{n\lambda}{8}}\left[\frac{e^{-j\beta(H+r_H)}}{r_H(r_H+H-z)} - \frac{e^{-j\beta r_o}}{r_o(r_o-z)}\right]\right\} \qquad (1\text{-}46)$$

referring to fig. 1-9

$$r_H = \sqrt{(x-x_1)^2 + y^2 + (z-H)^2}$$

$$r_o = \sqrt{(x-x_1)^2 + y^2 + z^2} \qquad (1\text{-}47)$$

5,6, Appendix II

Now if the current element is brought to coincide with L_1, in equation (1-46), n and x_1 must be:

$$n = 1$$

$$x_1 = 0$$

$$E_{xL_1} = \frac{Ii e^{jwt}(x)}{4\pi\epsilon_0 C}\left\{ e^{j\beta\frac{\lambda}{8}}\left[\frac{e^{j\beta(H-r_H)}}{r_H(r_H-H+z)} - \frac{e^{-j\beta r_0}}{r_o(r_o+z)}\right] + e^{-j\beta\frac{\lambda}{8}}\left[\frac{e^{-j\beta(H+r_H)}}{r_H(r_H+H-z)} - \frac{e^{-j\beta r_o}}{r_o(r_o-z)}\right]\right\} \quad (1\text{-}48)$$

where $r = \sqrt{x^2 + y^2 + (z-h)^2}$

If the point P is brought to the surface of L_{II} or L_{IV} of the reflecting loop, equation (1-48) represents the tangential induced emf on L_{II} or L_{IV} due to charges on L_1.

If the current element is brought to coincide with L_3 as shown in fig. 1-11 equation (1-43) is still valid, but x_1 and r must be changed to

$$x_1 = H$$

$$r = \sqrt{(x-H)^2 + y^2 + (z-h)^2}$$

and ℓ must be changed to

$$\ell = \frac{7\lambda}{8} - h$$

as was stated in section 1-5.

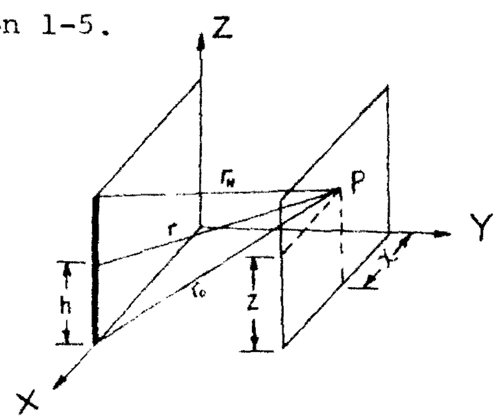

Fig. 1-11

Hence equation (1-43) becomes:

$$E_{xL_3} = \frac{I_i e^{jwt}}{4\pi\epsilon_0 C}(x-H)\left\{ e^{j\beta\frac{7\lambda}{8}}\int_0^H\left(\frac{j\beta}{r^2}+\frac{1}{r^3}\right)e^{-j\beta(h+r)}dh - e^{-j\beta\frac{7\lambda}{8}}\int_0^H\left(\frac{j\beta}{r^2}+\frac{1}{r^3}\right)e^{j\beta(h-r)}dh\right\} \quad (1\text{-}43a)$$

The first integrand of the equation (1-43a) turns out to to be a perfect differential of the form[7]:

$$\frac{d}{dh}\left[-\frac{e^{j\beta(h+r)}}{r(r+h-z)}\right] \quad (1\text{-}49)$$

and the second integrand of the equation (1-43a) turns out to be a perfect differential of the form[8]:

$$\frac{d}{dh}\left[\frac{e^{j\beta(h-r)}}{r(r-h+z)}\right] \quad (1\text{-}50)$$

Thus equation (1-43a) becomes

$$E_x = \frac{I_i e^{jwt}}{4\pi\epsilon_0 C}(x-H)\left\{ e^{j\beta\frac{7\lambda}{8}}\left[-\frac{e^{-j\beta(h+r)}}{r(r+h-z)}\right]_0^H - e^{-j\beta\frac{7\lambda}{8}}\left[\frac{e^{j\beta(h-r)}}{r(r-h+z)}\right]_0^H\right\}$$

or

$$E_{xL_3} = \frac{I_i e^{jwt}}{4\pi\epsilon_0 C}(x-H)\left\{ e^{j\beta\frac{7\lambda}{8}}\left[-\frac{e^{-j\beta(H+r_H)}}{r_H(r_H+H-z)}+\frac{e^{-j\beta r_o}}{r(r-h+z)}\right] + e^{-j\beta\frac{7\lambda}{8}}\left[-\frac{e^{j\beta(H-r_H)}}{r_H(r_H-H+z)}+\frac{e^{-j\beta r_o}}{r_o(r_o+z)}\right]\right\} \quad (1\text{-}51)$$

where $r = \sqrt{(x-H)^2 + y^2 + (z-h)^2}$

If the point P is brought to the surface of L_{II} or L_{IV},

---

7,8 Appendix II

the equation (1-51) represents the tangential induced emf on L_{II} or L_{IV} due to the charges on L_3.

1-8 <u>The Induced EMF in the z Direction at Point P Due to Charges Distributed Along the x Direction</u>

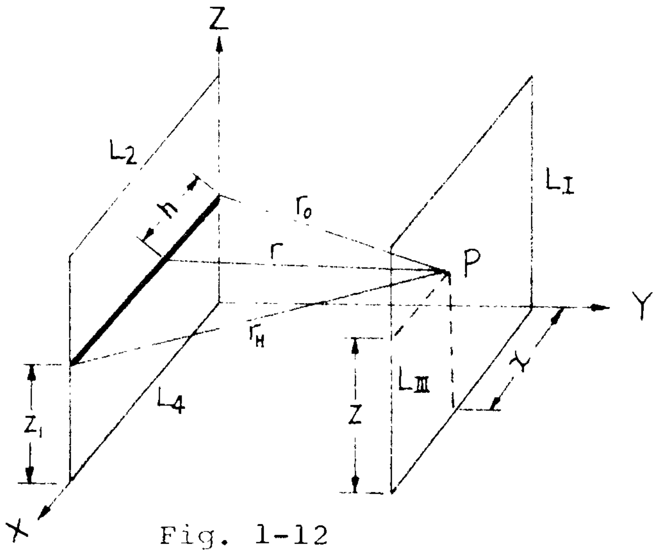

Fig. 1-12

The induced emf at point P is given by

$$\vec{E} = -\nabla V - \frac{\partial \vec{A}}{\partial t}$$

$$= -\left(\vec{a}_x \frac{\partial V}{\partial x} + \vec{a}_y \frac{\partial V}{\partial y} + \vec{a}_z \frac{\partial V}{\partial z}\right) - \frac{\partial}{\partial t}\left(\vec{a}_x A_x + \vec{a}_y A_y + \vec{a}_z A_z\right)$$

If only the z direction field intensity is required

$$E_z = -\frac{\partial V}{\partial z} - \frac{\partial A_z}{\partial t}$$

Since $A_z = o$

$$\therefore \qquad E_z = -\frac{\partial V}{\partial z} \tag{1-52}$$

Equation (1-52) shows that the z direction field intensity is caused by charges only. Introducing equation (1-13) into equation (1-52)

$$E_z = -\frac{jI_i e^{jwt}}{2\pi\epsilon_0 C}\int \sin\left[\beta\left(\frac{n\lambda}{8}+h\right)\right]\frac{\partial}{\partial z}\left(\frac{e^{-j\beta r}}{r}\right)dh \tag{1-53}$$

where $r = \sqrt{(x-h)^2 + y^2 + (z-z_1)^2}$

$$\frac{\partial}{\partial z}\left(\frac{e^{-j\beta r}}{r}\right) = -\frac{j\beta(Z-Z_1)e^{-j\beta r}}{r^2} - \frac{(z-z_1)e^{-j\beta r}}{r^3} \tag{1-54}$$

$$\sin\left[\beta\left(\frac{n\lambda}{8}+h\right)\right] = \sin(\beta\ell) = \frac{e^{j\beta\ell} - e^{-j\beta\ell}}{2j} \tag{1-55}$$

Substituting equations (1-54) and (1-55) into equation (1-53) yields

$$E_z = \frac{I_i e^{jwt}}{4\pi\epsilon_0 C}\int (e^{j\beta\ell} - e^{-j\beta\ell})\left(\frac{j\beta(z-z_1)e^{-j\beta r}}{r^2} + \frac{(Z-Z_1)e^{-j\beta r}}{r^3}\right)dh$$

$$= \frac{I_i e^{jwt}}{4\pi\epsilon_0 C}(z-z_1)\left[\int e^{j\beta(\ell-r)}\left(\frac{j\beta}{r^2}+\frac{1}{r^3}\right)dh - \int e^{-j\beta(\ell+r)}\left(\frac{j\beta}{r^2}+\frac{1}{r^3}\right)dh\right] \tag{1-56}$$

The field intensity due to the time-changing charges distributed on the current element of length H will be

$$E_z = \frac{I_i e^{jwt}}{4\pi\epsilon_0 C}(z-z_1)\left\{e^{j\beta\frac{n\lambda}{8}}\int_0^H\left(\frac{j\beta}{r^2}+\frac{1}{r^3}\right)e^{j\beta(h-r)}dh - e^{-j\beta\frac{n\lambda}{8}}\int_0^H\left(\frac{j\beta}{r^2}+\frac{1}{r^3}\right)e^{-j\beta(h+r)}dh\right\} \tag{1-57}$$

The first and the second integrands of equation (1-57) are two perfect differentials as shown in section 1-6 Hence, equation (1-57) becomes

$$E_z = \frac{I_i e^{jwt}}{4\pi\epsilon_o C}(z-z_1)\left\{ e^{j\beta\frac{n\lambda}{8}}\left[\frac{e^{j\beta(h-r)}}{r(r-h+x)}\right]_o^H + e^{-j\beta\frac{n\lambda}{8}}\left[\frac{e^{-j\beta(h+r)}}{r(r+h-x)}\right]_o^H \right\}$$

$$= \frac{I_i e^{jwt}}{4\pi\epsilon_o C}(z-z_1)\left\{ e^{j\beta\frac{n\lambda}{8}}\left[\frac{e^{j\beta(H-r_H)}}{r_H(r_H-H+x)} - \frac{e^{-j\beta r_o}}{r_o(r_o+x)}\right] + \left[\frac{e^{-j\beta(H+r_H)}}{r_H(r_H+H-x)} - \frac{e^{-j\beta r_o}}{r_o(r_o-x)}\right] e^{-j\beta\frac{n\lambda}{8}} \right\} \quad (1-58)$$

Now if the current element is brought to coincide with L_2 and the point P is brought to the surface of L_I or L_{III}, equation (1-58) becomes

$$E_{zL_2} = \frac{I_i e^{jwt}}{4\pi\epsilon_o C}(z-H)\left\{ e^{j\beta\frac{3\lambda}{8}}\left[\frac{e^{j\beta(H-r_H)}}{r_H(r_H-H+x)} - \frac{e^{-j\beta r_o}}{r_o(r_o+x)}\right] + \left[\frac{e^{-j\beta(H+r_H)}}{r_H(r_H+H-x)} - \frac{e^{-j\beta r_o}}{r_o(r_o-x)}\right] e^{-j\beta\frac{3\lambda}{8}} \right\} \quad (1-59)$$

where $r = \sqrt{(x-h)^2 + y^2 + (z-H)^2}$

Equation (1-59) represents the tangential induced emf on L_I or L_{III} due to the charges distributed along L_2.

Now if the current element is brought to coincide with L_4, as shown in fig. 1-9, equation (1-57) is still valid, but r and z_1 must be specified to

$$r = \sqrt{(x-h)^2 + y^2 + z^2} \quad ; \quad z_1 = o$$

and ℓ must be changed to

$$\ell = \frac{\lambda}{8} - h$$

as was stated in section 1-6. Substituting these conditions into equation (1-57) yields

$$E_{zL_4} = \frac{I_i e^{jwt}}{4\pi\epsilon_o C}(z)\left\{ e^{j\beta\frac{\lambda}{8}}\int_0^H \left(\frac{j\beta}{r^2} + \frac{1}{r^3}\right) e^{-j\beta(h+r)}\, dh - e^{-j\beta\frac{\lambda}{8}}\int_0^H \left(\frac{j\beta}{r^2} + \frac{1}{r^3}\right) e^{j\beta(h-r)}\, dh \right\} \quad (1-60)$$

$$E_{zL_4} = \frac{I_i e^{jwt}}{4\pi\epsilon_0 C}(z)\left\{e^{j\beta\frac{\lambda}{8}}\left[-\frac{e^{-j\beta(h+r)}}{r(r+h-x)}\right]_0^H - e^{-j\beta\frac{\lambda}{8}}\left[\frac{e^{j\beta(h-r)}}{r(r-h+x)}\right]_0^H\right\} \quad \text{---- (1-61)}$$

$$E_{zL_4} = \frac{I_i e^{jwt}}{4\pi\epsilon_0 C}(z)\left\{e^{j\beta\frac{\lambda}{8}}\left[-\frac{e^{-j\beta(H+r_H)}}{r_H(r_H+H-x)} + \frac{e^{-j\beta r_o}}{r_o(r_o-x)}\right]\right.$$
$$\left. + e^{-j\beta\frac{\lambda}{8}}\left[-\frac{e^{j\beta(H-r_H)}}{r_H(r_H-H+x)} + \frac{e^{-j\beta r_o}}{r_o(r_o+x)}\right]\right\} \qquad (1\text{-}62)$$

If the point P is brought to the surface of L_I or L_{III}, equation (1-62) represents the tangential induced emf on L_I or L_{III} due to the charges on L_4.

1-9 The Total Tangential Induced EMF at a Point P on L_I

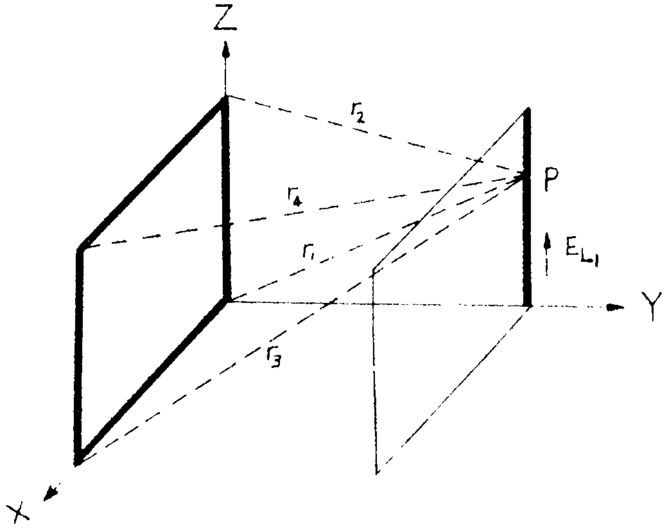

Fig. 1-13

Referring to fig. 1-13, the distances r_1, r_2, r_3 and r_4 are given by the following equations:

$$r_1 = \sqrt{y^2 + z^2} \qquad r_2 = \sqrt{y^2 + (z-H)^2}$$
$$r_3 = \sqrt{H^2 + y^2 + z^2} \qquad r_4 = \sqrt{H^2 + y^2 + (H-z)^2} \qquad (1\text{-}63)$$

where $H = \frac{\lambda}{4}$ for the cubical quad antenna.

Define $E_{I_I L_1}$ = the induced emf on the side L_I due to the charges and the current on L_1

$E_{L_I L_2}$ = the induced emf on the side L_I due to the charges on L_2

$E_{L_I L_3}$ = the induced emf on the side L_I due to the charges and the current on L_3

$E_{L_I L_4}$ = the induced emf on the side L_I due to the charges on L_4

The expression for $E_{L_I L_1}$ is given by equation (1-29). In this case $x = o$, $r_H = r_2$, $r_o = r_1$

$$\therefore \quad E_{L_I L_1} = \frac{jI_i e^{jwt}}{2\pi\epsilon_o C}\left[\frac{\sin\left[(\beta\frac{3\lambda}{8})\right]\ e^{-j\beta r_2}}{r_2} - \frac{\sin(\beta\frac{\lambda}{8})\ e^{-j\beta r_1}}{r_1}\right]$$

Separating the real and the imaginary parts

$$E_{L_I L_1} = \frac{jI_i e^{jwt}}{2\pi\epsilon_o C}\left\{\frac{\sin(\beta\frac{3\lambda}{8})\cos(\beta r_2)}{r_2} - \frac{\sin(\frac{\beta\lambda}{8})\cos(\beta r_1)}{r_1}\right.$$

$$\left. +j\left[\frac{-\sin(\beta\frac{3\lambda}{8})\sin(\beta r_2)}{r_2} + \frac{\sin(\frac{\beta\lambda}{8})\sin(\beta r_1)}{r_1}\right]\right\}$$

$$= \frac{I_i e^{jwt}}{2\pi\epsilon_o C}\left\{\frac{(\sin(\beta\frac{3\lambda}{8})\ \sin(\beta r_2)}{r_2} - \frac{\sin(\frac{\beta\lambda}{8})\sin(\beta r_1)}{r_1}\right.$$

$$\left. + j\left[\frac{\sin(\beta\frac{3\lambda}{8})\cos(\beta r_2)}{r_2} - \frac{\sin(\frac{\beta\lambda}{8})\cos(\beta r_1)}{r_1}\right]\right\} \quad (1\text{-}64)$$

The expression for $E_{L_I L_3}$ is given by equation (1-33).

In this case $x = o$, $r_H = r_4$, $r_o = r_3$

$$E_{L_I L_3} = \frac{jI_i e^{jwt}}{2\pi\epsilon_o C}\left[\frac{e^{-j\beta r_4}}{r_4}\sin(\beta\frac{5\lambda}{8}) - \frac{e^{-j\beta r_3}}{r_3}\sin(\beta\frac{7\lambda}{8})\right)$$

Separating the real and the imaginary parts

$$E_{L_I L_3} = \frac{jI_i e^{jwt}}{2\pi\epsilon_0 C}\left\{\frac{\sin(\frac{\beta 5\lambda}{8})\left[\cos(\beta r_4) - j\sin(\beta r_4)\right]}{r_4} - \frac{\sin(\beta\frac{7\lambda}{8})}{r_3}\left[\cos(\beta r_3) - j\sin(\beta r_3)\right]\right\}$$

$$= \frac{jI_i e^{jwt}}{2\pi\epsilon_0 C}\left\{\left(\frac{\sin(\beta\frac{5\lambda}{8})\cos(\beta r_4)}{r_4} - \frac{\sin(\frac{\beta 7\lambda}{8})\cos(\beta r_3)}{r_3}\right) - j\left(\frac{\sin(\beta\frac{5\lambda}{8})\sin(\beta r_4)}{r_4} - \frac{\sin(\beta\frac{7\lambda}{8})\sin(\beta r_3)}{r_3}\right)\right\}$$

$$= \frac{I_i e^{jwt}}{2\pi\epsilon_0 C}\left[\left(\frac{\sin(\beta\frac{5\lambda}{8})\sin(\beta r_4)}{r_4} - \frac{\sin(\beta\frac{7\lambda}{8})\sin(\beta r_3)}{r_3}\right) + j\left(\frac{\sin(\frac{\beta 5\lambda}{8})\cos(\beta r_4)}{r_4} - \frac{\sin(\frac{\beta 7\lambda}{8})\cos(\beta r_3)}{r_3}\right)\right] \quad (1\text{-}65)$$

The expression for $E_{L_I L_2}$ is given by equation (1-59).

In this case

$$x = 0,\ r_H = r_4,\ r_o = r_2$$

$$E_{L_I L_2} = \frac{I_i e^{jwt}}{4\pi\epsilon_0 C}(z-H)\left\{e^{j\beta\frac{3\lambda}{8}}\left(\frac{e^{j\beta(H-r_4)}}{r_4(r_4-H)} - \frac{e^{-j\beta r_2}}{r_2^2}\right) + e^{-j\beta\frac{3\lambda}{8}}\left(\frac{e^{-j\beta(H+r_4)}}{r_4(r_4+H)} - \frac{e^{-j\beta r_2}}{r_2^2}\right)\right\}$$

$$= \frac{I_i e^{jwt}}{4\pi\epsilon_0 C}(z-H)\left[\frac{e^{j\beta(\frac{5\lambda}{8} - r_4)}}{r_4(r_4-H)} + \frac{e^{-j\beta(\frac{5\lambda}{8} + r_4)}}{r_4(r_4+H)} - \frac{2e^{-j\beta r_2}}{r_2^2}\cos(\beta\frac{3\lambda}{8})\right]$$

$$= \frac{I_i e^{jwt}}{4\pi\epsilon_0 C}(z-H)\left\{e^{-j\beta r_4}\left(\frac{e^{j\beta\frac{5\lambda}{8}}\left[r_4 + H\right] + e^{-j\beta\frac{5\lambda}{8}}\left[r_4 - H\right]}{r_4(r_4^2 - H^2)}\right) - \frac{2e^{-j\beta r_2}}{r_2^2}\cos\frac{3\beta\lambda}{8}\right\}$$

$$E_{L_IL_2} = \frac{I_i e^{jwt}}{4\pi\epsilon_o C}(x-H)\left\{e^{-j\beta r_4}\left(\frac{2r_4\cos(\beta\frac{5\lambda}{8}) + 2jH\sin(\beta\frac{5\lambda}{8})}{r_4(r_4^2 - H^2)}\right)\right.$$

$$\left. - \frac{2e^{-j\beta r_2}}{r_2^2}\cos\left(\frac{3\beta\lambda}{8}\right)\right\}$$

$$= \frac{I_i e^{jwt}}{2\pi\epsilon_o C}(x-H)\left\{\left(\frac{r_4\cos(\beta r_4)\cos(\frac{5\beta\lambda}{8}) + H\sin(\beta r_4)\sin(\frac{5\beta\lambda}{8})}{r_4(r_4^2 - H^2)}\right.\right.$$

$$\left. - \frac{\cos(\frac{3\beta\lambda}{8})\sin(\beta r_2)}{r_2^2}\right)$$

$$+j\left(\frac{H\sin(\frac{5\beta\lambda}{8})\cos(\beta r_4) - r_4\cos(\frac{5\beta\lambda}{8})\sin(\beta r_4)}{r_4(r_4^2 - H^2)}\right.$$

$$\left.\left. + \frac{\cos(\frac{3\beta\lambda}{8})\sin(\beta r_2)}{r_2^2}\right)\right\} \qquad (1-66)$$

The expression for $E_{L_IL_4}$ is given by equation (1-62).

In this case

$$x - o, \quad r_H = r_3, \quad r_o = r_1$$

$$E_{L_IL_4} = \frac{I_i e^{jwt}}{4\pi\epsilon_o C}(x)\left\{e^{j\beta\frac{\lambda}{8}}\left[-\frac{e^{-j\beta(H+r_3)}}{r_3(r_3+H)} + \frac{e^{-j\beta r_1}}{r_1^2}\right]\right.$$

$$\left. + e^{-j\beta\frac{\lambda}{8}}\left[-\frac{e^{j\beta(H-r_3)}}{r_3(r_3-H)} + \frac{e^{-j\beta r_1}}{r_1^2}\right]\right\}$$

$$= \frac{I_i e^{jwt}}{4\pi\epsilon_o C}(x)\left\{-\left(\frac{e^{-j\beta(r_3+\frac{\lambda}{8})}}{r_3(r_3+H)} + \frac{e^{j\beta(\frac{\lambda}{8}-r_3)}}{r_3(r_3-H)}\right)\right.$$

$$\left. + \frac{2e^{-j\beta r_1}}{r_1^2}\cos\left(\frac{\beta\lambda}{8}\right)\right\}$$

$$
\begin{aligned}
E_{L_I L_4} &= \frac{I_i e^{jwt}}{4\pi\epsilon_0 C}(z)\left\{-e^{-j\beta r_3}\left[\frac{(r_3-H)e^{-j\beta\frac{\lambda}{8}} + (r_3+H)e^{j\beta\frac{\lambda}{8}} -}{r_3(r_3^2-H^2)}\right.\right. \\
&\qquad \left.\left. + \frac{2e^{-j\beta r_1}}{r_1^2}\cos\left(\beta\frac{\lambda}{8}\right)\right]\right\} \\
&= \frac{I_i e^{jwt}}{4\pi\epsilon_0 C}(z)\left\{-e^{-j\beta r_3}\,\frac{2r_3\cos\left(\frac{\beta\lambda}{8}\right) + 2Hj\sin\left(\frac{\beta\lambda}{8}\right)}{r_3(r_3^2-H^2)}\right. \\
&\qquad \left. + \frac{2e^{-j\beta r_1}}{r_1^2}\cos\left(\beta\frac{\lambda}{8}\right)\right\} \\
&= \frac{I_i e^{jwt}}{2\pi\epsilon_0 C}(z)\left\{-\frac{r_3\cos(\beta r_3)\cos\left(\frac{\beta\lambda}{8}\right) + H\sin(\beta r_3)\sin\left(\frac{\beta\lambda}{8}\right)}{r_3(r_3^2-H^2)}\right. \\
&\qquad + \frac{\cos(\beta r_1)\cos\left(\frac{\beta\lambda}{8}\right)}{r_1^2} \\
&\qquad + j\left[-\frac{r_3\sin(\beta r_3)\cos\left(\frac{\beta\lambda}{8}\right) - H\cos(\beta r_3)\sin\left(\beta\frac{\lambda}{8}\right)}{r_3(r_3^2-H^2)}\right. \\
&\qquad \left.\left. - \frac{\sin(\beta r_1)\cos\left(\beta\frac{\lambda}{8}\right)}{r_2^2}\right]\right\} \qquad (1\text{-}67)
\end{aligned}
$$

E_{L_I} = the total tangential induced emf at a point P on L_I

$$= E_{L_I L_1} + E_{L_I L_2} + E_{L_I L_3} + E_{L_I L_4} \qquad (1\text{-}68)$$

1-10 <u>The Total Tangential Induced EMF at a Point P on L_{II}</u>

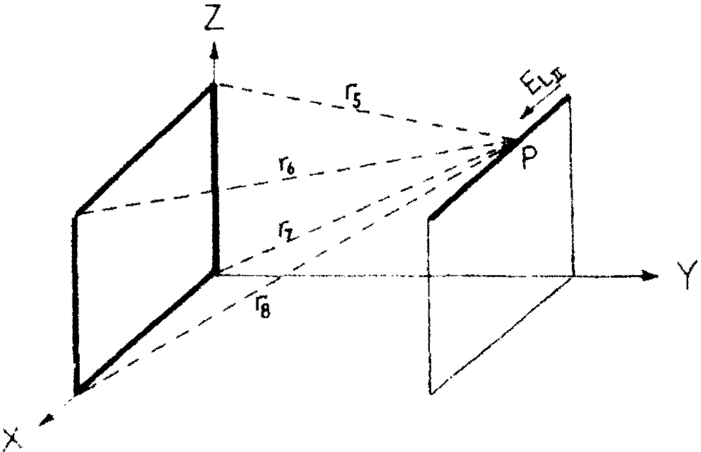

Fig. 1-14

Referring to fig. 1-14, the distances r_5, r_6, r_7 and r_8 are given by the following equations.

$$r_5 = \sqrt{x^2 + y^2} \qquad r_6 = \sqrt{(H-x)^2 + y^2}$$

$$r_7 = \sqrt{x^2 + y^2 + H^2} \qquad r_8 = \sqrt{(x-H)^2 + y^2 + H^2}$$

Define $E_{L_{II}L_1}$ = the tangential induced emf on L_{II} due to charges on L_1

$E_{L_{II}L_2}$ = the tangential induced emf on L_{II} due to charges and the current on L_2

$E_{L_{II}L_3}$ = the tangential induced emf on L_{II} due to charges on L_3

$E_{L_{II}L_4}$ = the tangential induced emf on L_{II} due to charges and the current on L_4

The expression for $E_{L_{II}L_1}$ is given by equation (1-48) where

$$Z = H, \quad r_H = r_5, \quad r_o = r_7$$

$$E_{L_{II}L_1} = \frac{I_i e^{jwt}}{4\pi\epsilon_o C}(x) \left\{ e^{j\beta\frac{\lambda}{8}} \left[\frac{e^{j\beta(H-r_5)}}{r_5^2} - \frac{e^{-j\beta r_7}}{r_7(r_7+H)} \right] + e^{-j\beta\frac{\lambda}{8}} \left[\frac{e^{-j\beta(H+r_5)}}{r_5^2} - \frac{e^{-j\beta r_7}}{r_7(r_7-H)} \right] \right\}$$

Separating the real and the imaginary parts

$$E_{L_{II}L_1} = \frac{I_i e^{jwt}}{4\pi\epsilon_o C}(x) \left\{ \frac{2e^{-j\beta r_5}}{r_5^2} \cos\left[\beta\left(\frac{\lambda}{8} + H\right)\right] - e^{-j\beta r_7} \left[\frac{2r_7 \cos(\beta\frac{\lambda}{8}) - 2jH \sin(\beta\frac{\lambda}{8})}{r_7(r_7^2 - H^2)} \right] \right\}$$

$$= \frac{I_i e^{jwt}}{2\pi\epsilon_o C}(x) \left\{ \frac{e^{-j\beta r_5}}{r_5^2} \cos\left(\beta\frac{3\lambda}{8}\right) - e^{-j\beta r_7} \left[\frac{r_7 \cos(\beta\frac{\lambda}{8}) - jH \sin(\beta\frac{\lambda}{8})}{r_7(r_7^2 - H^2)} \right] \right\}$$

$$E_{L_{II}L_1} = \frac{I_i e^{jwt}}{2\pi\epsilon_0 C}(x)\left\{\left(\frac{r_7\cos(\beta r_7)\cos(\beta\frac{\lambda}{8}) + H\sin(\beta r_7)\sin(\beta\frac{\lambda}{8})}{r_7(r_7^2 - H^2)} + \frac{\cos(\beta r_7)\cos(\beta\frac{3\lambda}{8})}{r_5^2}\right)\right.$$

$$\left. -j\left[\frac{r_7\sin(\beta r_7)\cos(\beta\frac{\lambda}{8}) + H\cos(\beta r_7)\sin(\beta\frac{\lambda}{8})}{r_7(r_7^2 - H^2)} - \frac{\sin(\beta r_5)\cos(\beta\frac{3\lambda}{8})}{r_5^2}\right]\right\} \quad (1\text{-}69)$$

The expression for $E_{L_{II}L_2}$ is given by equation (1-36) where

$Z = H,\ r_H = r_6,\ r_o = r_5$

$$E_{L_{II}L_2} = \frac{jI_i e^{jwt}}{2\pi\epsilon_0 C}\left\{\frac{e^{-j\beta r_6}}{r_6}\sin\left[\beta\left(\frac{3\lambda}{8}+H\right)\right] - \frac{e^{-j\beta r_5}}{r_5}\sin\left(\frac{3\lambda}{8}\beta\right)\right\}$$

$$= \frac{jI_i e^{jwt}}{2\pi\epsilon_0 C}\left\{\left(\frac{\sin(\beta\frac{5\lambda}{8})\cos(\beta r_6)}{r_6} - \frac{\sin(\beta\frac{3\lambda}{8})\cos(\beta r_5)}{r_5}\right) + j\left(\frac{-\sin(\beta\frac{5\lambda}{8})\sin(\beta r_6)}{r_6} + \frac{\sin(\beta\frac{3\lambda}{8})\sin(\beta r_5)}{r_5}\right)\right\}$$

$$= \frac{I_i e^{jwt}}{2\pi\epsilon_0 C}\left\{\left(\frac{\sin(\beta\frac{5\lambda}{8})\sin(\beta r_6)}{r_6} - \frac{\sin(\beta\frac{3\lambda}{8})\sin(\beta r_5)}{r_5}\right) + j\left(\frac{\sin(\beta\frac{5\lambda}{8})\cos(\beta r_6)}{r_6} - \frac{\sin(\beta\frac{3\lambda}{8})\cos(\beta r_5)}{r_5}\right)\right\} \quad (1\text{-}70)$$

The expression for $E_{L_{II}L_3}$ is given by equation (1-51) where

$Z = H,\ r_H = r_6,\ r_o = r_8,$

$$E_{L_{II}L_3} = \frac{I_i e^{jwt}}{4\pi\epsilon_o C}(x-H)\left\{ e^{j\beta\frac{7\lambda}{8}}\left[-\frac{e^{-j\beta(H+r_6)}}{r_6^2} + \frac{e^{-j\beta r_8}}{r_8(r_8-H)}\right] + e^{-j\beta\frac{7\lambda}{8}}\left[-\frac{e^{j\beta(H-r_6)}}{r_6^2} + \frac{e^{-j\beta r_8}}{r_8(r_8+H)}\right]\right\}$$

$$= \frac{I_i e^{jwt}}{4\pi\epsilon_o C}(x-H)\left\{ e^{-j\beta r_6}\left[-\frac{e^{j\beta\frac{5\lambda}{8}} + e^{-j\beta\frac{5\lambda}{8}}}{r_6^2}\right] + e^{-j\beta r_8}\left[\frac{(r_8+H)e^{j\beta\frac{7\lambda}{8}} + (r_8-H)e^{-j\beta\frac{7\lambda}{8}}}{r_8(r_8^2-H^2)}\right]\right\}$$

$$= \frac{I_i e^{jwt}}{4\pi\epsilon_o C}(x-H)\left\{ \frac{-2\cos(\frac{5\beta\lambda}{8})e^{-j\beta r_6}}{r_6^2} + e^{-j\beta r_8}\left(\frac{2r_8\cos(\beta\frac{7\lambda}{8}) + 2jH\sin(\beta\frac{7\lambda}{8})}{r_8(r_8^2-H^2)}\right)\right\}$$

$$= \frac{I_i e^{jwt}}{2\pi\epsilon_o C}(x-H)\left\{ -\frac{\cos(\beta\frac{5\lambda}{8})\cos(\beta r_6) - j\cos(\beta\frac{5\lambda}{8})\sin(\beta r_6)}{r_6^2} + \frac{r_8\cos(\beta\frac{7\lambda}{8})\cos(\beta r_8) + H\sin(\beta\frac{7\lambda}{8})\sin(\beta r_8)}{r_8(r_8^2-H^2)} + j\left(\frac{-r_8\cos(\beta\frac{7\lambda}{8})\sin(\beta r_8) + H\sin(\beta\frac{7\lambda}{8})\cos(\beta r_8)}{r_8(r_8^2-H^2)}\right)\right\}$$

$$= \frac{I_i e^{jwt}}{2\pi\epsilon_o C}(x-H)\left\{ -\frac{\cos(\beta\frac{5\lambda}{8})\cos(\beta r_6)}{r_6^2} + \frac{r_8\cos(\beta\frac{7\lambda}{8})\cos(\beta r_8) + H\sin(\beta\frac{7\lambda}{8})\sin(\beta r_8)}{r_8(r_8^2-H^2)} + j\left(\frac{\cos(\beta\frac{5\lambda}{8})\sin(\beta r_6)}{r_6^2} - \frac{r_8\cos(\beta\frac{7\lambda}{8})\sin(\beta r_8) - H\sin(\beta\frac{7\lambda}{8})\cos(\beta r_8)}{r_8(r_8^2-H^2)}\right)\right\} \qquad (1\text{-}71)$$

The expression for $\dot{E}_{L_{II}L_4}$ is given by equation (1-38) where

$$Z = H,\ r_H = r_8,\ r_o = r_7$$

$$E_{L_{II}L_4} = \frac{jI_i e^{jwt}}{2\pi\epsilon_o C}\left\{\frac{e^{-j\beta r_8}}{r_8}\sin\left(\frac{7\beta\lambda}{8}\right) - \frac{e^{-j\beta r_7}}{r_7}\sin\left(\frac{\beta\lambda}{8}\right)\right\}$$

$$= \frac{jI_i e^{jwt}}{2\pi\epsilon_o C}\left\{\frac{\sin\left(\frac{\beta 7\lambda}{8}\right)}{r_8}\left[\cos(\beta r_8) - j\sin(\beta r_8)\right] - \frac{\sin\left(\beta\frac{\lambda}{8}\right)}{r_7}\left[\cos(\beta r_7) - j\sin(\beta r_7)\right]\right\}$$

$$= \frac{jI_i e^{jwt}}{2\pi\epsilon_o C}\left\{\left(\frac{\sin\left(\beta\frac{7\lambda}{8}\right)\cos(\beta r_8)}{r_8} - \frac{\sin\left(\beta\frac{\lambda}{8}\right)\cos(\beta r_7)}{r_7}\right) - j\left(\frac{\sin\left(\beta\frac{7\lambda}{8}\right)\sin(\beta r_8)}{r_8} - \frac{\sin\left(\beta\frac{\lambda}{8}\right)\sin(\beta r_7)}{r_7}\right)\right\}$$

$$= \frac{I_i e^{jwt}}{2\pi\epsilon_o C}\left\{\frac{\sin\left(\beta\frac{7\lambda}{8}\right)\sin(\beta r_8)}{r_8} - \frac{\sin\left(\beta\frac{\lambda}{8}\right)\sin(\beta r_7)}{r_7} + j\left(\frac{\sin\left(\beta\frac{7\lambda}{8}\right)\cos(\beta r_8)}{r_8} - \frac{\sin\left(\beta\frac{\lambda}{8}\right)\cos(\beta r_7)}{r_7}\right)\right\} \tag{1-72}$$

$$E_{L_{II}} = E_{L_{II}L_1} + E_{L_{II}L_2} + E_{L_{II}L_3} + E_{L_{II}L_4} \tag{1-73}$$

1-11 <u>The Total Tangential Induced EMF at a Point P on L_{III}</u>

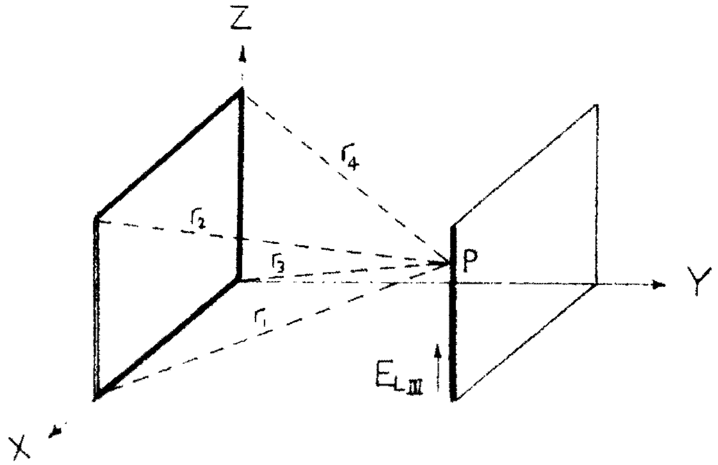

Fig. 1-15

Define $E_{L_{III}L_1}$ = the induced emf on L_{III} due to the charges and the current on L_L

$E_{L_{III}L_2}$ = the induced emf on L_{III} due to the charges on L_2

$E_{L_{III}L_3}$ = the induced emf on L_{III} due to the charges and the current on L_3

$E_{L_{III}L_4}$ = the induced emf on L_{III} due to the charges on L_4

The expression for $E_{L_{III}L_1}$ is given by equation (1-29) where

$$X = H, \quad r_H = r_4, \quad \text{or} \quad r_o = r_3$$

$$E_{L_{III}L_1} = \frac{jI_i e^{jwt}}{2\pi\epsilon_o C}\left[\frac{e^{-j\beta r_4}}{r_4}\sin(\beta\frac{3\lambda}{8}) - \frac{e^{-j\beta r_3}}{r_3}\sin(\beta\frac{\lambda}{8})\right]$$

$$= \frac{jI_i e^{jwt}}{2\pi\epsilon_o C}\left\{\frac{\sin(\frac{\beta 3\lambda}{8})\left[\cos(\beta r_4) - j\sin(\beta r_4)\right]}{r_4} - \frac{\sin(\frac{\beta\lambda}{8})\left[\cos(\beta r_3) - j\sin(\beta r_3)\right]}{r_3}\right\}$$

$$= \frac{jI_i e^{jwt}}{2\pi\epsilon_o C}\left\{\left(\frac{\sin(\beta\frac{3\lambda}{8})\cos(\beta r_4)}{r_4} - \frac{\sin(\beta\frac{\lambda}{8})\cos(\beta r_3)}{r_3}\right) - j\left(\frac{\sin(\beta\frac{3\lambda}{8})\sin(\beta r_4)}{r_4} - \frac{\sin(\beta\frac{\lambda}{8})\sin(\beta r_3)}{r_3}\right)\right\}$$

$$= \frac{I_i e^{jwt}}{2\pi\epsilon_o C}\left\{\left(\frac{\sin(\beta\frac{3\lambda}{8})\sin(\beta r_4)}{r_4} - \frac{\sin(\beta\frac{\lambda}{8})\sin(\beta r_3)}{r_3}\right) - j\left(\frac{\sin(\beta\frac{3\lambda}{8})\cos(\beta r_4)}{r_4} - \frac{\sin(\beta\frac{\lambda}{8})\cos(\beta r_3)}{r_3}\right)\right\} \tag{1-74}$$

The expression for $E_{L_{III}L_2}$ is given by equation (1-59) where

$$X = H, \quad r_H = r_2, \quad r_o = r_4$$

$$E_{L_{III}L_2} = \frac{I_i e^{jwt}}{4\pi\epsilon_o C}(z-H)\left\{ e^{j(\frac{\beta 3\lambda}{8})}\left[\frac{e^{j\beta(H-r_2)}}{r_2^2} - \frac{e^{-j\beta r_4}}{r_4(r_4+H)}\right]\right.$$

$$\left. + e^{-j\frac{\beta 3\lambda}{8}}\left[\frac{e^{-j\beta(H+r_2)}}{r_2^2} - \frac{e^{-j\beta r_4}}{r_4(r_4-H)}\right]\right\}$$

$$= \frac{I_i e^{jwt}}{4\pi\epsilon_o C}(z-H)\left\{ e^{-j\beta r_2}\left[\frac{e^{j\frac{\beta 5\lambda}{8}} + e^{-j\frac{\beta 5\lambda}{8}}}{r_2^2}\right]\right.$$

$$\left. - e^{-j\beta r_4}\left[\frac{(r_4-H)\,e^{j\frac{\beta 3\lambda}{8}} + (r_4+H)\,e^{-j\frac{\beta 3\lambda}{8}}}{r_4(r_4^2 - H^2)}\right]\right\}$$

$$= \frac{I_i e^{jwt}}{4\pi\epsilon_o C}(z-H)\left\{ \frac{2e^{-j\beta r_2}\cos(\beta\frac{5\lambda}{8})}{r_2^2}\right.$$

$$\left. - e^{-j\beta r_4}\left[\frac{2r_4\cos(\frac{\beta 3\lambda}{8}) - 2jH\sin(\frac{\beta 3\lambda}{8})}{r_4(r_4^2 - H^2)}\right]\right\}$$

$$= \frac{I_i e^{jwt}}{4\pi\epsilon_o C}(z-H)\left\{ \left(\frac{-r_4\cos(\beta r_4)\cos(\frac{\beta 3\lambda}{8}) + H\sin(\beta r_4)\sin(\beta\frac{3\lambda}{8})}{r_4\,(r_4^2 - H^2)}\right.\right.$$

$$\left. + \frac{\cos(\beta r_2)\cos(\beta\frac{5\lambda}{8})}{r_2^2}\right)$$

$$+ j\left(\frac{r_4\sin(\beta r_4)\cos(\beta\frac{3\lambda}{8}) + H\cos(\beta r_4)\sin(\beta\frac{3\lambda}{8})}{r_4(r_4^2 - H^2)}\right.$$

$$\left.\left. - \frac{\sin(\beta r_2)\cos(\beta\frac{5\lambda}{8})}{r_2^2}\right)\right\} \quad (1-75)$$

The expression for $E_{L_{III}L_3}$ is given by equation (1-33) where

$X = H,\ r_H = r_2,\ r_o = r_1$

$$E_{L_{III}L_3} = \frac{jI_i e^{jwt}}{2\pi\epsilon_o C}\left\{\frac{e^{-j\beta r_2}}{r_2}\sin(\frac{5\beta\lambda}{8}) - \frac{e^{-j\beta r_1}}{r_1}\sin(\frac{7\beta\lambda}{8})\right\}$$

$$E_{L_{III}L_3} = \frac{jI_i e^{jwt}}{2\pi\epsilon_o C}\left\{\frac{\sin(\beta\frac{5\lambda}{8})\left[\cos(\beta r_2)-j\sin(\beta r_2)\right]}{r_2} - \frac{\sin(\beta\frac{7\lambda}{8})\left[\cos(\beta r_1)-j\sin(\beta r_1)\right]}{r_1}\right\}$$

$$= \frac{jI_i e^{jwt}}{2\pi\epsilon_o C}\left\{\frac{\sin(\beta\frac{5\lambda}{8})\cos(\beta r_2)}{r_2} - \frac{\sin(\beta\frac{7\lambda}{8})\cos(\beta r_1)}{r_1} - j\left(\frac{\sin(\beta\frac{5\lambda}{8})\sin(\beta r_2)}{r_2} - \frac{\sin(\beta\frac{7\lambda}{8})\sin(\beta r_1)}{r_1}\right)\right\}$$

$$= \frac{I_i e^{jwt}}{2\pi\epsilon_o C}\left\{\left(\frac{\sin(\beta\frac{5\lambda}{8})\sin(\beta r_2)}{r_2} - \frac{\sin(\beta\frac{7\lambda}{8})\sin(\beta r_1)}{r_1}\right) + j\left(\frac{\sin(\beta\frac{5\lambda}{8})\cos(\beta r_2)}{r_2} - \frac{\sin(\beta\frac{7\lambda}{8})\cos(\beta r_1)}{r_1}\right)\right\} \qquad (1\text{-}76)$$

The expression for $E_{L_{III}L_4}$ is given by equation (1-62) where

$$X = H,\quad r_H = r_1,\quad r_o = r_3$$

$$E_{L_{III}L_4} = \frac{I_i e^{jwt}}{4\pi\epsilon_o C}(z)\left\{e^{j\beta\frac{\lambda}{8}}\left[\frac{e^{-j\beta r_3}}{r_3(r_3-H)} - \frac{e^{-j\beta(r_1+H)}}{r_1(r_1)}\right] + e^{-j\beta\frac{\lambda}{8}}\left[\frac{e^{-j\beta r_3}}{r_3(r_3+H)} - \frac{e^{+j\beta(H-r_1)}}{r_1^2}\right]\right\}$$

$$= \frac{I_i e^{jwt}}{4\pi\epsilon_o C}(z)\left\{e^{-j\beta r_3}\cdot\left(\frac{(r_3+H)e^{j\beta\frac{\lambda}{8}} + (r_3-H)e^{-j\beta\frac{\lambda}{8}}}{r_3(r_3^2-H^2)}\right) - e^{-j\beta r_1}\cdot\left(\frac{e^{-j\beta\frac{\lambda}{8}} + e^{j\beta\frac{\lambda}{8}}}{r_1^2}\right)\right\}$$

$$= \frac{I_i e^{jwt}}{4\pi\epsilon_o C}(z)\left\{e^{-j\beta r_3}\left(\frac{2r_3\cos(\beta\frac{\lambda}{8}) + 2jH\sin(\beta\frac{\lambda}{8})}{r_3(r_3^2-H^2)}\right) - \frac{2e^{-j\beta r_1}\cos(\beta\frac{\lambda}{8})}{r_1^2}\right\}$$

$$= \frac{I_i e^{jwt}}{2\pi\epsilon_o C}(z)\left\{\frac{r_3\cos(\beta r_3)\cos(\frac{\beta\lambda}{8}) + H\sin(\beta r_3)\sin(\beta\frac{\lambda}{8})}{r_3(r_3^2-H^2)} - \frac{\cos(\beta r_1)\cos(\beta\frac{\lambda}{8})}{r_1^2}\right.$$

$$+j\left(\frac{H\cos(\beta r_3)\sin(\frac{\beta\lambda}{8}) - r_3\sin(\beta r_3)\cos(\beta\frac{\lambda}{8})}{r_3(r_3^2 - H^2)} + \frac{\sin(\beta r_1)\cos(\beta\frac{\lambda}{8})}{r_1^2}\right)\Bigg\} \quad (1-77)$$

$E_{L_{III}}$ = the total tangential induced emf at a point P on L_{III}

$$= E_{L_{III}L_1} + E_{L_{III}L_2} + E_{L_{III}L_3} + E_{L_{III}L_4} \quad (1-78)$$

1-12 The Total Tangential Induced EMF at a Point P on L_{IV}

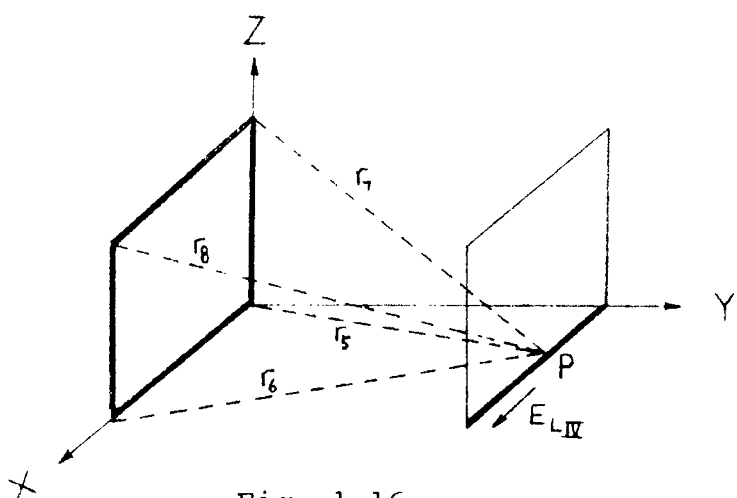

Fig. 1-16

Define $E_{L_{IV}L_1}$ = the induced emf on L_{IV} due to the charges on L_1

$E_{L_{IV}L_2}$ = the induced emf on L_{IV} due to the charges and the current on L_2

$E_{L_{IV}L_3}$ = the induced emf on L_{IV} due to the charges on L_3

$E_{L_{IV}L_4}$ = the induced emf on L_{IV} due to the charges and the current on L_4

The expression for $E_{L_{IV}L_1}$ is given by equation (1-48) where

$Z=0,\ r_H = r_7,\ r_o = r_5$

$$E_{L_{IV}L_1} = \frac{I_i e^{jwt}}{4\pi\epsilon_0 C}(x)\left\{ e^{j\beta\frac{\lambda}{8}}\left[\frac{e^{j\beta(H-r_7)}}{r_7(r_7-H)} - \frac{e^{-j\beta r_5}}{r_5(r_5)}\right] + e^{-j\beta\frac{\lambda}{8}}\left[\frac{e^{-j\beta(H+r_7)}}{r_7(r_7+H)} - \frac{e^{-j\beta r_5}}{r_5^2}\right]\right\}$$

or

$$E_{L_{IV}L_1} = \frac{I_i e^{jwt}}{4\pi\epsilon_0 C}(x)\left\{\frac{e^{j\beta(\frac{3\lambda}{8}-r_7)}}{r_7(r_7-H)} + \frac{e^{-j\beta(\frac{3\lambda}{8}+r_7)}}{r_7(r_7+H)} - \frac{2e^{-j\beta r_5}}{r_5^2}\cos(\beta\frac{\lambda}{8})\right\}$$

$$= \frac{I_i e^{jwt}}{4\pi\epsilon_0 C}(x)\left\{e^{-j\beta r_7}\left(\frac{2r_7\cos(\frac{\beta 3\lambda}{8}) + 2jH\sin(\frac{\beta 3\lambda}{8})}{r_7(r_7^2 - H^2)}\right) - \frac{2e^{-j\beta r_5}}{r_5^2}\cos(\frac{\beta\lambda}{8})\right\}$$

$$= \frac{I_i e^{jwt}}{2\pi\epsilon_0 C}(x)\left\{\left(\frac{r_7\cos(\beta r_7)\cos(\beta\frac{3\lambda}{8}) + H\sin(\beta r_7)\sin(\beta\frac{3\lambda}{8})}{r_7(r_7^2 - H^2)} - \frac{\cos(\beta r_5)\cos(\beta\frac{\lambda}{8})}{r_5^2}\right)\right.$$
$$\left. + j\left(\frac{H\cos(\beta r_7)\sin(\frac{\beta 3\lambda}{8}) - r_7\sin(\beta r_7)\cos(\frac{\beta 3\lambda}{8})}{r_7(r_7^2 - H^2)} + \frac{\sin(\beta r_5)\cos(\beta\frac{\lambda}{8})}{r_5^2}\right)\right\} \qquad (1\text{-}79)$$

The expression for $E_{L_{IV}L_2}$ is given by equation (1-36) where

$Z = 0,\ r_H = r_8,\ r_o = r_7$

$$E_{L_{IV}L_2} = \frac{jI_i e^{jwt}}{2\pi\epsilon_o C}\left[\frac{e^{-j\beta r_8}}{r_8}\sin(\beta\frac{5\lambda}{8}) - \frac{e^{-j\beta r_7}}{r_7}\sin(\beta\frac{3\lambda}{8})\right]$$

$$= \frac{jI_i e^{jwt}}{2\pi\epsilon_o C}\left[\left(\frac{\sin(\frac{5\beta\lambda}{8})\cos(\beta r_8)}{r_8} - \frac{\sin(\frac{3\beta\lambda}{8})\cos(\beta r_7)}{r_7}\right)\right.$$

$$\left. -j\left(\frac{\sin(\frac{5\beta\lambda}{8})\sin(\beta r_8)}{r_8} - \frac{\sin(\frac{\beta 3\lambda}{8})\sin(\beta r_7)}{r_7}\right)\right]$$

$$= \frac{I_i e^{jwt}}{2\pi\epsilon_o C}\left[\left(\frac{\sin(\frac{5\beta\lambda}{8})\sin(\beta r_8)}{r_8} - \frac{\sin(\frac{3\beta\lambda}{8})\sin(\beta r_7)}{r_7}\right)\right.$$

$$\left. +j\left(\frac{\sin(\beta\frac{5\lambda}{8})\cos(\beta r_8)}{r_8} - \frac{\sin(\frac{\beta 3\lambda}{8})\cos(\beta r_7)}{r_7}\right)\right] \quad (1-80)$$

The expression for $E_{L_{IV}L_3}$ is given by equation (1-51)

where

$$Z = 0, \quad r_H = r_8, \quad r_o = r_6$$

$$E_{L_{IV}L_3} = \frac{I_i e^{jwt}}{4\pi\epsilon_o C}(x-H)\left\{e^{+j\beta\frac{7\lambda}{8}}\left[-\frac{e^{-j\beta(H+r_8)}}{r_8(r_8+H)} + \frac{e^{-j\beta r_6}}{r_6^2}\right]\right.$$

$$\left. +e^{-j\beta\frac{7\lambda}{8}}\left[-\frac{e^{+j\beta(H-r_8)}}{r_8(r_8^2-H)} + \frac{e^{-j\beta r_6}}{r_6^2}\right]\right\}$$

$$= \frac{I_i e^{jwt}}{4\pi\epsilon_o C}(x-H)\left\{-e^{-j\beta r_8}\left[\frac{(r_8-H)e^{j\beta\frac{5\lambda}{8}} + (r_8+H)e^{-j\beta\frac{5\lambda}{8}}}{r_8(r_8^2-H^2)}\right]\right.$$

$$\left. + \frac{2e^{-j\beta r_8}}{r_6^2}\cos(\beta\frac{7\lambda}{8})\right\}$$

$$= \frac{I_i e^{jwt}}{4\pi\epsilon_o C}(x-H)\left\{-e^{-j\beta r_8}\left(\frac{2r_8\cos(\beta\frac{5\lambda}{8}) - 2jH\sin(\beta\frac{5\lambda}{8})}{r_8(r_8^2-H^2)}\right)\right.$$

$$\left. + \frac{2e^{-j\beta r_6}}{r_6^2}\cos(\beta\frac{7\lambda}{8})\right\}$$

$$E_{L_{IV}L_3} = \frac{I_i e^{jwt}}{2\pi\epsilon_0 C}(x-H)\left\{\frac{-r_8\cos(\beta r_8)\cos(\beta\frac{5\lambda}{8}) + H\sin(\beta r_8)\sin(\beta\frac{5\lambda}{8})}{r_8(r_8^2 - H^2)} + \frac{\cos(\beta r_6)\cos(\beta\frac{7\lambda}{8})}{r_6^2} + j\left(\frac{r_8\sin(\beta r_8)\cos(\frac{5\beta\lambda}{8}) + H\cos(\beta r_8)\sin(\beta\frac{5\lambda}{8})}{r_8(r_8^2 - H^2)} - \frac{\sin(\beta r_8)\cos(\beta\frac{7\lambda}{8})}{r_6^2}\right)\right\} \quad (1\text{-}81)$$

The expression for $E_{L_{IV}L_4}$ is given by equation (1-38)

where

$$Z = 0,\ r_H = r_6,\ r_o = r_5$$

$$E_{L_{IV}L_4} = \frac{jI_i e^{jwt}}{2\pi\epsilon_0 C}\left[\frac{e^{-j\beta r_6}}{r_6}\sin(\beta\frac{7\lambda}{8}) - \frac{e^{-j\beta r_5}}{r_5}\sin(\beta\frac{\lambda}{8})\right]$$

$$= \frac{jI_i e^{jwt}}{2\pi\epsilon_0 C}\left[\left(\frac{\cos(\beta r_6)\sin(\frac{\beta 7\lambda}{8})}{r_6} - \frac{\cos(\beta r_5)\sin(\beta\frac{\lambda}{8})}{r_5}\right) - j\left(\frac{\sin(\beta r_6)\sin(\beta\frac{7\lambda}{8})}{r_6} - \frac{\sin(\beta r_5)\sin(\beta\frac{\lambda}{8})}{r_5}\right)\right]$$

$$= \frac{I_i e^{jwt}}{2\pi\epsilon_0 C}\left[\left(\frac{\sin(\beta r_6)\sin(\beta\frac{7\lambda}{8})}{r_6} - \frac{\sin(\beta r_5)\sin(\beta\frac{\lambda}{8})}{r_5}\right) + j\left(\frac{\sin(\frac{7\beta\lambda}{8})\cos(\beta r_6)}{r_6} - \frac{\sin(\frac{\beta\lambda}{8})\cos(\beta r_5)}{r_5}\right)\right] \quad (1\text{-}82)$$

$E_{L_{IV}}$ = The total tangential induced emf at a point on L_{IV}

$$= E_{L_{IV}L_1} + E_{L_{IV}L_2} + E_{L_{IV}L_3} + E_{L_{IV}L_4} \qquad (1\text{-}83)$$

1-13 The derivation of the Mutual Impedance

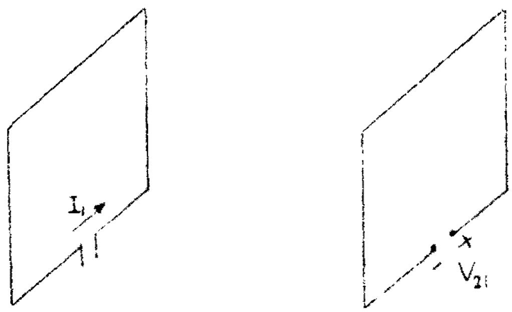

Fig. 1-17

The mutual impedance of the cubical quad antenna is defined as

$$Z_{21} = \frac{V_{21}}{I_1} \qquad (1\text{-}84)$$

where V_{21} is the open-circuit voltage at the terminals of the reflecting loop due to a base current, I_1, in the radiating loop. Now, the electric field intensity at all points along the reflecting loop has been calculated, and the problem is that of determining the open-circuit voltage at the terminals of the reflecting loop. This voltage is the resultant of the voltages induced in all the elemental lengths of the loop. The result may be obtained by an application of the reciprocity theorem.

Consider the reflecting loop with the radiating loop in place, but not radiating. A voltage $V_2 = I_2(0)Z_2$ applied at the terminals will produce a terminal current $I_2(o)$ and a current at any point, designated as $I_2(\ell)$. The impedance Z_2 is the impedance looking into the terminals of the reflecting loop. The reciprocity theorem states that if a voltage $I_2(o)Z_2$, applied at the terminals, produces a current $I_2(\ell)$ at a point along the reflecting loop, then a voltage $E(\ell)dh$, induced at ℓ, will produce a short circuit current at the terminals.

$$dI_{so} = \frac{E(\ell)\,dh}{I_2(o)Z_2}\, I_2(\ell) \tag{1-85}$$

The total short-circuit current at the terminals, due to the induced emf along the entire length of the reflecting loop, will be

$$I_{sc} = \frac{1}{I_2(o)Z_2} \oint E(\ell)\, I_2(\ell)\, dh \tag{1-86}$$

By Thevenin's theorem the open-circuit voltage at the terminals will be

$$V_{21} = -\, I_{sc} Z_2$$

$$= \frac{-1}{I_2(o)} \oint E(\ell)\, I_2(\ell)\, dh \tag{1-87}$$

the minus sign results from the fact that either I_{sc} or V_{21} will be opposite to the assumed positve direction when the reflecting loop is short-circuited. The expression for the mutual impedance of the cubical quad antenna is

$$Z_{21} = \frac{V_{21}}{I_1(o)} = - \frac{1}{I_1(0)\, I_2(0)} \oint E(\ell)\, I_2(\ell)\, dh \qquad (1\text{-}88)$$

where $I_1(o)$ = the terminal current of the radiating loop

$I_2(o)$ = the terminal current of the reflecting loop

$I_2(\ell)$ = the current distribution along the reflecting loop when fed by a voltage at the terminals and with the terminals of the radiating loop open-circuited.

$E(\ell)$ = the induced emf along the reflecting loop due to the time-changing current in the radiating loop.

Since the radiating and the reflecting loops are identical, $I_2(\ell)$ may be expressed as

$$I_2(\ell) = -2I_i' \cos\left[\beta\left(\frac{n\lambda}{8} + h\right)\right] e^{jwt} \qquad (1\text{-}89)$$

In equation (1-89), h is equal to z when the current element is set in the z direction; h will be equal to x when the current element is set in the x direction. As was stated in section 1-5, the current in L_{III} may be expressed as

$$I_2(\ell) = 2\, I_i' \cos\left[\beta\left(\frac{7\lambda}{8} - z\right)\right] e^{jwt}$$

and the current in L_{IV} may be expressed or

$$I_2(\ell) = 2\, I_i' \cos\left[\beta\left(\frac{\lambda}{8} - x\right)\right] e^{jwt}$$

$E(\ell)$ has been given by equations (1-68), (1-73), (1-78) and (1-83). Introducing these expressions into equation (1-88) yields

$$Z_{21} = \frac{2I_i' e^{jwt}}{I_1(o)\, I_2(o)} \left\{ \int_0^H E_{L_I} \cos\left[\beta\left(-\frac{\lambda}{8} + z\right)\right] + \int_0^H E_{L_{II}} \cos\left[\beta\left(\frac{3\lambda}{8} + x\right)\right] dx \right.$$

$$\left. - \int_0^H E_{L_{III}} \cos\left[\beta\left(\frac{7\lambda}{8} - z\right)\right] dz - \int_0^H E_{L_{IV}} \cos\left[\beta\left(-\frac{\lambda}{8} - x\right)\right] dx \right\} \quad (1\text{-}90)$$

where

$$I_1(o) = 2\, I_i e^{jwt}$$

$$I_2(o) = 2\, I_i' e^{jwt}$$

Equation (1-90) can be simplified to the form

$$Z_{21} = \frac{1}{I_1(o)} \left\{ \int_0^H E_{L_I} \cos\left[\beta\left(\frac{\lambda}{8} + z\right)\right] dz + \int_0^H E_{L_{II}} \cos\left[\beta\left(\frac{3\lambda}{8} + x\right)\right] dx \right.$$

$$- \int_0^H E_{L_{III}} \cos\left[\beta\left(\frac{7\lambda}{8} - z\right)\right] dz - \int_0^H E_{L_{IV}} \cos\left[\beta\left(\frac{\lambda}{8} - x\right)\right] dx \quad (1\text{-}91)$$

The integrals of equation (1-91) can best be evaluated by means of numerical integration.

1-14 A Computer Program For Evaluating The Mutual Impedance

A fortran II program was used to evaluate equation (1-91). The approximation method used in the program is called Simpson's rule. In the program the interval, $H = \frac{\lambda}{4}$, was divided into 50 equal parts. Each part was 0.005 wavelength, 0.5 cm, long. The output of the program represents the real part and the imaginary part of the mutual impedance.

The computer program is shown on the following pages.

```
C     REAL PART OF THE MUTUAL IMPEDANCE OF THE CUBICAL QUAD ANTENNA
C     IMAGINARY PART OF THE MUTUAL IMPEDANCE OF THE CUBICAL QUAD
C     ANTENNA
      DIMENSION VR1(51), VR2(51), VR3(51), VR4(51)
      DIMENSION V11(51), VI2(51), V13(51), V14(51)
      Y=0.01
1     X=0.0
      Z=0.0
      DO  2   I=1, 51
      A=SINF(0.7854)
      B=SINF(3.92698)
      C=COSF(0.7854)
      D=COSF(2.35619)
      Q=6.28318
      R1=sQRTF(Y**2+Z**2)
      R2=SQRTF(Y**2+(Z-0.25)**2)
      R3=SQRTF(0 .0625+Y**2+Z**2)
      R4=SQRTF(0.0625+Y**2+(Z-0.25)**2)
      R5=SQRTF(X**2+Y**2)
      R6=SQRTF((X-0.25)**2+Y**2)
      R7=SQRTF(X**2+Y**2+Y**2+0.0625)
      R8=SQRTF((X-0.25)**2+Y**2+0.0625
      Q1=Q*R1
      Q2=Q*R2
      Q3=Q*R3
      Q4=Q*R4
      Q5=Q*R5
      Q6=Q*R6
      Q7=Q*R7
      Q8=Q*R8
      SQ1=SINF(Q1)
      SQ2=SINF(Q2)
      SQ3=SINF(Q3)
      SQ4=SINF(Q4)
      SQ5=SINF(Q5)
      SQ6=SINF(Q6)
      SQ7=SINF(Q7)
      SQ8=SINF(Q8)
      CQ1=COSF(Q1)
      CQ2=COSF(Q2)
      CQ3=COSF(Q3)
      CQ4=COSF(Q4)
      CQ5=COSF(Q5)
      CQ6=COSF(Q6)
      CQ7=COSF(Q7)
      CQ8=COSF(Q8)
```

```
OVR1(I)=+30.0*(+SQ2*A/R2-SQ1*A/R1+B*SQ4/R4-B*SQ3/R3+(Z-0.25)
 *((+R4
1*CQ4*D+0.25*SQ4*B)/(R4*(R4**2-0.0625))-CQ2*D/R2**2)+Z*((-R3*
2CQ3*C-0.25*SQ3*A)/(R3*(R3**2-0.0625))+CQ1*C/R1**2))
3*COSF(0.7854+Q*Z)
OVR2(I)=+30.0*(-SQ7*A/R7+R7+SQ8*B/R8-SQ5*A/R5+SQ6*B/R6+X*((-R7
 *CQ7*C
1+0.25*SQ7*A)/(R7*(R7**2-0.0625))+CQ5*D/R5**2)+(X-0.25)*((+R8
2*CQ8*C+0.25*SQ8*B)/(R8*(R8**2-0.0625))-CQ6*D/R6**2))
3*COSF(2.35619+Q*X)
OVR3(I)=-30.*(A*(SQ4/R4-SQ3/R3)+B*(SQ2/R2-SQ1/R1)+Z*((0.25*A
1*SQ3+C*R3*CQ3)/(R3*(R3**2-0.0625))-C*CQ1/R1**2)+(Z-0.25)*((-D*R4
2*CQ4+0.25*A*SQ4)/(R4*(R4**2-0.0625))+D*CQ2/R2**2))
3*COSF(5.49778-Q*Z)
OVR4(I)=-30.0*(B*SQ6/R6-SQ5*A/R5-A*SQ7/R7+B*SQ8/R8+X*((+R7*CQ7*D
1+0.25*SQ7*A)/(R7*(R7**2-0.0625))-CQ5*C/R5**2)+(X-0.25)*((-R8*
2CQ8*D+0.25*SQ8*B)/(R8*(R8**2-0.0625))+CQ6*C/R6**2))*
3COSF(0.7854-Q*X)
OVI1(I)=+30.0*(A*(+CQ2/R2-CQ1/R1)+B*(CQ4/R4-CQ3/R3)+(Z-0.25)*
1((0.25*B*CQ4-D*R4*SQ4)/(R4*(R4**2-0.0625))+D*SQ2/R2**2)
2+Z*((+C*R3*SQ3-0.25*A*CQ3)/(R3*(R3**2-0.0625))-SQ1*C/R1**2))*
3COSF(0.7854+Q*Z)
OVI2(I)=+30.0*(B*CQ6/R6-A*CQ5/R5-A*CQ7/R7+B*CQ8/R8+(X-0.25)*((-C*
1R8*SQ8+0.25*B*CQ8)/(R8**R8**2-0.0625))+D*SQ6/R6**2)+X*((+C*R7
2*SQ7+A*0.25*CQ7)/(R7*(R7**2-0.0625))-D*SQ5/R5**2))*
3COSF(2.35619+Q*X)
OVI3(I)=-30.*(A*(CQ4/R4-CQ3/R3)+B*(-CQ1/R1+CQ2/R2)+Z*((-C*R3*SQ3
1+0.25*A*CQ3)/(R3*(R3**2-0.0625))+C*SQ1/R1**2)+(Z-0.25)*((+D*R4
2*SQ4+0.25*A*CQ4)/(R4*(R4**2-0.0625))-D*SQ2/R2**2))*
3COSF(5.49778-Q*Z)
OVI4(I)=-30.0*(B*CQ6/R6-A*CQ5/R5-A*CQ7/R7+B*CQ8/R8+(X-0.25)*((0.25*
1B*CQ8+D*SQ8)/(R8*(R8**2-0.0625))-C*SQ6/R6**2)+X*((+0.25*A*
2CQ7-D*R7*SQ7)/(R7*(R7***2-0.0625))+C*SQ5/R5**2))
3*COSF(0.7854-Q*X)
 X=X+0.005
2 Z=Z+0.005
 CALL  ZMVR(VR1)
 ZMVR1=ZMVR(VR1)
 ZMVR2=ZMVR(VR2)
 ZMVR3=ZMVR(VR3)
 ZMVR4=ZMVR(VR4)
 REZM+ZMVR1+ZMVR2+ZMVR3+ZMVR4
 PUNCH   4,Y,  ZMVR1,    ZMVR2,    ZMVR3,    ZMVR4,   REZM
 FORMAT  (F6.3,  5F13.5)
 CALL  ZMVI(VII)
```

```
      ZMVI1=ZMVI(VII)
      ZMVI2=ZMVI(VI2)
      ZMVI3=ZMVI(VI3)
      ZMVI4=ZMVI(VI4)
      ZMI=ZMVI1+ZMVI2+ZMVI3+ZMVI4
      PUNCH  9, Y,  ZMVI1,  ZMVI2,  ZMVI3,  ZMVI4,  ZMI
9     FORMAT   (F5.2, 5F12.5)
      Y+Y+0.02
      IF(Y-0.05)    1,   1,   5
5     Y+Y+0.03
      IF(Y-0.1)  1,  1,  6
6     Y+Y+0.05
      IF(Y-1.0)   1,  1,  29
29    STOP
      END
      FUNCTION   ZMVR(VR1)
      DIMENSION  VR1(51), VR2(51),  VR3(51),  VR4(51)
      ODD=0.0
      EVEN=0.0
      DO  3   I=2, 50,  2
3     EVEN=EVEN+VR1(I)
      DO  4   J=3,  49,  2
4     ODD=ODD+VR1(I)
      ZMVI=0.001666*(VI1(1)+4.0*EVEN+2.0*ODD+VI1(51))
      RETURN
      END
      FUNCTION   ZMVI(VI1)
      DIMENSION   VI1(51), VI2(51), VI3(51), VI4(51)
      ODD=0.0
      EVEN=0.0
      DO   7 I=2,  50, 2
7     EVEN=EVEN+VI1(I)
      DO    8   I=3,   49,   2
8     ODD=ODD+VI1(I)
      ZMVR=0.001666*(VRI(1)+4.0*EVEN+2.0*ODD+VR1(51)
      RETURN
      END
```

The output data is as follows:

Distances Between Two Loops Y (cm)	Mutual Impedances (ohms)	
	Rectangular Form	Polar Form
0.01	116.419-j137.876	180 $\angle -49.8^{\circ}$
0.03	115.567-j119.440	166 $\angle -45.9^{\circ}$
0.05	113.870-j105.523	155 $\angle -42.9^{\circ}$
0.10	106.074-j85.784	136 $\angle -39^{\circ}$
0.20	77.432-j80.953	112 $\angle -46.2^{\circ}$
0.30	37.954-j84.963	92.9 $\angle -65.9^{\circ}$
0.40	-2.493-j78.030	78.1 $\angle -91.8^{\circ}$
0.50	-34.350-j56.949	66.5 $\angle -121.1^{\circ}$
0.60	-50,989-j26.733	57.4 $\angle -152.4^{\circ}$
0.70	-50.396+j 4.232	50.2 $\angle -184.8^{\circ}$
0.80	-35.415+j27.755	45.0 $\angle -218.1^{\circ}$
0.90	-12,545+j38.575	40.5 $\angle -251.9^{\circ}$
1.00	10.251+j35.278	36.8 $\angle -286.2^{\circ}$

Table 1-1

Curves of the mutual impedance will be shown in the next chapter as a comparison with those obtained from measurements.

CHAPTER II

EXPERIMENTAL MEASUREMENT OF THE MUTUAL IMPEDANCE

The mutual impedance of the quad antenna was analyzed mathematically using a few assumptions that can not be realized in the practical system. The mathematical treatment, as was discussed in the previous chapter, is an approximation. The validity and usefulness of this approximation can best be determined by experimental means. Therefore, measurements of the mutual impedances were made to compare the results established in Chapter I.

2-1 General Considerations

Referring to fig. 2-1 the terminal impedances of two identical antennas are

$$Z_1 = Z_{self} + \frac{I_2}{I_1} Z_{mutual}$$

$$Z_2 = Z_{self} + \frac{I_1}{I_2} Z_{mutual}$$

where Z_{self} = the self-impedance of loop #1 or loop #2

Z_1 = the terminal impedance of loop #1 when loop #2 is in place.

Z_2 = the terminal impedance of loop #2 when loop #1 is in place.

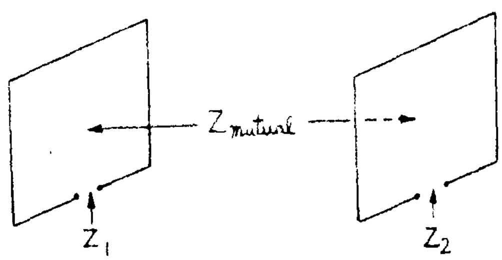

Fig. 2-1

Z_{self} is further defined as the limit of Z_1 as the current I_2 approaches zero at the terminals of the other loop. Z_{self} will in general depend on the spacing between the antennas since the current is not necessarily zero everywhere in the second loop, even though the current is zero at its terminals.[1]

Z_{mutual} may be obtained by short-circuiting loop #2 and measuring the terminal impedance Z_1 of loop #1. Thus

$$Z_1 = Z_{self} + \frac{I_2}{I_1} Z_{mutual}$$

$$Z_2 = 0 = Z_{self} + \frac{I_1}{I_2} Z_{mutual}$$

Then $$(Z_{mutual})^2 = Z_{self} (Z_{self} - Z_1) \qquad (2\text{-}1)$$

From equation (2-1) the value of the mutual impedance can be calculated from a knowledge of only the terminal impedance and the self-impedance. However, when taking antenna measurements, it is impossible to connect a measuring meter or a signal generator directly to the terminals of the antenna; a transmission line must be used. With the transmission line in place, the impedances read in the meter are not necessarily the terminal impedances of the antenna. If the transmission line is lossless and has no attenuation, the terminal impedances can be found from the meter readings by use of a Smith chart.

---

[1] Jasik, Antenna Engineering Handbook, McGraw-Hill Book Company, Inc., N.Y., 1961.

If the transmission line parameters are known, transmission line equations can be used. If the line parameters are not known, other techniques should be used. One of the possible techniques treats the line as a four terminal network.

2-2 Four Terminal Network

A transmission line can be represented by a circuit consisting of two terminals where power enters the circuit and two terminals where power leaves the circuit. The circuit is said to be passive, linear, and bilateral. It is passive because it contains no sources of electric energy, linear because impedances of its elements are independent of the amount of current passing through them, and bilateral because the impedances are independent of the direction of current. It can be shown that any linear, passive, and bilateral four-terminal network can be represented by either an equivalent "T" or a "π" circuit so far as measurements at the input or output terminals are concerned.

To find the relations between the sending-end and the receiving-end quantities, of a four terminal network, let us determine the voltage and current at the sending end of the unsymmetrical T circuit of fig. 2-2 in terms of the voltage and current at the receiving end.

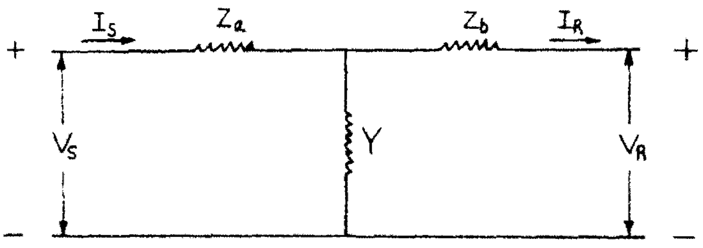

Fig. 2-2

The current at the sending end is

$$I_s = I_R + Y\,(V_R + I_R Z_b)$$

$$= YV_R + (1 + YZ_b)I_R \qquad (2\text{-}2)$$

The voltage at the receiving end is

$$V_s = V_R + I_R Z_b + I_s Z_a$$

$$= V_R + I_R Z_b + Z_a Y V_R + I_R Z_a + I_R Y Z_a Z_b$$

$$= (1 + YZ_a)\,V_R + (Z_a + Z_b + YZ_a Z_b)\,I_R \qquad (2\text{-}3)$$

The above equations are simplified in form by letting

$$A = 1 + YZ_a \qquad C = Y$$

$$B = Z_a + Z_b + YZ_a Z_b \qquad D = 1 + YZ_b \qquad (2\text{-}4)$$

If the network is symmetrical,

$$Z_a = Z_b$$

and hence A = D

and substituting equation (2-4) into equation (2-3)

$$V_s = AV_R + BI_R \qquad (2\text{-}5)$$

Substituting equation (2-4) into equation (2-2)

$$I_s = CV_R + DI_R \qquad (2\text{-}6)$$

Since the unsymmetrical T circuit is valid for measuring the end conditions of any passive, linear and bilateral four terminal network, equations (2-5) and (2-6) are valid for any such network. The constants A, B, C, D are called the generalized circuit constants.

Solving equations (2-5) and (2-6) for V_R and I_R

$$V_R = \frac{DV_S - BI_S}{AD - BC} \qquad (2\text{-}7)$$

$$I_R = \frac{AI_S - CV_S}{AD - BC} \qquad (2\text{-}8)$$

It can be shown that AD − BC = 1. Substituting this relation into equations (2-7) and (2-8)

$$V_R = DV_s - BI_s \qquad (2\text{-}9)$$

$$I_R = - CV_s + AI_s \qquad (2\text{-}10)$$

When a transmission line is chosen, the generalized circuit constants can be computed by making a few impedances measurements on the line. The impedances to be measured are:

Z_{SO} = the sending-end impedance with the receiving-end open-circuit

Z_{SS} = the sending-end impedance with the receiving-end short-circuit

Z_{Ro} = the receiving-end impedance with the sending-end open-circuit

Z_{Rs} = the receiving-end impedance with the sending-end short-circuit

The impedance measured from the sending-end can be determined in terms of A, B, C, D constants from equations (2-5) and (2-6). With I_R=0 the equations give

$$Z_{SO} = \frac{V_S}{I_S} = \frac{A}{C} \qquad (2\text{-}11)$$

and with $V_R = 0$

$$Z_{ss} = \frac{V_S}{I_S} = \frac{B}{D} \qquad (2\text{-}12)$$

To find the impedances measured from the receiving-end, equations (2-9) and (2-10) must be modified by changing the signs of all current terms. This change is necessary because, with the voltage applied at the receiving-end rather than at the sending-end, the direction of current flow assumed to be positive when measuring impedance is opposite to the direction shown in fig. 2-2 to which equations (2-9) and (2-10) apply. The equations become

$$V_R = DV_s + BI_s \qquad (2\text{-}13)$$

$$I_R = CV_s + AI_s \qquad (2\text{-}14)$$

From equations (2-13) and (2-14) with $I_s = 0$

$$Z_{Ro} = \frac{V_R}{I_R} = \frac{D}{C} \qquad (2\text{-}15)$$

and when $V_s = 0$

$$Z_{Rs} = \frac{V_R}{I_R} = \frac{B}{A} \qquad (2\text{-}16)$$

the values of the A B C D constants in terms of measured impedances are found as follows:

$$Z_{Ro} - Z_{Rs} = \frac{AD-BC}{AC} = \frac{1}{AC}$$

$$\frac{Z_{Ro}-Z_{Rs}}{Z_{so}} = \frac{1}{AC} \cdot \frac{C}{A} = \frac{1}{A^2}$$

$$A = \sqrt{\frac{Z_{so}}{Z_{Ro}-Z_{Rs}}} \qquad (2\text{-}17)$$

After "A" is computed, the other constants may be found by equations (2-11), (2-12) and (2-15); and then network elements

Z_a, Z_b and Y can be computed by equation (2-4). The accuracy of such a network depends on how closely the measured data approaches the actual conditions.

2-3 Equipment Used

The antenna under test was made of copper wire with a diameter of 0.133 cm. It was formed into two square loops measuring 25 cm per side, in other words, its circumference is 100 cm which is one wave-length for an electromagnetic wave of 300 megacycles propagating in vacuum. The antenna was fixed on a wood frame to make sure the two loops were parallel and had their centers on the same axis.

The radiating loop, which was a balanced device, was fed by a 300-ohm balanced transmission line. The other end of this line was connected to a balun transformer, which transforms the balanced system to an unbalanced detecting system.

The balun transformer was adjusted for proper operation at 300MC by means of adjustable stubs. This was done with the aid of an admittance meter.

The balun transformer and the admittance meter were linked by the type 874-LK constant-impedance adjustable line adjusted to an odd multiple of a quarter wavelength. Therefore, the admittance meter measured the resistance and reactance of the balanced circuit.

A crystal mixer was used to combine the 300 M.C. and 330 M.C. signals to produce a signal of 30 M.C. which was measured by the i-f amplifier. The block diagram of the system is shown in Fig. 2-3.

2-4 Experimental Results and the Corresponding Calculations

The impedances measured on the admittance meter of fig. 2-3 are the impedances appearing across the balun terminals; i.e. the impedances looking into the 300 ohm twin lead.

It was shown in section 2-2 that the equivalent circuit of the transmission line could be obtained by a few impedance measurements. They are:

	Impedances in ohms
Z_{so}	105 + j 475
Z_{ss}	40 - j 175
Z_{Ro}	100 + j 467.5
Z_{Rs}	37.5 - j 175

Table 2-1

The generalized circuit constant "A" may be obtained from equation (2-17)

$$A = \sqrt{\frac{Z_{so}}{Z_{Ro} - Z_{Rs}}}$$

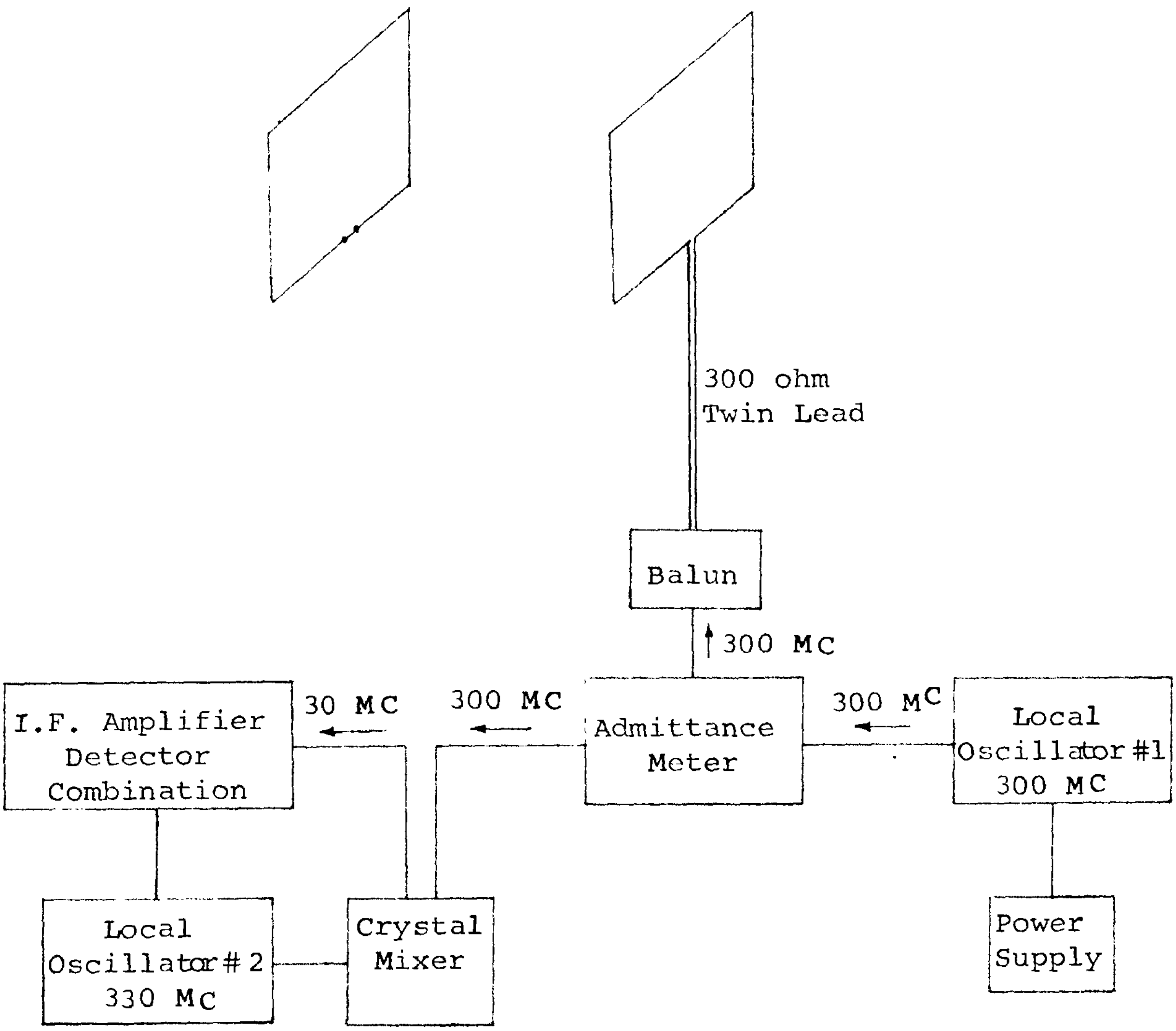

Fig. 2-3. Block diagram of circuit arrangement for impedance measurements

$$A = \sqrt{\frac{485 \angle 77.5^\circ}{62.5 + j642.5}}$$

$$= \sqrt{0.754 \angle -6.92^\circ}$$

$$= 0.87 \angle -3.46^\circ = 0.869 - j0.0525$$

and

$$C = \frac{A}{Z_{so}}$$

$$= \frac{0.87 \angle -3.46^\circ}{485 \angle 77.5^\circ}$$

$$= 0.00179 \angle -80.96^\circ$$

$$= 0.000282 - j\,0.00177$$

$$D = C \cdot Z_{Ro}$$

$$= 0.00179 \angle -80.96^\circ \cdot 477 \angle 77.9^\circ$$

$$= 0.854 \angle -3.06^\circ$$

$$= 0.852 - j\,0.0461$$

$$B = D \cdot Z_{ss}$$

$$= 0.854 \angle -3.06^\circ \cdot 179 \angle -77.1^\circ$$

$$= 152.7 \angle -80.16^\circ$$

Elements of the equivalent circuit may be obtained by substituting A, C, D constants into equation (2-4). Thus,

$$Y = C$$

$$= 0.000282 - j\,0.00177 \text{ mho}$$

$$Z_a = \frac{A-1}{Y}$$

$$= \frac{-0.131 - j0.0525}{0.00179 \angle -80.96^\circ}$$

$$= \frac{0.141 \angle 201.85^\circ}{0.00179 \angle -80.96^\circ}$$

$$= 78.8 \angle -77.19^\circ$$

$$= 17.5 - j\ 76.9 \quad \text{ohms}$$

$$Z_b = \frac{D-1}{Y}$$

$$= \frac{-0.148 - j\ 0.0461}{0.00179 \angle -80.96^\circ}$$

$$= 86.6 \angle 278.26^\circ$$

$$= 12.45 - j\ 85.75 \quad \text{ohms}$$

The equivalent circuit diagram is shown as follows:

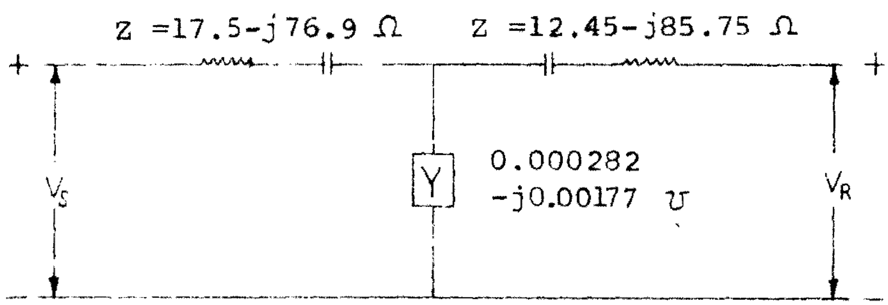

Fig. 2-4

When the cubical quad antenna was connected to the transmission line the impedances read from the admittance meter were as follows:

Distance Between Two Loops (cm)	Impedance	
	Resistance (ohms)	Reactance (ohms)
∞	228	-220
100	243	-212
90	226	-206
80	210	-216
70	226	-240
60	252	-250
50	263	-210
40	235	-165
30	190	-160
20	136	-178
10	74	-200

Table 2-2

Those impedances are the impedances appearing across the balun terminals. The impedances looking into the terminals of the radiating loop may be obtained by the following calculations.

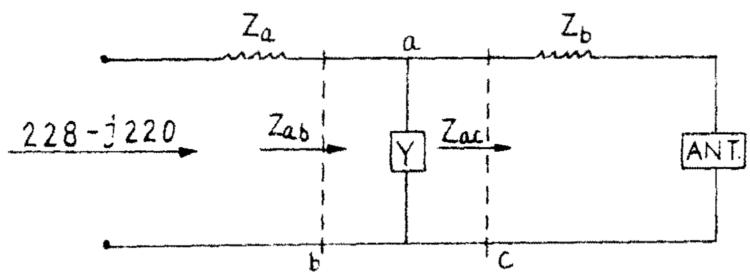

Fig. 2-5

Z_{ab} = 228-j220-Z_a

= 228-j22)-17.5 + j 76.9

= 210.5 - j 143.1

= 254.5 $\angle -34.2^{\circ}$ ohms

$Y_{ab} = \frac{1}{Z_{ab}}$ = 0.003925 $\angle 34.2^{\circ}$

= 0.003242 + j 0.002205 mho

$$Y_{ac} = Y_{ab} - Y$$

$$= 0.003242 + j\,0.002205 - 0.000282 + j\,0.00177$$

$$= 0.002960 + j\,0.003975$$

$$= 0.00495 \angle 53.35^{\circ} \quad \text{mho}$$

$$Z_{ac} = \frac{1}{Y_{ac}}$$

$$= 202 \angle -53.35^{\circ}$$

$$= 121 - j162 \quad \text{ohms}$$

$$Z_{self} = Z_{ac} - Z_{b}$$

$$= 121 - j162 - 12.45 + j\,85.75$$

$$= 108.55 - j\,76.25$$

$$= 132.5 \angle -35.1^{\circ} \quad \text{ohms}$$

Z_{self} if the self-impedance of the single loop. Other terminal impedances may be obtained in the same way. They are tabulated below:

Distance Between Two Loops (cm)	Terminal Impedances (ohms)
∞	108.55 - j76.25
100	117.85 - j77.25
90	112.55 - j67.75
80	100.45 - j67.25
70	99.55 - j84.75
60	108.55 - j95.25
50	128.55 - j85.75
40	135.37 - j50.40
30	109.0 - j12.65
20	69.67 - j19.90
10	22.35 - j20.45

Table 2-3

Knowing the self-impedance and terminal impedance, the mutual impedance between two loops of the antenna may be founded by equation (2-1). The calculation of Z_{M30} is typical. By equation (2-1)

$$(Z_{M30})^2 = Z_{self}\,(Z_{self} - Z_{30})$$

Where Z_{30} = the terminal impedance of the antenna when the distance between two loops is 30 cm.

Z_{M30} = the mutual impedance between two loops when the distance between them is 30 cm.

$$Z_{self} - Z_{30} = 108.55 - j76.25 - 109.0 + j12.65$$

$$= -0.45 - j63.6$$

$$= 63.60 \angle -90.42^{\circ} \quad \text{ohms}$$

$$Z_{M30} = \sqrt{132.5 \cdot 63.60 \angle -35.1^{\circ} - 90.42^{\circ}}$$

$$= \sqrt{8440} \angle -62.76^{\circ}$$

$$= 91.90 \angle -62.76^{\circ}$$

$$= 42.1 - j81.6 \quad \text{ohms}$$

Other mutual impedances may be founded in the same way. They are tabulated on the next page.

	Mutual Impedances (ohms)	
	Rectangular Form	Polar Form
Z_{self}	108.55 - j76.25	132.5 $\angle -35.1^{\circ}$
Z_{M10}	96.60 - j65.10	116.6 $\angle -34^{\circ}$
Z_{M20}	67.40 - j67.80	95.6 $\angle -45.15^{\circ}$
Z_{M30}	42.50 - j81.6	92.0 $\angle -62.5^{\circ}$
Z_{M40}	6.08 - j70.1	70.4 $\angle -85.05^{\circ}$
Z_{M50}	-27.30 - j46.8	54.2 $\angle -120.25^{\circ}$
Z_{M60}	-44.5 - j23.1	50.2 $\angle -152.6^{\circ}$
Z_{M70}	-40.5 - j 2.94	40.6 $\angle -184.15^{\circ}$
Z_{M80}	-31.4 + j26.6	40.1 $\angle -221.55^{\circ}$
Z_{M90}	- 8.6 + j32.5	33.6 $\angle -255.12^{\circ}$
Z_{M100}	13.88 + j33.4	36.2 $\angle -292.56^{\circ}$

Table 2-4

Curves of the mutual impedances obtained from calculations and from measurements are drawn in fig. 2-6 and fig. 2-7.

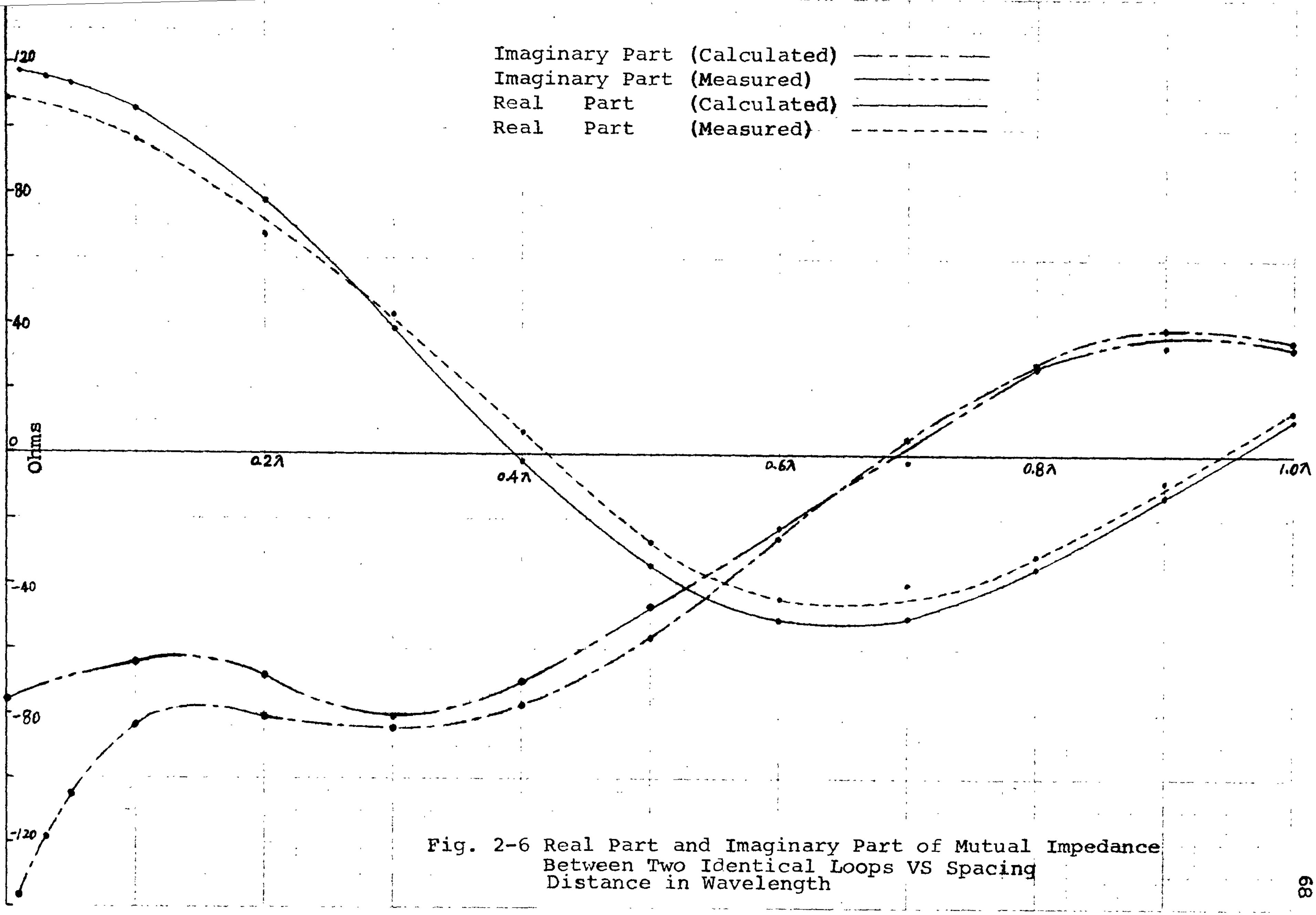

Fig. 2-6 Real Part and Imaginary Part of Mutual Impedance Between Two Identical Loops VS Spacing Distance in Wavelength

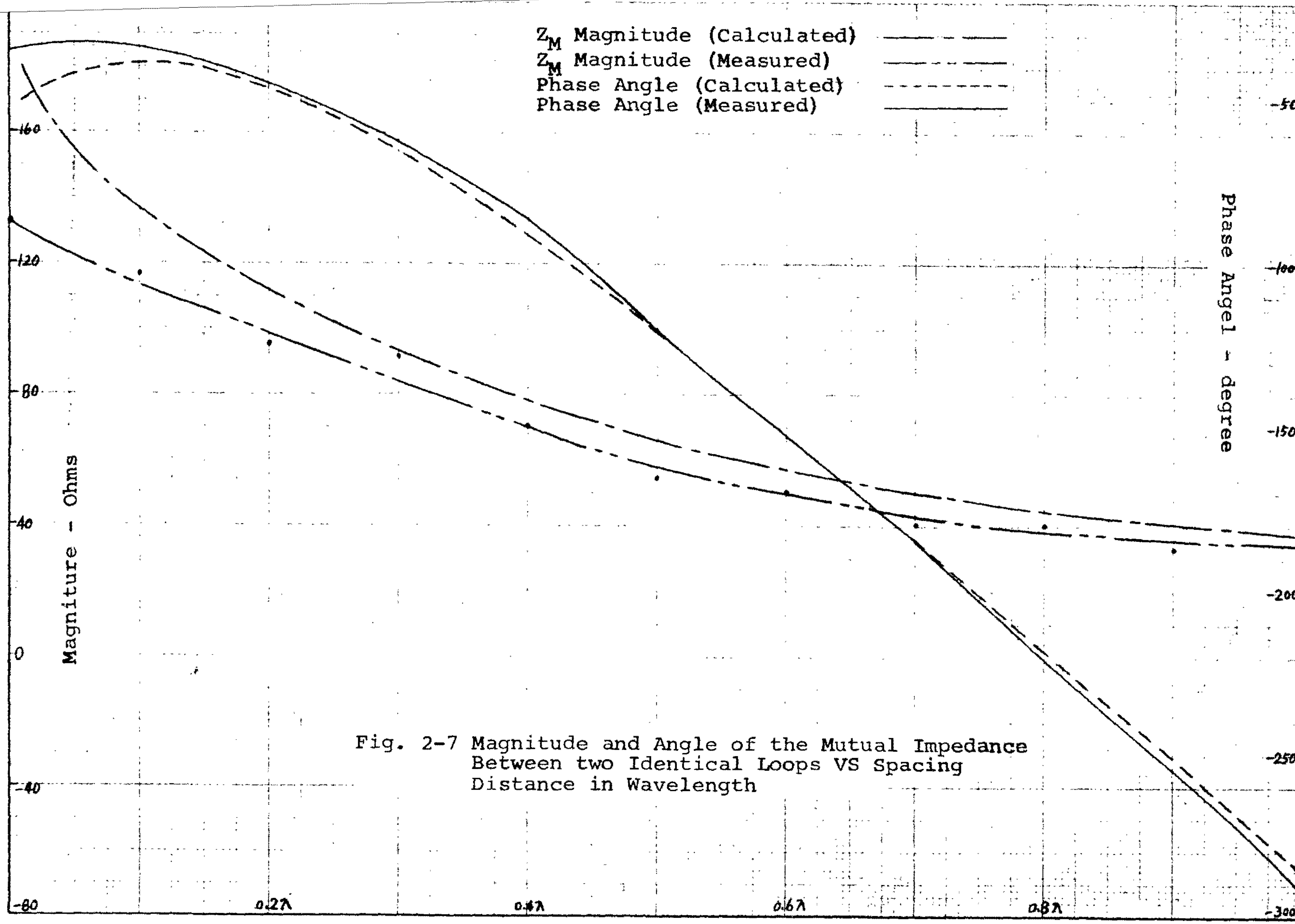

Fig. 2-7 Magnitude and Angle of the Mutual Impedance Between two Identical Loops VS Spacing Distance in Wavelength

2-5 Discussion

The mutual impedance between two loops is a measure of the voltage induced at the terminals of the second loop for one ampere of current flow into the terminals of the first loop. As the two loops are brought closer together, the voltage induced in the second loop becomes equal to the back or self-induced emf against which the current in the first loop must be driven. Therefore, it would be expected that the mutual impedance between two identical loops would approach the self-impedance of one as the loop spacing approaches zero. Hence, if the space between two loops is put equal to zero, the real part of the mutual impedance is equal to the radiation resistance. However, the reactance of a loop with a wire diameter of zero will be infinity (note: The last term of equation (1-64)). It is evident that in computing the reactance of the antenna, its wire diameter will have to be considered.

CHAPTER III

CONCLUSIONS AND SUGGESTIONS FOR FURTHER STUDY

3-1 Conclusion and Discussion

The results of this thesis show that the measured values follow closely the calculated values. The slight deviations are due to the following reasons:

(a) The antenna is not located at a place which is completely free from obstructions in all directions (the antenna is not in free space).

(b) The gap between the two terminals is not infinitely small.

(c) The ohmic losses in the antenna loops are not zero.

(d) For an antenna loop with losses, the velocity of wave propagation is not exactly $V_p = \frac{1}{\sqrt{LC}}$; it changes with frequency. Hence, the one meter length loop is not exactly a full wavelength around the periphery.

(e) The two loops are not exactly parallel and their centers are not exactly on the same axis.

(f) The 300 ohm twin lead is an unshielded transmission line; it effects the near fields of the antenna.

(g) The current in the radiating loop is not exactly a sinusoid.

(h) The equivalent T circuit does not exactly represent the 300 ohm line.

(A) When the line is open circuited, the fringing capacitance will effectively make the line appear to be longer than it really is. In the short circuited case, inductance in the short circuit strap will cause a similar error[1].

(B) In measuring Z_{RO} and Z_{RS} the test equipment should be located at the receiving end, where the antenna is to be connected; and the open circuit and the short circuit located at the sending end. It is, however, impossible to locate the test equipment at a height corresponding to the antenna height. The data for Z_{RO} and Z_{RS} were, of necessity, measured before the transmission line was put in place; the result being a slight error in the equivalent T circuit.

Some of the affects listed above can be avoided or reduced.

---

[1]Note: In U.H.F. type 874-WN short circuit termination and type 874-WO open circuit termination are used, but they do not fit the 300 ohm twin lead and the terminals of the balun.

For instance, if a completely shielded balanced line is used, errors due to terms (f) and (g) disappear. If a transmission line with low characteristic impedance is used, the reading of the admittance meter will fall into the maximum accuracy range; better results will be obtained.

The line used does not appear to be exactly symmetrical i.e. $Z_{RO} \neq Z_{SO}$ and $Z_{RS} \neq Z_{SS}$. This is due to poor manufacture and capacity differences along the line to the ground. Since Z_{RO}, Z_{RS}, Z_{SO} and Z_{SS} were measured this discrepancy does not introduce any error.

3-2 Suggestions for Further Study

The field pattern and the gain of the cubical antenna can be found with the knowledge of the pattern factor of the single loop and the data of this thesis.

The radiation resistance of a single loop was obtained by the extension of the Z_{MR} curve to the vertical axis. The radiation resistance of a single loop could be obtained by integrating the radical component of the Poynting vector over a large spherical surface. Similarly, the radiation resistance of the cubical quad antenna could be obtained by integrating the radical component of the Poynting vector over a large spherical surface.

The radiation loop need not necessarily be fed by a single power source or the current distribution along the radiating loop need not necessarily be a cosine wave. The kind of the current distribution that will yield maximum radiation or the kind of current distribution that will yield a desired field pattern are worthy of study.

APPENDIX I

Prove that

$$E_z = \frac{I_i e^{jwt}}{4\pi\epsilon_0 C}\left\{\int_0^H \frac{\partial}{\partial z}\left[\frac{e^{-j\beta(\ell+r)} - e^{j\beta(\ell-r)}}{r}\right] dh + j\beta\int_0^H \frac{e^{j\beta(\ell-r)} + e^{-j\beta(\ell+r)}}{r}\, dh\right\}$$

$$= \frac{I_i e^{jwt}}{4\pi\epsilon_0 C}\left\{\left[\frac{e^{j\beta(\ell-r)}}{r} - \frac{e^{-j\beta(\ell+r)}}{r}\right]_0^H\right\}$$

where $r = \sqrt{x^2 + y^2 + (z-h)^2}$, $\ell = \frac{\lambda}{8} + h$

proof:

$$\frac{\partial}{\partial z}\left[\frac{e^{-j\beta(\ell+r)} - e^{+j\beta(\ell-r)}}{r}\right]$$

$$= \frac{-j\beta(z-h)\, e^{-j\beta(\ell+r)}}{r^2} - \frac{(z-h)\, e^{-j\beta(\ell+r)}}{r^3}$$

$$+ \frac{j\beta(z-h)\, e^{j\beta(\ell-r)}}{r^2} + \frac{(z-h)\, e^{j\beta(\ell-r)}}{r^3}$$

set

$$F_1 = \int_0^H \frac{\partial}{\partial z}\left[\frac{e^{-j\beta(\ell+r)} - e^{j\beta(\ell-r)}}{r}\right] dh$$

$$= \int_0^H \frac{j\beta(z-h)}{r^2}\left[e^{j\beta(\ell-r)} - e^{-j\beta(\ell+r)}\right] dh$$

$$+ \int_0^H \frac{(Z-h)}{r^3}\left[e^{j\beta(\ell-r)} - e^{-j\beta(\ell+r)}\right] dh \qquad \text{(I)}$$

set $$F_2 = j\beta\int_0^H \frac{1}{r}\left[e^{j\beta(\ell-r)} + e^{-j\beta(\ell+r)}\right] dh \qquad \text{(II)}$$

but $$\frac{\partial}{\partial h}\left(\frac{e^{-j\beta(\ell+r)}}{r}\right) = \frac{-j\beta e^{-j\beta(\ell+r)}}{r} + \frac{j\beta(z-h)\, e^{-j\beta(\ell+r)}}{r^2}$$

$$+ \frac{(Z-h)\, e^{-j\beta(\ell+r)}}{r^3} \qquad \text{(III)}$$

$$\frac{\partial}{\partial h}\left(\frac{e^{j\beta(\ell-r)}}{r}\right) = \frac{j\beta e^{j\beta(\ell-r)}}{r} + \frac{j\beta(z-h)\, e^{j\beta(\ell-r)}}{r^2}$$

$$+ \frac{(z-h)\, e^{j\beta(\ell-r)}}{r^3} \qquad \text{(IV)}$$

Comparing (I), (II), (III) and (IV)

$$F_1 + F_2 = \int_0^H \left[\frac{\partial}{\partial h}\left(\frac{e^{j\beta(\ell-r)}}{r}\right) - \frac{\partial}{\partial h}\left(\frac{e^{-j\beta(\ell+r)}}{r}\right)\right] dh$$

$$= \left[\frac{e^{j\beta(\ell-r)}}{r} - \frac{e^{-j\beta(\ell+r)}}{r}\right]_0^H$$

or

$$E_z = \frac{I_i e^{jwt}}{4\pi\epsilon_0 C}\left\{\left[\frac{e^{j\beta(\ell-r)}}{r} - \frac{e^{j\beta(\ell+r)}}{r}\right]_0^H\right\} \quad \text{Q.E.D.}$$

APPENDIX II

5 Expanding the perfect differential of equation (1-44)

$$\frac{d}{dh}\,\frac{e^{j\beta(h-r)}}{r(r-h+z)} = \frac{1}{r^2(r-h+z)^2}\left\{ j\beta(r-h+z)\,r\left(1+\frac{z-h}{r}\right)e^{j\beta(h-r)} - \left[(r-h+z)\,\frac{h-z}{r} + r\left(\frac{-z+h}{r}-1\right)\right]e^{j\beta(h-r)} \right\}$$

$$= \frac{e^{j\beta(h-r)}}{r^2(r-h+z)^2}\left\{ j\beta(r-h+z)^2 + (r-h+z)\;\left(\frac{r-h+z}{r}\right)\right\}$$

$$= e^{j\beta(h-r)}\left\{\frac{j\beta}{r^2} + \frac{1}{r^3}\right\}$$

6 Expanding the differential of equation (1-45)

$$\frac{d}{dh}\,\frac{e^{-j\beta(h+r)}}{r(r+h-z)} = -\frac{1}{r^2(r+h-z)^2}\left\{ r(r+h-z)\,(-j\beta)\,\left(1+\frac{h-z}{r}\right)e^{-j\beta(h+r)} - \left[\frac{h-z}{r}(r+h-z) + r\left(\frac{h-z}{r} + 1\right)\right]e^{-j\beta(h+r)}\right\}$$

$$= -\frac{e^{-j\beta(h+r)}}{r^2(r+h-z)^2}\left\{ -j\beta(r+h-z)^2 - \frac{(r+h-z)^2}{r}\right\}$$

$$= e^{-j\beta(h+r)}\left\{\frac{j\beta}{r^2} + \frac{1}{r^3}\right\}$$

BIBLIOGRAPHY

1. Jasik, Antenna Engineering Handbook, McGraw-Hill Book Company, Inc., N.Y. 1961

2. Jordan, Electromagnetic Waves and Radiating Systems, Prentice-Hill, N.Y., 1950

3. Kraus, J. D., Antennas, McGraw-Hill Book Company, Inc., N.Y., 1950

4. King, W.P., Minmo, H.R., and Wing, A.H., Transmission Lines, Antennas and Wave Guides, McGraw-Hill Book Company, Inc., 1945

5. Skilling, Electric Transmission Lines, McGraw-Hill Book Company, Inc., 1951

6. Stevenson, W.D., Elements of Power System Analysis, 1955

7. The Radio Amateur's Handbook, American Radio Relay League, 39th Edition, 1962