NUMERICAL METHODS FOR FRACTIONAL OPTIMAL CONTROL AND PARAMETRIC PROBLEMS

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The Supervisory Committee certifies that this ***disquisition*** complies with North Dakota State University's regulations and meets the accepted standards for the degree of

MASTER OF SCIENCE

ABSTRACT

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Fractional derivatives (FDs) or derivatives of arbitrary order have attracted considerable interest in the past few decades, and almost every field of science and engineering has applications of fractional derivatives. Since fractional derivatives have such property as being non-local, it can be extremely challenging to find analytical solutions for fractional optimization problems, and in many cases, analytical solutions may not exist. Therefore, it is of great importance to develop approximate or numerical solutions for such problems.

The primary focus of this thesis is to develop numerical schemes to solve optimization problems in fractional orders. Numerical methods for integer order problems of Variational Calculus, using the Euler-Lagrange equation, have already been well established. A Fractional Variational Calculus Problem (FVCP) is a problem in which either the objective functional or the constraints or both contain at least one fractional derivative term. There is a critical need to develop numerical algorithms for solving FVCPs.

The main contributions of this thesis is to develop formulations and solution methods for various fractional order optimization problems, including fractional optimal control problems, linear functional minimization problems and isoperimetric problems in fractional orders. The FDs are defined in terms of the Riemann-Liouville or Caputo definitions. Numerical schemes have been developed to obtain the numerical results for various

problems. For each scheme, the rate of convergence and the convergence errors are analyzed to ensure that the algorithm yields stable results. Various fractional orders of derivatives are considered and as the order approaches the integer value of 1, the numerical solution recovers the analytical result of the corresponding integer order problem.

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NOMENCLATURE

Γ	Gamma function
g	Gravitational acceleration
${}^{RL}_{a}D^{\alpha}_{t}$	Left Riemann-Liouville fractional derivative
${}^{RL}_{t}D^{\alpha}_{b}$	Right Riemann-Liouville fractional derivative
${}^{C}_{a}D^{\alpha}_{t}$	Left Caputo fractional derivative
${}^{C}_{t}D^{\alpha}_{b}$	Right Caputo fractional derivative
${}^{GL}_{0}D^{\alpha}_{x}y$	Standard left Grünwald–Letnikov (GL) fractional derivative
${}^{GL,S}_{0}D^{\alpha}_{x}y$	Standard shifted left GL fractional derivative
${}^{MGL}_{0}D^{\alpha}_{x}y$	Modified left GL fractional derivative
α	Order of the fractional derivative
J	Performance index of Fractional Optimal Control Problems (FOCPs)
w	State variable for FOCPs
f	Control variable for FOCPs
λ	Lagrange multiplier
θ	Polar angle
φ	Azimuth angle
t	Time
J_n	n^{th} order Bessel function of the first kind
P_n	n^{th} order Legendre polynomial of the first kind
X	$cos\theta$
q_{nj}	State eigencoordinates for FOCPs
p_{nj}	Control eigencoordinates for FOCPs

k	Total no of eigenfunctions in radial directions for FOCPs in spherical coordinates
n	Total no of eigenfunctions in radial directions for FOCPs in cylindrical coordinates
m	Total no of eigenfunctions in angular or axial directions for FOCPs
R	Outer radius of spheres or cylinders for FOCPs
r	Radius
z	Axial direction
L	Length of the cylinder for FOCPs in cylindrical coordinates
Q'	Arbitrary function for FOCPs
R'	Arbitrary function for FOCPs
K	Integral constraint for fractional linear parametric problem
N	Total number of element
y_A	Analytical solution
h	Step size
r_c	The rate of convergence for parametric problems
R^{**}	Errors ratio for the parametric problem
I	Integral constraint for fractional order isoperimetric problem

CHAPTER 1. INTRODUCTION

1.1. HISTORY OF FRACTIONAL CALCULUS

The concept of Fractional Calculus (FC) is not new and it is as old as calculus itself (Ross, 1977a). The history of FC dates back to more than 300 years ago. In 1695, Leibniz asked a question to de l'Hôpital by exchanging a letter, "Can the meaning of derivatives with integer order be generalized to derivative with non-integer orders?" That revolutionary question aroused the curiosity of de l'Hôpital and on 30th September 1695 he replied to Leibniz with another question, "What if the order will be ½?" Leibniz replied, "Thus it follows that will be equal to $x\sqrt[2]{dx:x}$ an apparent paradox, from which one day useful consequences will be drawn." Nowadays, many scientists consider 30th September 1695 as the birthday of FC and Gottfried Wilhelm Von Leibniz as the father of FC (Ross, 1977a).

Following this nonconventional discussion, in 1730 Leonhard Euler mentioned FC when he studied the interpolation between integer orders of a derivative. Consequently, in 1772, Lagrange developed the exponents for differential operators of integer order (Lagrange, 1849):

$$\frac{d^m}{dx^m}\frac{d^n}{dx^n}y = \frac{d^{m+n}}{dx^{m+n}}y \tag{1.1}$$

This result can be expanded to arbitrary order.

However, the earliest systematic studies of FC were done at the beginning of the 19th century. In 1812 (Laplace, 1820), Laplace defined a fractional derivative (FD) for functions by means of an integral and it was first documented in a text in 1819 (Lacroix, 1819). Starting with $y = x^m$, where m is a positive integer, Lacroix developed the nth derivative

$$\frac{d^n y}{dx^n} = \frac{m!}{(m-n)!}x^{m-n}, \quad m \geq n \tag{1.2}$$

Using Legendre's symbol Γ, for the generalized factorial, he wrote

$$\frac{d^n y}{dx^n} = \frac{\Gamma(m+1)}{\Gamma(m-n+1)} x^{m-n} \tag{1.3}$$

Finally substituting $m = 1$ and $n = 1/2$, he obtained

$$\frac{d^{1/2} y}{dx^{1/2}} = \frac{\Gamma(2)}{\Gamma(3/2)} x^{1/2} = \frac{2\sqrt{x}}{\sqrt{\pi}} \tag{1.4}$$

Next more general definition of fractional operation was presented by J. B. J. Fourier in 1822 (Fourier, 1822). Fourier's definition of fractional operations can be obtained from an integral representation of $f(x)$:

$$f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\alpha) d\alpha \int_{-\infty}^{+\infty} cos\left[p(x-\alpha) + \frac{n\pi}{2}\right] dp \tag{1.5}$$

Using the present notation the final equation becomes

$$\frac{d^i}{dx^i} f(x) = \frac{1}{2\pi} \int_{-\infty}^{+\infty} f(\alpha) d\alpha \int_{-\infty}^{+\infty} p^i cos\left[p(x-\alpha) + \frac{i\pi}{2}\right] dp \tag{1.6}$$

From the above equation Fourier stated, "The number i which appears in the above equation can be regarded as any quantity whatever, positive or negative." However, both Lacroix's and Fourier's method of generalization of arbitrary order did not provide any hint for applications.

It wasn't until 1823 before fractional operations were used in any applications. Abel was the first who applied FC to solve an integral equation in the formulation of the *tautochrone problem* (Abel, 1826; 1881). The following example describes the *tautochrone problem* and its solution briefly.

Example: 1.1 (Tautochrone problem): The problem consists of determining the shape of the curve in the (x, y) plane such that the time required for an object to slide down

the curve to its lowest point under uniform gravity is independent of its initial position (x_o, y_o) on the curve.

If the particle slides without friction then from the principle of conservation of energy we can state that the loss of potential energy during the descent of the particle is equal to the gain of the kinetic energy of the particle. It can be written mathematically:

$$\frac{1}{2}m\left(\frac{ds}{dt}\right)^2 = mg(y_o - y), \tag{1.7}$$

where m is the mass of the particle, s is the arc length of the particle from the starting point along the curve and g is the gravitational acceleration. Now equation (1.7) becomes

$$\frac{ds}{\sqrt{(y_o - y)}} = \sqrt{2g}dt \tag{1.8}$$

Integrating (1.8) from the starting point at $t = 0$ to time $t = T$ gives

$$\sqrt{2g}T = \int_0^{y_o}(y_o - y)^{-1/2}ds \tag{1.9}$$

In that problem, the time of the slide (consider K) to reach its lowest point is constant. Denoting the arc length s as a function of the height y, we get $s = F(y)$. Taking the first order derivative of $s, ds = F'(y)dy$, changing variables $y_0 \to x$, $y \to t$ and denoting $F' = \sqrt{2g}\, f$, the tautochrone integral equation becomes

$$K = \int_0^x (x - t)^{-\frac{1}{2}} f(t)dt \tag{1.10}$$

where the function f needs to be determined. Dividing the above equation by $\Gamma\left(\frac{1}{2}\right) = \sqrt{\pi}$,

$$\frac{K}{\sqrt{\pi}} = \frac{1}{\sqrt{\frac{1}{2}}}\int_0^x (x - t)^{-\frac{1}{2}} f(t)dt = \frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}} f(x), \tag{1.11}$$

And then taking $\frac{d^{1/2}}{dx^{1/2}}$ of both sides of equation (1.11), the following equation is obtained:

$$\frac{1}{\sqrt{\pi}}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}K = \frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}\frac{d^{-\frac{1}{2}}}{dx^{-\frac{1}{2}}}f(x) = f(x) \tag{1.12}$$

Finally, the solution of the *tautochrone problem* is

$$f(x) = \frac{1}{\sqrt{\pi}}\frac{d^{\frac{1}{2}}}{dx^{\frac{1}{2}}}K, \tag{1.13}$$

which was the notable achievement by Abel in the FC.

The first major study of FC was done by Joseph Liouville who developed two different definitions of FDs in 1932. Liouville (1832) in his first definition, assumed that the arbitrary derivative of a function $f(x)$ can be expanded in the series

$$f(x) = \sum_{n=0}^{\infty} C_n e^{a_n x} \tag{1.14}$$

is

$$D^v f(x) = \sum_{n=0}^{\infty} C_n a_n^v e^{a_n x}, \tag{1.15}$$

where v is a derivative of arbitrary order that can be any number-rational, irrational or complex. This definition is restricted in choices of v for which the series (1.14) converges. In accordance with the awareness of this restriction, Liouville (1832) presented his second definition that does not have such restriction on v. However, in his second definition, there is a restriction on the type of function for which it is applicable. Liouville developed the definition of arbitrary order v for the functions of the type $f(x) = \frac{1}{x^a}$:

$$D^v x^{-a} = \frac{(-1)^{\alpha}\Gamma(a+\alpha)}{\Gamma(a)} x^{-a-\alpha} \tag{1.16}$$

Although both of his definitions have some restrictions, Liouville (1834) provided a number of applications in geometrical, physical and mechanical problems.

Additionally, Liouville did remarkable work on complementary functions. He (1834) stated that the ordinary differential equation $\frac{d^n y}{dx^n} = 0$ has the complementary solution $y_c = c_0 + c_1 x + c_2 x^2 + \cdots + c_{n-1} x^{n-1}$. Thus $\frac{d^u y}{dx^u} = 0$ (u is arbitrary) should have a corresponding solution. Later Riemann used complimentary functions and created a significant impact on the development of FC.

After Liouville, G. F. Berhard Riemann developed a different theory of fractional integration. He used a generalization of a Taylor series and deduced the definition

$$D^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_c^x (x-t)^{\alpha-1} f(t) dt + \Psi(x) \qquad (1.17)$$

Because of the ambiguity of the lower limit c of the integration, Riemann introduced the complementary function $\Psi(x)$. Substituting $c = 0$ and without the complementary function $\Psi(x)$, equation (1.17) is called the Riemann-Liouville (RL) fractional integral, the popular fractional integral in present days.

Since neither Riemann nor Liouville solved the problem of additional complementary function, a number of famous mathematicians focused their work on solving that problem. Utilizing Cauchy's integral formula, as a starting point, N. Y. Sonin (1869) started to reach differentiation to arbitrary order. Later, A. V. Letnikov (1872) extended the idea given by Sonin. They both used a close contour integral method. Using Cauchy's integral formula for integer order derivatives

$$f^{(n)}(z) = \frac{n!}{2\pi i} \int_C \frac{f(t)}{(t-z)^{n+1}} dt, \qquad (1.18)$$

they generalized the fractional order by substituting the factorial with Euler gamma function $n! = \Gamma(n+1)$. Consequently, instead of a close circuit, H. Laurent (1884) used a

contour given as an open circuit (*Laurent loop*) developed today's definition of the Riemann-Liouville fractional integral:

$$ {}_cD_x^{-\alpha} f(x) = \frac{1}{\Gamma(\alpha)} \int_c^x (x-t)^{\alpha-1} f(t) dt, \qquad Re(\alpha) > 0 $$

(1.19)

by the standard contour integral method. J. H. Davis (1924) first introduced notation ${}_cD_t^{-\alpha} f(x)$ as a fractional integral and replace α instead of $-\alpha$ as FDs. However, the change from fractional integral to FDs is not accurate and problems originate from the integral $\int_c^x (x-t)^{\alpha-1} f(t) dt$, which is divergent in general. By analytical continuation it can be shown that

$$ {}_cD_x^{\alpha} f(x) = {}_cD_x^{n-\beta} f(x) = {}_cD_x^n \, {}_cD_x^{-\beta} f(x) = \frac{d^n}{dx^n} \left(\frac{1}{\Gamma(\beta)} \int_c^x (x-t)^{\beta-1} f(t) dt \right) $$

(1.20)

which is now known as Riemann-Liouville fractional derivatives (RLFDs) (Ross, 1977a). Where n is the smallest integer greater than α and $0 < \beta = n - \alpha < 1$. ${}_cD_x^n$ is the ordinary differentiation operator of $\frac{d^n}{dx^n}$.

Grünwald (1867) and Letnikov (1868) proposed another popular definition of FDs which is frequently used today. Using the idea from Liouville, to use the limit of a difference quotient using differences of fractional order, Grünwald and Letnikov obtained

$$ {}^{GL}D_t^{\alpha} f(t) = \lim_{h \to 0} \frac{\sum_{j=0}^{\left[\frac{t-a}{h}\right]} (-1)^j \binom{\alpha}{j} f(t-jh)}{h^{\alpha}}, \quad \alpha > 0 \qquad (1.21) $$

which is the definition of FDs now called Grünwald–Letnikov (GL) fractional derivative. In Eq. (1.21), t and a are the upper and lower limit of the differentiation. $\binom{\alpha}{j}$ is the

generalized binomial coefficient. Under certain conditions the GL definition is with the same as the RL definition. Now the question may arise whether the different definition of fractional derivatives (RL and GL) are either the same or different since the concept of origination is the same. However, the RL definition is suited for analytical solution and the GL definition is suited for numerical calculation.

At the end of the nineteenth century, Oliver Heaviside (1892) showed how certain linear differential equations can be solved by the use of generalized operators. He denoted the operator d/dx by the letter p and in several applications, he used arbitrary power of p, mostly $p^{1/2}$. Today his collection of works is named as Heaviside operational calculus and his method is useful in the theory of the transmission of electrical currents in cables.

At the beginning of the twentieth century, FC had already grown to a broad mathematical field. Many famous mathematicians contributed to the development of FC, including H. Weyl, Marchaud, Hardy, Littlewood, Watanabe, M. Riesz, B. S. Nagy, Erdélyi, Kober and M. Caputo. Besides RL and GL definitions, in the last century several definitions of FDs and fraction integral have been proposed (Weilbeer, 2005), among which Caputo derivative has been often adopted in recent research. For a function f with $(n-1)$ continuous derivative, Caputo derivative of order $\alpha > 0$ is defined as (Caputo, 1967):

$$D_*^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_a^t (t-\tau)^{n-\alpha-1} \left(\frac{d}{d\tau}\right)^n f(\tau) d\tau. \qquad (1.22)$$

A Caputo fractional derivative (CFD) coincides with the RL fractional derivative (RLFD) under the same set of homogeneous initial conditions. CFD is more popular in application because the fractional differential equations can contain the initial conditions in such forms

as $f(0), f'(0), f''(0), \ldots, f^{(n)}(0)$; for RLFD, however, it is necessary to specify the FDs of the function at the initial point (Podlubny, 1999; Almeida and Torres, 2011).

FC and its applications have grown rapidly in the last few decades. Consequently, the first conference that solely contributed to the theory and applications of FC was held in New Haven in 1974 (Ross, 1977b). Several books were published in the field of FC in the last couple of decades (Kilbas et al., 2006; Miller and Ross, 1993; Minardi, 2009; Oldham and Spanier, 1974; PIdlubny, 1999; Samko et al., 1993). The journal "Fractional Calculus & Applied Analysis" is solely concerned with topics on the theory of FC and its applications (Weilbeer, 2005). From a simple question of l'Hôpital to Leibniz in 1695, FC has become an interesting topic of research for many researchers. It has significant application in many science and engineering fields.

1.2. APPLICATIONS OF FRACTIONAL CALCULUS

Although FC has a long history, it does not appear systematically in modern literature, and from the application point of view it is rigorously limited because it was considered to be extremely difficult to solve problems with FDs (Almeida and Torres, 2009). Difficulties may arise due to the lack of geometrical and physical interpretation of fractional operators (Löhle, 2008). However, based on the projection of a shadow on the wall, Podlubny (2002) suggested geometrical interpretation of RL fractional integral. He (2002) also suggested geometrical and physical interpretation of the RL fractional differentiation, the Caputo fractional differentiation, the Riesz potential, and the Feller potential. Additionally, Podlubny (2007) also presented the animation of their interpretation. Recently, based on the GL definition, Machado (2009) presented a probabilistic interpretation of FD.

FC and its applications have attracted considerable interest and it has proved to be valuable tool for the modeling of many physical phenomena (Shao, 2009). The main reason of this fact is that the realistic modeling of a physical phenomenon depends not only on the instant time, but also on the history of the previous time which can be incorporated by using FC (El-Ajou et al., 2010). In particular, processes associated with complex systems have non-local dynamics involving the long term memory effect, and the fractional integral and FD operators can describe well in this phenomena (Kilbas et al., 2006). Furthermore, most of the engineering phenomena possess complex microscopic behavior and their macroscopic dynamics cannot be characterized by classical integer order derivative models (Kilbas et al., 2006). In order to better understand the potential of FC, some examples are given as follows:

Viscoelasticity: The most extensive applications of FC are in the field of viscoelasticity due to its ability to model hereditary phenomena with long memory effect (Mainardi, 2009). FD is the best mathematical tool to describe the visco-elastic constitutive law (Di Paola and Pirotta, 2009). Di Paola and Pirrotta (2009) demonstrated that as the creep compliance exhibited power law decay, the FD appeared naturally of the stress and strain response and the various components, such as, intensity and order of fractional operator, can be directly evaluated by a curve fitting procedure on creep test.

According to Stiassnie (1979), constitutive equations for an elastic material and a viscous material in a given unidirectional stress σ and the respective strain ε can be written as follows:

$$\sigma = k\varepsilon(t) = k\ {}_0D_t^0\varepsilon(t) \tag{1.23}$$

$$\sigma = k\frac{d\varepsilon}{dt} = k\ {}_0D_t^1\varepsilon(t) \tag{1.24}$$

Equations (1.23) and (1.24) are the Hooke's law of elasticity and Newton's law of viscosity respectively. ${}_0D_t^{\alpha}$ represents differentiation of order α with respect to t. In this case, $\alpha = 0$ and $\alpha = 1$ correspond to purely elastic material and purely viscous materials, respectively. Since viscoelastic materials behave in between viscous and elastic, it becomes reasonable to express their properties by applying an intermediate order between 0 to 1 (Scott Blain, 1949):

$$\sigma = E\ {}_0D_t^{\alpha}\varepsilon(t) \quad (0 \leq \alpha \leq 1) \tag{1.25}$$

Viscoelastic materials like polymers generally show time dependent behavior, such as creep, relaxation and damping. An accurate material model is necessary in order to avoid expensive tests by numerical simulation. There are various classical models, such as the Maxwell model, the Kelvin Voigt model, the Burger model and the Standard Linear Solid model, to determine the stress and strain interaction of viscoelastic materials. It is well-known that traditional integer order models do not always fit the experimental data well (Di Paola and Pirrotta, 2009). Therefore, fractional order models (FOMs) have been developed to characterize viscoelastic materials.

The applications of FC in the field of viscoelasticity started in the early 20th century. Nutting (1921) indicated that the stress relaxation phenomenon of viscoelastic materials appeared to be proportional to fractional power of time, and Gemant (1936, 1938) stated that damping properties of viscoelastic materials fit much better by using fractional powers of frequency. Scott-Blair (1949) again suggested applying fractional time-derivatives to meet the observations of Nutting and Gemant (Schmidt and Gaul, 2002). Caputo was the first to apply fraction derivatives to model the behavior of viscoelastic geological strata

(Caputo, 1967, 1976). Later, Caputo and Mainardi (1971) validated their FOM with experimental observations of some metals and glasses.

The damping properties of viscoelastic materials show weak frequency dependence within a broad frequency range (Adolfsson et al., 2005). This weak frequency dependence is hard to describe in classical viscoelastic models that consider integer order rate laws, at least without an excessive number of material parameters (Adolfsson et al., 2005). On the other hand, only one single FD operator acting on both stress and strain is necessary to model the simplest uniaxial fractional order viscoelastic model. More or less FC of viscoelastic material based on the methodology of replacing the viscous dashpot by generalized element called 'spring-pot', a name given by Bagley (Bagley, 1983; Koeller, 1984) needs only one parameter.

Until the beginning of the 1980s, the application of FD s in viscoelasticity had been limited to curve-fitting methods. Later, Bagley and Torvik (1983) demonstrated the physical justification for the concept of FDs. In 1986, Bagley and Torvik (1986) developed constraints on parameters of a fractional 3-parameter model to ensure a nonnegative rate of energy dissipation and a nonnegative internal work. Padovan (1987) applied time integration algorithms to calculate responses of viscoelastic structures by using a single equation involving FD operators acting on both stress and strain (Enelund et al. 1999).

Recently, the application of FC to viscoelasticity has expanded further. Drozdov (1997) constructed fractional differential model analogs of the Kelvin-Voigt, Maxwell and Mawwell-Weichert constitutive models and verified these models with experimental data for viscoelastic solids. Welch et al. (1999) applied time based FC technique instead of frequency based technique for quasi-static viscoelastic response because time-domain

techniques are more suitable than frequency- domain techniques in quasi-static viscoelastic theory. Adolfsson (2004) also formulated a fractional order viscoelastic model for large deformation and developed an algorithm for the integration of the constitutive response. Additionally, Adolfsson et al. (2005) presented physical interpretations of the viscoelastic model by using viscoelastic functions. Hanyga (2007) developed a general constitutive nonlinear model of viscoelastic medium with singular memory by applying internal variable concept. The detailed study of FC in viscoelastic media was done by Rossikin and Shitikova (1997, 2010) who used FDs to study wave propagation in viscoelastic media. Recently, Sunny et al. (2010) developed a model based on FC to model the hysteresis in conductive polymer and the accuracy of that model was verified by comparison with the experimental results. More information of the applications of FC in the field of viscoelasticity can be found in (Samko et al., 1993; Alcoutlabi and Martinez-Vega, 1998; Gen-guo et al., 2001; Soczkiewicz, 2002; Surguladze, 2002; Pritz, 2003; Chen et al., 2004; Lu, 2006; Catania and Sorrentino, 2007; Nasuno et al, 2007; Reyes-Melo et al. 2008; Mainardi, 2010).

Bioengineering: In the bioengineering field, it is highly necessary to develop an efficient and highly precise materials model to simulate the stress response of biological materials (Carew et al., 2003). FC is useful for modeling of soft biological tissue (Carew et al., 2003; Cafagna, 2007). Moreover, some complex biological systems like modeling of connecting gene expression with protein structure and their function can be expressed well by FC (Magin, 2010).

Since the distribution relaxation process appears in human tissues and cells, FC naturally has an important role in describing the input-output behavior of biological

systems. The properties of tissues and cells depend on the storage modulus that describes the elastic property and the loss modulus that describes the ability to absorb energy. Both of these properties change with frequency and the classical model is not able to describe these properties well. Yuan et al. (1997, 2000) investigated the properties of lung tissues and found the necessity of implementing FOMs. Since then, the application of FC in modeling of soft biological tissues has been gradually increasing.

Carew et al. (2003) formulated a one-dimensional version of fractional-order viscoelastic equations called *quasilinear fractionalorder viscoelasticity*, and applied it to model the stress response of porcine aortic valve tissues. Later, Chen et al. (2004) applied FD model to characterize the dynamic viscoelasticity of the agarose gels used for culturing tissues, especially cartilage cells. Craiem and Armentotano (2007) also presented a fractional order Voigt model and validated that model by an experimental observation that was conducted over human arterial *in-vivo*. FOMs have also been applied to model the liver tissues (Taylor et al., 2002; Kiss et al., 2004) and brain tissue (Kohandel, 2005). Freed and Diethelm (2006) developed a fractional order viscoelastic model for isotropic biological tissue and applied to the fat pad of human heel.

The FOM is able to characterize the dynamic behavior of arterial wall and Craiem et al. (2008) applied a model (a spring in parallel with two springpots) to describe the viscoelastic properties of an arterial specimen behavior. Al-Mezel et al. (2009) also presented an approximate solution to the nonlinear FC model of the semilunar heart valve vibrations. Recently, Papoulia et al. (2010) developed rheological representations (discrete spectrum models) for the FD viscoelastic element and proved its accuracy by presenting

some computational results. Other applications of FC on biological tissues and processes can be found in (Glockle and Nonnenmacher, 2002; Suki et al., 1994).

FC has been used not only in modeling of soft tissues. The behavior of neural systems can also be described well by using FOMs (Magin, 2006). Anastasio (1994, 1998) modeled the vestibular- oculomotor system using the Laplace domain as s^k or s^{-k}, where $0 < k < 1$. Fractional order circuit, such as the impedance $Z = Z_0/(s)^\alpha$ or $Z = Z_0/(j\omega)^\alpha$ has been used to develop an electrical circuit model of complex processes like the electrode cardiac tissue interface of a pacemaker electrode (Grimnes, S., Martinsen, 2000; Magin and Obaida, 2008; Magin, 2010). FOM was able to predict the propagation of diseases and treatment specific parameters effectively (Meral et al., 2010). Sinkus et al. (2007) used a FOM to fit magnetic resonance elastography (MRE) data for breast tumors. Magin (2010) stated that if the structure in living systems was fractal or the measurement of signal displayed anomalous properties, one should suspect that the dynamics might better be described by a FOM.

Fractional order controller: In order to attain an effective control of the physical systems, the study of fractional-order controllers has attracted great interests. The Proportional Integral Derivative (PID) controllers have been used widely in industrial applications for many years for its simplicity of design and good performance including low percentage overshoot and small settling time for slow process plants (Astrom and Hagglund, 1995; Biswas et al. 2009). However, a major concern is to attain an optimum performance and researchers have been trying to obtain the optimum performance by using fractional order tool (Xue et al., 2006). The fractional order PID controller is a generalization of the classical one that can fulfill stricter contradictory design

specifications. In a fractional order $PI^{\lambda}D^{\mu}$ controller, the Integral (I) and the Derivative (D) being fractional have wider scope of design, and therefore besides proportional (K_p), derivative (T_d) and integral (T_i) constants, the fractional order $PI^{\lambda}D^{\mu}$ has two more parameters: the order of fractional integration (λ) and the fractional derivative (μ). Therefore, finding an optimal settings of K_p, T_d, T_i, λ and μ it is easy to meet the user specifications (Biswas, et al., 2009). Podlubly (1999b) proposed a generalization of PID controller which responded better in comparison with the classical PID controller. Consequently, Xue et al. (2006) designed a fractional order controller and demonstrated that if properly designed and implemented, the fractional order PID controller will outperform the conventional integer order PID controller.

Since the fractional order PID controller performs well, it has found extensive applications in real industrial processes (Vinagre et al., 2007; Maiti et al., 2008). Calderon et al. (2003a,b; 2006) successfully applied fractional order control strategy in the control of a power electronic buck converter. This fractional order converter is fast to reach an effective control of devices (Cafagna, 2007). The CRONE (French acronym for *Controle Robuste d'Ordre Non-Entier*) team conducted research on fractional order controller and successfully applied fractional order controller to various sections, including car suspension control (Oustaloup et al., 1996), flexible transmission (Oustaloup, et al., 1995), and hydraulic actuator (Lanusse, 2000). Chengbin and Hori (2004) proposed a design of a fractional order PID^{k} controller for a torsional system's backlash vibration suppression control which gave the possibility of directly tuning and enabled one to adjust the control system's frequency response. However, previous classical method failed in suppressing the vibrations caused by gear backlash nonlinearity (Cafagna, 2007). Using fractional order

proportional and integral controller (FOPI), Bhaskaran et al. (2007) presented a new practical tuning method and also demonstrated by experimental results that FOPI was not only valid for first order plus delay time but also applicable for other general class of plants. Recently, Ahn et al. (2009) established a new strategy by using a fractional order integral and derivative controller for a temperature profile tracking.

Different design methods for fractional order controller have been proposed, such as pole distribution (Petras, 1999), frequency domain approach (Vinagre et al., 2000), state-space design approach (Dorcak et al., 2001) and hybrid approach (Chengbin and Hori, 2004). Monje et al. (2004b) proposed a design of PI^{α} to ensure that the closed loop system is robust to gain variations and the step responses exhibit an iso-damping property. By using a fractional order PID controller for active reduction of vertical tail buffeting, an extensive work has been developed by Sanchez (1999). In recent years, particle swarm optimization (PSO) technique which is a revolutionary type global optimization algorithm has been used in fractional order Controller (Zamani et al., 2009; Gao and Hu, 2008, 2009). Some other interesting work on fractional order controllers can be found in (Matignon and d'Andrea-Novel, 1996; Matignon, 1998; Milos and Martin, 2006; Monje et al., 2004a; Oustaloup, 1981; Axtell and Bise, 1990).

Chaotic behavior: In recent years, the application of FC in the chaotic behavior and dynamic system has become an interesting research area (Ahmad and Sprott, 2003; Podlubny, 1999a; Hilfer, 2000; Odaibat, 2010). Chaos, which is nonlinear and cannot be predicted beforehand, is important in information processing, safe and sound communication, liquid mixing and also biological systems (Chen and Lee, 2004). However, synchronization of a chaotic system is a challenging issue (Deng et al., 2009) and recent

studies proved that chaotic fractional order system can be synchronized well (Deng and Lee, 2005; Lu and Chen, 2006; Li et al, 2003; Zhou et al., 2008).

It is common that chaos in a continuous system occur of total order not less than three (Petráš, 2006). The other conditions for chaos in a continuous-time system are: there must be nonlinear element in the system and it must be sensitive to initial conditions. For a continuous – time system, the most important condition for occurring chaos is highly dependent on initial condition that is not observed in order less than three (Baleanu et al., 2010). However, chaotic behavior observed for the FOM of order less than three (Sheu et al., 2007). Hartley et al (1995) showed that the fractional order Chua's circuit with order as low as 2.7 can generate a chaotic attractor. Ahmad and Sprott (2003) demonstrated that using the proper control parameters, both an electronic chaotic oscillator and a mechanical chaotic ''jerk'' model can produce a chaotic attractor with system with orders as low as 2.1. Li and Chen (2004) also studied the chaotic behaviors in the fractional-order Rössler equations and found the order of chaotic behaviors as low as 2.4 and hyperchaos behaviors as low as 3.8.

The chaotic behavior of fractional order have been studied by Arena et al. (1998) who showed that nonautonomous Duffing systems with order less than 2 could still behave in a chaotic manner. Ge (2007) demonstrated that chaos in a modified Duffing system existed for total systems of order were 1.8, 1.9, 2.0, and 2.1. Ahmad and Harba (2003) investigated the chaos control for fractional chaotic systems, where controllers have been designed using the ''backstepping'' method of nonlinear control design. Ahmed et al., in (2001) studied the fractional-order Wien bridge sinusoidal oscillator and demonstrated that limit cycle could be generated for any fractional order, with a proper value of the amplifier

gain. Tavazoei (2008) proposed techniques to suppress chaotic oscillations in 3-D chaotic system by using fractional differentiator and fractional integrator. Many other fractional-order nonlinear systems are chaotic, such as the fractional-order Arneodo's systems (Lu, 2005), a fractional-order rotational mechanical system with a centrifugal governor (Ge and Jhang, 2007), the fractional-order Chen-Lee system (Tam et al., 2008) and the fractional order Newton-Leipnik system (Wang, 2003).

Heat conduction and diffusion phenomena: Transient, particularly periodic, diffusion problems are very common in engineering fields (Kulish and Lage, 2000). Transient heat flux measurement are widely used to measure heat flux in harsh environments, such as high enthalpy plasma flows, fusion plasma and rocket motor combustion chambers based on solving the inverse heat conduction problem in a semi-infinite environment (L*ö*hle et al., 2008). However, transient heat measurement is a challenging issue in present days. Using Non-Integer System Identification (NISI), Löhle et al. (2008) proposed a method that provided a significant improvement of the rather classical heat flux measurement technique. Podlubny (1999a) reported a comparative analysis between the traditional integer and fractional for a reheating surface and concluded that the fractional model enables a better fit to experimental results for the same number of free constants because of the nature of the model. Aoki et al. (2008) presented through examples that the time-dependent temperature in a transient thermal system can be approximately modeled with a fractional order differential equation. Murio (2008) also analyzed Caputo's time fractional inverse heat conduction problem. Additionally, Agrawal (2004b) presented a fractional order derivative based approach to compute the surface temperature and the heat flux at the contact surface of disk brake by carrying out a transient

analysis of two neighboring points. Recently, Pineda et al. (2011) presented a solution based on FOM for a one-dimensional transient heat conduction problem.

Diffusion phenomena, particularly anomalous diffusion phenomena, are common in many physical systems, such as pollutant transport through porous media, electron transfer in semiconductor and nuclear proliferation (Metzler, et al., 2000; Reynolds, 2002; Plasmanter, 1991). In recent decades, anomalous diffusion phenomena attracted particular interest in biology, chemistry, environmental science and even in economics (Gorenflo and Mainardi, 2009; Sun et al., 2010). It has been proved by experimental observation and theoretical investigation that the anomalous diffusion processes exhibits the characteristics of history dependence and long-range correlation (Sokolov, et al., 2004; Sun et al., 2010). Such anomalous diffusion phenomena can be characterized by using FDs (Li and Deng, 2007). Anomalous behaviors of fractional diffusion equation have been investigated in fractional Brownian motion (Lim and Muniandy, 2002), anomalous diffusion with adsorption (Drazer et al., 2002), tracer advection (Zaslavsky et al., 1997).

Based on the assumptions of linear unidirectional heat transport within a semi-infinite domain with common constant initial and asymptotic boundary conditions, Oldham and Spanier (1972, 1974) were the first to reduce the classical diffusion equation, which consisted of second order spatial derivatives and first order time derivatives, to an equation involving a first order spatial derivative and a half order time derivative (Murio, 2007). Kulish and Lage (2000, 2002) demonstrated the application of FC to the solution of time-dependent, viscous-diffusion fluid mechanics problems. Since then, many engineers and scientists have developed the FOMs to represent diffusion phenomena. Kilbas et al. (2006) presented a FOM for the superdiffusion equation. Applications of fractional diffusion

process, such as super diffusion and non-Gaussian diffusion problem can be found in (Liu et al., 2004; Meerschaert and Tadjeran, 2004, 2006). Murio (2007, 2009) developed a stable numerical solution of a fractional-diffusion inverse heat conduction problem. Detailed applications of FC to the diffusion equations can be found in Debnath (2003).

Other applications of fractional calculus: The applications of FC are extensive. Miller and Rose (1993) mentioned that almost every field of science and engineering has the applications of FDs. Many processes in physics and engineering sciences would be governed more accurately by fractional order differential equations instead of traditional integer order differential equations (Oustaloup et al., 2000; Ray et al., 2005). With its emerging applications in a variety of fields, FC and its application has become an important topic for many researchers. In addition to aforementioned fields, some other fields of its applications include electrochemistry (Ichise et al., 1971; Sun et al., 1984a; Oldham, 2010; Debnath, 2003), dielectric polarization (Sun et al., 1984b), colored noise (Mandelbrot, 1967), finance (Scalas et al., 2000) and also physics (Hilfer, 2000). Detailed applications of FC can be found in (Oustaloup et al., 2000, Machado et al., 2009).

Although FC can describe many physical phenomena well, it still has some drawbacks. These are: (1) Integer order models have clear geometrical and physical meanings that fractional order definitions lack (2) There are lots of fractional operators like the Marchaud, the Riesz, the Caputo, the GL and the RL definitions. Since real problems involve different operators. For example, the Riesz operator is more amenable to obtain simpler solutions in an unbounded domain working in dynamics at steady state (Paola and Pirrotta, 2009). However, these many different fractional definitions could bring confusion in applications (3) FDs and integrals may not be solved by hands and they usually require

rigorous and laborious numerical calculations. The integer order derivative of a constant is always zero, the fractional order derivatives of a constant, however, is not always zero iv) Different symbols have been used to define the same fractional operator, and therefore, it can be difficult to recognize the fractional operator just from the symbol being used in various articles.

1.3. SCOPE OF THE THESIS

The primary focus of this thesis is to develop numerical schemes to solve optimization problems in fractional orders. Since FDs have such property as being non-local, it can be extremely challenging to find analytical solutions for fractional parametric optimization problems, and in many cases, analytical solutions may not exist. Therefore, it is of great importance to develop numerical methods for such problems. This thesis presents formulations and solution schemes of various optimization problems, including (1) fractional optimal control problems (2) linear fractional functional minimization problems and (3) nonlinear fractional functional minimization problems. The FDs are defined in terms of the RL or Caputo definition. The convergence of each method is analyzed to ensure stability of the algorithm. Various orders of FDs are considered and as the order approaches the integer value of 1, the numerical solution recovers the analytical result for the corresponding integer order problem.

1.4. OUTLINE OF THE THESIS

The organization of the thesis is as follows:

Chapter 1: Introduction: This chapter described the history and application of FC.

Chapter 2: Fractional optimal control problem in spherical and cylindrical coordinates: In this Chapter, we present a general formulation and numerical scheme for

Fractional Optimal Control Problems (FOCPs) of distributed systems in spherical and cylindrical coordinates. The FDs are expressed in the Caputo-Sense and the Fractional Optimal Control (FOC) equations are reduced to the Volterra-type integral equations. The time domain is discretized into several subintervals. Various orders of FDs are analyzed and compare the result with the analytical result. This work has been published in the following publications:

1. Hasan, M. M., Tangpong, X. W., Agrawal, O. P., 2011a "Fractional Optimal Control of Distributed Systems in Spherical and Cylindrical Coordinates," *Journal of Vibration and Control* (in press).
2. Hasan, M. M., Tangpong, X. W., Agrawal, O. P., 2011b, "A Formulation and Numerical Scheme for Fractional Optimal Control of Cylindrical Structures Subjected to General Initial Conditions," *Fractional Dynamics and Control*, editors, Dumitru Baleanu, J.A. Tenreiro Machado and Albert Luo, Springer.
3. Hasan, M. M., Tangpong, X. W., Agrawal, O. P., 2010, "Fractional Optimal Control of a Hollow Cylindrical Structure," *The 3rd Nonlinear Science and Complexity Conference*, Ankara, Turkey, July 28-31.

Statement of Joint authorship

Md. Mehedi Hasan (Candidate) derived the formulation of FOCPs in Cylindrical coordinates, developed the numerical codes in MATLAB and wrote the initial draft of the manuscript.

Xiangqiang W. Tangpong derived the formulation of FOCPs in spherical coordinates, directed and guided the work, assisted in the interpretation of results, paper preparation and revised/proofread the manuscript.

Om P. Agrawal assisted in the interpretation of results, directed the work and proofread the manuscript.

Chapter 3: A numerical scheme for a class of parametric problem of fractional variational calculus: This chapter presents a numerical scheme for a linear functional minimization problem that involves FD terms. The FD is defined in terms of the RL definition. The spatial domain is discretized into several subdomains and 2-node one-dimensional linear elements are adopted to approximate the solution and it's FD at point within the domain. Various fractional orders of derivative are considered and as the order approaches the integer value of 1, the solution recovers the analytical result for the corresponding integer order problem. In addition, convergence study and error analysis have been performed to ensure the stability and accuracy of the algorithms. This work has been accepted by the following publication:

1. Agrawal, O. P., Hasan, M. M., Tangpong, X. W., 2011, "A numerical scheme for a class of parametric problem of fractional variational calculus," *ASME 2011 International Design Engineering Technical Conferences & Computers and Information in Engineering Conference IDETC/CIE,* Washington, DC, September 28-31 (accepted).

Statement of Joint authorship

Md. Mehedi Hasan (Candidate) developed the numerical codes in MATLAB, interpreted the results and wrote the results and discussion part of the manuscript.

Xiangqiang W. Tangpong wrote the manuscript, assisted in the interpretation of results and acted as the corresponding author.

Om P. Agrawal derived the formulation and numerical scheme of that problem and proofread the manuscript.

Chapter 4: A numerical scheme for fractional-order isoperimetric problem: This chapter presents numerical schemes for solving a nonlinear functional minimization problem in fractional order. The integer order version of the problem is to determine the shape of a hanging chain in its equilibrium state with two fixed ends and constant length. We defined the FD in terms of the RL definition. The spatial domain is discretized into several subdomains. Different definitions of GL approximations are taken to approximate the FDs of fractional orders in between 0 and 1. The performances of different GL definitions are analyzed.

Chapter 5: Conclusions and future work: The main contribution of this thesis and results are summarized in this chapter. Finally, this chapter concludes with some prospective works.

CHAPTER 2. FRACTIONAL OPTIMAL CONTROL PROBLEMS IN SPHERICAL AND CYLINDRICAL COORDINATES

This Chapter presents a general formulation and numerical scheme for Fractional Optimal Control Problem (FOCP) of distributed systems in spherical and cylindrical coordinates (Hasan et al., 2010, 2011a, b). The materials presented in this chapter have been accepted for publications in (Hasan et al., 2010, 2011a, b). The FDs are expressed in the Caputo-Sense. The performance index of FOCP is considered as a function of both the state and the control variables and the dynamic constraints are expressed by a partial fractional differential equation. Method of separation of variables is employed to separate the time and space terms, and the eigenfunction approach is used to eliminate the terms containing space parameter and define the formulation in terms of countable number of infinite state and control variables. The Fractional Optimal Control (FOC) equations are reduced to the Volterra-type integral equations. For the problems considered, only a few eigenfunctions in each direction are sufficient for the calculations to converge. The time domain is discretized into several subintervals and the result is more stable for larger number of time segments. Various orders of fractional derivatives (FDs) are analyzed and the results converge toward those of integer optimal control problems as the order approaches the integer value of 1.

2.1. INTRODUCTION

The general definition of an optimal control problem requires the minimization of a functional over an admissible set of control functions subject to dynamic constraints on the state and control variables (Agrawal, 1989). Optimal control problems have found applications in many areas such as engineering, science and economics. A FOCP is an

optimal control problem in which either the performance index or the differential equations governing the dynamics of the system or both contain at least one FD term (Tangpong and Agrawal, 2009). As the demand for accurate and high precision systems increases, the demand for numerical formulation and solution scheme of FOC theories also increases (Agrawal, 2004a).

The formulation of FOCPs stems from the Fractional Variational Calculus (FVC) and this FVC is applied to deterministic and stochastic analysis of FOCPs (Baleanu et al., 2009). Riewe (1996, 1997) was among the earliest researchers to formulate a FVC, and used FC of variations to develop the Larangian, Euler-Lagrange equations, and other concepts for mechanics of nonconservative systems (Agrawal, 2004a). Later, Agrawal (2002) presented Euler-Lagrange equations for both unconstrained and constrained Fractional Variational Problems (FVPs). Klimek (2001) presented a model of fractional sequential mechanics with symmetric FDs. Klimek (2002) also presented stationary conservation laws for fractional differential equations with variable coefficients.

Integer order optimal controls (IOOCs) have already been well established and a significant amount of work has been done in the field of optimal control of integer order systems. Excellent textbooks are available in that field (Bryson et al., 1969; Sage and White, 1977; Hestenes, 1966; Gregory and Lin, 1992) and various methods have been employed to solve such problems (Agrawal, 1989; Gregory and Lin, 1992). A lot of work has been done in the area of Fractional Order Control (FORC) (Xue and Chen, 2002; Manabe, 2003; Bode, 1945; Oustaloup, 1983, 1991; Podlubny, 1999; Vinagre and Chen, 2002) without any discussions about FOCP. With the growing number of applications of FOCPs, it is necessary to establish solutions for FOCPs.

The outline of FOCPs has grown rapidly over the last decade. Agrawal (2004a) gave a general formulation of FOCPs in the Riemann-Liouville (RL) sense and described a solution scheme for FOCPs for classical optimal control problem that was based on variational virtual work coupled with the Lagrange multiplier technique. The works presented in (Tangpong and Agrawal, 2009; Agrawal, 2006a, 2008a) formulated FOCPs in terms of Caputo fractional derivatives (CFDs) instead of RL derivatives and an iterative numerical scheme was applied to solve the problem numerically where the time domain was discretized into small segments. CFDs allow one to incorporate the usual initial conditions in a simple manner, and therefore are popular choices for researchers. In (Agrawal and Baleanu, 2007), the FDs of the system were approximated using the Grunwald-Letnikov definition that led to a set of algebraic equations that can be solved using numerical techniques (Tricaud and Chen, 2010b). Agrawal (2005) presented a general scheme for stochastic analysis of FOCPs. In (Baleanu et al., 2009) a different solution scheme was proposed where a modified Grunwald-Letnikov definition was used to derive a central difference formula. Based on the expansion formula for FDs, a new solution scheme was proposed in (Jelicic and Petrovacki, 2009). Using the definitions of the FOCPs, Frederico and Torres (2006, 2008a,b) formulated a Noether-type theorem in the general context of the FOC in the sense of CFDs. Agrawal (2008b) considered a one dimensional distributed system and used the eigenfunction approach to solve the FOC problem. The eigenfunction expansion-based scheme was also used in (Ozdemir et al., 2009a) to formulate FOCP of a 2-dimensional distributed system.

In recent years, FOCPs have been addressed in polar coordinates. Ozdemir et al. (2009b) presented a formulation for a 2D distributed system in polar coordinates using the

separation of variables method. FOCPs of a 3D distributed system were investigated in cylindrical coordinate in (Ozdemir et al., 2009c). Fractional diffusion problems were discussed in polar coordinates (Ozdemir et al., 2009d) and in cylindrical (Qi and Liu, 2009) and spherical coordinates (Povstenko, 2008); however, those works did not discuss FOCPs. This chapter presents general formulations and numerical solution schemes for FOCPs in spherical and cylindrical coordinates. Three cases of problems are discussed in detail, which are 1) a sphere with axial symmetry, 2) a sphere with complete symmetry and 3) a hollow cylinder with axial symmetry. The FDs are defined in the Caputo sense and the separation of variable method is used to decouple the equations. The eigenfunction approach is used to eliminate the space parameters and it is indicated by the combination of state and control functions. For numerical solutions, the FD differential equations are converted into Volterra-type integral equations and the time domain is discretized into several segments. The formulation derived here is used to solve for various derivative orders and the calculation converges toward the analytical solution for integer order problems as the order approaches 1.

2.2. FORMULATION OF A FRACTIONAL OPTIMAL CONTROL PROBLEM

A FOCP is defined in terms of the left and the right Caputo fractional derivatives (CFDs) (Tangpong and Agrawal, 2009; Hasan et al., 2011a,b) that are given as the following:

The left Caputo fractional derivative (LCFD),

$$ {}_{a}^{C}D_{t}^{\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_{a}^{t} (t-\tau)^{n-\alpha-1} \left(\frac{d}{d\tau}\right)^{n} f(\tau) d\tau \tag{2.1} $$

and the right Caputo fractional derivative (RCFD),

$$ {}_t^C D_b^\alpha f(t) = \frac{1}{\Gamma(n-\alpha)} \int_t^b (\tau - t)^{n-\alpha-1} (-\frac{d}{d\tau})^n f(\tau) d\tau \quad (2.2) $$

where f (t) is a time dependent function and α is the order of the derivative in the range $n-1 < \alpha < n$. When α is an integer, the left (forward) and the right (backward) derivatives are replaced with D and $-D$, respectively, where D is the differential operator. Note that in the literature, the CFD generally means the LCFD.

Using the above definitions, the FOCP under consideration can be defined as follows. Find the optimal control $f(t)$ that minimizes the performance index

$$ J(f) = \int_0^1 F(w, f, t) dt \quad (2.3) $$

subject to the dynamic constraints

$$ {}_0^C D_t^\alpha w = G(w, f, t) \quad (2.4) $$

and the initial conditions

$$ w(0) = w_0, \quad (2.5) $$

where $w(t)$ and $f(t)$ are the state and control variables, respectively, F and G are two arbitrary functions, and w_0 represents the initial condition of the state variable. Note that equation (2.3) may also include additional terms containing state variable at the end points. The order of the derivative α is considered to be in between 0 and 1, and when $\alpha = 1$, the above problem reduces to a standard optimal control problem. Here the limits of the integration are taken as 0 and 1 considering a normalized case. An end point term can also be included in the performance index and any integration limits can be considered with any order of the derivative. The conditions considered here are for simplicity only.

To obtain the necessary equations, we combine equations (2.3) and (2.4) using a Lagrange multiplier technique, and then take the variations of resulting equation and apply integration by parts to modify the equation so that it does not contain variations of a derivative term. After imposing necessary terminal conditions and setting the coefficients of $\delta\lambda, \delta w$, and δf to zero, the following equations are obtained:

$$ {}_{0}^{C}D_{t}^{\alpha} w = G(w, f, t) \tag{2.6} $$

$$ {}_{t}^{C}D_{1}^{\alpha} \lambda = \frac{\delta F}{\delta w} + \left(\frac{\delta G}{\delta w}\right)^{T} \lambda, \tag{2.7} $$

$$ \frac{\delta F}{\delta f} + \left(\frac{\delta G}{\delta f}\right)^{T} \lambda = 0 , \tag{2.8} $$

$$ w(0) = w_0 \text{ and } \lambda(1) = 0 \tag{2.9} $$

where λ is the Lagrange multiplier also known as co-state or adjoint variable. The details of the derivation of equations (2.6)-(2.9) are given in (Agrawal, 2004a).

Equations (2.6)–(2.8) represent the Euler–Lagrange equations for the FOCP. These equations give the necessary conditions for the optimality of the FOCP considered here. They are very similar to the Euler–Lagrange Equations for classical optimal control problems, except that the resulting differential equations contain the left and the right FDs. Observe that equation (2.6) contains the LCFD whereas equation (2.7) contains the RCFD. This clearly indicates that the solution of such optimal control problems requires knowledge of not only forward derivatives but also of backward derivatives to account for all end conditions. In classical optimal control theories, such issue is either not discussed or not clearly stated largely because the backward derivative of order 1 is the negative of the forward derivative of order 1. It can be demonstrated that in the limit of $\alpha \to 1$,

equations (2.6)– (2.8) reduce to those obtained using the standard methods for optimal control problems.

The following section presents a formulation for FOC of a distributed system in spherical and cylindrical co-ordinates.

2.3. FORMULATIONS OF FOC OF SPHERICAL AND CYLINDRICAL STRUCTURES

This section presents the formulations of FOC of two spherical structures and two cylindrical structures.

2.3.1. Spherical structures

For spherical structures, the FOCP in consideration is as follows: Find the control $f(r,\theta,\varphi,t)$ that minimizes the cost functional

$$J(f) = \frac{1}{2}\int_0^1 \int_0^{2\pi} \int_0^{\pi} \int_0^R \left[Q' w^2(r,\theta,\varphi,t) + R' f^2(r,\theta,\varphi,t)\right] r^2 dr d\theta d\varphi dt \tag{2.10}$$

subjected to the system dynamic constraints

$$\frac{\partial^\alpha w}{\partial t^\alpha} = \beta \nabla^2 w(r,\theta,\varphi,t) + f(r,\theta,\varphi,t), \tag{2.11}$$

where

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2 Sin^2\theta}\frac{\partial^2}{\partial \varphi^2} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{Cot\theta}{r^2}\frac{\partial}{\partial \theta} \tag{2.12}$$

is the Laplacian in spherical coordinate. The azimuth angle φ is in the range of $[0, 2\pi]$, and the polar angle θ is in the range of $[0, \pi]$. Two cases of spherical structures: axial symmetry and complete symmetry—are discussed next.

2.3.1.1. Sphere with axial symmetry

For a sphere with axial symmetry, there is no variation in the azimuth angle φ, and equations (2.10) and (2.12) reduce to

$$J(f) = \frac{1}{2}\int_0^1 \int_0^\pi \int_0^R [Q'w^2(r,\theta,t) + R'f^2(r,\theta,t)]r^2 dr d\theta dt \tag{2.13}$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r} + \frac{1}{r^2}\frac{\partial^2}{\partial \theta^2} + \frac{Cot\theta}{r^2}\frac{\partial}{\partial \theta}. \tag{2.14}$$

The initial condition is

$$w(r,\theta,0) = w_0(r,\theta) \tag{2.15}$$

and the boundary conditions are

$$w(R,\theta,t) = w(0,\theta,t) = 0, \quad t > 0. \tag{2.16}$$

where $w(r,\theta,t)$ and $f(r,\theta,t)$ are the state and control functions that depend on radius r, angle θ and time t. $\frac{\partial^\alpha w}{\partial t^\alpha}$ is the left partial Caputo derivative of $w(r,\theta,t)$ with order α with respect to time t. Here we consider $0 < \alpha < 1$. Q' and R' are two arbitrary functions that may depend on time. R is the radius of the sphere. For convenience, the upper limit of time t is taken as 1.

The eigenfunction approach is used here to decouple the equations. The state and the control functions can be expressed as

$$w(r,\theta,t) = \sum_{n=1}^{k}\sum_{j=1}^{m} q_{nj}(t)P_n(X)J_{n+\frac{1}{2}}(u_j\frac{r}{R})/\sqrt{r}, \tag{2.17}$$

$$f(r,\theta,t) = \sum_{n=1}^{k}\sum_{j=1}^{m} p_{nj}(t)P_n(X)J_{n+\frac{1}{2}}(u_j\frac{r}{R})/\sqrt{r}, \tag{2.18}$$

where $J_{n+\frac{1}{2}}(u_j \frac{r}{R})$ is the $\left(n+\frac{1}{2}\right) - th$ order Bessel function of the first kind and u_j are the roots of this Bessel function. $P_n(X)$ is the Legendre polynomial of the first kind, and $X = cos\theta$. The total numbers of the eigenfunctions, k and m, should both go to infinity; however, for practical applications, k and m are taken as finite values through convergence studies. $q_{nj}(t)$ and $p_{nj}(t)$ are the state and control eigencoordinates. By substituting equations (2.17) and (2.18) into equation (2.13), we obtain

$$J(f) = \frac{R^2}{4}\int_0^1 \left[\sum_{n=1}^{k}\sum_{j=1}^{m}\frac{2}{2n+1}\left(Q' J^2_{n+\frac{3}{2}}(u_j) q^2_{in}(t) + R' J^2_{n+\frac{3}{2}}(u_j) p^2_{jn}(t)\right)\right] dt. \qquad (2.19)$$

Substituting equations (2.17) and (2.18) into equation (2.11), we get

$${}_0^C D_t^\alpha q_{nj}(t) = \left[n(n+1)r^{-2} - \left(\frac{u_j}{R}\right)^2 - 2X\frac{P'_n}{P_n}r^{-2} + (1-X^2)\frac{P''_n}{P_n}r^{-2}\right]\beta q_{jn}(t) + p_{jn}(t) \quad , \qquad (2.20)$$

and from equations (2.6)-(2.9), (2.19) and (2.20), we obtain

$${}_0^C D_t^\alpha p_{nj}(t) =$$

$$-\frac{Q'}{R'} q_{jn}(t) + \left[n(n+1)r^{-2} - \left(\frac{u_j}{R}\right)^2 - 2X\frac{P'_n}{P_n}r^{-2} + (1-X^2)\frac{P''_n}{P_n}r^{-2}\right]\beta p_{jn}(t) \qquad (2.21)$$

By substituting equation (2.17) into equation (2.15), and then multiplying the equation by $r^{\frac{3}{2}} J_{n+\frac{1}{2}}(\mu_j \frac{r}{R})$ on both sides and integrating it from 0 to R, we find the initial condition of the eigencoordinate,

$$q_{nj}(0) = \frac{2n+1}{R^2 J^2_{n+\frac{1}{2}}(u_j)}\int_{-1}^{1}\int_0^R r^{\frac{3}{2}} J_{n+\frac{1}{2}}\left(u_j \frac{r}{R}\right) w_0(r,\theta) P_n(X) dr dX. \qquad (2.22)$$

Equations (2.20) and (2.21) have $m+1$ sets of decoupled equations that can be solved separately. A numerical scheme that can be used to solve equations (2.20) and (2.21) is given in Section 2.5.

For $\alpha = 1$, equations (2.20) and (2.21) reduce to

$$\dot{q}_{nj}(t) = \left[n(n+1)r^{-2} - \left(\frac{u_j}{R}\right)^2 - 2X\frac{P'_n}{P_n}r^{-2} + (1-X^2)\frac{P''_n}{P_n}r^{-2}\right]q_{jn}(t) + p_{jn}(t), \quad (2.23)$$

$$\dot{p}_{nj}(t) = -\frac{Q'}{R'}q_{jn}(t) + \left[n(n+1)r^{-2} - \left(\frac{u_j}{R}\right)^2 - 2X\frac{P'_n}{P_n}r^{-2} + (1-X^2)\frac{P''_n}{P_n}r^{-2}\right]\beta p_{jn}(t) \quad (2.24)$$

Equations (2.23) and (2.24) represent a set of linear differential equations and the general solutions of these equations are given in (Agrawal, 2008b).

2.3.1.2. Sphere with complete symmetry

For a sphere with complete symmetry, there is also no variation in the polar angle θ, and equations (2.10) and (2.12) further reduce to

$$J(f) = \frac{1}{2}\int_0^1 \int_0^R [Q'w^2(r,t) + R'f^2(r,t)]r^2 dt, \quad (2.25)$$

$$\nabla^2 = \frac{\partial^2}{\partial r^2} + \frac{2}{r}\frac{\partial}{\partial r}. \quad (2.26)$$

The initial condition is

$$w(r,0) = w_0(r), \quad (2.27)$$

and the boundary condition is

$$w(R,t) = 0, \quad t > 0. \quad (2.28)$$

The eigenfunction method is used to decouple the equations, and state function and the control function can be written as

$$w(r,t) = \sum_{j=1}^{m} q_j(t) J_{\frac{1}{2}}(u_j \frac{r}{R})/\sqrt{r}\,, \tag{2.29}$$

$$f(r,t) = \sum_{j=1}^{m} p_j(t) J_{\frac{1}{2}}(u_j \frac{r}{R})/\sqrt{r}\,, \tag{2.30}$$

where $J_{\frac{1}{2}}(u_j \frac{r}{R})$ is the $\frac{1}{2} - th$ order Bessel function of the first kind and u_j are the roots of this Bessel function. The parameter m represents the number of eigenfunctions, and for practical reasons, m is taken as a finite value that is determined through convergence studies. $q_j(t)$ and $p_j(t)$ are the state and control eigencoordinates. By substituting equations (2.26) and (2.27) into (2.25), we obtain the cost function

$$J(f) = \frac{1}{4}\int_0^1 R^2 \left[\sum_{j=1}^{m}\left(Q' J^2_{n+\frac{3}{2}}(u_j) q_j^2 + R' J^2_{n+\frac{3}{2}}(u_j) p_j^2\right)\right] r^2 dt\,. \tag{2.31}$$

Next, by substituting equations (2.29) and (2.30) into equation (2.11) and equating the coefficients of $J_{\frac{1}{2}}(u_j \frac{r}{R})/\sqrt{r}$, we obtain

$${}_0^C D_t^{\alpha} q_j(t) = -\left(\frac{u_j}{R}\right)^2 \beta q_j(t) + p_j(t). \tag{2.32}$$

From equations (2.6)-(2.9), (2.31) and (2.32), the following equation is obtained,

$${}_t^C D_1^{\alpha} p_j(t) = -\frac{Q'}{R'} q_j(t) - \beta \left(\frac{u_j}{R}\right)^2 p_j(t). \tag{2.33}$$

We then obtain the initial condition of the eigencoordinate by substituting equation (2.29) into equation (2.27),

$$q_j(0) = \int_0^R w_0(r) r^{\frac{3}{2}} J_{\frac{1}{2}}\left(u_j \frac{r}{R}\right) dr \Big/ R^2 J^2_{n+\frac{3}{2}}(u_j). \tag{2.34}$$

Equations (2.32) and (2.33) can be solved by the numerical algorithm presented in Section 4.

In the case of $\alpha = 1$, equations (2.32) and (2.33) reduce to a set of linear differential equations for an integer optimal control problem, and the closed form solution can be found in (Agrawal, 2008b).

2.3.2. Cylindrical structures

For cylindrical structures, the FOCP in consideration is as follows: Find the control $f(r,\theta,z,t)$ that minimizes the cost functional

$$J(f) = \frac{1}{2}\int_0^1 \int_0^L \int_0^{2\pi} \int_0^R [Q'w^2(r,z,\theta,t) + R'f^2(r,z,\theta,t)]rdrd\theta dzdt \tag{2.35}$$

subject to the system dynamic constraints

$$\frac{\partial^\alpha w}{\partial t^\alpha} = \beta\left(\frac{\partial^2 w(r,z,\theta,t)}{\partial r^2} + \frac{1}{r}\frac{\partial w(r,z,\theta,t)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 w(r,z,\theta,t)}{\partial \theta^2} + \frac{\partial^2 w(r,z,\theta,t)}{\partial z^2}\right) + f(r,z,\theta,t). \tag{2.36}$$

For an axial symmetric case, there are no variations in θ, and therefore, Eqs. (2.35) and (2.36) become

$$J(f) = \frac{1}{2}\int_0^1 \int_0^R \int_0^L [Q'w^2(r,z,t) + R'f^2(r,z,t)]rdrdzdt\,, \tag{2.37}$$

$$\frac{\partial^\alpha w}{\partial t^\alpha} = \beta\left(\frac{\partial^2 w(r,z,t)}{\partial r^2} + \frac{1}{r}\frac{\partial w(r,z,t)}{\partial r} + \frac{\partial^2 w(r,z,t)}{\partial z^2}\right) + f(r,z,t), \tag{2.38}$$

where $\frac{\partial^\alpha w}{\partial t^\alpha}$ is the partial Caputo derivative of order α and $0 < \alpha < 1$. Q' and R' are the two arbitrary functions that may depend on time. R and L are respectively the cylinder's radius and length. For convenience, the upper limit of time t is taken as 1. The initial condition is represented by

$$w(r,z,0) = w_0(r,z). \tag{2.39}$$

Two cases of cylindrical structures: solid and hollow—are discussed next.

2.3.2.1. Solid cylinder with axial symmetry

The boundary conditions for FOC of a solid cylinder are considered as

$$w(0,z,t) = w(R,z,t) = w(r,0,t) = w(r,L,t) = 0, \quad t > 0. \tag{2.40}$$

The eigenfunction approach is used here to decouple the equations and the state and the control functions are found to be

$$w(r,z,t) = \sum_{i=1}^{n}\sum_{j=1}^{m} q_{ij}(t) J_0\left(u_j \frac{r}{R}\right) sin(i\pi \frac{z}{L}), \tag{2.41}$$

$$f(r,z,t) = \sum_{i=1}^{n}\sum_{j=1}^{m} p_{ij}(t) J_0\left(u_j \frac{r}{R}\right) sin(i\pi \frac{z}{L}), \tag{2.42}$$

where $J_0\left(u_j \frac{r}{R}\right)$ and $sin(i\pi \frac{z}{L})$ are the eigenfunctions in the radial direction and the axial direction respectively. The total numbers of eigenfunctions, n and m, are determined by convergence studies. J_0 is the zero-order Bessel function of the first kind and u_j are the roots of this Bessel function. $q_{ij}(t)$ and $p_{ij}(t)$ are the state and control eigencoordinates. Substituting equations (2.41) and (2.42) into (2.35), we obtain the cost function

$$J = \frac{R^2 L}{8} \int_0^1 \sum_{i=1}^{n}\sum_{j=1}^{m} {J_1}^2(u_j)\left[Q' {q_{ij}}^2(t) + R' {p_{ij}}^2(t)\right] dt. \tag{2.43}$$

By substituting equations (2.41) and (2.42) into equation (2.36) and equating the coefficients of $J_0\left(u_j \frac{r}{R}\right) sin(i\pi \frac{z}{L})$, we obtain

$${}_0^C D_t^\alpha q_{ij}(t) = -\beta\left(\left(\frac{u_j}{R}\right)^2 + \left(\frac{i\pi}{L}\right)^2\right) q_{ij}(t) + p_{ij}(t). \tag{2.44}$$

From equations (2.6)-(2.9), (2.43) and (2.44), we further obtain

$${}_t^C D_1^\alpha p_{ij}(t) = -\frac{Q'}{R'} q_{ij}(t) - \beta\left(\left(\frac{u_j}{R}\right)^2 + \left(\frac{i\pi}{L}\right)^2\right) p_{ij}. \tag{2.45}$$

Substituting Eq. (2.41) into equation (2.39), and then multiplying the equation by $rJ_0\left(u_j\frac{r}{R}\right)sin(i\pi\frac{z}{L})$ on both sides and integrate it, we find the initial condition of the eigencoordinates

$$q_{ij}(0) = \frac{4\int_0^L\int_0^R rw_0(r,z)J_0\left(u_j\frac{r}{R}\right)sin(i\pi\frac{z}{L})drdz}{R^2LJ_1{}^2(u_j)}, \tag{2.46}$$

where J_1 is the first-order Bessel function of the first kind.

2.3.2.2. Hollow cylinder with axial symmetry

The boundary conditions of a hollow cylinder is considered as

$$w(a,z,t) = w(R,z,t) = 0, \quad t > 0. \tag{2.47}$$

Using the eigenfunction approach, the state function and the control function are found to be

$$w(r,z,t) = \sum_{i=1}^n\sum_{j=1}^m q_{ij}(t)\,u_0(\lambda_j r)sin(i\pi\frac{z}{L}), \tag{2.48}$$

$$f(r,z,t) = \sum_{i=1}^n\sum_{j=1}^m p_{ij}(t)\,u_0(\lambda_j r)sin(i\pi\frac{z}{L}), \tag{2.49}$$

where

$$u_0(\lambda_j r) = Y_0(\lambda_j \mathrm{a})J_0(\lambda_j r) - J_0(\lambda_j \mathrm{a})Y_0(\lambda_j r) \tag{2.50}$$

are the eigenfunctions in the radial direction, and $sin(i\pi z/L)$ are the eigenfunctions in the axial direction. J_0 and Y_0 are the zero-order Bessel function of the first kind and the second kind, respectively, and λ_j are the roots of the characteristic equation for the eigenfunctions in the radial direction. Substituting Eqs. (2.48) and (2.49) into (2.10), we obtain the cost function

$$J = \frac{L}{4}\int_0^1\sum_{i=1}^n\sum_{j=1}^m\left(Q'q_{ij}{}^2 + R'p_{ij}{}^2\right)\left(\int_a^R[u_0(\lambda_j r)]^2 rdr\right)dt. \tag{2.51}$$

By substituting Eqs. (2.48) and (2.49) into Eq. (2.11) and equating the coefficients of $u_0(\lambda_j r)sin(i\pi z/L)$, we obtain

$$ {}_0^C D_t^\alpha q_{ij}(t) = -\beta\left(\left(\frac{\lambda_j}{R}\right)^2 + \left(\frac{i\pi}{L}\right)^2\right) q_{ij}(t) + p_{ij}(t). \tag{2.52} $$

From Eqs. (2.6)-(2.9), (2.51) and (2.52), we obtain

$$ {}_t^C D_1^\alpha p_{ij}(t) = -\frac{Q'}{R'} q_{ij}(t) - \beta\left(\left(\frac{\lambda_j}{R}\right)^2 + \left(\frac{i\pi}{L}\right)^2\right) p_{ij}. \tag{2.53} $$

Substituting Eq. (2.48) into Eq. (2.39), and then multiplying the equation by $ru_0(\lambda_j r)$ on both sides and integrate from a to R, we find the initial condition of the eigencoordinates

$$ q_{ij}(0) = \frac{2\int_0^L (\int_a^R r w_0(r) u_0(\lambda_j r) dr) Sin(i\pi \frac{z}{L}) dz}{L\int_a^b r[u_0(\lambda_j r)]^2 dr}. \tag{2.54} $$

A numerical scheme that can be used to solve Eqs. (2.52) and (2.53) is presented in the following section.

2.4. NUMERICAL ALGORITHM

This section briefly describes the numerical algorithm for the FOCPs, similar to that presented in (Tangpong and Agrawal 2009; Agrawal, 2006a). For simplicity in the discussions to follow, we consider the following generic form to represent the FOCPs:

$$ {}_0^C D_t^\alpha w = -\mathrm{A}w + \mathrm{B}f, \tag{2.55} $$

$$ {}_t^C D_1^\alpha f = -\mathrm{C}w - \mathrm{D}f, \tag{2.56} $$

$$ w(0) = w_0, \tag{2.57} $$

and

$$ f(1) = 0. \tag{2.58} $$

Equations (2.55) and (2.56) can be expressed in the Volterra integral form as follows.

$$w(t) = w_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{(\alpha-1)}\,(Bf(\tau) - Aw(\tau))d\tau, \tag{2.59}$$

$$f(t) = -\frac{1}{\Gamma(\alpha)}\int_t^1 (\tau - t)^{(\alpha-1)}\,(Df(\tau) + Cw(\tau))d\tau. \tag{2.60}$$

To develop the numerical algorithm, we divide the time domain [0, 1] into N equal intervals, and number the nodes from 0 to N. Here N is a positive integer. The time at node j is given as $t_j = jh$, $j = 0,\ldots,N$ and $h = 1/N$. Furthermore, approximating $w(t)$ and $f(t)$ between two successive temporal nodes linearly. Using the above definitions and approximations, equation (2.55) reduces to (Tangpong and Agrawal, 2009)

$$w(t_i) = w_0 - A\sum_{j=0}^{i} a_{ij}\, w(t_j) + B\sum_{j=0}^{i} a_{ij}\, f(t_j), \qquad i = 1,\ldots,N \tag{2.61}$$

where the coefficients a_{ij} are defined as

$$a_{ij} = d_1\begin{cases} (i-1)^\beta - i^\beta + \beta i^\alpha & if\ j = 0 \\ (k+1)^\beta + (k-1)^\beta - 2k^\beta & if\ 1 \le j \le i-1 \\ 1 & if\ j = i \end{cases} \tag{2.62}$$

Here $d_1 = h^\alpha/\Gamma(\alpha+2)$, $\beta = \alpha + 1$, and $k = i - j$. Following the same approach, the value of $f(t)$ at node i becomes

$$f(t_i) = -C\sum_{j=i}^{N} b_{ij}\, w(t_j) - D\sum_{j=i}^{N} b_{ij}\, f(t_j), \quad i = 0,\ldots,N-1, \tag{2.63}$$

where

$$b_{ij} = d_1\begin{cases} 1 & if\ j = i \\ (k+1)^\beta + (k-1)^\beta - 2k^\beta & if\ i+1 \le j \le N-1 \\ (M-1)^\beta - M^\beta + \beta M^\alpha & if\ j = N \end{cases} \tag{2.64}$$

Here $M = N - i$ and $k = j - i$. Equations (2.61) and (2.63) represent a set of $2N$ linear algebraic equations in terms of $2N$ unknowns, which can be solved using a standard linear algebraic equations solver.

2.5. NUMERICAL RESULTS AND DISCUSSIONS

This section presents simulation results of FOC of the four structures discussed in Section 2.4. For simplicity, we considered $Q' = R' = R = \beta = 1$ for all four cases, and furthermore, $a = 0.5$ and $L = 1$ for the hollow cylinder problem. For each case, we discretized the spatial dimensions and the time domain into several segments and took different values of α.

2.5.1. Sphere with axial symmetry

The initial condition for the sphere with axial symmetry is taken as

$$w_0(r, \theta) = r(1 - \frac{r}{R})\cos\theta. \tag{2.65}$$

We first conducted convergence study on the number of eigenfunctions in both the radial direction (m) and the angular direction (n). Different combinations of m and n values were taken to calculate the responses of the state and control variables at different locations across the time span of 0 to 1, and the maximum relative error from each iteration was recorded. The maximum relative error reached approximately 1% when $= n = 4$, and it continued decreasing as m and n took larger values. For computational efficiency, the simulation results presented in figures 2.1-2.8 were based on $m = n = 4$.

Figures 2.1 and 2.2 demonstrate the state and control variables as a function of time, and they both converged as the time steps were reduced. The maximum relative error of the state variable reached 0.2 % when the number of time steps $N = 100$ with an increase of

10 time steps for each iteration. The results shown in figures 2.1 and 2.2 are up to $N = 50$ for better presentation since the differences in results with larger values of N are not discernible. At $N = 50$, the maximum relative error was around 1%. Figures 2.3 and 2.4 show changes of the state and control variables as functions of time for various orders of derivative (α) and also compare the numerical result with the analytical result of integer order optimal control problem when $\alpha = 1$. In the limit of $\alpha = 1$, the numerical solution recovers the analytical solution of the integer order optimal control problem. This further verifies the accuracy of the numerical algorithm developed for the FOCPs.

Figures 2.5 and 2.6 are the 3D plots of the state and control variables in the radial direction, and figures 2.7 and 2.8 are the 3D plots of both variables in the polar angular direction. The state variables in figures 2.5 and 2.7 initially have different values across the radial or angular dimension due to the initial conditions; as the time progresses, the

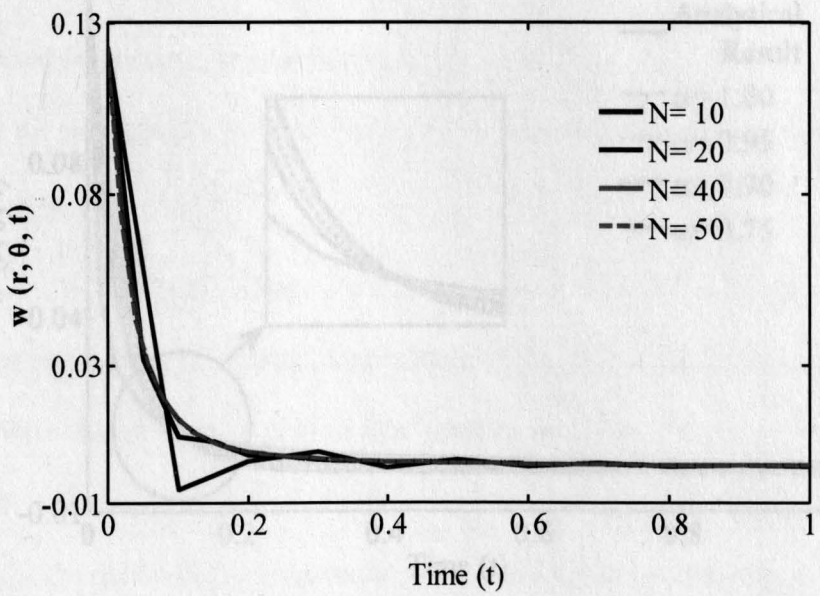

Figure 2.1. Sphere with axial symmetry: state variable $\boldsymbol{w(r, \theta, t)}$ for different number of time segments; $\boldsymbol{\alpha}$=0.90, $\boldsymbol{r}$=0.50 and $\boldsymbol{X = 0.50}$.

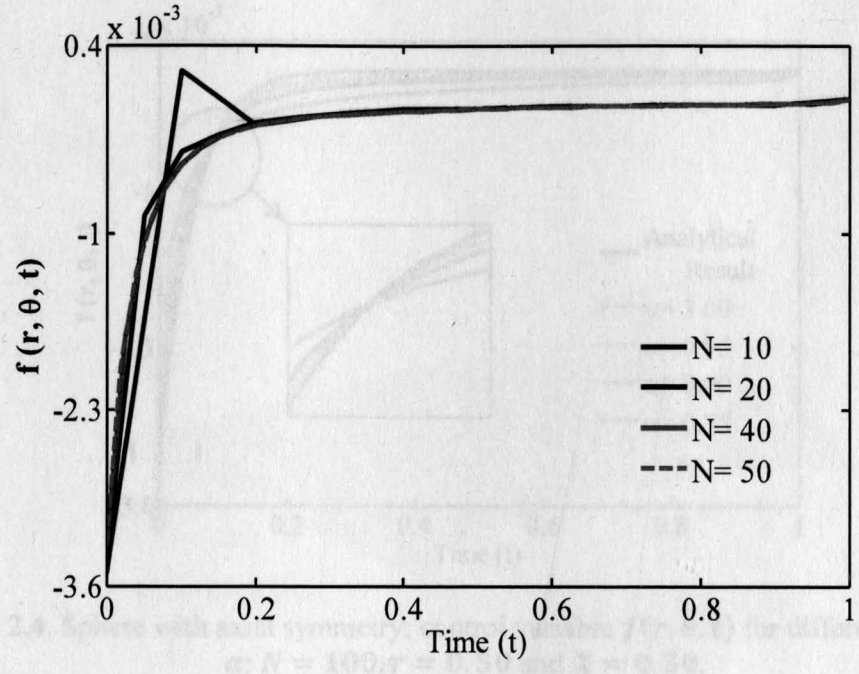

Figure 2.2. Sphere with axial symmetry: control variable $\boldsymbol{f(r,\theta,t)}$ for different number of time segments; $\boldsymbol{\alpha}$=0.90, $\boldsymbol{r}$=0.50 and $\boldsymbol{X = 0.50}$.

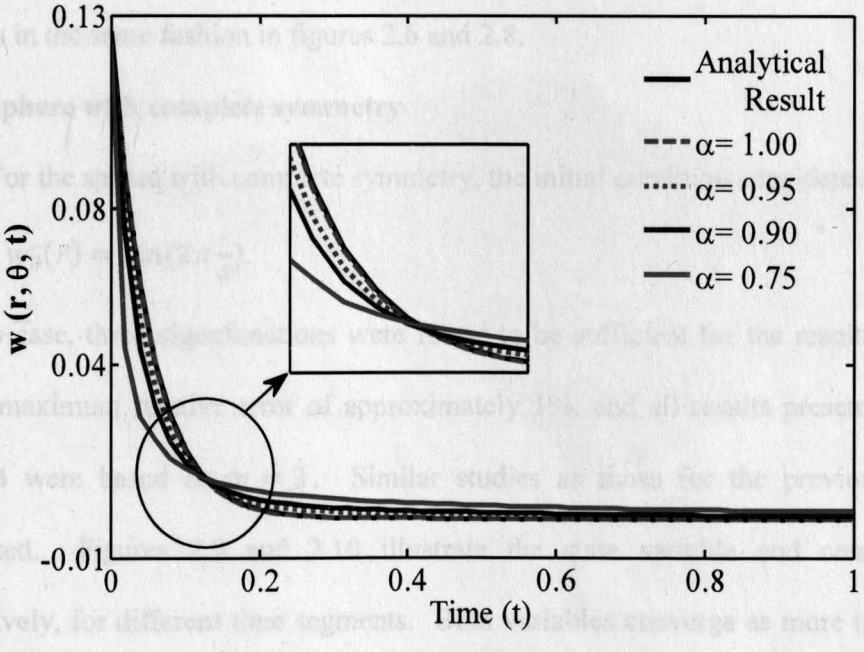

Figure 2.3. Sphere with axial symmetry: state variable $\boldsymbol{w(r,\theta,t)}$ for different values of $\boldsymbol{\alpha}$; $\boldsymbol{N = 100, r = 0.5}$ and $\boldsymbol{X = 0.50}$.

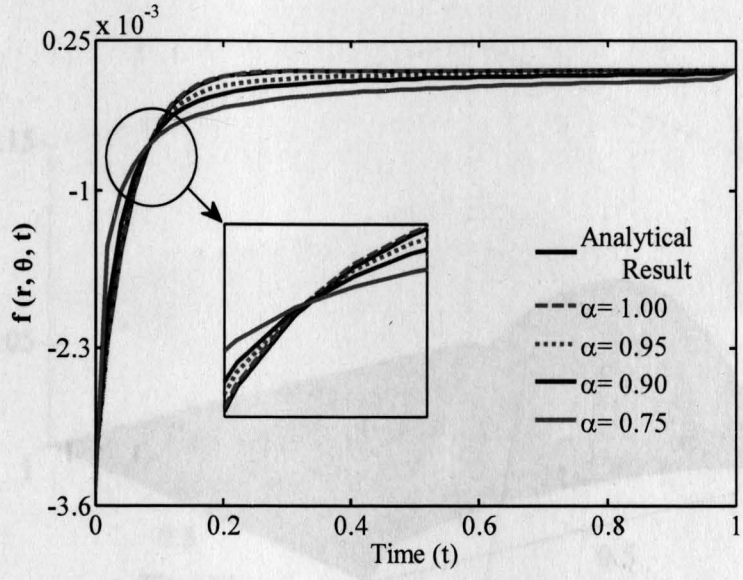

Figure 2.4. Sphere with axial symmetry: control variable $\boldsymbol{f(r,\theta,t)}$ for different values of $\boldsymbol{\alpha}$; $\boldsymbol{N = 100, r = 0.50}$ and $\boldsymbol{X = 0.50}$.

state variable reaches the same value across the radius or angular direction. The phenomena shown in figures 2.5 and 2.7 are typical of a diffusion process. The control variable changes in the same fashion in figures 2.6 and 2.8.

2.5.2. Sphere with complete symmetry

For the sphere with complete symmetry, the initial condition considered is

$$w_0(r) = sin(2\pi \frac{r}{R}). \tag{2.66}$$

For this case, three eigenfunctions were found to be sufficient for the results to converge with a maximum relative error of approximately 1%, and all results presented in figures 2.9-2.14 were based on $m = 3$. Similar studies as those for the previous case were conducted. Figures 2.9 and 2.10 illustrate the state variable and control variable, respectively, for different time segments. Both variables converge as more time segments are taken.

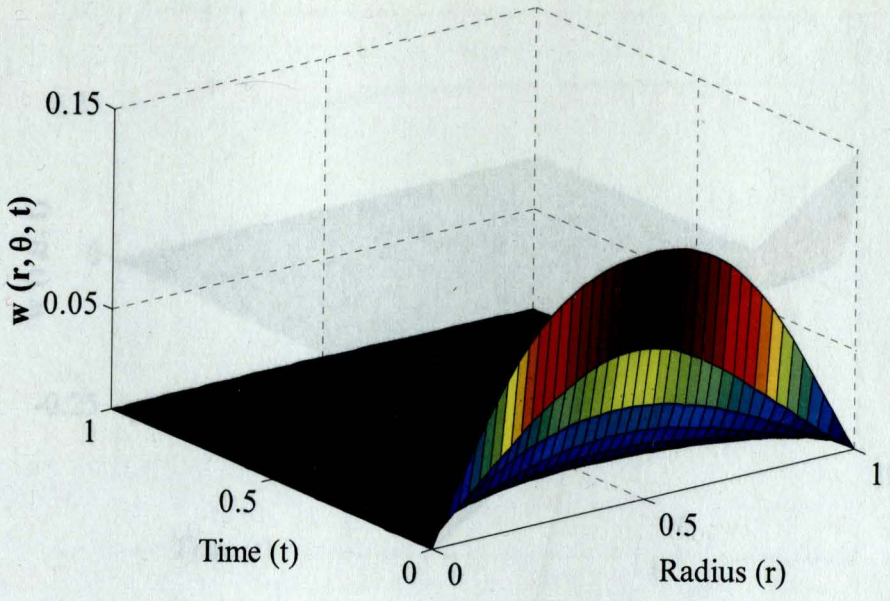

Figure 2.5. Sphere with axial symmetry: surface plot of the state variable $\boldsymbol{w}$ at $\boldsymbol{\theta = cos^{-1} 0.50}$ for $\boldsymbol{\alpha = 0.90}$ and $\boldsymbol{N = 50}$.

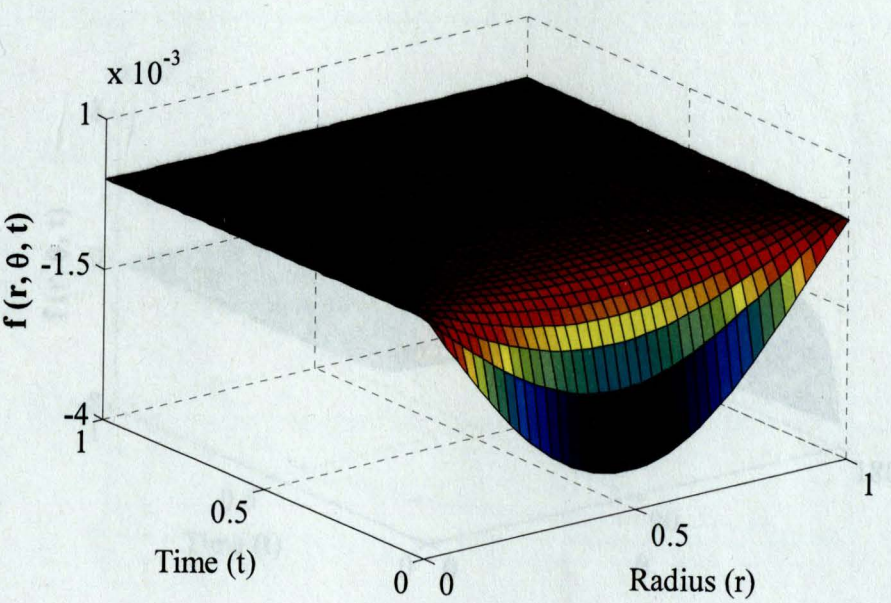

Figure 2.6. Sphere with axial symmetry: surface plot of the control variable $\boldsymbol{f}$ at $\boldsymbol{\theta = cos^{-1} 0.50}$ for $\boldsymbol{\alpha = 0.90}$ and $\boldsymbol{N = 50}$.

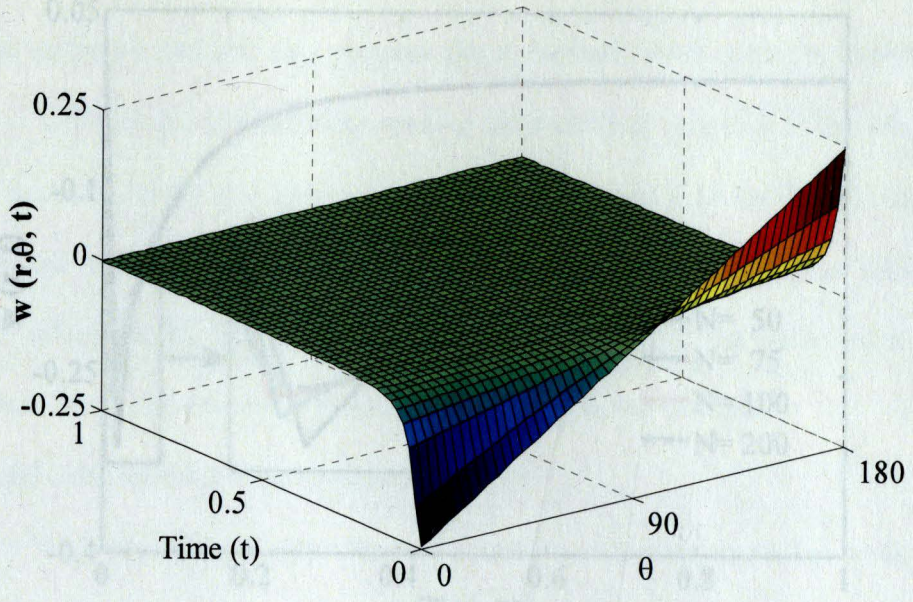

Figure 2.7. Sphere with axial symmetry: surface plot of the state variable $\boldsymbol{w}$ at $\boldsymbol{r = 0.50}$ for $\boldsymbol{\alpha = 0.90}$ and $\boldsymbol{N = 50}$.

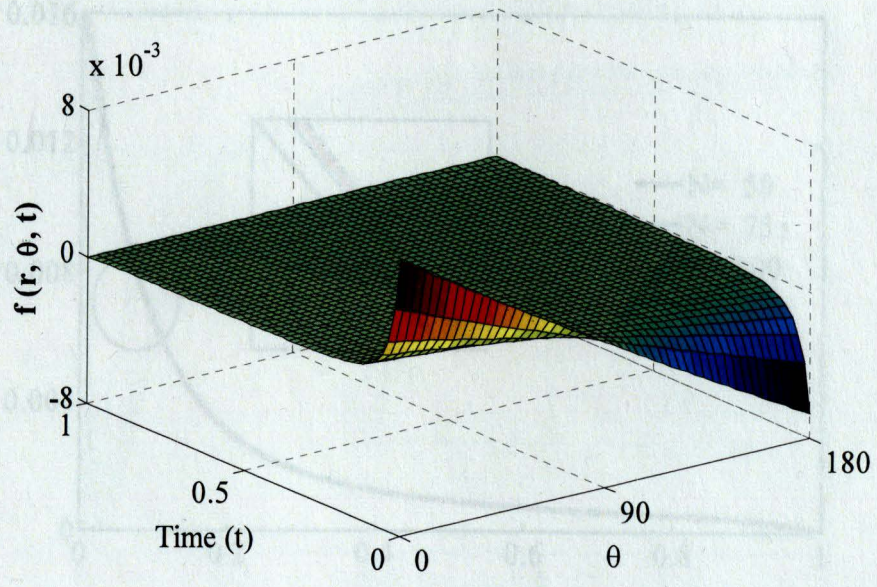

Figure 2.8. Sphere with axial symmetry: surface plot of the control variable $\boldsymbol{f}$ at $\boldsymbol{r = 0.50}$ for $\boldsymbol{\alpha = 0.90}$ and $\boldsymbol{N = 50}$.

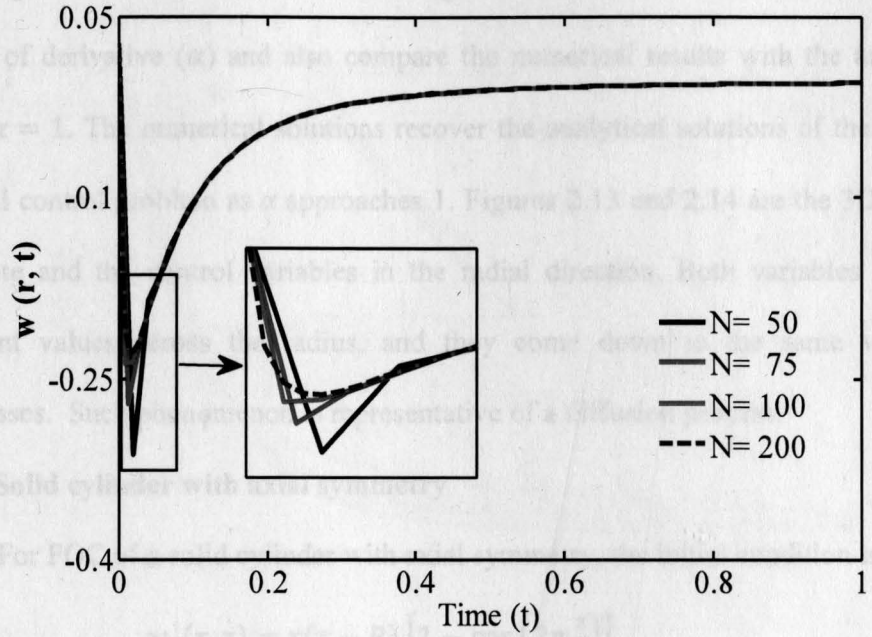

Figure 2.9. Sphere with complete symmetry: state variable $\boldsymbol{w(r = 0.5, t)}$ for different number of time segments for $\boldsymbol{\alpha = 0.90}$.

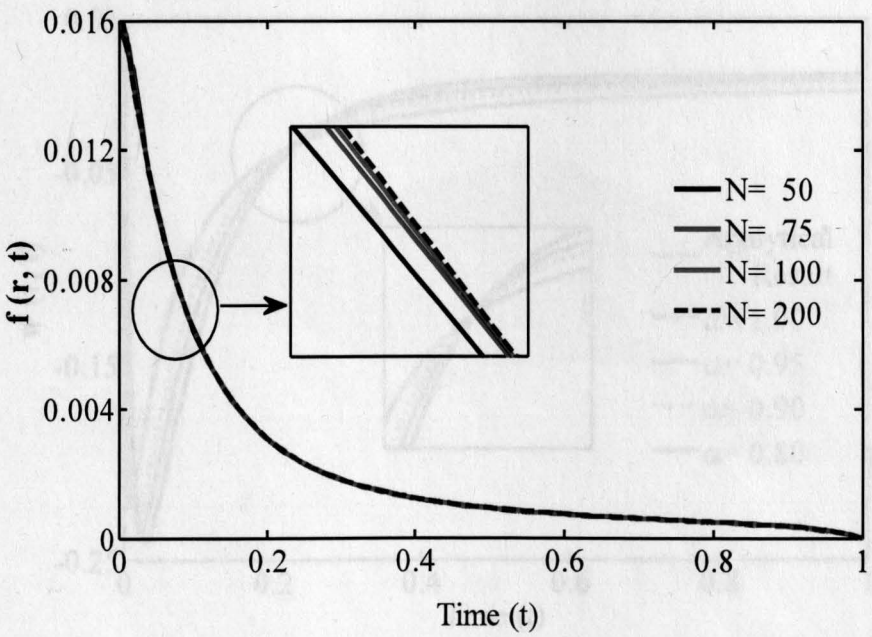

Figure 2.10. Sphere with complete symmetry: control variable $\boldsymbol{f(r = 0.5, t)}$ different number of time segments for $\boldsymbol{\alpha = 0.90}$.

Figures 2.11 and 2.12 show the changes of the state and control variables for various orders of derivative (α) and also compare the numerical results with the analytical ones when $\alpha = 1$. The numerical solutions recover the analytical solutions of the integer order optimal control problem as α approaches 1. Figures 2.13 and 2.14 are the 3D responses of the state and the control variables in the radial direction. Both variables initially have different values across the radius, and they come down to the same value as time progresses. Such phenomenon is representative of a diffusion process.

2.5.3. Solid cylinder with axial symmetry

For FOC of a solid cylinder with axial symmetry, the initial condition is taken as

$$w_0(r,z) = r(r-R)\left[1-\cos\left(2\pi\frac{z}{L}\right)\right], \tag{2.67}$$

The numbers of eigenfunctions in the radial and axial directions were determined through

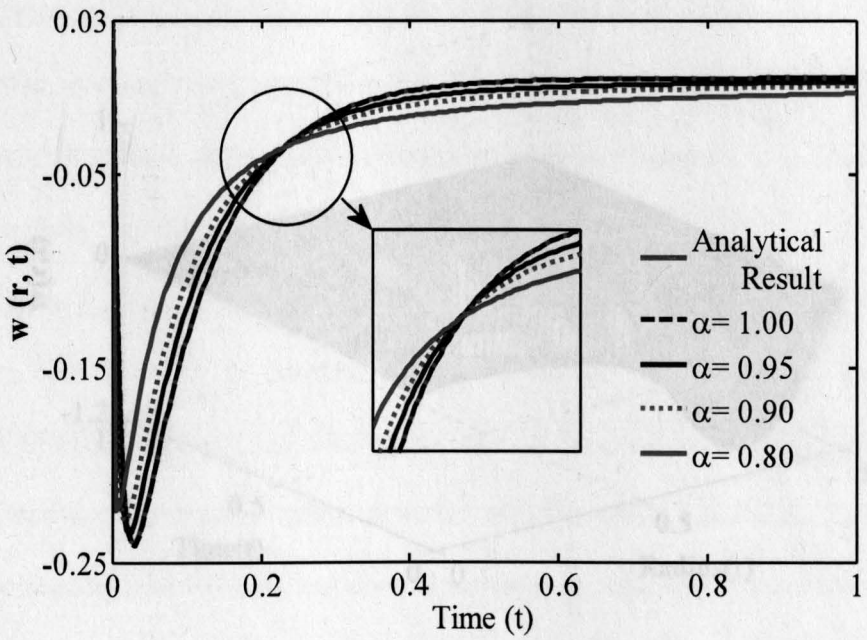

Figure 2.11. Sphere with complete symmetry: state variable $\boldsymbol{w(r = 0.5, t)}$ for different values of $\boldsymbol{\alpha}$; $\boldsymbol{N = 200}$.

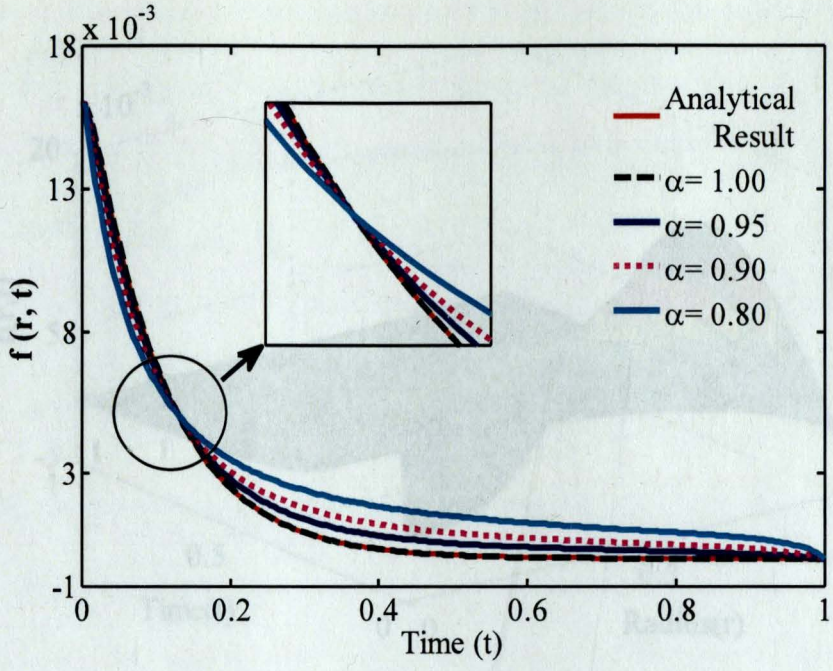

Figure 2.12. Sphere with complete symmetry: control variable $f(r = 0.5, t)$ for different values of α; $N = 200$.

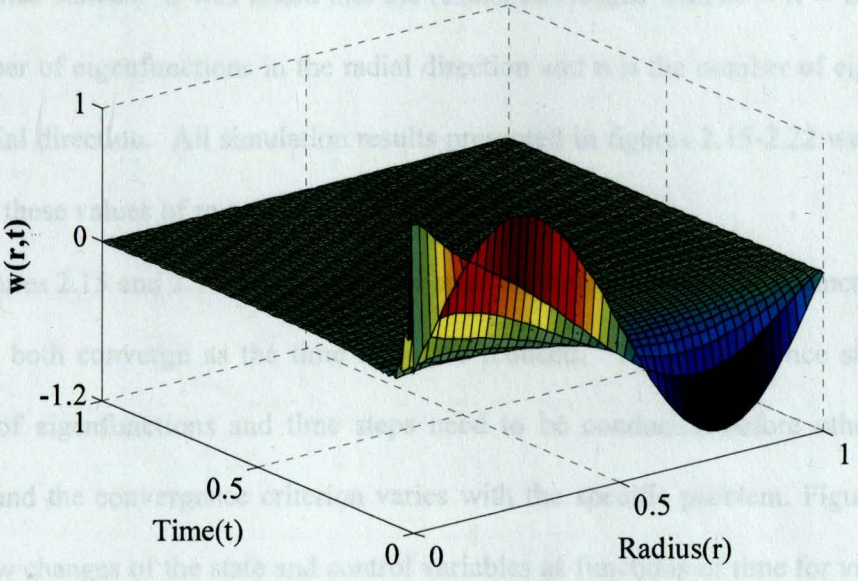

Figure 2.13. Sphere with complete symmetry: state variable $w(r, t)$ for $\alpha = 0.90$ and $N = 100$.

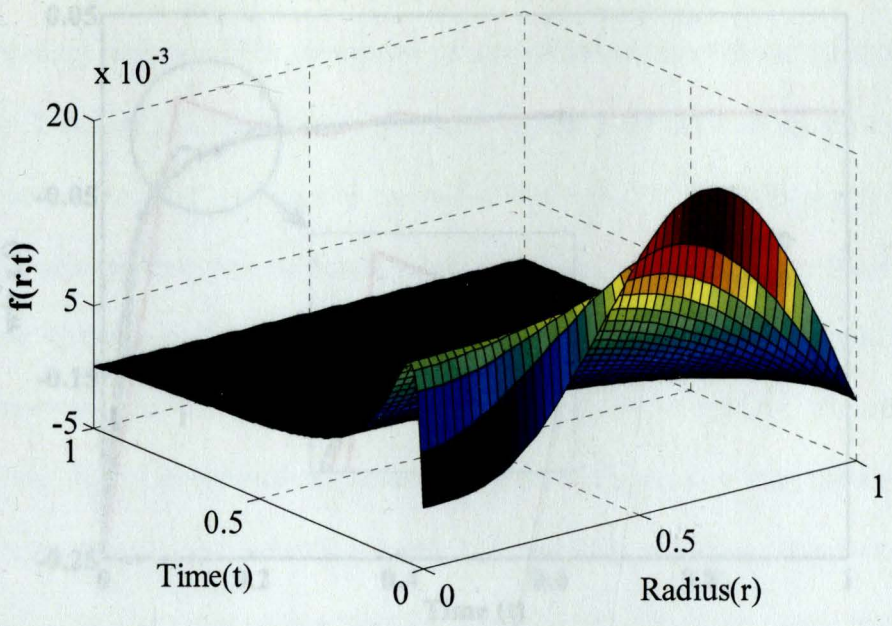

Figure 2.14. Sphere with complete symmetry: control variable $\boldsymbol{f(r,t)}$ for $\boldsymbol{\alpha = 0.90}$ and $\boldsymbol{N = 100}$.

convergence studies. It was found that the results converged with $m = n = 5$, where m is the number of eigenfunctions in the radial direction and n is the number of eigenfunctions in the axial direction. All simulation results presented in figures 2.15-2.22 were generated based on these values of m and n.

Figures 2.15 and 2.16 demonstrate the state and control variables as functions of time, and they both converge as the time steps are reduced. The convergence studies of the number of eigenfunctions and time steps need to be conducted before other parameter studies, and the convergence criterion varies with the specific problem. Figures 2.17 and 2.18 show changes of the state and control variables as functions of time for various orders of α and also compare the numerical results with the analytical results when $\alpha = 1$. In the

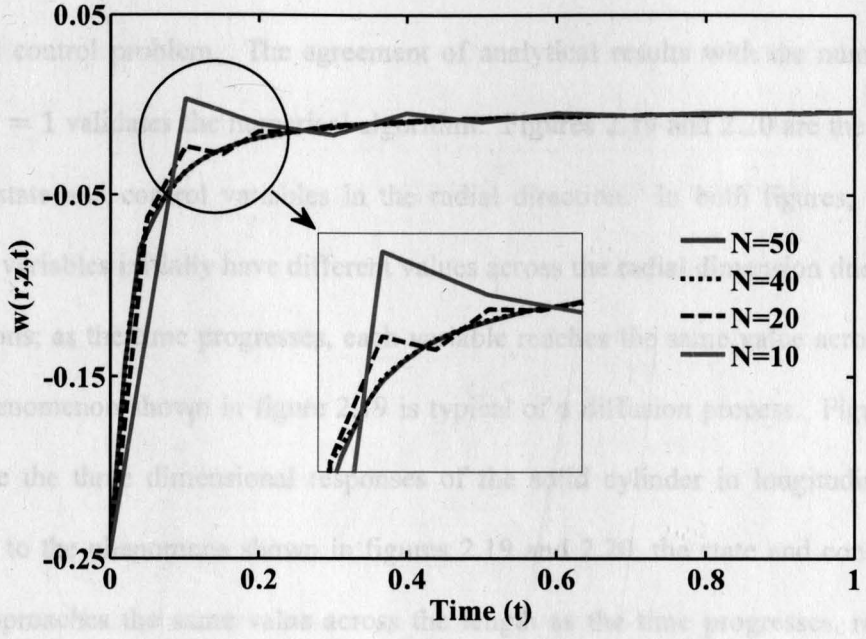

Figure 2.15. Solid cylinder with axial symmetry: convergence of the state variable $w(r = 0.5, z = 0.25, t)$ for different number of time segments for $\alpha = 0.80$.

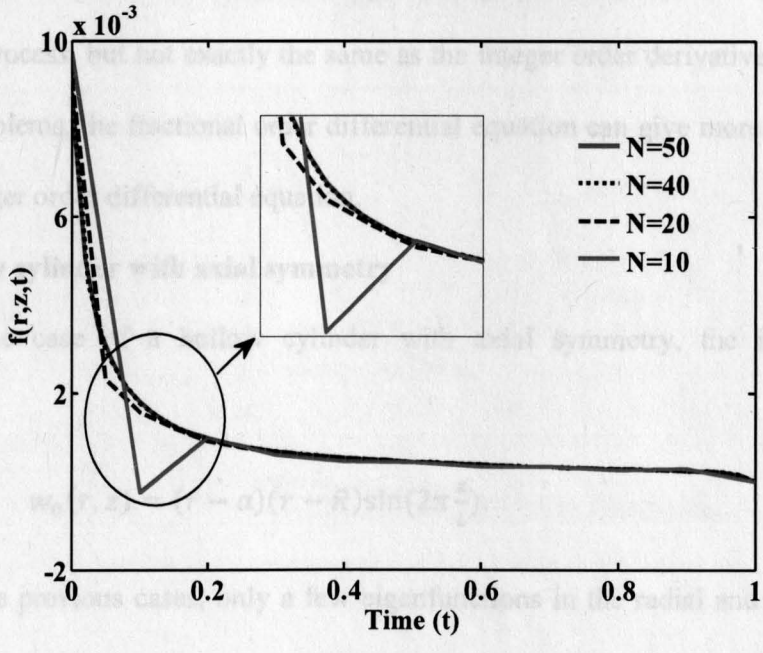

Figure 2.16. Solid cylinder with axial symmetry: convergence of the control variable $f(r = 0.5, z = 0.25, t)$ for different number of time segments for $\alpha = 0.80$.

limit of $\alpha = 1$, the numerical solutions recover the analytical solutions of the integer order optimal control problem. The agreement of analytical results with the numerical results when $\alpha = 1$ validates the numerical algorithm. Figures 2.19 and 2.20 are the surface plots of the state and control variables in the radial direction. In both figures, the state and control variables initially have different values across the radial dimension due to the initial conditions; as the time progresses, each variable reaches the same value across the radius. The phenomenon shown in figure 2.19 is typical of a diffusion process. Figures 2.21 and 2.22 are the three dimensional responses of the solid cylinder in longitudinal direction. Similar to the phenomena shown in figures 2.19 and 2.20, the state and control variables each approaches the same value across the length as the time progresses, representing a diffusion process. The dynamics constraint equation (2.38) becomes a heat diffusion equation when $\alpha = 1$; when $\alpha = 0.9$, the dynamics governed by equation (2.38) is close to a diffusion process, but not exactly the same as the integer order derivative case. For such dynamic problems, the fractional order differential equation can give more accurate results than the integer order differential equation.

2.5.4. Hollow cylinder with axial symmetry

For the case of a hollow cylinder with axial symmetry, the initial condition considered is

$$w_0(r,z) = (r-a)(r-R)\sin(2\pi\tfrac{z}{L}). \tag{2.68}$$

Similar to the previous cases, only a few eigenfunctions in the radial and axial directions are sufficient to model the system and the results presented in figures 2.23-2.30 were all

based on the number of eigenfunctions $m = 3$ and $n = 5$. We also discretized the space and time domain into several sub domains and took different values of α for the simulation.

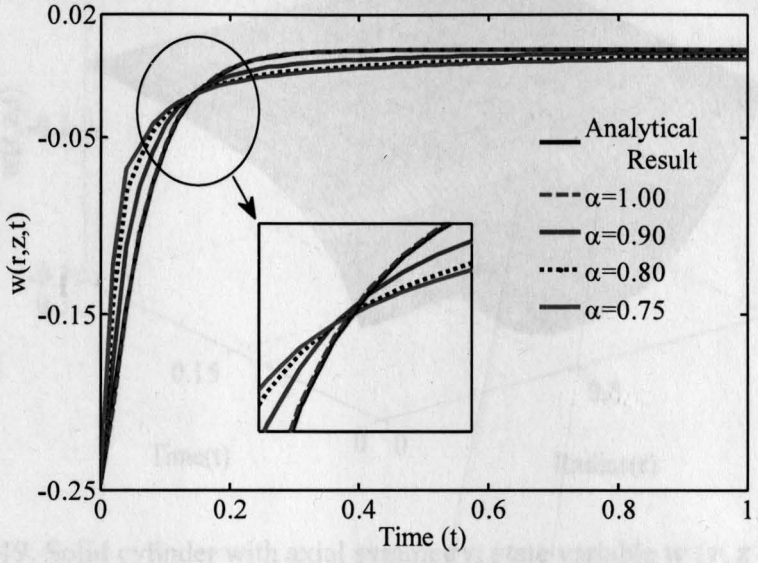

Figure 2.17. Solid cylinder with axial symmetry: state variable $\boldsymbol{w(r = 0.5, z = 0.25, t)}$ for different values of $\boldsymbol{\alpha}$ with $\boldsymbol{N = 50}$.

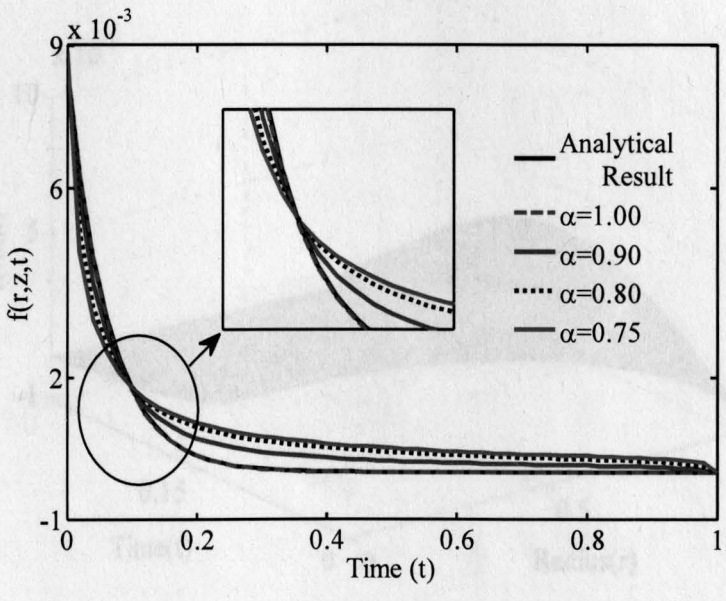

Figure 2.18. Solid cylinder with axial symmetry: control variable $\boldsymbol{f(r = 0.5, z = 0.25, t)}$ for different values of $\boldsymbol{\alpha}$ with $\boldsymbol{N = 50}$.

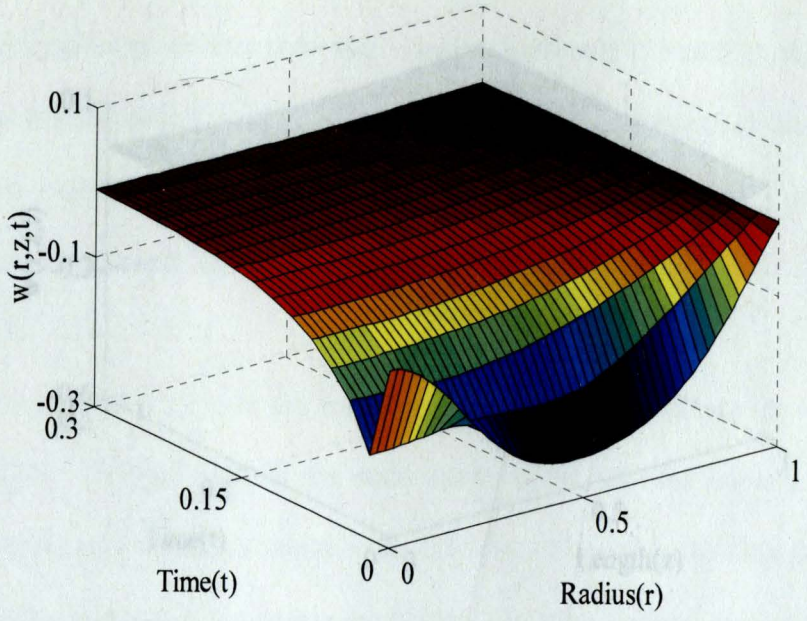

Figure 2.19. Solid cylinder with axial symmetry: state variable $\boldsymbol{w}\ (\boldsymbol{r}, \boldsymbol{z} = \mathbf{0.25}, \boldsymbol{t})$ for $\boldsymbol{N} = \mathbf{50}$ and $\boldsymbol{\alpha} = \mathbf{0.90}$.

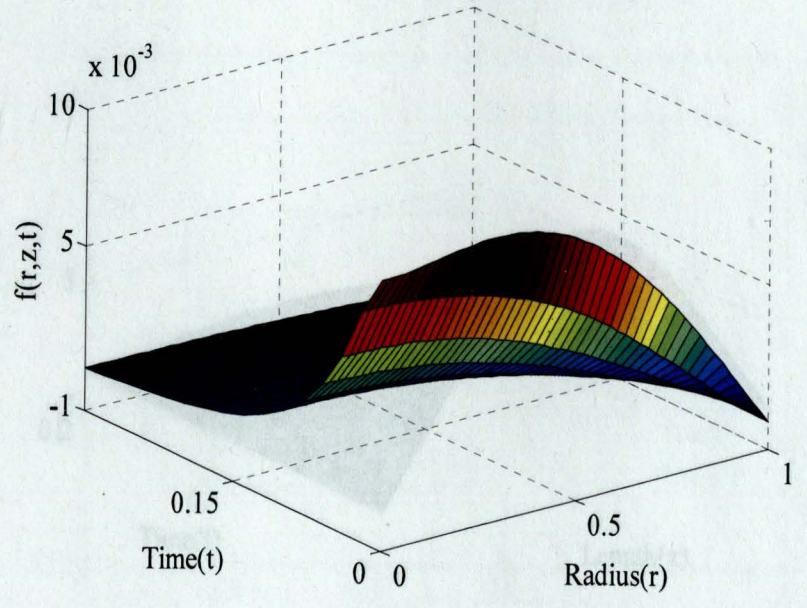

Figure 2.20. Solid cylinder with axial symmetry: control variable $\boldsymbol{f}\ (\boldsymbol{r}, \boldsymbol{z} = \mathbf{0.25}, \boldsymbol{t})$ for $\boldsymbol{N} = \mathbf{50}$ and $\boldsymbol{\alpha} = \mathbf{0.90}$.

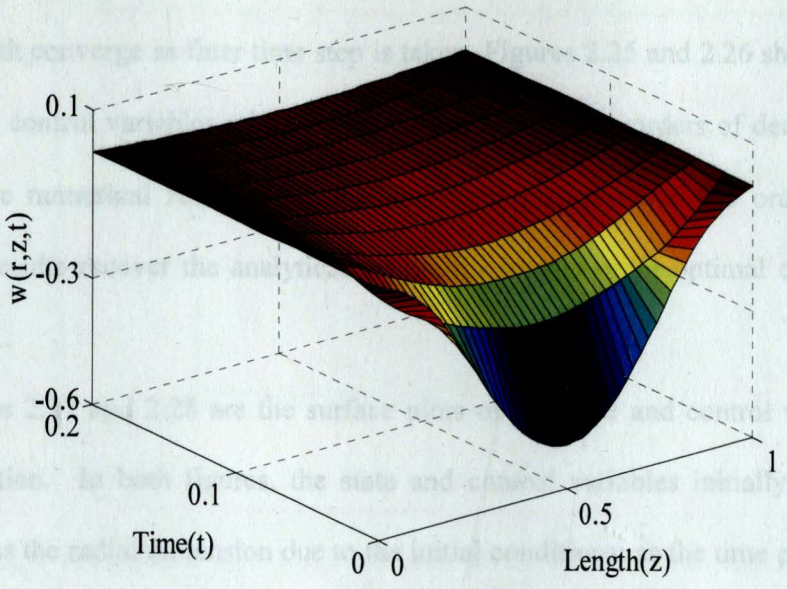

Figure 2.21. Solid cylinder with axial symmetry: state variable $\boldsymbol{w(r = 0.5, z, t)}$ for $\boldsymbol{N = 50}$ and $\boldsymbol{\alpha = 0.90}$.

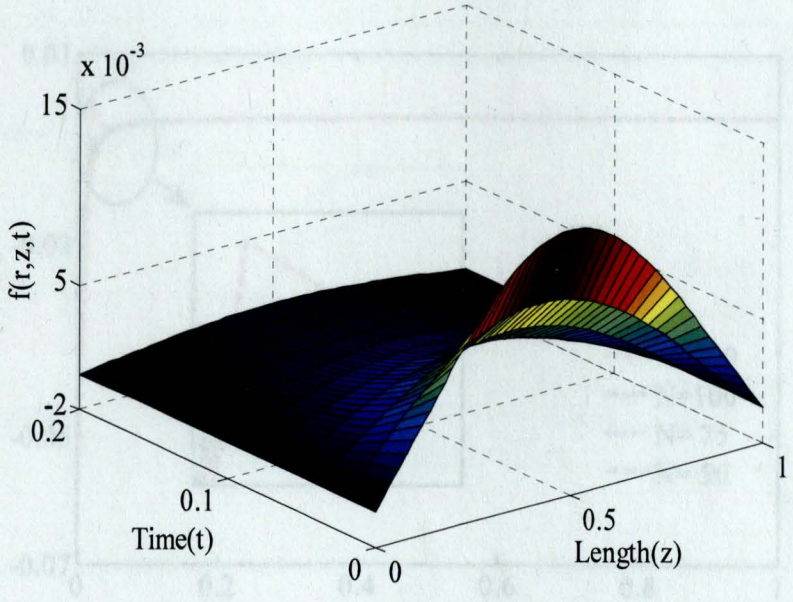

Figure 2.22. Solid cylinder with axial symmetry: control variable $\boldsymbol{f(r = 0.5, z, t)}$ for $\boldsymbol{N = 50}$ and $\boldsymbol{\alpha = 0.90}$.

Figures 2.23 and 2.24 demonstrate the state and control variables as functions of time and they both converge as finer time step is taken. Figures 2.25 and 2.26 show the changes of state and control variables with respect to time for various orders of derivative (α) and compare the numerical results with the analytical results when the order $\alpha = 1$. The numerical results recover the analytical results for integer order optimal control problem when $\alpha = 1$.

Figures 2.27 and 2.28 are the surface plots of the state and control variables in the radial direction. In both figures, the state and control variables initially have different values across the radial dimension due to the initial conditions; as the time progresses, each variable reaches the same value across the radius. The phenomenon shown in figure 2.27 is typical of a diffusion process. Figures 2.29 and 2.30 are the three dimensional responses of the hollow cylinder in longitudinal direction. Similar to the phenomena shown in figures

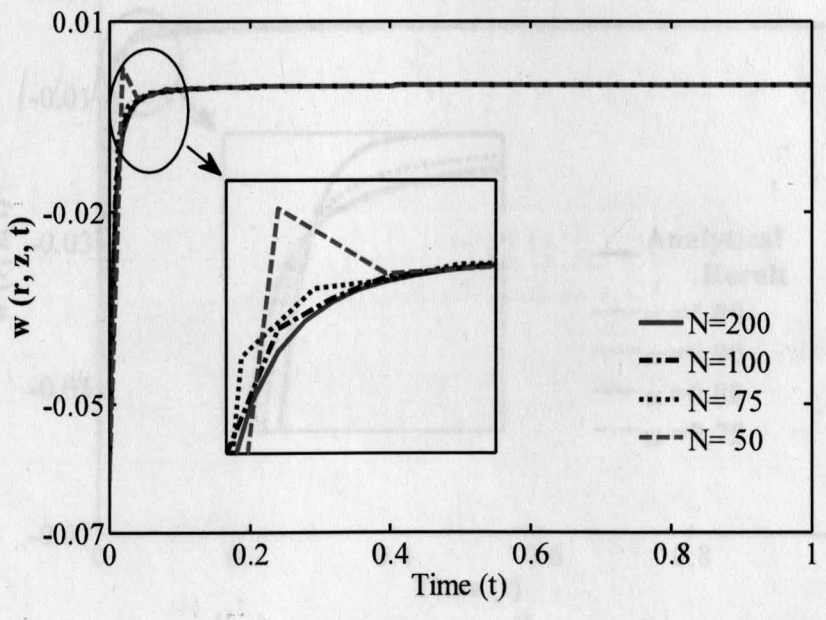

Figure 2.23. Hollow cylinder with axial symmetry: convergence of the state variable $\boldsymbol{w(r = 0.75, z = 0.25, t)}$ for different number of time segments for $\boldsymbol{\alpha = 0.90}$.

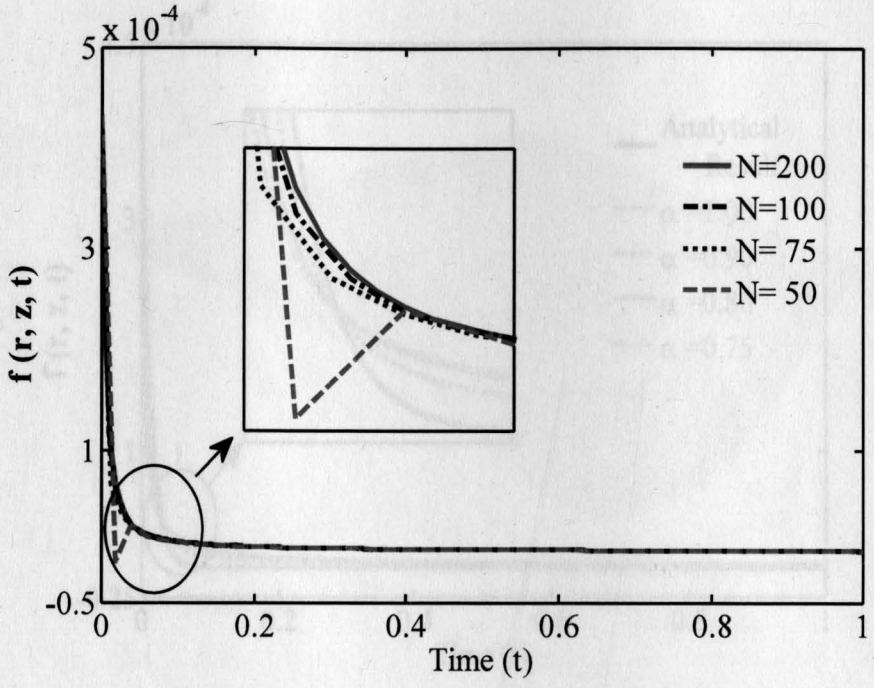

Figure 2.24. Hollow cylinder with axial symmetry: convergence of the control variable $f(r = 0.75, z = 0.25, t)$ for different number of time segments for $\alpha = 0.90$.

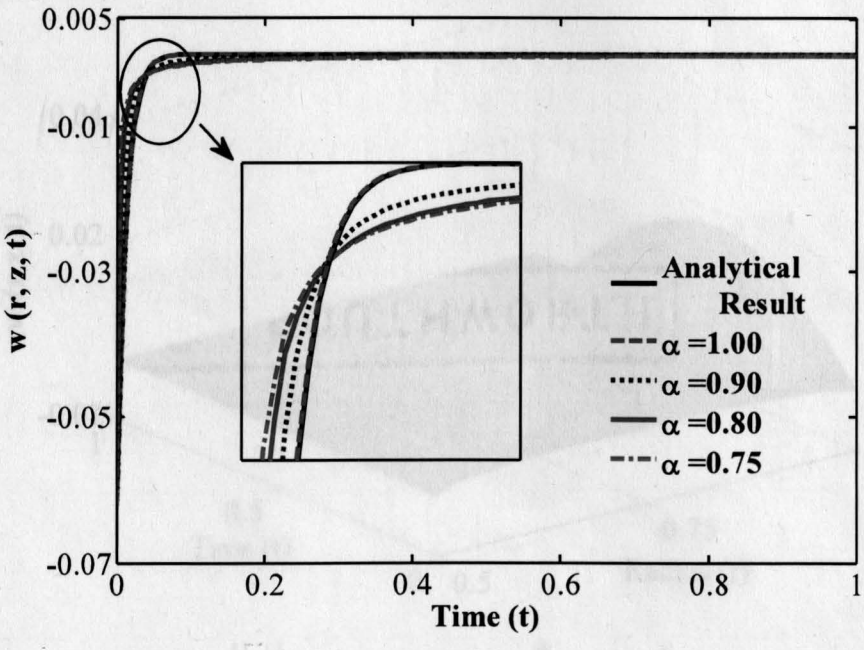

Figure 2.25. Hollow cylinder with axial symmetry: state variable $w(r = 0.75, z = 0.25, t)$ for different values of α with $N = 200$.

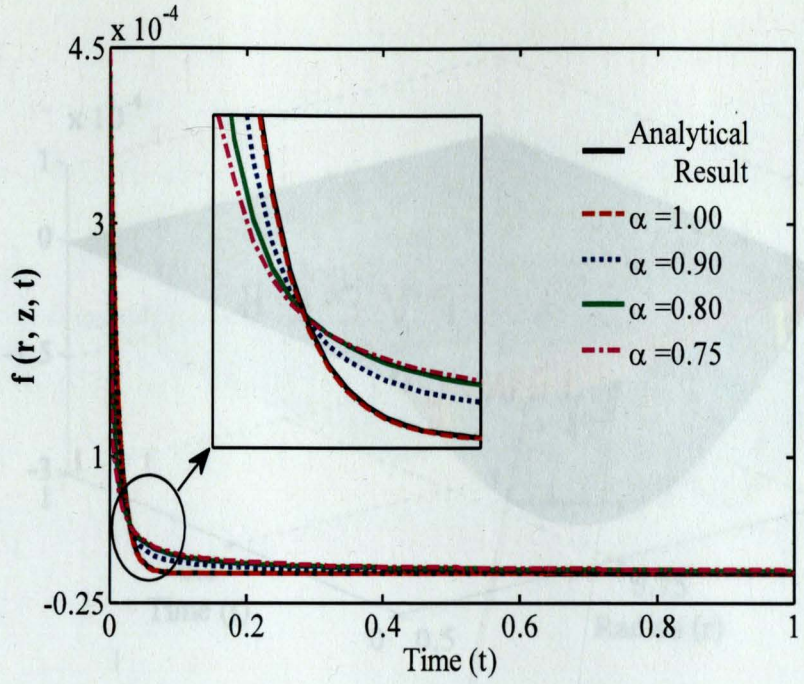

Figure 2.26. Hollow cylinder with axial symmetry: control variable $f(r = 0.75, z = 0.25, t)$ for different values of α with $N = 200$.

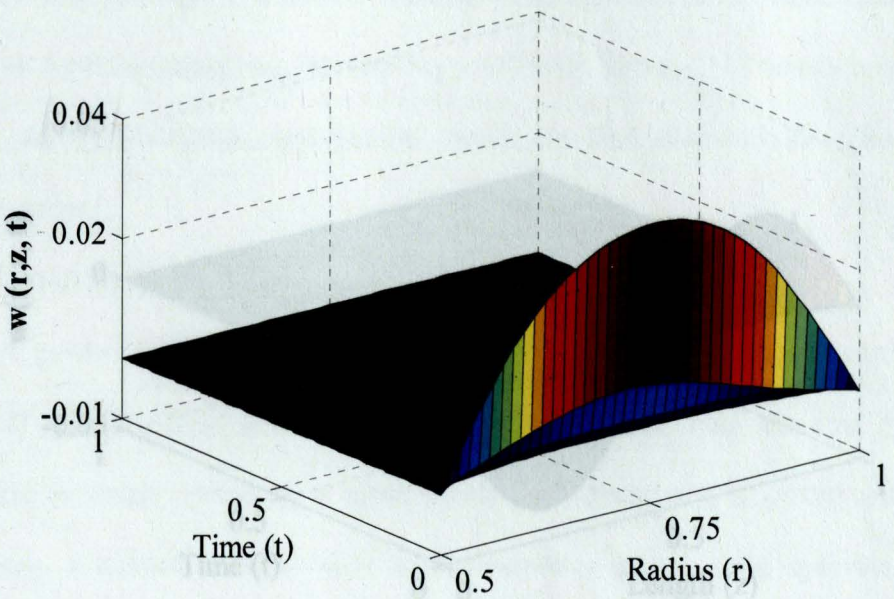

Figure 2.27. Hollow cylinder with axial symmetry: state variable $w(r, z = 0.9, t)$ for $N = 100$ and $\alpha = 0.90$.

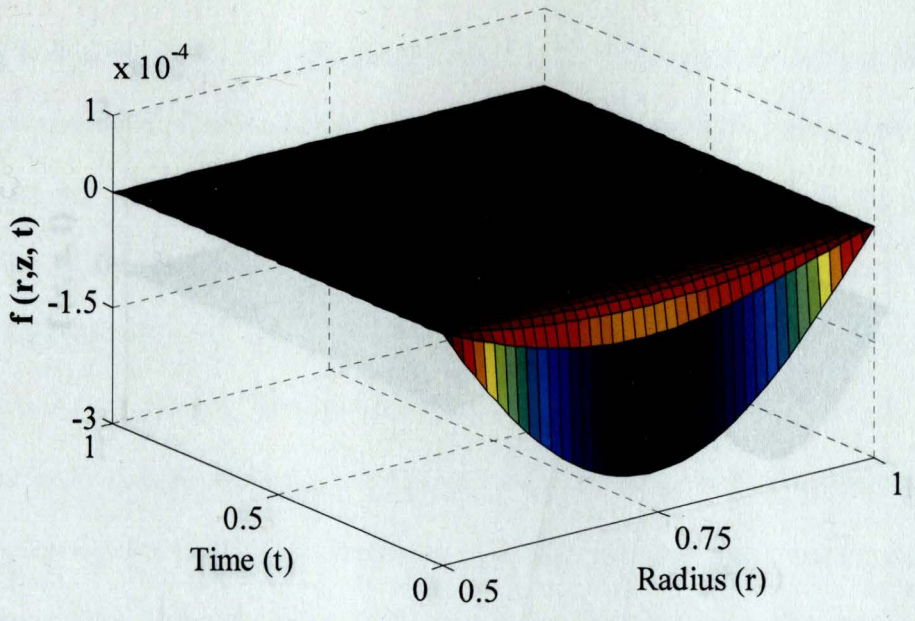

Figure 2.28. Hollow cylinder with axial symmetry: control variable $f(r, z = 0.9, t)$ for $N = 100$ and $\alpha = 0.90$.

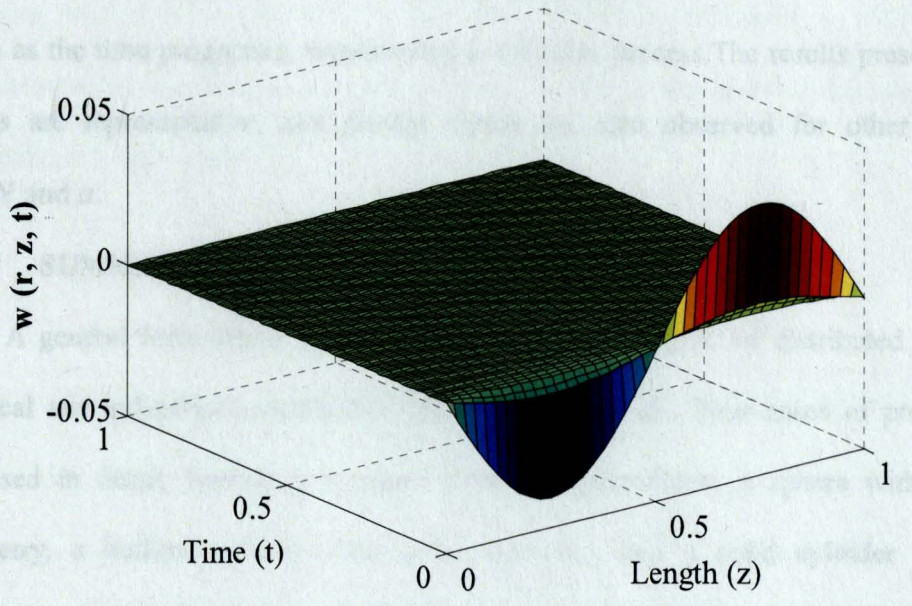

Figure 2.29. Hollow cylinder with axial symmetry: state variable $w(r = 0.9, z, t)$ for $N = 100$ and $\alpha = 0.90$.

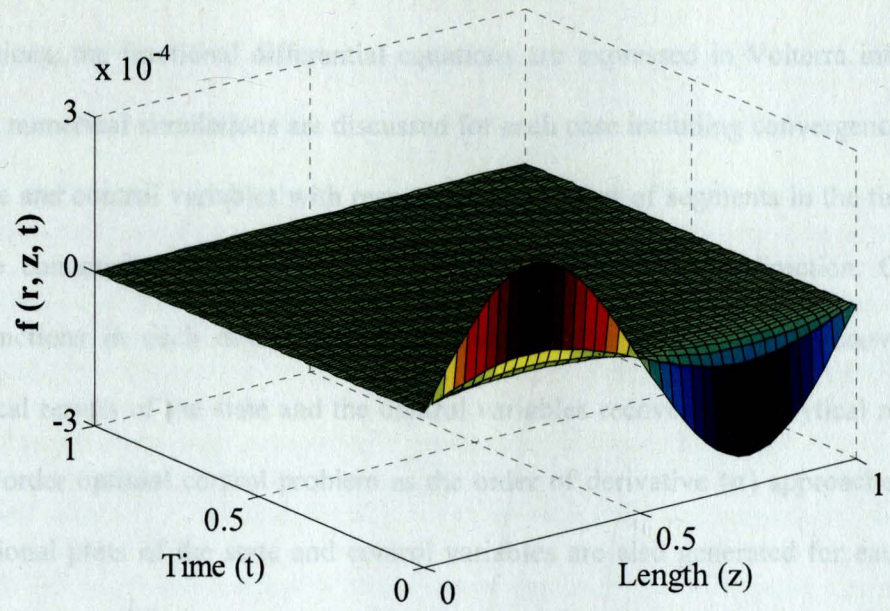

Figure 2.30. Hollow cylinder with axial symmetry: control variable $f(r = 0.9, z, t)$ for $N = 100$ and $\alpha = 0.90$.

2.27 and 2.28, the state and control variables each approaches the same value across the length as the time progresses, representing a diffusion process.The results presented in all figures are representative, and similar trends are also observed for other values of m, n, N and α.

2.6. SUMMARY

A general formulation and a numerical scheme for FOC of distributed systems in spherical and cylindrical coordinate system are presented. Four cases of problems are discussed in detail, including a sphere with axial symmetry, a sphere with complete symmetry, a hollow cylinder with axial symmetry and a solid cylinder with axial symmetry. Partial fractional time derivatives are defined in the Caputo sense and the performance index of the FOCP is defined as a function of both state and control variables. The separation of variable method and the eigenfunction approach are used to decouple the

equations and define the problem in terms of the state and control variables. For numerical calculations, the fractional differential equations are expressed in Volterra integral form. Several numerical simulations are discussed for each case including convergence studies of the state and control variables with respect to the number of segments in the time domain, and the convergence of the number of eigenfunctions in each direction. Only a few eigenfunctions in each direction are sufficient for the calculations to converge. The numerical results of the state and the control variables recover the analytical results of an integer order optimal control problem as the order of derivative (α) approaches 1. Three dimensional plots of the state and control variables are also generated for each case that clearly shows a diffusion process in the structure. When the order of derivative $\alpha = 1$, the governing dynamics equation represents a heat diffusion process and the analytical solution for such an integer order optimal control problem exists. In phenomena that are close to a diffusion process but not quite the same, a FOC model will be useful to model the system more accurately. The formulation and numerical algorithm presented in this paper will have applications to such systems.

CHAPTER 3. A NUMERICAL SCHEME FOR A CLASS OF PARAMETRIC PROBLEM OF FRACTIONAL VARIATIONAL CALCULUS

Fractional derivatives (FDs) or derivatives of arbitrary order have been used in many applications, and it is envisioned that in future they will appear in many functional minimization problems of practical interest (Agrawal, et al., 2011). The materials presented in this chapter have been accepted for publication in (Agrawal, et al., 2011). Since FDs have such property as being non-local, it can be extremely challenging to find analytical solutions for fractional parametric optimization problems, and in many cases, analytical solutions may not exist. Therefore, it is of great importance to develop numerical methods for such problems. This chapter presents a numerical scheme for a linear functional minimization problem that involves FD terms. The FD is defined in terms of the RL definition; however, the scheme will also apply to Caputo derivatives, as well as other definitions of FDs. In this scheme, the spatial domain is discretized into several subdomains and 2-node one-dimensional linear elements are adopted to approximate the solution and its FD at point within the domain. The fractional optimization problem is converted to an eigenvalue problem, the solution of which leads to fractional orthogonal functions. Convergence study of the number of elements and error analysis of the results ensure that the algorithm yields stable results. Various fractional orders of derivative are considered and as the order approaches the integer value of 1, the solution recovers the analytical result for the corresponding integer order problem.

3.1. INTRODUCTION

A Fractional Calculus of Variational (FCV) problem is a problem in which either the objective functional or the constraints or both contain at least one FD term (Agrawal, 2002). Significant amount of work has been done in the field of Integer Order Variational Calculus (IOVC). However, as demonstrated recently, many of the results of IOVC can be extended to FCV field. Riewe (1996, 1997) was the pioneer of the FCV who developed the nonconservative Lagrangian, Hamiltonian, and other concepts of classical mechanics by using FD and formulated a version of the Euler-Lagrange equation. Agrawal (2002, 2006b) and Klimek (2001, 2002) were among the earliest researchers who developed formulations and numerical schemes for various types of fractional variational problems (FVPs) in terms of RL and Caputo fractional derivatives (CFD). Other researchers further extended these work to other variational problems (Agrawal, 2007, 2010a; Baleanu and Avkar, 2004; Muslih and Baleanu, 2005; Herzallah and Baleanu, 2009; Almeida and Torres, 2009, 2011).

Integer order parametric problems can be solved analytically by using Euler-Lagrange differential equation. For fractional order parametric problems, however, it is extremely difficult to solve by using fractional Euler-Lagrange equation since the equation contains both left and right derivatives. Almeida and Torres (2009, 2011) provided necessary and sufficiency conditions of optimality for functionals containing fractional integrals and FD and for the fractional isoperimetric problem in the sense of RL and Caputo derivatives. Approximation of FDs is more complex than integer derivatives because fractional order derivatives are non-local (Sousa, 2010). Many fractional differential equations do not have exact analytical solution and therefore numerical or approximation techniques are

necessary to solve such equations (Odibat and Momani, 2008). Some approximate solutions of fractional differential equations can be found in (Momani and Odibat, 2007; Momani et al., 2008; Kumar and Agrawal, 2006; Diethelm et al., 2002; Diethelm et al. 2005).

In this chapter, a numerical scheme is developed to solve a fractional optimization problem using the finite element concept. The FD is taken in the RL sense. Since solutions of fractional differential equations in terms of RL require fractional initial condition (Podlubny, 1999), the initial conditions are taken as zero here as a simpler demonstration of the method. Therefore, the numerical scheme presented in this paper also applies to other definitions of FD such as Caputo derivatives. Various fractional orders of derivative are considered and analysis of the convergence error is performed. Analytical solutions for a large class of problems which arise in FCV have been presented in (Klimek, 2009). However, we are not aware of an analytical solution of the problem considered here.

It should be stated that though the algorithm has been developed for a specific problem, it can also be applied to an entire class of problems where the functions in cost functionals and integral constraints are defined as quadratic functions in terms of solutions and their FDs. Furthermore, the problem has been defined in terms of RLFDs, but the same technique could be applied to problems defined using other derivatives.

3.2. DEFINITIONS OF FRACTIONAL DERIVATIVE

The left Riemann-Liouville fractional derivative (RLFD) of order α, denoted as ${}_0D_t^{\alpha} f(t)$, is given by equation (1.19) and the left CFD of order α, denoted as ${}_a^C D_t^{\alpha} f(t)$is given by equation (1.20). The relation between RL and Caputo FDs is defined as follows (Podlubny, 1999; Kilbas, et. al, 2006).

$$ {}_{a}^{C}D_{t}^{\alpha} f(t) = {}_{a}D_{t}^{\alpha} f(t) - \sum_{k=0}^{n-1} \frac{f^{(k)}(a)}{\Gamma(k-\alpha+1)} (t-a)^{k-\alpha} \tag{3.1} $$

It is obvious from equation (3.1) that when $f(a) = f'(a) = \cdots = f^{(n-1)}(a) = 0$, ${}_{a}^{C}D_{t}^{\alpha} f(t) = {}_{a}D_{t}^{\alpha} f(t)$. Therefore, with zero initial conditions, RL and Caputo FDs become the same. The numerical scheme developed in this paper applies to problems that are in terms of both RL and Caputo FDs.

The problem discussed in the following section is defined in terms of the left RLFD only, however, for zero initial condition, the results of the problems defined in terms of the RL and CFDs would be the same. Furthermore, a problem defined in terms of Caputo derivatives could be cast into a problem in terms of RLFD if necessary.

3.3. NUMERICAL METHOD FOR A FRACTIONAL PARAMETRIC PROBLEM

Consider the following problem: among all functions $y(t)$ which satisfy the integral constraint

$$ K(y) = \int_0^1 y^2 dt = 2 \tag{3.2} $$

and the fixed terminal (boundary) conditions

$$ y(0) = y(1) = 0, \tag{3.3} $$

find the one that minimizes the cost functional

$$ J(y) = \int_0^1 ({}_0D_x^{\alpha} y)^2 dt \tag{3.4} $$

where $({}_0D_x^{\alpha})y$ represents the left RLFD of order α, $0 < \alpha < 1$. For simplicity, we have taken the limit of integration from 0 to 1 and the order of the derivative α between 0 and 1. In a generalized case, the limit of integration could be a to b, $a < b$, and α could be a positive number. When $\alpha = 1$, equation (3.4) becomes

$$J(y) = \int_0^1 \dot{y}^2 dt \,. \tag{3.5}$$

For the integer order optimization problem defined by equations (3.2), (3.3) and (3.5) an analytical solution exists, and it is given in the Section 3.4.2. A fractional power series solution was discussed in (Agrawal, 2010b) for the problem defined by equations (3.2-3.4). Here we present a numerical scheme using the finite element concept to solve this fractional parametric problem.

We divide the t domain $[0, 1]$ into N equal elements and number the nodes from 0 to N such that their coordinates are $y_i = i/N$, $i = 0, 1, \ldots, N$. Due to the terminal conditions specified in equation (3.3), $y_0 = y_N = 0$. Using the one dimensional two-node finite element and considering linear interpolation between the nodes, function $y(t)$ can then be expressed as

$$y(t) = \sum_{i=1}^{N-1} \phi_i(t) y_i = \Phi^T(t) Y, \tag{3.6}$$

where

$$\phi_i(t) = \begin{cases} 0, & t < t_{i-1} \\ \frac{t - t_{i-1}}{t_i - t_{i-1}}, & t_{i-1} < t < t_i \\ \frac{t_{i+1} - t}{t_{i+1} - t_i}, & t_i < t < t_{i+1} \\ 0, & t > t_{i+1} \end{cases}, \tag{3.7}$$

$$\Phi^T(t) = [\phi_1(t)\ \phi_2(t)\ \phi_3(t) \ldots \phi_{N-1}(t)], \tag{3.8}$$

and

$$Y = [\, y_1, y_2, \ldots, y_{N-1} \,]^T. \tag{3.9}$$

Note that the approximation of $y(t)$ by equation (3.6) does not contain the end nodes due to zero terminal conditions. Functions $\phi_i(t)$ are also known as the hat functions. Using the above approximation of $y(t)$, the FD of the function can be expressed as

$$ {}_0D_x^{\alpha} y(t) = {}_0D_x^{\alpha} \Phi^T(t) Y \,. \tag{3.10} $$

By substituting equation (3.10) into equation (3.4), the cost functional can be written as

$$ J \cong Y^T A Y, \tag{3.11} $$

where matrix A is

$$ A = \int_0^1 \left({}_0D_x^{\alpha} \Phi(t) \right) {}_0D_x^{\alpha} \Phi^{\mathrm{T}}(t)\, dt. \tag{3.12} $$

Similarly, by substituting equation (3.6) into equation (3.2), the integral constraint can be expressed as

$$ K \cong Y^T B Y = 2, \tag{3.13} $$

where matrix B is formed such that

$$ B = \int_0^1 \Phi(t) \Phi^{\mathrm{T}}(t)\, dt. \tag{3.14} $$

The Lagrangian can then be expressed as

$$ F = Y^T A Y - \lambda (Y^T B Y - 2). \tag{3.15} $$

By taking the derivative of F with respect to Y and setting it to 0, the following equation is obtain,

$$ AY = \lambda B Y, \tag{3.16} $$

which is an eigenvalue problem. Note that matrices A and B formed from equations (3.12) and (3.14) are positive definite matrices, and therefore, the eigenvalues $\lambda_i, i = 1,2, \ldots N - 1$, of equation (3.16) are positive real numbers. We arrange λ_i such that $\lambda_1 < \lambda_2 < \cdots < \lambda_{N-1}$. The eigenvectors $Y_i, i = 1,\ 2, \ldots ,\ N - 1$ of this problem are normalized such that

$$ Y_i^T B Y_j = \delta_{ij}, \tag{3.17} $$

$$ Y_i^T A Y_j = \lambda_i \delta_{ij}, \tag{3.18} $$

where δ_{ij} is the Kronecker delta function. Thus, an arbitrary vector Y can be expressed as

$$Y = \sum_{i=1}^{N-1} a_i Y_i. \tag{3.19}$$

Substituting equation (3.19) into the constraint function in equation (3.13), the integral constraint equation becomes

$$K(y) = \sum_{i=1}^{N-1} a_i^2 = 2, \tag{3.20}$$

or

$$a_1^2 = 2 - \sum_{i=2}^{N-1} a_i^2. \tag{3.21}$$

Substituting equation (3.19) into equation (3.11) and considering equation (3.21), the cost functional becomes

$$J(y) = 2\lambda_1 + \sum_{i=2}^{N-1} (\lambda_i - \lambda_1) a_i^2. \tag{3.22}$$

It is clear in equation (3.22) that in order to minimize the value of $J(y)$, a_i must be zero for $i = 2, 3, \dots, N-1$. Hence,

$$Y = a_1 Y_1, \tag{3.23}$$

and

$$K(y) = a_1^2 = 2 \text{ or } a_1 = \sqrt{2}. \tag{3.24}$$

It should be noted that $Y_i, i = 1, 2, \dots, N-1$, are orthonormal vectors. In the case of analytical solutions, they would turn out to be orthonormal functions. Such functions and vectors have not been studied for fractional operators, and they may lead to major applications.

3.4. RESULTS AND DISCUSSION

3.4.1. Integer order problem

An integer order optimization parametric problem defined by equations (3.2), (3.3) and (3.5) is solved by the numerical scheme presented in the previous section first. Since

the analytical solution exists for such a problem (Section 3.4.2), the numerical solution of this problem serves as a validation of the numerical algorithm.

Convergence study of the number of segments N is conducted. Figure 3.1 shows the results of $y(t)$ for different values of N. The results stabilize as N increases and recover the analytical solution at large number of N or as the step size reduces. To determine the rate of convergence, the following error calculations are performed: Let

$$e_h = y_A - y_{(h)} = Ch^{r_c} \tag{3.25}$$

be the error at a given nodal point, where y_A is the analytical solution, $y_{(h)}$ represents the numerical solution with step size h, C is a constant and r_c is the rate of convergence that is to be determined. As the step size is halved, the error for the new step size is

$$e_{h/2} = y_A - y_{(h/2)} = C\left(\frac{h}{2}\right)^{r_c}. \tag{3.26}$$

The value of r can be determined from the following ratio

$$R^* = \frac{e_h}{e_{h/2}} = 2^{r_c}. \tag{3.27}$$

Table 3.1 lists the values of the relative errors at various nodal points and the ratios of the relative errors as defined in equation (3.27) are presented in Table 3.2. The results indicate that the rate of convergence r_c is around 2.

When analytical solutions are not available, the rate of convergence can also be determined from equations (3.25) and (3.26). Subtracting equation (3.26) from equation (3.25),

$$\Delta e_h = e_h - e_{h/2} = y_{(h/2)} - y_{(h)} = Ch^{r_c}\left(1 - \frac{1}{2^r}\right). \tag{3.28}$$

As the step size gets halved,

$$\Delta e_{h/2} = e_{h/2} - e_{h/4} = y_{(h/4)} - y_{(h/2)} = C\left(\frac{h}{2}\right)^{r_c}\left(1 - \frac{1}{2^{r_c}}\right) \tag{3.29}$$

The rate of convergence c can then be determined from the ratio

$$R^{**} = \frac{\Delta e_h}{\Delta e_{h/2}} = 2^{r_c}. \tag{3.30}$$

Table 3.3 presents the error calculation of the same integer order parametric optimization problem using equation (3.28). Note that the results from taking two step sizes are needed to calculate the errors in each column. The ratios of these relative errors are presented in Table 3.4, and the values of R^{**} indicate that the rate of convergence r_c is also close to 2, same as the results demonstrated in Tables 3.1 and 3.2. The method outlined by equation (3.28-3.30) can therefore be used to obtain correct estimate of the rate of convergence using the numerical solution only.

3.4.2. Analytical solution

The analytical solution of this integer order problem is presented here which serves the validation of the numerical solution. For this problem, the Lagrangian is

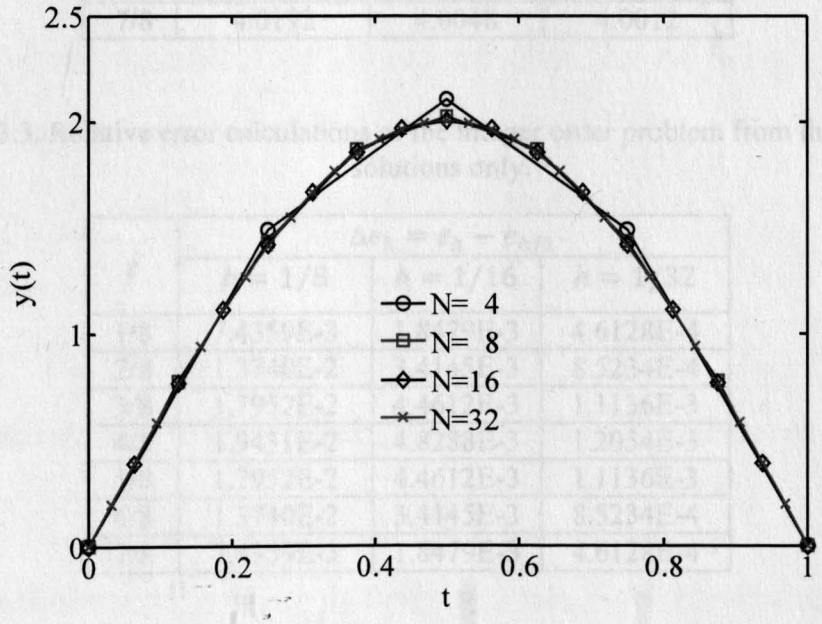

Figure 3.1. Function $\boldsymbol{y(t)}$ at different values of $\boldsymbol{N}$ for the integer order problem.

Table 3.1. Relative error calculations of the integer order problem from the analytical solution and the numerical solution.

t	$e_h = y_A - y_{(h)}$			
	$h = 1/8$	$h = 1/16$	$h = 1/32$	$h = 1/64$
1/8	0.98988E-2	2.4629E-3	0.61498E-3	1.53699E-4
2/8	1.82906E-2	4.5508E-3	1.13633E-3	2.83999E-4
3/8	2.38978E-2	5.9459E-3	1.4847E-3	3.71063E-4
4/8	2.5867E-2	6.4358E-3	1.60702E-3	4.01636E-4
5/8	2.3898E-2	5.9459E-3	1.4847E-3	3.71063E-4
6/8	1.82906E-2	4.5508E-3	1.1363E-3	2.83999E-4
7/8	0.98988E-2	2.4629E-3	0.61498E-3	1.53699E-4

Table 3.2. Ratios of the relative error of the integer order problem based on the data in Table 3.1.

t	$R^* = e_h/e_{h/2}$		
	$h = 1/8$	$h = 1/16$	$h = 1/32$
1/8	4.0192	4.0048	4.0012
2/8	4.0192	4.0048	4.0012
3/8	4.0192	4.0048	4.0012
4/8	4.0192	4.0048	4.0012
5/8	4.0192	4.0048	4.0012
6/8	4.0192	4.0048	4.0012
7/8	4.0192	4.0048	4.0012

Table 3.3. Relative error calculations of the integer order problem from the numerical solutions only.

t	$\Delta e_h = e_h - e_{h/2}$		
	$h = 1/8$	$h = 1/16$	$h = 1/32$
1/8	7.4359E-3	1.8479E-3	4.6128E-4
2/8	1.3740E-2	3.4145E-3	8.5234E-4
3/8	1.7952E-2	4.4612E-3	1.1136E-3
4/8	1.9431E-2	4.8288E-3	1.2054E-3
5/8	1.7952E-2	4.4612E-3	1.1136E-3
6/8	1.3740E-2	3.4145E-3	8.5234E-4
7/8	7.4359E-3	1.8479E-3	4.6128E-4

Table 3.4. Ratios of the relative error of the integer order problem based on the data in Table 3.3.

t	$R^{**} = \Delta e_h / \Delta e_{h/2}$	
	$h = 1/8$	$h = 1/16$
1/8	4.024	4.006
2/8	4.024	4.006
3/8	4.024	4.006
4/8	4.024	4.006
5/8	4.024	4.006
6/8	4.024	4.006
7/8	4.024	4.006

$$\bar{F} = \dot{y}^2 - \lambda y^2. \tag{3.31}$$

The differential equation is then obtained from applying Lagrange's method,

$$\ddot{y} + \lambda y = 0. \tag{3.32}$$

After imposing the boundary conditions given by equation (3.3), the solution for equation (3.32) is found,

$$y(x) = \sum_{n=1}^{\infty} B_n \sin(n\pi x). \tag{3.33}$$

After substituting equation (3.33) into equation (3.2), and due to orthogonality of the eigenfunctions, the constraint function is found to be

$$K(y) = \frac{1}{2}\sum_{n=1}^{\infty} B_n^2 = 2, \tag{3.34}$$

and therefore,

$$B_1^2 = 4 - \sum_{n=2}^{\infty} B_n^2. \tag{3.35}$$

Substituting equation (3.35) into equation (3.5), the cost functional becomes

$$J = \frac{\pi^2}{2}(4 + \sum_{n=2}^{\infty}(n^2 - 1)B_n^2). \tag{3.36}$$

In order to minimize the value of J, the following condition needs to be satisfied:

$$B_n = 0, \; n = 2,3, \ldots \infty. \tag{3.37}$$

Therefore, the solution is

$$y(x) = B_1 \sin(\pi x), \tag{3.38}$$

where B_1 can be determined from the integral constraint function equation (3.2), and it is

$$B_1 = \pm 2. \tag{3.39}$$

3.4.3. Fractional order problem

The algorithm developed in Section 3.3 is applied to the problem defined by equation (3.2) to (3.4). Figure 3.2 shows the convergence results of $y(t)$ for different values of N when the order of derivative $\alpha = 0.8$. The results stabilize as N increases.

To further demonstrate the convergence of the numerical algorithm, Table 3.5 presents the values of the cost functional J for problems of different orders of FD and at different step

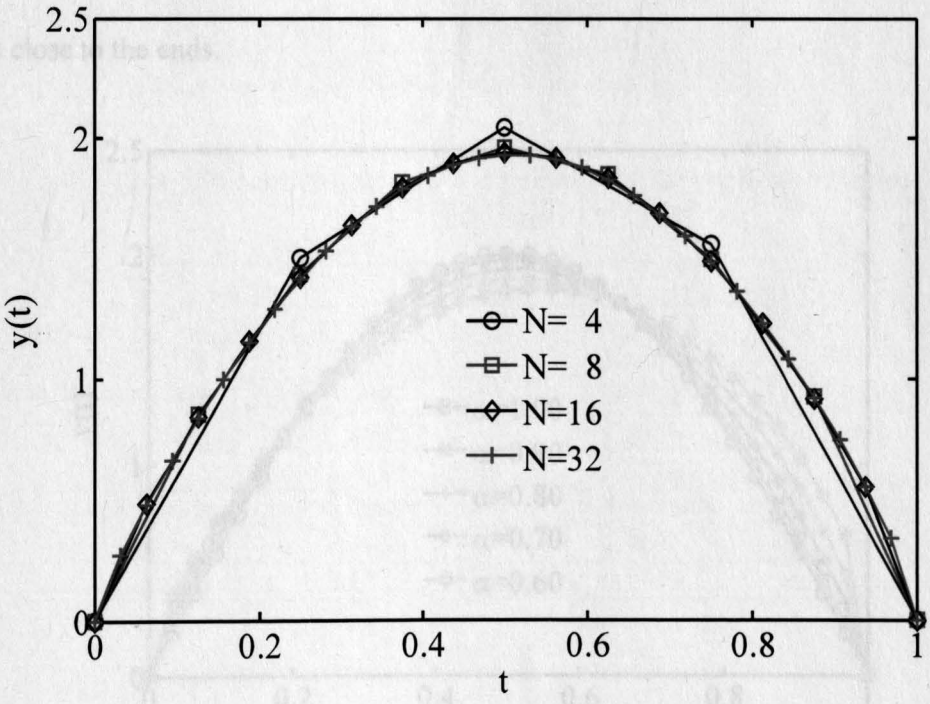

Figure 3.2. Function $\boldsymbol{y(t)}$ when $\boldsymbol{\alpha = 0.8}$ at different values of $\boldsymbol{N}$.

sizes. As the step size decreases, the value of J also decreases for such an optimization problem.

Figure 3.3 demonstrates the results for different values of α after performing the convergence study for each case. As α approaches the integer value of 1, the results recover the analytical solution of the integer order problem and therefore validates the numerical algorithm.

Since the analytical solution for this fractional order problem is not available to our best knowledge, the method outlined by equations (3.28-3.30) is used to calculate the rate of convergence for this problem. Table 3.6 presents the error calculation of the problem using equation (3.28), and the ratios of these relative errors are presented in Table 3.7. The values of R^* show that the rate of convergence r_c is also around 2 except for a couple of nodes close to the ends.

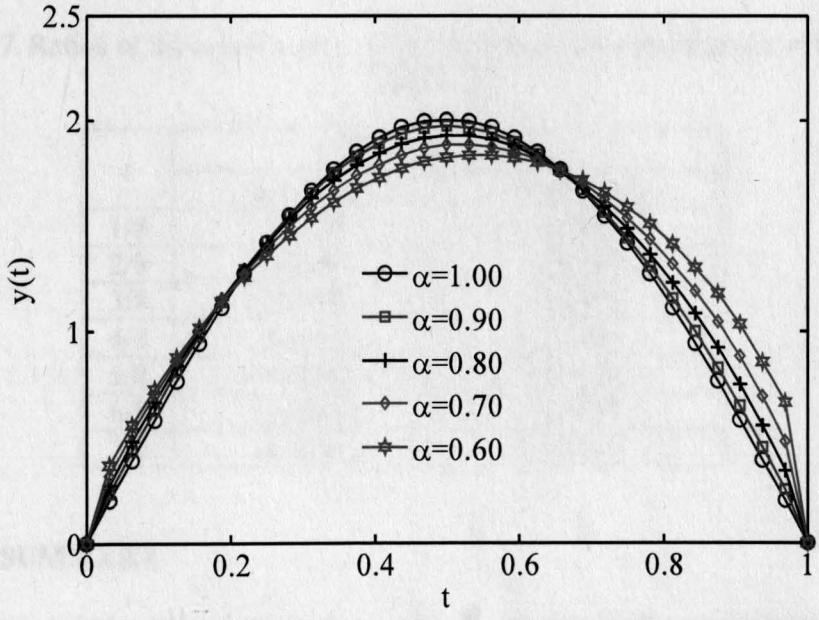

Figure 3.3. Function $\boldsymbol{y(t)}$ when $\boldsymbol{N = 32}$ at different values of $\boldsymbol{\alpha}$.

Table 3.5. Convergence of the cost functional J.

N \ α	0.9	0.8	0.7	0.6	0.5
4	14.0104	9.6897	6.8775	5.0402	3.8487
8	13.5788	9.3849	6.6174	4.8061	3.6482
16	13.4726	9.2798	6.4907	4.6656	3.5159
32	13.4404	9.2282	6.4103	4.5648	3.4166
64	13.4281	9.1983	6.3548	4.4880	3.3382

Table 3.6. Relative error for the fractional order problem ($\alpha = 0.9$) from the numerical solutions only.

t	$\Delta e_h = e_h - e_{h/2}$		
	$h = 1/8$	$h = 1/16$	$h = 1/32$
1/8	8.5274E-3	1.6614E-3	2.8359E-4
2/8	1.4114E-2	3.5565E-3	9.5113E-4
3/8	1.8829E-2	4.9023E-3	1.3986E-3
4/8	2.0358E-2	5.2764E-3	1.4831E-3
5/8	1.8370E-2	4.5813E-3	1.1756E-3
6/8	1.3304E-2	2.9767E-3	5.4283E-4
7/8	8.0109E-3	9.8585E-4	2.2164E-4

Table 3.7. Ratios of the relative error of the fractional order problem ($\alpha = 0.9$) based on Table 3.6.

t	$R^{**} = \Delta e_h / \Delta e_{h/2}$	
	$h = 1/8$	$h = 1/16$
1/8	5.1327	5.8583
2/8	3.9687	3.7392
3/8	3.8409	3.5052
4/8	3.8583	3.5576
5/8	4.0099	3.8970
6/8	4.4693	5.4837
7/8	8.1258	4.4479

3.5. SUMMARY

A numerical scheme for a linear fractional parametric optimization problem is developed using the finite element concept. The FD is expressed in terms of the RL

definition; however, the scheme also applies to other definitions of FD. The time domain $[0, 1]$ is discretized into N equal segments, and the 2-node one-dimensional element and linear interpolation are adopted in developing the matrix representations of the cost functional and the integral constraint function. After applying the Euler-Lagrange method, the problem becomes an eigenvalue problem that has real and positive eigenvalues and orthogonal eigenfunctions. Convergence study of the number of elements is conducted to ensure stability of the computation. Convergence error is analyzed and the rate of convergence is found to be close to 2. The problem is solved for different orders of derivative, and as the order approaches the value of 1, the result recovers the analytical result of the corresponding integer-order problem. The numerical scheme is easy to implement and can be extended to other fractional parametric optimization problems.

CHAPTER 4. NUMERICAL SCHEMES FOR FRACTIONAL ORDER ISOPERIMETRIC PROBLEM

This chapter presents numerical schemes to solve a functional minimization problem in fractional orders. The integer order version of the problem is to determine the shape of a hanging chain in its equilibrium state with two fixed ends and constant length. This problem is chosen because its analytical solution exists and therefore, it can serve as a validation of the numerical methods developed here for arbitrary orders of derivative when an integer order is considered. The FD is defined in terms of the RL definition; however, the scheme will also apply to Caputo derivatives, as the initial condition considered here is zero. In this scheme, the spatial domain of the equations is discretized into several small subdomains and the FD at each nodal point is approximated using the GL approach. Different definitions of GL approximations are taken to approximate the FDs of fractional orders in between 0 and 1. Convergence study with respect to different number of segments is conducted and problems with different orders of derivatives are solved. Results show that as the order of the derivatives approaches an integer order, the solution recovers the analytical result. The performances of different GL definitions are compared and analyzed.

4.1. INTRODUCTION

The isoperimetric problem is one of the most classical optimization problems (Sussmann and Willems, 1997). The isoperimetric problem is a problem of constrained optimization and it is now part of the calculus of variations (Odzijewicz and Torres, 2010). Integer order isoperimetric problems can be solved analytically using Euler-Lagrange differential equation. For fractional order isoperimetric problems, however, it is extremely difficult to solve using fractional Euler-Lagrange equation since the differential equation

contains both left and right derivatives. Almeida and Torres (2009, 2011) provided necessary and sufficient conditions of optimality for functionals containing fractional integrals and FDs in the sense of RL and Caputo derivatives. They (2011) also proved the optimality conditions for various variational functionals in terms of the left and right CFs and formulated the isoperimetric problem with an integral constraint in terms of Caputo derivatives. Additionally, Almeida and Torres (2011) provided a solution for an isoperimetric problem in fractional order for a special kind of function using fractional Euler-Lagrange equation. However, the exact solution of isoperimetric problems still remains unknown.

The main aim of this chapter is to find an approximation solution of fractional order isoperimetric problem. The FD is taken in the RL sense. Since solutions of fractional differential equations in terms of RL require fractional initial conditions (Podlubny, 1999a), the initial conditions are taken as zero here as a simpler demonstration of the method. Therefore, the numerical scheme presented in this paper also applies to other definitions of FDs such as Caputo derivatives. Approximation of FD is more complex than integer derivatives because fractional order derivative is non-local (Sousa, 2010). In this work, an iterative numerical scheme is developed where the horizontal distance is discretized into several small segments. Various orders of FDs are considered. Different definitions of GL approximations are used to represent the FDs and their performances are analyzed.

4.2. FORMULATION OF THE FRACTIONAL ORDER ISOPERIMETRIC PROBLEM

Let us consider the following problem: Among all curves $y(x)$ which satisfy the integral constraint

$$I = \int_0^1 \sqrt{1 + ({}_0D_x^{\alpha} y(x))^2}\, dx \tag{4.1}$$

and the fixed terminal (boundary) conditions

$$y(0) = y_0 \text{ and } y(1) = y_N, \tag{4.2}$$

find the curve $y(x) = y^*(x)$ that minimizes the functional

$$J(y) = \int_0^1 mg\{y\sqrt{1 + ({}_0D_x^{\alpha} y(x))^2}\}dx, \tag{4.3}$$

where ${}_0D_x^{\alpha} y(x)$ represents the left RLFD of order α. Here, for simplicity, we have taken the limits of integration from 0 to1 and the order of the derivative α in between 0 and 1. When $\alpha = 1$, equations (4.1) and (4.2) become

$$I = \int_0^1 \sqrt{1 + \left(\frac{dy}{dx}\right)^2}\, dx \tag{4.4}$$

and

$$J(y) = \int_0^1 mg\left(y\sqrt{1 + \left(\frac{dy}{dx}\right)^2}\right) dx. \tag{4.5}$$

For an integer order problem define by equations (4.2), (4.4) and (4.5), the analytical solution exists and it is given in Section 4.4. For the fractional order problem, we can represent the FDs by the GL approximation in order to find the solution. For convenience, we consider $y(0) = y(1) = 0$ and $mg = 1$. The FDs are approximated by different GL definitions described in the next section.

4.3. NUMERICAL SCHEMES FOR FRACTIONAL ORDER ISOPERIMETRIC PROBLEM

In this section, we present the numerical scheme to approximate the FDs by different GL definitions. The general definition of GL is given by equation (1.21), which is the generalization of the ordinary discretization formula for integer order derivatives. It

obviously converges for any $\alpha > 0$ and for every bound of $f(t)$ (Sousa, 2010). The definition of operator in the GL sense is equivalent to the definition of operator in the RL sense (Podlubly, 1999a). Nevertheless, the GL operator is more flexible and most straightforward in numerical calculations. There are various GL definitions to approximate the FDs that are discussed below.

To develop the numerical scheme, we divide the horizontal distance $[0, 1]$ into N elements of equal width of $h = 1/N$. The coordinates of the nodal points are $x_j = jh,\ j = 0, 1, 2, \ldots, N$ and $y(x_j) = y(jh) = y_j$. We further assume that $y(x)$ is linear over each element and the FD of each element $(({}_0^{GL}D_x^{\alpha} y)_e, e = 1, 2, \ldots, N)$ is constant over the element.

4.3.1. Standard and standard shifted GL methods

Using the standard GL approximation, the FDs of each element can be approximated by the following equation (Podlubly, 1999),

$$({}_0^{GL}D_x^{\alpha} y)_e \cong \left(\frac{1}{h}\right)^{\alpha} \sum_{j=0}^{e} \omega_j^{(\alpha)} y_{e-j} \tag{4.6}$$

where the coefficients of $\omega_j^{(\alpha)}, j = 0,1,2, \ldots\ldots, N$, are computed as follows

$$\omega_0^{(\alpha)} = 1,\ \omega_j^{(\alpha)} = \left(1 - \frac{\alpha+1}{j}\right) \omega_{j-1}^{(\alpha)},\ \ j = 1,2, \ldots\ldots, N. \tag{4.7}$$

However, the numerical approximation based on the standard (unshifted) GL approximation has limitations such as (1) it frequently generates unstable numerical methods, and (2) its order of accuracy is limited to 1 (Meerschaert and Tedjeran, 2004; Sousa, 2010). Therefore, in recent years, the shifted GL approximation has been used frequently (Meerschaert and Tedjeran, 2004; Yang, 2010). The standard shifted GL

definition for two sided FDs was presented by Meerschaert and Tedjeran (2004) and Tedjeran et al., (2006), and it was expressed as (Sousa, 2010)

$$\left({}^{GL,S}_{0}D^{\alpha}_{x}y\right)_{e} \cong \left(\frac{1}{h}\right)^{\alpha} \sum_{j=0}^{e} \omega_j^{(\alpha)}\, y_{e-j+1}, \tag{4.8}$$

where the coefficients $\omega_j^{(\alpha)}$, $j = 1,2,\ldots..,N$, can be computed by equation (4.7). The discrete representations of equations (4.1) and (4.3) become

$$I = h\sum_{e=1}^{N}\sqrt{1+\left[\left({}^{GL,S}_{0}D^{\alpha}_{x}y\right)_{e}\right]^{2}} \tag{4.9}$$

and

$$J(y) = h\sum_{e=1}^{N}\frac{(y_e+y_{e-1})}{2}\sqrt{1+\left[\left({}^{GL,S}_{0}D^{\alpha}_{x}y\right)_{e}\right]^{2}}. \tag{4.10}$$

Finally, equation (4.10) is minimized subject to the constraint of equation (4.9) and the boundary conditions given by equation (4.2).

4.3.2. Modified GL method

In the standard shifted GL definition, we approximate the FD at each nodal point. In the modified GL definition, we approximate the derivatives at the center of each element (Beleanu et. al, 2009). To develop the numerical scheme, we take $y(x)$ as an average value of the element; therefore, $y(x) = (y_j + y_{j-1})/2$. Let

$$z_{2i-1} = \frac{(y_{i-1}+y_i)}{2} \quad i = 1,2,\ldots..N \tag{4.11}$$

and by the modified GL definition, the FD of each element can be approximated as (Beleanu et. al, 2009):

$$\left({}^{MGL}_{0}D^{\alpha}_{x}y\right)_{e} = \left(\frac{2}{h}\right)^{\alpha} \sum_{j=0}^{2e-1} \omega_j^{(\alpha)}\, z_{2e-1-j}, \tag{4.12}$$

where the coefficients $\omega_j^{(\alpha)}$, $j = 1,2,\ldots\ldots,2N$, can be computed by equation (4.7). Substituting these approximations into equations (4.1) and (4.3), the discrete presentations of the two equations become

$$I = h\sum_{e=1}^{N}\sqrt{1+\left[\left({}^{MGL,S}_{0}D_x^{\alpha}y\right)_e\right]^2} \tag{4.13}$$

and

$$J(y) = h\sum_{e=1}^{N}\frac{(y_e+y_{e-1})}{2}\sqrt{1+\left[\left({}^{MGL,S}_{0}D_x^{\alpha}y\right)_e\right]^2}, \tag{4.14}$$

Equation (4.14) is then minimized subject to the constraint of the equation (4.13) and the boundary condition given by equation (4.2).

4.4. ANALYTICAL SOLUTION OF INTEGER ORDER PROBLEM

The analytical solution of the integer order problem is presented here which serves as the validation of the numerical algorithm. For this problem, the Lagrangian is

$$\bar{F} = (mgy - \lambda)\sqrt{1+y'^2}. \tag{4.15}$$

The differential equation is then obtained from applying Lagrange's method,

$$\frac{(mgy-\lambda)}{\sqrt{1+y'^2}} = C_1 \tag{4.16}$$

Rearranging equation (4.16), we obtain

$$\frac{dy}{\sqrt{(mgy-\lambda)^2-C_1^{\,2}}} = \frac{1}{C_1}dx, \tag{4.17}$$

which can be integrated to provide

$$y(x) = \frac{\lambda}{\rho g} + \frac{C_1}{\rho g}\cosh\frac{\rho g(x+C_2)}{C_1}, \tag{4.18}$$

where C_1 and C_2 are the two constant of integration.

4.5. NUMERICAL RESULTS

In this section, we present numerical results for the fractional order isoperimetric problem. We considered two types of GL definitions, the standard shifted GL definition and the modified GL definition, to approximate the FDs. The MATLAB optimization toolbox was used to solve this constraint minimization problem. The numerical result was compared with the analytical solution when the order of the derivative equaled to 1 to verify the performance of the numerical algorithms. We consider the constraint, $I = 1.4$.

Numerical results were obtained for various orders and at various step sizes. For each value of α, we first conducted convergence study of the number of elements N. Figure (4.1) shows the convergence results for the case of $\alpha = 0.9$ using the standard shifted GL definition. As the number of segment increases, the solution converges. Similarly, when the modified GL definition is employed, the result converges as well as Nincreases (figure 4.2). To further demonstrate the convergence of the numerical algorithm, Table 4.1 presents the values of the cost functional J for problems of different orders of FD and at different step sizes when standard shiftet GL definition was considered. As the step size decreases, the negative value of J also decreases for such an optimization problem. Same observation can be made when the modified GL definition was employed in Table 4.2.

We also obtained the results for various orders of the FDs at the same discretization ($N = 32$) using the standard shifted GL definition that are shown in figure 4.3. It is noted that as α approaches 1, the numerical solution converges toward the analytical solution and therefore validates the numerical algorithm. Same observation can be made when the modified GL definition was employed in figure 4.4.

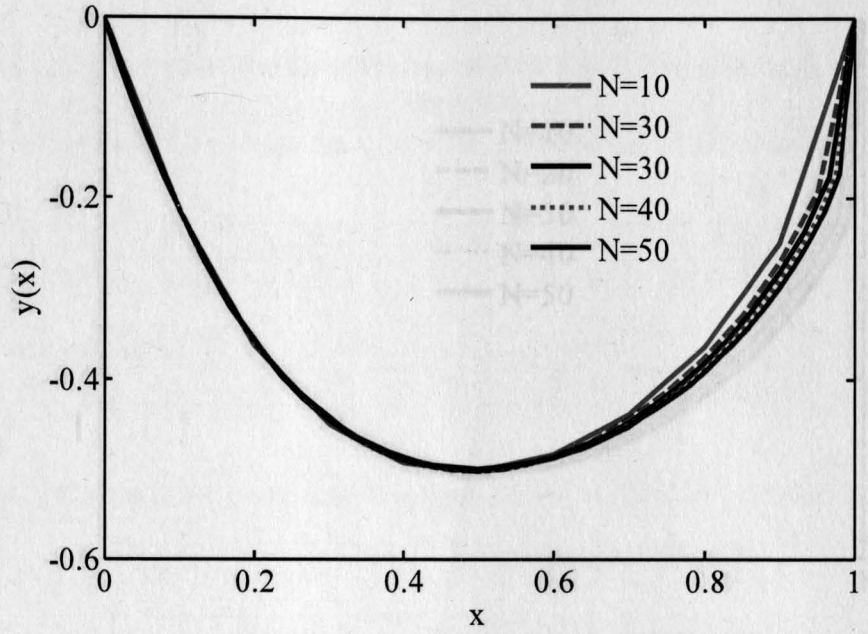

Figure 4.1. Function $y(x)$ when $\alpha = 0.9$ for different values of N when standard shifted GL definition was considered.

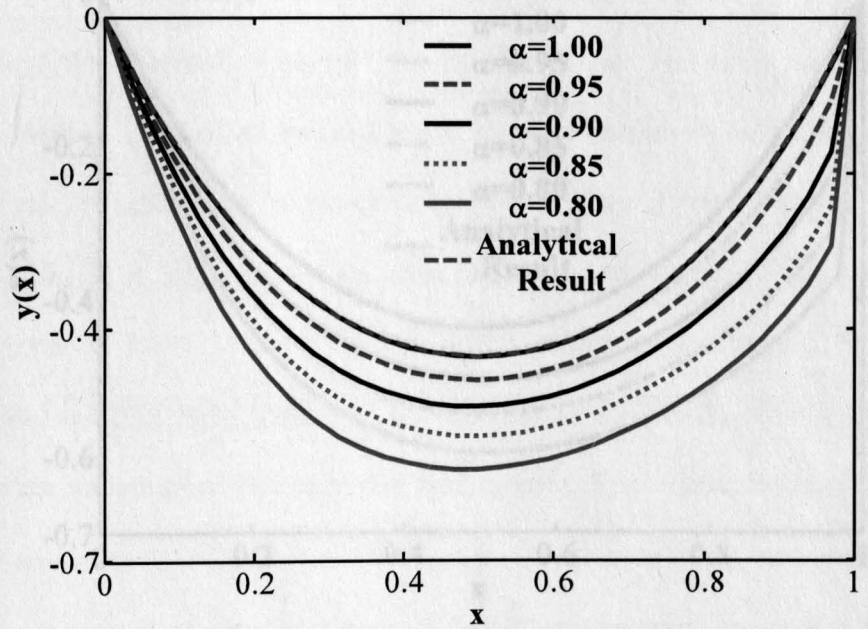

Figure 4.2. Function $y(x)$ when $N = 32$ and different values of α when standard shifted GL definition was considered.

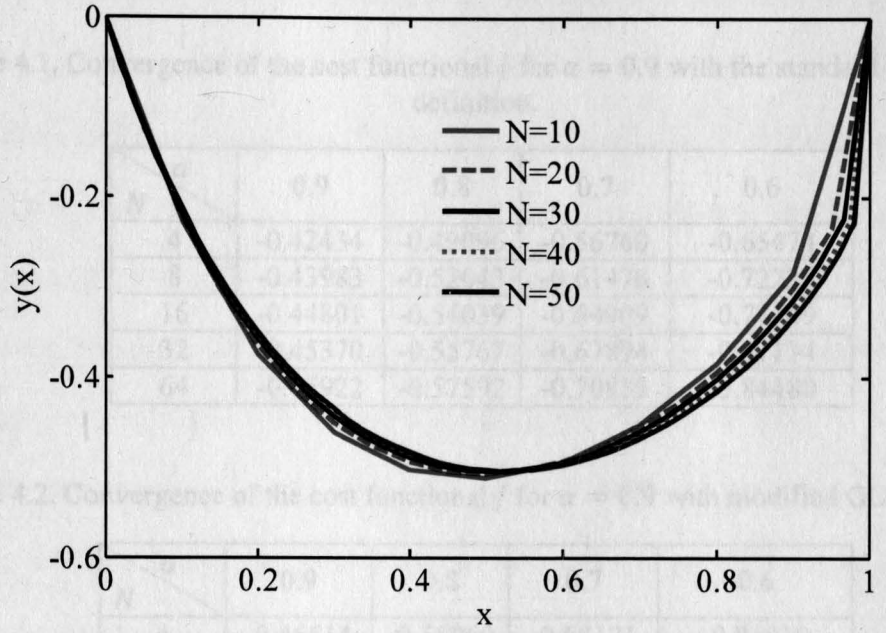

Figure 4.3. Function $y(x)$ when $\alpha = 0.9$ and different values of N when modified GL definition was considered.

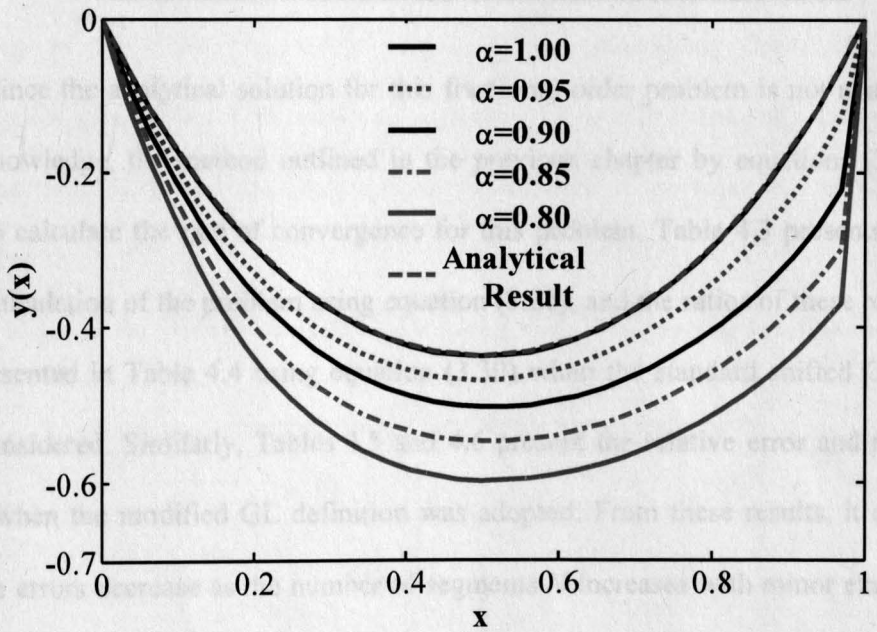

Figure 4.4. Function $y(x)$ when $N = 32$ and different values of α when modified GL definition was considered.

Table 4.1. Convergence of the cost functional J for $\alpha = 0.9$ with the standard shifted GL definition.

N \ α	0.9	0.8	0.7	0.6
4	-0.42434	-0.49096	-0.56760	-0.65474
8	-0.43983	-0.52043	-0.61470	-0.72225
16	-0.44801	-0.54039	-0.64909	-0.77149
32	-0.45370	-0.55767	-0.67894	-0.81134
64	-0.45922	-0.57592	-0.70833	-0.84480

Table 4.2. Convergence of the cost functional J for $\alpha = 0.9$ with modified GL definition.

N \ α	0.9	0.8	0.7	0.6
4	-0.45614	-0.55960	-0.68121	-0.81910
8	-0.45824	-0.56385	-0.68808	-0.82922
16	-0.46126	-0.57342	-0.70409	-0.85204
32	-0.46428	-0.58552	-0.72333	-0.87289
64	-0.46912	-0.60155	-0.74651	-0.88955

Since the analytical solution for this fractional order problem is not available to our best knowledge, the method outlined in the previous chapter by equations (3.28-3.30) is used to calculate the rate of convergence for this problem. Table 4.3 presents the relative error calculation of the problem using equation (3.28), and the ratios of these relative errors are presented in Table 4.4 using equation (3.30) when the standard shifted GL definition was considered. Similarly, Tables 4.5 and 4.6 present the relative error and relative error ratios when the modified GL definition was adopted. From these results, it can be noted that the errors decrease as the number of segments N increases with minor exceptions. We can also observe that the results at $N = 32$ have less errors than those at $N = 64$, which are represented by the relative errors in the columns of $h = 1/16$ in Tables 4.3 and 4.5. The

results in Tables 4.4 and 4.6 also indicated that the order of convergence is not constant for this problem. Kumar and Agrawal (2006) presented some numerical results for two types of fractional differential equations (FDEs), linear and nonlinear. Their results demonstrated that for the linear FDEs the rate of convergence was constant, however, the rate of convergence for the nonlinear FDEs was not constant and it varies for various order of derivatives. Similarly, the fractional order isoperimetric problem considered here is a nonlinear problem, it does not maintain a particular order of convergence.

We can see from figures 4.1-4.4, in cases of fractional orders, the shape of the function $y(x)$is no longer symmetric, and there is a sudden jump at the final element. The reason for the sudden jump is as follows: the FD has a property of being nonlocal and it incorporates all the results in the history of computation. Compared to the integer order case, the FD of the function at the final element is smaller. In order to satisfy the boundary condition at the right end, i.e. $y(1) = 0$, while minimizing the cost function, the optimization algorithm forces to take a large value of the FD at the end element and causes a sudden jump at the end Such non-symmetric behavior of the result is therefore due to the properties of the FD.

Table 4.3. Relative error analysis for $\alpha = 0.9$ with the standard shifted GL definition.

x	$\Delta e_h = e_h - e_{h/2}$		
	$h = 1/8$	$h = 1/16$	$h = 1/32$
1/8	5.6620E-4	7.5070E-4	4.0401E-3
2/8	3.7667E-3	1.3259E-3	8.6627E-3
3/8	3.6548E-3	1.9480E-3	7.1650E-3
4/8	1.6360E-3	9.6372E-4	2.0003E-3
5/8	2.1374E-3	1.6423E-3	5.7190E-3
6/8	8.0680E-3	6.8537E-3	1.4036E-2
7/8	1.7590E-2	1.6024E-2	2.5665E-2

Table 4.4. Ratios of the relative errors based on the data in Table 4.3.

t	$R^{**} = \Delta e_h / \Delta e_{h/2}$	
	$h = 1/8$	$h = 1/16$
1/8	0.7542	0.1858
2/8	2.8408	0.1531
3/8	1.8762	0.2719
4/8	1.6977	0.4818
5/8	1.3014	0.2872
6/8	1.1772	0.4883
7/8	1.0980	0.6243

Table 4.5. Relative error analysis for $\alpha = 0.9$ with modified GL definition.

x	$\Delta e_h = e_h - e_{h/2}$		
	$h = 1/8$	$h = 1/16$	$h = 1/32$
1/8	1.6494E-2	2.297E-3	8.738E-3
2/8	2.3907E-2	3.762E-4	1.122E-2
3/8	1.9238E-2	6.672E-4	7.799E-3
4/8	8.2441E-3	2.265E-3	2.445E-3
5/8	7.1852E-3	6.126E-4	5.083E-3
6/8	2.9011E-2	1.461E-3	1.349E-2
7/8	5.9419E-2	2.926E-3	2.489E-2

Table 4.6. Ratios of the relative errors based on the data in Table 4.5.

t	$R^{**} = \Delta e_h / \Delta e_{h/2}$	
	$h = 1/8$	$h = 1/16$
1/8	7.1806	0.2629
2/8	6.3549	0.0335
3/8	28.628	0.0855
4/8	3.6398	0.9264
5/8	1.1729	0.1205
6/8	1.9857	0.1083
7/8	2.0307	0.1176

4.6. CONCLUSIONS

In this chapter, two different definitions of GL approximations, the standard shifted GL definition and the modified GL approximation, have been presented to solve

isoperimetric problems in fractional order. The isoperimetric problem contains FD terms in both the minimization functional and the constraint equation. The FDs are expressed in terms of the RL definition; however, the scheme also applies to other definitions of FD such as Caputo derivatives since zero initial conditions are considered. The spatial domain $[0, 1]$ is discretized into N equal segments. As the order of the derivative α approaches the integer value of 1, the numerical results recover the analytical result. Convergence study of the number of elements is conducted to ensure the stability of the computation. The convergence errors are analyzed and the results do not suggest any particular order of convergence.

CHAPTER 5. CONCLUSIONS AND FUTURE WORK

5.1. CONCLUSIONS

This thesis presents numerical schemes for functional minimization problems that involve FD terms. Since FDs have such property as being non-local, it can be extremely challenging to find analytical solutions for fractional optimization problems, and in many cases, analytical solutions may not exist. Therefore, it is of great importance to develop numerical methods for such problems. Several numerical schemes were presented in that thesis where FDs were defined either in terms of RL or Caputo FDs. In all numerical schemes, the spatial or time domain was discretized in several subdomains and the rate of convergence and the convergence errors are analyzed to ensure that the algorithm yields stable results.

Chapter 2 presented formulations and numerical schemes for FOCPs of distributed systems in spherical and cylindrical coordinates. Partial fractional time derivatives were defined in the Caputo sense and the performance index of the FOCP was defined as a function of both state and control variables. The separation of variable method and the eigenfunction approach were used to decouple the equations and define the problem in terms of the state and control variables. For numerical simulations, the fractional differential equations were expressed in Volterra integral form. Several numerical simulations were discussed including convergence studies of the state and control variables with respect to the number of segments in the time domain. We observed that the numerical results of the state and the control variables recovered the analytical results as the order α approached 1. The three dimensional plots of the state and control variables were also generated that clearly indicated a diffusion process in the structures.

A numerical scheme for a linear functional minimization problem that involved FD terms was presented in Chapter 3. The FD was expressed in terms of the RL definition. The time domain $[0, 1]$ was discretized into N equal segments, and the 2-node one-dimensional element and linear interpolation were adopted in developing the matrix representations of the cost functional and the integral constraint function. The problem was solved for different orders of derivative, and as the order approached the value of 1, the result recovered the analytical result of the corresponding integer-order problem. That numerical scheme is easy to implement and can be extended to other fractional parametric optimization problems.

Chapter 4 presented the standard shifted GL definition and the modified GL definition to solve isoperimetric problems in fractional order. The isoperimetric problem contained FD terms in both the minimization functional and the constraint equation. The FDs were expressed in terms of the RL definition and the spatial domain $[0, 1]$ was discretized into N equal segments. Convergence study of the number of elements was conducted to ensure the stability of the computation. The numerical results of the problem recovered the analytical results as the order α approached 1.

5.2. RECOMMENDATIONS FOR FUTURE WORK

This research can be extended to several directions in the future. Some recommendations for future work are given below:

1. The numerical schemes for FOCPs were developed for only one FD term in the system dynamic constraint equation. Problems that involve multiple FD terms can be considered.

2. For the FOCPs, zero end conditions were considered. We can further specify other nonzero end conditions.

3. In all described optimization problems, we considered the order of the FDs between 0 and 1. The numerical schemes can be extended to other orders above 1.

4. In this thesis, the FDs were expressed in terms of the RL or Caputo derivatives. New types of derivatives such as the Riesz derivatives can be used to analyze the results.

5. For the linear functional minimization problem and fractional isoperimetric problem, the left FDs were considered. We can extend the problems to include the right FDs or take the average of both left and right FDs.

6. The fractional isoperimetric problem and linear optimization problem can be extended to incorporate end constraints that are functions of time.

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