NEW PERSPECTIVES ON PROMOTION AND ROWMOTION: GENERALIZATIONS AND TRANSLATIONS

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The supervisory committee certifies that this dissertation complies with North Dakota State University's regulations and meets the accepted standards for the degree of

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ABSTRACT

We define P-strict labelings for a finite poset P as a generalization of semistandard Young tableaux and show that promotion on these objects is in equivariant bijection with a toggle action on B-bounded Q-partitions of an associated poset Q. In many nice cases, this toggle action is conjugate to rowmotion. We apply this result to flagged tableaux, Gelfand–Tsetlin patterns, and symplectic tableaux, obtaining new cyclic sieving and homomesy conjectures. We then study cases in which P is a finite, graded poset other than a chain, yielding new results for products of chains and new perspectives on known conjectures. Additionally, we give resonance results for promotion on P-strict labelings and rowmotion on Q-partitions and demonstrate that P-strict promotion can be equivalently defined using Bender–Knuth and jeu-de-taquin perspectives. Finally, we explore conjectures, related and unrelated to our main theorems, on objects that promise beautiful dynamical properties.

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TABLE OF CONTENTS

LIST OF FIGURES

1. INTRODUCTION

One of the primary goals of *algebraic combinatorics* is to use *combinatorial objects* in order to gain a more intuitive understanding of complex algebraic structures. These objects, often able to be represented pictorially, allow abstract interactions to be illustrated as games with clear rules. In this way, we can arrive at algebraic results through a more straightforward manipulation of combinatorial objects, yielding proofs that are satisfying not only for their mathematical content, but for their visual, simple beauty as well. The objects that arise in these proofs are often compelling in their own right and inspire further questions without necessarily being tied to their algebraic counterpart. This is the perspective of this thesis. We will primarily draw on objects used in fields such as representation theory, but this context is not required for the enjoyment of the material. We derive our results using only the combinatorial objects and rules for their associated games, and will provide frequent figures in order to present these results in their intended manner.

More specifically, our results lie in the field of *dynamical algebraic combinatorics*, which aims to find beauty in the behavior of certain actions on combinatorial objects. We are most interested in objects that have some sort of canonical action associated with them. A good first question to consider is that of order: how many times must we apply an action to get back to the same object we started from? (In this thesis, we will only work with finite sets of objects and invertible actions, so this question will always have an answer). A similar idea is determining the cardinality of all *orbits*, or the partitioning of all the objects into sets where each element can be obtained from another by repeated application of the associated action. In addition to these periodicity questions, there are more subtle phenomena such as *homomesy* [40], which considers *combinatorial statistics* across orbits, or the *cyclic sieving phenomenon* [41]. Additionally, dynamical algebraic combinatorics suggests another desirable condition for the *explicit bijection* of enumerative combinatorics, which gives a clear process to correspond two disparate objects with each other: that the explicit bijection must also be *equivariant*. That is, the bijection additionally preserves the orbit structure of an action on one set of objects with that of the associated action on the corresponding set of objects. This notion of an equivariant bijection motivates the majority of the content of this thesis.

Briefly, this thesis is organized as follows: Chapters 2 and 3 are complete papers coauthored with J. Striker and C. Vorland. Chapter 2 introduces a new set of combinatorial objects, *P-strict labelings*, as well as an equivariant bijection to known objects, *Q-partitions*. This paper also finds applications of this bijections to other known objects, namely certain classes of *semistandard Young tableaux*. Chapter 3 aims to find additional cases where this bijection applies, especially for non-tableaux objects. Finally, Chapter 4 is a short exploration into independently pursued topics, still with a focus on bijections and dynamical results.

The purpose of the next two sections is to motivate the main results of Chapter 2 through two different paths. The first places our results a well-known combinatorial context, that of semistandard Young tableaux and Gelfand–Tsetlin patterns. The second frames the content of Chapters 2 and 3 as the fourth and fifth in a series of papers authored by Striker and collaborators. Because Chapters 2 and 3 are complete papers, they each contain some introduction to their respective content as well as preliminary definitions. So, we will use this "bonus" introduction to present this material alongside additional background information of use to those who do not work in this specific corner of the field. Additionally, some definitions in the following two sections may be stated differently than in their Chapter 2 and 3 counterparts. This is due to a difference of priority; the conventions chosen here are to describe the context in which we discovered our results, while the conventions in Chapters 2 and 3 are chosen both to serve the new definitions and results and avoid overlapping notation.

1.1. Semistandard Young tableaux and Gelfand–Tsetlin patterns

We begin with the definition of a popular combinatorial object, the semistandard Young tableau. See [48, Chapter 7] for a standard reference. For the material in this section, a **partition** λ is an ordered tuple $(\lambda_1, \lambda_2, \ldots, \lambda_n)$ with $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n$, and we will associate a partition with the shape of its **Young diagram**, consisting of n upper-left-justified rows of boxes with λ_i boxes in row i.

Definition 1.1.1. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ with all parts nonzero. A **semistandard Young tableau** of shape λ is a filling of the Young diagram of λ with positive integers such that the rows weakly increase from left to right and the columns strictly increase from top to bottom. Let $\mathrm{SSYT}(\lambda, m)$ denote the set of all semistandard Young tableau with entries at most m.

An example of a semistandard Young tableau can be found in the top left of Figure 1.3. These tableaux are notable for their use in providing a combinatorial definition of the symmetric **Schur polynomials**. In the definition below, we use the **weight** $\alpha = (\alpha_1, \ldots, \alpha_m)$ of a tableau $T \in \mathrm{SSYT}(\lambda, m)$, where α_i is the number of i entries in T. Then, we write $x^T = x^\alpha = x_1^{\alpha_1} x_2^{\alpha_2} \cdots x_m^{\alpha_m}$.

Definition 1.1.2. Define the **Schur polynomial** $s_\lambda(x)$ as

$$s_\lambda(x_1, x_2, \ldots, x_m) = \sum_{T \in \mathrm{SSYT}(\lambda, m)} x^T.$$

Of related interest, then, is the so-called Jacobi–Trudi identity given below, which gives the Schur polynomial above as a determinant in terms of the *homogeneous symmetric polynomials*. Here, the polynomial denoted by $h_k(x)$ is the sum of all monomials in the variables $x = (x_1, \ldots, x_m)$ with degree exactly k.

Theorem 1.1.3 (Jacobi–Trudi identity)**.** *Let* $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ *be a partition. Then*

$$s_\lambda(x) = \det \left[h_{\lambda_i - i + j}(x) \right]_{1 \le i, j \le n}.$$

Finally, while we certainly could enumerate $\mathrm{SSYT}(\lambda, m)$ by computing the above determinant with $x = (1, 1, \ldots, 1) := 1^m$, there is a nice product formula that we give below. Critical to this formula is the notion of the *hook length* $h(i, j)$ of a box (i, j) in the Young diagram of λ. The hook length of (i, j) is the number of boxes in row i to the right of (i, j) plus the number of boxes in column j below (i, j), plus 1 (to include the box itself). Note this is an entirely different use of the letter h than above.

Theorem 1.1.4 (Hook-content formula, see [48], Theorem 7.21.2)**.**

$$s_\lambda(1^m) = |\,\mathrm{SSYT}(\lambda, m)| = \prod_{(i,j) \in \lambda} \frac{m - i + j}{h(i, j)}$$

In the above, the numerator is sometimes written as $m - c$, where the *content* c is defined as $i - j$.

We can also define a class of Schur functions using semistandard Young tableaux of *skew* shape where, instead of an upper-left-justified Young diagram of λ, we consider the set of boxes in

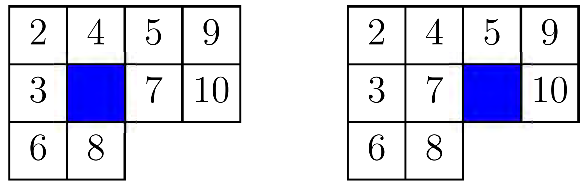


Figure 1.1. A jeu-de-taquin slide

a partition λ but not in a second partition $\mu \subset \lambda$, resulting in a shape such as in the bottom left of Figure 1.3. As before, we define these **skew semistandard Young tableaux** $\mathrm{SSYT}(\lambda/\mu, m)$ as fillings of this shape that weakly increase along rows and strictly increase down columns. Moreover, there is an associated Jacobi–Trudi identity, indicating that these tableaux are a natural generalization of $\mathrm{SSYT}(\lambda, m)$.

Importantly for dynamical algebraic combinatorics, there is a well-studied action on (skew) semistandard Young tableaux known as *promotion*. Promotion is often described in terms of *jeu-de-taquin* slides, which are easier to understand when we do not allow duplicate labels in our tableau. As such, we will first define promotion using jeu-de-taquin on **standard Young tableaux**, or fillings of a partition shape λ using each of the numbers $\{1, 2, \ldots, m\}$ exactly once that strictly increase along rows and columns, where m is the total number of boxes in λ. Denote these tableaux as $\mathrm{SYT}(\lambda)$, noting that the value m is implicit in λ.

Definition 1.1.5. Promotion on $T \in \mathrm{SYT}(\lambda)$ (denoted $\mathrm{Pro}(T)$) is performed as follows. First, remove the 1 label from the top left corner. Next, we perform a *jeu-de-taquin slide* (see [48, Chapter 7: Appendix 1]): determine which of the filled boxes adjacent to the right and below the empty box contains the lower value, and then swap the two boxes. (After the first slide, there will be a 2 in the top left corner and an empty box either to its right or below). Continue performing these slides until there are no filled boxes to the right or below the empty box. Fill this box with $m + 1$. Finally, subtract 1 from all entries.

Figure 1.1 demonstrates a single jeu-de-taquin slide in the process of promotion on the given tableau. Because the physical notion of a "slide" breaks down somewhat when working with more complex objects, we give an alternate definition of promotion using Bender–Knuth involutions. First, we will define them in the context of standard Young tableaux, and then, shortly after, provide the full original definition on semistandard Young tableaux.

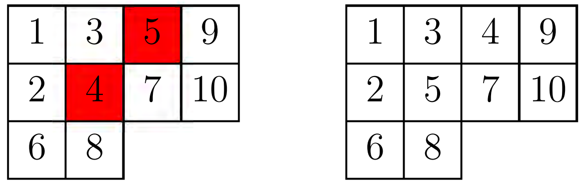


Figure 1.2. The Bender–Knuth involution ρ_4

Definition 1.1.6. Let $T \in \mathrm{SYT}(\lambda)$. Let the ith **Bender–Knuth involution** ρ_i act on T by increasing the label i to $i+1$ and decreasing the label $i+1$ to i, if the resulting tableau would be in $\mathrm{SYT}(\lambda)$.

Figure 1.2 demonstrates a Bender–Knuth involution on a standard Young tableau. Now, we give the equivalent definition of promotion using ρ_i.

Definition 1.1.7. Promotion on $T \in \mathrm{SYT}(\lambda)$ is given by $\mathrm{Pro}(T) = \rho_{m-1} \circ \cdots \circ \rho_2 \circ \rho_1(T)$.

With a slight adjustment in the definition of ρ_i, this new definition of promotion is the same as the definition of promotion on $\mathrm{SSYT}(\lambda/\mu, m)$ (and, indeed on any object for which we can devise an appropriate definition of ρ_i). The Bender–Knuth involutions on $\mathrm{SSYT}(\lambda/\mu, m)$ are given below.

Definition 1.1.8. Let $T \in \mathrm{SSYT}(\lambda/\mu, m)$. Let the ith Bender–Knuth involution ρ_i act on T as follows: consider all i and $i+1$ labels. If $i+1$ does not appear directly below an i, that is, if i could be incremented in its column to $i+1$ without violating the strict increase condition, call this i label "free." Similarly, call an $i+1$ label "free" if i does not appear directly above the $i+1$ label. Suppose there are a free i labels and b free $i+1$ labels in row k, necessarily all adjacent. Then, for each row, ρ_i replaces these $a+b$ total labels with b i labels and a $i+1$ labels.

To give a taste of the sort of result we look for, we present the following widely-known fact that entirely classifies the size of promotion orbits acting on rectangular semistandard Young tableaux.

Theorem 1.1.9 ([42])**.** *Let λ be of rectangular shape and let $T \in \mathrm{SSYT}(\lambda, m)$. Then $\mathrm{Pro}^m(T) = T$.*

A nice periodicty result such as this is somewhat rare. For example, promotion on the tableaux $\mathrm{SSYT}((5,5,3,2),5)$, (an example of such a tableaux is pictured in Figure 1.3) has an orbit of size 40, much larger than 5, as well as an orbit of size 7, relatively prime to 5.

Next, we take a look at another, related, combinatorial object called a Gelfand–Tsetlin pattern.

Definition 1.1.10. [48, Chapter 7] A **Gelfand–Tsetlin pattern** (GT-pattern) is a triangular array G of nonnegative integers indexed as follows:

$$\begin{array}{ccccccccc} a_{n1} & & a_{n2} & & a_{n3} & & \cdots & & a_{nn} \\ & \ddots & & \ddots & & \iddots & & \iddots & \\ & & & a_{21} & & a_{22} & & & \\ & & & & a_{11} & & & & \end{array}$$

with the additional conditions that G weakly decreases along rows and the value of a_{ij} is weakly between its two upward neighbors. That is, $a_{i,j+1} \leq a_{i-1,j} \leq a_{ij}$ whenever these numbers are defined.

There is a straightforward bijection between GT-patterns and semistandard Young tableaux. For $T \in \mathrm{SSYT}(\lambda, m)$, let $\lambda_{\leq i}$ be the partition shape consisting of boxes with labels less than or equal to i. Then, in the associated GT-pattern G, the ith row is given by $\lambda_{\leq i}$, appending zeroes as necessary to create a triangular array. See Figure 1.3 for an example of this bijection.

With this bijection in mind, we can slightly revise the definition of GT-patterns such that it accommodates skew semistandard tableaux. For this, we consider a GT-pattern as a parallelogram-shaped (or trapezoidal, as in Chapter 2) array with $n+1$ rows, including a bottom row indexed $a_{01}, \ldots, a_{0n}$, with the same inequalities as before. Now, $T \in \mathrm{SSYT}(\lambda/\mu, m)$ corresponds to a parallelogram-shaped GT-pattern with top row λ and bottom row μ, where each row is made the same length by appending zeroes as needed.

Such a bijection is an example of an *explicit bijection*, demonstrating that $\mathrm{SSYT}(\lambda, m)$ and GT-patterns with m rows whose top row is λ are equinumerous. As noted earlier, we aim to find *equivariant bijections* that additionally preserve the behavior of particular actions on both involved objects. It turns out that due to work by A. Kirillov and A. Berenstein [31], we have such a bijection, involving the earlier-defined action of Bender–Knuth involutions on skew semistandard Young tableaux and elementary transformations on GT-patterns.

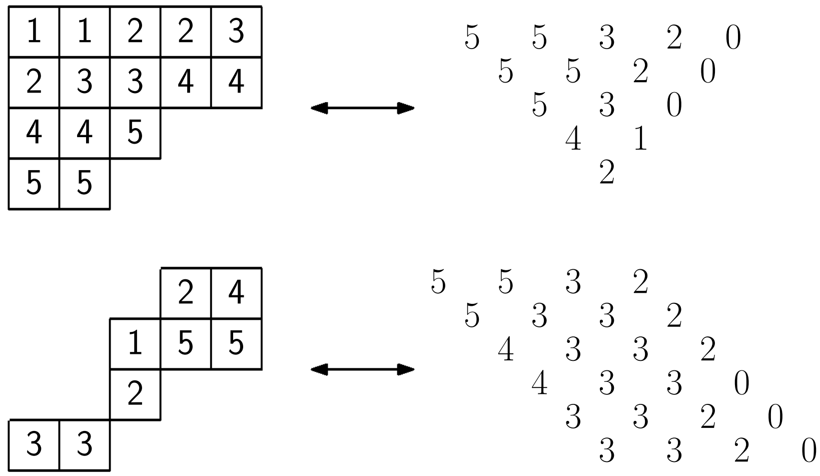


Figure 1.3. Top: an element of $\mathrm{SSYT}(\lambda, 5)$ for $\lambda = (5, 5, 3, 2)$ and its associated GT-pattern. Bottom: an element of $\mathrm{SSYT}(\lambda/\mu, 5)$ with the same λ and $\mu = (3, 2, 2)$ and its associated (parallelogram-shaped) GT-pattern.

Definition 1.1.11. Let G be a GT-pattern. For $1 \leq k \leq n-1$, define the **elementary transformation** $t_k(G)$ over all elements of G by

$$t_k(a_{ij}) = \begin{cases} a_{ij} & i \neq k \\ \min(a_{i+1,j}, a_{i-1,j-1}) + \max(a_{i+1,j+1}, a_{i-1,j}) - a_{ij} & \text{otherwise,} \end{cases}$$

where we ignore any undefined arguments of max and min.

The following proposition, labeled so because of its original source, gives us our equivariant bijection.

Proposition 1.1.12 ([31, Proposition 2.2])**.** *Let $T \in \mathrm{SSYT}(\lambda/\mu, m)$ and G the corresponding GT-pattern. The Bender–Knuth involution ρ_k on T corresponds to t_k on G.*

What could we try to generalize from this context? One direction could consider restrictions on the values of T other than just bounding all entries by m. For example, *flagged tableaux* impose a bound for each row of the tableau. Another possible generalization lies in changing the entire construction of a tableau shape. Instead of pasting strictly increasing columns together to form weakly increasing rows, the columns could be a different shape, such as an array, that strictly increases in some way (more specifically, a *poset* shape, defined in the next section). Then, from these generalizations on the tableaux side, what adjustments can be made to keep a similar

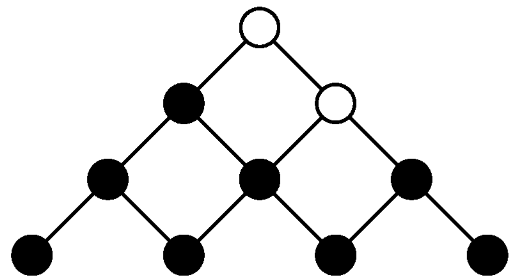


Figure 1.4. An element of $J(\triangle_4)$ displayed on a Hasse diagram

equivariant bijection to the above, and what new GT-pattern-like objects arise on the other side? Our main results in Chapter 2 address these sorts of inquiries.

1.2. A series of papers

While the previous section describes an approach to our results using widely-known objects and actions, our path of discovery to the main results of Chapter 2 follows a line of thinking owed to a series of papers [54, 12, 13]. We will give a brief overview of each of these, and conclude by describing the generalization idea that eventually led to Chapter 2. Note that we will only extract results and ideas that are critical to Chapter 2, so the following subsections should not be considered a comprehensive review of these papers.

As a quick preliminary, following [51, Chapter 3], a **partially ordered set** P, or **poset**, is a set together with a reflexive, antisymmetric, and transitive partial order relation $\leq_P$. If $p \leq_P p'$ and no q exists such that $p \leq_P q \leq_P p'$, then we say p' **covers** p and write $p \lessdot_P p'$. We represent a poset by its **Hasse diagram**, a graph whose vertices are poset elements, whose edges are covering relations, and is drawn such that p is lower than p' whenever $p \leq_P p'$.

An important combinatorial object for this section is an **order ideal** I of a poset P. An **order ideal** is a set of elements in P that is closed downwards, that is, if $p' \in I$ and $p \leq_P p'$, then $p \in I$. Denote the set of all order ideals of P as $J(P)$. We often draw order ideals on a Hasse diagram by representing elements in the order ideal as filled dots and elements outside of the order ideal as empty circles. Figure 1.4 shows an order ideal of the triangle-shaped poset $\triangle_4$ drawn on the Hasse diagram. For an exact definition of $\triangle_n$, also known as the type A root poset A_n, see Definition 2.4.26.

Lastly, we use the notation $[n]$ to represent the **chain poset** $1 \lessdot 2 \lessdot \cdots \lessdot n$, and the direct product notation $P \times Q$ to denote poset with elements $\{(p,q) \mid p \in P, q \in Q\}$ where $(p,q) \leq (p',q')$ if $p \leq_P p'$ and $q \leq_Q q'$. We will commonly consider products of posets with chains.

1.2.1. Promotion and rowmotion

In [54], Striker and N. Williams devise a novel proof of the following theorem, which provides the perspective for the later papers in this series.

Theorem 1.2.1. *Let* $\lambda = (n+1, n+1)$. *Then there is an equivariant bijection between* $\mathrm{SYT}(\lambda)$ *under promotion and* $J(\triangle_n)$ *under rowmotion.*

Striker and Williams coined the term *rowmotion* to give a name to a known action on order ideals with previously non-descriptive notation.

Definition 1.2.2. Let $I \in J(P)$. Let **rowmotion**, denoted $\mathrm{Row}(I)$, be the order ideal generated by the minimal elements of P that are not in I.

From this definition, it is not clear why the notion of a "row" is emphasized. To see why, we first redefine rowmotion using toggles, following Cameron and Fon-der-Flaass [8].

Definition 1.2.3. Let I in $J(P)$. Define the **toggle** at $p \in P$ as an action on I as follows:

$$t_p(I) = \begin{cases} I \cup \{p\} & p \notin I \text{ and } I \cup \{p\} \in J(P) \\ I \setminus \{p\} & p \in I \text{ and } I \setminus \{p\} \in J(P) \\ I & \text{otherwise.} \end{cases}$$

A **linear extension** of a poset P is an ordering of the elements of P such that if $p \leq_P p'$, then p' occurs later in the ordering.

Definition 1.2.4. Let $I \in J(P)$. Then $\mathrm{Row}(I) = t_{p_1} \circ t_{p_2} \circ \cdots \circ t_{p_n}(I)$, where $p_1, p_2, \ldots, p_n$ is a linear extension of P. That is, rowmotion toggles the poset "top to bottom."

If a poset is ranked, it has rows. Rowmotion toggles these rows top to bottom, hence the name.

The crucial result of this paper is the observation that, for a poset with a defined notion of "rows" and "columns," toggling the rows top to bottom (rowmotion) is conjugate (has the same orbit structure) with toggling the columns left to right. Note that the equivariant bijection indicated in the statement of Theorem 1.2.1 is not given explicitly. However, there is an explicit bijection equivariant with toggling $\triangle_n$ from left to right. The authors name this toggle order Pro in keeping with its association to promotion on $\mathrm{SYT}(\lambda)$.

Thus, the big idea of this paper is not only that promotion on some objects is equivariant with rowmotion on others, but that we can prove these results using explicit equivariant bijections and demonstrate that these explicit actions are conjugate with rowmotion, a naturally-defined action.

1.2.2. Resonance in orbits of plane partitions and increasing tableaux

The main idea we will look at from [12] of K. Dilks, O. Pechenik, and Striker is another case in which promotion on a tableau object is in explicit equivariant bijection with a toggle order on a particular set of order ideals, which is conjugate to rowmotion.

A **plane partition** is a stacking of $1 \times 1 \times 1$ boxes in the lower corner of an $a \times b \times c$ box, and can therefore be represented as an element of $J([a] \times [b] \times [c])$. The authors define a map from these order ideals to rectangular increasing tableaux.

Definition 1.2.5. An **increasing tableau** is a filling of the partition shape λ with positive integers such that the entries strictly increase along rows and down columns.

Now, we can map $I \in J([a] \times [b] \times [c])$ to an increasing tableau T with shape $a \times b$ by setting the entry $T(i,j)$ equal to the number of elements of the chain subposet $(i,j) \times [c]$ not in I, and then adding $i + j - 1$, the rank of (i,j) in $[a] \times [b]$ plus one. In this way, T is a increasing tableau with upper bound $m = a+b+c-1$. This map is shown in Figure 1.5. Moreover, the authors prove the following equivariance result, where Pro on increasing tableaux is defined using Bender–Knuth involutions analogously to Definition 1.1.7.

Theorem 1.2.6. *$J([a] \times [b] \times [c])$ under* Row *is in equivariant bijection with increasing tableau of shape $a \times b$ with upper bound $a + b + c - 1$ under* Pro*.*

As in [54], this theorem is not proved by giving an explicit bijection. Rather, Pro on increasing tableaux corresponds directly with toggling $J([a] \times [b] \times [c])$ using a particular hyperplane sweep that is conjugate to the top to bottom hyperplane sweep of rowmotion. The authors addi-

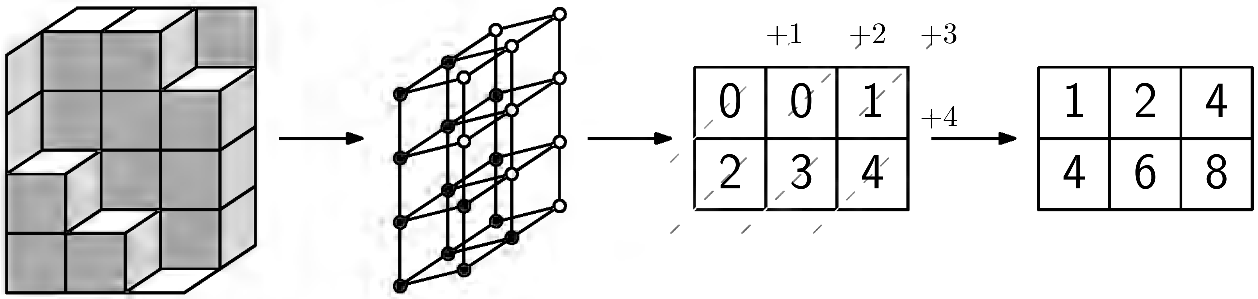


Figure 1.5. An illustration of the equivariant bijection from $J([a] \times [b] \times [c])$ to increasing tableaux of shape $a \times b$ with entries no greater than $a + b + c - 1$. Here, $a = 2, b = 3$, and $c = 4$. We also include, on the far left, the plane partition corresponding to the order ideal.

tionally show that multiple hyperplane sweeps share this characteristic, demonstrating a variety of "promotions" conjugate with rowmotion.

1.2.3. Rowmotion and increasing labeling promotion

In [13], Dilks, Striker, and C. Vorland generalize the result stated in the previous subsection. Beginning on the increasing tableaux side, we first generalize to increasing labelings, which are not tied to partition shapes.

Definition 1.2.7. Let P be a poset. We call a labeling f of P with positive integers no greater than m an **increasing labeling** of P if $f(p) < f(p')$ whenever $p <_P p'$. Denote the set of these increasing labelings as $\mathrm{Inc}^m(P)$.

Pursuing full generality, the authors additionally define increasing labelings in which one can specify exactly which values are permissible for a particular poset element. We call this specification of values a **restriction function** $R : P \to \mathcal{P}(\mathbb{Z})$, and the set of all increasing labelings of P satisfying this restriction function $\mathrm{Inc}^R(P)$.

After generalizing this half of the bijection of [12], what remains is to determine the order ideals in correspondence with $\mathrm{Inc}^R(P)$. The poset $\Gamma(P, R)$, defined in detail in Chapter 2, is constructed in order to be precisely the poset whose order ideals are in bijection with $\mathrm{Inc}^R(P)$. An example of an order ideal of $\Gamma(P, R)$ along with its associated increasing labeling are shown in Figure 1.6

Most importantly, the bijection to order ideals of $\Gamma(P, R)$ is equivariant. For an adjusted definition of Bender–Knuth involutions, Pro on $\mathrm{Inc}^R(P)$ is in equivariant bijection with $J(\Gamma(P, R))$

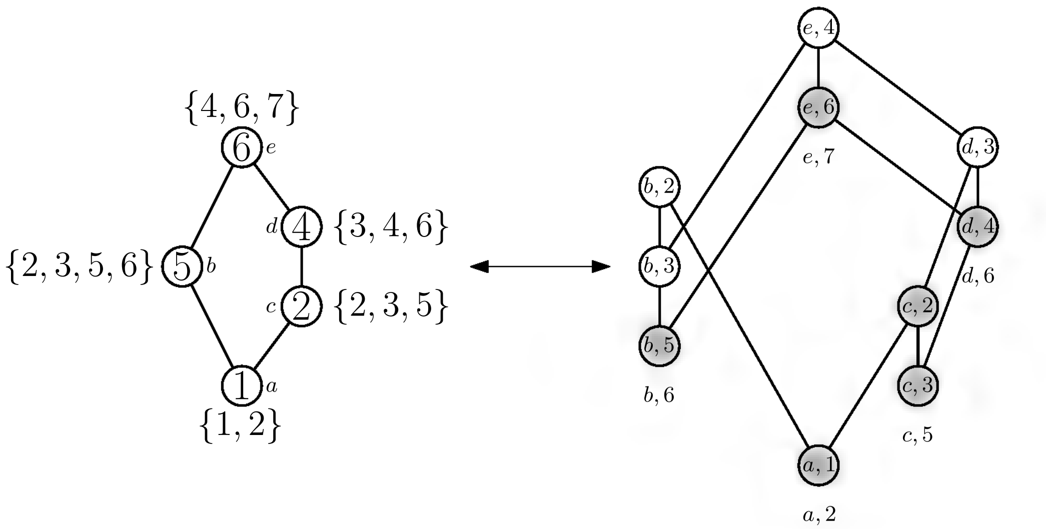


Figure 1.6. On the left is an increasing labeling of P whose elements are accompanied by the restriction function $R(p)$, and on the right is the order ideal of $\Gamma(P, R)$ in equivariant bijection with the increasing labeling.

under *toggle promotion*, or toggling all the elements of $\Gamma(P, R)$ of the form $(p, 1)$, then $(p, 2)$ and so on. Clearly, toggle promotion is not equivalent to rowmotion in general, and could not be considered a natural action on an arbitrary poset. However, this paper also provides a concrete characterization of when a particular order of toggling is conjugate to rowmotion, called a *column toggle order*. This notion is explained further in Chapter 2, under the slightly more general idea of *column adjacency*. Briefly, if the order of toggling shares the same relationships between consecutively toggled elements as if they were in columns of a poset such as $\triangle_n$, then this column toggle order is conjugate with rowmotion. The hyperplane sweeps of [12] are exactly column toggle orders of $[a] \times [b] \times [c]$, so these toggle orders are, as proved, conjugate with rowmotion.

1.2.4. Piecewise-linear toggles and Q-partitions

This final subsection introduces two definitions that provide a natural ramp to Chapter 2, given the context of the series of papers introduced in the previous three subsections. First, we look at what could be considered a generalization of order ideals, Q-partitions. This is a good place to note that these objects are widely known as P-partitions (see, e.g., [51, Chapter 3.15]), and sometimes notated $\mathcal{PP}$, but, given our widespread use of P in the upcoming chapters to refer to a poset on which we are not considering (P)-partitions, we choose to call them Q-partitions in general.

Definition 1.2.8. A **Q-partition** is an order-preserving labeling of a poset Q using nonnegative integers that weakly increase up the poset.

We can consider these as a generalization of order ideals, since a Q-partition with a maximum label of 1 corresponds to the order ideal with elements labeled by zero. We also have a notion of toggling on posets labeled with integers, *piecewise-linear toggling*, in which toggling a particular element "flips" its value in the interval determined by the minimum label of its upper covers and the maximum label of its lower covers. This is rigorously defined in Chapter 2, as it is exactly this toggle action that drove our interest. Because of [13], we understand order ideal toggling on $\Gamma(P, R)$ and its relation to $\text{Inc}^R(P)$. So, if we instead considered Q-partitions with $Q = \Gamma(P, R)$, what action on which objects would piecewise-linear toggles correspond with? This is the question that is answered in depth in the chapters to follow.

2. P-STRICT PROMOTION AND B-BOUNDED ROWMOTION[1]

2.1. Introduction

This paper builds on the papers [54, 12, 13] investigating ever more general domains in which promotion on tableaux (or tableaux-like objects) and rowmotion on order ideals (or generalizations of order ideals) correspond. In [54], N. Williams and the second author proved a general result about rowmotion and toggles which yielded an equivariant bijection between promotion on $2 \times n$ standard Young tableaux and rowmotion on order ideals of the triangular poset $\triangle_{n-1}$ (by reinterpreting the Type A case of a result of D. Armstrong, C. Stump, and H. Thomas [2] as a special case of a general theorem they showed about toggles). In [12], the second author, with K. Dilks and O. Pechenik, found a correspondence between $a \times b$ increasing tableaux with entries at most $a+b+c-1$ under K-promotion and order ideals of $[a] \times [b] \times [c]$ under rowmotion. In [13], the second and third authors with Dilks broadened this correspondence to generalized promotion on increasing labelings of any finite poset P with restriction function R on the labels and rowmotion on order ideals of a corresponding poset.

In this paper, we generalize from rowmotion on order ideals to rowmowtion on *B-bounded Q-partitions* and determine the corresponding promotion action on tableaux-like objects we call *P-strict labelings* (named in analogy to column-strict tableaux). This general theorem includes all of the previously known correspondences between promotion and rowmotion and gives new corollaries relating P-strict promotion on flagged or symplectic tableaux to B-bounded rowmotion on nice Q-partitions. Our main results also specialize to include a result of A. Kirillov and A. Berenstein [31] which states that Bender-Knuth involutions on semistandard Young tableaux correspond to piecewise-linear toggles on the corresponding Gelfand-Tsetlin pattern.

[1]The material in this chapter was coauthored by J. Bernstein, J. Striker, and C. Vorland and is published as *P-strict promotion and B-bounded rowmotion, with applications to tableaux of many flavors*. *Combinatorial Theory*, 1 (2021). The coauthors worked collaboratively on the main results of the paper. Bernstein had primary responsibility for drafting the proofs of the main theorems and corollaries of the main theorems in Sections 2.2 and 2.4. Section 2.3 is largely the work of Vorland. All coauthors revised and proofread this chapter.

The paper is structured as follows. The introduction begins in Section 2.1.1 with a motivating example. Then we define our new objects, P-strict labelings, and a corresponding promotion action in Section 2.1.2. In Section 2.1.3, we define B-bounded Q-partitions and the associated toggle and rowmotion actions. In Section 2.1.4, the final section of our introduction, we summarize the main results of this paper. Section 2.2 proves our main theorems relating P-strict promotion, toggles, and B-bounded rowmotion. Section 2.3 studies further properties of promotion and evacuation on P-strict labelings, including a jeu de taquin characterization of promotion for special P-strict labelings. Finally, Section 2.4 applies our main theorem to many special cases of interest.

2.1.1. An example

To motivate our main results, we begin with an example (see the remaining subsections of the introduction for definitions). In [13], Dilks and the second two authors found as an application of their main results an equivariant bijection between promotion on increasing labelings of a chain $P = [n] := p_1 \lessdot p_2 \lessdot \cdots \lessdot p_n$ with the label $f(p_j)$ restricted as $j \leq f(p_j) \leq 2j$ and rowmotion on order ideals of the positive root poset $\triangle_n$. The idea for the current paper arose from the question of what happens in the above correspondence when order ideals of $\triangle_n$ are replaced by $\triangle_n$-partitions of height ℓ (that is, weakly increasing labelings of $\triangle_n$ with labels in $\{0, 1, \ldots, \ell\}$).

In this paper, we give a bijection to the following: take ℓ copies of $P = [n]$ to form the poset $P \times [\ell] = \{(p, i) \mid p \in P \text{ and } 0 \leq i \leq \ell\}$ and consider labelings $f : P \times [\ell] \to \mathbb{N}$ that are strictly increasing in each copy of P, weakly increasing along each copy of $[\ell]$, and obey the restriction $j \leq f(p_j, i) \leq 2j$ as before (call this restriction R). We call these *P-strict labelings* of $P \times [\ell]$ with restriction function R. In this special case, under a mild transformation (represented by the top arrow of Figure 2.1), these are flagged tableaux of shape ℓ^n with flag $(2, 4, \ldots, 2n)$ (that is, semistandard tableaux with entries in row j at most $2j$). The rightmost arrow of Figure 2.1 represents the bijection from the first main result of this paper, Theorem 2.2.8.

Our second main result, Theorem 2.2.20, implies that P-strict promotion (also called *flagged* promotion in this case) on these flagged tableaux is in equivariant bijection with B-bounded rowmotion (also called *piecewise-linear* rowmotion) on these $\triangle_n$-partitions with labels at most ℓ. Then we deduce by a theorem of D. Grinberg and T. Roby [23, Corollary 66] on *birational* rowmotion that promotion on these flagged tableaux is, surprisingly, of order $2(n+1)$. Note there is no dependence on the number of columns ℓ! We discuss this and other applications to flagged tableaux in more detail in Section 2.4.2. See Corollaries 2.4.28 and 2.4.30 for these specific results and Figure 2.1 for an example of the bijection.

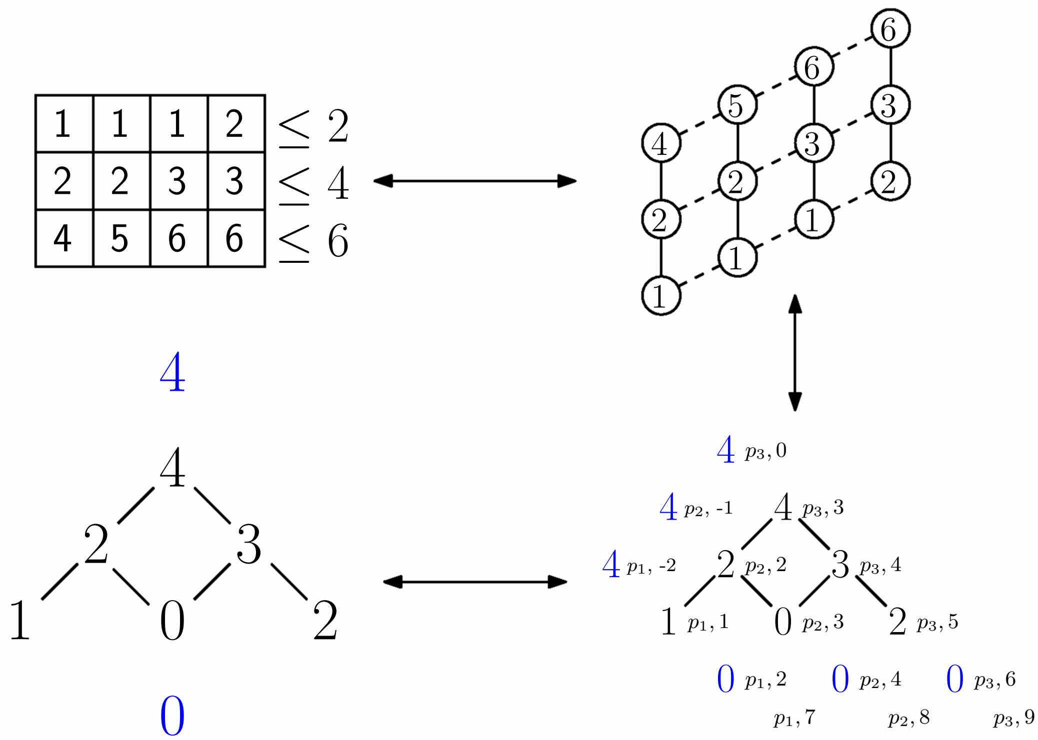


Figure 2.1. A motivating example of the bijection of this paper, relating flagged tableaux of shape ℓ^n with flag $(2, 4, 6, \ldots, 2n)$ to $\triangle_n$-partitions with labels at most ℓ. See Corollaries 2.4.28 and 2.4.30, which imply the order of promotion on these flagged tableaux is 8.

2.1.2. Promotion on P-strict labelings

Promotion is a well-loved action defined by M.-P. Schützenberger on linear extensions of a finite poset [46]. On a partition-shaped poset, linear extensions are equivalent to standard Young tableaux. Promotion has been defined on many other flavors of tableaux and labelings of posets using *jeu de taquin* slides and their generalizations. Equivalently (as shown in [20, 50, 12, 13]), promotion may be defined by a sequence of involutions, introduced by E. Bender and D. Knuth on semistandard Young tableaux [4]. This will be our main perspective; we discuss the jeu de taquin viewpoint further in Section 2.3.

Below, we define *P-strict labelings*, which generalize both semistandard Young tableaux and increasing labelings. We extend the definition of promotion in terms of Bender-Knuth involutions to this setting. We show in Theorem 2.3.10 in which cases promotion may be equivalently defined using jeu de taquin.

Definition 2.1.1. In this paper, P represents a finite poset with partial order $\leq_P$, $\lessdot$ indicates a covering relation in a poset, ℓ and q are positive integers, $[\ell]$ denotes a chain poset (total order) of ℓ elements (whose elements will be named as indicated in context), and $P \times [\ell] = \{(p, i) \mid p \in P, i \in \mathbb{N}, \text{ and } 1 \leq i \leq \ell\}$ with the usual Cartesian product poset structure.

Below, we define P-strict labelings on *convex subposets* of $P \times [\ell]$. A **convex** subposet is a subposet such that if two comparable poset elements a and b are in the subposet, then so is the entire interval $[a, b]$. This level of generality is necessary to, for instance, capture the case of promotion on semistandard Young tableaux of non-rectangular shape.

Definition 2.1.2. Given S a convex subposet of $P \times [\ell]$, let $L_i = \{(p, i) \in S \mid p \in P\}$ be the ith **layer** of S and $F_p = \{(p, i) \in S \mid 1 \leq i \leq \ell\}$ be the pth **fiber** of S.

Convex subposets of $P \times [\ell]$ have a predictable structure, as we show in the following proposition.

Definition 2.1.3. Let $u : P \to \{0, 1, \ldots, \ell\}$ and $v : P \to \{0, 1, \ldots, \ell\}$ with $u(p) + v(p) \leq \ell$ for all $p \in P$ and $v(p_1) \leq v(p_2)$ and $u(p_1) \geq u(p_2)$ whenever $p_1 \leq_P p_2$. Then define $P \times [\ell]_u^v$ as the subposet of $P \times [\ell]$ given by $\{(p, i) \in P \times [\ell] \mid u(p) < i < \ell + 1 - v(p)\}$.

Proposition 2.1.4. *Let S be a convex subposet of $P \times [\ell]$. Then there exist u and v such that $S = P \times [\ell]_u^v$.*

Proof. Since S is convex, along any fiber F_p we have $(p, i) \in S$ with $i_0 < i < i_1$ for some $i_0 \geq 0$ and some $i_1 \leq \ell + 1$. If $F_p \neq \emptyset$, let $u(p) = i_0$ and $v(p) = \ell + 1 - i_1$. If $\omega \gtrdot_P p$, then $u(\omega) \leq u(p)$, otherwise $(p, u(\omega)), (\omega, u(\omega) + 1) \in S$ but $(\omega, u(\omega)) \notin S$, contradicting the convexity of S. Similarly, $v(\omega) \geq v(p)$. If $F_p = \emptyset$, then $F_\omega = \emptyset$ for all $\omega \gtrdot_P p$ by convexity. For all $p \in P$ with $F_p = \emptyset$, set $u(p) = \min\{u(q) \mid F_q \neq \emptyset\}$ and $v(p) = \ell - u(p)$. Thus $u(p) + v(p) = \ell$ and, over all of P, $u(p_1) \geq u(p_2)$ when $p_1 \lessdot_P p_2$. Moreover, since for all p with $F_p \neq \emptyset$ we have $v(p) < \ell - u(p)$, $v(p_1) \leq v(p_2)$ for all $p_1 \lessdot_P p_2$. $\square$

Example 2.1.5. Let $\lambda = (\lambda_1, \ldots, \lambda_n)$ and $\mu = (\mu_1, \ldots, \mu_n)$ be partitions. Consider the case where $P = [n]$, $u(p) = \mu_p$, and $v(p) = \ell - \lambda_p$ for all $p \in P$. In this case, the convex subposet is a skew tableau shape λ/μ that fits inside an $n \times \ell$ rectangle.

Definition 2.1.6. Let $\mathcal{P}(\mathbb{Z})$ represent the set of all nonempty, finite subsets of $\mathbb{Z}$. A **restriction function on** P is a map $R : P \to \mathcal{P}(\mathbb{Z})$.

In this paper, R will always represent a restriction function.

Definition 2.1.7. We say that a function $f : P \times [\ell]_u^v \to \mathbb{Z}$ is a P**-strict labeling of** $P \times [\ell]_u^v$ **with restriction function** R if f satisfies the following on $P \times [\ell]_u^v$:

1. $f(p_1, i) < f(p_2, i)$ whenever $p_1 <_P p_2$,

2. $f(p, i_1) \leq f(p, i_2)$ whenever $i_1 \leq i_2$,

3. $f(p, i) \in R(p)$.

That is, f is strictly increasing inside each copy of P (layer), weakly increasing along each copy of the chain $[\ell]$ (fiber), and such that the labels come from the restriction function R.

Let $\mathcal{L}_{P \times [\ell]}(u, v, R)$ denote the set of all P-strict labelings on $P \times [\ell]_u^v$ with restriction function R. If the convex subposet is $P \times [\ell]$ itself, i.e. $u(p) = v(p) = 0$ for all $p \in P$, we use the notation $\mathcal{L}_{P \times [\ell]}(R)$.

The following definition says that R is consistent if every possible label is used in some P-strict labeling.

Definition 2.1.8. Let $R : P \to \mathcal{P}(\mathbb{Z})$. We say R is **consistent** with respect to $P \times [\ell]_u^v$ if, for every $p \in P$ and $k \in R(p)$, there exists some P-strict labeling $f \in \mathcal{L}_{P \times [\ell]}(u, v, R)$ and $u(p) < i < \ell + 1 - v(p)$ such that $f(p, i) = k$.

We denote the consistent restriction function induced by (either global or local) upper and lower bounds as R_a^b, where $a, b : P \to \mathbb{Z}$. In the case of a global upper bound q, our restriction function will be R_1^q, that is, we take a to be the constant function 1 and b to be the constant function q. Since a lower bound of 1 is used frequently, we suppress the subscript 1; that is, if no subscript appears, we take it to be 1.

Remark 2.1.9. If $\ell = 1$, $\mathcal{L}_{P\times[\ell]}(R) = \mathrm{Inc}^R(P)$ from [13]. A notion of consistent R for this case was defined. This coincides with the above definition.

We will use the following two definitions in Definition 2.1.12.

Definition 2.1.10. Let $R(p)_{>k}$ denote the smallest label of $R(p)$ that is larger than k, and let $R(p)_{<k}$ denote the largest label of $R(p)$ less than k.

Definition 2.1.11. Say that a label $f(p,i)$ in a P-strict labeling $f \in \mathcal{L}_{P\times[\ell]}(u,v,R)$ is **raisable (lowerable)** if there exists another P-strict labeling $g \in \mathcal{L}_{P\times[\ell]}(u,v,R)$ where $f(p,i) < g(p,i)$ ($f(p,i) > g(p,i)$), and $f(p',i') = g(p',i')$ for all $(p',i') \in P \times [\ell]_u^v$, $p' \neq p$.

It is important to note that the above definition is analogous to the increasing labeling case of [13], so raisability (lowerability) is thought of with respect to the layer, not the entire P-strict labeling.

Definition 2.1.12. Let the action of the k**th Bender–Knuth involution** ρ_k on a P-strict labeling $f \in \mathcal{L}_{P\times[\ell]}(u,v,R)$ be as follows: identify all raisable labels $f(p,i) = k$ and all lowerable labels $f(p,i) = R(p)_{>k}$ (if $k = \max R(p)$, then there are no raisable or lowerable labels on the fiber F_p). Call these labels 'free'. Suppose the labels $f(F_p)$ include a free k labels followed by b free $R(p)_{>k}$ labels; ρ_k changes these labels to b copies of k followed by a copies of $R(p)_{>k}$. **Promotion** on P-strict labelings is defined as the composition of these involutions: $\mathrm{Pro}(f) = \cdots \circ \rho_3 \circ \rho_2 \circ \rho_1 \circ \cdots (f)$. Note that since R induces upper and lower bounds on the labels, only a finite number of Bender-Knuth involutions act nontrivially.

We compute promotion on a P-strict labeling in Figure 2.2. We continue this example in Figure 2.5.

Example 2.1.13. Consider the action of ρ_1 in Figure 2.2. In the fiber F_a, neither of the 1 labels can be raised to $R(a)_{>1} = 3$, since they are restricted above by the 3 labels in the fiber F_b. However, the 3 label in F_a can be lowered to a 1, and so the action of ρ_1 takes the one free 3 label and replaces it with a 1. Similarly, in F_c, the 2 is lowered to a 1. In F_b, the 1 can be raised to a 3 and the 3 can be lowered to a 1. Because there is one of each, ρ_1 makes no change in F_b.

After applying ρ_2, we look closer at the action of ρ_3. In F_a, there are no 3 labels or $R(a)_{>3} = 4$ labels, so we do nothing. In F_b, however, there are three 3 labels that can be raised to

$R(b)_{>3} = 5$ and one 5 that can be lowered to 3. Thus ρ_3 replaces these four free labels with one 3 and three 5 labels.

Remark 2.1.14. In the case $\ell = 1$, $\mathcal{L}_{P\times[\ell]}(R)$ equals $\mathrm{Inc}^R(P)$, the set of increasing labelings of P with restriction function R. So the above definition specializes to *generalized Bender-Knuth involutions* and *increasing labeling promotion* IncPro, as studied in [13]. If, in addition, P is (skew-)partition shaped, these increasing labelings are equivalent to *(skew-)increasing tableaux*, and the above definition specializes to *K-Bender-Knuth involutions* and *K-Promotion*, as in [12].

If we restrict our attention to *linear extensions* of P, the above definition specializes to usual Bender-Knuth involutions and promotion, as studied in [50].

If $P = [n]$ and ℓ is arbitrary, $\mathcal{L}_{P\times[\ell]}(R^q)$ is equivalent to the set of semistandard Young tableaux of shape an $n \times \ell$ rectangle and entries at most q, and $\mathcal{L}_{P\times[\ell]}(u, v, R^q)$ is the set of (skew-)semistandard Young tableaux with shape corresponding to $P \times [\ell]_u^v$ and entries at most q. In these cases, the above definition specializes to usual Bender-Knuth involutions and promotion. We give more details on this specialization in Section 2.4.1.

Given that Definition 2.1.12 specializes to the right thing in each of these cases (including linear extensions and semistandard Young tableaux), we will no longer use the prefixes K-, increasing labeling, or generalized, and rather call all these actions 'Bender–Knuth involutions' and 'promotion', letting the object acted upon specify the context.

2.1.3. Rowmotion on Q-partitions

Rowmotion is an intriguing action that has recently generated significant interest as a prototypical action in dynamical algebraic combinatorics; see, for example, the survey articles [43, 53]. Rowmotion was originally defined on hypergraphs by P. Duchet [14] and generalized to order ideals $J(Q)$ of an arbitrary finite poset $(Q, \leq_Q)$ by A. Brouwer and A. Schrijver [7]. P. Cameron and D. Fon-der-Flaass [8] then described it in terms of toggles; thereafter, Williams and the second author [54] related it to promotion and gave it the name 'rowmotion'. Rowmotion was further generalized to piecewise-linear and birational domains by D. Einstein and J. Propp [15, 16]. In this paper, we discuss toggling and rowmotion on Q-partitions, as a rescaling of the piecewise-linear version.

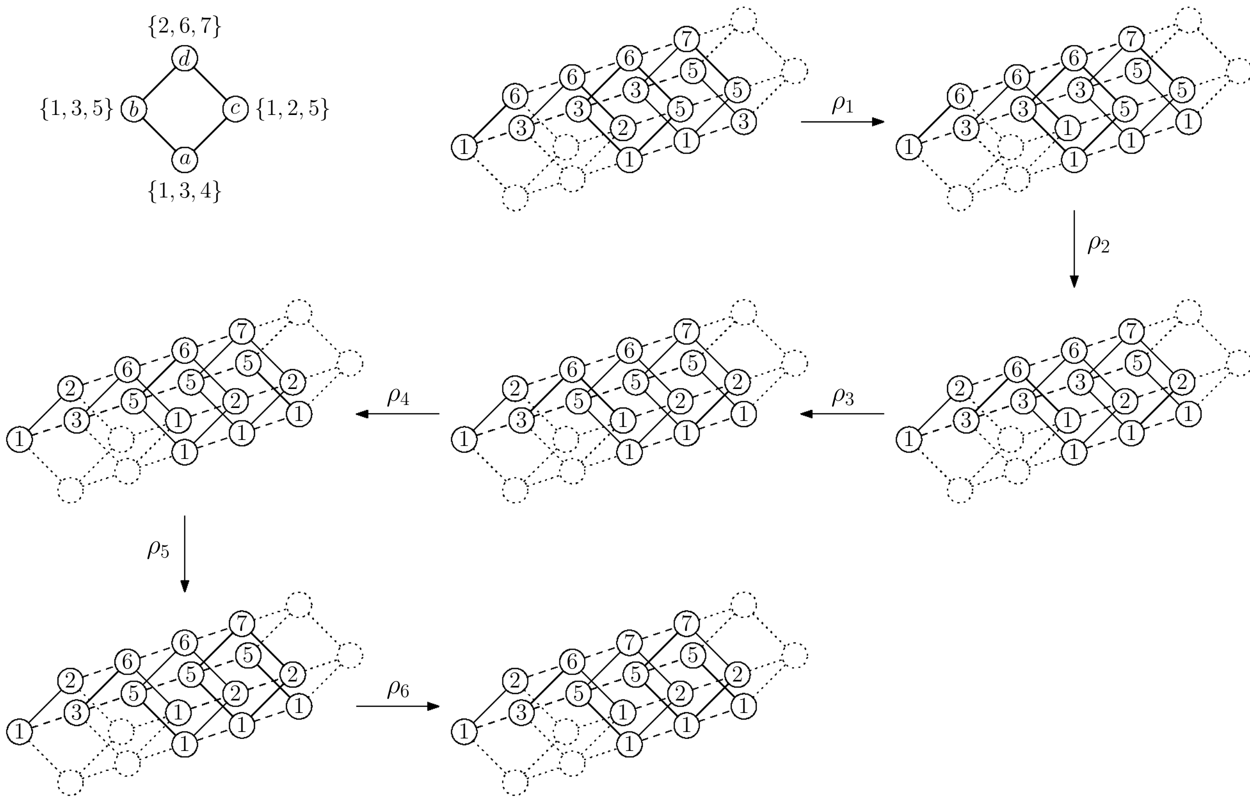


Figure 2.2. Promotion on a P-strict labeling of a convex subposet of $P \times [5]$, where the poset $P = \{a, b, c, d\}$ along with the restriction function R are given at the top. Each Bender-Knuth involution ρ_i is shown.

In light of our use of P for P-strict labelings, we use Q rather than P when referring to an arbitrary finite poset associated with the definitions of this section.

Definition 2.1.15. A **Q-partition** is a map $\sigma : Q \to \mathbb{N}_{\geq 0}$ such that if $x \leq_Q x'$, then $\sigma(x) \leq \sigma(x')$. Let $\hat{Q}$ denote Q with $\hat{0}$ added below all elements and $\hat{1}$ added above all elements. Let $\mathcal{A}^\ell(Q)$ denote the set of all $\hat{Q}$-partitions σ with $\sigma(\hat{0}) = 0$ and $\sigma(\hat{1}) = \ell$.

Remark 2.1.16. In [49], Stanley uses the reverse convention: that a Q-partition is order-*reversing* rather than order-*preserving*. We choose our convention to match with the order-preserving nature of points in the order polytope, on which the toggles of Einstein and Propp act [15, 16].

In Definition 2.1.17, we generalize Definition 2.1.15 by specifying bounds element-wise. Then in Definition 2.1.19, we define our main objects of study in this section: B-bounded Q-partitions.

Definition 2.1.17. Let $\delta, \epsilon \in \mathcal{A}^\ell(Q)$. Let $\mathcal{A}_\epsilon^\delta(Q)$ denote the set of all Q-partitions $\sigma \in \mathcal{A}^\ell(Q)$ with $\epsilon(x) \le \sigma(x) \le \delta(x)$. Call these **$(\delta,\epsilon)$-bounded Q-partitions**.

Remark 2.1.18. If $\delta(x) = \ell$ and $\epsilon(x) = 0$ for all $x \in Q$, then $\mathcal{A}_\epsilon^\delta(Q) = \mathcal{A}^\ell(Q)$.

Definition 2.1.19. Let $B \in \mathcal{A}^\ell(W)$ where W is a subset of Q that includes all maximal and minimal elements. Let $\mathcal{A}^B(Q)$ denote the set of all Q-partitions $\sigma \in \mathcal{A}^\ell(Q)$ with $\sigma(x) = B(x)$ for all $x \in W$. Call these **B-bounded Q-partitions**. We refer to the subset W as $\mathrm{dom}(B)$, the domain of B.

The next two remarks note that Definition 2.1.19 contains Definitions 2.1.15 and 2.1.17 as special cases.

Remark 2.1.20. If B is defined as $B(\hat{0}) = 0$, $B(\hat{1}) = \ell$, then $\mathcal{A}^B(\hat{Q})$ is equivalent to $\mathcal{A}^\ell(Q)$.

Remark 2.1.21. Let Q' be the poset Q with two additional elements added for each $x \in Q$: a minimal element $\hat{0}_x$ covered by x and a maximal element $\hat{1}_x$ covering x. If B is defined as $B(\hat{0}_x) = \epsilon(x)$, $B(\hat{1}_x) = \delta(x)$, then $\mathcal{A}^B(Q')$ is equivalent to $\mathcal{A}_\epsilon^\delta(Q)$.

Remark 2.1.22. Note that B-bounded Q-partitions correspond to rational points in a certain *marked order polytope*, though this perspective is not necessary for this paper.

In Definitions 2.1.23 and 2.1.25 below, we define toggles and rowmotion. In the case of $\mathcal{A}^\ell(Q)$, these definitions are equivalent (by rescaling) to those first given by Einstein and Propp on the order polytope [15, 16]. By the above remarks, it is sufficient to give the definitions of toggles and rowmotion for $\mathcal{A}^B(Q)$.

Definition 2.1.23. For $\sigma \in \mathcal{A}^B(Q)$ and $x \in Q \setminus \mathrm{dom}(B)$, let $\alpha_\sigma(x) = \min\{\sigma(y) \mid y \in Q \text{ covers } x\}$ and $\beta_\sigma(x) = \max\{\sigma(z) \mid z \in Q \text{ is covered by } x\}$. Define the **toggle** $\tau_x : \mathcal{A}^B(Q) \to \mathcal{A}^B(Q)$ by

$$\tau_x(\sigma)(x') := \begin{cases} \sigma(x') & x \neq x' \\ \alpha_\sigma(x') + \beta_\sigma(x') - \sigma(x') & x = x'. \end{cases}$$

Remark 2.1.24. By the same reasoning as in the case of order ideal toggles, the τ_x satisfy:

1. $\tau_x^2 = 1$, and

2. τ_x and $\tau_{x'}$ commute whenever x and x' do not share a covering relation.

Definition 2.1.25. **Rowmotion** on $\mathcal{A}^B(Q)$ is defined as the toggle composition Row $:= \tau_{x_1} \circ \tau_{x_2} \circ \cdots \circ \tau_{x_m}$ where $x_1, x_2, \ldots, x_m$ is any linear extension of $Q \setminus \mathrm{dom}(B)$.

Remark 2.1.26. It may be argued that we should call these actions piecewise-linear toggles and piecewise-linear rowmotion as defined in [15, 16], but as in the case of promotion on tableaux and labelings, unless clarification is needed, we choose to leave the names of these actions adjective-free, allowing the objects acted upon to indicate the context.

2.1.4. Summary of main results

Our first main theorem gives a correspondence between P-strict labelings $\mathcal{L}_{P\times[\ell]}(u, v, R)$ under promotion and specific $\hat{B}$-bounded Q-partitions $\mathcal{A}^{\widehat{B}}(Q)$ under a composition of toggles, namely, the *toggle-promotion* TogPro of Definition 2.2.6. Here Q is the poset $\Gamma(P, \hat{R})$ constructed in Section 2.2.1 and $\hat{B}$ depends on u, v, and R. The bijection map Φ is given in Definition 2.2.9. See Figure 2.5 for an illustration of this theorem and Figure 2.6 for an example of Φ.

Theorem 2.2.8. *The set of P-strict labelings $\mathcal{L}_{P\times[\ell]}(u, v, R)$ under* Pro *is in equivariant bijection with the set $\mathcal{A}^{\widehat{B}}(\Gamma(P,\hat{R}))$ under* TogPro*. More specifically, for $f \in \mathcal{L}_{P\times[\ell]}(u, v, R)$, $\Phi(\mathrm{Pro}(f)) = \mathrm{TogPro}(\Phi(f))$.*

Our second main theorem specifies cases in which toggle-promotion is conjugate in the toggle group to rowmotion, namely, when $\mathcal{A}^{\widehat{B}}(\Gamma(P, \hat{R}))$ is *column-adjacent* (see Definition 2.2.19).

Theorem 2.2.20. *If $\mathcal{A}^{\widehat{B}}(\Gamma(P, \hat{R}))$ is column-adjacent, then $\mathcal{A}^{\widehat{B}}(\Gamma(P, \hat{R}))$ under* Row *is in equivariant bijection with $\mathcal{L}_{P\times[\ell]}(u, v, R)$ under* Pro.

Column-adjacency holds in many cases of interest, including the case of restriction functions induced by global or local bounds, such as the various sets of tableaux discussed in Section 2.4.

Our third main theorem states that in the case of a global upper bound q, P-strict promotion can be equivalently defined in terms of jeu de taquin; see Definition 2.3.1 and Figure 2.7.

Theorem 2.3.10. *For $f \in \mathcal{L}_{P\times[\ell]}(u, v, R^q)$,* $\mathrm{JdtPro}(f) = \mathrm{Pro}(f)$.

In this same special case, we define and study P-strict evacuation; see Section 2.3.2.

We highlight some corollaries of our main theorems. The first is a correspondence that has been noted before (see Remark 2.4.16) between promotion on rectangular semistandard Young tableaux $\mathrm{SSYT}(\ell^n, q)$ and rowmotion on Q-partitions $\mathcal{A}^\ell(Q)$, where Q is a product of two chains poset (see Figure 2.12). Such correspondences are often of interest since they provide immediate translation of results, such as Rhoades' *cyclic sieving* theorem on $\mathrm{SSYT}(\ell^n, q)$ [42], from one domain to the other.

Corollary 2.4.12. *The set of semistandard Young tableaux* $\mathrm{SSYT}(\ell^n, q)$ *under* Pro *is in equivariant bijection with the set* $\mathcal{A}^\ell([n] \times [q-n])$ *under* Row*.*

We also recover the following result of Kirillov and Berenstein relating Bender–Knuth involutions ρ_k on semistandard Young tableaux $\mathrm{SSYT}(\lambda/\mu, q)$ with *elementary transformations* t_k on *Gelfand–Tsetlin patterns* $\mathrm{GT}(\tilde{\lambda}, \tilde{\mu}, q)$ (see Figure 2.11).

Corollary 2.4.6 ([31, Proposition 2.2])**.** *The set* $\mathrm{SSYT}(\lambda/\mu, q)$ *is in bijection with* $\mathrm{GT}(\tilde{\lambda}, \tilde{\mu}, q)$*, where* $\tilde{\lambda}_i := \lambda_1 - \mu_{n-i+1}$ *and* $\tilde{\mu}_i := \lambda_1 - \lambda_{n-i+1}$*. Moreover,* ρ_k *on* $\mathrm{SSYT}(\lambda/\mu, q)$ *corresponds to* t_{q-k} *on* $\mathrm{GT}(\tilde{\lambda}, \tilde{\mu}, q)$*.*

Theorem 2.2.20 specializes to the following two corollaries on *flagged tableaux* $\mathrm{FT}(\lambda, b)$ of shape λ and flag b. The first was our motivating example of Subsection 2.1.1 and Figure 2.1; the second involves flagged tableaux of staircase shape that have appeared in the literature [10, 47].

Corollary 2.4.28. *The set of flagged tableaux* $\mathrm{FT}(\ell^n, (2, 4, \ldots, 2n))$ *under* Pro *is in equivariant bijection with* $\mathcal{A}^\ell(\triangle_n)$ *under* Row*.*

Corollary 2.4.41. *Let* $b = (\ell+1, \ell+2, \ldots, \ell+n)$*. There is an equivariant bijection between* $\mathrm{FT}\,(sc_n, b)$ *under* Pro *and* $\mathcal{A}^\delta_\epsilon([n] \times [\ell])$ *under* Row*, where for* $(i,j) \in [n] \times [\ell]$*,* $\delta(i,j) = n$ *and* $\epsilon(i,j) = i - 1$*.*

The first corollary enables us to translate an existing cyclic sieving conjecture on $\mathcal{A}^\ell(\triangle_n)$ [28] to these flagged tableaux (see Conjecture 2.4.32). The second allows us to translate an existing cyclic sieving conjecture on flagged tableaux [10, 47] to (δ, ϵ)-bounded Q-partitions $\mathcal{A}^\delta_\epsilon(Q)$, where Q is a product of two chains poset (see Conjecture 2.4.42 and Figure 2.13). These translations provide new perspectives on the conjectures, which may be helpful for proving them.

We also present a new conjecture regarding *homomesy* on $\mathcal{A}^\ell(\triangle_n)$ and use our main theorem to translate it to flagged tableaux in Conjecture 2.4.37.

Conjecture 2.4.35. *The triple* $\left(\mathcal{A}^\ell(\triangle_n), \mathrm{TogPro}, \mathcal{R}\right)$ *is* 0*-mesic when* n *is even and* $\frac{\ell}{2}$*-mesic when* n *is odd, where* $\mathcal{R}$ *is the rank-alternating label sum statistic.*

Finally, we obtain the following correspondence between promotion on *symplectic tableaux* of staircase shape sc_n and rowmotion on (δ, ϵ)-bounded Q-partitions $\mathcal{A}^\delta_\epsilon(Q)$, where Q is the triangular poset $\diagdown_n$ (see Figure 2.14).

Corollary 2.4.50. *There is an equivariant bijection between* $\mathrm{Sp}(sc_n, 2n)$ *under* Pro *and* $\mathcal{A}^\delta_\epsilon(\diagdown_n)$ *under* Row*, where for* $(i,j) \in \diagdown_n$, $\delta(i,j) = \min(j,n)$ *and* $\epsilon(i,j) = i-1$.

This correspondence shows the cardinality of $\mathcal{A}^\delta_\epsilon(\diagdown_n)$ is 2^{n^2}, as a consequence of the symplectic hook-content formula of P. Campbell and A. Stokke [9].

2.2. P-strict promotion and rowmotion

In this section, we prove our first two main theorems. Theorem 2.2.8 relates promotion on P-strict labelings with restriction function R and toggle-promotion on B-bounded Q-partitions, where Q is the poset $\Gamma(P, \hat{R})$, whose construction we discuss in the next subsection. Theorem 2.2.20 extends this correspondence to rowmotion in the case when our poset is *column-adjacent*.

2.2.1. Preliminary definitions

Below we give some definitions needed for the objects of our main theorems. Recall R is a restriction function (see Definition 2.1.6).

Definition 2.2.1 ([13, Definition 2.10])**.** For $p \in P$, let $R(p)^*$ denote $R(p)$ with its largest element removed.

Definition 2.2.2 ([13, Definition 2.11])**.** Let P be a poset and $R : P \to \mathcal{P}(\mathbb{Z})$ a (not necessarily consistent) map of possible labels. Then define $\Gamma(P,R)$ to be the poset whose elements are (p,k) with $p \in P$ and $k \in R(p)^*$, and covering relations given by $(p_1,k_1) \lessdot (p_2,k_2)$ if and only if either

1. $p_1 = p_2$ and $R(p_1)_{>k_2} = k_1$ (i.e., k_1 is the next largest possible label after k_2), or
2. $p_1 \lessdot p_2$ (in P), $k_1 = R(p_1)_{<k_2} \neq \max(R(p_1))$, and no greater k in $R(p_2)$ has $k_1 = R(p_1)_{<k}$. That is to say, k_1 is the largest label of $R(p_1)$ less than k_2 ($k_1 \neq \max(R(p_1))$), and there is no greater $k \in R(p_2)$ having k_1 as the largest label of $R(p_1)$ less than k.

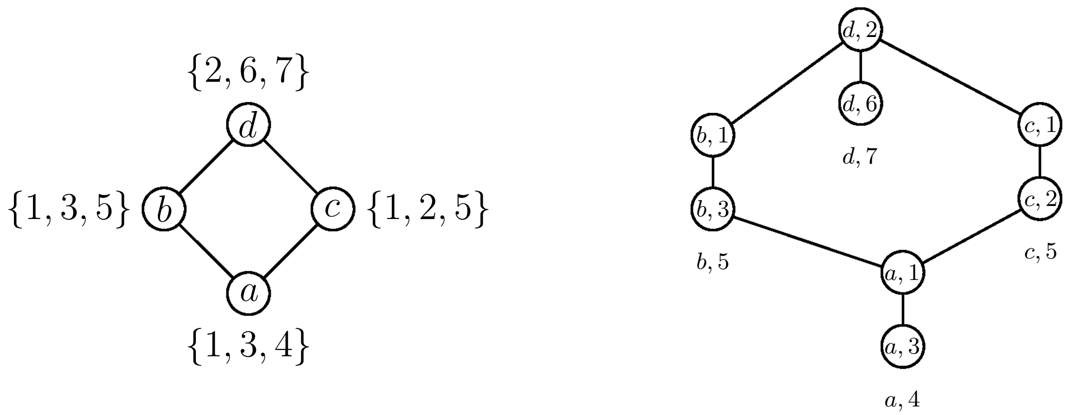


Figure 2.3. The diamond-shaped poset $P = \{a, b, c, d\}$ is shown on the left along with a consistent restriction function R, where $R(p)$ is displayed as a set next to the corresponding element. The poset $\Gamma(P, R)$ defined in Definition 2.2.2 is shown on the right.

Example 2.2.3. Refer to Figure 2.3. The poset $\Gamma(P, R)$ consists of four chains corresponding to each element $a, b, c,$ and d, where each chain contains one less element than $R(p)$. For instance, $R(a) = \{1, 3, 4\}$, so, by (1) in Definition 2.2.2, $\Gamma(P, R)$ contains the chain $(a, 3) \lessdot (a, 1)$. There is no element $(a, 4)$ since $4 = \max R(a)$ and is therefore not in $R(a)^*$. We indicate this omission by writing $a, 4$ beneath the element $(a, 3)$. The covering relations between the elements in these chains are described by (2) in Definition 2.2.2. For example, $(b, 1) \lessdot (d, 2)$ since $b \lessdot d$ and 1 is the greatest element of $R(b)$ that is strictly less than 2. Note $(d, 6)$ does not cover $(b, 3)$ since $5 \in R(b)$ is the greatest element less than 6, not 3.

In [13, Theorem 4.31] it is shown that if R consistent on P, increasing labelings on P under *increasing labeling promotion* are in equivariant bijection with order ideals of $\Gamma(P, R)$ under *toggle-promotion*. This correspondence drives our first main theorem. In order to apply this result from [13] to P-strict labelings, we need a restriction function that is consistent on P, not just on $P \times [\ell]_u^v$. The next definition constructs such a restriction function.

Definition 2.2.4. Suppose R is a consistent restriction function on $P \times [\ell]_u^v$. Denote the number of elements less than or equal to p in a maximum length chain containing p as $h(p)$ and the number of elements greater than or equal to p in a maximum length chain containing p as $\tilde{h}(p)$. Define a new restriction function $\hat{R}$ on P given by

$$\hat{R}(p) = R(p) \cup \left\{ \min \bigcup_{q \in P} R(q) - \tilde{h}(p),\ \max \bigcup_{q \in P} R(q) + h(p) \right\}.$$

Proposition 2.2.5. *If R is a consistent restriction function on $P \times [\ell]_u^v$, then $\hat{R}$ is consistent on P.*

Proof. If $p_1 <_P p_2$, then $\min \bigcup_{q \in P} R(q) - \tilde{h}(p_1)$, an element of $\hat{R}(p_1)$, is less than all elements of $\hat{R}(p_2)$ and $\max \bigcup_{q \in P} R(q) + h(p_2)$, an element of $\hat{R}(p_2)$, is greater than all elements of $\hat{R}(p_1)$. Thus, for any $p' \in P$ and any element k of $\hat{R}(p')$, the labeling f of P given by

$$f(p) = \begin{cases} k & p = p' \\ \min \bigcup_{q \in P} R(q) - \tilde{h}(p) & p' < p \\ \max \bigcup_{q \in P} R(q) + h(p) & p' > p \end{cases}$$

is an element of $\mathcal{L}_{P \times [1]}(\hat{R}) = \mathrm{Inc}^{\widehat{R}}(P)$ (see Remark 2.1.9). Since for all $p \in P$ and $k \in \hat{R}(p)$ there exists a labeling f with $f(p) = k$, $\hat{R}$ is consistent on P. □

We use the structure of $\Gamma(P, \hat{R})$ in our main result. While any consistent restriction function on P constructed by adding a new minimum and maximum element to each $R(p)$ would serve our purposes, we choose to use $\hat{R}$ for the sake of consistency.

2.2.2. First main theorem: P-strict promotion and toggle-promotion

Below, we state and prove our first main result, Theorem 2.2.8. First, we define an action on B-bounded $\Gamma(P, \hat{R})$-partitions.

Definition 2.2.6. Toggle-promotion on $\mathcal{A}^B(\Gamma(P, \hat{R}))$ is defined as the toggle composition $\mathrm{TogPro} := \cdots \circ \tau_2 \circ \tau_1 \circ \tau_0 \circ \tau_{-1} \circ \tau_{-2} \circ \cdots$, where τ_k denotes the composition of all the $\tau_{(p,k)}$ over all $p \in P$ such that $(p, k) \notin \mathrm{dom}(B)$.

This composition is well-defined since the toggles within each τ_k commute by Remark 2.1.24.

Definition 2.2.7. Given $\mathcal{L}_{P \times [\ell]}(u, v, R)$, define $\hat{B}$ (on $\Gamma(P, \hat{R})$) as $\hat{B}(p, \min \hat{R}(p)^*) = \ell - u(p)$ and $\hat{B}(p, \max \hat{R}(p)^*) = v(p)$.

To see an example of toggle-promotion on a $\hat{B}$-bounded $\Gamma(P,\hat{R})$-partition, refer to Figure 2.4. See Figure 2.5 for an example illustrating Theorem 2.2.8.

In Theorem 2.2.8 below, Φ is the bijection map given in Definition 2.2.9.

Theorem 2.2.8. *The set of P-strict labelings $\mathcal{L}_{P\times[\ell]}(u,v,R)$ under* Pro *is in equivariant bijection with the set $\mathcal{A}^{\widehat{B}}(\Gamma(P,\hat{R}))$ under* TogPro. *More specifically, for $f \in \mathcal{L}_{P\times[\ell]}(u,v,R)$, $\Phi(\mathrm{Pro}(f)) = \mathrm{TogPro}(\Phi(f))$.*

The proof will use the following definitions and lemmas. We first define the bijection map.

Definition 2.2.9. We define the map $\Phi : \mathcal{L}_{P\times[\ell]}(u,v,R) \to \mathcal{A}^{\widehat{B}}(\Gamma(P,\hat{R}))$ as the composition of three intermediate maps ϕ_1, ϕ_2, and ϕ_3. Start with a P-strict labeling $f \in \mathcal{L}_{P\times[\ell]}(u,v,R)$. Let $\phi_1(f) = \hat{f} \in \mathcal{L}_{P\times[\ell]}(\hat{R})$ where $\hat{f}$ is given by:

$$\hat{f}(p,i) = \begin{cases} \min \hat{R}(p) & i \le u(p) \\ f(p,i) & u(p) < i < \ell + 1 - v(p) \\ \max \hat{R}(p) & \ell + 1 - v(p) \le i. \end{cases}$$

Next, ϕ_2 sends $\hat{f}$ to the multichain $\mathcal{O}_\ell \le \mathcal{O}_{\ell-1} \le \cdots \le \mathcal{O}_1$ in $J(\Gamma(P,\hat{R}))$ layer by layer, that is, $\hat{f}(L_i)$ is sent to its associated order ideal $\mathcal{O}_i \in J(\Gamma(P,\hat{R}))$, where $\mathcal{O}_i$ is generated by the set $\{(p,k) \mid p \in P, \hat{f}(p,i) = k\}$. Lastly, ϕ_3 maps the above multichain to a $\Gamma(P,\hat{R})$-partition σ, where $\sigma(p,k) = |\{i \mid (p,k) \notin \mathcal{O}_i\}|$, the number of order ideals not including (p,k). Let $\Phi = \phi_3 \circ \phi_2 \circ \phi_1$.

The map ϕ_2 in Definition 2.2.9 is the main bijection used in [13, Theorem 2.14], and the map ϕ_3 is the usual bijection between multichains of $J(Q)$ and Q-partitions (see [49]).

Lemma 2.2.10. *The map Φ is well-defined and invertible.*

Proof. Since $P \times [\ell]_u^v$ is a convex subposet of $P \times [\ell]$, $\hat{f} \in \mathcal{L}_{P\times[\ell]}(\hat{R})$. Therefore, since $\hat{f}$ is weakly increasing across layers, $\mathcal{O}_\ell \le \mathcal{O}_{\ell-1} \le \cdots \le \mathcal{O}_1$ is a multichain in $J(\Gamma(P,\hat{R}))$.

For invertibility, ϕ_1 is invertible by removing the labels of $\hat{f}$ that are not in R, and ϕ_2 is invertible by [13]. Given $\sigma \in \mathcal{A}^{\widehat{B}}(\Gamma(P,\hat{R}))$ we can recover the associated multichain by $\mathcal{O}_i = \{(p,k) \mid \sigma(p,k) > i\}$, so ϕ_3 is invertible.

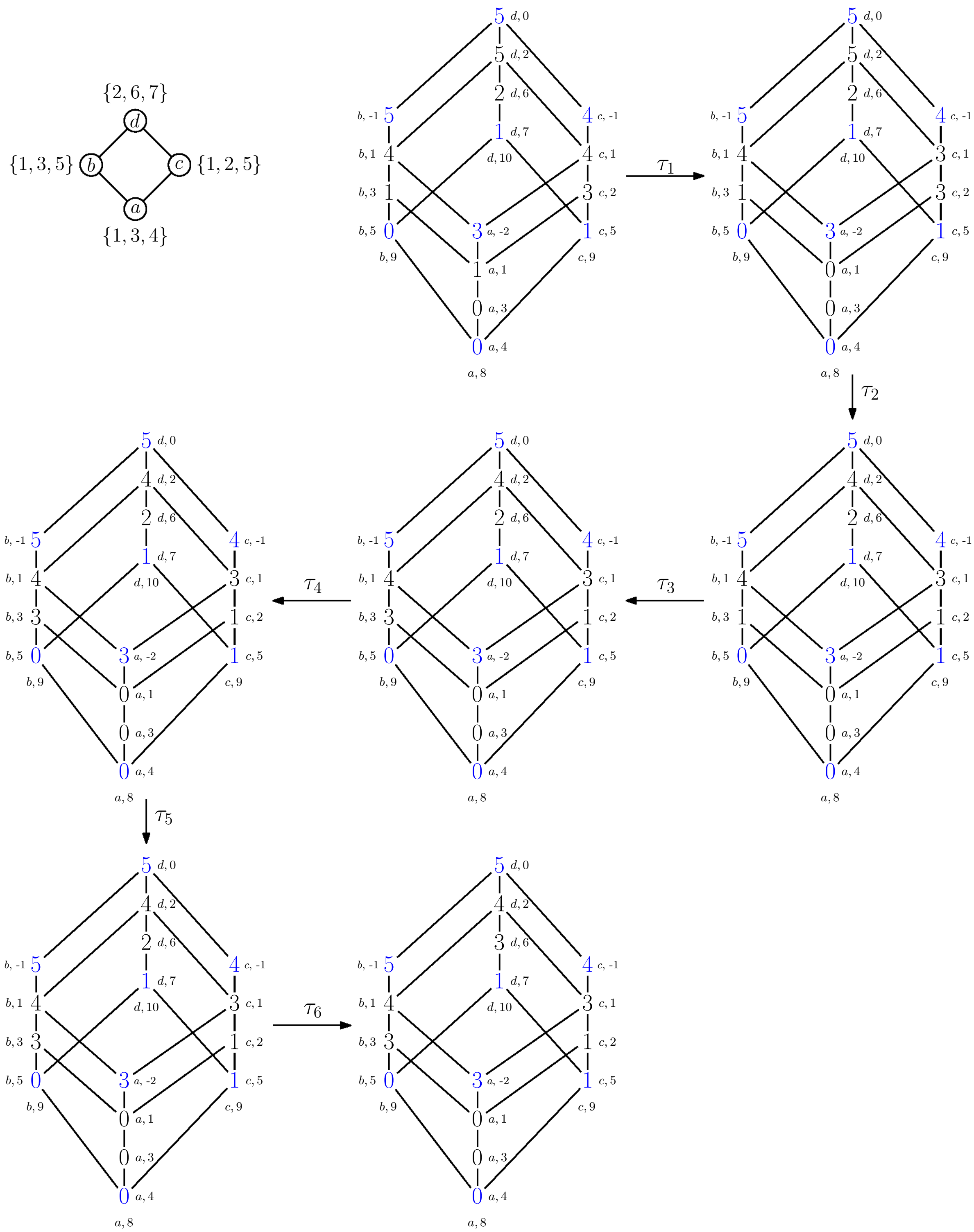


Figure 2.4. Toggle-promotion on a $\hat{B}$-bounded $\Gamma(P, \hat{R})$-partition, where the poset $P = \{a, b, c, d\}$ along with the restriction function R are given at the top. Each toggle τ_i is shown.

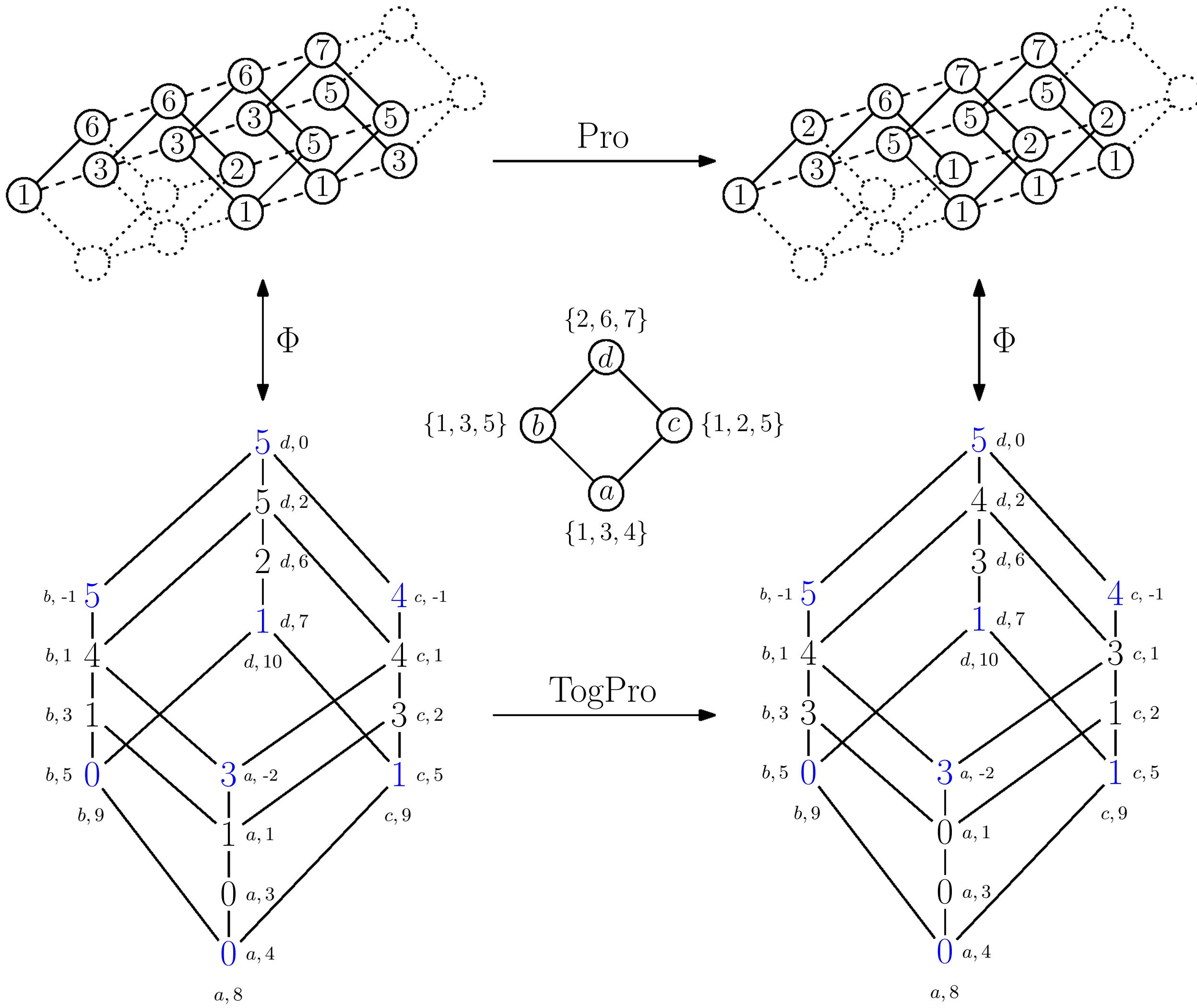


Figure 2.5. An illustration of Theorem 2.2.8. Promotion on the P-strict labeling of Figure 2.2 corresponds to toggle-promotion on a $\hat{B}$-bounded $\Gamma(P,\hat{R})$-partition. The poset $P = \{a,b,c,d\}$ along with the restriction function R are shown in the center. See Figure 2.4 for the steps in calculating TogPro in this example.

What remains to show is that $\Phi(f) \in \mathcal{A}^{\widehat{B}}(\Gamma(P,\hat{R}))$ for all $f \in \mathcal{L}_{P\times[\ell]}(u,v,R)$. We verify $\Phi(f)(p, \min \hat{R}(p)) = \ell - u(p)$ and $\Phi(f)(p, \max \hat{R}(p)^*) = v(p)$ for all $p \in P$. Suppose $\phi_1(f) = \hat{f}$ and $\phi_2(\hat{f})$ is the multichain $\mathcal{O}_\ell \le \cdots \le \mathcal{O}_1$. From the definition of ϕ_1, the number of $\min \hat{R}(p)$ labels in $f(F_p)$, or the number of order ideals in $\phi_2(\hat{f})$ containing $(p, \min \hat{R}(p))$, is $u(p)$. Therefore, $\Phi(f)(p, \min \hat{R}(p)) = \ell - u(p)$. Next, $(p, \max R(p)^*)$ is included in every order ideal associated to a layer where p is not labeled by $\max \hat{R}(p)$. Since there are $v(p)$ such layers, $v(p)$ order ideals do not contain $(p, \max R(p)^*)$, so $\Phi(f)(p, \max \hat{R}(p)^*) = v(p)$. $\square$

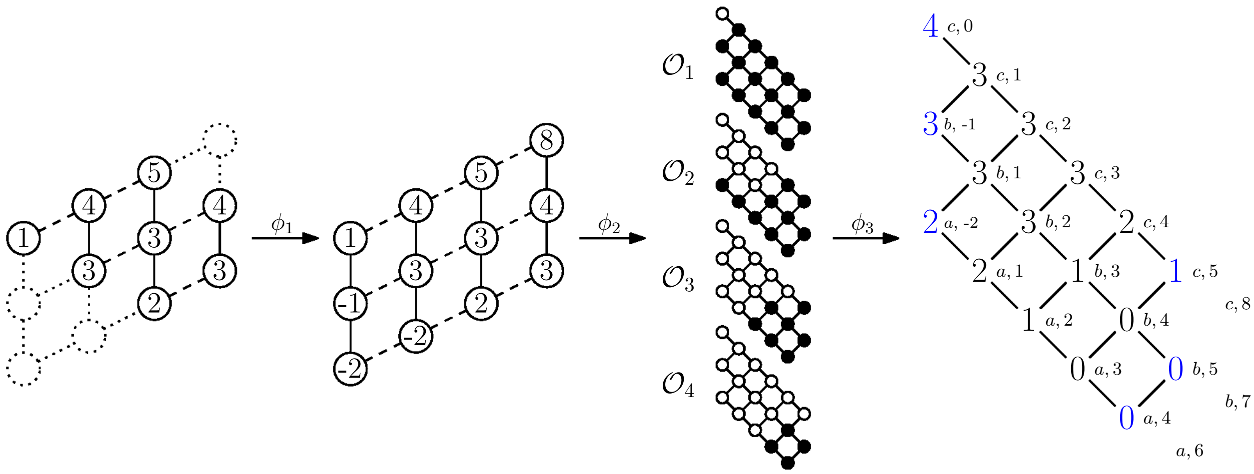


Figure 2.6. An example of the map $\Phi = \phi_3 \circ \phi_2 \circ \phi_1$ beginning with $f \in \mathcal{L}_{P\times[4]}(u, v, R^5)$ on the left and ending with $\sigma \in \mathcal{A}^{\widehat{B}}(\Gamma(P, \widehat{R^5}))$ on the right, where P is the chain $a \lessdot b \lessdot c$, $u(a, b, c) = (2, 1, 0)$, and $v(a, b, c) = (0, 0, 1)$.

Lemma 2.2.11. *The bijection map Φ equivariantly takes the generalized Bender-Knuth involution ρ_k to the toggle operator τ_k.*

The following notation will be useful for the proof of this lemma.

Definition 2.2.12. We consider the label at $(p, i) \in P \times [\ell]_u^v$ to be in **position** i, and the first (last) position satisfying a particular condition is the least (greatest) such position.

Definition 2.2.13. For $f \in \mathcal{L}_{P\times[\ell]}(u, v, R)$ and $k \in \mathbb{Z}$, let

$$j_k^p = \begin{cases} \min\{j \mid f(p, j) > k\} & \exists b \in f(F_p) \text{ such that } b > k \\ \ell - v(p) + 1 & \text{otherwise} \end{cases}.$$

That is, j_k^p is the first position in the fiber F_p with label greater than k, where we may consider a label at $(p, \ell - v(p) + 1)$ that is greater than all other labels.

Example 2.2.14. In Figure 2.6, we have $j_4^c = 3$, $j_4^a = 5$, and $j_{-1}^b = 2$.

We can now write the bijection Φ in terms of j_k^p.

Lemma 2.2.15. *Let $f \in \mathcal{L}_{P\times[\ell]}(u, v, R)$ and $\Phi(f) = \sigma \in \mathcal{A}^{\widehat{B}}(\Gamma(P, \hat{R}))$. Then $\sigma(p, k) = \ell + 1 - j_k^p$*

Proof. From the definition of Φ, in the multichain $\phi_2(\phi_1(f))$, $(p, k) \in \mathcal{O}_i$ for $1 \le i < j_k^p$ and $(p, k) \notin \mathcal{O}_i$ for $j_k^p \le i \le \ell$. Thus $\sigma(p, k) = \#\{i \mid (p, k) \notin \mathcal{O}_i\} = \ell + 1 - j_k^p$. □

Proof of Lemma 2.2.11. We prove this lemma by showing Φ equivariantly takes the action of ρ_k on $f(F_p)$ to the toggle τ_k at $(p,k) \in \Gamma(P,\hat{R})$. Let $f \in \mathcal{L}_{P\times[\ell]}(u,v,R)$ and $\Phi(f) = \sigma \in \mathcal{A}^B(\Gamma(P,\hat{R}))$.

Consider the action of ρ_k on $f(F_p)$. If $k \notin R(p)^*$, then ρ_k acts as the identity on $f(F_p)$ and τ_k acts as the identity on $\sigma(p,k)$, so we are done. Therefore, let $k \in R(p)^*$. We aim to count the number of raisable k labels and lowerable $\hat{R}(p)_{>k}$ labels in $f(F_p)$. We begin with finding the number of raisable k.

Using Lemma 2.2.15, the total number of (not necessarily raisable) k labels is given by

$$\begin{aligned} j_k^p - j_{\hat{R}(p)_{<k}}^p &= (\ell + 1 - \sigma(p,k)) - (\ell + 1 - \sigma(p, \hat{R}(p)_{<k})) \\ &= \sigma(p, \hat{R}(p)_{<k}) - \sigma(p,k). \end{aligned} \tag{1}$$

Now, we determine which of these labels are raisable. We consider three cases based on the upper covers of (p,k) in $\Gamma(P,\hat{R})$ associated to a different element of P. We denote this set in the upcoming cases by $\mathcal{U} = \{(\omega,c) \in \Gamma(P,\hat{R}) \mid \omega \gtrdot_P p \text{ and } (\omega,c) \gtrdot_{\Gamma(P,\hat{R})} (p,k)\}$.

Case $\mathcal{U} \neq \emptyset$: For $(\omega,c) \in \mathcal{U}$, by construction of $\Gamma(P,\hat{R})$, $k = \hat{R}(p)_{<c}$ and c is the largest such $c \in \hat{R}(\omega)$. Equivalently, c is the greatest element of $\hat{R}(\omega)$ that is less than or equal to $\hat{R}(p)_{>k}$. Thus, since $\hat{R}(\omega)_{>c} > \hat{R}(p)_{>k}$, the first position in $f(F_p)$ that is not restricted above by labels in $f(F_\omega)$ is j_c^ω. Therefore the first position in $f(F_\omega)$ that can be raised to $\hat{R}(p)_{>k}$ is $\max_{(\omega,c)\in\mathcal{U}} j_c^\omega$, so the number of labels in $f(F_p)$ that can be raised to $\hat{R}(p)_{>k}$ (that are necessarily less than $\hat{R}(p)_{>k}$) is

$$\begin{aligned} j_k^p - \max_{(\omega,c)\in\mathcal{U}} (j_c^\omega) &= (\ell + 1 - \sigma(p,k)) - \max_{(\omega,c)\in\mathcal{U}} (\ell + 1 - \sigma(\omega,c)) \\ &= -\sigma(p,k) - \max_{(\omega,c)\in\mathcal{U}} (-\sigma(\omega,c)) \\ &= \min_{(\omega,c)\in\mathcal{U}} (\sigma(\omega,c)) - \sigma(p,k). \end{aligned}$$

Case $\mathcal{U} = \emptyset$ and $\omega \gtrdot_P p$ for some $\omega \in P$: This implies that $k \neq \hat{R}(p)_{<c}$ for any $c \in \hat{R}(\omega)$ for any $\omega \gtrdot_P p$. Thus, if $c > k$, then we also have $c > \hat{R}(p)_{>k}$. Since f is strict on layers, if $f(p,i) = k$, all $f(\omega,i)$ are greater than $R(p)_{>k}$. Therefore all k labels in $f(F_p)$ are raisable.

Case $\mathcal{U} = \emptyset$ and p has no upper covers in P: In this case, $f(F_p)$ is not restricted above, and again all k labels in $f(F_p)$ are raisable.

The number of raisable k is the lesser of the number of k labels and the number of labels less than $\hat{R}(p)_{>k}$ that can be raised to $\hat{R}(p)_{>k}$. Let $Y = \{y \mid y \text{ covers } (p,k) \text{ in } \Gamma(P,\hat{R})\}$. Then, by the above cases, the number of raisable k in $f(F_p)$ is given by

$$\min_{y\in Y}(\sigma(x)) - \sigma(p,k).$$

If $Z = \{z \mid z \text{ is covered by } (p,k) \text{ in } \Gamma(P,\hat{R})\}$, by a similar argument we obtain that the number of lowerable $\hat{R}(p)_{>k}$ labels in $f(F_p)$ is

$$\sigma(p,k) - \max_{z\in Z}(\sigma(z)).$$

Suppose there are a raisable k and b lowerable $\hat{R}(p)_{>k}$ in $f(F_p)$. Apply ρ_k to $f(F_p)$, and let σ_k^p be the $\Gamma(P,\hat{R})$-partition corresponding to this new P-strict labeling. For all $d \neq k$, the first position in $f(F_p)$ with a label greater than d is unchanged after applying ρ_k. Thus the only label that differs between σ and σ_k^p is the label at (p,k). Since there are b raisable k in $\rho_k(f(F_p))$, with Y and Z defined as above we have

$$\sigma(p,k) - \max_{z\in Z}(\sigma(z)) = b = \min_{y\in Y}(\sigma(y)) - \sigma_k^p(p,k).$$

Therefore,

$$\sigma_k^p(p,k) = \min_{y\in Y}(\sigma(y)) + \max_{z\in Z}(\sigma(z)) - \sigma(p,k),$$

which is exactly $\tau_{(p,k)}(\sigma)(p,k)$.

Thus, ρ_k on f corresponds to toggling on σ over all elements (p,k) with $p \in P$. □

In the following, we give an example of each case from the previous proof.

Example 2.2.16. Refer to the poset $\Gamma(P,\hat{R})$ from Figure 2.5. For the element $(a,1)$ we have $\mathcal{U} = \{(b,3),(c,2)\}$, for the element $(a,3)$ we have $a \lessdot b$ and $a \lessdot c$ but $\mathcal{U} = \emptyset$, and for the element $(d,6)$ we have $\mathcal{U} = \emptyset$ and d has no upper covers in P.

Example 2.2.17. Figure 2.6 shows an example of the bijection map; the number of 3 labels in $f(F_b)$ is $2 = \sigma(b,2) - \sigma(b,3)$ and the number of 4 labels is $1 = \sigma(b,3) - \sigma(b,4)$. The number of positions where a 3 could be raised to a 4 is $1 = \sigma(c,4) - \sigma(b,3)$ and the number of positions where a 4 could be lowered to a 3 is $0 = \sigma(b,3) - \sigma(a,2)$.

We now prove our first main theorem.

Proof of Theorem 2.2.8. By Lemma 2.2.10, Φ is a bijection. By Lemma 2.2.11, $\Phi(\text{Pro}(f)) = \Phi(\cdots \circ \rho_2 \circ \rho_1 \circ \rho_0 \circ \rho_{-1} \circ \rho_{-2} \circ \cdots (f)) = \cdots \circ \tau_{-2} \circ \tau_{-1} \circ \tau_0 \circ \tau_1 \circ \tau_2 \circ \cdots (\Phi(f)) = \text{TogPro}(\Phi(f))$. □

2.2.3. Second main theorem: P-strict promotion and rowmotion

Our next main result, Theorem 2.2.20, says that for certain kinds of restriction functions, promotion on P-strict labelings of $P \times [\ell]_u^v$ with restriction function R is equivariant with rowmotion on B-bounded $\Gamma(P, \hat{R})$-partitions.

Definition 2.2.18. We call an element $p \in P$ **fixed in** $\mathcal{A}^B(P)$ if there exists some value a such that $\sigma(p) = a$ for all $\sigma \in \mathcal{A}^B(P)$.

Definition 2.2.19. We say that $\mathcal{A}^B(\Gamma(P,R))$ is **column-adjacent** if whenever $(p_1,k_1) \lessdot (p_2,k_2)$ in $\Gamma(P,R)$ and neither of (p_1,k_1) nor (p_2,k_2) are fixed in $\mathcal{A}^B(\Gamma(P,R))$, then $|k_2 - k_1| = 1$.

We call this column-adjacent because it implies that the non-fixed poset elements (p,k) of $\Gamma(P,R)$ can be partitioned into subsets indexed by k, called *columns*, whose elements have covering relations with other non-fixed elements only when they are in adjacent columns. For many nice cases, including the posets considered in Section 2.4, the word *column* is visually appropriate.

Theorem 2.2.20. *If* $\mathcal{A}^{\widehat{B}}(\Gamma(P,\hat{R}))$ *is column-adjacent, then* $\mathcal{A}^{\widehat{B}}(\Gamma(P,\hat{R}))$ *under* Row *is in equivariant bijection with* $\mathcal{L}_{P\times[\ell]}(u,v,R)$ *under* Pro.

Proof. Let $\tilde{\Gamma}(P,\hat{R})$ be the poset with elements $\Gamma(P,\hat{R}) \setminus \{(p,k) \mid (p,k) \text{ is fixed in } \mathcal{A}^{\widehat{B}}(\Gamma(P,\hat{R}))\}$ where $(p,k) \lessdot (p',k')$ in $\tilde{\Gamma}(P,\hat{R})$ if and only if $(p,k) \lessdot (p',k')$ in $\Gamma(P,\hat{R})$. To any $\sigma \in \mathcal{A}^{\widehat{B}}(\Gamma(P,\hat{R}))$ we associate a $\tilde{\Gamma}(P,\hat{R})$-partition $\tilde{\sigma}$ in $\mathcal{A}^{\ell}(\tilde{\Gamma}(P,\hat{R}))$ where $\tilde{\sigma}(p,k) = \sigma(p,k)$. We define the toggle $\tilde{\tau}_{(p,k)}$ on $\mathcal{A}^{\ell}(\tilde{\Gamma}(P,\hat{R}))$ as usual with the added restriction that, if $(p',k') \gtrdot (p,k)$ in $\Gamma(P,\hat{R})$ and (p',k') is fixed in $\mathcal{A}^{\widehat{B}}(\Gamma(P,\hat{R}))$ with $\sigma(p',k') = a$ for all σ, then the minimum value of the upper covers of (p,k)

may not exceed a, and, similarly, if $(p',k') \lessdot (p,k)$ in $\Gamma(P,\hat{R})$ and (p',k') is fixed as a in $\mathcal{A}^{\widehat{B}}(\Gamma(P,\hat{R}))$, the maximum value of the lower covers must be at least a. Thus, $\tilde{\tau}_{(p,k)}(\tilde{\sigma})(p,k) = \tau_{(p,k)}(\sigma)(p,k)$.

Since these toggles on $\tilde{\Gamma}(P,\hat{R})$-partitions share the same commutation relations as toggles on $J(\tilde{\Gamma}(P,\hat{R}))$, as noted in Remark 2.1.24, we can apply a conjugation result from [13] as follows. Because $\mathcal{A}^{\widehat{B}}(\Gamma(P,\hat{R}))$ is column-adjacent, if $(p,k) \lessdot (p',k')$ in $\tilde{\Gamma}(P,\hat{R})$, then $|k'-k| = 1$. Now, if $\tilde{\tau}_k$ is the composition over all $p \in P$ of $\tilde{\tau}_{(p,k)}$, $\text{TogPro} = \cdots \circ \tilde{\tau}_1 \circ \tilde{\tau}_0 \circ \tilde{\tau}_{-1} \circ \cdots$ is conjugate to Row on $\tilde{\Gamma}(P,\hat{R})$ by [13, Theorem 4.19]. Therefore, TogPro is also conjugate to Row on $\mathcal{A}^{\widehat{B}}(\Gamma(P,\hat{R}))$, and we obtain the result by Theorem 2.2.8. □

Remark 2.2.21. As long as a toggle order is a *column toggle order*, as defined in [13], the composition of toggles will be equivariant with rowmotion, so there are many more toggle orders besides that of TogPro that are conjugate to rowmotion. We do not need this full level of generality of toggle orders.

We show in the following proposition that for the case where our restriction function is induced by upper and lower bounds for each element (this includes the case of a global bound q), we have the column-adjacent property, so Theorem 2.2.20 yields Corollary 2.2.24.

Proposition 2.2.22. *$\mathcal{A}^{\widehat{B}}(\Gamma(P,\widehat{R^b_a}))$ is column-adjacent.*

The proof of the above uses the following lemma.

Lemma 2.2.23. *If $k \in R^b_a(p)$ and $k+1 \notin R^b_a(p)$, then (p,k) is fixed in $\mathcal{A}^{\widehat{B}}(\Gamma(P,\widehat{R^b_a}))$.*

Proof. Let $\sigma \in \mathcal{A}^{\widehat{B}}(\Gamma(P,\widehat{R^b_a}))$ and $f = \Phi^{-1}(\sigma) \in \mathcal{L}_{P\times[\ell]}(u,v,R^b_a)$. Suppose $k \in R^b_a(p)$ and $k+1 \notin R^b_a(p)$. If $k+1 > \max R^b_a(p)$, then $k = \max \widehat{R^b_a}(p)^*$, so (p,k) is fixed by the definition of $\hat{B}$. Suppose, then, that $k+1 < \max R^b_a(p)$. Then there exists $p' \gtrdot_P p$ such that $k+1 \in R^b_a(p')$. Otherwise, for all $p' \gtrdot_P p$, either $k+1 > \max R^b_a(p')$, $k+1 < a_{p'}$, or there exists $k' \geq k+2 \in R^b_a(p')$. In all cases, we could have $f(p,i) = k+1$ wherever $f(p,i) = k$, a contradiction.

If $k+1$ and $k+2 \in R^b_a(p')$, then, because $P \times [\ell]^v_u$ is a convex subposet, any position in the fiber $f(F_{p'})$ that can be labeled by $k+1$ can also be labeled by $k+2$. Thus, if $k+1$ and $k+2 \in R^b_a(p')$ for all covers $p' \gtrdot_P p$ with $k+1 \in R^b_a(p')$, then $k+1 \in R^b_a(p)$. Therefore, there must exist p_1 of the covers p' such that $k+2 \notin R(p_1)$. Moreover, if $\sigma(p,k) < \sigma(p_1,k+1)$, by Lemma 2.2.15, the first position greater than $k+1$ in $f(F_{p_1})$ occurs before the first position greater than k in $f(F_p)$. In this position, any values greater than $k+1$ and less than $R_a^b(p_1)_{\geq k+1}$, including $k+2$, would be possible, a contradiction. Thus $\sigma(p,k) = \sigma(p_1,k+1)$. Now, either $(p_1,k+1)$ is fixed by $\hat{B}$ and we are done, or, by the above reasoning, there exists $p_2 \gtrdot_P p_1$ such that $\sigma(p_1,k+1) = \sigma(p_2,k+2)$ and $k+3 \notin R_a^b(p_2)$. We continue this until there exists a maximal $p_m \in P$ such that $\sigma(p,k) = \sigma(p_1,k+1) = \cdots = \sigma(p_m,k+m)$ and $k+m+1 \notin R_a^b(p_m)$. Since p_m is maximal, $k+m+1 \notin R_a^b(p_m)$ only if $k+m = \max R_a^b(p_m)$, so $\sigma(p_m,k+m)$ is fixed by $\hat{B}$. Therefore $\sigma(p,k)$ is always equal to the value of a fixed element, and since σ was arbitrary, (p,k) is fixed in $\mathcal{A}^{\widehat{B}}(\Gamma(P,\widehat{R_a^b}))$. □

Proof of Proposition 2.2.22. We show that if $(p_1,k_1) \lessdot (p_2,k_2)$ in $\Gamma(P,\widehat{R_a^b})$ and $|k_2-k_1|>1$, then either (p_1,k_1) or (p_2,k_2) is fixed. Without loss of generality, let $k_2-k_1>1$. If $p_1=p_2$, then $k_1+1 \notin R_a^b(p_1)$, so (p_1,k_1) is fixed by Lemma 2.2.23. If $p_1 \lessdot_P p_2$, then $k_1+1 \notin R_a^b(p_1)$, otherwise k_1 would not be the greatest element in $R_a^b(p_1)$ less than k_2 by definition of $\Gamma(P,\widehat{R_a^b})$. Thus, by Lemma 2.2.23 again, (p_1,k_1) is fixed. □

Corollary 2.2.24. *The set of P-strict labelings $\mathcal{L}_{P\times[\ell]}(u,v,R_a^b)$ under* Pro *is in equivariant bijection with the set $\mathcal{A}^{\widehat{B}}(\Gamma(P,\widehat{R_a^b}))$ under* Row*.*

Proof. This follows from Theorem 2.2.20 and Proposition 2.2.22. □

Remark 2.2.25. Note that if $R_a^b(p)$ is a non-empty interval for all $p \in P$, then we obtain Corollary 2.2.24 by Corollary 4.22 in [13]. However, even though R_a^b is induced by lower and upper bounds, this is not always the case. The requirement that R_a^b be consistent on $P \times [\ell]_u^v$ can result in gaps in a particular $R_a^b(p)$ depending on u and v. As an example, consider the semistandard Young tableau with shape $(4,4,4,4,2,2,2)/(2,2,2)$ and global maximum 5 (that is, $P=[7]$ with restriction function R_1^5 and u,v determined by the shape). In this case, the fourth row of the tableau can only be labeled by elements of $\{1,2,4,5\}$.

2.2.4. Special cases of $\mathcal{A}^{\widehat{B}}(\Gamma(P,\hat{R}))$

In this subsection, we consider cases in which $\mathcal{A}^{\widehat{B}}(\Gamma(P,\hat{R}))$ from our main theorem can be more nicely described by restricting certain parameters. We begin with two propositions that show when $\mathcal{A}^{\widehat{B}}(\Gamma(P,\hat{R}))$ is equivalent to $\mathcal{A}^{\ell}(\Gamma(P,R))$ or $\mathcal{A}_{\epsilon}^{\delta}(\Gamma(P,R))$ from Definitions 2.1.15 and 2.1.17, and conclude with a corollary of our main theorem in the case where $\mathcal{A}^{\widehat{B}}(\Gamma(P,\hat{R}))$ is simply the product of the poset P with a chain. We use these results several times in Section 2.4.

Proposition 2.2.26. *If R is consistent on P, then $\mathcal{A}^{\widehat{B}}(\Gamma(P,\hat{R}))$ is equivalent to $\mathcal{A}_{\epsilon}^{\delta}(\Gamma(P,R))$, where $\delta(p,k)=\ell-u(p)$ and $\epsilon(p,k)=v(p)$.*

Proof. We first consider the covering relations of the elements $(p,\hat{k})$ in $\Gamma(P,\hat{R})$ given by condition (2) in Definition 2.2.2, where $\hat{k}\in\hat{R}(p)^*\setminus R(p)^*=\{\min\hat{R}(p)^*,\max\hat{R}(p)^*\}$. If $p_1\lessdot_P p_2$, then $\min\hat{R}(p_1)^*=\hat{R}(p_1)_{<\min\hat{R}(p_2)^*}$ and, since R is consistent on P, $\hat{R}(p_1)_{<\min R(p_2)}>\min\hat{R}(p_1)^*$, so $(p_1,\min\hat{R}(p_1)^*)\lessdot(p_2,\min\hat{R}(p_2)^*)$ in $\Gamma(P,\hat{R})$. Similarly, $\max\hat{R}(p_1)^*$ is necessarily $\hat{R}(p_1)_{<\max\hat{R}(p_2)^*}$, and since there is no larger $k\in\hat{R}(p_2)^*$ than $\max\hat{R}(p_2)^*$ we have $(p_1,\max\hat{R}(p_1)^*)\lessdot(p_2,\max\hat{R}(p_2)^*)$. Thus, if $(p_1,k_1)\lessdot(p_2,k_2)$ in $\Gamma(P,\hat{R})$ with $p_1\neq p_2$, then either $(p_1,k_1),(p_2,k_2)\in\Gamma(P,R)$ or $(p_1,k_1),(p_2,k_2)\in\Gamma(P,\hat{R})\setminus\Gamma(P,R)$.

Therefore the only covering relations in $\Gamma(P,\hat{R})$ between elements of $\Gamma(P,\hat{R})\setminus\Gamma(P,R)$ and $\Gamma(P,R)$ are given by (1) in Definition 2.2.2. Specifically, these are $(p,\min R(p))\lessdot(p,\min\hat{R}(p)^*)$ and $(p,\max\hat{R}(p)^*)\lessdot(p,\max R(p))$ for all $p\in P$.

The above shows that $(p_1,k_1)\lessdot(p_2,k_2)$ in $\Gamma(P,R)$ if and only if $(p_1,k_1)\lessdot(p_2,k_2)$ in $\Gamma(P,\hat{R})$. Let $\sigma\in\mathcal{A}^{\widehat{B}}(\Gamma(P,\hat{R}))$. Then, since $\sigma(p,\max\hat{R}(p)^*)=v(p)$, $\sigma(p,\min\hat{R}(p)^*)=\ell-u(p)$, and $(p,\max\hat{R}(p)^*)<_{\Gamma(P,\hat{R})}(p,k)<_{\Gamma(P,\hat{R})}(p,\min\hat{R}(p)^*)$ for all $k\in R(p)^*$, we have $v(p)\leq\sigma(p,k)\leq\ell-u(p)$ for all $(p,k)\in\Gamma(P,R)$. Thus the restriction of σ to $\Gamma(P,R)$ is an element of $\mathcal{A}_{\epsilon}^{\delta}(\Gamma(P,R))$ where $\delta(p,k)=\ell-u(p)$ and $\epsilon(p,k)=v(p)$, and, since this restriction only omits the fixed values of σ, restriction to $\Gamma(P,R)$ is a bijection and we have the desired equivalence. □

Proposition 2.2.27. *If $u(p)=v(p)=0$ for all $p\in P$, then $\mathcal{A}^{\widehat{B}}(\Gamma(P,\hat{R}))$ is equivalent to $\mathcal{A}^{\ell}(\Gamma(P,R))$.*

Proof. Since $u(p)=v(p)=0$ for all $p\in P$, $P\times[\ell]_u^v=P\times[\ell]$ by Definition 2.1.3. Since R is consistent on $P\times[\ell]$ it must also be consistent on P, and we can apply Proposition 2.2.26 where $\delta(p)=\ell$ and $\epsilon(p)=0$ for all $p\in P$, which, by Remark 2.1.18, gives the result. □

See Figures 2.1 and 2.12 for examples of this equivalence.

For the following lemma and corollary of our main theorem, we consider a poset P to be *graded* of rank n if all maximal chains of P have $n+1$ elements.

Lemma 2.2.28. *Let P be a graded poset of rank n. Then $\Gamma(P, R^q)$ is isomorphic to $P \times [q-n-1]$ as a poset.*

Proof. Write $P \times [q-n-1]$ as $\{(i,j) \mid 1 \le i \le n, 1 \le j \le q-n-1\}$, where $(p,j) \lessdot (p',j')$ if and only if $p = p'$ and $j = j'-1$ or $j = j'$ and $p \lessdot p'$ (that is, the usual ordering $(p,j) \le (p',j')$ if and only if $p \le_P p'$ and $j \le j'$).

Recall the definition of $h(p)$ from Definition 2.2.4. Since $u = v = 0$, R^p is consistent on P, and, since P is graded, for all $p \in P$ we have $R^q(p) = \{h(p), h(p)+1, \ldots, q-n+h(p)-1\}$. By definition of Γ (as noted in [13, Thm. 2.21]), $(p,k) \lessdot (p',k')$ in $\Gamma(P,R^q)$ if and only if either $p = p'$ and $k-1 = k'$ or $p \lessdot p'$ and $k+1 = k'$. Consider the map that takes $(p,k) \in \Gamma(P,R^q)$ to $(p, q-n+h(p)-k-1) \in [P] \times [q-n-1]$. This map is a bijection to the elements of $[n] \times [q-n-1]$, since $h(p) \le k \le q-n+h(p)-2$ implies $1 \le q-n+h(p)-k-1 \le q-n-1$. Moreover, the covers of (p,k) in $\Gamma(P,R^q)$ correspond exactly to the covers of $(p, q-n+i-k-1)$ in $P \times [q-n-1]$, as $(p, q-n+h(p)-k-1) \lessdot x \in P \times [q-n-1]$ if and only if $x = (p, q-n+h(p)-(k-1))$ or $x = (p', q-n+(h(p)+1)-(k+1))$, where $p \lessdot p'$ (and thus $h(p)+1 = h(p')$). Therefore $\Gamma(P,R^q)$ is isomorphic as a poset to $P \times [q-n-1]$. □

Corollary 2.2.29. *Let P be a graded poset of rank n. Then $\mathcal{L}_{P\times[\ell]}(R^q)$ under* Pro *is in equivariant bijection with $\mathcal{A}^\ell(P \times [q-n-1])$ under* Row*.*

Proof. By Corollary 2.2.24 and Proposition 2.2.27, $\mathcal{L}_{P\times[\ell]}(R^q)$ under Pro is in equivariant bijection with $\mathcal{A}^\ell(\Gamma(P,R^q))$ under Row which, by Lemma 2.2.28, is exactly $\mathcal{A}^\ell(P \times [q-n-1])$. □

2.3. P-strict promotion and evacuation

In this section, we define promotion on P-strict labelings $\mathcal{L}_{P\times[\ell]}(u,v,R^q)$ via jeu de taquin and prove Theorem 2.3.10, which shows this is equivalent to our promotion via Bender–Knuth involutions from Definition 2.1.12. We also define evacuation on P-strict labelings and show some properties of evacuation in this setting.

2.3.1. Third main theorem: P-strict promotion via jeu de taquin

We begin with the definition of jeu de taquin promotion on P-strict labelings $\mathcal{L}_{P\times[\ell]}(u,v,R^q)$.

Definition 2.3.1. Let $\mathbb{Z}_\square(P \times [\ell]_u^v)$ denote the set of labelings $g : P \times [\ell]_u^v \to (\mathbb{Z} \cup \square)$. Define the ith *jeu de taquin slide* $\mathrm{jdt}_i : \mathbb{Z}_\square(P \times [\ell]_u^v) \to \mathbb{Z}_\square(P \times [\ell]_u^v)$ as follows:

$$\mathrm{jdt}_i(g)(p,k) = \begin{cases} i & g(p,k) = \square \text{ and } g(p',k) = i \text{ for some } p' \gtrdot_P p & \text{(2.1a)} \\ i & g(p,k) = \square,\ g(p,k+1) = i, \text{ and } g(p',k+1) \neq \square \\ & \text{for any } p' \lessdot_P p & \text{(2.1b)} \\ \square & g(p,k) = i \text{ and } g(p',k) = \square \text{ for some } p' \lessdot_P p & \text{(2.1c)} \\ \square & g(p,k) = i,\ g(p,k-1) = \square, \text{and } g(p',k-1) \neq i \\ & \text{for any } p' \gtrdot_P p & \text{(2.1d)} \\ g(p,k) & \text{otherwise.} & \text{(2.1e)} \end{cases}$$

In words, $\mathrm{jdt}_i(g)$ replaces a label $\square$ at (p,k) with i if i is the label of a cover of (p,k) in its layer, or if i is the label of a cover of (p,k) in its fiber and this cover does not also cover an element within its own layer labeled by $\square$. Furthermore, $\mathrm{jdt}_i(g)$ replaces a label i by $\square$ if (p,k) covers an element in its layer labeled by $\square$, or replaces a label i by $\square$ if (p,k) covers an element in its fiber labeled by $\square$, provided said element is not covered by an element in its layer labeled with i. Aside from these cases, $\mathrm{jdt}_i(g)$ leaves all other labels unchanged.

Let $\mathrm{jdt}_{i\to j} : \mathbb{Z}_\square(P) \to \mathbb{Z}_\square(P)$ be defined as

$$\mathrm{jdt}_{i\to j}(g)(x) = \begin{cases} j & g(x) = i \\ g(x) & \text{otherwise.} \end{cases}$$

In words, $\mathrm{jdt}_{i\to j}(g)(x)$ replaces all labels i by j.

For $f \in \mathcal{L}_{P\times[\ell]}(u,v,R^q)$, let $jdt(f) = \mathrm{jdt}_{\square\to(q+1)} \circ (\mathrm{jdt}_q)^\ell \circ (\mathrm{jdt}_{q-1})^\ell \circ \cdots \circ (\mathrm{jdt}_3)^\ell \circ (\mathrm{jdt}_2)^\ell \circ \mathrm{jdt}_{1\to\square}(f)$. That is, first replace all 1 labels with $\square$. Then perform the ith jeu de taquin slide jdt_i ℓ times for each $2 \leq i \leq q$. Next, replace all labels $\square$ with $q+1$. Define *jeu de taquin promotion* on f as $\mathrm{JdtPro}(f)(x) = jdt(f)(x) - 1$.

Example 2.3.2. Figure 2.7 shows an example of JdtPro being applied to a P-strict labeling. In this example, P is the Y-shaped poset on four elements and $\ell = 5$. We perform JdtPro on the P-strict labeling $f \in \mathcal{L}_{P\times[5]}(u,v,R^3)$ where $u(a,b,c,d) = (4,1,0,1)$ and $v(a,b,c,d) = (0,0,0,1)$. Observe that as part of JdtPro, we perform $\mathrm{jdt}_{\square\to4} \circ (\mathrm{jdt}_3)^5 \circ (\mathrm{jdt}_2)^5 \circ \mathrm{jdt}_{1\to\square}(f)$. However, in this example, we do not show the applications of jdt_2 and jdt_3 that have no effect on the labeling. The final step of JdtPro is to subtract every label by 1, yielding the new P-strict labeling $\mathrm{JdtPro}(f)$.

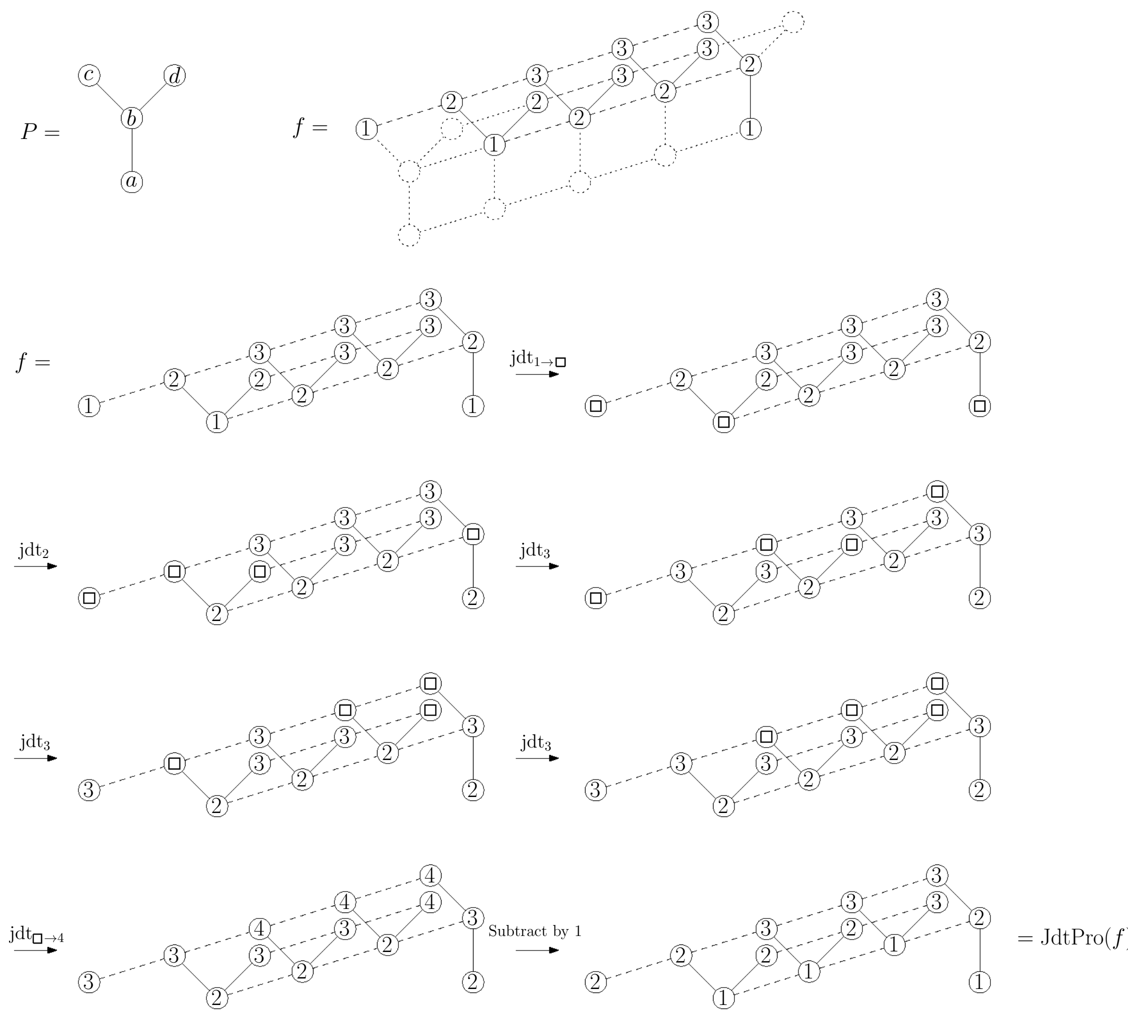


Figure 2.7. We perform JdtPro on the P-strict labeling $f \in \mathcal{L}_{P\times[5]}(u, v, 3)$ where $u(a, b, c, d) = (4, 1, 0, 1)$ and $v(a, b, c, d) = (0, 0, 0, 1)$. For the sake of brevity, we do not show the applications of jdt_2 and jdt_3 that do nothing.

In Proposition 2.3.5, we show that if we begin with a P-strict labeling f, $\mathrm{JdtPro}(f)$ is always a P-strict labeling. In order to prove this, we need Lemmas 2.3.3 and 2.3.4, which give us conditions that a labeling cannot violate when performing jeu de taquin slides.

Lemma 2.3.3. *Let $f \in \mathcal{L}_{P\times[\ell]}(u, v, R^q)$. When performing a jeu de taquin slide of* $\mathrm{JdtPro}(f)$*, no integer labels can violate the P-strict labeling order relations.*

Proof. Because we apply all jeu de taquin slides jdt_2, then all jeu de taquin slides jdt_3, and so on for each jdt_i where $2 \le i \le q$, each time $\square$ is replaced by a number, that number is the smallest label of its covers. As a result, no integer labels can violate the order relations after performing a jeu de taquin slide. $\square$

Lemma 2.3.4. *Let* $f \in \mathcal{L}_{P\times[\ell]}(u, v, R^q)$. *If* $g \in \mathbb{Z}_\square(P \times [\ell]_u^v)$ *is obtained by performing jeu de taquin slides on* f*, we can never have* $\mathrm{jdt}_i(g)(p, k) = \mathrm{jdt}_i(g)(p', k) = \square$ *when* $p' >_P p$.

Proof. We show the claim by contradiction. Suppose $\mathrm{jdt}_i(g)(p, k) = \mathrm{jdt}_i(g)(p', k) = \square$ for some $p' >_P p$. Furthermore, assume this is the first application of a jeu de taquin slide for which this occurs. In other words, we do not have two comparable elements within the same layer that both have a label of $\square$ prior to this application of jdt_i. Suppose this occurs from (2.1c) of Definition 2.3.1. This implies $g(p'', k) = \square$ for some $p'' \lessdot_P p$, which cannot occur by our assumption that $\mathrm{jdt}_i(g)(p, k) = \mathrm{jdt}_i(g)(p', k) = \square$ is the first application of a jeu de taquin slide for which we have comparable elements within the same layer that are both labeled with $\square$.

Now assume $\mathrm{jdt}_i(g)(p, k) = \mathrm{jdt}_i(g)(p', k) = \square$ occurs after applying (2.1d) of Definition 2.3.1. For this to occur, we would need either $g(p, k) = i$ and $g(p', k) = \square$, or $g(p, k) = \square$ and $g(p', k) = i$. However, by assumption, any element between (p, k) and (p', k) cannot be labeled with $\square$. Furthermore, by Lemma 2.3.3, we cannot have any integer labels violate the order relations, so any element between (p, k) and (p', k) cannot be labeled with i. As a result, we may assume $p' \gtrdot_P p$. We can eliminate $g(p, k) = \square$ and $g(p', k) = i$ as a possibility, as (2.1a) of Definition 2.3.1 would be applied to $g(p, k)$, resulting in $\mathrm{jdt}_i(g)(p, k) = i$. Therefore, we may assume $g(p, k) = i$ and $g(p', k) = \square$. We may also assume $g(p, k - 1) = \square$ in order for (2.1d) of Definition 2.3.1 to be invoked. However, by our assumption, this means $g(p', k - 1)$ cannot have label $\square$, implying that $g(p', k - 1) = i$. By definition, (2.1d) of Definition 2.3.1 cannot be applied. We obtained a contradiction with each of (2.1c) and (2.1d) of Definition 2.3.1, implying that we cannot have $\mathrm{jdt}_i(g)(p, k) = \mathrm{jdt}_i(g)(p', k) = \square$ for some $p' >_P p$. $\square$

Proposition 2.3.5. *For* $f \in \mathcal{L}_{P\times[\ell]}(u, v, R^q)$, $\mathrm{JdtPro}(f) \in \mathcal{L}_{P\times[\ell]}(u, v, R^q)$.

Proof. By construction, $\mathrm{JdtPro}(f)$ is a labeling of $P \times [\ell]_u^v$ with integers in $\{1, \ldots, q\}$. By the definition of $\mathrm{JdtPro}(f)$, we perform each jeu de taquin slide ℓ times. Note that we only need to perform each jdt_i until the $\square$ labels are above the i labels in every fiber where both appear. This is guaranteed to happen if we perform it ℓ times, as every fiber is of length at most ℓ. We only need to verify that JdtPro(f) has the order relations of a P-strict labeling. By Lemma 2.3.3, no integer labels of JdtPro(f) can violate the order relations after performing a jeu de taquin slide. Additionally, by Lemma 2.3.4, if $g \in \mathbb{Z}_{\square}(P \times [\ell]_u^v)$ is obtained by performing jeu de taquin slides on f, we can never have $\mathrm{jdt}_i(g)(p,k) = \mathrm{jdt}_i(g)(p',k) = \square$ when $p' >_P p$. Because of this, we guarantee that no $q+1$ labels violate the order relations after performing $\mathrm{jdt}_{\square \to (q+1)}$ as part of JdtPro. As a result, this means the strict order relations of the P-strict labeling will be satisfied when we perform $\mathrm{jdt}_{\square \to (q+1)}$. ☐

Our goal is Theorem 2.3.10, which states that jeu de taquin promotion from Definition 2.3.1 coincides with our definition of promotion by Bender-Knuth involutions. The crux of the proof is Lemmas 2.3.6 and 2.3.8. The idea of Lemma 2.3.6 is as follows. By definition, when performing JdtPro(f), we perform each jeu de taquin slide ℓ times. We observe that for $f \in \mathcal{L}_{P\times[\ell]}(u,v,R^q)$, when we apply jdt_i, cases (2.1a) and (2.1c) of Definition 2.3.1 can only be invoked on the first application of jdt_i.

Lemma 2.3.6. *For $f \in \mathcal{L}_{P\times[\ell]}(u,v,R^q)$, when applying jdt_i in JdtPro(f) for any $2 \leq i \leq q$, (2.1a) and (2.1c) of Definition 2.3.1 can only be invoked on the first application of jdt_i.*

Proof. We begin by proving the result for jdt_q. Suppose $g(p,k) = \square$. If there is a cover (p',k) of (p,k) in the kth layer of $P \times [\ell]_u^v$, then we must have $g(p',k) = q$, as $g(p',k)$ could not be less than q nor could it be $\square$ by Lemma 2.3.4. Furthermore, if there does not exist a cover (p',k) of (p,k) in the kth layer, neither (2.1a) nor (2.1c) is invoked on $\square$ from $g(p,k)$ when applying jdt_q. Therefore, we may assume a cover of (p,k) in the kth layer has a label of q. In other words, we assume there exists a $p' \gtrdot p$ such that $g(p',k) = q$. When applying jdt_q, the first application of jdt_q will invoke (2.1a) and (2.1c), resulting in $g(p',k)$ being labeled with $\square$ for any labels $g(p',k)$ such that $p' \gtrdot p$ and $g(p',k) = q$. However, on subsequent applications of jdt_q, (2.1a) cannot be invoked to result in a $\square$ for any $g(p'',k)$ where $p'' \gtrdot p'$. This is because $g(p'',k)$, a label for a cover of (p',k) in the kth layer, would need to be labeled with either q or $\square$, neither of which are possible due to Lemma 2.3.4. This means there does not exist a cover (p'',k) of (p',k) in the kth layer at all, as $g(p'',k)$ also cannot be less than q.

We might be concerned that subsequent invocations of (2.1b) or (2.1d) within the fiber $F_{p'}$ results in a $\square$ appearing in a layer with which (2.1c) can be invoked for a second time. However, because there is no $(p'', k) \in P \times [\ell]_u^v$, there cannot be an element $(p'', k') \in P \times [\ell]_u^v$ in any layer k' where $k' > k$ by definition of v. Hence, subsequent invocations of (2.1b) or (2.1d) cannot position a $\square$ into a separate layer such that (2.1c) can be invoked for a second time. As a result, for this case, the label $\square$ of $g(p, k)$ can affect the label of a separate fiber only on the first application of jdt_q via (2.1a) and (2.1c). An analogous argument shows that if we begin with $g(p, k) = q$, the label of q can only affect the label of a separate fiber on the first application of jdt_q.

We have shown that when applying jdt_q in $\mathrm{JdtPro}(f)$, (2.1a) and (2.1c) of Definition 2.3.1 can only be invoked on the first application of jdt_q. To show the result for any jdt_i, let $f_{\leq i}$ with restriction R^i denote the P-strict labeling f restricted to the subposet of elements with labels less than or equal to i. Because $f_{\leq i}$ has restriction function R^i, (2.1a) and (2.1c) of Definition 2.3.1 can only be invoked on the first application of jdt_i in $\mathrm{JdtPro}(f_{\leq i})$, which means these cases can only be invoked on the first application of jdt_i in $\mathrm{JdtPro}(f)$. $\square$

In order to state Lemma 2.3.8, we need the following definition.

Definition 2.3.7. For $f \in \mathcal{L}_{P\times[\ell]}(u, v, R^q)$, define $\mathrm{JdtPro}_i(f)$ to be the result of freezing all labels of f which are at least $i+1$, then performing jeu de taquin slides on the elements with labels less than or equal to i. In other words, perform $\mathrm{jdt}_{\square\to(i+1)}(\mathrm{jdt}_i)^\ell \circ (\mathrm{jdt}_{i-1})^\ell \circ \cdots \circ (\mathrm{jdt}_3)^\ell \circ (\mathrm{jdt}_2)^\ell \circ \mathrm{jdt}_{1\to\square}(f)$, then reduce all unfrozen labels by 1. We clarify that boxes labeled $i+1$ from the step $\mathrm{jdt}_{\square\to(i+1)}$ are considered unfrozen.

To prove Theorem 2.3.10, it will be sufficient to show that applying JdtPro_{q-1} and the Bender–Knuth involution ρ_q yields the same result as JdtPro itself.

Lemma 2.3.8. *For $f \in \mathcal{L}_{P\times[\ell]}(u, v, R^q)$, $\mathrm{JdtPro}(f) = \rho_{q-1} \circ \mathrm{JdtPro}_{q-1}(f)$.*

Proof. Both $\mathrm{JdtPro}(f)$ and $\rho_{q-1} \circ \mathrm{JdtPro}_{q-1}(f)$ begin by applying $(\mathrm{jdt}_{q-1})^\ell \circ (\mathrm{jdt}_{q-2})^\ell \circ \cdots \circ (\mathrm{jdt}_3)^\ell \circ (\mathrm{jdt}_2)^\ell \circ \mathrm{jdt}_{1\to\square}$ to f. Let $f' \in \mathbb{Z}_\square(P \times [\ell]_u^v)$ denote the labeling obtained after performing these jeu de taquin slides. What remains to be shown is that performing $\mathrm{jdt}_{\square\to(q+1)} \circ (\mathrm{jdt}_q)^\ell(f')$ and subtracting 1 from all labels results in the same P-strict labeling as performing $\mathrm{jdt}_{\square\to(q)}(f')$, subtracting 1 from all unfrozen labels, then performing the Bender-Knuth involution ρ_{q-1}.

First, consider the case that there are no boxes $\square$ in f'. This implies that there were no elements labeled 1 in f, so $\mathrm{JdtPro}(f)$ reduces all labels by 1. On the other hand, $\mathrm{JdtPro}_{q-1}(f)$ will reduce all labels by 1 except labels that are q, as these labels are frozen. However, after reducing unfrozen labels, there are no elements with a label of $q-1$, which means ρ_{q-1} changes all labels of q to $q-1$. The cumulative effect is that all labels in f are reduced by 1. Therefore, in this case, we have $\mathrm{JdtPro}(f) = \rho_{q-1} \circ \mathrm{JdtPro}_{q-1}(f)$.

We now consider the case where f' has at least one element labeled $\square$. When applying jdt_q, a label can only change if it is $\square$ or q. By Lemma 2.3.6, when applying jdt_q, (2.1a) and (2.1c) of Definition 2.3.1 can only be invoked on the first application of jdt_q. We now show that when applying jdt_q, the first application of jdt_q places the correct number of elements labeled q and $\square$ in each fiber. Suppose F_p has a elements labeled with $\square$ and b elements labeled with q. Additionally, suppose x of the elements that are labeled with $\square$ have a cover in a separate fiber labeled with q and suppose y of the elements that are labeled with q cover an element in a separate fiber labeled with a $\square$. When performing jdt_q, the x labels of $\square$ in F_p change to q and the y labels of q in F_p change to $\square$. Observe that the application of jdt_q may cause some labels of q and $\square$ to change positions within F_p. However, in Definition 2.3.1, jdt_q prioritizes (2.1a) and (2.1c), so this might not occur. Because we know a label remains in its fiber after the first application of jdt_q, the remaining applications of jdt_q results in all labels $\square$ above all labels of q in F_p. Additionally, we can determine that there are $a-x+y$ elements labeled $\square$ and $b+x-y$ labeled q in F_p. After performing $(\mathrm{jdt}_q)^\ell$ for all fibers, we apply $\mathrm{jdt}_{\square\to(q+1)}$ to replace all labels of $\square$ with $q+1$, then reduce every label by 1. The result in F_p is that we now have $b+x-y$ elements labeled $q-1$ and $a-x+y$ elements labeled q.

To determine what happens when we apply $\rho_{q-1} \circ \mathrm{JdtPro}_{q-1}(f)$, we begin by performing $\mathrm{jdt}_{\square\to(q)}(f')$ and subtracting 1 from all unfrozen labels. F_p will have a elements labeled with $q-1$ and b elements labeled with q. Furthermore, we know that x of the elements that are labeled with $q-1$ will have a cover in a separate fiber labeled with a q and that y of the elements that are labeled with q will cover an element in a separate fiber that is labeled with a $q-1$. This means F_p has $a-x$ labels of $q-1$ that are free and $b-y$ labels of q that are free. Performing ρ_{q-1} switches these into $a-x$ elements labeled with q and $b-y$ elements labeled $q-1$. Combining this with the x fixed labels of $q-1$, we obtain $b+x-y$ elements labeled $q-1$. Similarly, with the y fixed labels of q, we obtain $a-x+y$ elements labeled q. This matches the $\mathrm{JdtPro}(f)$ case, allowing us to conclude that $\mathrm{JdtPro}(f) = \rho_{q-1} \circ \mathrm{JdtPro}_{q-1}(f)$. $\square$

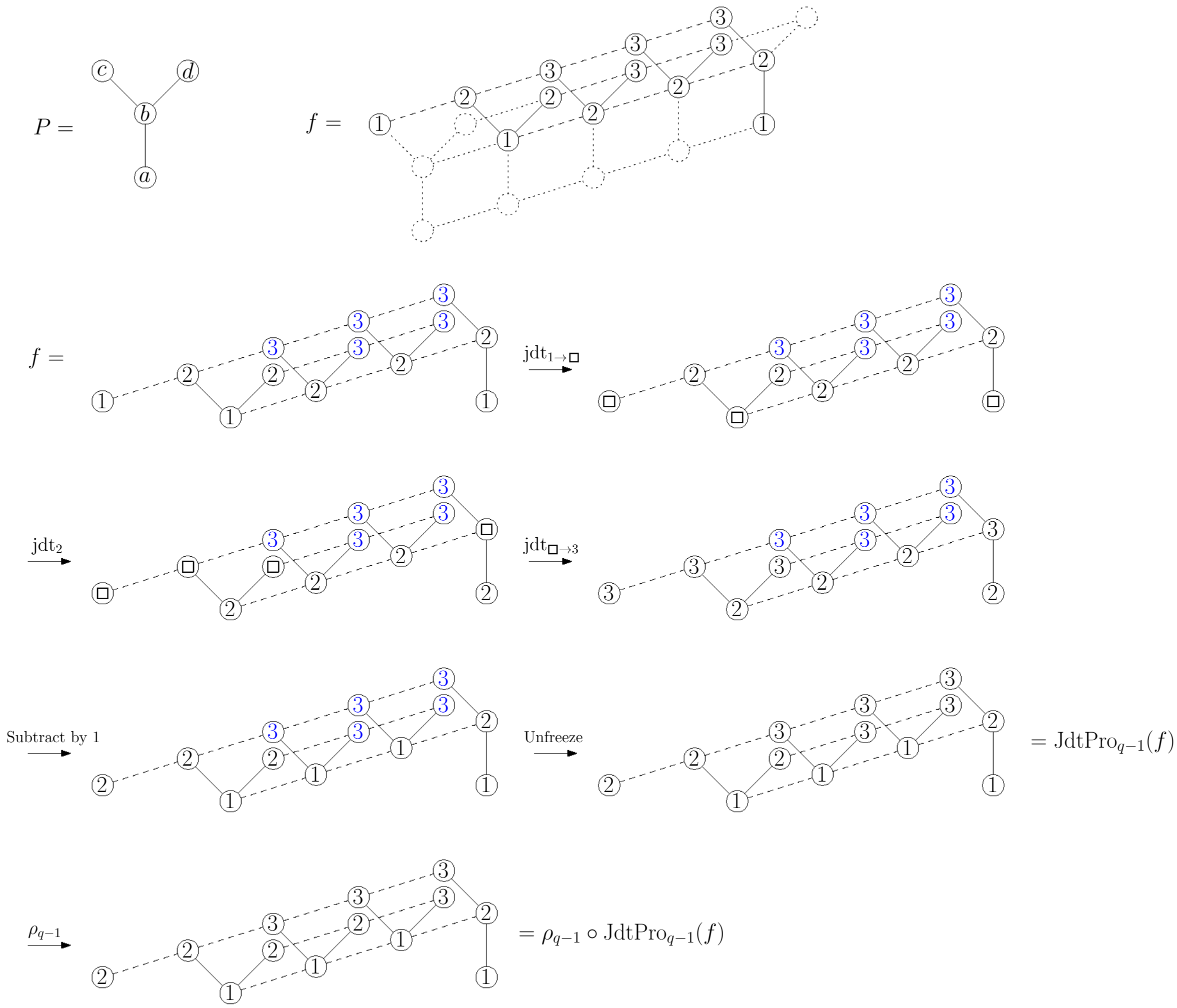


Figure 2.8. We perform $\rho_{q-1} \circ \mathrm{JdtPro}_{q-1}(f)$ on $f \in \mathcal{L}_{P\times[5]}(u, v, 3)$ from Figure 2.7. Labels colored blue are frozen. For the sake of brevity, we do not show the applications of jdt_2 that do nothing.

Before presenting the main result of this section, we first give an example demonstrating $\rho_{q-1} \circ \mathrm{JdtPro}_{q-1}$ and the result of Lemma 2.3.8.

Example 2.3.9. Figure 2.8 shows an example of $\rho_{q-1} \circ \mathrm{JdtPro}_{q-1}$ being applied to the same P-strict labeling from Figure 2.7 and Example 2.3.2. To perform JdtPro_2, we first freeze all labels that are greater than 2. In Figure 2.8, these frozen labels are colored blue. We then apply $\mathrm{jdt}_{\square\to 3} \circ (\mathrm{jdt}_2)^5 \circ \mathrm{jdt}_{1\to\square}(f)$. Note that in Figure 2.8, we do not show applications of jdt_2 that do nothing. Following this, we subtract all unfrozen labels by 1. After this step, we have finished applying JdtPro_2, so all labels are now considered unfrozen. We conclude by applying the Bender-Knuth involution ρ_2. Observe that the resulting P-strict labeling in Figure 2.8 is identical to the

P-strict labeling in Figure 2.7 obtained by applying JdtPro. Lemma 2.3.8 ensures that this will always be the case.

We proceed to the main theorem of this section, which states that P-strict promotion via jeu de taquin and P-strict promotion via Bender-Knuth toggles are equivalent. Our proof uses Lemma 2.3.8 and an inductive argument.

Theorem 2.3.10. *For* $f \in \mathcal{L}_{P\times[\ell]}(u, v, R^q)$*,* $\mathrm{JdtPro}(f) = \mathrm{Pro}(f)$*.*

Proof. Let $f_{\leq i}$ with restriction R^i denote the P-strict labeling f restricted to the subposet of elements with labels less than or equal to i. Observe that by Lemma 2.3.8, we have $\mathrm{JdtPro}(f_{\leq 2}) = \rho_1 \circ \mathrm{JdtPro}_1(f_{\leq 2}) = \rho_1(f_{\leq 2})$. Now suppose $\mathrm{JdtPro}(f_{\leq i}) = \rho_{i-1} \circ \cdots \circ \rho_1(f_{\leq i})$. By applying Lemma 2.3.8, we obtain $\mathrm{JdtPro}(f_{\leq i+1}) = \rho_i \circ \mathrm{JdtPro}_i(f_{\leq i+1})$. Observe that $\mathrm{JdtPro}_i(f_{\leq i+1})_{\leq i} = \mathrm{JdtPro}(f_{\leq i})$. This implies that $\mathrm{JdtPro}_i(f_{\leq i+1}) = \rho_{i-1} \circ \cdots \circ \rho_1(f_{\leq i+1})$, as none of $\rho_1, \ldots, \rho_{i-1}$ affect $i+1$. Therefore, $\mathrm{JdtPro}(f_{\leq i+1}) = \rho_i \circ \rho_{i-1} \circ \cdots \circ \rho_1(f_{\leq i+1})$. By induction, we know this holds for $i = q-1$, yielding $\mathrm{JdtPro}(f_{\leq q}) = \rho_{q-1} \circ \rho_{q-2} \circ \cdots \circ \rho_1(f_{\leq q})$, which is the desired result. □

2.3.2. P-strict evacuation

Evacuation has been well studied on both standard tableaux and semistandard tableaux. In [6], Bloom, Pechenik, and Saracino provide explicit statements and proofs for several evacuation results on semistandard tableaux. We define evacuation on P-strict labelings and investigate which of those results can be generalized and which cannot.

Definition 2.3.11. For $f \in \mathcal{L}_{P\times[\ell]}(u, v, R^q)$, we define **evacuation** in terms of Bender-Knuth involutions:

$$\mathcal{E} = (\rho_1) \circ (\rho_2 \circ \rho_1) \circ \cdots \circ (\rho_{q-2} \circ \cdots \circ \rho_2 \circ \rho_1) \circ (\rho_{q-1} \circ \cdots \circ \rho_2 \circ \rho_1)$$

Additionally, define **dual evacuation**:

$$\mathcal{E}' = (\rho_{q-1}) \circ (\rho_{q-2} \circ \rho_{q-1}) \circ \cdots \circ (\rho_2 \circ \cdots \circ \rho_{q-2} \circ \rho_{q-1}) \circ (\rho_1 \circ \cdots \circ \rho_{q-2} \circ \rho_{q-1})$$

Evacuation and dual evacuation have a special relation on rectangular semistandard Young tableaux. We generalize that relation here.

Definition 2.3.12. Fix the notation for the product of chains poset as: $[a_1] \times [a_2] \times \cdots \times [a_k] = \{(i_1, i_2, \ldots, i_k) \mid 1 \leq i_j \leq a_j, 1 \leq j \leq k\}$.

Definition 2.3.13. Given $(i_1, i_2, \ldots, i_k) \in [a_1] \times [a_2] \times \cdots \times [a_k]$, let $(a_1 + 1 - i_1, a_2 + 1 - i_2, \ldots, a_k + 1 - i_k)$ be the **antipode** of $(i_1, i_2, \ldots, i_k)$.

Definition 2.3.14. Suppose $P = [a_1] \times [a_2] \times \cdots \times [a_k]$. For $f \in \mathcal{L}_{P\times[\ell]}(R^q)$, we obtain a new labeling by interchanging each label with the label of its antipode, then replacing each label i with $q + 1 - i$. Denote this new labeling as f^+.

Lemma 2.3.15. *Let $P = [a_1] \times [a_2] \times \cdots \times [a_k]$ and $f \in \mathcal{L}_{P\times[\ell]}(R^q)$. Then $\mathcal{E}'(f) = \mathcal{E}(f^+)^+$.*

Proof. This follows from the definitions of evacuation and dual evacuation as a product of Bender-Knuth involutions. □

Since P-strict labelings generalize both increasing labelings and semistandard Young tableaux, a natural aim would be to generalize results from these domains. Bloom, Pechenik, and Saracino found a *homomesy* result on semistandard Young tableaux under promotion [6, Theorem 1.1]. A natural generalization to investigate would be to P-strict labelings under promotion, where P is a product of two chains and $\ell = 2$. We find that the result does not generalize due to several evacuation results failing to hold. We note below two statements on evacuation which do generalize and two examples showing statements that do not generalize.

Proposition 2.3.16. *Let P be a poset. For $f \in \mathcal{L}_{P\times[\ell]}(u, v, R^q)$, we have the following:*

1. $\mathcal{E}^2(f) = f$
2. $\mathcal{E} \circ \text{Pro}(f) = \text{Pro}^{-1} \circ \mathcal{E}(f)$

Proof. Both parts rely only on the commutation relations of toggles (see Remark 2.1.24), and therefore follow using previous results on the toggle group. □

Remark 2.3.17. $\text{Pro}^q(f) = f$ does not hold for general $f \in \mathcal{L}_{([a]\times[b])\times[2]}(R^q)$. The P-strict labeling $f \in \mathcal{L}_{([3]\times[2])\times[2]}(R^7)$ from Figure 2.9 gives a counterexample.

Remark 2.3.18. $\mathcal{E}(f) = f^+$ does not hold for general $f \in \mathcal{L}_{([a]\times[b])\times[2]}(R^q)$. The P-strict labeling $f \in \mathcal{L}_{([3]\times[2])\times[2]}(R^7)$ from Figure 2.10 gives a counterexample.

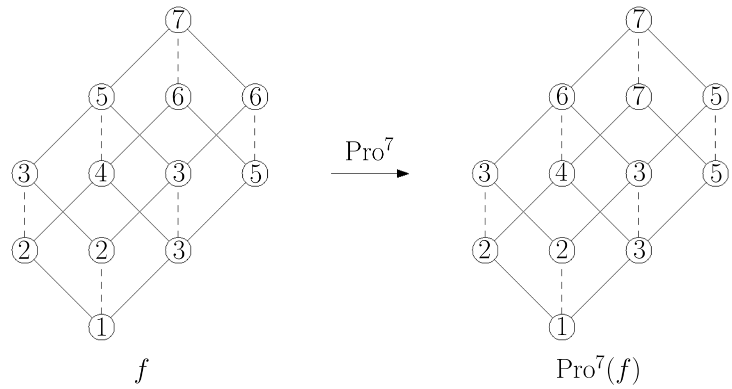


Figure 2.9. By applying Pro^7 to the P-strict labeling f on the left, we obtain the P-strict labeling on the right. We see that these are not equal and so $\mathrm{Pro}^q(f) = f$ does not hold in general.

2.4. Applications of the main theorems to tableaux of many flavors

In this section, we apply Theorems 2.2.8 and 2.2.20 to the case in which P is a chain; in the subsections, we specialize to various types of tableaux. We translate results and conjectures from the domain of P-strict labelings to B-bounded $\Gamma(P, \hat{R})$-partitions and vice versa.

2.4.1. Semistandard tableaux

First, we specialize Theorem 2.2.8 to skew semistandard Young tableaux in Corollary 2.4.3. We relate this to Gelfand-Tsetlin patterns and show how a proposition of Kirillov and Berenstein, Corollary 2.4.6, follows from our bijection. Finally, we state some known cyclic sieving and homomesy results and use Corollary 2.4.3 to translate between the two domains.

We begin by defining skew semistandard Young tableaux.

Definition 2.4.1. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$ be partitions with non-zero parts such that $\mu \subset \lambda$. Where applicable, define $\mu_j := 0$ for $j > m$. Let λ/μ denote the skew partition shape defined by removing the (upper-left justified, in English notation) shape μ from λ. A **skew semistandard Young tableau** of shape λ/μ is a filling of λ/μ with positive integers such that the rows weakly increase from left to right and the columns strictly increase from top to bottom. Let $\mathrm{SSYT}(\lambda/\mu, q)$ denote the set of semistandard Young tableaux of skew shape λ/μ with entries at most q. In the case $\mu = \emptyset$, the adjective 'skew' is removed.

In this and the next subsections, fix the chain $[n] = p_1 \lessdot p_2 \lessdot \cdots \lessdot p_n$. We also use the notation ℓ^n for the partition whose shape has n rows and ℓ columns.

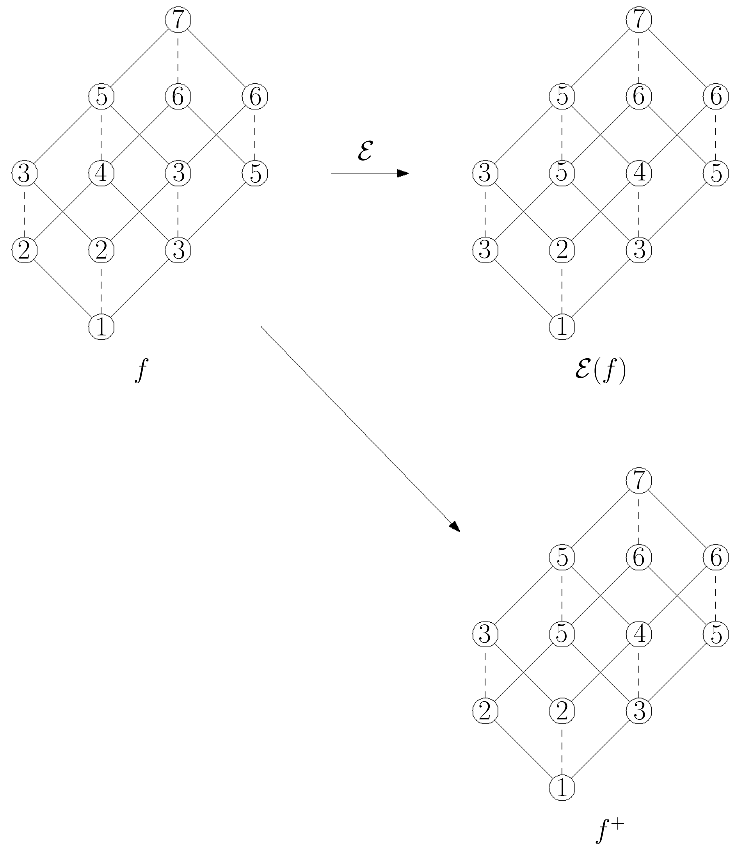


Figure 2.10. By applying $\mathcal{E}$ to the P-strict labeling f in the upper left, we obtain the P-strict labeling in the upper right. Comparing $\mathcal{E}(f)$ to f^+, shown in the bottom right, we see that these P-strict labelings are not equal and so $\mathcal{E}(f) = f^+$ does not hold in general.

Proposition 2.4.2. *The set of semistandard Young tableaux* $\mathrm{SSYT}(\lambda/\mu, q)$ *is equivalent to* $\mathcal{L}_{[n]\times[\lambda_1]}(u, v, R^q)$*, where* $u(p_i) = \mu_i$ *and* $v(p_i) = \lambda_1 - \lambda_i$ *for all* $1 \le i \le n$*.*

Proof. Each box (i, j) of a tableau in $\mathrm{SSYT}(\lambda/\mu, q)$ corresponds exactly to the element (p_i, j) in $P \times [\ell]_u^v$. The weakly increasing condition on rows and strictly increasing condition on columns in $\mathrm{SSYT}(\lambda/\mu, q)$ corresponds to the weak increase on fibers and strict increase on layers, respectively, in $\mathcal{L}_{[n]\times[\lambda_1]}(u, v, R^q)$. □

We now specify the B-bounded $\Gamma(P, \hat{R})$-partitions in bijection with $\mathrm{SSYT}(\lambda/\mu, q)$. Recall $\hat{B}$ from Definition 2.2.7.

Corollary 2.4.3. $\mathrm{SSYT}(\lambda/\mu, q)$ *under* Pro *is in equivariant bijection with* $\mathcal{A}^{\widehat{B}}(\Gamma([n], \widehat{R^q}))$ *under* Row*, with* $\ell = \lambda_1$*,* $u(p_i) = \mu_i$*,* $v(p_i) = \lambda_1 - \lambda_i$ *for all* $1 \le i \le n$*. Moreover, for* $T \in \mathrm{SSYT}(\lambda/\mu, q)$*,* $\Phi(\mathrm{Pro}(T)) = \mathrm{TogPro}(\Phi(T))$*.*

Proof. By Proposition 2.4.2, P-strict labelings $\mathcal{L}_{[n]\times[\lambda_1]}(u, v, R^q)$ with u and v as above are exactly semistandard Young tableaux of shape λ/μ with largest entry q, $\mathrm{SSYT}(\lambda/\mu, q)$. Therefore, the first claim follows from Corollary 2.2.24, where $a(p_i) = 1$ and $b(p_i) = q$ for all $1 \leq i \leq n$. The second claim follows directly from Theorem 2.2.8. □

When $P = [n]$, the lemma underlying our first main theorem is equivalent to a result of Kirillov and Berenstein regarding the correspondence between Bender-Knuth involutions on semistandard Young tableaux and *elementary transformations* on *Gelfand-Tsetlin patterns*. We define these objects below and then state their result, Corollary 2.4.6, in our notation. We then prove a more general result from which this follows, Theorem 2.4.8, as a corollary of our first main theorem.

Definition 2.4.4. Given partitions $\lambda = (\lambda_1, \ldots, \lambda_n)$, $\mu = (\mu_1, \ldots, \mu_m)$ such that $\mu \subset \lambda$, and q, a **Gelfand–Tsetlin pattern from μ to λ with $q + 1$ rows** is a trapezoidal array of nonnegative integers $a = \{a_{ij}\}_{0 \leq i \leq q, 1 \leq j \leq i+m}$ satisfying the following whenever the indices are defined:

1. $a_{0j} = \mu_j$,
2. $a_{ij} \geq a_{i-1,j}$,
3. $a_{ij} \geq a_{i+1,j+1}$, and
4. $a_{qj} = \lambda_j$, where if $j > |\lambda|$, we say $\lambda_j = 0$

Let the set of Gelfand-Tsetlin patterns from μ to λ with $q + 1$ rows be denoted $\mathrm{GT}(\lambda, \mu, q)$.

Definition 2.4.5. Let $a \in \mathrm{GT}(\lambda, \mu, q)$. For $1 \leq k \leq q-1$, define the **elementary transformation** $t_k(a) : \mathrm{GT}(\lambda, \mu, q) \to \mathrm{GT}(\lambda, \mu, q)$ as

$$t_k(a_{ij}) := \begin{cases} a_{i,j} & i \neq k \\ \min(a_{i-1,j-1}, a_{i+1,j}) + \max(a_{i-1,j}, a_{i+1,j+1}) - a_{ij} & \text{otherwise}, \end{cases}$$

where we consider $a_{ij} = \infty$ if $j < 1$ and $a_{ij} = 0$ if $j > i + m$.

We use the mechanism of our main theorem to prove Theorem 2.4.8, which yields the following result. We prove this corollary right before Remark 2.4.11.

Corollary 2.4.6 ([31, Proposition 2.2])**.** *The set* $\mathrm{SSYT}(\lambda/\mu, q)$ *is in bijection with* $\mathrm{GT}(\tilde{\lambda}, \tilde{\mu}, q)$*, where* $\tilde{\lambda}_i := \lambda_1 - \mu_{n-i+1}$ *and* $\tilde{\mu}_i := \lambda_1 - \lambda_{n-i+1}$*. Moreover,* ρ_k *on* $\mathrm{SSYT}(\lambda/\mu, q)$ *corresponds to* t_{q-k} *on* $\mathrm{GT}(\tilde{\lambda}, \tilde{\mu}, q)$*.*

To put this corollary in the language of our main theorem, we show that $\mathrm{GT}(\tilde{\lambda}, \tilde{\mu}, q)$ is equivalent to $\mathcal{A}^{\overline{B}}(\Gamma([n], \overline{R}))$, where the restriction function $\overline{R}$ and the bounding function $\overline{B}$ are defined below.

Definition 2.4.7. For any convex subposet $P \times [\ell]_u^v$ and global bound q, let $\overline{R}$ be the (not necessarily consistent) restriction function on P given by $\overline{R}(p) = \{0, 1, \ldots, q+1\}$ for all $p \in P$, and let $\overline{B}$ be defined on $\Gamma(P, \overline{R})$ as $\overline{B}(p, 0) = \ell - u(p)$ and $\overline{B}(p, q) = v(p)$.

Thus the structure of $\Gamma(P, \overline{R})$ consists of the chains $(p, 0) \gtrdot (p, 1) \gtrdot \cdots \gtrdot (p, q)$ and we have $(p, k) \lessdot (p', k+1)$ whenever $p \lessdot_P p'$ and $0 \leq k \leq q-1$. As we will see in the proof, these covering relations provide the inequality conditions (2) and (3) from Definition 2.4.4 in $\mathcal{A}^{\overline{B}}(\Gamma(P, \overline{R}))$ when $P = [n]$, and $\overline{B}$ gives conditions (1) and (4).

By generalizing semistandard tableaux to P-strict labelings, we are able to prove the equivariance result of Corollary 2.4.6 for any poset P. In this way, $\mathcal{A}^{\overline{B}}(\Gamma(P, \overline{R}))$ can be considered a generalization of Gelfand-Tsetlin patterns.

Theorem 2.4.8. *The set* $\mathcal{L}_{P \times [\ell]}(u, v, R^q)$ *is in bijection with the set* $\mathcal{A}^{\overline{B}}(\Gamma(P, \overline{R}))$ *and* ρ_k *on* $\mathcal{L}_{P \times [\ell]}(u, v, R^q)$ *corresponds to* τ_k *on* $\mathcal{A}^{\overline{B}}(\Gamma(P, \overline{R}))$*.*

We first define the bijection map using the value j_k^p from Definition 2.2.13. Recall from Definition 2.2.12 that we consider the label $f(p, i)$ to be in *position* i.

Definition 2.4.9. Let $\Psi : \mathcal{L}_{P \times [\ell]}(u, v, R^q) \to \mathcal{A}^{\overline{B}}(\Gamma(P, \overline{R}))$ where $\Psi(f)(p, k) = \ell + 1 - j_k^p$. We can treat $\Psi(f)(p, k)$ as the number of positions j in the fiber F_p such that $f(p, j)$ is larger than k, where we consider $f(p, i) > k$ in the positions $\ell + 1 - v(p) \leq i \leq \ell$ for which f is not defined.

Refer to Figure 2.11 for an example of the map Ψ.

Lemma 2.4.10. Ψ *is a bijection.*

Proof. We begin by verifying that $\Psi(f) \in \mathcal{A}^{\overline{B}}(\Gamma(P, \overline{R}))$. For $1 \leq k \leq q$, $(p, k) \lessdot (p, k-1)$. Since f is weakly increasing on fibers, we have $\Psi(f)(p, k) \leq \Psi(f)(p, k-1)$, as there must be at least as many positions greater than $k-1$ as are greater than k. If $p \lessdot_P p'$ and $0 \leq k \leq q-1$, then $(p,k) \lessdot_{\Gamma(P,R^q)} (p',k+1)$. Since there are $\Psi(f)(p,k)$ positions greater than k in $f(F_p)$, there must be at least as many positions greater than $k+1$ in $f(F_{p'})$ in order to accommodate those values in $f(F_p)$, as f is strictly increasing on layers. Thus $\Psi(f)(p,k) \leq \Psi(f)(p',k+1)$, so $\Psi(f)(p,k)$ respects all covering relations in $\Gamma(P,\overline{R})$. Moreover, $\Psi(f)(p,0) = \ell - u(p)$ since the first position greater than zero is at $f(p,u(p)+1)$ for all p, and $\Psi(f)(p,q) = v(p)$ since the only positions considered greater than q are those after the end of the fiber. Thus $\Psi(f) \in \mathcal{A}^{\overline{B}}(\Gamma(P,\overline{R}))$.

For the reverse map, let $\overline{\sigma} \in \mathcal{A}^{\overline{B}}(\Gamma(P,\overline{R}))$ and let $\Psi^{-1}(\overline{\sigma})(p,\ell+1-i) = k$ for i such that $\overline{\sigma}(p,k) < i \leq \overline{\sigma}(p,k-1)$. Since $\overline{\sigma}(p,k) \leq \overline{\sigma}(p,k-1)$, $\Psi^{-1}(\overline{\sigma})$ is weakly increasing on fibers, and because $\overline{\sigma}(p,k) \geq \overline{\sigma}(p',k-1)$ for all $p' \lessdot_P p$, if $\Psi^{-1}(\overline{\sigma})(p,j) = k$ then $\Psi^{-1}(\overline{\sigma})(p',j) \leq k-1$, so $\Psi^{-1}(\overline{\sigma})$ is strictly increasing across layers. Thus $\Psi^{-1}(\overline{\sigma}) \in \mathcal{L}_{P\times[\ell]}(u,v,R^q)$.

Now $\Psi(\Psi^{-1}(\overline{\sigma}))(p,k) = \overline{\sigma}(k)$ since there are $\overline{\sigma}(p,k)$ positions greater than k in $\Phi^{-1}(\overline{\sigma})(F_p)$, and, if $f(p,i) = k$, $\Psi^{-1}(\Psi(f))(p,i) = k$ since $\Psi(f)(p,k) < \ell+1-(\ell+1-i) = i$. Therefore Ψ is a bijection. □

Proof of Theorem 2.4.8. Via the maps Φ from Definition 2.2.9 and Ψ from Definition 2.4.9, the set $\mathcal{A}^{\overline{B}}(\Gamma(P,\overline{R}))$ is in bijection with $\mathcal{A}^{\widehat{B}}(\Gamma(P,\widehat{R^q}))$. We wish to show this bijection $\Phi\Psi^{-1}$ is equivariant under the action of $\tau_{(p,k)}$.

Let $\overline{\sigma} \in \mathcal{A}^{\overline{B}}(\Gamma(P,\overline{R}))$, $f = \Psi^{-1}(\overline{\sigma}) \in \mathcal{L}_{P\times[\ell]}(u,v,R^q)$, and $\sigma = \Phi(f) \in \mathcal{A}^{\widehat{B}}(\Gamma(P,\widehat{R^q}))$. By Lemma 2.2.15, $\sigma(p,k) = \ell+1-j_k^p = \overline{\sigma}(p,k)$ where $k \in R^q(p)^*$ (that is, for $(p,k) \in \Gamma(P,\widehat{R^q})\setminus \mathrm{dom}\hat{B}$). Suppose $k \notin R^q(p)^*$. If $k < \min R^q(p)$, then $f(p,i)$ is always greater than k, so $\overline{\sigma}(p,k) = \ell - u(p)$. If $k \geq \max R^q(p)^*$, then $f(p,i)$ is always less than or equal to k, so $\overline{\sigma}(p,k) = v(p)$. Finally, if k_1 is the largest value in $R^q(p)^*$ such that $k_1 < k$, then $\overline{\sigma}(p,k) = \overline{\sigma}(p,k_1)$, since the number of positions greater than k_1 must be the same as the number of positions greater than k. By Lemma 2.2.23, (p,k_1) is fixed in $\mathcal{A}^{\widehat{B}}(\Gamma(P,\widehat{R^q}))$ and therefore in $\mathcal{A}^{\overline{B}}(\Gamma(P,\overline{R}))$ since $k_1 \in R^q(p)^*$. Thus, whenever $k \notin R^q(p)^*$, $\overline{\sigma}(p,k)$ is fixed, so $\tau_{(p,k)}$ acts as the identity on $\mathcal{A}^{\overline{B}}(\Gamma(P,\overline{R}))$. Now, for equivariance, we need only show that $\tau_{(p,k)}(\sigma)(p,k) = \tau_{(p,k)}(\overline{\sigma})(p,k)$ whenever $k \in R^q(p)^*$.

Let $k \in R^q(p)^*$. If (p,k) covers and is covered by the same elements in $\Gamma(P,\widehat{R^q})$ as in $\Gamma(P,\overline{R})$, then we are done, so we will consider the cases in which these covers differ. Suppose $k_1 > k+1$ and either $(p,k) \gtrdot (p,k_1)$ in $\Gamma(P,\widehat{R^q})$ or there exists $p' \gtrdot_P p$ such that $(p,k) \lessdot (p',k_1)$. In each case, by definition of Γ, $k+1 \notin R^q(p)^*$ so, by Lemma 2.2.23, (p,k) is fixed in $\mathcal{A}^{\widehat{B}}(\Gamma(P,\widehat{R^q}))$ and therefore in $\mathcal{A}^{\overline{B}}(\Gamma(P,\overline{R}))$. Now suppose $k_1 < k-1$ and either $(p,k) \lessdot (p,k_1)$ or there exists $p' \lessdot_P p$ such that $(p,k) \gtrdot (p',k_1)$. In the first case, $\sigma(p,k_1) = \overline{\sigma}(p,k_1) = \overline{\sigma}(p,k-1)$. In the second case, $k-1 \notin R^q(p')$, otherwise we would have $(p,k) \gtrdot (p',k-1)$, so $\sigma(p',k_1) = \overline{\sigma}(p',k_1) = \overline{\sigma}(p',k-1)$.

In both cases where the covers in $\Gamma(P,\widehat{R^q})$ differ from $\Gamma(P,\overline{R})$, the minimum value of the upper covers and the maximum value of the lower covers of (p,k) is unchanged between $\mathcal{A}^{\widehat{B}}(\Gamma(P,\widehat{R^q}))$ and $\mathcal{A}^{\overline{B}}(\Gamma(P,\overline{R}))$. Thus, $\tau_{(p,k)}(\sigma)(p,k) = \tau_{(p,k)}(\overline{\sigma})(p,k)$.

By the above, τ_k on $\mathcal{A}^{\overline{B}}(\Gamma(P,\overline{R}))$ is equivalent to τ_k on $\mathcal{A}^{\widehat{B}}(\Gamma(P,\widehat{R^q}))$. Thus, by Lemma 2.2.11, τ_k on $\mathcal{A}^{\overline{B}}(\Gamma(P,\overline{R}))$ corresponds to ρ_k on $\mathcal{L}_{P\times[\ell]}(u,v,R^q)$. □

In the following proof of the Kirillov and Berenstein result, we consider a Gelfand-Tsetlin pattern as a parallelogram-shaped array $\{a_{ij}\}_{0\le i\le q, 1\le j\le n}$ with the same properties described in Definition 2.4.4.

Proof of Corollary 2.4.6. Following Proposition 2.4.2, given $\mathrm{SSYT}(\lambda/\mu,q)$, define $u(p_i) = \mu_i$ and $v(p_i) = \lambda_1 - \lambda_i$ for all $1 \le i \le n$. Then $\mathrm{SSYT}(\lambda/\mu,q)$ is equivalent to $\mathcal{L}_{[n]\times[\lambda_1]}(u,v,R^q)$. Thus, to apply Theorem 2.4.8, consider $\mathcal{A}^{\overline{B}}(\Gamma([n],\overline{R}))$ where $\overline{B}$ is defined using the u and v above, that is, $\overline{B}(p_i,0) = \lambda_1 - \mu_i$ and $\overline{B}(p_i,q) = \lambda_1 - \lambda_i$ for $1 \le i \le n$.

Let $\overline{\sigma} \in \mathcal{A}^{\overline{B}}(\Gamma([n],\overline{R}))$. Then the array given by $a_{ij} = \overline{\sigma}(p_{n+1-j}, q-i)$ for $0 \le i \le q$ and $1 \le j \le n$ satisfies the inequalities $a_{ij} \ge a_{i-1,j}$ and $a_{ij} \ge a_{i+1,j+1}$, since $(p_{n+1-j}, q-i) \gtrdot (p_{n+1-j}, q-i+1)$ and $(p_{n+1-j}, q-i) \gtrdot (p_{n-j}, q-i-1)$ in $\Gamma([n],\overline{R})$. Additionally, $a_{0j} = \overline{\sigma}(p_{n+1-j}, q) = \lambda_1 - \mu_{n+1-j}$ and $a_{qj} = \overline{\sigma}(p_{n+1-j}, 0) = \lambda_1 - \lambda_{n+1-j}$. Thus $\{a_{ij}\} \in \mathrm{GT}(\tilde{\lambda},\tilde{\mu},q)$. Since the map $\overline{\sigma} \mapsto \{a_{ij}\}$ given above is invertible (as it simply "rotates" the $\Gamma([n],\overline{R})$-partition $\overline{\sigma}$), $\mathcal{A}^{\overline{B}}(\Gamma([n],\overline{R}))$ is equivalent to $\mathrm{GT}(\tilde{\lambda},\tilde{\mu},q)$.

By their respective definitions, the toggle $\tau_{(p_i,k)}$ at (p_i,k) on $\mathcal{A}^{\overline{B}}(\Gamma([n],\overline{R}))$ is exactly the elementary transformation t_{q-k} at $a_{q-k,q-i}$ on $\mathrm{GT}(\tilde{\lambda},\tilde{\mu},q)$, so τ_k on $\mathcal{A}^{\overline{B}}(\Gamma([n],\overline{R}))$ corresponds to t_{q-k} on $\mathrm{GT}(\tilde{\lambda},\tilde{\mu},q)$. Thus, by Theorem 2.4.8, t_{q-k} on $\mathrm{GT}(\tilde{\lambda},\tilde{\mu},q)$ corresponds to ρ_k on $\mathcal{L}_{[n]\times[\lambda_1]}(u,v,R^q) = \mathrm{SSYT}(\lambda/\mu,q)$. □

Remark 2.4.11. Note, Kirillov and Berenstein [31, Proposition 2.2] actually gave a bijection between $\mathrm{SSYT}(\lambda/\mu,q)$ and $\mathrm{GT}(\lambda,\mu,q)$. Our bijection is dual to theirs, but this is an artifact of our conventions, not a substantive difference. See also [27] (Appendix A, especially Proposition A.7) and [18].

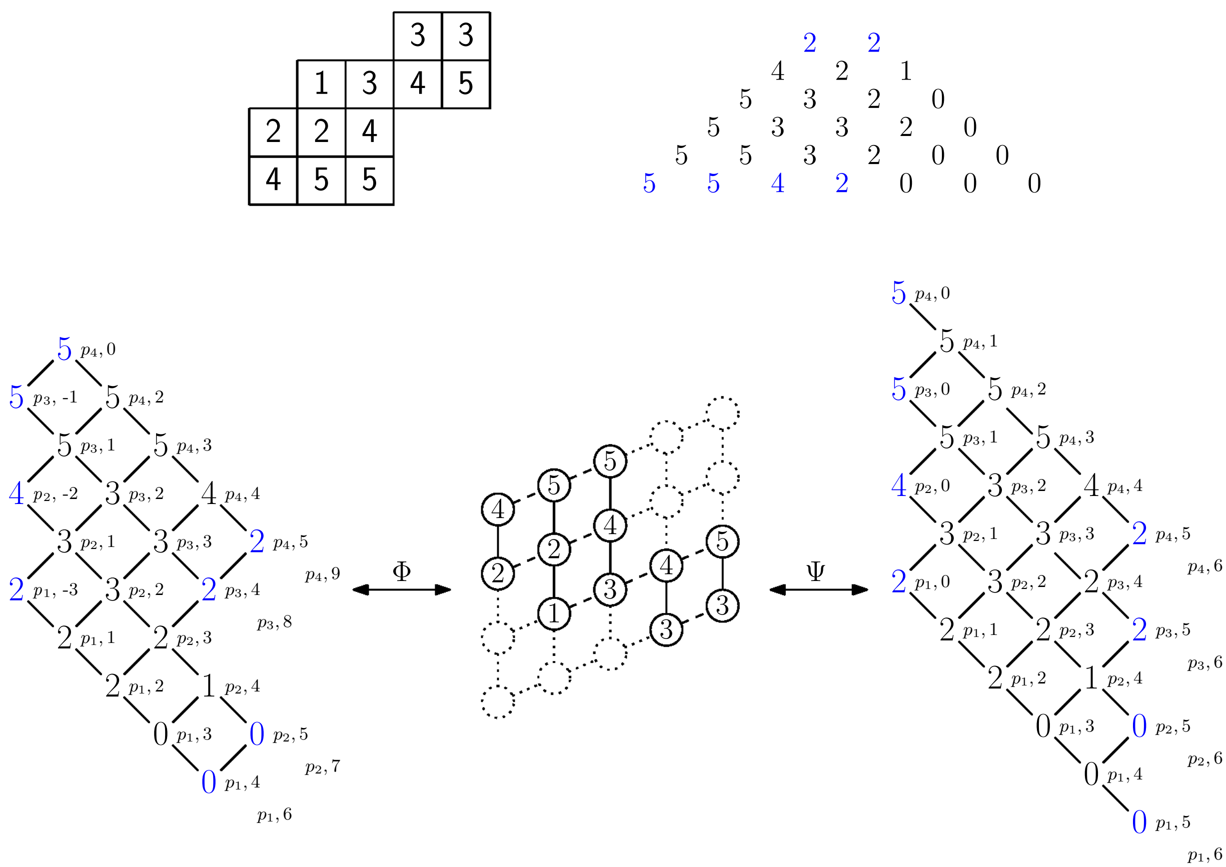


Figure 2.11. The top row shows a skew semistandard tableau with maximum entry 5 and its corresponding Gelfand–Tsetlin pattern in $\mathrm{GT}(\tilde{\lambda}, \tilde{\mu}, 5)$ where $\mu = (3,1)$, $\tilde{\mu} = (2,2)$, $\lambda = (5,5,3,3)$, and $\tilde{\lambda} = (5,5,4,2)$. In the bottom row, the left is an element of $\mathcal{A}^{\widehat{B}}(\Gamma([n], \widehat{R^5}))$ from our main theorem, and on the right is an element of $\mathcal{A}^{\overline{B}}(\Gamma([n], \overline{R}))$ from Theorem 2.4.8. If we rotate this $\overline{B}$-bounded $\Gamma([n], \overline{R})$-partition 90 degrees counterclockwise, the labels coincide with those of the Gelfand–Tsetlin pattern above.

In the case where $\mu = \emptyset$ and λ is a rectangle, Corollary 2.4.3 specializes nicely.

Corollary 2.4.12. *The set of semistandard Young tableaux* $\mathrm{SSYT}(\ell^n, q)$ *under* Pro *is in equivariant bijection with the set* $\mathcal{A}^{\ell}([n] \times [q-n])$ *under* Row*.*

Proof. By Proposition 2.4.2, $\mathrm{SSYT}(\ell^n, q)$ under Pro is equivalent to $\mathcal{L}_{[n]\times[\ell]}(R^q)$ which, by Corollary 2.2.29, is in equivariant bijection with $\mathcal{A}^{\ell}([n] \times [q-n])$ under Row. □

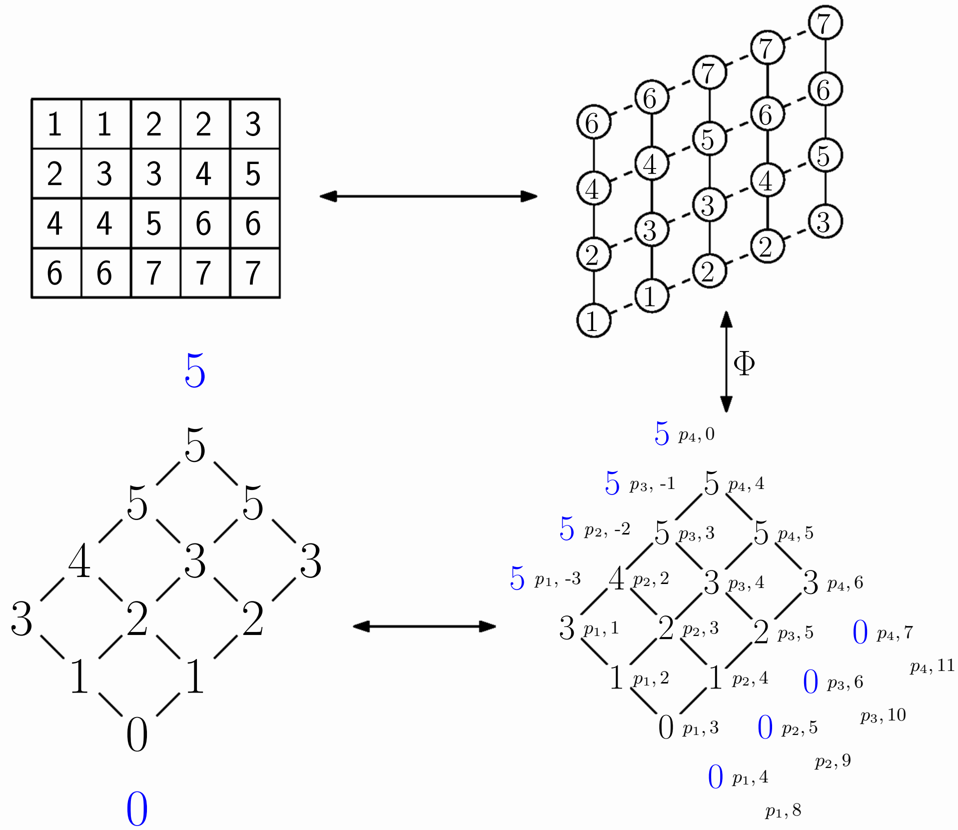


Figure 2.12. The correspondence in the top row is that of Proposition 2.4.2, the bijection in the right column is our main theorem, and the bottom row more clearly shows the element of $\mathcal{A}^{\widehat{B}}(\Gamma([4], \widehat{R^7}))$ as an element of $\mathcal{A}^5([4] \times [3])$. To emphasize the underlying shape of $\Gamma(P, \hat{R})$, in this and the following figures we do not draw covering relations between the elements of $\Gamma(P, \hat{R})$ fixed by $\hat{B}$ and grey out the covering relations between those elements and the rest of the poset.

We now discuss a cyclic sieving result of B. Rhoades on rectangular semistandard Young tableaux and its translation via Corollary 2.4.12.

Definition 2.4.13 ([41])**.** Let C be a finite cyclic group acting on a finite set X and let c be a generator of C. Let $\zeta \in \mathbb{C}$ be a root of unity having the same multiplicative order as c and let $g \in \mathbb{Q}[x]$ be a polynomial. The triple (X, C, g) exhibits the **cyclic sieving phenomenon** if for any integer $d \geq 0$, the fixed point set cardinality $|X^{c^d}|$ is equal to the polynomial evaluation $g(\zeta^d)$.

Theorem 2.4.14 ([42, Theorem 1.4])**.** *The triple* $(\mathrm{SSYT}(\ell^n, q), \langle \mathrm{Pro} \rangle, X(x))$ *exhibits the cyclic sieving phenomenon, where*

$$X(x) := \prod_{i=1}^{\ell} \prod_{j=1}^{n} \frac{1 - x^{i+j+q-n-1}}{1 - x^{i+j-1}}$$

Corollary 2.4.15. *Let $1 \leq n \leq q$. Then the triple $(\mathcal{A}^\ell([n] \times [q-n]), \langle \mathrm{Row} \rangle, X(x))$ exhibits the cyclic sieving phenomenon.*

Proof. This follows from Theorem 2.4.14 and Corollary 2.4.12. Note that $X(x)$ is MacMahon's generating function for plane partitions which fit inside a box having dimensions ℓ by n by $q-n$. These are in simple bijection with $\mathcal{A}^\ell([n] \times [q-n])$. □

Remark 2.4.16. Corollary 2.4.15 has been noted in the literature, for example, by Hopkins [27] and Frieden [18]. Note the fact that the order of rowmotion on $\mathcal{A}^\ell([n] \times [q-n])$ divides q (implicit in the statement of cyclic sieving) also follows from the order of *birational rowmotion* on the poset $[n] \times [q-n]$. This was proved first by D. Grinberg and T. Roby [23] with a more direct proof by G. Musiker and Roby [33].

We now turn our attention toward several *homomesy* results. Rather than present the most general definition, this definition is given for actions with finite orbits, as this is the only case we consider.

Definition 2.4.17 ([40])**.** Given a finite set S, an action $\tau : S \to S$, and a statistic $f : S \to k$ where k is a field of characteristic zero, we say that (S, τ, f) exhibits **homomesy** if there exists $c \in k$ such that for every τ-orbit $\mathcal{O}$

$$\frac{1}{|\mathcal{O}|} \sum_{x \in \mathcal{O}} f(x) = c$$

where $|\mathcal{O}|$ denotes the number of elements in $\mathcal{O}$. If such a c exists, we will say the triple is c**-mesic**.

We state two known theorems below and prove their equivalence as a corollary of Theorem 2.2.8.

Theorem 2.4.18 ([6, Theorem 1.1])**.** *Let S be a set of boxes in the rectangle ℓ^n that is fixed under $180°$ rotation and Σ_S denote the sum of entries in the boxes of S. Then $(\mathrm{SSYT}(\ell^n, q), \mathrm{Pro}, \Sigma_S)$ exhibits homomesy.*

Recall Definition 2.3.12, which specifies notation for $[a] \times [b]$, and Definition 2.3.13 of antipode.

Definition 2.4.19. A subset S of $[a] \times [b]$ is **antipodal** if S contains the antipode of each of its elements.

Theorem 2.4.20 ([15] [16, Theorem 3.4])**.** *Let S be an antipodal subset of $[n] \times [q-n]$ and Σ_S denote the sum of labels of S. Then $(\mathcal{A}^\ell([n] \times [q-n]), \text{TogPro}, \Sigma_S)$ exhibits homomesy.*

Corollary 2.4.21. *The previous two results, Theorem 2.4.18 and Theorem 2.4.20, imply each other.*

Proof. By Corollary 2.4.12, $\text{SSYT}(\ell^n, q)$ under promotion is in equivariant bijection with $\mathcal{A}^\ell([n] \times [q-n])$ under rowmotion, and also TogPro, by conjugacy. By Corollary 2.4.3, for $T \in \text{SSYT}(\ell^n, q)$, $\Phi(\text{Pro}(T)) = \text{TogPro}(\Phi(T))$. Furthermore, we claim that $\Sigma_S(T) = \Sigma_S(\Phi(T)) + \frac{\ell n(n+1)}{2}$. To show this claim, observe that if T is the tableau with all 1's in the first row, all 2's in the second row, and so on, then $\Sigma_S(T) = \frac{\ell n(n+1)}{2}$. Additionally, the corresponding Q-partition $\Phi(T)$, where $Q = [n] \times [q-n]$, is such that every label is 0. Increasing the entry of a box in T by 1 increases the label of an element in $\Phi(T)$ by 1, showing the claim. Because the statistic Σ_S under the bijection differs by a constant, the corollary statement follows. □

2.4.2. Flagged tableaux

In this section, we first specialize Theorem 2.2.8 to flagged tableaux and use this correspondence to enumerate the corresponding set of $\hat{B}$-bounded $\Gamma(P, \hat{R})$-partitions. Then, we state some recent cyclic sieving and new homomesy conjectures and use Theorem 2.2.8 to translate these conjectures between the two domains.

Definition 2.4.22. Let $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_n)$ and $\mu = (\mu_1, \mu_2, \dots, \mu_m)$ be partitions with $\mu \subset \lambda$ and let $b = (b_1, b_2, \dots, b_n)$ where b_i is a positive integer and $b_1 \leq b_2 \leq \dots \leq b_n$. A **flagged tableau** of shape λ/μ and flag b is a skew semistandard Young tableau of shape λ/μ whose entries in row i do not exceed b_i. Let $\text{FT}(\lambda/\mu, b)$ denote the set of flagged tableaux of shape λ/μ and flag b.

Note that, depending on context, b represents either the increasing sequence of positive integers $(b_1, \dots, b_n)$ or the function $b : [n] \to \mathbb{Z}^+$ with $b(p_i) = b_i$.

Proposition 2.4.23. *The set of flagged tableaux $\text{FT}(\lambda/\mu, b)$ is equivalent to $\mathcal{L}_{[n]\times[\lambda_1]}(u, v, R^b)$ where $u(p_i) = \mu_i$ and $v(p_i) = \lambda_1 - \lambda_i$ for all $1 \leq i \leq n$.*

Proof. Since $\mathcal{L}_{[n]\times[\lambda_1]}(u, v, R^b) \subset \mathcal{L}_{[n]\times[\lambda_1]}(u, v, R^{b_n})$, by Proposition 2.4.2 we have that $[n]$-strict labelings in $\mathcal{L}_{[n]\times[\lambda_1]}(u, v, R^b)$ correspond to semistandard Young tableaux whose entries in row i are restricted above by b_i, which is exactly $\text{FT}(\lambda/\mu, b)$. □

We now specify the $\hat{B}$-bounded $\Gamma(P, \hat{R})$-partitions in bijection with $\mathrm{FT}(\lambda/\mu, b)$. Recall $\hat{B}$ from Definition 2.2.7.

Corollary 2.4.24. *The set* $\mathrm{FT}(\lambda/\mu, b)$ *under* Pro *is in equivariant bijection with* $\mathcal{A}^{\widehat{B}}(\Gamma([n], \widehat{R^b}))$ *under* Row*, with* $\ell = \lambda_1$*,* $u(p_i) = \mu_i$*,* $v(p_i) = \lambda_1 - \lambda_i$ *for all* $1 \leq i \leq n$*. Moreover, for* $T \in \mathrm{FT}(\lambda/\mu, b)$*,* $\Phi(\mathrm{Pro}(T)) = \mathrm{TogPro}(\Phi(T))$.

Proof. This follows from Proposition 2.4.23, Corollary 2.2.24, and Theorem 2.2.8. □

Remark 2.4.25. Flagged tableaux are enumerated by an analogue of the Jacobi-Trudi formula due to I. Gessel and X. Viennot [22] with an alternative proof by M. Wachs [56]. Thus the bijection of Corollary 2.4.24 allows one to translate this to enumerate $\mathcal{A}^{\widehat{B}}(\Gamma([n], \widehat{R^b}))$.

In the rest of this subsection, we apply Corollary 2.4.24 to some specific sets of flagged tableaux, obtaining Corollaries 2.4.28 and 2.4.41 along with further corollaries and conjectures. Our first corollary involves the triangular poset from the following definition. This poset is isomorphic to the *Type* A_n *positive root poset* from Coxeter theory. Though this algebraic interpretation is what has generated interest surrounding this poset, we will not need it here.

Definition 2.4.26. Let $\triangle_n$ denote the subposet of $[n] \times [n]$ given by $\{(i,j) \mid 1 \leq i \leq n, n-i < j \leq n\}$.

As noted in Section 2.1.1 as our motivating example, we have the following correspondence in the case of flagged tableaux of shape ℓ^n and flag $b = (2, 4, \ldots, 2n)$. Following the procedure of Corollary 2.4.12, we first show that $\Gamma([n], R^b)$ has the desired shape.

Lemma 2.4.27. *Let* $b = (2, 4, \ldots, 2n)$*. Then, if* R^b *is consistent on* $[n]$*,* $\Gamma([n], R^b)$ *and* $\triangle_n$ *are isomorphic as posets.*

Proof. The restriction function on $[n]$ induced by b is given by $R^b(p_i) = \{i, i+1, \ldots, 2i\}$. By definition of Γ (as noted in [13, Thm. 2.21]), $(p_{i_1}, k_1) \lessdot (p_{i_2}, k_2)$ if and only if either $i_1 = i_2$ and $k_1 - 1 = k_2$ or $i_1 + 1 = i_2$ and $k_1 + 1 = k_2$. Define a map from $\Gamma([n], R^b)$ to $\triangle_n$ by $(p_i, k) \mapsto (i, n-k+i)$. Since $i \leq k \leq 2i-1$ we have $n-i+1 \leq n-k+i \leq n+1$, so the above map is a bijection to $\{(i,j) \mid i+j > n\}$. Because $(i,j) \lessdot (i',j') \in [n] \times [n]$ if and only if $i = i'$ and $j+1 = j'$ or $i+1 = i'$ and $j = j'$, the covers of (p_i, k) in $\Gamma([n], R^b)$ correspond exactly to the covers of $(i, n-k+i)$ in $\triangle_n$. Thus $\Gamma([n], R^b)$ and $\triangle_n$ are isomorphic as posets. □

Corollary 2.4.28. *The set of flagged tableaux* $\mathrm{FT}(\ell^n, (2, 4, \ldots, 2n))$ *under* Pro *is in equivariant bijection with* $\mathcal{A}^\ell(\triangle_n)$ *under* Row.

Proof. Let $b = (2, 4, \ldots, 2n)$. By Corollary 2.4.24, $\mathrm{FT}(\ell^n, b)$ under Pro is in equivariant bijection with $\mathcal{A}^{\widehat{B}}(\Gamma([n], \widehat{R^b}))$ under Row where $u(p_i) = v(p_i) = 0$ for all $p_i \in [n]$. By Proposition 2.2.27, $\mathcal{A}^{\widehat{B}}(\Gamma([n], \widehat{R^b}))$ is equivalent to $\mathcal{A}^\ell(\Gamma([n], R^b))$ which, by Lemma 2.4.27, is exactly $\mathcal{A}^\ell(\triangle_n)$. $\square$

D. Grinberg and T. Roby proved a result on the order of birational rowmotion on $\triangle_n$, which implies the following.

Theorem 2.4.29 ([23, Corollary 66])**.** Row *on* $\mathcal{A}^\ell(\triangle_n)$ *is of order dividing* $2(n+1)$.

We then obtain the following as a corollary of this theorem and Corollary 2.4.28.

Corollary 2.4.30. Pro *on* $\mathrm{FT}(\ell^n, (2, 4, \ldots, 2n))$ *is of order dividing* $2(n+1)$.

Note, the order does not depend on ℓ. Therefore, the order of promotion in this case is independent of the number of columns.

J. Propp conjectured the following instance of the cyclic sieving phenomenon (see Definition 2.4.13) on $\mathcal{A}^\ell(\triangle_n)$ under rowmotion with a polynomial analogue of the Catalan numbers. S. Hopkins recently extended this conjecture to positive root posets of all coincidental types (see [28, Conj 4.23], [27, Remark 5.5]).

Conjecture 2.4.31. *The triple* $\left(\mathcal{A}^\ell(\triangle_n), \langle \mathrm{Row} \rangle, Cat_\ell(x)\right)$ *exhibits the cyclic sieving phenomenon, where*

$$Cat_\ell(x) := \prod_{j=0}^{\ell-1} \prod_{i=1}^{n} \frac{1 - x^{n+1+i+2j}}{1 - x^{i+2j}}.$$

Thus, Corollary 2.4.28 implies the equivalence of this conjecture and the following.

Conjecture 2.4.32. *The triple* $(\mathrm{FT}(\ell^n, (2, 4, \ldots, 2n)), \langle \mathrm{Pro} \rangle, Cat_\ell(x))$ *exhibits the cyclic sieving phenomenon.*

We conjecture the following homomesy statement (Conjecture 2.4.35), which was proved in the case $\ell = 1$ by S. Haddadan [25, 26].

Definition 2.4.33. We say a poset P is **ranked** if there exists a rank function $\mathrm{rk} : P \to \mathbb{Z}$ such that $p_1 \lessdot_P p_2$ implies $\mathrm{rk}(p_2) = \mathrm{rk}(p_1) + 1$.

Definition 2.4.34. Let P be a ranked poset and let $\sigma \in \mathcal{A}^\ell(P)$. Define *rank-alternating label sum* to be $\mathcal{R}(\sigma) = \sum_{p \in P} (-1)^{\mathrm{rk}(p)} \sigma(p)$.

For the following conjecture, we use the rank function of $\triangle_n$ defined by $\mathrm{rk}(p) = 0$ if p is a minimal element.

Conjecture 2.4.35. *The triple* $\left(\mathcal{A}^\ell(\triangle_n), \mathrm{TogPro}, \mathcal{R}\right)$ *is* 0*-mesic when* n *is even and* $\frac{\ell}{2}$*-mesic when* n *is odd.*

Using Sage [52], we have checked this conjecture for $n \leq 6$ and $\ell \leq 3$. We have also verified that a similar statement fails to hold for the Type B/C case when $n = 2$ and $\ell = 1$, and the Type D case when $n = 4$ and $\ell = 1$.

We use Corollary 2.4.28 to translate this to a conjecture on flagged tableaux.

Definition 2.4.36. Suppose $T \in \mathrm{FT}(\ell^n, (2, 4, \ldots, 2n))$. Let R_O denote the boxes in the odd rows of T and let R_E denote the boxes in the even rows of T. Furthermore, let O denote the set of boxes in T containing an odd integer and E denote the set of boxes in T containing an even integer. Then $\sum |R_O \cap E| - \sum |R_E \cap O|$ denotes the difference of the number of boxes in odd rows of T that contain an even integer and the number of boxes in even rows of T that contain an odd integer.

Conjecture 2.4.37. $(\mathrm{FT}(\ell^n, (2, 4, \ldots, 2n)), \mathrm{Pro}, \sum |R_O \cap E| - \sum |R_E \cap O|)$ *is* 0*-mesic when* n *is even and* $\frac{\ell}{2}$*-mesic when* n *is odd.*

Theorem 2.4.38. *The previous two conjectures, Conjecture 2.4.35 and Conjecture 2.4.37, imply each other.*

Proof. Corollary 2.4.24 shows that $\mathcal{A}^{\widehat{B}}(\Gamma([n], \widehat{R^b}))$ under TogPro is in equivariant bijection with $\mathrm{FT}(\ell^n, (2, 4, \ldots, 2n))$ under Pro. Furthermore, recall that $\Gamma([n], R^b)$ and $\triangle_n$ are isomorphic as posets by Lemma 2.4.27. As a result, by Proposition 2.2.27, the objects and the actions in these conjectures are equivalent. What remains to be shown is that the rank-alternating label sum statistic $\mathcal{R}$ on $\mathcal{A}^\ell(\triangle_n)$ corresponds to the statistic $\sum |R_O \cap E| - \sum |R_E \cap O|$ on $\mathrm{FT}(\ell^n, (2, 4, \ldots, 2n))$.

Let $T \in \mathrm{FT}(\ell^n, (2, 4, \ldots, 2n))$ and consider an even row, say row $2m$, of T. The allowable entries in the boxes of row $2m$ are $\{2m, 2m + 1, \ldots, 4m\}$. Using the notation of Definition 2.2.13, we can compute the negation of the number of boxes that contain odd entries in row $2m$ as:

$$-(j^{2m}_{4m-1} - j^{2m}_{4m-2}) - (j^{2m}_{4m-3} - j^{2m}_{4m-4}) - \cdots - (j^{2m}_{2m+1} - j^{2m}_{2m}).$$

By (1) from the proof of Lemma 2.2.11, the corresponding calculation on $\sigma = \Phi(T) \in \mathcal{A}^{\widehat{B}}(\Gamma([n], \widehat{R^b}))$ is:

$$\sigma(2m, 4m-1) - \sigma(2m, 4m-2) + \sigma(2m, 4m-3) - \sigma(2m, 4m-2) + \cdots + \sigma(2m, 2m+1) - \sigma(2m, 2m),$$

which is the statistic $\mathcal{R}$ on the diagonal $i = 2m$ in $\triangle_n$.

Now consider an odd row, say row $2m+1$, of T. The allowable entries in the boxes of row $2m+1$ are $\{2m+1, 2m+2, \ldots, 4m+2\}$. We can compute the number of boxes that contain even entries in row $2m+1$ as:

$$(j^{2m+1}_{4m+2} - j^{2m+1}_{4m+1}) + (j^{2m+1}_{4m} - j^{2m+1}_{4m-1}) + \cdots + (j^{2m+1}_{2m+2} - j^{2m+1}_{2m+1}).$$

By (1) from the proof of Lemma 2.2.11, this computation on the corresponding σ is:

$$(\sigma(2m+1, 4m+1) - \sigma(2m+1, 4m+2)) + (\sigma(2m+1, 4m-1) - \sigma(2m+1, 4m)) + \ldots$$
$$+ (\sigma(2m+1, 2m+1) - \sigma(2m+1, 2m+2)).$$

However, by construction we have $\sigma(2m+1, 4m+2) = \widehat{B}(2m+1, 4m+2) = 0$. Thus, we obtain

$$\sigma(2m+1, 4m+1) - \sigma(2m+1, 4m) + \sigma(2m+1, 4m-1) - \sigma(2m+1, 4m-2) + \ldots$$
$$- \sigma(2m+1, 2m+2) + \sigma(2m+1, 2m+1),$$

which is the statistic $\mathcal{R}$ on the diagonal $i = 2m+1$ in $\triangle_n$. As a result, by summing the statistic $\sum |R_O \cap E| - \sum |R_E \cap O|$ over all rows in T, we observe the corresponding statistic is $\mathcal{R}$, summed over all diagonals of the poset $\triangle_n$. $\square$

Another set of flagged tableaux of interest in the literature is that of staircase shape $sc_n = (n, n-1, \ldots, 2, 1)$ with flag $b = (\ell+1, \ell+2, \ldots, \ell+n)$. The Type A case of a result of C. Ceballos, J.-P. Labbé, and C. Stump [10] on multi-cluster complexes along with a bijection of L. Serrano and Stump [47] yields the following result on the order of promotion on these flagged tableaux.

Theorem 2.4.39 ([10, Theorem 8.8], [47, Theorem 4.7])**.** *Let* $b = (\ell+1, \ell+2, \ldots, \ell+n)$*.* Pro *on* $\mathrm{FT}(sc_n, b)$ *is of order dividing* $n+1+2\ell$*.*

The following conjecture is given in terms of flagged tableaux in [47] and in terms of multi-cluster complexes in [10].

Conjecture 2.4.40 ([47, Conjecture 1.7],[10, Open Problem 9.2])**.** *Let* $b = (\ell+1, \ell+2, \ldots, \ell+n)$ *and* $Cat_\ell(x)$ *be as in Conjecture 2.4.31.* $(\mathrm{FT}(sc_n, b), \langle \mathrm{Pro} \rangle, Cat_\ell(x))$ *exhibits the cyclic sieving phenomenon.*

Note this is a set of flagged tableaux with different shape and flag but the same cardinality as the flagged tableaux in Corollary 2.4.28, the same conjectured cyclic sieving polynomial, and a different order of promotion. The case $\ell = 1$ follows from a result of S.P. Eu and T.S. Fu [17] on cyclic sieving of faces of generalized cluster complexes, but for $\ell > 1$ this conjecture is still open.

We can translate this conjecture to rowmotion on Q-partitions with the following corollary of Theorem 2.2.8. Recall Definition 2.3.12, which specifies notation for $[a] \times [b]$.

Corollary 2.4.41. *Let* $b = (\ell+1, \ell+2, \ldots, \ell+n)$*. There is an equivariant bijection between* $\mathrm{FT}(sc_n, b)$ *under* Pro *and* $\mathcal{A}_\epsilon^\delta([n] \times [\ell])$ *under* Row*, where for* $(i,j) \in [n] \times [\ell]$*,* $\delta(i,j) = n$ *and* $\epsilon(i,j) = i-1$*.*

Proof. By Proposition 2.4.23, $\mathrm{FT}(sc_n, b)$ is equivalent to $\mathcal{L}_{[n]\times[n]}(u, v, R^b)$ where $u(p_i) = 0$ and $v(p_i) = i-1$ for all $1 \leq i \leq n$. The restriction function R^b consistent on $[n] \times [n]_u^v$ is given by $R^b(p_i) = \{i, i+1, \ldots \ell+i\}$, and so R^b is also consistent on $[n]$. Now, by Proposition 2.2.26 and Corollary 2.4.24, $\mathrm{FT}(sc_n, b)$ under Pro is equivalent to $\mathcal{A}_\epsilon^\delta(\Gamma([n], R^b))$ under Row where $\delta(p_i, k) = n$ and $\epsilon(p_i, k) = i-1$. Thus what remains to show is that $\Gamma([n], R^b)$ is isomorphic to $[n] \times [\ell]$ as a poset, and, in order to respect the bounds δ and ϵ, for a given i we have $(p_i, k) \in \Gamma([n], R^b)$ in correspondence with $(i,j) \in [n] \times [\ell]$ for some j.

R^b is exactly the restriction function $R^{\ell+n}$ on $[n]$ induced by the global bound $\ell+n$, so, by the map $(p_i, k) \mapsto (p, q-(n-1)+h(p)-k-1)$ from Lemma 2.2.28, $\Gamma([n], R^b)$ is isomorphic to $[n] \times [(\ell+n)-n] = [n] \times [\ell]$ and we have the desired correspondence of elements. Therefore $\mathcal{A}^{\delta}_{\epsilon}(\Gamma([n], R^b))$ is equivalent to $\mathcal{A}^{\delta}_{\epsilon}([n] \times [\ell]))$ where $\delta(i,j) = n$ and $\epsilon(i,j) = i-1$ for all i. $\square$

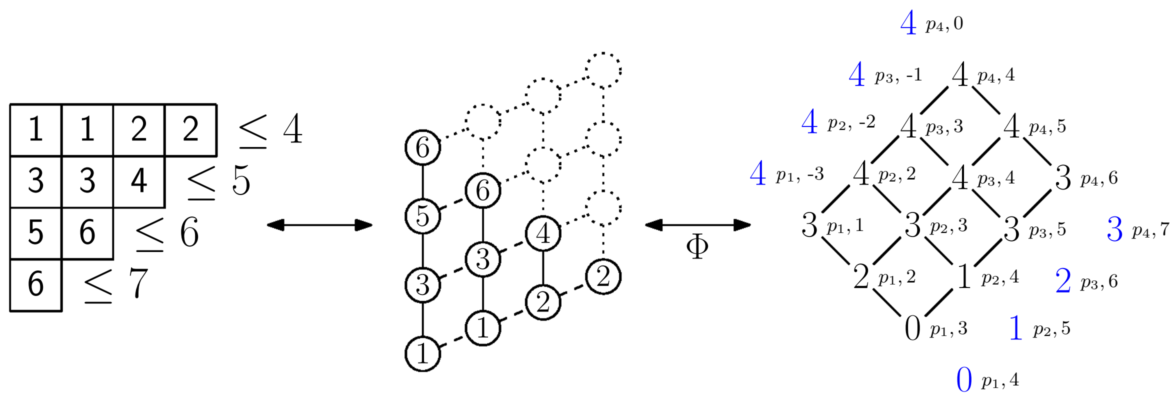


Figure 2.13. On the left is an element of $\mathrm{FT}\,(sc_4, b)$ and, in the center, its equivalent [4]-strict labeling. The corresponding (δ,ϵ)-bounded $[n]\times[\ell]$-partition is shown on the right, using the poset labels of $\Gamma([n], \widehat{R^b})$.

Corollary 2.4.41 implies the equivalence of this conjecture and the following new conjecture.

Conjecture 2.4.42. $\big(\mathcal{A}^{\delta}_{\epsilon}([n] \times [\ell])), \langle \mathrm{Row} \rangle, Cat_{\ell}(x)\big)$ *exhibits the cyclic sieving phenomenon, where* $\delta(i,j) = n$ *and* $\epsilon(i,j) = i-1$ *for all* i*.*

2.4.3. Symplectic tableaux

We begin by defining semistandard symplectic Young tableaux, following the conventions of [9].

Definition 2.4.43. Let $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$ and $\mu = (\mu_1, \mu_2, \ldots, \mu_m)$ be partitions with non-zero parts such that $\mu \subset \lambda$. (Let $\mu_j := 0$ for $j > m$.) A **skew semistandard symplectic (Young) tableau** of shape λ/μ is a filling of λ/μ with entries in $\{1, \overline{1}, 2, \overline{2}, 3, \overline{3}, \ldots\}$ such that the rows increase from left to right and the columns strictly increase from top to bottom, with respect to the ordering $1 < \overline{1} < 2 < \overline{2} < 3 < \overline{3} < \ldots$, and such that the entries in the ith row are greater than or equal to i. Let $\mathrm{Sp}(\lambda/\mu, 2q)$ denote the set of semistandard symplectic tableaux of skew shape λ/μ with entries at most $\bar{q}$.

Proposition 2.4.44. *The set of symplectic tableaux* $\mathrm{Sp}(\lambda/\mu, 2q)$ *is equivalent to* $\mathcal{L}_{[n]\times[\lambda_1]}(u, v, R_a^{2q})$ *where* $a = (1, 3, 5, \ldots, 2n-1)$*,* $u(p_i) = \mu_i$ *and* $v(p_i) = \lambda_1 - \lambda_i$ *for all* $1 \le i \le n$*.*

Proof. Since $\mathcal{L}_{[n]\times[\lambda_1]}(u,v,R_a^{2q}) \subset \mathcal{L}_{[n]\times[\lambda_1]}(u,v,R^{2q})$, by Proposition 2.4.2 we have that $[n]$-strict labelings in $\mathcal{L}_{[n]\times[\lambda_1]}(u,v,R_a^{2q})$ correspond to semistandard Young tableaux whose entries in row i are restricted below by $2i-1$. Then, sending $2k$ to $\overline{k}$ and $2k-1$ to k for each $1 \le k \le q$, we have exactly $\mathrm{Sp}(\lambda/\mu, 2q)$. □

We now specify the $\hat{B}$-bounded $\Gamma(P,\hat{R})$-partitions in bijection with $\mathrm{Sp}(\lambda/\mu, 2q)$. Recall $\hat{B}$ from Definition 2.2.7.

Corollary 2.4.45. *The set* $\mathrm{Sp}(\lambda/\mu, 2q)$ *under* Pro *is in equivariant bijection with* $\mathcal{A}^{\widehat{B}}(\Gamma([n],\widehat{R_a^{2q}}))$ *under* Row*, where* $a = (1,3,5,\ldots,2n-1)$ *and* $\ell = \lambda_1$*,* $u(p_i) = \mu_i$*,* $v(p_i) = \lambda_1 - \lambda_i$ *for all* $1 \le i \le n$*.*

Proof. This follows from Proposition 2.4.44 and Corollary 2.2.24. □

Remark 2.4.46. Symplectic tableaux in the case $\mu = \emptyset$ are enumerated by an analogue of the Jacobi-Trudi formula, due to M. Fulmek and C. Krattenthaler [19]. Thus the bijection of Corollary 2.4.24 allows one to translate this to enumerate $\mathcal{A}^{\widehat{B}}(\Gamma([n],\widehat{R_a^{2q}}))$.

There is also a hook-content formula for symplectic tableaux, due to P. Campbell and A. Stokke [9]. They proved a symplectic Schur function version of this formula, but we will not need that here.

Theorem 2.4.47 ([9, Corollary 4.6])**.** *The cardinality of* $\mathrm{Sp}(\lambda, 2q)$ *is*

$$\prod_{(i,j)\in[\lambda]} \frac{2q + r_\lambda(i,j)}{h_\lambda(i,j)}$$

where $h_\lambda(i,j)$ *is the hook length* $h_\lambda(i,j) = \lambda_i + \lambda_j^t - i - j + 1$ *and* $r_\lambda(i,j)$ *is defined to be*

$$r_\lambda(i,j) = \begin{cases} \lambda_i + \lambda_j - i - j + 2 & \text{if } i > j \\ i + j - \lambda_i^t - \lambda_j^t & \text{if } i \le j \end{cases}$$

We use this formula to enumerate symplectic tableaux of staircase shape, finding a particularly simple formula.

Corollary 2.4.48. *The cardinality of* $\mathrm{Sp}(sc_n, 2n)$ *is* 2^{n^2}*.*

Proof. This follows from Theorem 2.4.47 above. For $\lambda = sc_n = (n, n-1, \dots 1)$, we have $\lambda_i = \lambda_i^t = n - i + 1$. First, we calculate the product of the numerator, where we always take $(i,j) \in [\lambda]$, i.e. $1 \le i \le n$ and $1 \le j \le n - i + 1$.

$$\begin{aligned}\prod_{(i,j)\in[\lambda]} 2n + r_\lambda(i,j) &= \left(\prod_{i>j} 2n + r_\lambda(i,j)\right)\left(\prod_{i\le j} 2n + r_\lambda(i,j)\right)\\ &= \left(\prod_{i>j} 2(2n-i-j+2)\right)\left(\prod_{i\le j} 2(i+j-1)\right)\end{aligned}$$

We now rewrite by considering the products over the columns $j \le \lfloor \frac{n}{2} \rfloor$ or the rows $i \le \lceil \frac{n}{2} \rceil$:

$$\begin{aligned}&= 2^{\binom{n}{2}}\left(\prod_{j\le\lfloor\frac{n}{2}\rfloor}\ \prod_{j<i\le n-j+1} 2n-i-j+2\right)\left(\prod_{i\le\lceil\frac{n}{2}\rceil}\ \prod_{i\le j\le n-i+1} i+j-1\right)\\ &= 2^{\binom{n}{2}}\left(\prod_{j\le\lfloor\frac{n}{2}\rfloor}\frac{(2n-2j+1)!}{n!}\right)\left(\prod_{i\le\lceil\frac{n}{2}\rceil}\frac{n!}{(2i-2)!}\right)\end{aligned}$$

Next, we find the product of the hook lengths, considered over the rows $1 \le i \le n$:

$$\begin{aligned}\prod_{(i,j)\in[\lambda]} h_\lambda(i,j) &= \prod_{1\le i\le n}\ \prod_{1\le j\le n-i+1} 2n-2i-2j+3\\ &= \prod_{1\le i\le n}(2n-2i+1)(2n-2i-1)\cdots 3\cdot 1\\ &= \prod_{1\le i\le n}\frac{(2n-2i+1)!}{2^{n-i}(n-i)!} = \frac{1}{2^{\binom{n-1}{2}}}\prod_{1\le i\le n}\frac{(2n-2i+1)!}{(n-i)!}\end{aligned}$$

Finally,

$$\begin{aligned}&\prod_{(i,j)\in[\lambda]}\frac{2n+r_\lambda(i,j)}{h_\lambda(i,j)}\\ &= 2^{\binom{n}{2}+\binom{n-1}{2}}\left(\prod_{k\le\lfloor\frac{n}{2}\rfloor}\frac{(2n-2k+1)!}{n!}\right)\left(\prod_{k\le\lceil\frac{n}{2}\rceil}\frac{n!}{(2k-2)!}\right)\Bigg/\prod_{1\le k\le n}\frac{(2n-2k+1)!}{(n-k)!}\\ &= 2^{n^2}(n!)^{\lceil\frac{n}{2}\rceil-\lfloor\frac{n}{2}\rfloor}\prod_{1\le k\le n}(n-k)!\Bigg/\left[\left(\prod_{\lfloor\frac{n}{2}\rfloor<k\le n}(2n-2k+1)!\right)\left(\prod_{1\le k\le\lceil\frac{n}{2}\rceil}(2k-2)!\right)\right]\end{aligned}$$

$$= 2^{n^2}(n!)^{\lceil\frac{n}{2}\rceil-\lfloor\frac{n}{2}\rfloor}\prod_{1\le k\le n}(n-k)!\Big/\prod_{1\le k\le\lceil\frac{n}{2}\rceil}(2k-1)!(2k-2)!$$

$$= 2^{n^2}(n!)^{\lceil\frac{n}{2}\rceil-\lfloor\frac{n}{2}\rfloor}\prod_{1\le k\le n}(n-k)!\Big/\left((n!)^{\lceil\frac{n}{2}\rceil-\lfloor\frac{n}{2}\rfloor}\prod_{1\le k\le n}(n-k)!\right) = 2^{n^2}.$$

□

In the rest of this subsection, we apply Corollary 2.4.45 to staircase-shaped symplectic tableaux, obtaining Corollaries 2.4.50 and 2.4.51. This involves the poset in the following definition. This poset is isomorphic to the dual of the *Type B_n positive root poset*. As before, we will not need this algebraic motivation here. See Figure 2.14.

Definition 2.4.49. Let $\diagdown_n$ denote the subposet$[n] \times [2n-1]$ given by $\{(i,j) \mid i \le j, i+j \le 2n\}$.

We obtain the following correspondence, as a corollary of our main results.

Corollary 2.4.50. *There is an equivariant bijection between* $\mathrm{Sp}(sc_n, 2n)$ *under* Pro *and* $\mathcal{A}^{\delta}_{\epsilon}(\diagdown_n)$ *under* Row*, where for* $(i,j) \in \diagdown_n$*,* $\delta(i,j) = \min(j,n)$ *and* $\epsilon(i,j) = i-1$*.*

Proof. Let $a = (1,3,5,\ldots,2n-1)$ and define δ and ϵ as above. Then, by Corollary 2.4.45, $\mathrm{Sp}(sc_n, 2n)$ under Pro is in equivariant bijection with $\mathcal{A}^{\hat{B}}(\Gamma([n],\widehat{R^{2n}_a}))$ under Row with $\ell = n$, $u(p_i) = 0$, $v(p_i) = i-1$ for all $1 \le i \le n$. We show $\mathcal{A}^{\hat{B}}(\Gamma([n],\widehat{R^{2n}_a}))$ is equivalent to $\mathcal{A}^{\delta}_{\epsilon}(\diagdown_n)$.

The restriction function R^{2n}_a consistent on $\mathrm{Sp}(sc_n, 2n)$ is given by $R^{2n}_a(p_i) = \{2i-1,\ldots,2n\}$. Consider the poset structure of the elements of $\Gamma([n],\widehat{R^{2n}_a})$ that are not fixed by $\hat{B}$, that is $\{(p_i,k) \mid k \in R^{2n}_a(p_i)^*\}$. By definition of Γ (as noted in [13, Thm. 2.21]), $(p_i,k_1) \lessdot (p_j,k_2)$ if and only if either $i=j$ and $k_1-1=k_2$ or $i+1=j$ and $k_1+1=k_2$. By the map $(p_i,k) \to (i, 2n-1+i-k)$, the subposet $\Gamma([n],\widehat{R^{2n}_a}) \setminus \mathrm{dom}(\hat{B})$ is a subposet of $[n]\times[2n-1]$, since $1 \le i \le n$, $2i-1 \le k \le 2n-1$ implies $i \le 2n-1+i-k \le 2n-i$, and the above covering relations imply $(i_1, 2n-1+i_1-k_1) \le (i_2, 2n-1+i_2-k_2)$ if and only if $i_1 \le i_2$ or $2n-1+i_1-k_1 \le 2n-1+i_2-k_2$. Moreover, this subposet of $[n]\times[2n-1]$ is exactly $\diagdown_n$, since the range $i \le 2n-1+i-k \le 2n-i$ of the second component satisfies Definition 2.4.49. Therefore, $\hat{B}$-bounded $\Gamma([n],\widehat{R^{2n}_a})$-partitions are exactly elements of $\mathcal{A}^n(\diagdown_n)$ with per-element bounds on the labels induced by the elements fixed by $\hat{B}$.

Finally, we determine these upper and lower bounds on the label of any element $(i, j) \in \nwarrow_n$ by determining the corresponding bounds on the label $\sigma(p_i, k)$ where $\sigma \in \mathcal{A}^{\widehat{B}}(\Gamma([n], \widehat{R_a^{2n}}))$ and $(p_i, k) \in \Gamma([n], \widehat{R_a^{2n}}) \setminus \mathrm{dom}(\hat{B})$. For the elements $(p_i, \min \widehat{R_a^{2n}}(p_i)^*)$ that are fixed, we have $\hat{B}(p_i, \min \widehat{R_a^{2n}}(p_i)^*) = n - u(p_i) = n$, so these elements induce an upper bound of n on all $\sigma(p_i, k)$. Next, the fixed elements $(p_i, \max \widehat{R_a^{2n}}(p_i)^*) = (p_i, 2n)$ induce a lower bound $v(p_i) = i - 1$ on all $\sigma(p_i, k)$ and an equivalent upper bound on $\sigma(p_{i'}, k')$, where $(p_{i'}, k') < (p_i, 2n)$, which is the case whenever $i' < i$ and $k' \geq 2n - (i - i')$. Therefore, a generic $\sigma(p_i, k)$ is bounded below by $i - 1$ and above by at most n and, if $k = 2n - (i' - i)$ for any $i < i' \leq n$, then $\sigma(p_i, k)$ is bounded above by $i' - 1$. Translating to $\mathcal{A}^n(\nwarrow_n)$, $\sigma(i, j) = \sigma(p_i, 2n - 1 + i - j)$ (we keep the notation σ due to the equivalence shown above) so $\sigma(i, j)$ is bounded below by $i - 1$ and above by at most n. We have $2n - 1 + i - j = 2n - (j + 1 - i)$, so $\sigma(i, j)$ is bounded above by j for $1 \leq j \leq n - 1$. Thus, if $\delta(i, j) = \min(j, n)$ and $\epsilon(i, j) = i - 1$, then $\mathcal{A}^{\widehat{B}}(\Gamma([n], \widehat{R_a^{2n}}))$ is equivalent to $\mathcal{A}_\epsilon^\delta(\nwarrow_n)$. □

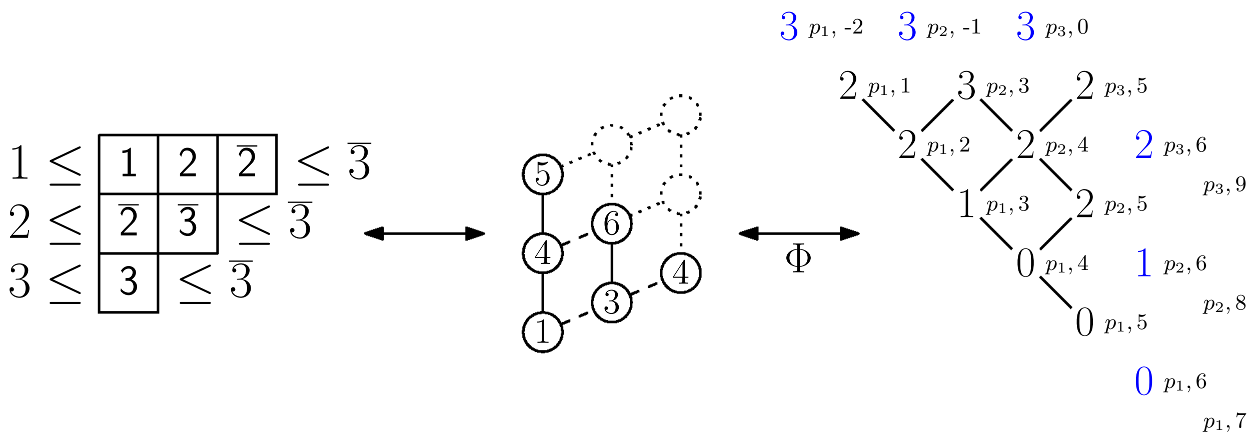


Figure 2.14. On the left is an element of $\mathrm{Sp}(sc_3, 6)$ (with entries in $\{1, \overline{1}, 2, \overline{2}, 3, \overline{3}\}$), and in the center is the equivalent [3]-strict labeling (with labels in $\{1, 2, 3, 4, 5, 6\}$). The corresponding (δ, ϵ)-bounded $\nwarrow_n$-partition is given on the right, shown as the equivalent element of $\mathcal{A}^{\widehat{B}}(\Gamma([3], \widehat{R_a^6}))$. Here, the poset element $(p_1, 5) \in \Gamma([3], \widehat{R_a^6})$ corresponds to $(1, 1) \in \nwarrow_n$, $(p_1, 4)$ corresponds to $(1, 2)$, and so on.

The corollary below follows directly from Corollaries 2.4.48 and 2.4.50.

Corollary 2.4.51. *The cardinality of* $\mathcal{A}_\epsilon^\delta(\nwarrow_n)$ *with* $\delta(i, j) = \min(j, n)$ *and* $\epsilon(i, j) = i - 1$ *is* 2^{n^2}.

It would be interesting to see whether one can find a set of symplectic tableaux that exhibit the cyclic sieving phenomenon with respect to promotion. A nice counting formula is generally a necessary first step.

3. THE GRADED CASE[1]

3.1. Introduction

Semistandard Young tableaux and *Gelfand–Tsetlin patterns* are well-loved combinatorial objects with a nice, statistic-preserving bijection between them. In [5], we generalized this correspondence to objects we called *P-strict labelings of* $P \times [\ell]$ (analogous to semistandard tableaux) for a finite poset P and $\ell \in \mathbb{N}$, and *B-bounded Q-partitions* (analogous to Gelfand–Tsetlin patterns) for related posets Q and bounding function B. In addition, we showed this bijection is *equivariant*, mapping the well-studied *promotion* action on tableaux to a *piecewise-linear toggle group action* on Q-partitions, which under a certain condition is equivalent to *rowmotion*, an important action in dynamical algebraic combinatorics.

Informally, P-strict labelings of $P \times [\ell]$ are labelings of $P \times [\ell]$ with positive integers that strictly increase on each copy of P and weakly increase along each copy of $[\ell]$. Additional parameters include a *restriction function* R, that specifies which labels are allowed in the P-strict labeling, as well as functional parameters u and v. The case $P = [n]$ corresponds to (skew) semistandard Young tableaux of n rows, where ℓ is the number of tableau columns, and u and v determine the shape of the skew tableau by specifying which partitions to remove from the upper left and lower right of the $n \times \ell$ bounding rectangle (the case of $u = v = 0$ corresponds to rectangular tableaux). While the bijection of [5] allows for all of these parameters in full generality, our applications of the general formula in that paper were in the case $P = [n]$. These included interesting results and conjectures on flagged and symplectic tableaux.

In this paper, we apply the results of [5] to cases of interest beyond $P = [n]$, with an eye toward translating objects with known dynamical behavior through our bijection in order to obtain new results and new perspectives on conjectures. Along with periodicity, we aim to describe the behavior of promotion and rowmotion through the notions of homomesy and resonance. The *homomesy phenomenon*, first described in [40], is where the average value of a given combinatorial

[1]The material in this chapter was coauthored by J. Bernstein, J. Striker, and C. Vorland. The preprint can be found at `https://arxiv.org/abs/2205.04938`. The coauthors worked collaboratively on the main results of the paper. Bernstein had primary responsibility for reframing the main theorems of Chapter 2 as well as for Subsections 3.4.2 and 3.4.3. Subsections of Section 3.3 which Vorland composed mostly independently have been omitted (subsections 3.3, 3.4, and 3.5 in the preprint). All coauthors revised and proofread this chapter.

statistic is the same across all orbits. For an overview of this phenomenon, including many examples, see the survey of T. Roby [43]. *Resonance*, defined in [12], is a way to capture the structure of orbits without requiring predictable periodicity, by projecting a more complicated object and action to a simpler set that demonstrates cyclic behavior. The article [53] contains a more detailed description of resonance, as well as a general survey of dynamical algebraic combinatorics, including homomesy.

This paper is organized as follows. In Section 3.2, we give relevant background definitions and theorems from [5] at the level of generality necessary for this paper ($u = v = 0$) and state Corollaries 3.2.25 and 3.2.26, which are consequences of these general results when P is graded. In Section 3.3, we apply these results to the case where P is a product of chains, obtaining results on symmetry. Section 3.4 considers the dynamics of P-strict labelings in cases where either P is graded but not equal to a product of chains or the labels of $P \times [\ell]$ are restricted by *flags* instead of by a global bound. Finally, in Section 3.5, we prove Theorem 3.5.4, a resonance result on general P-strict labelings with global bound q.

3.2. Background and general results

In this section, we give relevant background and general results from [5]. In Subsections 3.2.1-3.2.3, we state definitions and theorems in the slightly less general case of $u = v = 0$. Then in Subsection 3.2.4, we restrict our attention to the case where P is graded and state the consequences of these theorems that will be of use in the remainder of the paper. For the definitions and theorems in full generality, see [5].

3.2.1. Promotion on P-strict labelings

The definitions in this section are adapted from [5, §1.2]. We begin with the following preliminary definitions.

Definition 3.2.1. In this paper, P and Q represent a finite posets with partial orders $\leq_P$ and $\leq_Q$, respectively. Also, $\lessdot$ indicates a covering relation in a poset, $J(P)$ is the poset of order ideals of P ordered by containment, ℓ and q are positive integers, $[\ell]$ denotes a chain poset of ℓ elements, and $P \times [\ell] = \{(p, i) \mid p \in P, i \in \mathbb{N}, \text{ and } 1 \leq i \leq \ell\}$. $\mathcal{P}(\mathbb{Z})$ represents the set of all nonempty, finite subsets of $\mathbb{Z}$.

Let L_i represent the ith **layer** of $P \times [\ell]$, that is, the set of $(p, i) \in P \times [\ell]$ where p ranges over all possible values and i is fixed. Let F_p denote the pth **fiber** of $P \times [\ell]$, that is, the set of $(p, i) \in P \times [\ell]$ where i ranges over all possible values and p is fixed.

Definition 3.2.2. We say that a function $f : P \times [\ell] \to \mathbb{Z}$ is a **P-strict labeling of $P \times [\ell]$ with restriction function** $R : P \to \mathcal{P}(\mathbb{Z})$ if f satisfies the following:

1. $f(p_1, i) < f(p_2, i)$ whenever $p_1 <_P p_2$,
2. $f(p, i_1) \leq f(p, i_2)$ whenever $i_1 \leq i_2$,
3. $f(p, i) \in R(p)$.

That is, f is strictly increasing inside each copy of P (layer), weakly increasing along each copy of the chain $[\ell]$ (fiber), and such that the labels come from the restriction function R. Let $\mathcal{L}_{P\times[\ell]}(R)$ denote the set of all P-strict labelings on $P \times [\ell]$ with restriction function R.

Definition 3.2.3. A restriction function R is **consistent** with respect to $P \times [\ell]$ if, for all $p \in P$ and $k \in R(p)$, there exists some P-strict labeling f of $P \times [\ell]$ with $f(p, i) = k$, $1 \leq i \leq \ell$.

Due to the strictly increasing condition on layers, a generic $R : P \to \mathcal{P}(\mathbb{Z})$ may fail to be consistent. This occurs when a set $R(p)$ contains a label that can not be attained on the pth fiber by any P-strict labeling.

We denote the consistent restriction function induced by (either global or local) upper and lower bounds as R_α^β, where $\alpha, \beta : P \to \mathbb{Z}$. In the case of a global upper bound q, our restriction function will be R_1^q, that is, we take α to be the constant function 1 and β to be the constant function q. Since a lower bound of 1 is used frequently, we suppress the subscript 1; that is, if no subscript appears, we take it to be 1.

Example 3.2.4. The objects on the left half of Figure 3.1 are elements of $\mathcal{L}_{P\times[4]}(R^6)$, where P is the X-shaped poset shown in the center. The labels on each of the four layers are strictly increasing (from 1 up to the global maximum $q = 6$) and are connected by solid lines, while the labels along each of the five fibers are weakly increasing and connected by dotted lines.

Definition 3.2.5. Let $R(p)_{>k}$ denote the smallest label of $R(p)$ that is larger than k, and let $R(p)_{<k}$ denote the largest label of $R(p)$ less than k.

Say that a label $f(p,i)$ in a P-strict labeling $f \in \mathcal{L}_{P\times[\ell]}(R)$ is **raisable (lowerable)** if there exists another P-strict labeling $g \in \mathcal{L}_{P\times[\ell]}(R)$ where $f(p,i) < g(p,i)$ ($f(p,i) > g(p,i)$), and $f(p',i') = g(p',i')$ for all $(p',i') \in P \times [\ell]$, $p' \neq p$.

Definition 3.2.6. Let the action of the k**th Bender-Knuth involution** ρ_k on a P-strict labeling $f \in \mathcal{L}_{P\times[\ell]}(R)$ be as follows: identify all raisable labels $f(p,i) = k$ and all lowerable labels $f(p,i) = R(p)_{>k}$. Call these labels 'free'. Suppose the labels $f(F_p)$ include a free k labels followed by b free $R(p)_{>k}$ labels; ρ_k changes these labels to b copies of k followed by a copies of $R(p)_{>k}$. **Promotion** on P-strict labelings is defined as the composition of these involutions: $\text{Pro}(f) = \cdots\circ\rho_3\circ\rho_2\circ\rho_1\circ\cdots(f)$. Note that since R induces upper and lower bounds on the labels, only a finite number of Bender-Knuth involutions act nontrivially.

3.2.2. Rowmotion on Q-partitions

The definitions in this section are adapted from [5, §1.3].

Definition 3.2.7. A Q**-partition** is a map $\sigma : Q \to \mathbb{N}_{\geq 0}$ such that if $x \leq_Q x'$, then $\sigma(x) \leq \sigma(x')$. Let $\hat{Q}$ denote Q with $\hat{0}$ added below all elements and $\hat{1}$ added above all elements. Let $\mathcal{A}^\ell(Q)$ denote the set of all $\hat{Q}$-partitions σ with $\sigma(\hat{0}) = 0$ and $\sigma(\hat{1}) = \ell$.

Note we use Q for a generic poset in this context rather than P to avoid confusion when we later relate these objects to the objects of the previous subsection.

Example 3.2.8. The objects on the right half of Figure 3.1 are elements of $\mathcal{A}^4(P \times [3])$, where P is the X-shaped poset shown in the center. In our visualizations, we omit the elements $\hat{0}$ and $\hat{1}$.

Remark 3.2.9. When $\ell = 1$, $\mathcal{A}^\ell(Q) = J(Q)$, the set of order ideals (or lower sets) of Q. The set of order ideals forms a *distributive lattice* ordered by containment. This is the setting in which rowmotion and toggles were originally studied [8, 54].

In Definitions 3.2.10 and 3.2.12 below, we define toggles and rowmotion in what is often called the *piecewise-linear* context. These definitions are equivalent (by rescaling) to those first given by Einstein and Propp on the order polytope [15, 16].

Definition 3.2.10. For $\sigma \in \mathcal{A}^\ell(Q)$ and $x \in Q$, let $\nabla_\sigma(x) = \min\{\sigma(y) \mid y \in \hat{Q} \text{ covers } x\}$ and $\Delta_\sigma(x) = \max\{\sigma(y) \mid y \in \hat{Q} \text{ is covered by } x\}$. Define the **toggle**, $\tau_x : \mathcal{A}^\ell(Q) \to \mathcal{A}^\ell(Q)$ by

$$\tau_x(\sigma)(x') := \begin{cases} \sigma(x') & x \neq x' \\ \nabla_\sigma(x') + \Delta_\sigma(x') - \sigma(x') & x = x'. \end{cases}$$

Remark 3.2.11. By the same reasoning as in the case of combinatorial toggles, the τ_x satisfy:

1. $\tau_x^2 = 1$, and
2. τ_x and $\tau_{x'}$ commute whenever x and x' do not share a covering relation.

Definition 3.2.12. Rowmotion on $\mathcal{A}^\ell(Q)$ is defined as the toggle composition $\text{Row} := \tau_{x_1} \circ \tau_{x_2} \circ \cdots \circ \tau_{x_m}$ where $x_1, x_2, \ldots, x_m$ is any linear extension of Q.

3.2.3. P-strict promotion and $\Gamma(P, R)$ rowmotion

In this subsection, we give the main results of [5] in slightly less generality (the case where $u = v = 0$), which is all we need for this paper. Since the statements do not match exactly the statements in [5], we give the specific references of each statement from [5] for comparison. In the next subsection, we further specialize these results to the case where P is graded.

Definition 3.2.13 ([13, Definition 2.10])**.** For $p \in P$, let $R(p)^*$ denote $R(p)$ with its largest element removed.

Definition 3.2.14 ([13, Definition 2.11])**.** Let $R : P \to \mathcal{P}(\mathbb{Z})$ be a consistent map of possible labels. Then, define the **gamma poset** $\Gamma(P, R)$ to be the poset whose elements are (p, k) with $p \in P$ and $k \in R(p)^*$, and covering relations given by $(p_1, k_1) \lessdot (p_2, k_2)$ if and only if either

1. $p_1 = p_2$ and $R(p_1)_{>k_2} = k_1$ (i.e., k_1 is the next largest possible label after k_2), or
2. $p_1 \lessdot p_2$ (in P), $k_1 = R(p_1)_{<k_2} \neq \max(R(p_1))$, and no greater k in $R(p_2)$ has $k_1 = R(p_1)_{<k}$. That is, k_1 is the largest label of $R(p_1)$ less than k_2 ($k_1 \neq \max(R(p_1))$), and there is no greater $k \in R(p_2)$ having k_1 as the largest label of $R(p_1)$ less than k.

Definition 3.2.15 ([5, Definition 2.6])**. Toggle-promotion** on $\mathcal{A}^\ell(\Gamma(P, R))$ is defined as the toggle composition $\text{TogPro} := \cdots \circ \tau_2 \circ \tau_1 \circ \tau_0 \circ \tau_{-1} \circ \tau_{-2} \circ \cdots$, where τ_k denotes the composition of all the $\tau_{(p,k)}$ over all $p \in P$, $(p, k) \in \Gamma(P, R)$.

This composition is well-defined since the toggles within each τ_k commute by Remark 3.2.11.

The first main result of [5], which we state below as Theorem 3.2.17, gives an equivariant bijection between P-strict labelings and certain Q-partitions, via the map of Definition 3.2.16.

Definition 3.2.16 ([5, Definition 2.9])**.** We define the map $\Phi : \mathcal{L}_{P\times[\ell]}(R) \to \mathcal{A}^{\ell}(\Gamma(P,R))$ as the composition of two intermediate maps ϕ_2 and ϕ_3. Start with a P-strict labeling $f \in \mathcal{L}_{P\times[\ell]}(R)$. First, ϕ_2 sends f to the multichain $\mathcal{O}_\ell \leq \mathcal{O}_{\ell-1} \leq \cdots \leq \mathcal{O}_1$ in $J(\Gamma(P,R))$ where, for $1 \leq i \leq \ell$ and L_i the ith layer of $P \times [\ell]$, ϕ_2 sends $f(L_i)$ to its associated order ideal $\mathcal{O}_i \in J(\Gamma(P,R))$ via the bijection of [13, Theorem 2.14]. Then, ϕ_3 maps the above multichain to a $\Gamma(P,R)$-partition σ as seen in [49, p. 11], where $\sigma(p,k) = \#\{i \mid (p,k) \notin \mathcal{O}_i\}$, the number of order ideals not including (p,k). Let $\Phi = \phi_3 \circ \phi_2$.

Theorem 3.2.17 ([5, Theorem 2.8])**.** *The set* $\mathcal{L}_{P\times[\ell]}(R)$ *under* Pro *is in equivariant bijection with* $\mathcal{A}^{\ell}(\Gamma(P,R))$ *under* TogPro*. More specifically, for* $f \in \mathcal{L}_{P\times[\ell]}(R)$*,* $\Phi(\mathrm{Pro}(f)) = \mathrm{TogPro}(\Phi(f))$*.*

The equivalence is not only of the actions Pro and TogPro, but is proved by corresponding Bender-Knuth involutions with toggles.

Lemma 3.2.18 ([5, Lemma 2.11])**.** *The bijection map* Φ *equivariantly takes the generalized Bender-Knuth involution* ρ_k *to the toggle operator* τ_k*.*

The second main result of [5], Theorem 3.2.20, involves rowmotion and requires the following definition.

Definition 3.2.19 ([5, Definition 2.19])**.** We say that $\mathcal{A}^{\ell}(\Gamma(P,R))$ is **column-adjacent** if whenever $(p_1,k_1) \lessdot (p_2,k_2)$ in $\Gamma(P,R)$, then $|k_2 - k_1| = 1$.

Theorem 3.2.20 ([5, Theorem 2.20])**.** *If* $\mathcal{A}^{\ell}(\Gamma(P,R))$ *is column-adjacent, then* $\mathcal{A}^{\ell}(\Gamma(P,R))$ *under* Row *is in equivariant bijection with* $\mathcal{L}_{P\times[\ell]}(R)$ *under* Pro*.*

While the following proposition in full generality is proved in [5], for the specific case given below (where $u = v = 0$), it is also implicit in [13]. See [5, Remark 2.25] for a more detailed description of this difference.

Proposition 3.2.21 ([5, Proposition 2.22])**.** $\mathcal{A}^{\ell}(\Gamma(P,R_\alpha^\beta))$ *is column-adjacent.*

This proposition, combined with Theorem 3.2.20, yields the following corollary.

Corollary 3.2.22 ([5, Corollary 2.24])**.** *$\mathcal{A}^\ell(\Gamma(P, R_\alpha^\beta))$ under* Row *is in equivariant bijection with $\mathcal{L}_{P\times[\ell]}(R_\alpha^\beta)$ under* Pro*.*

3.2.4. The graded case: P-strict promotion and $P \times [q-n-1]$ rowmotion

In the rest of the paper (with the exception of Section 3.5), we concentrate on the case where P is a **graded** poset of **rank** n, meaning all maximal chains of P have $n+1$ elements. This condition simplifies matters significantly; in particular, it determines $\Gamma(P, R^q)$, the gamma poset with restriction function induced by a global bound q. While Lemma 3.2.23 and Corollary 3.2.26 were stated and proved in [5], the other results of the subsection are new statements that follow from the general theorems of [5]. These will be easier to apply to our cases of interest.

Lemma 3.2.23 ([5, Lemma 2.28])**.** *Let P be a graded poset of rank n. Then $\Gamma(P, R^q)$ is isomorphic to $P \times [q-n-1]$ as a poset.*

Remark 3.2.24. The proof of the above uses the bijection from $\Gamma(P, R^q)$ to $P \times [q-n-1]$ that sends (p, k) to $(p, q-n+\mathrm{rank}(p)-k)$, where the rank of a minimal poset element is zero.

We now state the versions of our main theorems we will use.

Corollary 3.2.25. *Let P be a graded poset of rank n. Then $\mathcal{L}_{P\times[\ell]}(R^q)$ under* Pro *is in equivariant bijection with $\mathcal{A}^\ell(P \times [q-n-1])$ under* TogPro*. More specifically, for $f \in \mathcal{L}_{P\times[\ell]}(R)$, $\Phi(\mathrm{Pro}(f)) = \mathrm{TogPro}(\Phi(f))$.*

Proof. This follows directly from Theorem 3.2.17 and Lemma 3.2.23. □

Corollary 3.2.26 ([5, Corollary 2.29])**.** *Let P be a graded poset of rank n. Then $\mathcal{L}_{P\times[\ell]}(R^q)$ under* Pro *is in equivariant bijection with $\mathcal{A}^\ell(P \times [q-n-1])$ under* Row*.*

Proof. This follows directly from Theorem 3.2.20 and Lemma 3.2.23. □

Figure 3.1 showcases an example of the bijection from Corollary 3.2.25.

Remark 3.2.27. In the case $\ell = 1$, the bijection degenerates to a correspondence from [13] between order ideals $\mathcal{A}^1(P \times [a]) = J(P \times [a])$ and $\mathcal{L}_{P\times[1]}(R^{a+n+1})$, which has fibers of size 1 and is thus equivalent to the set of strictly increasing labelings on P with labels between 1 and $a+n+1$. When

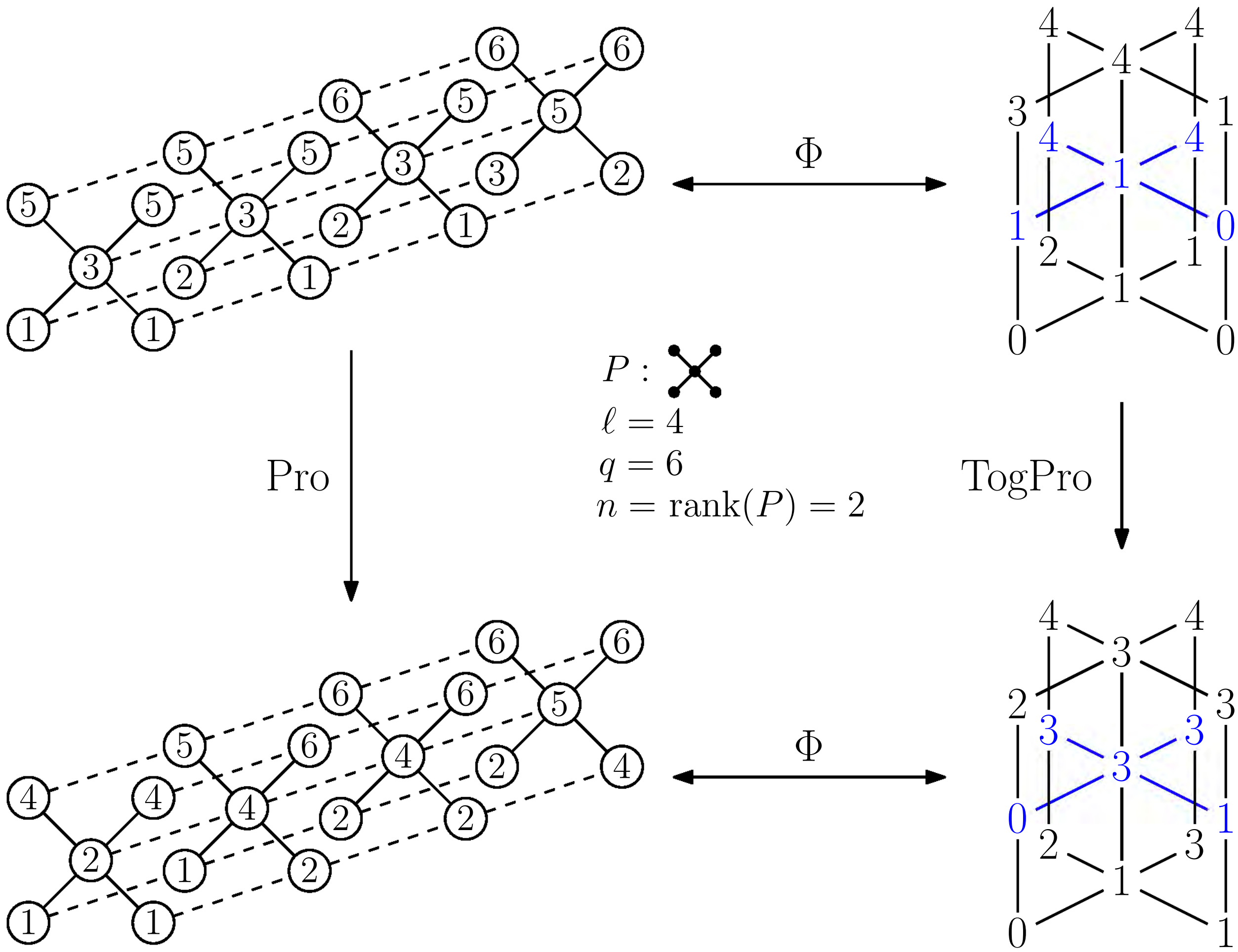


Figure 3.1. An illustration of Corollary 3.2.25. Promotion on the P-strict labeling in the upper left corresponds to toggle-promotion on the $\Gamma(P, R^q)$-partition in the upper right. The poset P of rank n along with the parameters ℓ and q are shown in the center. Since P is ranked, $\Gamma(P, R^q)$ is isomorphic to $P \times [q-n-1] = P \times [3]$. The blue elements of $P \times [3]$ are the elements of the subposet $P \times \{2\}$ described in the statement of Proposition 3.2.29.

P is graded, this is equivalent (by subtracting i from the labels in rank i) to the correspondence of [51, Prop. 3.5.1] between order ideals and order-preserving maps. We will mainly focus on the case $\ell > 1$.

The characterization of the gamma poset in Lemma 3.2.23 allows us to give an explicit description of the toggles in TogPro without reference to the gamma poset labeling.

Lemma 3.2.28. *Let P be a graded poset of rank n. Then* TogPro *on $\Gamma(P, R^q)$ is equivalent to the toggle composition* TogPro $:= \tau_{q-1} \circ \tau_{q-2} \circ \cdots \circ \tau_2 \circ \tau_1$ *on $\mathcal{A}^\ell(P \times [q-n-1])$, where τ_k denotes the composition of all the $\tau_{(p,i)}$ over all $p \in P$, where $i = q - n + rank(p) - k$.*

Proof. Apply the bijection from Remark 3.2.24 to Definition 3.2.15. ☐

The following gives an equivalent toggle order for TogPro, which will be helpful in Corollary 3.3.4.

Proposition 3.2.29. *Let P be a graded poset of rank n. Denote by $P \times \{j\}$ the subposet of $P \times [q-n-1]$ consisting of all elements of the form (p, j), $p \in P$. Then* TogPro *on $\Gamma(P, R^q)$ is equivalent to* $\mathrm{Row}^{-1}(P\times\{1\})\circ\mathrm{Row}^{-1}(P\times\{2\})\circ\cdots\circ\mathrm{Row}^{-1}(P\times\{q-n-2\})\circ\mathrm{Row}^{-1}(P\times\{q-n-1\})$.

Proof. Let P_m denote the set of elements of P with rank m and, for some $j \in [q-n-1]$, let $\tau_{(P_m,j)}$ be the composition of all the toggles $\tau_{(p,j)}$ where $p \in P_m$. Then $\mathrm{Row}^{-1}(P \times \{j\})$ is given by the toggle composition $\tau_{(P_n,j)} \circ \tau_{(P_{n-1},j)} \circ \cdots \circ \tau_{(P_1,j)} \circ \tau_{(P_0,j)}$, and we have that $\tau_{(P_m,j)}$ commutes with $\tau_{(P_{m'},j')}$ whenever $|m - m'| + |j - j'| > 1$.

Now, the τ_k in Lemma 3.2.28 are given by

$$\begin{aligned}
\tau_1 &= \tau_{(P_0,q-n-1)} \\
\tau_2 &= \tau_{(P_1,q-n-1)} \circ \tau_{(P_0,q-n-2)} \\
\tau_3 &= \tau_{(P_2,q-n-1)} \circ \tau_{(P_1,q-n-2)} \circ \tau_{(P_0,q-n-3)} \\
&\vdots \\
\tau_n &= \tau_{(P_{n-1},q-n-1)} \circ \tau_{(P_{n-2},q-n-2)} \circ \cdots \circ \tau_{(P_0,q-2n)} \\
\tau_{n+1} &= \tau_{(P_n,q-n-1)} \circ \tau_{(P_{n-1},q-n-2)} \circ \cdots \circ \tau_{(P_0,q-2n-1)} \\
\tau_{n+2} &= \tau_{(P_n,q-n-2)} \circ \tau_{(P_{n-1},q-n-3)} \circ \cdots \circ \tau_{(P_0,q-2n-2)} \\
\tau_{n+3} &= \tau_{(P_n,q-n-3)} \circ \tau_{(P_{n-1},q-n-4)} \circ \cdots \circ \tau_{(P_0,q-2n-3)} \\
&\vdots \\
\tau_{q-3} &= \tau_{(P_n,3)} \circ \tau_{(P_{n-1},2)} \circ \tau_{(P_{n-2},1)} \\
\tau_{q-2} &= \tau_{(P_n,2)} \circ \tau_{(P_{n-1},1)} \\
\tau_{q-1} &= \tau_{(P_n,1)},
\end{aligned}$$

so we can rewrite TogPro in terms of the $\tau_{(P_m,i)}$. After doing so, we note that, for all $m > 0$, $\tau_{(P_m,q-n-1)}$ commutes with everything to its right except for $\tau_{(P_{m-1},q-n-1)}$. Therefore, we can reorder TogPro such that all of the $\tau_{(P_m,q-n-1)}$ are on the right as follows:

$$\begin{aligned}\text{TogPro} =&\tau_{(P_n,1)} \circ \cdots \circ \tau_{(P_0,q-n-2)} \circ \tau_{(P_n,q-n-1)} \circ \tau_{(P_{n-1},q-n-1)} \circ \cdots \circ \tau_{(P_1,q-n-1)} \circ \tau_{(P_0,q-n-1)} \\ =&\tau_{(P_n,1)} \circ \cdots \circ \tau_{(P_0,q-n-2)} \circ \text{Row}^{-1}(P \times \{q-n-1\}).\end{aligned}$$

Similarly, $\tau_{(P_m,q-n-2)}$ commutes with everything to its right except for $\tau_{(P_{m-1},q-n-2)}$ and $\text{Row}^{-1}(P \times \{q-n-1\})$, so

$$\text{TogPro} = \tau_{(P_n,1)} \circ \cdots \circ \tau_{(P_0,q-n-3)} \circ \text{Row}^{-1}(P \times \{q-n-2\}) \circ \text{Row}^{-1}(P \times \{q-n-1\}),$$

and so on until we have

$$\text{TogPro} = \text{Row}^{-1}(P\times\{1\})\circ\text{Row}^{-1}(P\times\{2\})\circ\cdots\circ\text{Row}^{-1}(P\times\{q-n-2\})\circ\text{Row}^{-1}(P\times\{q-n-1\})$$

as desired. □

3.3. Products of chains and multifold symmetry

Our motivation in this section is to explore the bijection of Corollary 3.2.25 further when P is a product of chains. In Subsection 3.3.1, we begin with Corollaries 3.3.4 and 3.3.6, which specialize Corollary 3.2.25 when P is a product of chains and states TogPro in terms of the hyperplane toggle definition of [12]. In the rest of the section, we use symmetry to prove equivalences of P-strict labelings and apply the resulting bijections to obtain order and homomesy results on the P-strict labelings $\mathcal{L}_{([a]\times[b])\times[\ell]}(R^{a+b})$. Specifically, in Subsection 3.3.2, we give multifold symmetry results on P-strict labelings in Theorems 3.3.7 and 3.3.8.

3.3.1. Application of main result to $P = [a] \times [b]$

Using the hyperplane toggle definition of [12], we can determine which hyperplane sweep on the product of chains poset corresponds to TogPro.

Definition 3.3.1. [12, Definition 3.13] We say that an *n-dimensional lattice projection* of a ranked poset P is an order and rank preserving map $\pi : P \to \mathbb{Z}^n$, where the rank function on $\mathbb{Z}^n$ is the sum of the coordinates and $x \leq y$ in $\mathbb{Z}^n$ if and only if the componentwise difference $y - x$ is in $(\mathbb{Z}_{\geq 0})^n$.

Definition 3.3.2. [12, Definition 3.14] Let Q be a poset with an n-dimensional lattice projection π and let $v \in \{\pm 1\}^n$. Let $T^i_{\pi,v}$ be the product of toggles t_x for all elements x of Q that lie on the affine hyperplane $\langle \pi(x), v\rangle = i$. If there is no such x, then this is the empty product, considered to be the identity. Define *promotion with respect to* π *and* v as the (finite) toggle product $\text{Pro}_{\pi,v} = \ldots T^{-2}_{\pi,v} T^{-1}_{\pi,v} T^{0}_{\pi,v} T^{1}_{\pi,v} T^{2}_{\pi,v} \ldots$.

Remark 3.3.3. For a Q-partition $\mathcal{A}^\ell([a_1] \times \cdots \times [a_k])$, we will frequently use the identity map for π; the identity map will be denoted with id. For Q-partitions obtained from a Γ poset construction, the last coordinate decreases as we traverse up the poset. As a result, we use a non-identity map π to compensate for this labeling. See Corollaries 3.3.4 and 3.3.6 for instances of this.

Corollary 3.3.4. $\mathcal{L}_{([a]\times[b])\times[\ell]}(R^{a+b+c-1})$ *under* Pro *is in equivariant bijection with* $\mathcal{A}^\ell([a]\times[b]\times[c])$ *under* Row*. More specifically, for* $f \in \mathcal{L}_{([a]\times[b])\times[\ell]}(R^{a+b+c-1})$, $\Phi(\text{Pro}(f)) = \text{Pro}_{\pi,(-1,-1,1)}(\Phi(f))$ *where* $\pi((i,j),k) = (i,j,i+j-k+c-1)$.

Proof. By setting $P = [a] \times [b]$ and $q = a+b+c-1$, the existence of an equivariant bijection follows directly from Corollary 3.2.25. To show that $\Phi(\text{Pro}(f)) = \text{Pro}_{\pi,(-1,-1,1)}(\Phi(f))$, we will show that TogPro and $\text{Pro}_{\pi,(-1,-1,1)}$ have the same toggle order on $\Gamma(P, R^q)$. By Theorem 23 of [55], the toggles of $\text{Pro}_{\pi,(-1,-1,1)}$ can be reordered to $\text{Row}^{-1}([a]\times[b]\times\{1\}) \circ \text{Row}^{-1}([a]\times[b]\times\{2\}) \circ \cdots \circ \text{Row}^{-1}([a]\times[b]\times\{c-1\}) \circ \text{Row}^{-1}([a]\times[b]\times\{c\})$. By Proposition 3.2.29, this is equivalent to TogPro. It follows that $\Phi(\text{Pro}(f)) = \text{Pro}_{\pi,(-1,-1,1)}(\Phi(f))$ from Corollary 3.2.25. □

Remark 3.3.5. In [12], K. Dilks, O. Pechenik, and Striker showed that if I is an order ideal of $[a]\times[b]\times[c]$, the action $\text{Pro}_{(1,1,-1)}$ on I corresponds to K-promotion on a corresponding increasing tableaux. The reason $\text{Pro}_{(1,1,-1)}$ is used in their result is because of labeling conventions: an element in an order ideal is labeled 1 and an element not in an order ideal is labeled 0. With Q-partitions, we are using the reverse convention, which is the reason for $\text{Pro}_{\pi,(-1,-1,1)}$ appearing in Corollary 3.3.4.

The following corollary extends Corollary 3.3.4 to products of more than two chains. Its proof is nearly identical to the proof of Corollary 3.3.4, so we omit it here. By setting $q = \sum_i^k a_i - 1$, the existence of an equivariant bijection follows directly from Corollary 3.2.25.

Corollary 3.3.6. *Let* $P = [a_1] \times [a_2] \times \cdots \times [a_{k-1}]$*. Then* $\mathcal{L}_{P\times[\ell]}(R^{\sum_i^k a_i - 1})$ *under* Pro *is in equivariant bijection with* $\mathcal{A}^\ell(P \times [a_k])$ *under* Row*. More specifically, for* $f \in \mathcal{L}_{P\times[\ell]}(R^{\sum_i^k a_i - 1})$*,* $\Phi\left(\mathrm{Pro}(f)\right) = \mathrm{Pro}_{(-1,\ldots,-1,1)}\left(\Phi(f)\right)$ *where* $\pi((i_1, i_2, \ldots, i_{k-1}), i_k) = (i_1, i_2, \ldots, i_{k-1}, i_1 + \cdots + i_{k-1} - i_k + a_k - 1)$*.*

3.3.2. Multifold symmetry

In this subsection, we use the product of chains results of Corollaries 3.3.4 and 3.3.6 to show P-strict trifold symmetry and P-strict multifold symmetry in Theorems 3.3.7 and 3.3.8, respectively. We also state a similar, more general equivalence in Theorem 3.3.9.

Theorem 3.3.7. *There are promotion-equivariant bijections among the sets* $\mathcal{L}_{([a]\times[b])\times[\ell]}(R^{a+b+c-1})$*,* $\mathcal{L}_{([a]\times[c])\times[\ell]}(R^{a+b+c-1})$*, and* $\mathcal{L}_{([b]\times[c])\times[\ell]}(R^{a+b+c-1})$*.*

Proof. By Corollary 3.3.4, $\mathcal{L}_{([a]\times[b])\times[\ell]}(R^{a+b+c-1})$ under Pro is in equivariant bijection with $\mathcal{A}^\ell([a] \times [b] \times [c])$ under Row, $\mathcal{L}_{([a]\times[c])\times[\ell]}(R^{a+b+c-1})$ under Pro is in equivariant bijection with $\mathcal{A}^\ell([a] \times [c] \times [b])$ under Row, and $\mathcal{L}_{([b]\times[c])\times[\ell]}(R^{a+b+c-1})$ under Pro is in equivariant bijection with $\mathcal{A}^\ell([b] \times [c] \times [a])$ under Row. The set of linear extensions of $[a] \times [b] \times [c]$ is the same under any permutation of a, b, c. As a result, all three sets under their respective promotion actions are in equivariant bijection with $\mathcal{A}^\ell([a] \times [b] \times [c])$ under Row, and so there are equivariant bijections between the three sets under their respective promotion actions. □

We can extend this same idea to a more general product of chains in the theorem below. Since the proof is nearly identical to the proof of Theorem 3.3.7, we give it in condensed form.

Theorem 3.3.8. *Let* $P_i = [a_1] \times [a_2] \times \cdots \times [a_{i-1}] \times [a_{i+1}] \times \cdots \times [a_k]$*. For any* $1 \leq i, j \leq k$*, there is an equivariant bijection between the sets* $\mathcal{L}_{P_i\times[\ell]}(R^{\sum_i^k a_i - 1})$ *and* $\mathcal{L}_{P_j\times[\ell]}(R^{\sum_i^k a_i - 1})$ *under their respective promotion actions.*

Proof. Both sets are in equivariant bijection with $\mathcal{A}^\ell([a_1] \times \cdots \times [a_k])$ under Row by Corollary 3.3.6. □

In the case of the product of an arbitrary graded poset with a chain, we have a similar symmetry, but the equivalence is between only two sets rather than three or more.

Theorem 3.3.9. *Let P be a graded poset of rank n. Then there is an equivariant bijection between $\mathcal{L}_{(P\times[a])\times[\ell]}(R^{a+b+n})$ under* Pro *and $\mathcal{L}_{(P\times[b])\times[\ell]}(R^{a+b+n})$ under* Pro*.*

Proof. By Corollary 3.2.25, both sets are in equivariant bijection with $\mathcal{A}^\ell([P]\times[a]\times[b])$ under Row. □

Remark 3.3.10. When $a = 1$, R^{1+b+n} restricts the labels of $\mathcal{L}_{(P\times[b])\times[\ell]}(R^{1+b+n})$ to only two possible values at any (p, i). Thus, we can think of the layers of $f \in \mathcal{L}_{(P\times[b])\times[\ell]}(R^{1+b+n})$ as order ideals of $P \times [b]$ by considering all elements labeled by the lower of their two values as elements of the corresponding order ideal. Then, f itself corresponds to a multichain of order ideals, and we can therefore interpret the $a = 1$ case of Theorem 3.3.9 as an application of the intermediate bijection ϕ_2 from Definition 3.2.16.

3.4. Beyond the product of chains

The previous section studied P-strict promotion where P is a product of chains and the restriction function is induced by a global bound. In this section, we apply our main theorem to examples of interest where P is not a product of chains or the restriction function is not induced by a global bound. In Subsection 3.4.1, we study the case of P a minuscule poset. In Subsection 3.4.2, we let $P = [a] \times [b]$ but impose flags on our restriction function. Finally, Subsection 3.4.3 discusses the case where P is the three-element poset V.

3.4.1. Minuscule posets

Minuscule posets are interesting families of posets arising from Lie theory. The product of two chains $[a] \times [b]$ is a *Type A minuscule poset*, and thus, it is natural to ask which results of the previous section extend to other types. The answer is: not many. A special feature of the Type A minuscule poset is that it is constructed as a Cartesian product of chains, and many of the results in Section 3.3 rely on the multifold symmetry of Corollary 3.3.7. Since the other minuscule posets are not products of chains, we do not have an analogous result to use. But, our main bijection is useful for translating results on Q-partitions where Q is the Cartesian product of a minuscule poset and a chain to obtain new results on P-strict labelings of minuscule posets. The new such result of this section is Corollary 3.4.4.

Since the focus of this section is on translating known results about minuscule posets across our bijection, we choose to give explicit descriptions of the minuscule posets rather than the full

Lie-theoretic definitions. Rather, we refer the reader to the cited papers for further explanation of the algebraic meaning.

The minuscule posets are the following three infinite families followed by two exceptional posets. We follow the notation and definitions of minuscule posets found in [35]. For each, we also give the *Coxeter number* h of the associated Lie algebra, which will be used in statements of results. Let J^k denote the order ideal functor applied k times. For example, $J^2(P) = J(J(P))$ is the poset of order ideals of the poset of order ideals of P.

1. Rectangles: $[k] \times [m]$, $h = k + m$,

2. Shifted staircases: $S_k := \{(x, y) \mid x \leq y \in [k]\}$, $h = 2k$

3. Propellers: $J^k([2] \times [2])$, $h = 2(k + 2)$,

4. Cayley–Moufang: $J^2([3] \times [2])$, $h = 12$

5. Freudenthal: $J^3([3] \times [2])$, $h = 18$.

First, since all minuscule posets are graded, we can use Corollary 3.2.26 to construct an equivariant bijection.

Corollary 3.4.1. *Let P be a minuscule poset of rank n. Then $\mathcal{A}^\ell(P \times [a])$ under* Row *is in equivariant bijection with $\mathcal{L}_{P\times[\ell]}(R^{a+n+1})$ under* Pro.

Proof. This follows directly from Corollary 3.2.26 and the fact that any minuscule poset is graded. □

We state some known results on $\mathcal{A}^\ell(P \times [a])$ and/or $\mathcal{L}_{P\times[\ell]}(R^{a+n+1})$ and their translations via Corollary 3.4.1.

In the case $\ell = 1$, the bijection of Corollary 3.4.1 was already known and used by H. Mandel and Pechenik to obtain results on order and cyclic sieving in minuscule posets. We state these below, together with previous results of D. Rush and X. Shi which used the order ideal perspective. (For more on the $\ell = 1$ case for general P, see also Remark 3.2.27.)

Cyclic sieving is a dynamical phenomenon in which evaluation of a polynomial at certain roots of unity completely describes the orbit structure of an action. Since we state no new cyclic sieving results here, we refer the reader to the foundational paper [41] and the papers cited in the theorem below for the precise definition.

Theorem 3.4.2 ([44, $a = 1, 2$ on order ideals], [32, other cases])**.** *Let P be a minuscule poset associated with a minuscule weight λ of $\mathfrak{g}$, and let n denote the rank of P. Each of*

- *the set of order ideals $J(P \times [a])$ under* Row *and*
- *the set of increasing labelings $\mathcal{L}_{P\times[1]}(R^{a+n+1})$ under* Pro

have order dividing h and exhibit the cyclic sieving phenomenon with respect to the cardinality generating function of $J(P \times [a])$ for the values of a given below:

1. *Rectangles: $a = 1, 2$,*
2. *Shifted staircases: $a = 1, 2$,*
3. *Propellers: all a,*
4. *Cayley–Moufang: all a,*
5. *Freudenthal: $a \leq 4$.*

The next theorem and translation allow for arbitrary ℓ, but specify $a = 1$. The theorem has appeared in the literature in several places; for more on these references, see the discussion in [34, p. 3-5].

Theorem 3.4.3 ([21, 23, 24, 34])**.** *Let P be a minuscule poset associated with a minuscule weight λ of $\mathfrak{g}$, and let n denote the rank of P. Then $\mathcal{A}^\ell(P)$ has order h under* Row*, where h is the Coxeter number of $\mathfrak{g}$.*

The following corollary is a translation of the above theorem, using our main bijection.

Corollary 3.4.4. *Let P be a minuscule poset associated with a minuscule weight λ of $\mathfrak{g}$, and let n denote the rank of P. Then the set of P-strict labelings $\mathcal{L}_{P\times[\ell]}(R^{n+2})$ has order h under* Pro*, where h is the Coxeter number of $\mathfrak{g}$.*

Proof. By Corollary 3.4.1, $\mathcal{A}^\ell(P \times [1])$ is in bijection with $\mathcal{L}_{P\times[\ell]}(R^{n+2})$. Then the result follows by Theorem 3.4.3. □

In addition, [34] includes discussion of several homomesy results, which one could translate, but this would require more background on minuscule posets than what we have stated in this section.

One might hope for some nice results on the order of rowmotion on $\mathcal{A}^\ell(P \times [a])$ when both a and ℓ are greater than one (and P is a minuscule poset). But our SageMath computations show that for the propeller $P = J^2([2] \times [2])$, $\mathcal{A}^2(P \times [2])$ has rowmotion orbits of sizes 6, 9, 17, and 44. Since this is one of the simplest minuscule posets, a nice general result involving order or cyclic sieving is unlikely.

3.4.2. Flagged product of two chains

A *flagged tableau* of n rows with flag $\beta = (\beta_1, \beta_2, \ldots, \beta_n)$ is a semistandard tableau whose entries in row i do not exceed β_i. Given that semistandard tableaux are P-strict labelings with $P = [n]$, it is natural to consider a flagged P-strict labeling with $P \neq [n]$, where the flags on the fibers (the "rows" of our P-strict labeling) are given by the function $\beta : P \to \mathbb{Z}^+$. In this section, we explore a case where a particular flagging of $[a] \times [b]$ results in a familiar structure for the gamma poset.

The poset we consider is the following.

Definition 3.4.5. Let $\triangle_n$ denote the triangle-shaped subposet of $[n] \times [n]$ given by the elements $\{(i,j) \mid 1 \leq i \leq n, n-i+1 \leq j \leq n\}$.

Theorem 3.4.6. *For $(i,j) \in [a] \times [b]$, let $\beta(i,j) = b + 2i - 1$. Then $\mathcal{L}_{([a]\times[b])\times[\ell]}(R^\beta)$ under* Pro *is in equivariant bijection with $\mathcal{L}_{\triangle_a \times [\ell]}(R^{a+b})$ under* Pro.

To prove Theorem 3.4.6, we use the following lemma, which proves the poset isomorphism shown in Figure 3.2.

Lemma 3.4.7. *Let β be defined as above. Then $\Gamma([a]\times[b], R^\beta)$ is isomorphic as a poset to $\triangle_a \times [b]$.*

Proof. We have that for any (i,j), $R^\beta(i,j) = \{i+j-1, i+j, \ldots, b+2i-1-(b-j)\} = \{i+j-1, \ldots, 2i+j-1\}$, so $\Gamma([a] \times [b], R^\beta)$ consists of the elements $\{((i,j),k) | \ 1 \leq i \leq a, 1 \leq j \leq b, i+j-1 \leq k \leq 2i+j-2\}$. By definition of Γ (as noted in [13, Thm. 2.21]), $((i_1,j_1),k_1) \lessdot ((i_2,j_2),k_2)$ in $\Gamma([a] \times [b], R^\beta)$ if and only if either

(1) $(i_1,j_1) = (i_2,j_2)$ and $k_1 - 1 = k_2$, or

(2) $k_1 + 1 = k_2$ and $(i_1, j_1) \lessdot_{[a]\times[b]} (i_2, j_2)$.

Consider the map from $\Gamma([a]\times[b], R^\beta)$ to $\triangle_a \times [b]$ defined by $((i,j),k) \mapsto ((i, i+j-k+a-1), j) := ((i,u),j)$. Then, using the bounds on k, we have

$$i + j - (2i + j - 2) + a - 1 \leq u \leq i + j + a - (i + j - 1) - 1,$$

implying $a-i+1 \leq u \leq a$. Together with the bounds $1 \leq i \leq a$ and $1 \leq j \leq b$, we conclude our map is a bijection between the elements of $\Gamma([a] \times [b], R^\beta)$ and $\triangle_a \times [b]$. Moreover, $((i_1, u_1), j_1) \lessdot_{\triangle_a\times[b]} ((i_2, u_2), j_2)$ if and only if either $i_1+1 = i_2$, $u_1+1 = u_2$, or $j_1+1 = j_2$, and the other coordinates are equal. Each of these conditions correspond exactly with the covering relations in $\Gamma([a] \times [b], R^\beta)$ as follows: if $i_1 + 1 = i_2$ or $j_1 + 1 = j_2$, then $(i_1, j_1) \lessdot_{[a]\times[b]} (i_2, j_2)$ and, since $u_1 = i_1 + j_1 - k_1 + a - 1 = u_2 = i_2+j_2-k_2+a-1$, it must be the case that $k_1+1 = k_2$. If $u_1+1 = u_2$ and $i_1 = i_2$, $j_1 = j_2$, then $-k_1 - 1 + 1 = -k_2 - 1$, so $k_1 - 1 = k_2$. Thus, the above map bijects the elements of $\Gamma([a] \times [b], R^\beta)$ to those of $\triangle_a \times [b]$ and preserves covering relations, so $\Gamma([a] \times [b], R^\beta)$ is isomorphic as a poset to $\triangle_a \times [b]$. □

Proof of Theorem 3.4.6. Since $\mathcal{L}_{([a]\times[b])\times[\ell]}(R^\beta)$ under Pro is in equivariant bijection with $\mathcal{A}^\ell(\Gamma([a]\times[b], R^\beta))$ under Row by Corollary 3.2.22, we have that it is also in equivariant bijection with $\mathcal{A}^\ell(\triangle_a \times [b])$ under Row by Lemma 3.4.7. Since $\triangle_a$ has rank $a - 1$, $\mathcal{A}^\ell(\triangle_a \times [b])$ under Row is in equivariant bijection with $\mathcal{L}_{\triangle_a\times[\ell]}(R^{a+b})$ under Pro by Corollary 3.2.25, and we have the desired result. □

Remark 3.4.8. Lemma 3.4.7 appears as [5, Lemma 4.27] for the case $b = 1$.

Remark 3.4.9. There is an explicit bijection between $\mathcal{L}_{([a]\times[b])\times[\ell]}(R^\beta)$ and $\mathcal{L}_{A_a\times[\ell]}(R^{a+b})$ using Φ from Definition 3.2.16 with the appropriate restriction functions. However, this is not the same as the equivariant bijection from the above theorem, since the toggle orders associated to their respective gamma posets are different (but both conjugate to rowmotion).

Remark 3.4.10. As a result of D. Grinberg and T. Roby [23, Corollary 66], we have that the order of rowmotion on $\mathcal{A}^\ell(\triangle_a \times [1])$ is given by $2(a+1)$, and it is conjectured (see Conjecture 3.4.13) that $\mathcal{A}^\ell(\triangle_2 \times [b])$ also has predictable order under rowmotion. Additionally, based on some preliminary calculations in SageMath, there may be a resonance result (see Definition 3.5.1) on the order ideals $\mathcal{A}^1(\triangle_a \times [b])$. Unfortunately, any change to the parameters beyond those of the sets listed above results in a breakdown of the orbit structure. As early as $\mathcal{A}^2(\triangle_3 \times [2])$, we find an orbit of length 94. It could be said, however, that *many* of the orbits are of length 10, a notion which may be worth future consideration.

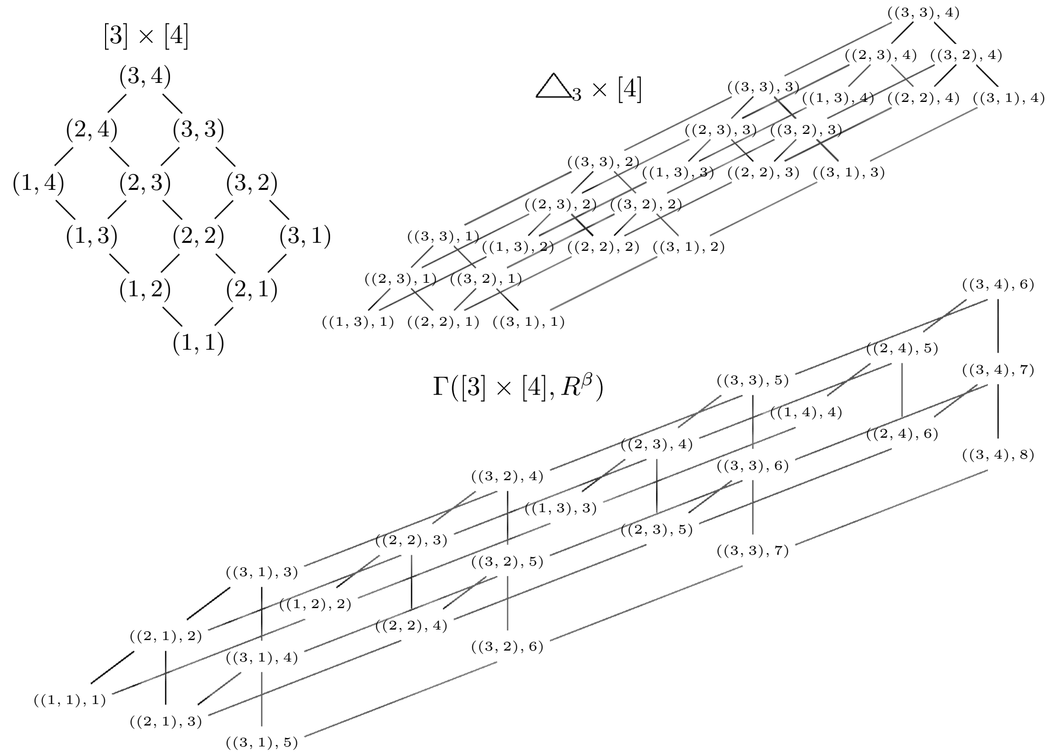


Figure 3.2. An example of the posets in Lemma 3.4.7 with $a = 3$ and $b = 4$. The consistent restriction function R^β on $[3] \times [4]$ is induced by the flags β, where $\beta(1, j) = 5$, $\beta(2, j) = 7$, and $\beta(3, j) = 9$. Note that this flag is not necessarily the greatest element of the induced $R^\beta(i, j)$ because of the strictly increasing requirement on layers in $\mathcal{L}_{([a]\times[b])\times[\ell]}(R^\beta)$.

3.4.3. The V poset

Corollaries 3.2.25 and 3.2.26 give us reason to pursue any cases where P is graded and Row on $\mathcal{A}^\ell(P \times [m])$ has noteworthy dynamical properties, as the translation through our bijection produces an example in which Pro on $\mathcal{L}_{P\times[\ell]}(R^q)$ is also of dynamical interest. One such poset that has caught recent attention is $P = V \times [m]$, where V is defined below.

Definition 3.4.11. Let V be the three-element poset $\{a, b, c\}$ with $a \lessdot b$ and $a \lessdot c$.

As a result of M. Plante, we know the order ideals of $V \times [m]$ have nice order under rowmotion.

Theorem 3.4.12 ([38]). Row *on* $J(V \times [m])$ *has order dividing* $2(m+2)$.

Moreover, it is believed to be the case that piecewise-linear rowmotion on $(V \times [m])$-partitions shares the same order.

Conjecture 3.4.13 ([29]). Row *on* $\mathcal{A}^{\ell}(V \times [m])$ *has order dividing* $2(m+2)$.

This conjecture has been experimentally verified for small m and ℓ, and we have used our SageMath code to check many particular examples with larger ℓ and m.

We apply Corollary 3.2.25 to Conjecture 3.4.13 (see Figure fig:VstrictVrow),noting that the poset V is of rank $n = 1$, to obtain the following translated conjecture on V-strict labelings:

Conjecture 3.4.14. Pro *on* $\mathcal{L}_{V\times[\ell]}(R^q)$ *is of order dividing* $2q$.

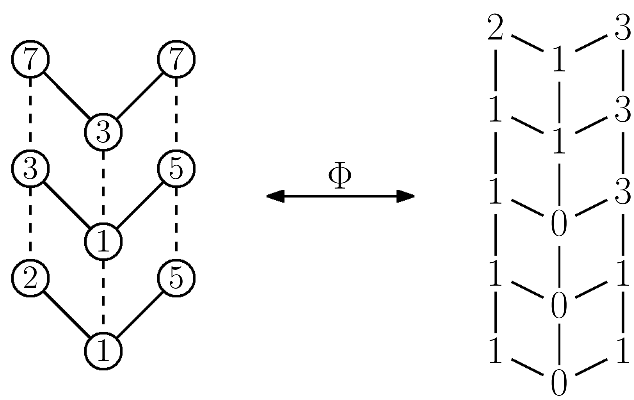


Figure 3.3. An element of $\mathcal{L}_{V\times[3]}(R^7)$ and its corresponding $(V \times [5])$-partition

Remark 3.4.15. Hopkins and M. Rubey show in [30] that the order of promotion on linear extensions of $V \times [\ell]$ is 6ℓ. Just as in the linear extension case, it seems to be true (experimentally) that applying promotion q times to some $f \in \mathcal{L}_{V\times[\ell]}(R^q)$ results in a reflection of the labels of f across the vertical axis of symmetry. Therefore, it may be possible to devise a method similar to that in [30] to accommodate V-strict labelings, proving Conjecture 3.4.14, and, in turn, Conjecture 3.4.13.

Finally, Conjecture 3.4.13 can be translated once more to an earlier case of interest, by applying Theorem 3.4.6 with $a = 2$, noting $\triangle_2$ is the dual poset of V.

Conjecture 3.4.16. *For* $(i,j) \in [2] \times [b]$*, let* $\beta(i,j) = b + 2i - 1$*. Then* Pro *on* $\mathcal{L}_{([2]\times[b])\times[\ell]}(R^{\beta})$ *is of order dividing* $2(b+2)$*.*

3.5. Resonance on P-strict labelings

In dynamical algebraic combinatorics, we are often interested in when an action, such as promotion or rowmotion, has a small, predictable order. Actions which do not have such an order may still exhibit some nice dynamical behavior, such as orbit sizes that may be multiples of a predictable number (or divisors of it). In [12], *resonance* was defined to explain this numerical phenomenon when the action in question projects to an action with a small, predictable order.

Definition 3.5.1 ([12])**.** Suppose $G = \langle g \rangle$ is a cyclic group acting on a set X, $C_\omega = \langle c \rangle$ a cyclic group of order ω acting nontrivially on a set Y, and $\varphi : X \to Y$ a surjection. We say the triple (X, G, φ) exhibits **resonance** with **frequency** ω if, for all $x \in X$, $c \cdot \varphi(x) = \varphi(g \cdot x)$.

A prototypical example of resonance given in [12] was increasing tableaux under K-promotion where the projection map φ was the binary content of the tableau. Here we give an analogue of that theorem in the more general setting of P-strict labelings.

Though the previous sections of this paper deal only with the case $\mathcal{L}_{P\times[\ell]}(R)$, which in the notation of [5] is $\mathcal{L}_{P\times[\ell]}(u, v, R)$ with $u = v = 0$, the theorems of this section apply to the case with general u and v. As the values of u and v play no role in the proof, we state the first few results of this section at this greater level of generality. See [5, Definition 1.7] for the definition of $\mathcal{L}_{P\times[\ell]}(u, v, R^q)$.

We first extend the definition of binary content from increasing tableaux to P-strict labelings.

Definition 3.5.2. Define the **binary content** of a P-strict labeling $f \in \mathcal{L}_{P\times[\ell]}(u, v, R^q)$ to be the sequence $\mathrm{Con}(f) = (a_1, a_2, \ldots, a_q)$, where $a_i = 1$ if $f(p, i) = i$ for some $(p, i) \in P \times [\ell]$ and 0 otherwise.

We now give Lemma 3.5.3 showing that promotion cyclically shifts the binary content of a P-strict labeling. This is an analogue of [12, Lemma 2.1]. Note the proof of Lemma 3.5.3 is not directly analogous to the proof of [12, Lemma 2.1], as we use the Bender-Knuth definition of P-strict promotion (Definition 3.2.6) rather than an analogue of *jeu de taquin*. In [5, Definition 3.1], we gave a definition of P-strict promotion via an analogue of jeu de taquin when the restriction function is R^q and showed these definitions are indeed equivalent [5, Theorem 3.10]. So we could have used this approach instead, at the cost of stating this alternate definition for P-strict promotion.

Lemma 3.5.3. *Let* $f \in \mathcal{L}_{P\times[\ell]}(u, v, R^q)$*. If* $\mathrm{Con}(f) = (a_1, a_2, \ldots, a_q)$*, then* $\mathrm{Con}(\mathrm{Pro}(f))$ *is the cyclic shift* $(a_2, \ldots, a_q, a_1)$*.*

Proof. Suppose $f \in \mathcal{L}_{P\times[\ell]}(u, v, R^q)$, $\mathrm{Con}(f) = (a_1, a_2, \ldots, a_q)$, and $\mathrm{Con}(\mathrm{Pro}(f)) = (b_1, b_2, \ldots, b_q)$. We wish to show $b_i = a_{i+1}$ for all $1 \leq i < q$ and $b_q = a_1$.

Suppose $a_1 = 0$, meaning 1 is not used as a label in f. Then all 2 labels in f will be lowerable, leaving no 2 labels after the application of ρ_1. Likewise, there are no $i + 1$ labels in $\rho_i\rho_{i-1}\cdots\rho_1(f)$, and there are no q labels in $\mathrm{Pro}(f) = \rho_{q-1}\cdots\rho_2\rho_1$. Thus q is not used as a label in $\mathrm{Pro}(f)$, therefore $b_q = 0 = a_1$.

Suppose $a_1 = 1$, meaning 1 is used as a label in f. Any 1 label is either fixed, meaning there is a 2 above it in its layer, or free. If it is free, then either there are no 2's in its fiber and the 1 changes to a 2, or there are 2's in the fiber, meaning there will still be at least one 2 in the fiber after the application of ρ_1. The same reasoning holds for i, $2 \leq i \leq q - 1$. So there will be at least one q in $\mathrm{Pro}(f)$, therefore $b_q = 1 = a_1$.

Likewise, suppose $i > 1$ and $a_i = 0$, meaning i is not used as a label in f. Then all $i - 1$ labels in f will be raisable, leaving no $i - 1$ labels in $\rho_i\rho_{i-1}\cdots\rho_1(f)$, and therefore in $\mathrm{Pro}(f)$. So $b_{i-1} = 0 = a_i$.

Finally, suppose $i > 1$ and $a_i = 1$, so there is at least one i used as a label in f. Thus, i is fixed in $\rho_{i-2}\cdots\rho_1(f)$ with respect to ρ_{i-1}, meaning there is an $i - 1$ below it in its layer. In that case, there is an $i - 1$ in $\mathrm{Pro}(f)$. Alternatively, the i is free, meaning it will either change into an $i - 1$ when ρ_i is applied, or there will be a $i - 1$ labels and b i labels that change to b $i - 1$ labels and a i labels. In either case, there is an $i - 1$ in $\mathrm{Pro}(f)$. As a result, $b_{i-1} = 1 = a_i$. □

Theorem 3.5.4. $(\mathcal{L}_{P\times[\ell]}(u, v, R^q), \mathrm{Pro}, \mathrm{Con})$ *exhibits resonance with frequency* q*.*

Proof. This follows directly from Lemma 3.5.3. □

We now translate this resonance theorem to the realm of Q-partitions via the bijections of Theorems 3.2.17 and 3.2.20. Though we could state this result for arbitrary u, v through the general bijection of [5], it would use too much excess notation. Thus, we restrict to the case $u = v = 0$. First, we define the analogue of Con in this realm.

Definition 3.5.5. For $\sigma \in \mathcal{A}^{\ell}(\Gamma(P, R^q))$, let $\mathrm{Diff}(\sigma)$ be the sequence $(a_1, a_2, \ldots, a_q)$ where

$$a_k = \begin{cases} 0 & \text{if } \sigma(p, k-1) = \sigma(p, k) \text{ for all } p \in P \\ 1 & \text{otherwise,} \end{cases}$$

where we consider $\sigma(p, j) = \ell$ if $j < \min R^q(p)$ and $\sigma(p, j) = 0$ if $j > \max R^q(p)$.

Corollary 3.5.6. $(\mathcal{A}^{\ell}(\Gamma(P, R^q)), \mathrm{Row}, \mathrm{Diff})$ *exhibits resonance with frequency* q.

Proof. This follows from Corollary 3.2.22, Theorem 3.5.4, and the fact that, for any $\Gamma(P, R^q))$-partition σ, the difference $\sigma(p, k-1) - \sigma(p, k)$ gives the number of k labels in the fiber F_p for the corresponding P-strict labeling f. □

In the case where P is graded, that is, where the poset $\Gamma(P, R^q)$ is isomorphic to $P \times [q-n-1]$, we can restate Corollary 3.5.6 in terms of the elements of $P \times [q-n-1]$ through a reinterpretation of Diff.

Definition 3.5.7. Let P be graded with rank n and let H_k be the set of all elements (p, i) in $P \times \mathbb{Z}$ with $q - n - i + \mathrm{rank}(p) = k$. For $\sigma \in \mathcal{A}^{\ell}(P \times [q-n-1])$, let $\mathrm{Diff}(\sigma)$ be the sequence $(a_1, a_2, \ldots, a_q)$ where

$$a_k = \begin{cases} 0 & \text{if } \sigma(p, i) = \sigma(p, i+1) \text{ for all } (p, i) \in H_k \\ 1 & \text{otherwise,} \end{cases}$$

where we consider $\sigma(p, i) = \ell$ if $i > q - n - 1$ and $\sigma(p, i) = 0$ if $i < 1$.

Corollary 3.5.8. $(\mathcal{A}^{\ell}(P \times [q-n-1]), \mathrm{Row}, \mathrm{Diff})$ *exhibits resonance with frequency* q.

Proof. We have that $\Gamma(P, R^q))$ is isomorphic to $P \times [q-n-1]$ as a poset by Lemma 3.2.23. The conditions for $a_k = 0$ and $a_k = 1$ are identical to those in Corollary 3.5.6 under the bijection in Remark 3.2.24, noting that $H_{k'}$ in $P \times [q-n-1]$ corresponds to the set $\{(p, k) \in \Gamma(P, R^q) \mid k = k'\}$. □

Remark 3.5.9. In the case when P is graded, we can think of $\mathrm{Diff}(\sigma)$ as indicating when the elements of consecutive hyperplanes in a lattice projection of $P \times [q-n-1]$ share the same labels. For example, in the top right of Figure 3.1, the two 4 labels at the top of $P \times \{3\}$, the center 1 label in $P \times \{2\}$, and the two 0 labels in $P \times \{1\}$ label the elements of H_3, and these labels are the same as those of H_4 (where the elements "below" the poset are considered to be labeled by 0). In this case, we have $\mathrm{Diff}(\sigma) = (1, 1, 1, 0, 1, 1)$.

4. OTHER RESULTS AND CONJECTURES

In this final chapter, we introduce an assortment of other results and conjectures that arose, directly or indirectly, from working through the material in Chapters 2 and 3. We begin with an enumeration result, and conclude with results with more dynamical flavor.

4.1. A bijection on tableaux

We introduce a bijection on semistandard Young tableaux as a solution to the enumeration problem of Corollary 4.1.6, though the map is given in larger generality that is not explored here. For a semistandard Young tableau T, let $T(i,j)$ be the entry in the box in the i^{th} row and j^{th} column of T, and let λ' denote the **conjugate** (see [48]) of a partition $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_n)$, that is, the partition whose Young diagram is the transpose of the Young diagram of λ. We write the set of all semistandard Young tableau with entries no more than m as $\mathrm{SSYT}(\lambda, m)$.

Definition 4.1.1. Define a map F on $\mathrm{SSYT}(\lambda, m)$ as $F(T) = T'$, where $T'(j,i) = T(i,j) - i + j$. In other words, F as subtracts i from the i^{th} row of T, conjugates, and then adds j to the j^{th} row of this result.

Proposition 4.1.2. *F is a map from SSYT of shape λ to SSYT of shape λ'.*

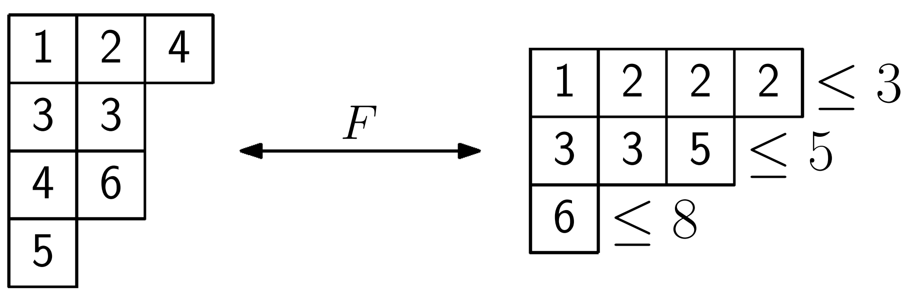


Figure 4.1. An example of the map F with $\lambda = (3,2,2,1)$. These two tableaux are in bijection by Theorem 4.1.3. The tableau on the left is an element of $\mathrm{SSYT}(\lambda, 6)$ and the tableau on the right is an element of $\mathrm{FT}(\lambda', (3,5,8))$

Proof. Let $T \in \mathrm{SSYT}(\lambda, m)$ with $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_k)$ and let $F(T) = T'$. By definition of F, T' is a filling of the partition shape λ'. Now, for $1 \le j \le \lambda_i - 1$,

$$T'(j,i) = F\left(T(i,j)\right) = T(i,j) - i + j \le T(i,j+1) - i + j < T(i,j+1) - i + (j+1)$$
$$= F\left(T(i,j+1)\right) = T'(j+1,i)$$

so we have $T'(j,i) < T'(j+1,i)$. Thus T' is strictly increasing down columns. For $1 \le i \le k-1$,

$$T'(j,i) = F\left(T(i,j)\right) = T(i,j) - i + j < T(i+1,j) - i + j$$
$$T(i,j) - i + j \le T(i+1,j) - (i+1) + j = F\left(T(i+1,j)\right) = T'(j,i+1)$$

so we have $T'(j,i) \le T'(j,i+1)$. Thus T' is weakly increasing across rows.

Since T' weakly increases in rows and strictly increases in columns, T' is a SSYT. □

Recall from Definition 2.4.22 that we denote set of flagged tableaux of shape λ and flags b by $FT(\lambda, b)$ (here, $b = (b_1, \ldots, b_n)$, associating a b_i with each part of λ).

Theorem 4.1.3. *F is a bijection between* $\mathrm{SSYT}(\lambda, m)$ *and* $\mathrm{FT}(\lambda', b)$, *where* $b_i = i + m - \lambda'_i$.

Proof. For a tableau T, let $F(T) = T'$. Then $F\left(F\left(T(i,j)\right)\right) = F\left(T'(j,i)\right) = T'(j,i) - j + i = (T(i,j) - i + j) - j + i = T(i,j)$. Thus F is an involution.

Suppose $T \in \mathrm{SSYT}(\lambda, m)$. Because T' is a SSYT by Proposition 4.1.2, to determine the largest possible entries in the rows of T' (its flags) we need only consider the last, and therefore greatest, entry in row i of T'. So,

$$T'(i, \lambda'_i) = F\left(T(\lambda'_i, i)\right) = T(\lambda'_i, i) - \lambda'_i + i \le m - \lambda'_i + i$$

thus the largest possible entry in row i of T' is $i + m - \lambda'_i$, so $T' \in \mathrm{FT}(\lambda', b)$.

Next, suppose $T' \in \mathrm{FT}(\lambda', b)$ for some λ. Then, for any (i,j) of the shape λ,

$$T(i,j) = F\left(T'(j,i)\right) = T'(j,i) - j + i \le \left(j + m - \lambda'_j\right) - j + i \le m - \lambda'_j + \lambda'_j = m$$

since the maximum value of i for any given j is λ'_j. Therefore the maximum entry of $T(i,j)$ is no greater than m, so $T \in \mathrm{SSYT}(\lambda, m)$.

We have $F^{-1} = F$, $F(\mathrm{SSYT}(\lambda, m)) \subset \mathrm{FT}(\lambda', b)$, and $F(\mathrm{FT}(\lambda', b))) \subset \mathrm{SSYT}(\lambda, m)$, so F is a bijection. $\square$

Consult Figure 4.1 for an example of this theorem. Now, we look at a case where this bijection yields an interesting enumeration result. Let sc_n denote the staircase partition shape $(n, n-1, \ldots, 1)$. The following corollary is a direct application of Theorem 4.1.3.

Corollary 4.1.4. $\mathrm{SSYT}(sc_n, n+1)$ *is in bijection with* $\mathrm{FT}(sc_n, (2, 4, \ldots, 2n))$.

We can enumerate the semistandard side of this bijection nicely using a hook-content formula (Theorem 1.1.4), yielding the following.

Proposition 4.1.5. $\mathrm{SSYT}(sc_n, n+1)$ *has cardinality* $2^{\binom{n}{2}}$.

Corollary 4.1.6. $\mathrm{FT}(sc_n, (2, 4, \ldots, 2n))$ *has cardinality* $2^{\binom{n}{2}}$.

The flags $(2, 4, \ldots, 2n)$ made an earlier appearance in Corollary 2.4.28, though in the case when λ is a rectangle. While this application is the extent of our use of the bijection F, it would be interesting to know if it is useful in other contexts.

4.2. Order ideals of $P \times [n]$

The material in this section arose largely from the pursuit of Conjecture 3.4.13, but includes some related objects. We demonstrate new proofs of results implicit in other work, with the notion that the included ideas represent small steps toward possible resolutions of more complicated conjectures. We begin with a more thorough explanation of Remark 3.4.15 after first reiterating the conjecture in question.

Conjecture 3.4.13. Row *on* $\mathcal{A}^{\ell}(V \times [m])$ *has order dividing* $2(m+2)$.

Because of the main results of Chapter 2, we have a new perspective on piecewise-linear rowmotion on $(V \times [m])$-partitions, namely, that it is equivariant with Bender–Knuth promotion on V-strict labelings of $V \times [\ell]$. It is often the case that promotion, rather than rowmotion, admits a rotational representation, so it seems likely that progress towards proving Conjecture 3.4.13 will come from analyzing promotion on $\mathcal{L}_{V\times[\ell]}(R^q)$ (Definition 2.1.7). Note that, in this correspondence, we have $q = m + \text{rank}(V) + 1 = m + 2$.

A first step in this direction is in the case $\ell = 1$, corresponding to rowmotion on order ideals of $V \times [m]$. Here, V-strict labelings of V, $\mathcal{L}_V(R^q)$ in our notation, are the same as increasing labelings of V, notated $\text{Inc}^q(V)$. As stated in Theorem 3.4.12, the order of rowmotion in this case is known to be $2(m+2)$ and is proved by Plante using what are described in [38] as *center seeking snakes*, though this idea can be generalized past the use of the word "center" and may thus appear differently in future work of Plante [39]. However, we can arrive at the same conclusion on the promotion side by using the method of Hopkins and Rubey from [30].

Theorem 4.2.1. Pro *on* $\text{Inc}^q(V)$ *has order dividing* $2q$.

The proof of this theorem uses exactly the process from [30], except on $\text{Inc}^q(V)$ instead of on the two linear extensions of V. To that end, we introduce the relevant definitions and results.

First, we call a word consisting of the letters A, B, and C (and later, blanks) a **Kreweras word** w if there are equal amounts of A, B, and C and the number of the letters B or C never exceeds that of A reading left to right. A nine letter Kreweras word consisting of three each of A, B, and C, such as $\omega =$ AACBCABCB, encodes a linear extension of $V \times [3]$ by letting A represent the minimal element of V and B and C represent one each of the maximal elements. For the example AACBCABCB, the three elements of $V \times [3]$ that are the minimal element of their respective V poset would have labels 1, 2, and 6, ascending. For an idea of this encoding in greater generality, see Figure 4.4.

From a Kreweras word we can construct a *Kreweras bump diagram* $\mathcal{D}_w$ by drawing blue arcs atop the Kreweras word corresponding to the noncrossing matching indicated by the A entries together with the B entries and drawing crimson arcs for the noncrossing matching indicated by the A and C entries. "Crimson" (instead of the more typical "red") is a reminder that these arcs connect with the letter C. Conveniently, "blue" already suffices. See Figure 4.3 for examples of bump diagrams, though in the context we will discuss later.

Finally, from $\mathcal{D}_w$ we can define the following, where S_n denotes the symmetric group.

Definition 4.2.2 ([30])**.** Define the **trip permutation** σ_w by assocating the numbers $\{1, \ldots, q\}$, in order, to the places of w, and then creating a permutation by taking a trip along the nodes

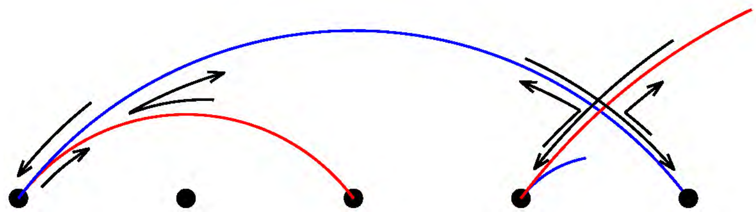


Figure 4.2. The rules of the road when taking a trip in a Kreweras bump diagram. The directions at an internal crossing are shown on the right side and at a boundary on the left.

and arcs of $\mathcal{D}_w$ in the following manner. First, if we start at an A node, we travel right along the innermost path to whichever is closest of the associated B or C and continue until we end at another node. If we start at a B or C node, we travel left along the arc toward the associated A, following the *rules of the road* shown in Figure 4.2 until we end at a node. For the purposes of this section, we additionally note that a trip starting at a blank "−" begins and ends in the same location. Then, $\sigma_w(i)$ is the value at the termination of the trip beginning in position i.

Note that we can recover the arcs, but not their color, just from σ_w. To make a bijection, we also need the following definition.

Definition 4.2.3. Define the map $\varepsilon_w \colon \{1, \ldots, q\} \to \{\mathrm{B}, \mathrm{C}, -\}$ as

$$\varepsilon_w(i) := \begin{cases} w_{\sigma_w(i)} & \text{if } w_{\sigma_w(i)} \neq \mathrm{A}; \\ w_{\sigma_w(\sigma_w(i))} & \text{if } w_{\sigma_w(i)} = \mathrm{A}, \end{cases}$$

Now, we give the crucial lemma. Here, for $\sigma \in S_n$ and using cycle notation, $\mathrm{Rot}(\sigma) = (1, 2, \ldots, n)^{-1} \circ \sigma \circ (1, 2, \ldots, n)$ and we consider $-\mathrm{B} = \mathrm{C}$, $-\mathrm{C} = \mathrm{B}$.

Lemma 4.2.4 ([30])**.** *Let w be a Kreweras word of length $3n$. Then,*

1. $\sigma_{\mathrm{Pro}(w)} = \mathrm{Rot}(\sigma_w)$;

2. $\varepsilon_{\mathrm{Pro}(w)} = [\varepsilon_w(2), \varepsilon_w(3), \ldots, \varepsilon_w(q), -\varepsilon_w(1)]$.

This lemma is proved in [30] for Kreweras words without blanks. We reiterate the lemma for our required case, considering the negation of a blank to be a blank.

Lemma 4.2.5. *Let w be a word of length q in the alphabet $\{A, B, C, -\}$ containing exactly one each of A, B, and C, where B and C occur after the letter A. Then,*

1. $\sigma_{\mathrm{Pro}(w)} = \mathrm{Rot}(\sigma_w)$;

2. $\varepsilon_{\mathrm{Pro}(w)} = [\varepsilon_w(2), \varepsilon_w(3), \ldots, \varepsilon_w(q), -\varepsilon_w(1)]$.

Proof. The introduction of the blank letter "$-$" does not interfere with the method of proof for this lemma shown in [30]. We will record the differences that working on $\mathrm{Inc}^q(V)$ adds, but will not reiterate the content in their paper.

Because Pro moves blanks cyclically one step to the left, the fixed points of the permutation σ_w induced by these blanks still satisfy 1 and 2. If $w(1) = -$, promotion acts by shifting all entries cyclically to the left, again satisfying 1 and 2. Finally, if a label is repeated in the maximal elements of V, the entire case reduces to promotion on $\mathrm{Inc}^q([2] \times [1])$, which has order q. In the context of $\mathcal{D}_w$, we have a single arc connecting A to B, ignoring C, and 1 and 2 hold. □

Now, for $\sigma_w \in S_q$, Rot has order q and Pro on ε_w has order $2q$, giving us Theorem 4.2.1. See Figure 4.3 for a demonstration of promotion using bump diagrams. Note that this proof also shows that the labels "reflect" across the vertical axis of symmetry, an idea we will revisit later in this section.

This bump diagram approach can also be used on natural generalizations of the V poset: p-stars.

Definition 4.2.6. Let $S(p)$ denote the ***p*-star poset**, the poset with one minimal element covered by p maximal elements.

In this notation, V is a 2-star. What, then, is the order of promotion on $\mathrm{Inc}^q(S(p))$?

Proposition 4.2.7. Pro *on an increasing labeling* $f \in \mathrm{Inc}^q(S(p))$*, in which the maximal elements are labeled by* k *unique labels, has order dividing* kq*.*

Remark 4.2.8. Instead of an explicit proof, we appeal again to Hopkins and Rubey. Without loss of generality, we can consider a p-star labeled by p unique maximal labels, as a smaller number k of unique labels reduces to the above case with a k-star. In this case the "Kreweras word" will consist of A and p more letters, which cycle to the next letter in line when they move from position 1 to position q in ε_ω under promotion. The usual "rules of the road" apply, so promotion acts on the trip permutation by rotation, and thus the order of promotion must divide pq.

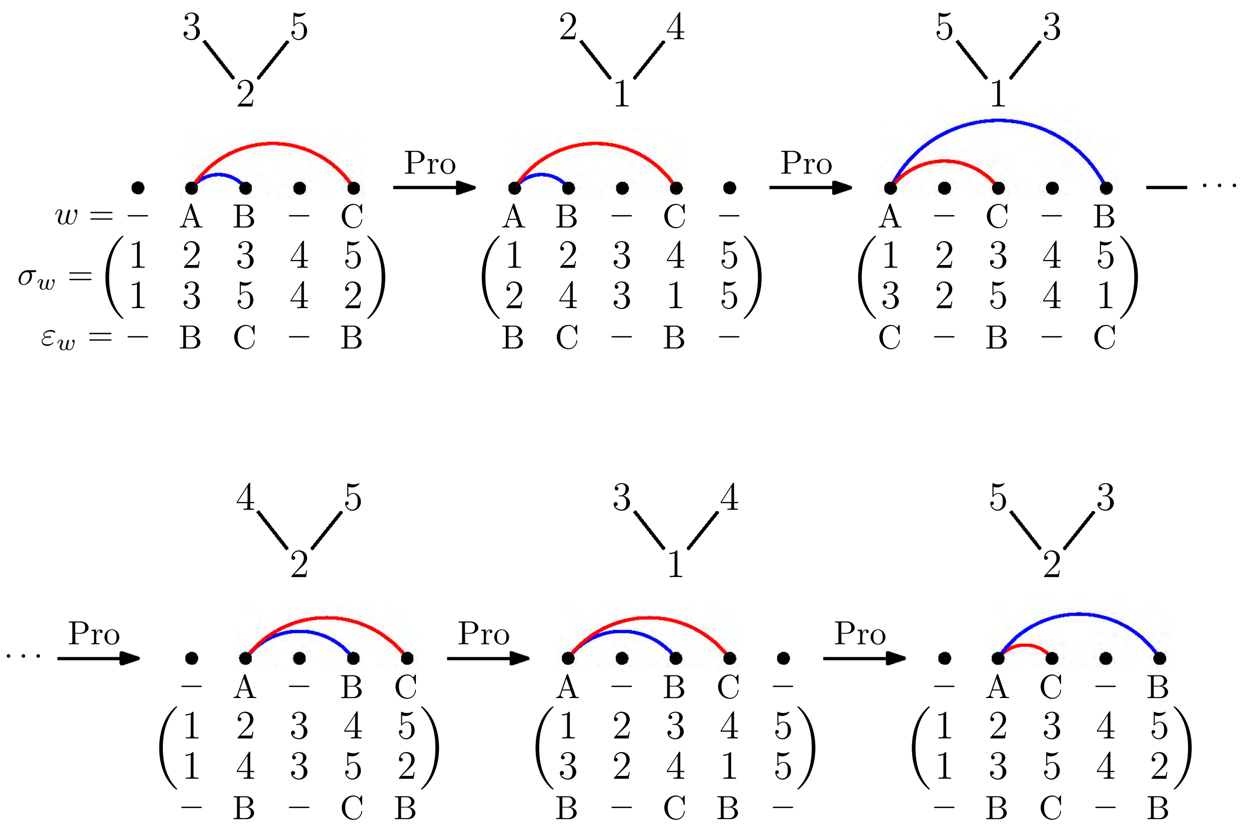


Figure 4.3. Shown here is half of an orbit of an element of $\mathrm{Inc}^5(V)$. Also included are the associated Kreweras word w (with blanks), the trip permutation σ_w in two-line notation, and the map ε_w.

Remark 4.2.9. The approach of Plante through rowmotion also applies nicely on $S(p) \times [m]$, and so the rowmotion translation of Proposition 4.2.7 will appear in future work of Plante [39].

The bump diagrams of [30] can also be used to biject linear extensions of $V \times [n]$ to *webs*, and it is these webs that visually rotate. The optimism of Remark 3.4.15 stems from this perspective; that a rotation object could be found to represent elements of $\mathcal{L}_{V\times[\ell]}(R^q)$ instead of just linear extensions. However, an initial foray into this idea hints that this may be challenging, as the "rules of the road" that are sufficient for linear extensions and the $\ell = 1$ case do not appear to be sufficient for a more general V-strict labeling (see Figure 4.4).

It would be reasonable to conclude that the difficulty stated above is due to $\mathcal{L}_{V\times[\ell]}(R^q)$ being the incorrect generalization of linear extensions. The labelings $\mathcal{L}_{V\times[\ell]}(R^q)$ seem to skip over one degree of weakening, as increasing labelings $\mathrm{Inc}^q(V)$ maintain the strictly increasing condition of linear extensions while allowing repeated labels, while V-strict labelings allow weak increase along fibers. However, the order of promotion on $\mathrm{Inc}^q(V \times [3])$ does not divide $2q$ (see below). This

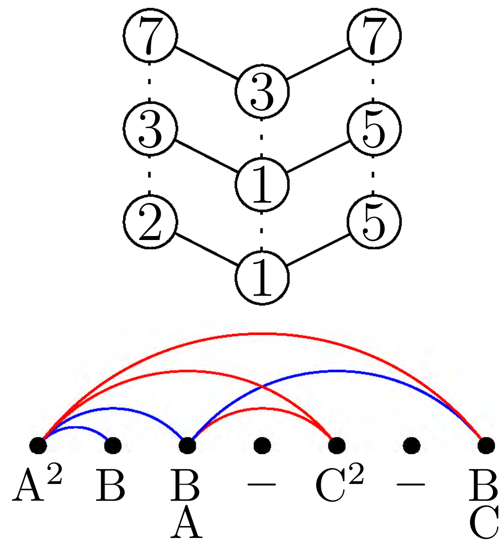


Figure 4.4. A bump diagram for a generic element of $\mathcal{L}_{V\times[3]}(R^7)$. This object is conjectured to have promotion orbits of cardinality 14.

parallels the known results on rectangles, where $\mathrm{SSYT}(\ell^n, q)$ under promotion has order dividing q, while $\mathrm{Inc}^q([a]\times[b])$ does not for a or b greater than 3.

It is also possible to prove Theorem 4.2.1 using the remarkable result of H. Mandel and Pechenik [32] which allows for the calculation of the period of promotion by analyzing only the *packed* labelings, or those which use all integers in a particular range $\{1,\ldots,q\}$. Since there are finitely many packed labelings for a given poset, the order of all increasing labelings is a matter of computation. In this way, we get the following proposition.

Proposition 4.2.10. *Promotion on* $\mathrm{Inc}^q(V\times[2])$ *has order* $2q$.

As a conclusion to the V poset, we note that there is an orbit of size 6 of promotion on $\mathrm{Inc}^7(V\times[3])$, notable as 6 is relatively prime to 7. This, as stated above, demonstrates that the nice promotion order of linear extension appears to extend to $\mathcal{L}_{V\times[\ell]}(R^q)$ instead of $\mathrm{Inc}^q(V\times[n])$. In this light, Proposition 4.2.10 seems to be a coincidence stemming from its small size.

For the remainder of this section, recall our definition of $\triangle_n$ as a subset of $[n]\times[n]$ (see Definition 3.4.5).

Besides p-stars and $V\times[n]$, another way of generalizing the V poset is to consider it as $\triangle_2$, and see if the nice promotion order extends to $\mathrm{Inc}^q(\triangle_n)$. While the order of promotion on $\mathrm{Inc}^q(\triangle_n)$ is not, in fact, $2q$, it does share a similar trait in that the labels on the V-shaped "boundary" elements of $\triangle_n$ reflect across the vertical axis of symmetry after q applications of promotion. The following definitions and conjectures state this observation more rigorously.

Definition 4.2.11 ([37])**.** The **top tree** $\mathcal{TT}(P)$ is the order filter consisting of all elements of P which generate a principal order filter that is a chain.

Definition 4.2.12. Define the **reflection** Refl on a labeling f of $\triangle_n$ as $\mathrm{Refl}\,(f(i,j)) = f(j,i)$.

The conjectures below are known to Pechenik, and should appear in forthcoming work of his [36].

Conjecture 4.2.13. *Let* $f \in \mathrm{Inc}^q(\triangle_n)$*. Then* $f\left(\mathcal{TT}(\triangle_n)\right) = \mathrm{Refl}\left(\mathrm{Pro}^q(f)\left(\mathcal{TT}(\triangle_n)\right)\right)$.

The following conjecture is determined computationally, though it is related to the conjecture above. It does not strictly follow from the above as projecting onto $\mathcal{TT}(\triangle_n)$ results in a substantial loss of information for large n.

Conjecture 4.2.14. *The order of* Pro *on* $\mathrm{Inc}^q(\triangle_n)$ *divides* $2q$ *or is a multiple of* $2q$*.*

To add a little weight to this conjecture, we note the following, again proved by computation.

Proposition 4.2.15. *The order of* Pro *on* $\mathrm{Inc}^q(\triangle_3)$ *divides* $2q$*.*

Now, let B_n denote the type B root poset, obtained from $\triangle_{2n-1}$ by removing the elements (i,j) with $i < j$. Since we can treat labelings of B_n as labelings f of $\triangle_{2n-1}$ with $f = \mathrm{Refl}(f)$, we obtain the following conjecture from the above discussion, after first defining an important notion of "nice" order from [37].

Definition 4.2.16 ([37])**.** A finite graded poset P has **NRP** ("not relatively prime") **rowmotion** if, for all positive c, no rowmotion orbit of $J(P\times c)$ has cardinality relatively prime to $\mathrm{rank}(P)+c+1$.

In terms of increasing labelings, the above definition translates as: P has NRP rowmotion if no promotion orbit of $\mathrm{Inc}^q(P)$ has cardinality relatively prime to q.

Conjecture 4.2.17. B_n *has NRP rowmotion.*

It is an ongoing issue to classify the aesthetics of the orbit sizes of promotion and rowmotion. Conjecture 4.2.14 is not a very strong statement, but it is an attempt to capture that $\triangle_n$ has nicer behavior than, for example, a six element fence poset $1 \lessdot 2 \gtrdot 3 \lessdot 4 \gtrdot 5 \lessdot 6$, which has promotion orbits of cardinality 17 and 22 with $q = 5$. It may be the case that more restrictive definitions such as NRP will have the most success while other cases of interest will rely on pattern-finding such as Conjecture 4.2.13 to demonstrate their beauty.

4.3. Order ideals of Raney posets

We begin this section by defining a new poset.

Definition 4.3.1. Let the **Raney poset** $\mathcal{R}(k, p, r)$ be the subposet of $[k] \times [(p-1)(k-2)+r-1]$ with elements $\{(i, j) \mid j > (p-1)(k-i)\}$.

See Figure 4.6 for examples of these posets and Figure 4.5 for the motivation behind the definition. Note that the labels are not as described above, but of the poset $\Gamma([n-1], R^b)$ described in Proposition 4.3.5.

The posets $\mathcal{R}(k, p, r)$ are enumerated by a known class of numbers that generalize Catalan numbers (see Remark rem:FCnum).

Definition 4.3.2. The Raney numbers are given by

$$R_{p,r}(k) = \frac{r}{kp+r}\binom{kp+r}{k}.$$

Theorem 4.3.3. *$J(\mathcal{R}(k, p, r))$ is enumerated by $R_{p,r}(k)$.*

While we do not claim this result is new or particularly difficult to demonstrate, we provide a proof below for the sake of completeness. This proof bijects the order ideals $J(\mathcal{R}(k, p, r))$ with **coral diagrams**, which we will describe as in [57]. A **coral diagram** of type (k, p, r) is constructed by repeatedly placing a total of k p-stars (see Definition 4.2.6) atop terminal vertices, starting with a single r-star as a base. By [3, Theorem 2.5], the number of coral diagrams of type (k, p, r) is given by $R_{p,r}(k)$.

Proof. First, we note that the poset $\mathcal{R}(k, p, r)$ can be represented by the whole 1×1 boxes in the rectangle $[0, (p-1)k + r - 1] \times [0, n+1] \subset \mathbb{R}^2$ that lie above, or have a corner touching, the line segments connecting $(0, 0)$ to $(r-1, 1)$ and connecting $(r-1, 1)$ to $((p-1)k + r - 1, k + 1)$ (see Figure 4.5). From this perspective, we can represent an order ideal of $\mathcal{R}(k, p, r)$ as a lattice path from $(0, 0)$ to $((p-1)k + r - 1, k + 1)$ that stays above or on the aforementioned line segments.

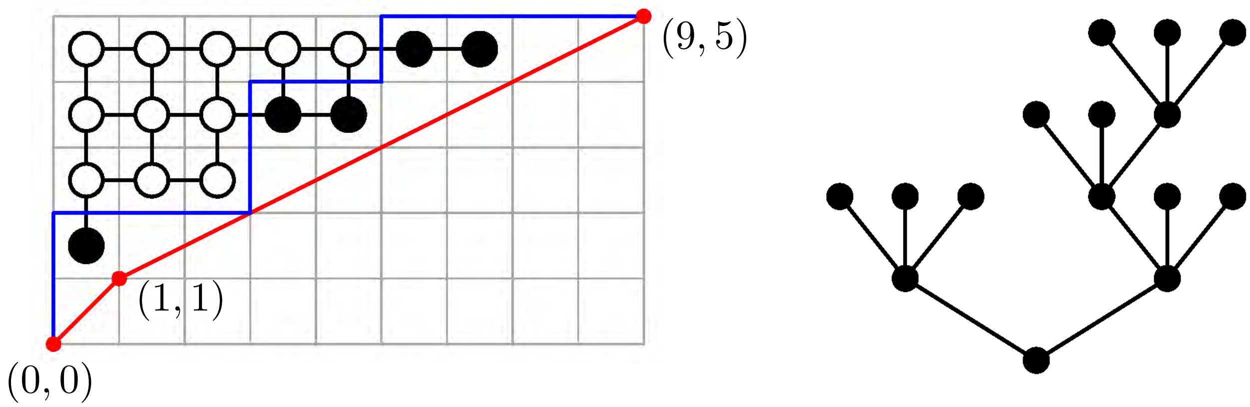


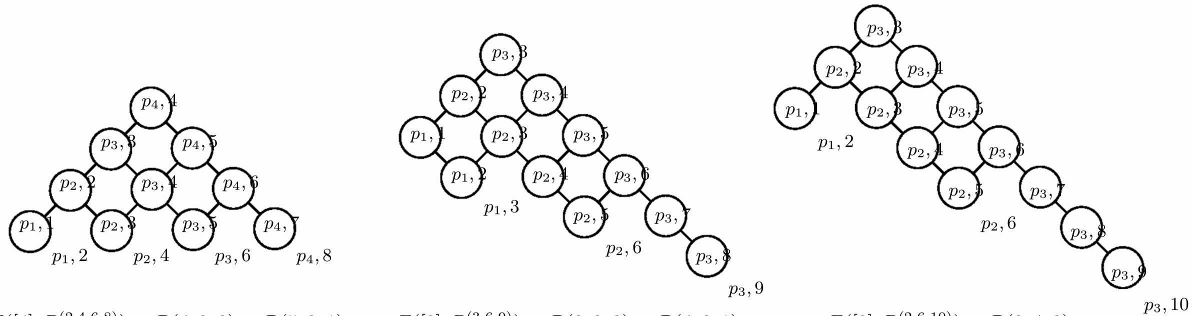
Figure 4.5. On the left is an element of $J(R(4,3,2))$ and on the right is its associated coral diagram.

What remains is to show these lattice paths and coral diagrams are equinumerous. A slight tweak to the usual bijection from binary trees to Dyck paths (as in [48, Chapter 5.3]) suffices . Starting from the base vertex, traverse the leftmost edge to the next vertex, and again take the leftmost edge. Upon reaching a terminal vertex, backtrack to the closest vertex with untraversed children and take the path immediately to the right. If no such path exists, backtrack to the next vertex, and so on. In this way, the entire coral diagram is traversed, ending at the rightmost terminal vertex. Ignoring backtracking, this results in $n+1$ leftmost edges and $(p-1)n+r-1$ non-leftmost edges traveled in a particular order. In this order, each traversal upon a leftmost edge is a $(0,1)$-step and all other edges are a $(1,0)$-step of our lattice path. This construction is easily reversed to associate a coral diagram to a given lattice path with the above restrictions. Thus, we have $|J(\mathcal{R}(k,p,r))| = R_{p,r}(k)$. □

See Figure 4.5 for an example of this bijection.

Remark 4.3.4. When $r = 1$, $R_{p,r}(k)$ equals the *Fuss–Catalan numbers*, or the number of lattice paths from $(0,0)$ to $((p-1)k, k)$ staying on or above the diagonal. The case when $r = 1$ and $p = 2$ is the usual Catalan numbers. Note that $R_{p,r}(k)$ is a different generalization of the Fuss–Catalan numbers than the *rational Catalan* numbers, which enumerate lattice paths above the single line segment connecting $(0,0)$ with any point (a,b). A geometric way to consider this difference is that Raney numbers are Fuss–Catalan numbers with a shift, while rational Catalan numbers are Fuss–Catalan numbers with any rational slope.

Recall the definition of the poset $\Gamma(P, R^b)$ from Definition 2.2.2.



$\Gamma([4], R^{(2,4,6,8)}) \cong \mathcal{R}(4,2,2) \cong \mathcal{R}(5,2,1)$ $\quad$ $\Gamma([3], R^{(3,6,9)}) \cong \mathcal{R}(3,3,3) \cong \mathcal{R}(4,3,1)$ $\quad$ $\Gamma([3], R^{(2,6,10)}) \cong \mathcal{R}(3,4,2)$

Figure 4.6. Three Raney posets labeled according to their interpretation as $\Gamma([k], R^b)$.

Proposition 4.3.5. *$\Gamma([k], R^b)$ is isomorphic to $\mathcal{R}(k,p,r)$ as a poset, where b denotes the flags $(r, r+p, r+2p, ...r+(k-1)p)$.*

This follows directly from the definitions of the respective posets. The details of this proof are similar to that of the proofs of Lemmas 2.2.28, 3.4.7, and Corollary 2.4.41. See Figure 4.6 for an illustration of this proposition.

Remark 4.3.6. Because there are $r-1$ elements in the chain corresponding to p_1 of $\mathcal{R}(k,p,1)$, there are no such elements when $r = 1$, and so $\mathcal{R}(k,p,1) \cong \mathcal{R}(k-1,p,p)$.

The following corollary of Theorem 2.2.20 translates $\mathcal{R}(k,p,r)$-partitions (and order ideals) through the main bijection of Chapter 2 to flagged tableaux.

Corollary 4.3.7. *Let $b = (r, r+p, r+2p, ...r+(k-1)p)$. Then $FT(\ell^n, b)$ under* Pro *is in equivariant bijection with $\mathcal{A}^\ell(\mathcal{R}(k,p,r))$ under* Row*.*

The next corollary is a specific case of the above.

Corollary 4.3.8. *Let $b = (r, r+p, r+2p, ...r+(k-1)p)$. Then $\mathrm{Inc}^{R^b}([k])$ under* Pro *is in equivariant bijection with $J(\mathcal{R}(k,p,r))$ under* Row*.*

Our aim is to determine, what, if any, interesting dynamics exist concerning the posets $\mathcal{R}(k,p,r)$. We have already seen what may be the most general result, expressed earlier in terms of $\triangle_n$ as Theorem 2.4.29:

Theorem 4.3.9 ([23, Corollary 66])**.** Row *on $\mathcal{A}^\ell(\mathcal{R}(k,2,1))$ has order dividing $2k$.*

Experimentally, it appears that $\mathcal{A}^\ell(\mathcal{R}(k,p,r))$ does not have predictable order in general, or, in fact, even for small parameter values. However, the following conjecture has a similar flavor to the above theorem, if for a less interesting poset.

Conjecture 4.3.10. Row *on* $\mathcal{A}^\ell(\mathcal{R}(2,2,r))$ *with* $r \geq 3$ *has order dividing* $(r+1)(r+2)$.

Translating this conjecture to a two-rowed flagged tableaux may open the door to a proof using promotion.

Conjecture 4.3.11 (equivalent to 4.3.10)**.** Pro *on* $FT(\ell^2,(r,r+2))$ *with* $r \geq 3$ *has order dividing* $(r+1)(r+2)$.

Remark 4.3.12. Note that $r = 2$ is covered under Theorem 4.3.9 and $r = 1$ reduces to the $k-1$ case with $r = 2$. Any increase in k or p results in a failure as early as $\ell = 2$. For clarity, the poset $\mathcal{R}(2,2,r)$ is exactly $[2] \times [r]$ with the minimal element removed.

For the order ideal case, $\ell = 1$, there is experimentally always one orbit of size $r+1$, and, if r is even, there is exactly one orbit of size $\frac{r}{2}+1$. Finally, there are $\lceil \frac{r}{2} \rceil - 1$ orbits of size $r+2$. We have tested this through $r = 50$. For the general case, we have tested through $r = 9$ for $\ell \leq 4$.

We conclude with a known result and a new conjecture about rowmotion on $J(\mathcal{R}(k,p,r))$.

Proposition 4.3.13 ([1, Proposition 5.2])**.** Pro *on* $J(\mathcal{R}(k,p,1))$ *has order dividing* kp.

This is a statement about rowmotion as well as promotion, since the left to right toggle action of Pro on $J(\mathcal{R}(k,p,1))$ is conjugate to Row as a result of [54].

The above proposition was proved using a rational Catalan perspective, and associated promotion on the lattice paths with the rotation of a *noncrossing partition*. We again note that this is a different generalization of the (Fuss–)Catalan numbers than the Raney numbers, though it may be possible that their method of proof could be used for the following conjecture.

Conjecture 4.3.14. *Let* $r \mid p-1$*. Then* Row *on* $J(\mathcal{R}(k,p,r))$ *has order dividing* $kp+r-1$.

Remark 4.3.15. Other values of r (i.e. $r \nmid p-1$ or $r > p$) do not have predictable order, except in the case of Conjecture 4.3.10. It is interesting to note that the periodicity in the above conjecture does not coincide with Proposition 4.3.13 in the case where $r = p$. For this conjecture, we have tested up to $p = 13$ for $k = 2,3$, and up to $p = 7$ for $k = 4,5$, as well as many random large examples.

4.4. Future work

The previous two sections outline possible avenues of future exploration. Foremost from Section 4.2 is resolution of Conjecture 3.4.13. Since many results on rowmotion can be described rotationally by promotion on a corresponding object, this approach seems promising, even if the bump diagram method is not tractable. However, given the work of Plante on rowmotion of order ideals, and, for that matter, Grinberg and Roby on birational rowmotion, we should not discount the idea of a direct rowmotion proof.

Besides this more general rowmotion result, there is still much to discover concerning rowmotion on $J(P \times [m])$ with P graded, or, equivalently, promotion on increasing labelings of P. Two phenomena, observed experimentally, need not only explanation, but a concrete definition. The first being a firm categorization of "nice" orbit structure besides a strict periodicity result. Notions such as resonance and NRP rowmotion begin to capture this, but are not well-explored. The second gap in classification comes when "many" orbits have the expected size (where notions of "expected" are a bit more robust!). While it is possible that this phenomenon occurs only for the small values we are able to test, there may be objects on which "many orbits" has definable meaning or gives rise to an asymptotic result.

Lastly, the conjectures that conclude Section 4.3 appear reasonable in nature, and can hopefully be proved with only slight modification to existing methods. Unexplored in this section is the possibility of homomesy results, which is worth future consideration.

As for the content of Chapters 2 and 3, future work should seek to firmly establish P-strict labelings in the combinatorial object canon. Developing algebraic structure similar to that of the semistandard Young tableaux they generalize (as well as flagged and symplectic tableaux), including their associated symmetric functions, possible Jacobi–Trudi identity, or even an RSK-like correspondence, would be of interest. Or, if promotion on P-strict labelings and the main bijection of Chapter 2 could be used to prove a rowmotion result, such as for the $\mathcal{A}^{\ell}(V \times [m])$ case, then P-strict labelings could enjoy wider appeal.

Finally, considering the main bijection as a generalization of Gelfand–Tsetlin patterns allows for possible exploration in that realm. Adapting known results on GT-patterns, such as the construction of their associated polytope, to their B-bounded counterparts could prove fruitful.

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