THE FIRST EXIT-TIME ANALYSIS OF AN APPROXIMATE BARNDORFF-NIELSEN AND SHEPHARD MODEL, WITH DATA SCIENCE-BASED APPLICATIONS IN THE COMMODITY MARKET

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The supervisory committee certifies that this dissertation complies with North Dakota State University's regulations and meets the accepted standards for the degree of

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ABSTRACT

In this dissertation, an approximate version of the Barndorff-Nielsen and Shephard model, driven by a Brownian motion and a Lévy subordinator, is formulated. The first-exit time of the log-return process for this model is analyzed. It is shown that with a certain probability, the first-exit time process of the log-return is decomposable into the sum of the first exit time of the Brownian motion with drift, and the first exit time of a Lévy subordinator with drift. Subsequently, the probability density functions of the first exit time of some specific Lévy subordinators, connected to stationary, self-decomposable variance processes, are studied. Analytical expressions of the probability density function of the first-exit time of three such Lévy subordinators are obtained in terms of various special functions. The results are implemented to empirical S&P 500 dataset.

After this exit time analysis, in this dissertation, we propose a model for the soybean export market share dynamics and analyze the empirical data using machine and deep learning algorithms. We justify the proposed general model and provide several theoretical analyses related to a special case of the general model. The empirical data set is a time series with weekly observations over the period January 6, 2012, through January 3, 2020. This is a period of growing intense competition, and during which a trade war had influenced the results. The target variable is the share of soybean exports made from the US Gulf to China. We implement machine and deep learning-based techniques to analyze the empirical data. Various numerical results are obtained. The results indicate that export market shares, which are otherwise highly volatile, can be effectively explained (predicted) using machine/deep learning methodologies and a set of logical feature variables.

We conclude this dissertation with an analysis of option pricing and implied volatility in the case when the market is driven by a jump-stochastic volatility model. We find the price of the European call option in this case. In addition, we implement Malliavin calculus to analyze the implied volatility.

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DEDICATION

This thesis is dedicated to my parents and grandparents.

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1. INTRODUCTION

1.1. General Introduction

The time required for a stochastic process, starting at a given initial state, to reach a threshold for the first time is referred to as the first-exit time or the first hitting time. It is typically very useful in determining expected lifetime of mechanical devices. The first-exit time processes are very useful for understanding various financial sectors, especially the insurance industry and investment firms. The first-exit time processes arise naturally in the studies of various disciplines. For example, it is used [56] to model the death probability density function for a decaying stochastic process that represents either the end of functionality for a machine, or a zero health state for an organism. The paper [30] provided an expanded first-exit time density function that expresses the human death distribution. The first-exit time analysis of a two-dimensional symmetric stable process was discussed in detail in the paper [18]. This was further developed in [32, 61] where the first-exit time process of an inverse Gaussian Lévy process is considered. The one-dimensional distribution function of the first-exit time process is obtained. The first-exit time analysis related to a geophysical data wass provided in [22]. The paper [43], provides generalized notions and analysis methods for the exit-time of random walks on graphs.

The first-exit time process of the standard Brownian motion is well-studied in the literature (see [4, 14]). The paper [35] studied the first-exit time of Brownian motion for a parabolic domain. In [19], the Fokker-Planck equation is solved for the Brownian motion with drift, in the presence of a fixed initial point and elastic boundaries. An explicit expression was obtained for the density of the first-exit time. The paper [27] studied the first-exit time problem for the solutions of some stochastic differential equations for bounded or unbounded intervals. Studies [39, 59, 60] discussed the first-exit time process for strictly increasing Lévy processes. In the pioneering paper [31], the authors study the first-exit-time to flat boundaries for a double exponential jump diffusion process. The related stochastic process consists of a continuous Brownian motion-driven part, and a jump part with jump sizes given by a double exponential distribution. In general, with the help of a fluctuation identity, the paper [1] provided, a generic link between a number of known identities for the first-exit time and the overshoot above/below a fixed level of a Lévy process. In [42], a class of increasing Lévy processes, perturbed by an independent Brownian motion was considered, and the problem of determining the distribution of the first-exit time is addressed. The first-exit time analysis of the Ornstein-Uhlenbeck (OU) process to a boundary was a long-standing problem with no known closed-form solution for the general case. In [37] a general mean-reverting process is considered to investigate the long-and short-time asymptotics using a combination of Hopf-Cole and Laplace transform techniques.

Many problems in finance are related to the first-exit time processes. A deeper understanding of such processes leads to a wiser estimation of fluctuations in the market. In [57], the first-exit time distributions of stock price returns in different time windows were analyzed. The probability distribution obtained by such analysis was compared with those obtained from different models for stock market evolution. The paper [26] showed that for continuous time transformations, independent of the Brownian motion, analytical results for the double-barrier problem can be obtained via the Laplace transform of the time change. The analysis provides a power series representation for the resulting first-exit time probabilities. In [36], explicit analytical characterizations were provided for the first-exit time densities for the Cox-Ingersoll-Ross (CIR) and OU diffusions. Such characterizations were obtained in terms of the relevant Sturm-Liouville eigenfunction expansions. In [66], a doubly skewed CIR process is studied. A modified spectral expansion was used to obtain the first-exit time distribution of a doubly skewed CIR process. A detailed study of the first-exit times of diffusion processes and their applications to finance is provided in [34]. The studies in [46, 47] discuss the first-exit time analysis related to some financial processes from a data-science and sequential hypothesis testing perspective. In [12], the authors provide a solution to the optimal stopping problem of a Brownian motion subject to the constraint that the stopping time's distribution was a given measure consisting of finitely many atoms. The distribution constraints lead to an application in mathematical finance to model-free super-hedging with an outlook on volatility.

Some analytically tractable formulas were available for the density of the first-exit time process (see [61]). However, in general, an explicit expression for the density of the first-exit time process for a financial model is mostly unknown. In this thesis, we analyze the first-exit time processes in connection to the Barndorff-Nielsen and Shephard (BN-S) model, a popularly used stochastic volatility model for financial analysis. We provide various analytical formulas related the distribution of the first-exit time processes in connection to an approximate version of the BN-S model. For this study we used various properties of the Laplace transform and their relations to special functions. In particular, the first-exit time processes for some well-known self-decomposable Lévy subordinators were analyzed.

1.2. Application in Agribusiness : An Introduction

Artificial intelligence (AI), and specifically deep learning (DL) are particularly attractive for analyzing competition in international markets. There are a multitude of reasons for this, and despite its attractiveness, there have been few reported studies using these methods to analyze export competition. In this paper we use deep learning models to analyze export competition in soybean market shares for shipments to China.

Briefly, one of the fastest growing commodity markets in world trade is soybean imports by China, which is the dominant buyer. Chinese imports of soybeans increased from near nil to 100 million metric tons/year in recent years, at a growth rate of about 18% per year. Intense competition in this market is dominated by the United States (primarily the US Gulf which is the focus of this study) and Brazil as major exporters. Over time, the US Gulf has gone from being dominant to now being replaced by Brazil.

A number of important attributes in this competition that motivates use of DL. Factors include the impact of growth in exports and the seasonality of competition with Brazil dominating during February to July. In addition, important quality differentials focused around EAA (Essential Amino Acids) and foreign material specifications in feed manufacture (see [25, 62]), can be challenging to meet as crop quality vary with quantity varies with weather conditions. In addition, soybeans that are genetically modified, which are not accepted by all buyers, varies with their adoption rate in the different countries. Logistics is also an important factor impacting costs. In the United States, rail and barge shipping costs vary through time and are random. In Brazil there has been inadequate infrastructure and capacity, which has an impact on wait times and costs for vessels arriving to load soybeans. Each of these persists, despite that they have improved over time. Finally, there are periodic interventions impacting trade. The most recent was the trade war during 2018-2019 coinciding with the Trump administration trade war.

Traditionally, economics trade is modeled as spatial equilibrium problems (see [58]) or using arbitrage models (e.g., see [55]). An important underlying assumptions of these models is that the supply and demand functions are known, can be represented as equations, are in equilibrium, and do not change. Implicit is that learning in these models is limited. In contrast, as noted above, these functions are not known, change through time, have a high degree of randomness, and quality differentials and shipping costs that vary randomly. Further, given the dynamics (growth, seasonal and dominance by one buyer) of this sector, and that the structural factors are random, deep learning models provide a more appropriate framework to analyze competition in this sector.

Deep learning models are appropriate for analyzing this market. The reasons for this are in part due to the large number of factor impacting trade, and that these are changing over time and some variables are sporadic, and that, in practice, market participants learn over time. One of the applications of this dissertation was to model the soybean export market share price dynamics; and analyze the related empirical data using machine and deep learning algorithms.

2. MATHEMATICAL PRELIMINARIES

2.1. Brownian Motion

There are many phenomena in nature that are seemingly random. Ranging from the diffusion of organisms into habitable land, to the price of a stock or commodity, to the behavioral patterns of humans, stochastic processes have an abundance of applications. One of the most classic examples of stochastic process is the Brownian motion. The motivation for such a process is a continuous symmetric random walk.

Paraphrasing Chapter 3 of the book [54], let $\omega = \{\omega_1, \omega_2, \ldots\}$ be a sequence of results of fair coin tosses. That is, ω_i is the outcome of the ith toss, H or T. Define

$$X_i = \begin{cases} 1 \text{ if } \omega_i = H \\ -1 \text{ if } \omega_i = T \end{cases},$$

and $M_n = \sum_{i=1}^{n} X_i$. Then $\{0, M_1, M_2, \ldots\}$ is a *symmetric random walk*. In particular, symmetric random walks are martingales that have independent increments. A *martingale* is a stochastic process that has expected value equal to the given present value for all times in the future. To have *independent increments* means for each $n \in \mathbb{N}$ (set of Natural numbers) and each $0 \leq t_1 \leq t_2 \leq \ldots < t_{n+1} < \infty$, the random variables $(M(t_{j+1}) - M(t_j), 1 \leq j \leq n)$ are independent.

Another crucial property of a symmetric random walk is its non-zero quadratic variation. In general, the *quadratic variation* of a discrete stochastic process M is

$$[M, M]_k := \sum_{j=1}^{k} (M_j - M_{j-1})^2,$$

which simplifies quite conveniently in our case to k. Note that while $[M, M]_k = \mathrm{Var}(M_k) = k$ for a symmetric random walk, this is not true in general. One varies dramatically for changes in the probabilities of each coin toss, while the other, the quadratic variation, remains constant.

Continuing toward the goal of a continuous random walk, define the scaled symmetric random walk by

$$W_t^{(n)} = \frac{1}{\sqrt{n}} M_{nt},$$

where $nt \in \mathbb{Z}$. Otherwise, define $W_t^{(n)}$ by a linear interpolation of its values for the closest integers. This new process is similarly a martingale with independent increments and quadratic variation, for $nt \in \mathbb{Z}$,

$$[W^{(n)}, W^{(n)}]_t = t.$$

Finally, we obtain a standard Brownian motion as the limit of this sequence of scaled random walks. We provide the following theorem from [41].

Theorem 2.1.1 (Kolmogorov)**.** *For all $t_1, \cdots, t_k \in T$, $k \in \mathbb{N}$, let $\nu_{t_1,\cdots,t_k}$ be probability measures on $\mathbb{R}^{nk}$ and Borel sets F_i such that*

$$\nu_{t_{\sigma(1)},\cdots,t_{\sigma(k)}}(F_1 \times \cdots \times F_k) = \nu_{t_1,\cdots,t_k}(F_{\sigma^{-1}(1)} \times \cdots \times F_{\sigma^{-1}(k)}) \tag{2.1}$$

for all permutations σ on 1,2,$\cdots$,k and

$$\nu_{t_1,\cdots,t_k}(F_1 \times \cdots \times F_k) = \nu_{t_1,\cdots,t_k,t_{k+1},\cdots,t_{k+m}}(F_1 \times \cdots \times F_k \times \mathbb{R}^n \times \cdots \times \mathbb{R}^n) \tag{2.2}$$

for all $m \in \mathbb{N}$, where the set on right hand side has a total of $k+m$ factors.
Then there exists a probability space (Ω,$\mathcal{F}$,P) and a stochastic process X_t on Ω, X_t: $\Omega \longrightarrow \mathbb{R}^n$, s.t.

$$\nu_{t_1,\cdots,t_k}(F_1 \times \cdots \times F_k) = P[X_{t_1} \in F_1, \cdots, X_{t_k} \in F_k], \tag{2.3}$$

for all $t_i \in T$, $k \in \mathbb{N}$ and all Borel sets F_i.

Fix $x \in \mathbb{R}^n$ and define

$$p(t,x,y) = (2\pi t)^{\frac{-n}{2}} . \exp\left(\frac{-|x-y|^2}{2t}\right) \quad for \quad y \in \mathbb{R}^n, t > 0. \tag{2.4}$$

If $0 \leq t_1 \leq t_2 \leq \cdots \leq t_k$, define a measure $\nu_{t_1,\cdots,t_k}$ on $\mathbb{R}^{nk}$ by

$$\nu_{t_1,\cdots,t_k}(F_1 \times \cdots \times F_k) = \int_{F_1\times\cdots\times F_k} p(t_1,x,x_1)p(t_2-t_1,x_1,x_2)\cdots p(t_k-t_{k-1},x_{k-1},x_k)dx_1\cdots dx_k, \tag{2.5}$$

where we use the notation $dy = dy_1 \cdots dy_k$ for Lebesgue measure and the convention that $p(0,x,y)dy = \delta_x(y)$, the unit point mass at x.

Since $\int_{\mathbb{R}^n} p(t,x,y)dy = 1$ for all $t \geq 0$, (2.2) holds, so by Kolmogorov's theorem there exists a probability space $(\Omega,\mathcal{F},P)$ and stochastic process W_t, $(t \geq 0)$ on Ω such that the finite-dimensional distributional of W_t are given by (2.5), *i.e.*,

$$P^x(W_{t_1} \in F_1, \cdots, W_{t_k} \in F_k) = \int_{F_1\times\cdots\times F_k} p(t_1,x,x_1)\cdots p(t_k-t_{k-1},x_{k-1},x_k)dx_1\cdots dx_k. \tag{2.6}$$

Such a process is called the Brownian motion starting at x.

Definition 2.1.2. *Let $(\Omega,\mathcal{F},P)$ be a probability space. For each $\omega \in \Omega$, suppose there is a continuous function $W : [0,\infty) \to \mathbb{R}$ that satisfies $W(0) = 0$ and that depends on ω. Then $\{W(t), t \geq 0\}$ is a* Brownian motion *if for all $0 = t_0 < t_1 < \ldots < t_m$, the increments*

$$W(t_1) - W(t_0), W(t_2) - W(t_1), \ldots, W(t_m) - W(t_{m-1})$$

are independent and each is normally distributed with

$$E[W(t_{i+1}) - W(t_i)] = 0,$$

$$\mathrm{Var}[W(t_{i+1}) - W(t_i)] = t_{i+1} - t_i.$$

2.2. Lévy Processes

While Brownian motions are classic and powerful tool for modeling a wide range of phenomena, sometimes the processes take on a more sudden nature, and a process with discontinuous capabilities is more adequate. Lévy processes are a general class of such processes. In [4], we have the following definition of a Lévy process:

Definition 2.2.1. *Let* $(X(t), t \geq 0)$ *be a stochastic process defined on a probability space* $(\Omega, \mathcal{F}, P)$. *We say that it has* stationary increments *if each* $X(t_{j+1}) - X(t_j) \overset{d}{=} X(t_{j+1} - t_j) - X(0)$, *where* $\overset{d}{=}$ *means the equality in distribution.*

We say that $X(t)$ *is a* Lévy process *if* $X(0) = 0$ *(a.s.);* X *has independent and stationary increments; and* X *is* stochastically continuous*; i.e., for all* $a > 0$ *and for all* $s \geq 0$,

$$\lim_{t \to s} P(|X(t) - X(s)| > a) = 0.$$

Having a quick way of classifying Lévy processes is crucial to the remainder of this dissertation. To do so, we use the following definition and theorem from [4]:

Definition 2.2.2. *Let* ν *be a Borel measure defined on* $\mathbb{R}^d \setminus \{0\}$. *We say that it is a* Lévy measure *if*

$$\int_{\mathbb{R}^d \setminus \{0\}} (|y|^2 \wedge 1) \nu(dy) < \infty,$$

where $a \wedge b := \min\{a, b\}$ *for any* $a, b \in \mathbb{R}$.

Theorem 2.2.3. (Lévy-Itô decomposition) *Let* $(X_t)_{t \geq 0}$ *be a Lévy process on* $\mathbb{R}$ *and* ν *its Lévy measure. Then*

1. ν *is a random measure on* $\mathbb{R} \setminus \{0\}$ *and verifies:* $\int_{|x| \leq 1} |x|^2 \nu(dx) < \infty$, *and* $\int_{|x| \geq 1} \nu(dx) < \infty$.

2. *The jump measure of* X, *denoted by* J_X, *is a Poisson random measure on* $[0, \infty) \times \mathbb{R}$ *with intensity measure* $\nu(dx)dt$.

3. *There exist* $\gamma, \sigma \in \mathbb{R}$, *with* $\sigma > 0$, *and a Brownian motion* $(W_t)_{t \geq 0}$ *such that*

$$X_t = \gamma t + \sigma W_t + X_t^l + \lim_{\epsilon \to 0} \tilde{X}_t^\epsilon, \tag{2.7}$$

   where $X_t^l = \int_{|x| \geq 1, s \in [0,t]} x J_X(ds \times dx)$, *and* $\tilde{X}_t^\epsilon = \int_{\epsilon \leq |x| < 1, s \in [0,t]} x \left(J_X(ds \times dx) - \nu(dx)ds\right)$.

4. *The terms in* (2.7) *are independent and the convergence in* $\tilde{X}_t^\epsilon$ *is almost sure and uniform in* $t \in [0, T]$.

In particular, every Lévy process is *uniquely* determined by its characteristic triplet (γ, σ, ν) in the decomposition above. Many of the novel theorems in this dissertation rely on manipulations of these characteristic triplets.

There are a host of familiar processes that can be represented with these characteristics. The following are examples, along with representative sample paths. These show that Lévy processes are suited to model a wide variety of phenomena, including the prices of commodities, as considered in Chapter 3.

- **Example 1** Brownian motion: $\gamma = 0$, $\sigma = 1$, $\nu(dx) = 0$.

- **Example 2** Poisson Process: $\gamma = 0$, $\sigma = 0$, $\nu(dx) = \lambda\delta_1(dx)$, where $\lambda > 0$ and δ_1 is point mass.

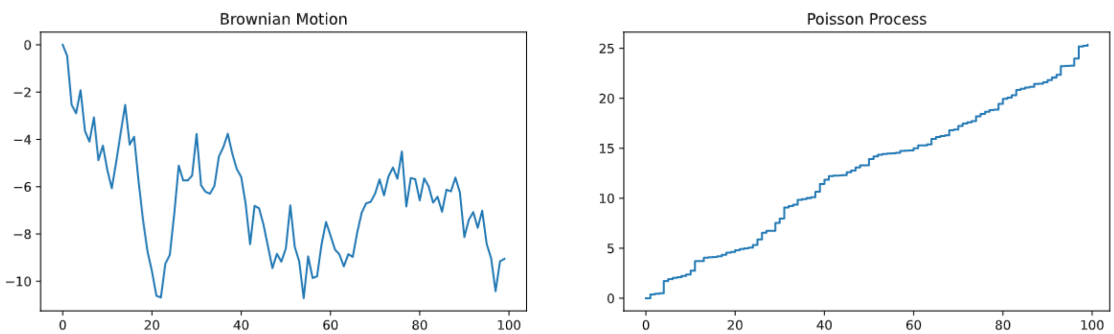

Figure 2.1. Sample Paths of a Standard Brownian Motion and of a Poisson Process with $\lambda = 1/4$.

- **Example 3** Gamma Process: $\gamma = -\int_0^1 x\nu(dx)$, $\sigma = 0$, $\nu(dx) = \beta x^{-1} e^{-\alpha x} 1_{x \geq 0}\, dx$, where $\alpha, \beta > 0$.

- **Example 4** Cauchy Process: $\gamma = 0$, $\sigma = 0$, $\nu(dx) = f(x)dx$, where $f(x) = |x|^{-2}$, $x \neq 0$.

- **Example 5** Wiener Process: $\gamma = m$, $\sigma = s$, $\nu(dx) = 0$, where $m \in \mathbb{R}, s > 0$.

- **Example 6** Subordinator Jump Process: $\gamma = 0$, $\sigma = 0$, $\nu(dx) = f(x)dx$, where $f(x) \geq 0$: $x > 0$ and $f(x) = 0$: $x < 0$.

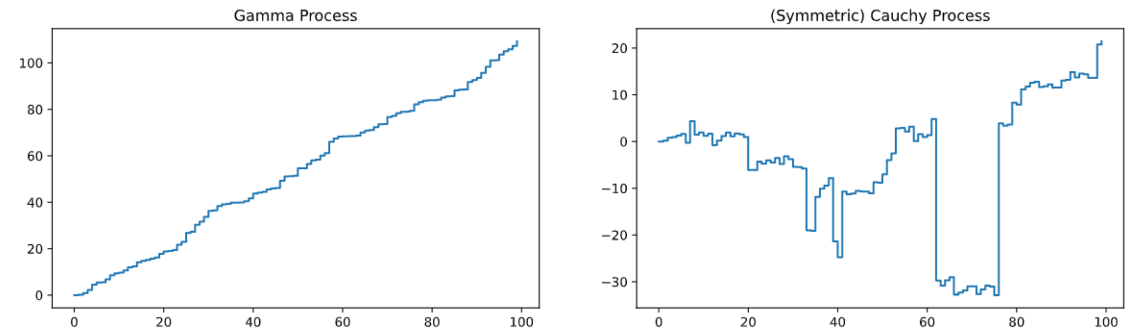

Figure 2.2. Sample Paths of a Gamma Process with Mean and Variance 1 and of a Cauchy Process.

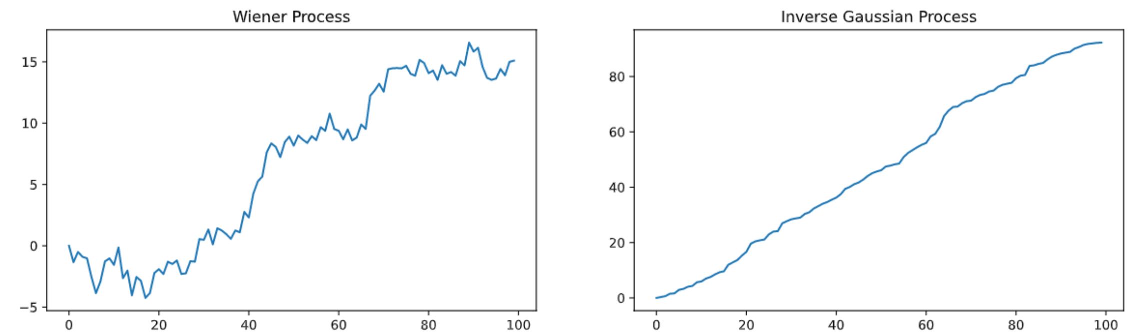

Figure 2.3. Sample Paths of a Wiener Process with $\gamma = 0.2$ and $\sigma = 1$ and of a Subordinator Process, an Inverse Gaussian Process with Mean 1.

2.3. Itô Calculus

The rest of the dissertation uses multiple concepts of integration. In particular, we often integrate with respect to some stochastic process. As such, it is important to understand the following definition from [41]:

Definition 2.3.1. *Let W_t be a Brownian motion and ϕ be a* simple cádlág (right-continuous with left limits) process *with partition $\pi = (0 = T_0, T_1, ..., T_{n+1} = T)$; i.e.,*

$$\phi_t = \phi_0 1_{t=0} + \sum_{i=0}^{n} \phi_i 1_{[T_i, T_{i+1})}.$$

Then the Brownian stochastic integral $\int \phi dW$ is defined as

$$\int_0^T \phi_t dW_t = \sum_{i=0}^{n} \phi_i (W_{T_{i+1}} - W_{T_i}).$$

This definition gives rise to another definition of a class of processes called Itô processes, by which we will define another integral.

Definition 2.3.2. ([54]) *Let $W_t, t \geq 0$ be a Brownian motion and $\mathcal{F}(t), t \geq 0$ be an associated filtration. An* Itô process *is a stochastic process of the form*

$$X(t) = X(0) + \int_0^t \Delta(t) dW_u + \int_0^t \Theta(u) du,$$

where $X(0)$ is nonrandom, and Δ and Θ are adapted stochastic processes.

Theorem 2.3.3. *In particular, Itô processes have quadratic variation*

$$[X, X](t) = \int_0^t \Delta(u)^2 du.$$

Naturally, the previous definition inspires the integral

$$\int_0^t \Gamma(u) dX(u) := \int_0^t \Gamma(u)\Delta(u) dW_u + \int_0^t \Gamma(u)\Theta(u) du,$$

for an adapted process Γ and Itô process X.

Finally, we can state the following:

Theorem 2.3.4. (Itô formula) *Let $X(t), t \geq 0$ be an Itô process and let $f(t, x)$ define a function for which partial derivatives f_t, f_x, and f_{xx} are defined and continuous. Then for every $T \geq 0$,*

$$\begin{aligned} f(T, X(T)) =& f(0, X(0)) + \int_0^T f_t(t, X(t)) dt + \int_0^T f_x(t, X(t)) \Delta(t) dW_t \\ &+ \int_0^T f_x(t, X(t)) \Theta(t) dt + \frac{1}{2} \int_0^T f_{xx}(t, X(t)) \Delta(t)^2 dt, \end{aligned}$$

which may be written, for convenience,

$$df(t, X(t)) = f_t(t, X(t)) dt + f_x(t, X(t)) dX(t) + \frac{1}{2} f_{xx}(t, X(t)) dX(t) dX(t).$$

This is often the case for the remainder of this dissertation: technical integrals are written in differential notation for convenience.

The Itô formula permits us to solve a large number of stochastic differential equations and is crucial in a thorough understanding of the Barndorff-Neilsen and Shephard model (BN-S model), which is investigated in further sections, but for now, let us state a uniqueness and existence theorem for stochastic differential equations:

Theorem 2.3.5. *The system*

$$dX(t) = \alpha(t, X(t))dt + \sigma(t, X(t))dW_t + \int_0^t \int_{\mathbb{R}^n} \gamma(s, X(s^-), z) \left(J_X(ds \times dx) - \nu(dx)ds\right)$$

with $X(0) = x_0 \in \mathbb{R}^n$ *and where* $\alpha : [0,T] \times \mathbb{R}^n \to \mathbb{R}^n$, $\sigma : [0,T] \times \mathbb{R}^n \to \mathbb{R}^{n\times m}$, *and* $\gamma : [0,T] \times \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^{n\times l}$ *and* $\nu = \nu_1 \times \ldots \times \nu_n$ *satisfy the conditions:*

1. *There exists a constant* $C_1 < \infty$ *such that*

$$\|\sigma(t,x)\|^2 + |\alpha(t,x)|^2 + \int_{\mathbb{R}} \sum_{k=1}^{l} |\gamma_k(t,x,z)|^2 \nu_k(dz_k) \leq C_1(1 + |x|^2)$$

   for all $x \in \mathbb{R}^n$.

2. *There exists a constant* $C_2 < \infty$ *such that*

$$\|\sigma(t,x) - \sigma(t,y)\|^2 + |\alpha(t,x) - \alpha(t,y)|^2$$

$$+ \sum_{k=1}^{l} \int_{\mathbb{R}} |\gamma^{(k)}(t,x,z_k) - \gamma^{(k)}(t,y,z_k)|^2 \nu_k(dz_k) \leq C_2|x-y|^2$$

   for all $x, y \in \mathbb{R}^n$,

has a unique cádlág adapted solution $X(t)$ *such that*

$$E[|X(t)|^2] < \infty \text{ for all } t.$$

In the time homogeneous case, when the coefficients do not depend on t, the solutions are called *jump diffusions*.

Now, we introduce Mallivian Calculus.

2.4. Mallivian Calculus

We start with the following definitions (see [41]):

Definition 2.4.1. *A real function* $g : [0,T]^n \longrightarrow \mathbb{R}$ *is called symmetric if* $g(t_{\sigma 1}, \cdots, t_{\sigma n}) = g(t_1, \cdots, t_n)$ *for all permutations* $\sigma = (\sigma_1, \cdots, \sigma_n)$ *of* $(1, 2, \cdots, n)$.

Let $\mathbb{L}^2([0,T]^n)$ be the standard space of square integrable Borel real functions on $[0,T]^n$ such that

$$|g|^2_{\mathbb{L}^2([0,T]^n)} := \int_{[0,T]^n} g^2(t_1, \cdots, t_n) dt_1, \cdots, dt_n < \infty. \tag{2.8}$$

Let $\tilde{\mathbb{L}}^2([0,T]^n) \subset \mathbb{L}^2([0,T]^n)$ be the space of symmetric square integrable Borel real functions on $[0,T]^n$. Let us consider the set

$$S_n = (t_1, \cdots, t_n) \in [0,T]^n : 0 \le t_1 \le t_2 \cdots \le t_n \le T.$$

If $g \in \tilde{\mathbb{L}}^2([0,T]^n)$, then $g\,|_{S_n} \in \mathbb{L}^2(S_n)$ and $|g|^2_{\mathbb{L}^2([0,T]^n)} = n!\,|g|^2_{\mathbb{L}^2(S_n)}$
if f is a real function on $[0,T]^n$,then its symmetrization $\tilde{f}$ is defined as

$$\tilde{f}(t_1, \cdots, t_n) = \frac{1}{n!} \sum_{\sigma} f(t_{\sigma_1}, \cdots, t_{\sigma_n}),$$

where the sum is taken over all permutations σ of $(1, \cdots, n)$.

Definition 2.4.2. *Let f be a deterministic function defined on* $S_n, (n \ge 1)$ *such that*

$$|f|^2_{\mathbb{L}^2(S_n)} := \int_{S_n} f^2(t_1, \cdots, t_n) dt_1, \cdots, dt_n.$$

Then we can define the n-fold iterated Itô integral as

$$J_n(f) = \int_0^T \int_0^{t_n} \cdots \int_0^{t_3} \int_0^{t_2} f(t_1, \cdots, t_n) dW_{t_1} \cdots dW_{t_n}.$$

Definition 2.4.3. *If* $g \in \tilde{\mathbb{L}}^2([0,T]^n)$,*we define*

$$I_n(g) := \int_{[0,T]^n} g(t_1, \cdots, t_n) dW_{t_1} \cdots dW_{t_n} := n! J_n(g). \tag{2.9}$$

We also call n-fold iterated Itô integrals the $I_n(g)$ here above.

Theorem 2.4.4. *Let ξ be an $\mathcal{F}_T$ measurable random variable in $\mathbb{L}^2(P)$. Then there exists a unique sequence $(f_n)_{n=0}^{\infty}$ of functions $f_n \in \tilde{\mathbb{L}}^2([0,T]^n)$ such that*

$$\xi = \sum_{n=0}^{\infty} I_n(f_n),$$

where the convergence is in $\mathbb{L}^2(P)$. Moreover, we have the following isometry

$$|\xi|^2_{\mathbb{L}^2(P)} = \sum_{n=0}^{\infty} n!\, |f_n|^2_{\mathbb{L}^2([0,T]^n)}\,. \tag{2.10}$$

Proof. A proof can be found in [41]. □

Let $u = u(t,\omega)$, $t \in [0,T]$,$\omega \in \Omega$ be a measurable stochastic process such that, for all $t \in [0,T]$, u(t) is a $\mathcal{F}_t$ measurable random variable and $E[u^2(t)] < \infty$.
Then, for each $t \in [0,T]$, we can apply the Wiener-Itô chaos expansion to the random variable $u(t) = u(t,\omega)$, $\omega \in \Omega$, and thus there exist the symmetric functions $f_{n,t} = f_{n,t}(t_1, \cdots, t_n, t) := f_{n,t}(t_1, \cdots, t_n)$.

Definition 2.4.5. *Let u(t), $t \in [0,T]$, be a measurable stochastic process such that for all $t \in [0,T]$ the random variable u(t) is $\mathcal{F}_t$- measurable and $E[\int_0^T u^2(t)dt] < \infty$. Let its Wiener-Itô chaos expansion be*

$$u(t) = \sum_{n=0}^{\infty} I_n(f_{n,t}).$$

Then we define the Skorohod integral of u by

$$\delta(u) = \int_0^T u(t)\delta W(t) = \sum_{n=0}^{\infty} I_{n+1}(\tilde{f_n}), \tag{2.11}$$

when convergent in $\mathbb{L}^2(P)$.Here $\tilde{f_n}$, $n = 1, 2, \cdots$, are the symmetric functions derived from $f_n(.,t), n = 1, 2, \cdots$. We say that u is Skorohod integrable,we write $u \in Dom(\delta)$ if the series in (2.11) converges to $\mathbb{L}^2(P)$

a stochastic process u belongs to Dom(δ) iff

$$E[\delta^2(u)] = \sum_{n=0}^{\infty}(n+1)! \left|\tilde{f_n}\right|^2_{\mathbb{L}^2([0,T]^{n+1})} < \infty.$$

Lemma 2.4.6. *For any $u \in Dom(\delta)$ the Skorohod integral has zero expectation, that is,*

$$E[\delta(u)] = 0.$$

Proof. A proof can be found in [41]. □

Lemma 2.4.7. *Let $u = u(t)$, $t \in [0,T]$, be a measurable stochastic process such that, for all $t \in [0,T]$, the random variable u(t) is $\mathcal{F}_t$- measurable and $E[u^2(t)] < \infty$. Let*

$$u(t) = \sum_{n=o}^{\infty} I_n(f_n(.,t)),$$

be its Wiener-Itô chaos expansion. Then u is $\mathcal{F}$- adapted iff

$$f_n(t_1, \cdots, t_n, t) = 0, \tag{2.12}$$

for every $t < max(t_i)$, $1 \leq i \leq n$.

Proof. A proof can be found in [41]. □

Theorem 2.4.8. *Let $u = u(t), t \in [0,T]$, be a measurable $\mathcal{F}$-adapted stochastic process such that*

$$E\left[\int_0^T u^2(t)dt\right] < \infty. \tag{2.13}$$

Then $u \in Dom(\delta)$ and its Skorohod integral coincides with the Itô Integral

$$\int_0^T u(t)\delta W(t) = \int_0^T u(t)dW(t). \tag{2.14}$$

Proof. A proof can be found in [41]. □

2.5. Mallivian Derivative

The Mallivian Calculus was originally created as a tool for studying the regularity of densities of solutions of stochastic differential equations. Today, the range of applications has extended even further to include numerical methods,stochastic control, and not just for systems driven by Brownian motion, but for systems driven by general Lévy process.

Definition 2.5.1. *Let $F \in \mathbb{L}^2(P)$ be $\mathcal{F}_t$ measurable with chaos expansion*

$$F = \sum_{n=0}^{\infty} I_n(f_n), \tag{2.15}$$

where $f_n \in \tilde{\mathbb{L}}^2([0,T]^n), n = 1, 2, \cdots$.

We say that $F \in \mathcal{D}_{1,2}$ if

$$\left|\tilde{F}\right|^2_{\mathcal{D}_{1,2}} = \sum_{n=1}^{\infty} nn! \left|f_n\right|^2_{L^2([0,T]^n)}$$

Definition 2.5.2. *If $F \in \mathcal{D}_{1,2}$, we define the Mallivian derivative $\mathcal{D}_t F$ of F at time t as the expansion*

$$\mathcal{D}_t F = \sum_{n=1}^{\infty} n I_{n-1}(f_n(.,t)), \quad t \in [0,T].$$

where $I_{n-1}(f_n(.,t))$ is the (n-1)- fold iterated integral of $f_n(t_1, \cdots, t_{n-1}, t)$ with respect to the first $(n-1)$ variables $t_1, \cdots, t_{n-1}$ and $t_n = t$ left as parameter.

Theorem 2.5.3. *Let $G \in \mathcal{D}_{1,2}$ and $g \in \mathcal{C}^1(\mathcal{R})$ with bounded derivative. Then $g(G) \in \mathcal{D}_{1,2}$ and*

$$\mathcal{D}_t g(G) = g^{'}(G) \mathcal{D}_t G. \tag{2.16}$$

Proof. A proof can be found in [41]. □

3. THE BARNDORFF-NIELSEN AND SHEPHARD MODEL AND A GENERALIZATION

3.1. Barndorff-Nielsen and Shephard Model, Self-Decomposability, and an Approximation

Financial time series of different assets share many common features which are successfully captured by the stochastic model introduced in various works of Ole Barndorff-Nielsen and Neil Shephard. The model is known in modern literature as the Barndorff-Nielsen and Shephard (BN-S) model (see [8, 10, 11]). This model is revised and refined in various recent works in literature such as [49, 50]. This model is successfully implemented in the commodity markets as well (see [52, 64]). Recently, this model is improved using various machine-learning driven algorithms (see [51, 53]).

For the BN-S model, a frictionless financial market is considered where a risk-less asset with constant interest rate r, and a stock, are traded up to a fixed horizon date T. It is assumed that the price process of the stock $S = \{S_t\}_{t\geq 0}$ is defined on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$ and is given by:

$$S_t = S_0 \exp(X_t), \tag{3.1}$$

where the log-return X_t is given by

$$dX_t = (\mu_1 + \beta_1 \sigma_t^2)\, dt + \sigma_t\, dW_t + \rho\, dZ_{\lambda t}, \tag{3.2}$$

with the variance process

$$d\sigma_t^2 = -\lambda \sigma_t^2\, dt + dZ_{\lambda t}, \quad \sigma_0^2 > 0, \tag{3.3}$$

where the parameters $\mu_1, \beta_1 \in \mathbb{R}$ with $\lambda > 0$ and $\rho < 0$. In (3.2) and (3.3), W_t and Z_t are a Brownian motion and a Lévy subordinator, respectively. The Lévy subordinator Z is referred to as the background driving Lévy process (BDLP). Also W and Z are assumed to be independent and $(\mathcal{F}_t)$ is assumed to be the usual augmentation of the filtration generated by the pair (W, Z). Without loss of generality, we assume $W_0 = Z_0 = 0$.

We assume Z satisfies the assumptions described in [40]. We describe the assumptions below:

Assumption 3.1.1. *Z has no deterministic drift and its Lévy measure has a Lévy density.*

Assumption 3.1.2. *Denote the cumulant transforms $\kappa(\theta) = \log E[e^{\theta Z_1}]$, and $\hat{\theta} = \sup\{\theta \in \mathbb{R} : \kappa(\theta) < +\infty\}$. Then $\hat{\theta} > 0$.*

Assumption 3.1.3. $\lim_{\theta \to \hat{\theta}} \kappa(\theta) = +\infty$.

It follows that the cumulant transform, where it exists, takes the form $\kappa(\theta) = \int_{\mathbb{R}_+} (e^{\theta x} - 1) w(x)\, dx$, where $w(x)$ is the Lévy density for Z. It is shown in [40] (Theorem 3.2) that there exists an equivalent martingale measure (EMM) $\mathbb{Q}$, under which equations (3.2) and (3.3) can be written as:

$$dX_t = b_t\, dt + \sigma_t\, dW_t + \rho\, dZ_{\lambda t}, \quad \text{with} \quad b_t = (r - \lambda\kappa(\rho) - \frac{1}{2}\sigma_t^2), \tag{3.4}$$

$$d\sigma_t^2 = -\lambda\sigma_t^2\, dt + dZ_{\lambda t}, \quad \sigma_0^2 > 0, \tag{3.5}$$

where W_t and Z_t are a Brownian motion and a Lévy subordinator respectively with respect to $\mathbb{Q}$. For the rest of this thesis we assume that the risk-neutral dynamics (with respect to $\mathbb{Q}$) of the stock price is given by (3.17), (3.4) and (3.5). It is trivial to show that the solution of (3.5) is given by

$$\sigma_t^2 = e^{-\lambda t}\sigma_0^2 + \int_0^t e^{-\lambda(t-s)}\, dZ_{\lambda s}. \tag{3.6}$$

From (3.6), the positivity of the process σ_t^2 is obvious. In fact, σ_t^2 is bounded below by the deterministic function $e^{-\lambda t}\sigma_0^2$. In addition, the instantaneous variance of log-return X_t is given by $(\sigma_t^2 + \rho^2\lambda\mathrm{Var}[Z_1])\, dt$. Consequently, the continuous realized variance in the interval $[0, T]$, denoted as σ_R^2, is given by $\sigma_R^2 = \frac{1}{T}\int_0^T \sigma_t^2\, dt + \rho^2\lambda\mathrm{Var}[Z_1]$. Therefore, by (3.6) we obtain

$$\sigma_R^2 = \frac{1}{T}\left(\lambda^{-1}(1 - e^{-\lambda T})\sigma_0^2 + \lambda^{-1}\int_0^T \left(1 - e^{-\lambda(T-s)}\right) dZ_{\lambda s}\right) + \rho^2\lambda\mathrm{Var}[Z_1]. \tag{3.7}$$

We state some results for the analysis of the variance process σ_t^2, when the process is stationary and self-decomposable. The results are motivated by [23, 24, 28]. The pricing formulas for various derivatives are dependent on the variance process.

Definition 3.1.4. *The distribution of a random variable X is said to be self-decomposable if for any constant c, $0 < c < 1$, there exists an independent random variable $X^{(c)}$, such that $X \stackrel{d}{=} cX + X^{(c)}$, where $\stackrel{d}{=}$ stands for the equality in the distribution.*

For self-decomposable laws the associated densities are unimodal (see [17, 48]). It is proved in [9, 65] that, if X is self-decomposable then there exists a *stationary stochastic process* $\{\sigma^2(t)\}_{t\geq 0}$, and a Lévy subordinator $\{Z_t\}_{t\geq 0}$, independent of σ_0^2, such that $\sigma_t^2 \stackrel{d}{=} X$ for all $t \geq 0$ and

$$\sigma_t^2 = \exp(-\lambda t)\sigma_0^2 + \int_0^t \exp\left(-\lambda(t-s)\right) dZ_{\lambda s}, \quad \text{for all } \lambda > 0.$$

Conversely, if $\{\sigma_t^2\}_{t\geq 0}$, is a *stationary stochastic process* and $\{Z_t\}_{t\geq 0}$ is a Lévy subordinator independent of σ_0^2, such that $\{\sigma_t^2\}$ and $\{Z_t\}$ satisfy

$$d\sigma_t^2 = -\lambda\sigma_t^2\, dt + dZ_{\lambda t}, \quad \sigma_0^2 > 0,$$

for all $\lambda > 0$, then σ_t^2 is self-decomposable.

It is clear from [48] (Theorem 17.5(ii)) that for any self-decomposable law D there exists a Lévy subordinator Z such that the process of OU type driven by Z has invariant distribution given by D. The following theorem (see [23, 24, 50]) gives the relation between the Lévy densities of such process generated by σ_t^2 and Z in (3.5).

Theorem 3.1.5. *A random variable X has law in L if and only if X has a representation of the form $X = \int_0^\infty e^{-t}\, dZ_t$, where Z_t is a Lévy subordinator. In this case the Lévy measure U and W of X and Z_1 are related by $U(dx) = \int_0^\infty W(e^t\, dx)\, dt$. In addition, if $u(x)$, the Lévy density of U is differentiable, then the Lévy measure W has a density w, and u and w are related by*

$$w(x) = -u(x) - xu'(x). \tag{3.8}$$

There are many known self-decomposable distributions, such as inverse Gaussian (IG), Gamma, positive tempered stable (PTS), etc.

Consequently, if the stationary distribution of σ_t^2 is given by IG(δ_1, γ) law, with the Lévy density $u(x) = \frac{1}{\sqrt{2\pi}}\delta_1 x^{-3/2}\exp(-\gamma^2 x/2)$, $x > 0$, then by (3.8), the Lévy density of Z_1 is given by

$w(x) = \frac{\delta_1}{2\sqrt{2\pi}} x^{-\frac{3}{2}}(1+\gamma^2 x)e^{-\frac{1}{2}\gamma^2 x}$, $x > 0$. Alternatively, if the stationary distribution of σ_t^2 is given by gamma law $\Gamma(\nu, \alpha)$, where the Lévy density of $\Gamma(\nu, \alpha)$ is given by $u(x) = \nu x^{-1} e^{-\alpha x}$, $x > 0$, then by (3.8) we obtain $w(x) = \nu\alpha e^{-\alpha x}$, $x > 0$.

A three-parameter self-decomposable process is positive tempered stable (PTS) process (see [15, 16]). It is denoted as $\mathrm{PTS}(\kappa, \delta, \gamma)$, where $\beta > 0$, $0 < \gamma < 1$, and $k \geq 0$. For $\mathrm{PTS}(\kappa, \delta, \gamma)$ process the Lévy density is simple and is given by (see [23, 24])

$$u(x) = \beta k^{-2\gamma} \frac{\gamma}{\Gamma(\gamma)\Gamma(1-\gamma)} x^{-\gamma-1} \exp\left(-\frac{1}{2}k^2 x\right), \quad x > 0.$$

If the stationary distribution of σ_t^2 is given by $\mathrm{PTS}(\kappa, \delta, \gamma)$ law, then by (3.8) we obtain that the Lévy density of Z_1 is given by

$$w(x) = \frac{\beta k^{-2\gamma} \gamma x^{-\gamma-1} e^{\frac{-k^2 x}{2}}}{\Gamma(\gamma)\Gamma(1-\gamma)} \left(\gamma + \frac{k^2 x}{2}\right), \quad x > 0. \tag{3.9}$$

In the above discussions we find that the distribution of Z is analytically tractable when the stationary distribution of σ_t^2 in (3.5) is given by a stationary, self-decomposable distribution. We denote (as σ_t^2 is stationary),

$$\sigma = E^{\mathbb{Q}}(\sigma_1^2), \tag{3.10}$$

and

$$\mu = r - \lambda\kappa(\rho) - \frac{1}{2}\sigma^2. \tag{3.11}$$

We approximate (3.4) by

$$dX_t = \mu\, dt + \sigma\, dW_t + \rho\, dZ_{\lambda t}. \tag{3.12}$$

We refer to (3.17) and (3.12), as an *approximation of the BN-S model* (3.17), (3.4), and (3.5). For most of the empirical financial data $\mu \leq 0$.

We write $X_t = \mu t + \sigma W_t + \rho Z_t$, with $\mu \in \mathbb{R}$, $\sigma > 0$, and $\rho < 0$, $t > 0$. For financial applications $\mu \leq 0$. For the subsequent sections we develop a general procedure to compute the first-exit time of the stochastic process X_t.

3.2. A Generalization of the BN-S Model for Application in the Soyabean Export Market

We assume that the soybean export market share price S_t is given by

$$S_t = S_0 e^{X_t}, \quad \text{where} \quad dX_t = b_t\,dt + \sum_{i=1}^{n} \theta_t^{(i)} \left(\sigma_t\,dW_t^{(i)} + dJ_t^{(i)} \right), \tag{3.13}$$

where b_t is a deterministic function of t, $W_t^{(i)}$, $i = 1, \ldots, n$, are independent Brownian motions, and $J_t^{(i)}$ is the jump process with intensities λ_i, $i = 1, \ldots, n$. We assume that $W_t^{(i)}$ and $J_t^{(i)}$, for $i = 1, \ldots, n$, are independent. The coefficients $\theta_t^{(i)}$, at every t satisfy $\sum_{i=1}^{n} (\theta_t^{(i)})^2 = 1$. In addition to that, σ_t is assumed to be stochastic, and its dynamics is governed by

$$d\sigma_t^2 = F(\sigma_t^2, \beta_t^{(1)} H_t^{(1)}, \beta_t^{(2)} H_t^{(2)}, \ldots, \beta_t^{(n)} H_t^{(n)}), \tag{3.14}$$

for an appropriate function F, where $H_t^{(j)}$, for $j = 1, \ldots, n$, are jump processes with intensities μ_j, $j = 1, \ldots, n$. The coefficients $\beta_t^{(j)}$, at every t satisfy $\sum_{j=1}^{n} (\beta_t^{(j)})^2 = 1$. For simplicity, for the rest of the paper, we assume $\theta^{(i)} = \beta^{(i)}$, for $i = 1, \ldots, n$.

There are several justifications of modeling a market share for soybeans with (3.13) and (6.4), over existing models in the literature. First of all, most exiting models use a single jump term for the dynamics of the log-return process X_t of the market share for soybeans. However, given the involved nature of jumps in a market share for soybeans, it is unlikely to be modeled by a single jump-term. Consequently, the proposed model provides a great deal of flexibility in terms of modeling. Secondly, the coefficients $\theta^{(i)}$, $i = 1, \ldots, n$, will aid in extracting various important features of a market share for soybeans dynamics. This is obviously not the case for a single jump (or, no jump) model. Finally, the proposed model in fact incorporates most of the existing models.

We consider the export market share for soybeans exported from the US Gulf (USG) for a given period of time. Figure 1 is a graphical representation of the data. From the empirical data, it is clear that a single jump term for "'big" fluctuations is very unlikely. Consequently, we propose to use the model as given by (3.13).

We consider that for the feature variables the individual dynamics are given by $e^{Y_t^{(i)}}$ where

$$dY_t^{(i)} = \sigma_t \, dW_t^{(i)} + dJ_t^{(i)}, \quad i = 1, \ldots, n.$$

Thus, from (3.13), we obtain $dX_t = b_t \, dt + \sum_{i=0}^{n} \theta_t^{(i)} dY_t^{(i)}$. We call $\theta_t^{(i)}$ as the "importance factor" for the i-th feature component, for $i = 1, \ldots, n$. We observe, that if $\sum_{i=1}^{n} (\theta_t^{(i)})^2 = 1$, then $\sum_{i=1}^{n} \theta_t^{(i)} \, dW_t^{(i)}$ can be represented by dB_t, where B_t is a Brownian motion. Consequently, (3.13) can be written as

$$S_t = S_0 e^{X_t}, \quad \text{where} \quad dX_t = b_t \, dt + \sigma_t dB_t + \sum_{i=1}^{n} \theta_t^{(i)} dJ_t^{(i)}. \tag{3.15}$$

The expression (3.15) provides an alternative explanation for the coefficients "importance factors". Those represent the significance in terms of big fluctuations (or "'jumps") of the i-th feature component $Y_t^{(i)}$. We write $J_t^{(i)}$ in terms of integral with respect to Poisson random measures $N^{(i)}(dt, dx)$, for $i = 1, \ldots, n$. Consequently,

$$J_t^{(i)} = \int_0^t \int_{\mathbb{R}} x N^{(i)}(dt, dx).$$

Hence (3.15) can be written as

$$S_t = S_0 e^{X_t}, \quad \text{where} \quad dX_t = b_t \, dt + \sigma_t dB_t + \sum_{i=1}^{n} \theta_t^{(i)} \int_{\mathbb{R}} x N^{(i)}(dt, dx). \tag{3.16}$$

We consider a special case of this model for developing some mathematical analysis. The model is the Barndorff-Nielsen & Shephard model (BN-S model, see [10, 11, 28, 23]), where the soybean export market share price $S = (S_t)_{t \geq 0}$ on some filtered probability space $(\Omega, \mathcal{G}, (\mathcal{G}_t)_{0 \leq t \leq T}, \mathbb{P})$ is given by

$$S_t = S_0 \exp(X_t), \tag{3.17}$$

$$dX_t = (\mu + \beta \sigma_t^2) \, dt + \sigma_t \, dW_t + \rho \, dZ_{\lambda t}, \tag{3.18}$$

$$d\sigma_t^2 = -\lambda\sigma_t^2\, dt + dZ_{\lambda t}, \quad \sigma_0^2 > 0, \tag{3.19}$$

where the parameters $\mu, \beta, \rho, \lambda \in \mathbb{R}$ with $\lambda > 0$ and $\rho \leq 0$ and r is the risk free interest rate where a stock or commodity is traded up to a fixed horizon date T. In the above model W_t is a Brownian motion, and the process $Z_{\lambda t}$ is a subordinator. Also W and Z are assumed to be independent, and $(\mathcal{G}_t)$ is assumed to be the usual augmentation of the filtration generated by the pair (W, Z).

The BN-S model is a special case of (3.16), where $dZ_s^{(i)} = \frac{1}{\rho}\int_0^\infty xN^{(i)}(ds, dx), \quad i = 1, \ldots, n,$ are subordinators. The BN-S model has been successfully implemented to oil in various cenent works (see [47, 47, 52]). Making a scaling in the time variable, we define $s = \lambda t$, for $\lambda > 0$. Then, we obtain, $dZ_{\lambda t}^{(i)} = \frac{1}{\rho}\int_0^\infty xN^{(i)}(\lambda\, dt, dx), \quad i = 1, \ldots, n,$ are subordinators. Consequently, we consider $S = (S_t)_{t\geq 0}$ on some filtered probability space $(\Omega, \mathcal{F}, (\mathcal{F}_t)_{0\leq t\leq T}, \mathbb{P})$, is given by (3.16). Thus we obtain the dynamics of X_t as

$$dX_t = (\mu + \beta\sigma_t^2)\, dt + \sigma_t\, dB_t + \rho\sum_{i=1}^n \theta_t^{(i)} dZ_{\lambda t}^{(i)}, \tag{3.20}$$

where $Z^{(i)}$, $i = 1, \ldots, n$ are independent subordinators. Machine learning algorithms can be implemented to determine the value of θ. The processes $Z^{(i)}$ have various intensities. Also, we assume that B_t, and $Z^{(i)}$, $i = 1, \ldots, n$, are independent, and $(\mathcal{F}_t)$ is assumed to be the usual augmentation of the filtration generated by $(W, Z^{(i)})$, $i = 1, \ldots, n$. In this case (3.19) will be given by

$$d\sigma_t^2 = -\lambda\sigma_t^2\, dt + \sum_{i=1}^n \theta_t^{(i)} dZ_{\lambda t}^{(i)}, \quad \sigma_0^2 > 0. \tag{3.21}$$

The solution of (3.21) can be explicitly written as

$$\sigma_t^2 = e^{-\lambda t}\sigma_0^2 + \int_0^t e^{-\lambda(t-s)} \sum_{i=1}^n \theta_s^{(i)} dZ_{\lambda s}^{(i)}. \tag{3.22}$$

The *integrated variance* over the time period $[t, T]$ is given by $\sigma_I^2 = \int_t^T \sigma_s^2\, ds$, and a straight-forward calculation shows

$$\sigma_I^2 = \epsilon(t, T)\sigma_t^2 + \int_t^T \epsilon(s, T) \sum_{i=1}^n \theta_s^{(i)} dZ_{\lambda s}^{(i)}, \tag{3.23}$$

where

$$\epsilon(s,T) = (1 - \exp(-\lambda(T-s)))/\lambda, \quad t \le s \le T. \tag{3.24}$$

We derive a general expression for the characteristic function of the conditional distribution of the market share for soybeans process appearing in the stochastic model given by equations (3.17), (3.20) and (3.21).

As shown in [51], the advantages of the dynamics given by (3.17), (3.20), and (3.21) over the existing models are significant. The following theorem is proved in [51]. From this result, it is clear that as θ is constantly adjusted, for a fixed s, the value of t always has an upper limit. Consequently, $\mathrm{Corr}(X_t, X_s)$ never becomes very small, and thus *long-range dependence* is incorporated in the model.

Theorem 3.2.1. *If the jump measures associated with the subordinators Z and $Z^{(b)}$ are J_Z and $J_Z^{(b)}$ respectively, and $J(s) = \int_0^s \int_{\mathbb{R}^+} J_Z(\lambda d\tau, dy)$, $J^{(b)}(s) = \int_0^s \int_{\mathbb{R}^+} J_Z^{(b)}(\lambda d\tau, dy)$; then for the log-return of the market share for soybeans for the improved BN-S model given by* (3.17), (3.20), *and* (3.21),

$$Corr(X_t, X_s) = \frac{\int_0^s \sigma_\tau^2 d\tau + \rho^2(1-\theta)^2 J(s) + \rho^2\theta^2 J^{(b)}(s)}{\sqrt{\alpha(t)\alpha(s)}}, \tag{3.25}$$

for $t > s$, where $\alpha(\nu) = \int_0^\nu \sigma_\tau^2 d\tau + \nu\rho^2\lambda((1-\theta)^2\, Var(Z_1) + \theta^2\, Var(Z_1^{(b)}))$.

We denote the cumulant transforms as $\kappa^{(i)}(\theta) = \log E^{\mathbb{P}}[e^{\theta Z_1^{(i)}}]$. In this work, we make the following assumption similar to [40, 50].

Assumption 3.2.2. *Assume that $\hat{\theta}^{(i)} = \sup\{\theta \in \mathbb{R} : \kappa^{(i)}(\theta) < +\infty\} > 0$, for $i = 1, \ldots, n$.*

We state the following well-known result from [40, 50] and denote the real part and imaginary part of $z \in \mathbb{C}$ as $\Re(z)$ and $\Im(z)$, respectively.

Theorem 3.2.3. *Let Z be a subordinator with cumulant transform κ, and let $f : \mathbb{R}_+ \to \mathbb{C}$ be a complex-valued, left continuous function such that $\Re(f) \le 0$. Then*

$$E\left[\exp\left(\int_0^t f(s)\, dZ_{\lambda s}\right)\right] = \exp\left(\lambda \int_0^t \kappa(f(s))\, ds\right). \tag{3.26}$$

The above formula still holds if $Z = Z^{(i)}$ *satisfies Assumption 3.2.2 and* f *is such that* $\Re(f) \leq \frac{\hat{\theta}^{(i)}}{(1+\epsilon)}$, $i = 1, \ldots, n$, *for* $\epsilon > 0$.

The *Laplace transform* of $X_{T|t}$, the conditional distribution of X_T given the information up to time $t \leq T$, is given by $\phi(z) = E^{\mathbb{P}}[\exp(zX_T)|\mathcal{F}_t]$, for $z \in \mathbb{C}$ such that the expectation is well-defined.

Theorem 3.2.4. *In the case of the stochastic model as described in equations* (3.17), (3.20) *and* (3.21), *the Laplace transform* $\phi(z) = E[\exp(zX_T)|\mathcal{F}_t]$ *of* $X_{T|t}$ *is given by*

$$\phi(z) = \exp\left(z(X_t + \mu(T-t)) + \frac{1}{2}(z^2 + 2\beta z)\epsilon(t,T)\sigma_t^2 + \lambda \sum_{i=1}^{n} \int_t^T G^{(i)}(s,z)\, ds\right), \tag{3.27}$$

where $G^{(i)}(s,z) = \kappa^{(i)}\left((\rho z + \frac{1}{2}(z^2 + 2\beta z)\epsilon(s,T))\theta_s^{(i)}\right)$.

The transform $\phi(z)$ *is well defined in the open strip* $\mathcal{S} = \{z \in \mathbb{C} : \Re(z) \in (\theta_-, \theta_+)\}$, *where*

$$\theta_-^{(i)} = \sup_{t \leq s \leq T}\left[\left(-\beta - \frac{\rho}{\epsilon(s,T)} - \sqrt{\Delta_1^{(i)}}\right)\theta_s^{(i)}\right], \quad \theta_- = \max_i \theta_-^{(i)}$$

and

$$\theta_+^{(i)} = \inf_{t \leq s \leq T}\left[\left(-\beta - \frac{\rho}{\epsilon(s,T)} + \sqrt{\Delta_1^{(i)}}\right)\theta_s^{(i)}\right], \quad \theta_+ = \min_i \theta_+^{(i)}$$

where $\Delta_1^{(i)} = (\beta + \frac{\rho}{\epsilon(s,T)})^2 + 2\frac{\hat{\theta}^{(i)}}{\epsilon(s,T)}$.

Proof. We obtain from equation (3.20)

$$X_T = \zeta + \beta\sigma_I^2 + \int_t^T \sigma_s\, dW_s + \rho \int_t^T \sum_{i=1}^{n} \theta_s^{(i)} dZ_{\lambda s}^{(i)},$$

where $\zeta = X_t + \mu(T-t)$. Let $\mathcal{G}$ denote the σ-algebra generated by $Z^{(i)}$, $i = 1,\ldots,n$, up to time T by $\mathcal{F}_t$. Then, proceeding by iterated conditional expectations, we obtain

$$\begin{aligned}
\phi(z) &= E^{\mathbb{P}}[\exp(zX_T)|\mathcal{F}_t] \\
&= E^{\mathbb{P}}\left[E^{\mathbb{P}}\left[\exp\left(z(\zeta+\beta\sigma_I^2+\int_t^T \sigma_s\, dW_s + \rho\int_t^T \sum_{i=1}^n \theta_s^{(i)} dZ_{\lambda s}^{(i)})\right)|\mathcal{G}\right]|\mathcal{F}_t\right] \\
&= E^{\mathbb{P}}\left[\exp\left(z(\zeta+\beta\sigma_I^2+\rho\int_t^T \sum_{i=1}^n \theta_s^{(i)} dZ_{\lambda s}^{(i)})\right)E^{\mathbb{P}}\left[\exp(z\int_t^T \sigma_s\, dW_s)|\mathcal{G}\right]|\mathcal{F}_t\right] \\
&= E^{\mathbb{P}}\left[\exp\left(z(\zeta+\beta\sigma_I^2+\rho\int_t^T \sum_{i=1}^n \theta_s^{(i)} dZ_{\lambda s}^{(i)}) + \frac{1}{2}\sigma_I^2 z^2\right)|\mathcal{F}_t\right].
\end{aligned}$$

Using (3.23) we obtain

$$\phi(z) = \exp\left(\zeta z + \frac{1}{2}\epsilon(t,T)\sigma_t^2(z^2+2\beta z)\right)E^{\mathbb{P}}\left[\exp\left(\int_t^T \left(\rho z + \frac{1}{2}(z^2+2\beta z)\epsilon(s,T)\right)\sum_{i=1}^n \theta_s^{(i)} dZ_{\lambda s}^{(i)}\right)\right].$$

Using the independence of the processes $Z^{(i)}$, $i = 1,\ldots,n$, we obtain

$$\begin{aligned}
&E^{\mathbb{P}}\left[\exp\left(\int_t^T \left(\rho z + \frac{1}{2}(z^2+2\beta z)\epsilon(s,T)\right)\sum_{i=1}^n \theta_s^{(i)} dZ_{\lambda s}^{(i)}\right)\right] \\
&= \Pi_{i=1}^n E^{\mathbb{P}}\left[\exp\left(\int_t^T \left(\rho z + \frac{1}{2}(z^2+2\beta z)\epsilon(s,T)\right)\theta_s^{(i)} dZ_{\lambda s}^{(i)}\right)\right].
\end{aligned}$$

Clearly if $z \in \mathcal{S}$, then $\Re((\rho z + \frac{1}{2}(z^2+2\beta z)\theta_s^{(i)}) < \hat{\theta}^{(i)}$. Thus the result follows from (3.26). $\square$

In order to study the characteristic of the "importance factors" ($\theta_t^{(i)}$, $i = 1,\ldots,n$), the following result is useful. In is providing a decomposition of first exit-time (3.20) in terms of the individual subordinators. For simplicity, we assume that $\sigma_t = \sigma$ is constant and the positive factors ($\theta_t^{(i)}$, $i = 1,\ldots,n$) are incorporated in the subordinators $Z^{(i)}$, $i = 1,\ldots,n$.

4. THE FIRST EXIT TIME ANALYSIS

4.1. First-Exit Time for a Combination of a Brownian Motion and a Lévy Subordinator

In this section, we develop a couple of results related to the first-exit time analysis of log-return processes [5] of the form (3.12). At first, we develop the result related to the first-exit time of a simpler process $W_t + Y_t$, where Y is a Lévy subordinator, with $W_0 = Y_0 = 0$. If X_1 and X_2 are independent random variables, we denote $X_1 \perp X_2$.

Theorem 4.1.1. *For a Brownian motion W_t and a Lévy subordinator Y_t, and $a, b > 0$,*

$$\inf\{\tau > 0 : W_\tau + Y_\tau \geq a + b\} = \inf\{t > 0 : W_t \geq a\} + \inf\{\alpha > 0 : Y_\alpha \geq b\}, \tag{4.1}$$

with probability

$$P = \int_0^\infty \int_0^\infty \int_{-\infty}^\infty P_1(\epsilon; t, \alpha) P_2(\epsilon; t, \alpha)\, d\epsilon\, dt\, d\alpha, \tag{4.2}$$

where

$$P_1(\epsilon; t, \alpha) = \int_{-\infty}^\infty \frac{e^{\frac{-\tau^2}{2\alpha}}}{\sqrt{2\pi\alpha}} \left(\int_{\max(a, a-\epsilon-\tau)}^\infty \frac{e^{\frac{-s^2}{2t}}}{\sqrt{2\pi t}} ds \right) d\tau, \tag{4.3}$$

and

$$P_2(\epsilon; t, \alpha) = \int_0^\infty f_{Y_t}(\beta) \left(\int_{\max(\max(b, b+\epsilon-\beta), 0)}^\infty f_{Y_\alpha}(s) ds \right) d\beta, \tag{4.4}$$

where the probability density function of Y_t is given by $f_{Y_t}(\cdot)$.

Proof. The first-exit time of a combination of W_t and Y_t, in the sense that its value is more than $a + b$, is given by

$$\begin{aligned} &\inf\{\tau > 0 : W_\tau + Y_\tau \geq a + b\} \\ &= \inf\{t + \alpha > 0 : W_{t+\alpha} + Y_{t+\alpha} \geq a + b, t > 0, \alpha > 0\}. \end{aligned}$$

For a fixed $\epsilon \in \mathbb{R}$, we define

$$P_1(\epsilon; t, \alpha) = P(W_{t+\alpha} \geq a - \epsilon, W_t \geq a), \tag{4.5}$$

$$P_2(\epsilon; t, \alpha) = P(Y_{t+\alpha} \geq b + \epsilon, Y_\alpha \geq b). \tag{4.6}$$

We proceed to compute $P_1(\epsilon; t, \alpha)$ and $P_2(\epsilon; t, \alpha)$. We observe,

$$\begin{aligned}
P_1(\epsilon; t, \alpha) &= P(W_{t+\alpha} \geq a - \epsilon, W_t \geq a) \\
&= P(W_{t+\alpha} - W_t \geq a - \epsilon - W_t, W_t \geq a) \\
&= P(W_t \geq a - \epsilon - (W_{t+\alpha} - W_t), W_t \geq a) \\
&= P\left(W_t \geq \max(a, a - \epsilon - \chi)\right), \quad \chi \sim \mathcal{N}(0, \alpha), \quad \chi \perp W_t \\
&= \int_{-\infty}^{\infty} \frac{e^{\frac{-\tau^2}{2\alpha}}}{\sqrt{2\pi\alpha}} \left(\int_{\max(a, a-\epsilon-\tau)}^{\infty} \frac{e^{\frac{-s^2}{2t}}}{\sqrt{2\pi t}} \, ds \right) d\tau.
\end{aligned}$$

On the other hand,

$$\begin{aligned}
P_2(\epsilon; t, \alpha) &= P(Y_{t+\alpha} \geq b + \epsilon, Y_\alpha \geq b) \\
&= P(Y_{t+\alpha} - Y_\alpha \geq b + \epsilon - Y_\alpha, Y_\alpha \geq b) \\
&= P(Y_\alpha \geq b + \epsilon - (Y_{t+\alpha} - Y_\alpha), Y_\alpha \geq b) \\
&= P(Y_\alpha \geq \max(b, b + \epsilon - \eta)), \quad \eta \perp Y_\alpha.
\end{aligned}$$

As the probability density function of Y_t is given by $f_{Y_t}(\cdot)$, therefore we obtain

$$P_2(\epsilon; t, \alpha) = \int_0^{\infty} f_{Y_t}(\beta) \left(\int_{\max(\max(b, b+\epsilon-\beta), 0)}^{\infty} f_{Y_\alpha}(s) ds \right) d\beta.$$

Clearly, $\{t > 0 : W_t \geq a\} + \{\alpha > 0 : Y_\alpha \geq b\} = \{t + \alpha > 0 : W_{t+\alpha} + Y_{t+\alpha} \geq a + b, t > 0, \alpha > 0\}$, with probability P, where P is given by (4.2), and $P_1(\epsilon; t, \alpha)$ and $P_2(\epsilon; t, \alpha)$ are obtained by (4.3) and (4.4), respectively. This leads to (4.1). $\square$

Next, we generalize the result in Theorem 4.1.1 for the log-return stochastic process (3.12) in the *approximation of the BN-S model*. In the BN-S model $\rho < 0$ is assumed in order to incorporate the *leverage effect* of the market. Typically in a derivative market, a significant fluctuation always corresponds to a "big-downward-movement" of the asset prices. Consequently, for the next theorem we focus on the first-exit time corresponding to a "downward-movement" of the log-return process (3.12). For the following theorem we assume $W_0 = Z_0 = 0$.

Theorem 4.1.2. *For a Brownian motion W_t and a Lévy subordinator Z_t, if $\mu \in \mathbb{R}$, $\sigma > 0$, $\rho < 0$, and $a, b > 0$, then*

$$\begin{aligned}
&\inf\{\tau > 0 : \mu\tau + \sigma W_\tau + \rho Z_\tau \leq -a - b\} \\
&= \inf\{t_1 > 0 : \mu t_1 + \sigma W_{t_1} \leq -a\} + \inf\{t_2 > 0 : \mu t_2 + \rho Z_{t_2} \leq -b\},
\end{aligned} \tag{4.7}$$

with probability

$$P = \int_0^\infty \int_0^\infty \left(\int_{-\infty}^\infty P_1(\epsilon; t_1, t_2) P_2(\epsilon; t_1, t_2)\, d\epsilon \right) dt_1\, dt_2, \tag{4.8}$$

where

$$P_1(\epsilon; t_1, t_2) = \int_{-\infty}^\infty \frac{e^{\frac{-\tau^2}{2t_2}}}{\sqrt{2\pi t_2}} \left(\int_{-\infty}^{\min\left(\frac{(-a-\mu t_1)}{\sigma}, \frac{(-a-\epsilon)}{\sigma} - \tau - \frac{\mu(t_1+t_2)}{2\sigma}\right)} \frac{e^{\frac{-s^2}{2t_1}}}{\sqrt{2\pi t_1}} ds \right) d\tau, \tag{4.9}$$

and

$$P_2(\epsilon; t_1, t_2) = \int_0^\infty f_{Z_{t_1}}(\beta) \left(\int_{\max\left(\max\left(\frac{(-b-\mu t_2)}{\rho}, \frac{(-b+\epsilon)}{\rho} - \beta - \frac{\mu(t_2+t_1)}{2\rho}\right), 0\right)}^\infty f_{Z_{t_2}}(s) ds \right) d\beta, \tag{4.10}$$

where the probability density function of Z_t is given by $f_{Z_t}(\cdot)$.

Proof. For fixed $\epsilon \in \mathbb{R}$, we define and compute the following joint probabilities. At first, we compute, for $a > 0$:

$$\begin{aligned}
P_1(\epsilon; t_1, t_2) &= P(W_{t_1+t_2} + \frac{\mu(t_1+t_2)}{2\sigma} \leq \frac{-a}{\sigma} - \frac{\epsilon}{\sigma}, W_{t_1} + \frac{\mu t_1}{\sigma} \leq \frac{-a}{\sigma}) \\
&= P(W_{t_1+t_2} - W_{t_1} + \frac{\mu t_1}{2\sigma} \leq \frac{-a}{\sigma} - \frac{\epsilon}{\sigma} - W_{t_1} - \frac{\mu t_2}{2\sigma}, W_{t_1} + \frac{\mu t_1}{\sigma} \leq \frac{-a}{\sigma}) \\
&= P(W_{t_1} + \frac{\mu t_1}{2\sigma} \leq \frac{-a-\epsilon}{\sigma} - (W_{t_1+t_2} - W_{t_1}) - \frac{\mu t_2}{2\sigma}, W_{t_1} + \frac{\mu t_1}{\sigma} \leq \frac{-a}{\sigma}) \\
&= P(W_{t_1} \leq \frac{-a-\epsilon}{\sigma} - (W_{t_1+t_2} - W_{t_1}) - \frac{\mu(t_1+t_2)}{2\sigma}, W_{t_1} \leq \frac{-a-\mu t_1}{\sigma}) \\
&= P\left(W_{t_1} \leq \min\left(\frac{(-a-\mu t_1)}{\sigma}, \frac{(-a-\epsilon)}{\sigma} - \chi - \frac{\mu(t_1+t_2)}{2\sigma} \right) \right), \\
&= \int_{-\infty}^\infty \frac{e^{\frac{-\tau^2}{2t_2}}}{\sqrt{2\pi t_2}} \left(\int_{-\infty}^{\min\left(\frac{(-a-\mu t_1)}{\sigma}, \frac{(-a-\epsilon)}{\sigma} - \tau - \frac{\mu(t_1+t_2)}{2\sigma}\right)} \frac{e^{\frac{-s^2}{2t_1}}}{\sqrt{2\pi t_1}} ds \right) d\tau,
\end{aligned}$$

where in the second to last step $\chi \sim \mathcal{N}(0,t_2)$, and $\chi \perp W_{t_1}$. With $\rho < 0$, we compute for $b > 0$,

$$\begin{aligned}
P_2(\epsilon;t_1,t_2) &= P(Z_{t_1+t_2} + \frac{\mu(t_1+t_2)}{2\rho} \geq \frac{-b}{\rho} + \frac{\epsilon}{\rho}, Z_{t_2} + \frac{\mu t_2}{\rho} \geq \frac{-b}{\rho}) \\
&= P(Z_{t_1+t_2} + \frac{\mu(t_1+t_2)}{2\rho} - Z_{t_2} \geq \frac{-b}{\rho} + \frac{\epsilon}{\rho} - Z_{t_2}, Z_{t_2} + \frac{\mu t_2}{\rho} \geq \frac{-b}{\rho}) \\
&= P\left(Z_{t_2} \geq \frac{(-b+\epsilon)}{\rho} - (Z_{t_1+t_2} - Z_{t_2} + \frac{\mu(t_2+t_1)}{2\rho}), Z_{t_2} \geq \frac{-b-\mu t_2}{\rho}\right) \\
&= P\left(Z_{t_2} \geq \max\left(\frac{(-b-\mu t_2)}{\rho}, \frac{(-b+\epsilon)}{\rho} - \eta - \frac{\mu(t_2+t_1)}{2\rho}\right)\right),
\end{aligned}$$

where Since $\eta \perp Z_{t_2}$, therefore we obtain (4.10). For $a, b > 0$, we define a set

$$\begin{aligned}
A &= \{\tau > 0 : \mu\tau + \sigma W_\tau + \rho Z_\tau \leq -a - b\} \\
&= \{t_1 + t_2 > 0 : \mu(t_1+t_2) + \sigma W_{t_1+t_2} + \rho Z_{t_1+t_2} \leq -a - b, t_1 > 0, t_2 > 0\}.
\end{aligned}$$

Consequently, we obtain

$$\begin{aligned}
A &= \{t_1 + t_2 > 0 : \mu(t_1+t_2) + \sigma W_{t_1+t_2} + \rho Z_{t_1+t_2} \leq -a - b\} \\
&= \{t_1 > 0 : \mu t_1 + \sigma W_{t_1} \leq -a\} + \{t_2 > 0 : \mu t_2 + \rho Z_{t_2} \leq -b\},
\end{aligned}$$

with probability P given by (4.8). Consequently,

$$\inf A = \inf\{t_1 > 0 : \mu t_1 + \sigma W_{t_1} \leq -a\} + \inf\{t_2 > 0 : \mu t_2 + \rho Z_{t_2} \leq -b\}.$$

This proves (4.7). $\square$

The purpose of Theorem 4.1.1 and Theorem 4.1.2 is to decompose the first-exit time process of a linear combination of a Brownian motion and a Lévy subordinator into the individual first-exit time processes of a Brownian motion and a Lévy subordinator. However, as observed in both of the theorems, such decomposition holds only with certain probability.

Remark 4.1.3. *It is well known (see [4, 32]) that for the process $G_t = \inf\{s > 0 : W_s + \gamma s \geq \delta_1 t\}$, with $\gamma, \delta_1 > 0$, known as the inverse Gaussian (IG) process, G_t follows an $IG(\delta_1 t, \gamma)$ distribution. As the process $W_s + \gamma s$ is continuous, we also have $G_t = \inf\{s > 0 : W_s + \gamma s = \delta_1 t\}$. The* *distribution* $IG(\delta_1, \gamma)$ *is concentrated on* $\mathbb{R}_+$ *and has probability density:*

$$p(x) = \frac{1}{\sqrt{2\pi}} \delta_1 e^{\delta_1 \gamma} x^{-3/2} \exp\left(-\frac{\delta_1^2 x^{-1} + \gamma^2 x}{2}\right), \quad \gamma, \delta_1 > 0.$$

Consequently, for Theorem 4.1.2 with $\mu < 0$ *and* $a > 0$*, the first term on the right hand side of* (4.37) *has the distribution* $\inf\{t_1 > 0 : \mu t_1 + \sigma W_{t_1} \leq -a\} \stackrel{d}{=} \inf\{t_1 > 0 : -\mu t_1 + \sigma W_{t_1} \geq a\} \sim IG\left(\frac{a}{\sigma}, \frac{-\mu}{\sigma}\right)$*.*

The case is not the same if the Brownian motion does not have any drift term. In that case, it is known (see [7]) that $\inf\{s > 0 : \sigma W_s \geq a\}$*, with* $a > 0$*, satisfies a Lévy distribution with the probability density function*

$$\frac{a}{\sigma\sqrt{2\pi x^3}} \exp\left(-\frac{a^2}{2\sigma^2 x}\right), \quad x > 0.$$

Consequently, for Theorem 4.1.1, the first term on the right hand side of (4.1)*, i.e.,* $\inf\{t > 0 : W_t \geq a\} = \inf\{t > 0 : W_t = a\}$*, with* $a > 0$*, has the probability density function* $\frac{a}{\sqrt{2\pi x^3}} \exp\left(-\frac{a^2}{2x}\right)$*,* $x > 0$*. A similar result holds for the first term on the right hand side of* (4.7) *in Theorem 4.1.2 with* $\mu = 0$*.*

Note that, for the case when $\mu = 0$ *and* $a > 0$*,* $\inf\{s > 0 : \sigma W_s \leq -a\} = \inf\{s > 0 : \sigma W_s = -a\} \stackrel{d}{=} \inf\{s > 0 : \sigma W_s = a\} = \inf\{s > 0 : \sigma W_s \geq a\}$*.*

We note that for Theorem 4.1.1, if $a, b \leq 0$, then (4.1) is trivially satisfied. Similarly, for Theorem 4.1.2, if $a, b \leq 0$, then (4.7) is trivially satisfied. As $W_0 = Z_0 = 0$, therefore all the related first-exit times are zero in those cases.

4.2. First-Exit Time Distribution For Some Self-Decomposable Processes

Consider the log-return dynamics X_t given by (3.12), in the *approximation of the BN-S model* (3.17) and (3.12). In Theorem 4.1.2, it is shown that with certain probability, the first-exit time process $\inf\{t > 0 : X_t \leq -a - b\}$, is decomposable into the sum of the first exit time of two processes- (1) the Brownian motion with drift, and (2) a Lévy subordinator with drift. We denote three stochastic processes: $A_{a+b} = \inf\{t > 0 : X_t \leq -a - b\} = \inf\{t > 0 : \mu t + \sigma W_t + \rho Z_t \leq -a - b\}$, $B_a = \inf\{t > 0 : \mu t + \sigma W_t \leq -a\}$, and $C_b = \inf\{t > 0 : \mu t + \rho Z_t \leq -b\}$, with $\rho < 0$, and $a, b > 0$. In these expressions σ and μ are given by (3.10) and (3.11), respectively. Thus, $\sigma > 0$. Also, in general, for financial applications $\mu \leq 0$. With these notations, from Theorem 4.1.2 we obtain that $A_{a+b} = B_a + C_b$.

The probability density function of the process B, with $\mu \leq 0$, is discussed in Remark 4.1.3. In this section we discuss the probability density function of the process C for some special cases. Accordingly, with probability P given by (4.8), the probability density function of the process A is equal to the convolution of the probability density functions of the processes B and C.

The goal of this section is to analyze the first-exit time distribution for the Lévy subordinator in the decompositions provided in Theorem 4.1.1 and Theorem 4.1.2. For simplicity we assume $\mu = 0$. We consider the distribution of the corresponding process $C_b = \inf\{s > 0 : Z_s \geq \frac{-b}{\rho}\}$, for three self-decomposable distributions. As $b > 0$ and $\rho < 0$, in general, C can be written as the stochastic process $T_t = \inf\{s > 0 : Z_s \geq t\}$, $t > 0$.

In Subsection 4.2.1, we describe some results related to special functions and Laplace transforms that are implemented for the subsequent analysis. Subsections 4.2.2, 4.2.3, and 4.2.4, deal with various analysis of T_t in relation to Gamma, IG, and PTS subordinators, respectively.

4.2.1. Laplace Transform and Some Relevant Special Functions

At first, we describe some special functions necessary for the development of the rest of this paper.

- The MacRobert E-function is denoted as

$$E(m; a_1 : n; b_j : x) = E(a_1, \cdots, a_m : b_1, \cdots, b_n : x).$$

For $m \geq n+1$, with $|x| < 1$, the MacRobert E-function is defined as

$$\sum_{i=1}^{m} \frac{\prod_{j=1}^{m} \tilde{*}\Gamma(a_j - a_i)\Gamma(a_i)x^{a_i}}{\prod_{k=1}^{n} \Gamma(b_k - a_i)} {}_{n+1}\tilde{F}_{m-1}A(x), \tag{4.11}$$

where $A(x) = \left[\begin{matrix} a_i, a_i - b_1 + 1, \cdots, a_i - b_n + 1; \\ a_i - a_1 + 1, \cdots, \tilde{*}, \cdots, a_i - a_m + 1; \end{matrix} (-1)^{m+n} x\right]$. For $m \leq n+1$, with $|x| > 1$, the MacRobert E-function is defined as

$$\frac{\prod_{i=1}^{m} \Gamma(a_i)}{\prod_{j=1}^{n} \Gamma(b_j)} {}_m\tilde{F}_n \left[\begin{matrix} a_1, \cdots, a_m; \\ b_1, \cdots, b_n; \end{matrix} \frac{-1}{x}\right]. \tag{4.12}$$

For $n = 0$, the notation $E(\cdot :: \cdot)$ is used. The $\tilde{*}$ denotes that the term containing $a_j - a_i$ corresponding to $j = i$ is omitted. Here ${}_m\tilde{F}_n[\cdot]$ is generalized hypergeometric functions, defined as

$${}_m\tilde{F}_n \left[\begin{matrix} a_1, \cdots, a_m; \\ b_1, \cdots, b_n; \end{matrix} x\right] = \sum_{n=0}^{\infty} \frac{(a_1)_1 \cdots (a_m)_n x^n}{(b_1)_1 \cdots (b_n)_n n!},$$

where $(\cdot)_n$ is the Pochhammer symbol.

- The Gauss hypergeometric function ${}_2F_1(a, b, c; x)$ is defined as

$${}_2F_1(a, b, c; x) = \sum_{n=0}^{\infty} \frac{(a)_n (b)_n x^n}{(c)_n n!},$$

where $(\cdot)_n$ is the Pochhammer symbol, $c \neq 0, -1, -2, \ldots;$, and $|x| \leq 1$. For $x \in \mathbb{C}$, with $|x| \geq 1$, the series can be analytically continued along any path in the complex plane that avoids the branch points 1 and infinity. An integral representation of the hypergeometric function is given by ${}_2F_1(a, b, c; x) = \frac{\Gamma(c) \int_0^1 t^{\,b-1} (1-t)^{\,c-b-1} (1-xt)^{\,-a} dt}{\Gamma(b)\Gamma(c-b)}$.

- Modified Bessel functions are solutions of the modified Bessel equation. The modified Bessel function of the first kind is defined by

$$I_\nu(z) = i^{-\nu} J_\nu(iz),$$

with $\nu \in \mathbb{R}$, and $J_\nu(\cdot)$ is the Bessel function of the first kind.

- Upper incomplete gamma function is given by

$$\Gamma(a,x)=\int_x^\infty t^{a-1}e^{-t}dt.$$

For $x>0$, $\Gamma(a,x)$ converges for all real a. In particular, $\Gamma(0,x)$ is the *exponential integral* $\int_x^\infty t^{-1}e^{-t}\,dt$.

Next, we describe some results related to the Laplace transform. For $t\geq 0$, and $s\in\mathbb{C}$, we denote the Laplace transform of $f(t)$ by $\mathcal{L}(f(t))=F(s)$, where $f(t)$ is piecewise continuous function on every finite interval in $[0,\infty)$ satisfying $|f(t)|<Me^{at}$, for some $M>0$ and for all $t\in[0,\infty)$. The Laplace transform and the inverse Laplace transform are related by:

$$F(s)=\int_0^\infty f(t)e^{-st}\,dt,$$

and

$$f(t)=\frac{1}{2\pi i}\int_{x_0-i\infty}^{x_0+i\infty}e^{st}\;F(s)ds,$$

for some $x_0\in\mathbb{R}$, where x_0 is greater than the real part of all singularities of $F(s)$, and $F(s)$ is bounded on the line $\mathrm{Re}(s)=x_0$ in the complex-plane. We list some useful properties related to the Laplace transform. The following result is elementary and can be found in [45].

Lemma 4.2.1. *The following results hold: (1)* $\mathcal{L}^{-1}\left(aF(as-b)\right)=e^{\frac{bt}{a}}f(\frac{t}{a})$, *with* $a>0$, $b\in\mathbb{R}$*; (2)* $\mathcal{L}^{-1}\left(-\frac{dF(s)}{ds}\right)=tf(t)$*; (3)* $\mathcal{L}^{-1}\left(\frac{F(s)}{s}\right)=\int_0^t f(u)du$*; (4)* $\mathcal{L}^{-1}\left(sF(s)-f(0)\right)=\frac{df(t)}{dt}$.

The following results provide various relations between the Laplace transform and special functions. These results can be found in [45].

Lemma 4.2.2. *The following results hold.*

(1) $\mathcal{L}\left(t^{\frac{-3}{2}}\int_0^\infty ue^{\frac{-u^2}{4t}}f(u)du\right)=2(\sqrt{\pi})F(\sqrt{s})$.

(2) $\mathcal{L}^{-1}\left(\frac{e^{\frac{a}{s}}}{s}\right)=I_0(2\sqrt{at})$, *where* $I_0(x)$ *is the modified Bessel function of the first kind, and* $Re(s)>0$.

(3) $\mathcal{L}^{-1}\left(e^{-a\sqrt{s}}\right)=\frac{ae^{\frac{-a^2}{4t}}}{2\sqrt{\pi}t^{\frac{3}{2}}}$, $Re(a^2)>0$, $Re(s)>0$.

(4) $\mathcal{L}\left(\Gamma(v, at)\right) = \frac{\Gamma(v)}{s}[1-(1+\frac{s}{a})^{-v}]$, *where* $\Gamma(v, at)$ *is the upper incomplete gamma function, and* $Re(\nu) > 0, \quad Re(s) > -Re(a)$.

(5) $\mathcal{L}\left((\sqrt{at})\right) = \frac{\sqrt{a}}{s\sqrt{s+a}}$, *where* $(x) = \frac{2}{\sqrt{\pi}}\int_0^x e^{-t^2}\,dt$, $Re(s) > \max(0, -Re(a))$.

(6) $\mathcal{L}^{-1}\left(\frac{1}{\sqrt{s+a}}\right) = \frac{e^{-at}}{\sqrt{\pi t}}$, $Re(s) > -Re(a)$.

(7) $\mathcal{L}^{-1}\left(\frac{\sqrt{s+a}}{s}\right) = \frac{e^{-at}}{\sqrt{\pi t}} + \sqrt{a}[\sqrt{at}]$, $Re(s) > \max(0, -Re(a))$.

(8)

$$\begin{aligned}&\mathcal{L}^{-1}\left(s^{c-1}e^{-(bs)^{\frac{1}{m}}}\right)\\ &= \frac{m^{\frac{1}{2}+mc}}{(2\pi)^{\frac{m+1}{2}}b^c}\sum_{i,-i}\frac{1}{i}E\left(c, c+\frac{1}{m}, \ldots, c+\frac{m-1}{m} :: \frac{be^{i\pi}}{m^m t}\right),\end{aligned}$$

where $E(\cdot : \cdot : \cdot)$ *is the MacRobert E-function,* $Re(s) > 0, Re(c) > 0, Re(b) > 0, m = 2, 3, \ldots$. *In the above expression* $\sum_{i,-i}$ *denotes that in expression following the summation sign,* i *is to be replaced by* $-i$ *and two expressions are to be added.*

In the two-dimension, for $x \in \mathbb{R}$, let $F(x,s) = \int_0^\infty f(x,t)e^{-st}dt$, be the Laplace transform of function $f(x,t)$ with respect to the t variable. Note that, for a subordinator X_t, with probability density function $f_{X_t}(\cdot)$, and Lévy measure π_X, the Lévy-Khinchin representation gives (see[13])

$$\int_0^\infty e^{-zt}f_{X_s}(t)dt = e^{-s\psi_X(z)} \tag{4.13}$$

where $\psi_X(\cdot)$ is the Laplace exponent of X and is given by $\psi_X(z) = \int_0^\infty(1-e^{-zu})\pi_X(du)$, where π_X is the Lévy measure of X. The following result can be found in [13].

Theorem 4.2.3. *The Lévy density* $w(x)$ *and Lévy measure* $\pi_X(t,\infty)$ *of the subordinator* X *(with* $\pi_X(t,\infty) = \int_t^\infty w(x)dx$*) satisfy* $\mathcal{L}(\pi_X(t,\infty)) = \frac{\psi_X(s)}{s}$, *where* $\psi_X(s)$ *is the Laplace exponent of the subordinator* X.

The following results are proved in [61].

Theorem 4.2.4. *Let $X = \{X_t\}_{t>0}$ be a subordinator with the probability density function $p(x,t)$. Suppose $p(x,t)$ admits continuous partial derivatives. Let $T_t = \inf\{\tau > 0 : X_\tau \geq t\}$, for $t > 0$, represents the first-exit time process of X. Denote the probability density function of T_t by $h_t(\cdot) = h(\cdot, t)$. Then,*

$$\mathcal{L}(h(x,t)) = \frac{\psi_X(s)e^{-x\psi_X(s)}}{s}, \tag{4.14}$$

where $\psi_X(\cdot)$ is the Laplace exponent of the subordinator X.

Theorem 4.2.5. *Denote the q-th moment of the first-exit time of the subordinator X by $M_q(x,t)$. Then,*

$$\mathcal{L}(M_q(x,t)) = \frac{q\Gamma(1+q)}{s(\psi_X(s))^q}. \tag{4.15}$$

4.2.2. Gamma Subordinators

Let X_t be a Gamma subordinator with Lévy density given by $w_X(x) = \frac{\nu e^{-\alpha x}}{x}$, $x > 0$, with $\nu, \alpha > 0$. In this case, the Laplace exponent of X is given by $\psi_X(s) = \nu \ln\left(1 + \frac{s}{\alpha}\right)$ (see [23, 40]).

Theorem 4.2.6. *For $x\nu = n+1$, $n = 0, 1, 2, \dots$, the probability density function of the first-exit time of X is given by*

$$h(x,t) = \int_0^t \frac{e^{-u\alpha}\alpha^{xc}\left(\nu(-u)^{\nu x}{}_2F_1\left(-\nu x, -\nu x, 1-\nu x; 1\right) + \nu u^{\nu x}\right)}{(x\nu - 1)!}\, du, \tag{4.16}$$

where ${}_2F_1\left(-\nu x, -\nu x, 1-\nu x; 1\right)$ is the hypergeometric function.

Proof. By Theorem 4.2.4, the Laplace transform of probability density of the first-exit time of Gamma subordinator is given as

$$\mathcal{L}(h(x,t)) = \frac{\ln(1+\frac{s}{\alpha})^\nu}{s(1+\frac{s}{\alpha})^{x\nu}} = \frac{K(x,s)}{s}, \tag{4.17}$$

where $K(x,s) = F(x,s)G(x,s)$, with $F(x,s) = \nu\ln(1+\frac{s}{\alpha})$,and $G(x,s) = \frac{1}{(1+\frac{s}{\alpha})^{x\nu}}$. Then $\mathcal{L}(h(x,t)) = \frac{K(x,s)}{s}$. Let the inverse Laplace transforms for $F(x,s)$ and $G(x,s)$ be $f(x,t)$ and $g(x,t)$, respectively.

Note that $\mathcal{L}^{-1}(\ln(1+s)) = -\frac{e^{-t}}{t}$, (see [45]). Using Lemma 4.2.1(1), we obtain,

$$f(x,t) = -\frac{\nu e^{-t\alpha}}{t}.$$

For $x\nu = n+1$, where n is a non-negative integer, $\mathcal{L}^{-1}\left(-\frac{1}{(s+1)^{x\nu}}\right) = \frac{t^{x\nu-1}e^{-t}}{(x\nu-1)!}$. Hence, by using Lemma 4.2.1(1), we obtain $g(x,t) = \frac{\alpha^{x\nu}t^{x\nu-1}e^{-t\alpha}}{(x\nu-1)!}$. Consequently, by standard convolution procedure, we obtain

$$\begin{aligned} k(x,t) &= \int_0^t f(x,\tau)g(x,t-\tau)d\tau = \int_0^t -\frac{\nu e^{-\tau\alpha}\alpha^{x\nu}(t-\tau)^{x\nu-1}e^{-\alpha(t-\tau)}}{(x\nu-1)!}d\tau \\ &= \frac{e^{-t\alpha}\alpha^{x\nu}[\nu(-t)^{\nu x}{}_2F_1\left(-\nu x, -\nu x, 1-\nu x; 1\right) + \nu t^{\nu x}]}{(x\nu-1)!}. \end{aligned}$$

Hence, with the application of Lemma 4.2.1(3), we obtain (4.16). □

The next result provides the first and the second order moment of the first-exit time of Gamma subordinator.

Theorem 4.2.7. *The first order moment (mean) the first-exit time of Gamma subordinator X_t is given by*

$$m(x,t) = \int_0^t \int_0^\infty \frac{\alpha e^{-\lambda\alpha}(\lambda\alpha)^{u-1}\, du\, d\lambda}{\nu\Gamma(u)}. \tag{4.18}$$

Proof. Using Theorem 4.2.5, we obtain the Laplace transform of the q-th moment of the first-exit time of the Gamma subordinator as $\frac{q\Gamma(1+q)}{s(\psi_X(s))^q}$. Consequently, the Laplace transform of the first order moment of the first-exit time of the Gamma subordinator is given by

$$M(x,s) = \frac{\Gamma(2)}{s\nu\ln(1+\frac{s}{\alpha})}. \tag{4.19}$$

We observe that $\mathcal{L}^{-1}\left(\frac{\Gamma(2)}{\ln(s)}\right) = \int_0^\infty \frac{\Gamma(2)t^{u-1}}{\Gamma(u)}du$. Consequently, using Lemma 4.2.1(2), we obtain

$$\mathcal{L}^{-1}\left(\frac{\Gamma(2)}{\nu\ln(1+\frac{s}{\alpha})}\right) = \int_0^\infty \frac{\alpha\Gamma(2)e^{-t\alpha}(t\alpha)^{u-1}}{\nu\Gamma(u)}\, du.$$

Consequently, $\mathcal{L}^{-1}(M(x,s))$ can be computed using Lemma 4.2.1(3) to obtain (4.18). □

We conclude this subsection by considering the case when the subordinator Z, that appears in (3.12) and (3.5), is related to the Gamma subordinator in the BN-S model. As observed in Section 3.1, if the stationary distribution of σ_t^2 is given by gamma law $\Gamma(\nu,\alpha)$, then the Lévy density of Z_1 is given by $w(x) = \nu\alpha e^{-\alpha x}$, $x > 0$.

Theorem 4.2.8. *The probability density function of the first-exit time of a subordinator Z with Lévy density $w(x) = \nu\alpha e^{-\alpha x}$, is given by*

$$h(x,t) = \nu e^{-x\nu} I_0\left(2\sqrt{x\nu\alpha t}\right) e^{-\alpha t},$$

where $I_0(\cdot)$ is the modified Bessel function of the first kind.

Proof. For this case, the Lévy measure of Z is given by $\pi_Z(t,\infty) = \int_t^\infty \nu\alpha e^{-\alpha x} dx = \nu e^{-\alpha t}$. Using Theorem 4.2.3, we obtain $\frac{\psi_Z(s)}{s} = \frac{\nu}{s+\alpha}$. Consequently, $\psi_Z(s) = \frac{\nu s}{s+\alpha}$. The Laplace transform of the probability density function of the first-exit time of Z is given by $H(x,s) = \frac{\nu e^{\frac{-x\nu s}{s+\alpha}}}{s+\alpha}$. Consequently, the probability density function of the first-exit time of Z is given by $h(x,t) = \mathcal{L}^{-1}(H(x,s))$, where

$$H(x,s) = \frac{\nu e^{\frac{-x\nu s}{s+\alpha}}}{s+\alpha} = \frac{\nu e^{-x\nu(1-\frac{\alpha}{s+\alpha})}}{s+\alpha} = \frac{\nu e^{-x\nu} e^{\frac{x\nu\alpha}{s+\alpha}}}{s+\alpha}. \tag{4.20}$$

Using Lemma 4.2.1(1) and Lemma 4.2.2(2), we obtain

$$h(x,t) = \nu e^{-x\nu} I_0\left(2\sqrt{x\nu\alpha t}\right) e^{-\alpha t}.$$

□

4.2.3. Inverse Gaussian Subordinators

The first-exit time of IG processes is described in [61]. In this subsection we consider the subordinator Z, that appears in (3.12), is related to the IG subordinator in the BN-S model.If the stationary distribution of σ_t^2 is given by IG(δ_1,γ) law, then the Lévy density of Z_1 is given by $w(x) = \frac{\delta_1}{2\sqrt{2\pi}} x^{-\frac{3}{2}}(1+\gamma^2 x)e^{-\frac{1}{2}\gamma^2 x}$, $x > 0$, and $\delta_1, \gamma > 0$.

For the results in Subsections 4.2.3 and 4.2.4, we define the *convolution* of two functions $p(x,t)$ and $q(x,t)$ by

$$p(x,t) * q(x,t) = \int_0^t p(x,\tau) q(x,t-\tau) d\tau.$$

Consequently, for three functions $p(x,t)$, $q(x,t)$, and $r(x,t)$,

$$(p(x,t) * q(x,t)) * r(x,t) = \int_0^t \int_0^u p(x,\tau) q(x,u-\tau) r(x,t-u)\, d\tau\, du.$$

Theorem 4.2.9. *The probability density function of the first-exit time of a subordinator Z with Lévy density $w(x) = \frac{\delta_1}{2\sqrt{2\pi}} x^{-\frac{3}{2}}(1+\gamma^2 x)e^{-\frac{1}{2}\gamma^2 x}$, is given by $h(x,t) = (p(x,t) * q(x,t)) * r(x,t)$, where*

$$p(x,t) = \frac{-\delta_1\gamma(\frac{\gamma\sqrt{t}}{\sqrt{2}})}{2} + \frac{\delta_1\gamma}{2} + \frac{\Gamma(\frac{-1}{2}, \frac{\gamma^2 t}{2})\delta_1\gamma}{4\sqrt{\pi}}, \tag{4.21}$$

$$q(x,t) = \frac{e^{\frac{-x\gamma\delta_1}{2}} e^{\frac{-t\gamma^2}{2}} t^{\frac{-3}{2}}}{2\sqrt{\pi}} \int_0^\infty u e^{\frac{-u^2}{4t}} \left(I_0'\left(2\sqrt{\frac{x\gamma^2\delta_1}{2\sqrt{2}}u}\right)\left(\sqrt{\frac{x\gamma^2\delta_1}{2\sqrt{2}u}}\right) + \delta(u)\right) du, \tag{4.22}$$

where $\delta(\cdot)$ is the Dirac delta function, $I_0(\cdot)$ is the modified Bessel function of the first kind, and

$$r(x,t) = \frac{x\delta_1\gamma^6 e^{\frac{-\gamma^2 t}{2}} e^{\frac{x\delta_1\gamma}{2}} e^{\frac{-\delta_1^2 x^2\gamma^4}{32t}}}{8\sqrt{\pi}(2t)^{\frac{3}{2}}}. \tag{4.23}$$

Proof. We obtain the Lévy measure for Z as

$$\pi_Z(t,\infty) = \int_t^\infty w(x)dx = \int_t^\infty \frac{\delta_1 x^{\frac{-3}{2}} e^{\frac{-\gamma^2 x}{2}} + \delta_1(\gamma^2)x^{\frac{-1}{2}} e^{\frac{-\gamma^2 x}{2}}}{2\sqrt{2\pi}} dx$$
$$= \frac{-\delta_1\gamma(\frac{\gamma\sqrt{t}}{\sqrt{2}})}{2} + \frac{\delta_1\gamma}{2} + \frac{\Gamma(\frac{-1}{2}, \frac{\gamma^2 t}{2})\delta_1\gamma}{4\sqrt{\pi}}.$$

Using Theorem 4.2.3, Lemma 4.2.2(4), and Lemma 4.2.2(5) we obtain,

$$\mathcal{L}(\pi_Z(t,\infty)) = \frac{\psi_Z(s)}{s} = \frac{\delta_1\gamma}{2s} - \frac{\delta_1\gamma^2}{2\sqrt{2}s(\sqrt{s+\frac{\gamma^2}{2}})} - \frac{\delta_1\gamma[1-\sqrt{(1+\frac{2s}{\gamma^2})}]}{2s}. \tag{4.24}$$

Consequently, by Theorem 4.2.4, we obtain that the Laplace transform of the probability density function of the first-exit time of Z is given by

$$H(x,s) = \left(\frac{\delta_1\gamma}{2s} - \frac{\delta_1\gamma^2}{2\sqrt{2}s(\sqrt{s+\frac{\gamma^2}{2}})} - \frac{\delta_1\gamma[1-\sqrt{(1+\frac{2s}{\gamma^2})}]}{2s}\right) e^{-xB(s)}$$
$$= P(x,s)Q(x,s)R(x,s),$$

where

$$B(s) = \left(\frac{\delta_1\gamma}{2} - \frac{\delta_1\gamma^2}{2\sqrt{2}\sqrt{s+\frac{\gamma^2}{2}}} - \frac{\delta_1\gamma[1-\sqrt{(1+\frac{2s}{\gamma^2})}]}{2}\right),$$

$$P(x,s) = \frac{\delta_1\gamma}{2s} - \frac{\delta_1\gamma^2}{2\sqrt{2}s\sqrt{s+\frac{\gamma^2}{2}}} - \frac{\delta_1\gamma[1-\sqrt{(1+\frac{2s}{\gamma^2})}]}{2s}, \tag{4.25}$$

$$Q(x,s) = \exp\left(\frac{-x\gamma\delta_1}{2} + \frac{x\gamma^2\delta_1}{2\sqrt{2}\sqrt{s+\frac{\gamma^2}{2}}}\right),$$

and

$$R(x,s) = \exp\left(\frac{x\delta_1\gamma}{2} - \frac{x\delta_1\gamma\sqrt{(1+\frac{2s}{\gamma^2})}}{2}\right).$$

We denote the inverse Laplace transforms of $P(x,s)$, $Q(x,s)$, and $R(x,s)$ by $p(x,t)$, $q(x,t)$, and $r(x,t)$, respectively.

We have $p(x,t) = \mathcal{L}^{-1}\left(\frac{\delta_1\gamma}{2s} - \frac{\delta_1\gamma^2}{2\sqrt{2}s(\sqrt{s+\frac{\gamma^2}{2}})} - \frac{\delta_1\gamma[1-\sqrt{(1+\frac{2s}{\gamma^2})}]}{2s}\right)$. From this, comparing with (4.24), we note that $p(x,t) = \mathcal{L}^{-1}(\frac{\psi_Z(s)}{s})$. Hence $p(x,t)$ is given by (4.21).

Next, we compute $q(x,t)$ using Lemma 4.2.2(2), Lemma 4.2.2(1), and Lemma 4.2.1(4). Lemma 4.2.2(2) gives $\mathcal{L}^{-1}\left(\frac{e^{\frac{a}{s}}}{s}\right) = I_0(2\sqrt{at})$.

With $L(s) = \frac{e^{\frac{a}{s}}}{s}$, we find $l(t) = \mathcal{L}^{-1}(L(s)) = I_0(2\sqrt{at})$. We notice $I_0(0) = 1$. Consequently, using Lemma 4.2.1(4), we have $\mathcal{L}^{-1}(sL(S) - l(0)) = l'(t)$. Hence, we obtain, $\mathcal{L}^{-1}(e^{\frac{a}{s}}) - \mathcal{L}^{-1}(1) = I_0'(2\sqrt{at})(\sqrt{\frac{a}{t}})$, and thus $\mathcal{L}^{-1}(e^{\frac{a}{s}}) = I_0'(2\sqrt{at})(\sqrt{\frac{a}{t}}) + \delta(t)$, where $\delta(\cdot)$ is the Dirac delta-function.

Using Lemma 4.2.2(1), we obtain

$$\mathcal{L}^{-1}(e^{as^{-1/2}}) = \frac{t^{\frac{-3}{2}}}{2\sqrt{\pi}}\int_0^\infty ue^{\frac{-u^2}{4t}}(I_0'(2\sqrt{au})(\sqrt{\frac{a}{u}}) + \delta(u))du.$$

Therefore, using Lemma 4.2.1(1) we obtain

$$q(x,t) = \frac{e^{\frac{-x\gamma\delta_1}{2}}e^{\frac{-t\gamma^2}{2}}t^{\frac{-3}{2}}}{2\sqrt{\pi}}\int_0^\infty ue^{\frac{-u^2}{4t}}\left(I_0'\left(2\sqrt{\frac{x\gamma^2\delta_1}{2\sqrt{2}}u}\right)\left(\sqrt{\frac{x\gamma^2\delta_1}{2\sqrt{2}u}}\right) + \delta(u)\right)du. \tag{4.26}$$

Finally,

$$r(x,t) = \mathcal{L}^{-1}(R(x,s)) = \mathcal{L}^{-1}\left(e^{\frac{x\delta_1\gamma}{2}}e^{-\frac{x\delta_1\gamma\sqrt{(1+\frac{2s}{\gamma^2})}}{2}}\right).$$

Using Lemma 4.2.2(3), we obtain $\mathcal{L}^{-1}(e^{\frac{x\delta_1\gamma}{2}}e^{-\frac{x\delta_1\gamma\sqrt{s}}{2}}) = \frac{e^{\frac{x\delta_1\gamma}{2}}(x\delta_1\gamma)e^{\frac{-\delta_1^2x^2\gamma^2}{16t}}}{4\sqrt{\pi}t^{\frac{3}{2}}}$. Consequently, using Lemma 4.2.1(1), we obtain (4.23).

Finally, if $h(x,t)$ is the probability density function of the first-exit time of Z, then $h(x,t) = \mathcal{L}^{-1}(H(x,s)) = \mathcal{L}^{-1}(P(x,s)Q(x,s)R(x,s)) = (p(x,t) * q(x,t)) * r(x,t)$. $\square$

4.2.4. Positive Tempered Stable Subordinators

Let X_t be a positive tempered stable (PTS) subordinator with Lévy density given by

$$u(x) = \beta k^{-2\gamma} \frac{\gamma}{\Gamma(\gamma)\Gamma(1-\gamma)} x^{-\gamma-1} \exp\left(-\frac{1}{2}k^2 x\right), \quad x > 0,$$

with $\beta > 0$, $0 < \gamma < 1$, and $k \geq 0$.

Theorem 4.2.10. *The probability density function of the first-exit time of X is given by $h(x,t) = p(x,t) * q(x,t)$, where*

$$p(x,t) = a\Gamma\left(-\gamma, \frac{k^2 t}{2}\right), \tag{4.27}$$

where $a = \frac{\beta\gamma}{2^\gamma \Gamma(\gamma)\Gamma(1-\gamma)}$, and

$$q(x,t) =$$
$$e^{-xa\Gamma(-\gamma)} e^{\frac{-k^2 t}{2}} \frac{\left(\frac{1}{\gamma}\right)^{\frac{\gamma+2}{2\gamma}}}{(2\pi)^{\frac{\gamma+1}{2\gamma}} (-xa(\frac{2}{k^2})^\gamma \Gamma(-\gamma))^{\frac{1}{\gamma}}} \sum_{i,-i} \frac{1}{i} E\left(1, 1+\gamma, \ldots, 2-\gamma :: C(x)\right), \tag{4.28}$$

where $C(x) = \frac{((\frac{2}{k^2})^\gamma (-xa)\Gamma(-\gamma))^{\frac{1}{\gamma}} e^{i\pi}}{\gamma^{\frac{-1}{\gamma}} t}$. In (4.28), $E(\cdot : : \cdot : \cdot)$ is the MacRobert E-function, and $\sum_{i,-i}$ denotes that in expression following the summation sign, i is to be replaced by $-i$ and two expressions are to be added.

Proof. We have

$$\pi_X(t) = \int_t^\infty u(x)dx = \int_t^\infty \frac{\beta k^{-2\gamma} \gamma x^{-\gamma-1} e^{\frac{-k^2 x}{2}}}{\Gamma(\gamma)\Gamma(1-\gamma)} dx = \frac{\beta\gamma\Gamma(-\gamma, \frac{k^2 t}{2})}{\Gamma(\gamma)\Gamma(1-\gamma)2^\gamma}.$$

We compute $\mathcal{L}(\pi_X(t))$ using Theorem 4.2.3 to obtain the Laplace exponent of density function of X as

$$\psi_X(s) = \frac{\beta\gamma\Gamma(-\gamma)[1 - (1 + \frac{2s}{k^2})^\gamma]}{\Gamma(\gamma)\Gamma(1-\gamma)2^\gamma}.$$

Now using Theorem 4.2.4, we obtain the Laplace transform of the probability density function of the first-exit time of X as

$$H(x,s) = \mathcal{L}(h(x,t)) = \left(\frac{a\Gamma(-\gamma)[1-(1+\frac{2s}{k^2})^\gamma]}{s}\right) e^{-ax\Gamma(-\gamma)[1-(1+\frac{2s}{k^2})^\gamma]},$$

where $a = \frac{\beta\gamma}{\Gamma(\gamma)\Gamma(1-\gamma)2^\gamma}$. To compute $h(x,t)$, the probability density function of the first-exit time of X, we find

$$p(x,t) = \mathcal{L}^{-1}\left(\frac{a\Gamma(-\gamma)[1-(1+\frac{2s}{k^2})^\gamma]}{s}\right),$$

and

$$q(x,t) = \mathcal{L}^{-1}\left(e^{-ax\Gamma(-\gamma)[1-(1+\frac{2s}{k^2})^\gamma]}\right),$$

and use the convolution result. By using Lemma 4.2.2(4), we obtain, the expression of $p(x,t)$ as (4.27).

Next, compute $q(x,t)$. Denote $Q(x,s) = \exp\left(-ax\Gamma(-\gamma)[1-(1+\frac{2s}{k^2})^\gamma]\right)$. We observe

$$\begin{aligned} Q(x,s) &= e^{-xa\Gamma(-\gamma)} \exp\left(-\left((-xa\Gamma(-\gamma))^{\frac{1}{\gamma}} + \frac{2s}{k^2}(-xa\Gamma(-\gamma))^{\frac{1}{\gamma}}\right)^\gamma\right) \\ &= e^{-xa\Gamma(-\gamma)} \exp\left(-\left((-xa\Gamma(-\gamma))^{\frac{1}{\gamma}} + s\left(-xa\left(\frac{2}{k^2}\right)^\gamma \Gamma(-\gamma)\right)^{\frac{1}{\gamma}}\right)^\gamma\right). \end{aligned}$$

Hence, by using Lemma 4.2.2(8) and Lemma 4.2.1(1), we obtain the expression of $q(x,t)$ as (4.28).

$\square$

We conclude this subsection by considering a subordinator Z related to the PTS subordinator in the BN-S model. If the stationary distribution of σ_t^2 is given by PTS(κ,δ,γ) law, then that the Lévy density of Z_1 is given by

$$w(x) = \frac{\beta k^{-2\gamma}\gamma x^{-\gamma-1}e^{\frac{-k^2x}{2}}}{\Gamma(\gamma)\Gamma(1-\gamma)}\left(\gamma + \frac{k^2x}{2}\right), \quad x>0, \quad \beta>0, 0<\gamma<1, k\geq 0. \tag{4.29}$$

As in the previous sections, Z is the subordinator that appears in (3.12).

Theorem 4.2.11. *The probability density function of the first-exit time of a subordinator Z, with Lévy density (4.29), is given by $h(x,t) = (p(x,t) * q(x,t)) * r(x,t)$, where*

$$p(x,t) = a\left(\gamma\Gamma(-\gamma, \frac{k^2t}{2}) + \Gamma(1-\gamma, \frac{k^2t}{2})\right), \tag{4.30}$$

where $a = \frac{\beta\gamma}{2^\gamma\Gamma(\gamma)\Gamma(1-\gamma)}$, and

$$q(x,t) = e^{-xa\gamma\Gamma(-\gamma)}e^{\frac{-k^2t}{2}}\frac{(\frac{1}{\gamma})^{\frac{\gamma+2}{2\gamma}}}{(2\pi)^{\frac{\gamma+1}{2\gamma}}(-xa\gamma(\frac{2}{k^2})^\gamma\Gamma(-\gamma))^{\frac{1}{\gamma}}}S_1, \tag{4.31}$$

where

$$S_1 = \sum_{i,-i}\frac{1}{i}E\left(1, 1+\gamma, \dots, 2-\gamma :: \frac{((\frac{2}{k^2})^\gamma(-xa\gamma)\Gamma(-\gamma))^{\frac{1}{\gamma}}e^{i\pi}}{\gamma^{\frac{-1}{\gamma}}t}\right),$$

and

$$r(x,t) = e^{-xa\Gamma(1-\gamma)}e^{-\frac{k^2t}{2}}\frac{(\frac{1}{\gamma-1})^{\frac{\gamma+1}{2\gamma-2}}}{(2\pi)^{\frac{\gamma}{2\gamma-2}}(-xa(\frac{2}{k^2})^{\gamma-1}\Gamma(1-\gamma))^{\frac{1}{\gamma-1}}}S_2, \tag{4.32}$$

where

$$S_2 = \sum_{i,-i}\frac{1}{i}E\left(1, \gamma, \dots, 3-\gamma :: \frac{((\frac{2}{\gamma^2})^{\gamma-1}(-xa)\Gamma(1-\gamma))^{\frac{1}{\gamma-1}}e^{i\pi}}{(\gamma-1)^{\frac{-1}{\gamma-1}}t}\right).$$

In the expressions of S_1 and S_2, $E(\cdot : : : \cdot)$ is the MacRobert E-function, and $\sum_{i,-i}$ denotes that in expression following the summation sign, i is to be replaced by $-i$ and two expressions are to be added.

Proof. We obtain $\pi_Z(t,\infty)$ as

$$\begin{aligned}\pi_Z(t,\infty) &= \int_t^\infty w(x)dx = \int_t^\infty \frac{\beta k^{-2\gamma}\gamma x^{-\gamma-1}e^{\frac{-k^2x}{2}}}{\Gamma(\gamma)\Gamma(1-\gamma)}(\gamma + \frac{k^2x}{2})\,dx \\ &= \frac{\beta\gamma}{2^\gamma\Gamma(\gamma)\Gamma(1-\gamma)}\left(\Gamma(-\gamma, \frac{k^2t}{2})\gamma + \Gamma(1-\gamma, \frac{k^2t}{2})\right).\end{aligned}$$

We use Theorem 4.2.3 to obtain

$$\frac{\psi_Z(s)}{s} = \frac{\beta\gamma}{2^\gamma\Gamma(\gamma)\Gamma(1-\gamma)}\left(\frac{\gamma\Gamma(-\gamma)}{s}(1-(1+\frac{2s}{k^2})^\gamma) + \frac{\Gamma(1-\gamma)}{s}(1-(1+\frac{2s}{k^2})^{\gamma-1})\right). \tag{4.33}$$

Consequently,

$$\psi_Z(s) = a\left(\gamma\Gamma(-\gamma)(1-(1+\frac{2s}{k^2})^{\gamma}) + \Gamma(1-\gamma)(1-(1+\frac{2s}{k^2})^{\gamma-1})\right),$$

where $a = \frac{\beta\gamma}{2^{\gamma}\Gamma(\gamma)\Gamma(1-\gamma)}$. Using Theorem 4.2.4, we obtain the Laplace transform of the probability density function of the first-exit time of Z as

$$H(x,s) = \mathcal{L}(h(x,t)) = P(x,s)Q(x,s)R(x,s),$$

where

$$P(x,s) = a\left(\frac{\gamma\Gamma(-\gamma)}{s} - \frac{\gamma\Gamma(-\gamma)}{s}(1+\frac{2s}{k^2})^{\gamma} + \frac{\Gamma(1-\gamma)}{s} - \frac{\Gamma(1-\gamma)}{s}(1+\frac{2s}{k^2})^{\gamma-1}\right), \tag{4.34}$$

$$Q(x,s) = \exp\left(-xa\left(\gamma\Gamma(-\gamma)(1-(1+\frac{2s}{k^2})^{\gamma})\right)\right),$$

and

$$R(x,s) = \exp\left(-xa\left(\Gamma(1-\gamma)(1-(1+\frac{2s}{k^2})^{\gamma-1})\right)\right).$$

We denote the inverse Laplace transform for $P(x,s)$, $Q(x,s)$, and $R(x,s)$, by $p(x,t)$, $q(x,t)$, and $r(x,t)$, respectively. Using Lemma 4.2.2(4), we obtain

$$\mathcal{L}^{-1}(\frac{\gamma\Gamma(-\gamma)}{s}(1-(1+\frac{2s}{k^2})^{\gamma}) = \gamma\Gamma(-\gamma,\frac{k^2t}{2}).$$

Also, using Lemma 4.2.2(4), we obtain $\mathcal{L}^{-1}(\frac{\Gamma(1-\gamma)}{s} - \frac{\Gamma(1-\gamma)}{s}(1+\frac{2s}{k^2})^{\gamma-1}) = \Gamma(1-\gamma,\frac{k^2t}{2})$. Hence, we obtain $p(x,t)$ as given by (4.30).

Next, we observe that $Q(x,s)$ can be written as:

$$\begin{aligned} Q(x,s) &= e^{-xa\gamma\Gamma(-\gamma)}\exp\left(-\left((-xa\gamma\Gamma(-\gamma))^{\frac{1}{\gamma}} + \frac{2s}{k^2}(-xa\gamma\Gamma(-\gamma))^{\frac{1}{\gamma}}\right)^{\gamma}\right) && (4.35)\\ &= e^{-xa\gamma\Gamma(-\gamma)}\exp\left(-\left((-xa\gamma\Gamma(-\gamma))^{\frac{1}{\gamma}} + s\left(-xa\gamma\left(\frac{2}{k^2}\right)^{\gamma}\Gamma(-\gamma)\right)^{\frac{1}{\gamma}}\right)^{\gamma}\right). && (4.36)\end{aligned}$$

Hence by using Lemma 4.2.2(8) and Lemma 4.2.1(1), we obtain (4.31). Finally, we observe that $R(x, s)$ can be written as

$$R(x,s) = e^{-xa\Gamma(1-\gamma)} \exp\left(- \left((-xa\Gamma(1-\gamma))^{\frac{1}{\gamma-1}} + \frac{2s}{k^2}(-xa\Gamma(1-\gamma))^{\frac{1}{\gamma-1}} \right)^{\gamma-1} \right)$$
$$= e^{-xa\Gamma(1-\gamma)} \exp\left(-D(x)^{\gamma-1}\right),$$

where $D(x) = \left((-xa\Gamma(1-\gamma))^{\frac{1}{\gamma-1}} + s\left(-xa\left(\frac{2}{k^2}\right)^{\gamma-1}\Gamma(1-\gamma)\right)^{\frac{1}{\gamma-1}} \right)$. Hence by using Lemma 4.2.2(8) and Lemma 4.2.1(1), we obtain (4.32). Finally, by convolution theorem, we obtain the probability density function of the first-exit time of Z as $h(x,t) = \mathcal{L}^{-1}(H(x,s)) = (p(x,t) * q(x,t)) * r(x,t)$. □

4.3. A Generalized Result

At this point, we prove a generalized version of Theorem 4.1.2. This can be implemented for the analysis of the commodity market as described in a previous chapter.

Theorem 4.3.1. *For a Brownian motion W_t and a Lévy subordinators $Z_t^{(i)}$, $i = 1, \ldots, n$, if $\mu \in \mathbb{R}$, $\sigma > 0$, $\rho < 0$, and $a, b_1, \ldots, b_n > 0$, then*

$$\inf\{\tau > 0 : \mu\tau + \sigma W_\tau + \rho \sum_{i=1}^{n} Z_\tau^{(i)} \leq -a - \sum_{i=1}^{n} b_i\}$$
$$= \inf\{t_1 > 0 : \mu t_1 + \sigma W_{t_1} \leq -a\} + \sum_{i=1}^{n} \inf\{t_2 > 0 : \mu t_2 + \rho Z_{t_2}^{(i)} \leq -b_i\}, \tag{4.37}$$

with probability

$$P = \int_0^\infty \int_0^\infty \left(\int_{-\infty}^{\infty} P_1(\epsilon; t_1, t_2) \prod_{i=1}^{n} P_2^{(i)}(\epsilon; t_1, t_2)\, d\epsilon \right) dt_1\, dt_2, \tag{4.38}$$

where

$$P_1(\epsilon; t_1, t_2) = \int_{-\infty}^{\infty} \frac{e^{\frac{-\tau^2}{2t_2}}}{\sqrt{2\pi t_2}} \left(\int_{-\infty}^{\min\left(\frac{(-a-\mu t_1)}{\sigma}, \frac{(-a-\epsilon)}{\sigma} - \tau - \frac{\mu(t_1+t_2)}{2\sigma}\right)} \frac{e^{\frac{-s^2}{2t_1}}}{\sqrt{2\pi t_1}} ds \right) d\tau, \tag{4.39}$$

and for $i = 1, \ldots, n$,

$$P_2^{(i)}(\epsilon; t_1, t_2) = \int_0^\infty f_{Z_{t_1}^{(i)}}(\beta) \left(\int_{\max\left(\max\left(\frac{(-nb_i-\mu t_2)}{\rho n}, \frac{(-nb_i+\epsilon)}{n\rho} - \beta - \frac{\mu(t_2+t_1)}{2\rho n}\right), 0\right)}^{\infty} f_{Z_{t_2}^{(i)}}(s) ds \right) d\beta, \tag{4.40}$$

where the probability density function of $Z_t^{(i)}$ is given by $f_{Z_t}^{(i)}(\cdot)$.

Proof. For fixed $\epsilon \in \mathbb{R}$, we define and compute the following joint probabilities. At first, we compute, for $a > 0$:

$$\begin{aligned}
P_1(\epsilon; t_1, t_2) &= P(W_{t_1+t_2} + \frac{\mu(t_1+t_2)}{2\sigma} \leq \frac{-a}{\sigma} - \frac{\epsilon}{\sigma}, W_{t_1} + \frac{\mu t_1}{\sigma} \leq \frac{-a}{\sigma}) \\
&= P(W_{t_1+t_2} - W_{t_1} + \frac{\mu t_1}{2\sigma} \leq \frac{-a}{\sigma} - \frac{\epsilon}{\sigma} - W_{t_1} - \frac{\mu t_2}{2\sigma}, W_{t_1} + \frac{\mu t_1}{\sigma} \leq \frac{-a}{\sigma}) \\
&= P(W_{t_1} + \frac{\mu t_1}{2\sigma} \leq \frac{-a-\epsilon}{\sigma} - (W_{t_1+t_2} - W_{t_1}) - \frac{\mu t_2}{2\sigma}, W_{t_1} + \frac{\mu t_1}{\sigma} \leq \frac{-a}{\sigma}) \\
&= P(W_{t_1} \leq \frac{-a-\epsilon}{\sigma} - (W_{t_1+t_2} - W_{t_1}) - \frac{\mu(t_1+t_2)}{2\sigma}, W_{t_1} \leq \frac{-a-\mu t_1}{\sigma}) \\
&= P\left(W_{t_1} \leq \min\left(\frac{(-a-\mu t_1)}{\sigma}, \frac{(-a-\epsilon)}{\sigma} - \chi - \frac{\mu(t_1+t_2)}{2\sigma}\right)\right), \\
&= \int_{-\infty}^{\infty} \frac{e^{\frac{-\tau^2}{2t_2}}}{\sqrt{2\pi t_2}} \left(\int_{-\infty}^{\min\left(\frac{(-a-\mu t_1)}{\sigma}, \frac{(-a-\epsilon)}{\sigma} - \tau - \frac{\mu(t_1+t_2)}{2\sigma}\right)} \frac{e^{\frac{-s^2}{2t_1}}}{\sqrt{2\pi t_1}} ds\right) d\tau,
\end{aligned}$$

where in the second to last step $\chi \sim \mathcal{N}(0, t_2)$, and $\chi \perp W_{t_1}$. For $i = 1, \ldots, n$, with $\rho < 0$, we compute for $b_i > 0$,

$$\begin{aligned}
P_2^{(i)}(\epsilon; t_1, t_2) &= P(Z_{t_1+t_2}^{(i)} + \frac{\mu(t_1+t_2)}{2\rho n} \geq \frac{-b_i}{\rho} + \frac{\epsilon}{\rho n}, Z_{t_2}^{(i)} + \frac{\mu t_2}{n\rho} \geq \frac{-b_i}{\rho}) \\
&= P(Z_{t_1+t_2}^{(i)} + \frac{\mu(t_1+t_2)}{2\rho n} - Z_{t_2}^{(i)} \geq \frac{-b_i}{\rho} + \frac{\epsilon}{\rho n} - Z_{t_2}^{(i)}, Z_{t_2}^{(i)} + \frac{\mu t_2}{n\rho} \geq \frac{-b_i}{\rho}) \\
&= P\left(Z_{t_2}^{(i)} \geq \frac{(-nb_i + \epsilon)}{\rho n} - (Z_{t_1+t_2}^{(i)} - Z_{t_2}^{(i)} + \frac{\mu(t_2+t_1)}{2\rho n}), Z_{t_2}^{(i)} \geq \frac{-nb_i - \mu t_2}{n\rho}\right) \\
&= P\left(Z_{t_2}^{(i)} \geq \max\left(\frac{(-nb_i - \mu t_2)}{\rho n}, \frac{(-nb_i + \epsilon)}{n\rho} - \eta^{(i)} - \frac{\mu(t_2+t_1)}{2\rho n}\right)\right),
\end{aligned}$$

where $\eta^{(i)} \sim$ the distribution of $Z_{t_1}^{(i)}$. Since $\eta^{(i)} \perp Z_{t_2}^{(i)}$, therefore we obtain (4.40). For $a, b > 0$, we define a set

$$\begin{aligned}
A &= \{\tau > 0 : \mu\tau + \sigma W_\tau + \rho \sum_{i=1}^{n} Z_\tau^{(i)} \leq -a - \sum_{i=1}^{n} b_i\} \\
&= \{t_1 + t_2 > 0 : \mu(t_1+t_2) + \sigma W_{t_1+t_2} + \rho \sum_{i=1}^{n} Z_{t_1+t_2}^{(i)} \leq -a - \sum_{i=1}^{n} b_i, t_1 > 0, t_2 > 0\}.
\end{aligned}$$

Consequently, we obtain

$$
\begin{aligned}
A &= \{t_1 + t_2 > 0 : \mu(t_1 + t_2) + \sigma W_{t_1+t_2} + \rho \sum_{i=1}^{n} Z^{(i)}_{t_1+t_2} \le -a - \sum_{i=1}^{n} b_i\} \\
&= \{t_1 > 0 : \mu t_1 + \sigma W_{t_1} \le -a\} + \sum_{i=1}^{n} \{t_2 > 0 : \mu t_2 + \rho Z^{(i)}_{t_2} \le -b_i\},
\end{aligned}
$$

with probability P given by (4.38). Consequently,

$$
\inf A = \inf\{t_1 > 0 : \mu t_1 + \sigma W_{t_1} \le -a\} + \sum_{i=1}^{n} \inf\{t_2 > 0 : \mu t_2 + \rho Z^{(i)}_{t_2} \le -b_i\}.
$$

This proves (4.37). □

The purpose of Theorem 4.1.2 is to decompose the first-exit time process of a linear combination of a Brownian motion and Lévy subordinators into the individual first-exit time processes of a Brownian motion and Lévy subordinators. However, as observed in the theorem, such decomposition is attained only with certain probability.

5. DATA ANALYSIS

5.1. Data Description and Analytical Framework

The data set in our study consistsed of 418 observations and 34 features. The data set is a time series with weekly observations over the period 2012 to 2020. This captures a period of growing intensity in competition, and during which a trade war had influenced the results. The target variable was the share of world soybean exports shipped from the US Gulf to China. This value was derived from USDA soybean market reports (see [67, 68, 69, 70]). A set of feature variables were developed as having potential impacts on the target variable. These variables depicted costs, time, logistics in addition to binary variable by month and for the duration of the trade war. The variables are summarized in Table 5.1 along with their source. In the following table "Grain TR" stands for "Grain Transportation Report".

Table 5.1. Variables and Data Sources.

Variable Name	Description	Source
DCV	Rail car values	Tradewest
Velocity	Rail car cycle time	BNSF
Outstanding Sales	Export sales	USDA Grain TR
Pnw in port	Ships in port at the PNW	USDA Grain TR
Wtime santos	Ship wait time Brazil	USDA Grain TR
Gulf duenext10d	Ships due in US Gulf next 10 days	USDA Grain TR
barge mm	Monthly barge rate	USDA Grain TR
gulf in port	Ships in port at the US Gulf	USDA Grain TR
gulf loaded post 7 days	Ships loaded in past 7 days	USDA Grain TR
usg b	Basis at the US Gulf	Thompson Reuters Eikon
pnw b	Basis at the PNW	Thompson Reuters Eikon
braz fre	Ocean shipping from Brazil	Thompson Reuters Eikon
arg fre	Ocean shipping from Argentina	Thompson Reuters Eikon
twar dummy	Trade war	
usg fre	Ocean shipping from US Gulf	Thompson Reuters Eikon
Pnw fre	Ocean shipping from PNW	Thompson Reuters Eikon
exrate arg	Exchange rate Argentina	Thompson Reuters Eikon
exrate bra	Exchange rate Brazil	Thompson Reuters Eikon
Monthly binary variables		

Our goal in this study is to predict the US Gulf soybean export market shares using Random Forest (RF) and Recurrent Neural Network (RNN) techniques. Once a prediction was obtained with sufficient accuracy, we used variable importance plot for RF, and LIME for RNN to examine which features are important in interpreting the prediction of shares of USA. The main motivation behind using variable importance plot for RF, and LIME for RNN stems from the fact that former technique helps in understanding the global interpretation while later one focuses more on local interpretation. Global interpretation helps in understating the inputs and and their relationship with prediction target, while local interpretation helps us understanding model predictions for single observation or group of similar observations.

We note that the Random Forest method is an ensemble method for classification and regression. Output from Random Forest is based on ensemble of decisions made by various decision trees. Studies have shown that ensemble method significantly improves predictive performances. On the other hand, Recurrent Neural Network (RNN) is a neural network algorithm that is used for time series data. The common problem that RNN faces is vanishing gradient, to solve this issue, Long Short Term Memory (LSTM) is used. The improvement of RNN can be further incorporated by the "LIME" technique.

LIME, that stands for Local Interpretable Model-agnostic Explanations, is a method for visualizing individual predictions. LIME uses an interpretable model locally around the prediction to clarify any classifier's predictions in an interpretable and faithful manner. It is model agnostic, which means that it will work with every supervised regression or classification model. LIME is based on the premise that any complex model is linear on a local scale and that it is possible to solve it. The simple model can then be used to illustrate the more complicated model's predictions locally. LIME is useful when there is possibility of dataset shift. LIME is also very useful when there is chance of data leakage. LIME can explain predictions of any machine learning algorithm, by approximating it locally with an interpretable model. SP-LIME uses submodular optimization to address the trusting the model issue.

Unlike in econometrics or linear policy modeling analytics such as hypothesis testing, it is not possible to take a partial derivative of an exogenous variable's coefficient in a deep learning model's functional form with respect to the endogenous. The ability to do this in a regression model allows for a pure ceteris paribus interpretation of the model. In contrast, in Deep Learning there are often a large number of variables and an often-attributed presence of multicollinearity, each variable having complex relations among other variables' "marginal" influences. Additionally, the presence of dynamic interrelations within the network, not only with feature scale, but with relational feature distance and differentials, a linear interpretation fails to capture these complex and nonlinear relations, notwithstanding nonstationary assumptions. As an alternative, it is possible to evaluate model determination and feature importance through non-conventional methods including Variable in Importance (VIP) or through Local Interpretable Model-Agnostic Explanation (LIME).

LIME as presented by [44] is a novel technique that explains classifiers or regression models at a local prediction. In doing this the method develops a linear, interpretable, and understandable model that defines the observation. It is exceptionally effective at identifying local model interpretation, but in deriving a global interpretation of a deep learning model, it is important to evaluate perturbations at varying points in the sample dataset. This is because it is entirely possible that if evaluated at differing observations, the ranking and signs of the variables may vary. Global interpretation is difficult, even if model weights and biases can be evaluated.

A method of Submodular Pick Local Interpretable Model-Agnostic Explanation (SP-LIME) was developed in [44] as a method to attempt a global perspective of model interpretability. Through sub-modular optimization, by selecting a few individual instances and corresponding predictions of the observation set, in such a way that they are representative of the model upon the individual, local predictions. In using SP-LIME, a feature attribution matrix is developed through perturbation search and estimation. By averaging this matrix, one can observe the global feature attribution. Though this method approaches linear interpretation, marginal essentialism cannot be attained like it can in statistical and linear approaches which is accomplished by taking partial derivatives of the function.

5.2. Numerical Results and Conclusion

For the analysis [6], 300 observation (numbered from 1 to 300) were used for training data, while 118 (numbered from 1 to 118) were used for testing. For the training data, the number 1 represents June 1, 2012, while the number 300 represents September 22, 2017. For the testing data (Figures 2 and 4), the number 1 (in the x-axis) represents September 29, 2017, while the number 118 represents March 1, 2020. Variables were chosen based on a priori expectations and some were excluded due to multicollinearity. The RF model is used, and based on variable importance plot the following variables were considered as important variables.

The data analysis is connected to the mathematical model. For all the feature variables we assign a binary "importance factor" $\theta^{(i)}$ as either 0 or 1. We list the features that are obtained with $\theta^{(i)} = 1$. We discard the features with $\theta^{(i)} = 0$. Variable Importance Plot is based on IncNodePurity. IncNodePurity relates to loss function by which best split is chosen in Table 4.2.

Loss function is our analysis is Mean Square Error. More useful variables achieve higher increases in node purities, that is to find a split which has a high inter node variance and a small intra-node variance.

Plots between Predicted shares and Actual shares are presented below in Figures 4.1 and 4.3. These plots show that the RF model has fairly well. In RNN, LSTM was used with 50 nodes and 1 dense layer. For optimization "Adam" was used. RNN was trained for 100 epochs. Batch size of 72 was used in training of RNN model. For RNN model, we made sure that all the variables were normalized so-that variables with higher variance do not unnecessarily dominate in model. For training of RNN, we used timestamp of value 1.

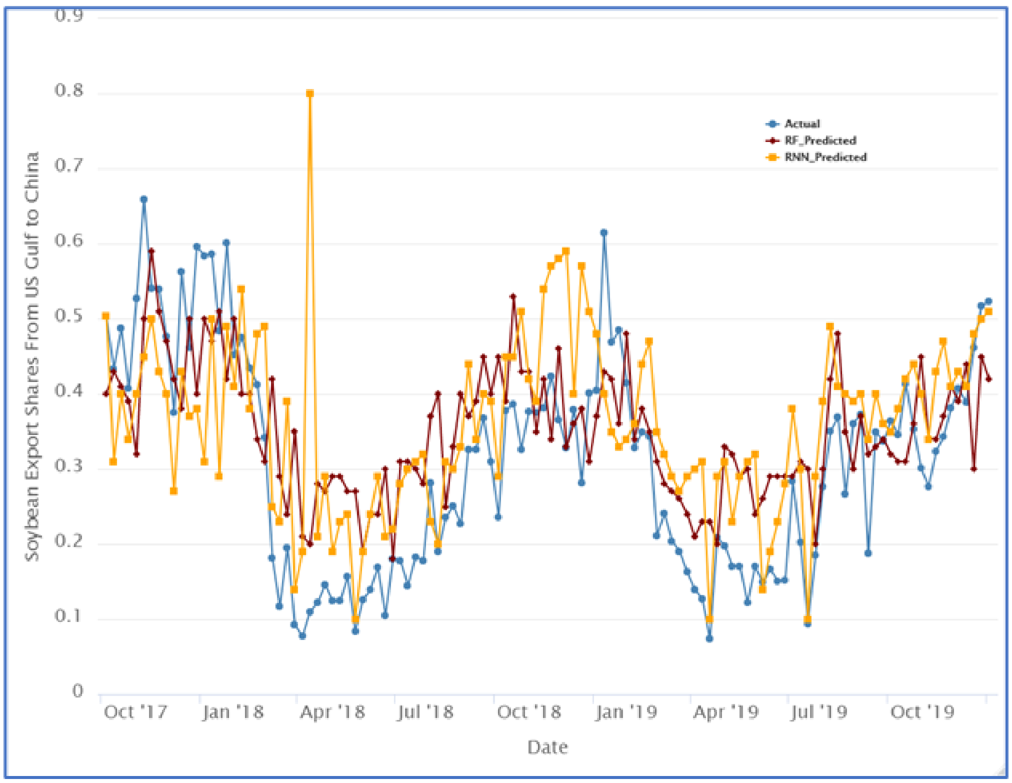

Figure 5.1. Soyabean Export Shares from US Gulf to China.

Table 5.2. Variable Importance Plot for Shares of USA.

Variable Name	IncNodePurity
DCV	0.22
velocity	0.82
Outstanding Sales	0.70
Pnw in port	0.27
Wtime santos	0.71
Gulf duenext10d	2.29
barge mm	2.39
gulf in port	0.72
gulf loaded post 7 days	3.21
usg b	0.20
pnw b	0.24
braz fre	0.41
arg fre	0.33
usg fre	0.34
Pnw fre	0.48
santos_b	1.92
arg_b	0.81
exrate arg	1.24
exrate bra	0.85
jan dummy	0.73
feb dummy	0.30
mar dummy	0.29
apr dummy	0.10
sep dummy	0.18
nov dummy	0.11
dec dummy	0.20

Results from RF: Variable importance plot for shares of USA is presented in Table 4.2. We can infer that “gulf loaded post 7 days”, “barge mm”, “Gulf duenext10d”, “Wtime santos”, “exrate arg” are the important variables on average. Figure 5.2 provides a description of predicted shares vs. actual shares using the RF algorithm.

Result from RNN:

In the Figure 5.2, orange colored variables support the feature variables, while blue colored variables oppose the feature variables.

LIME plot in Figure 5.2, shows that “twar dummy”, “Wtime santos”, “braz fre”, “arg fre”, “gulf loaded post 7 days”, “Pnw fre” variables positively correlates with USA shares, while “Velocity”, “gulf in port”, “usg fre”, “Pnw in port”, “arg fre” negatively correlates with USA share. Comparison between predicted shares and actual shares is given below in Figure 5.3.

Feature	Value	Feature	Value
exrate_arg	0.00	jan_dummy	0.00
pnw_fre	0.25	nov_dummy	0.00
velocity	0.73	exrate_bra	0.00
gulf_in_port	0.32	gulf_duenext10d	0.24
oct_dummy	0.00	feb_dummy	1.00
sep_dummy	0.00	santos_b	0.75
dcv	0.08	braz_fre	0.70
usg_fre	0.68	aug_dummy	0.00
dec_dummy	0.00	jul_dummy	0.00
pnw_in_port	0.32	pnw_b	0.37
wtime_santos	0.17	apr_dummy	0.00
arg_b	0.49	gulf_loadedpast7d	0.60
mar_dummy	0.00	jun_dummy	0.00
twar_dummy	0.00	arg_fre	0.64
outstand_sales	0.04	may_dummy	0.00
-	-	usg_b	0.23

Figure 5.2. LIME Plot.

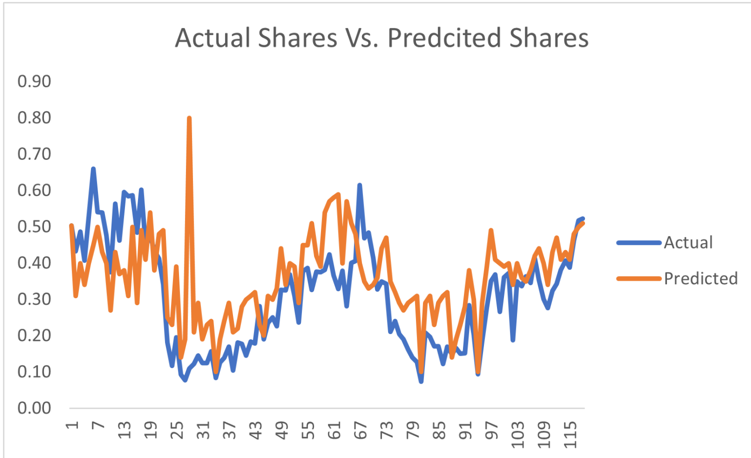

Figure 5.3. Predicted Shares Vs. Actual Shares using RNN.

Next, we incorporate Theorem 4.1.2 in the empirical data analysis. In the plots in Figure 5.4, 5.5, and 5.6, we provide the histograms corresponding to the first-exit time of the variables "exrate arg", "Gulf duenext10d", "gulf loaded post 7 days", and , respectively, for the empirical dataset for various values of t. Along with the histograms, we use Gamma-type subordinators (Z) described with Lévy density $w(x) = \nu\alpha e^{-\alpha x}$. After finding appropriate parameter values, in those plots, we plot the probability density functions of $\inf\{s > 0 : Z_s \geq t\}$, for various values of t. This is motivated by Theorem 4.1.2, and the analysis in [5].

In Figure 5.4 we use $t = 1, 2, 3$, and in Figure 5.5 we use $t = 1, 2, 3$. In Figure 5.6 we use $t = 1, 2, 3$. From these figures, it is clear that for the time duration when there is no *big fluctuation* of the empirical dataset, $\inf\{s > 0 : W_s = t\}$ plays the dominant role in determining the distribution of $\inf\{s > 0 : X_s \geq t\}$. However, for the time duration of *big fluctuation* of the empirical dataset, $\inf\{s > 0 : Z_s \geq t\}$ plays the dominant role in determining the distribution of $\inf\{s > 0 : X_s \geq t\}$.

Deep learning models have advantages in providing analysis of competition, particularly in this case. Many features of competition influence this sector which limits or constrains the ability of traditional "equilibrium" models normally specified in economics. Most important are the randomness of most variables, changes over time in the underlying relationships and functions, as well as periodic interventions. In this paper we developed a deep learning model of competition in soybean exports to China, the largest and fastest growing market in the world.

The results indicate that export market shares, which are otherwise highly volatile, can be effectively explained (predicted) using deep learning methodologies and a set of association variables. Some of the variables have significant influences, particularly using the Variable Importance. These factors include “Gulf duenext10D”, “barge mm”, “gulf loaded past 7 days”, in addition to a number of other variables including monthly binary variables.

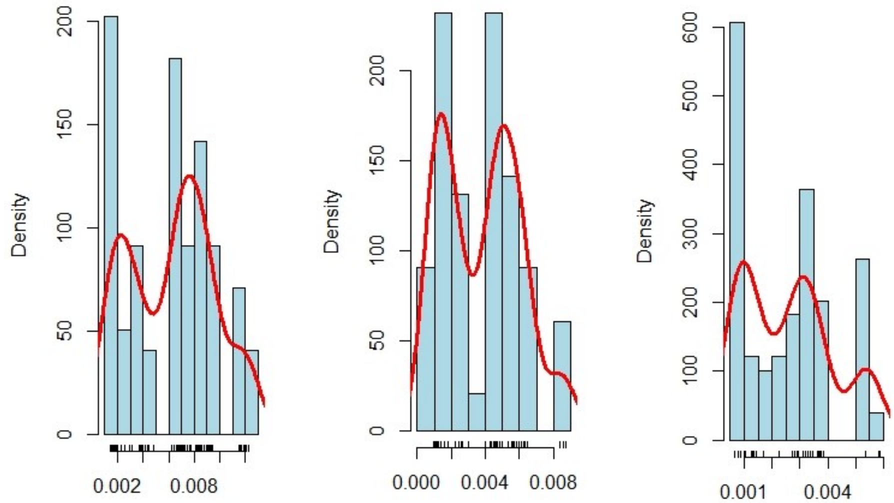

Figure 5.4. First-Exit Time for “exrate arg”.

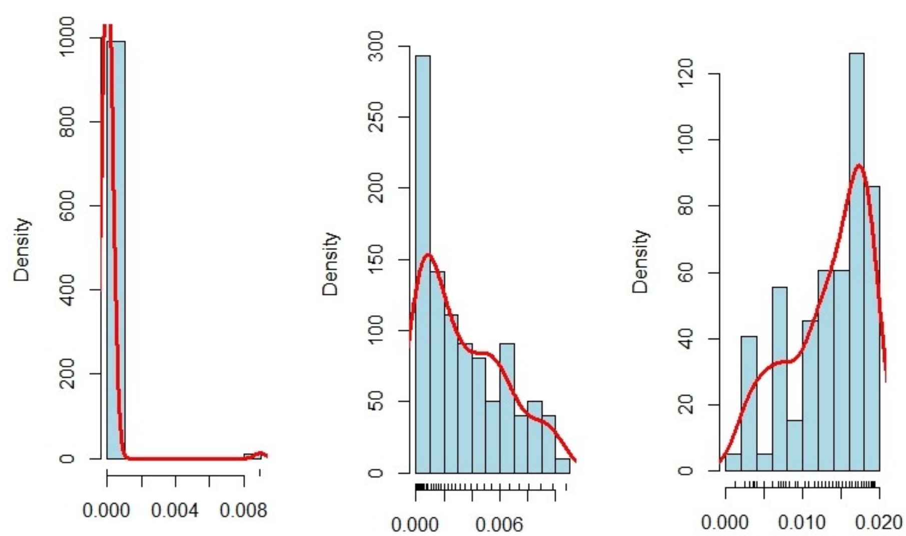

Figure 5.5. First-Exit Time for “Gulf duenext10d”.

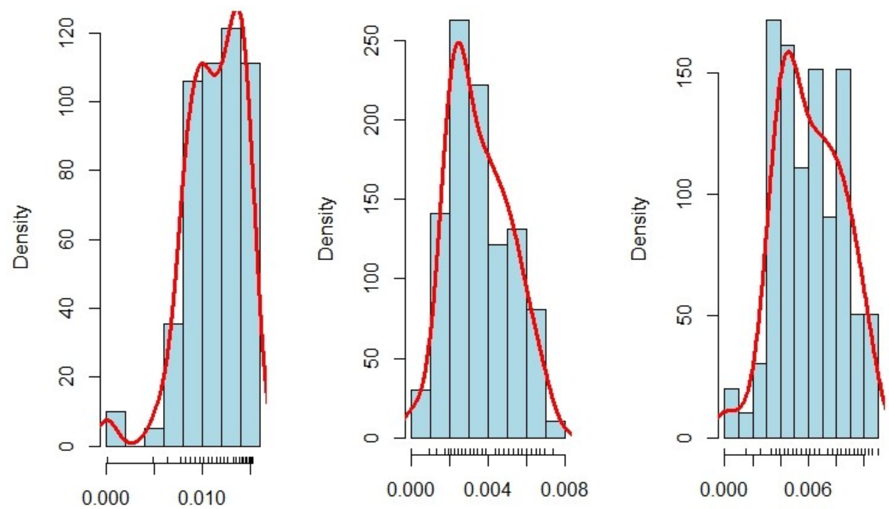

Figure 5.6. First-Exit Time for "gulf loaded post 7 days".

5.3. S&P 500 Data Analysis

Similar thing can be done for stock price too,we use the S&P 500 daily close price dataset for the period May 11, 2010 to May 8, 2020. Table 5.3 summarizes some features of this empirical dataset.

Table 5.3. Properties of the Empirical Dataset.

	S&P 500 Daily Close Price
Mean	2027.003
Median	2036.709
Maximum	3386.149
Minimum	1022.580

Figure 5.7 shows a line plot of the empirical dataset. The log-return process for the corresponding dataset is shown in Figure 5.8. Figure 5.9 and Figure 5.10 show the histograms of the S&P 500 daily close price, and corresponding log-returns respectively.

For the empirical dataset we consider the log-return process X_t, with $X_0 = 0$. For the first-exit time process of the log-return, $\inf\{s > 0 : X_s \geq t\}$, we consider the associated first-exit time processes of the Brownian motion $\inf\{s > 0 : W_s = t\}$, and the Lévy subordinator $\inf\{s > 0 : Z_s \geq t\}$. In the plots in Figure 5.11, we provide the histograms corresponding to the first-exit time of X_t for the empirical dataset for various values of t. In the plots of Figure

5.12, we use Remark 4.1.3 to plot the probability density functions of $\inf\{s > 0 : W_s = t\}$, for $t = 1, 2, 3, 4$. Finally, we use Gamma-type subordinators described in Section 4.2.2 with Lévy density $w(x) = \nu\alpha e^{-\alpha x}$. After finding appropriate parameter values, in the plots of Figure 5.13,we use Theorem 4.2.8 to plot the probability density functions of $\inf\{s > 0 : Z_s \geq t\}$, for $t = 1, 2, 3, 4$. From these figures, it is clear that for the time duration when there is no *big fluctuation* of the empirical dataset, $\inf\{s > 0 : W_s = t\}$ plays the dominant role in determining the distribution of $\inf\{s > 0 : X_s \geq t\}$. However, for the time duration of *big fluctuation* of the empirical dataset, $\inf\{s > 0 : Z_s \geq t\}$ plays the dominant role in determining the distribution of $\inf\{s > 0 : X_s \geq t\}$.

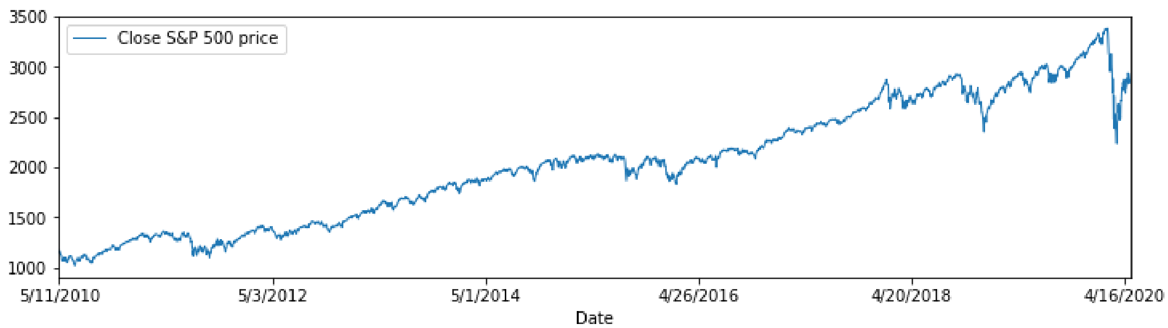

Figure 5.7. S&P 500 Daily Close Price from May, 2010 -May, 2020.

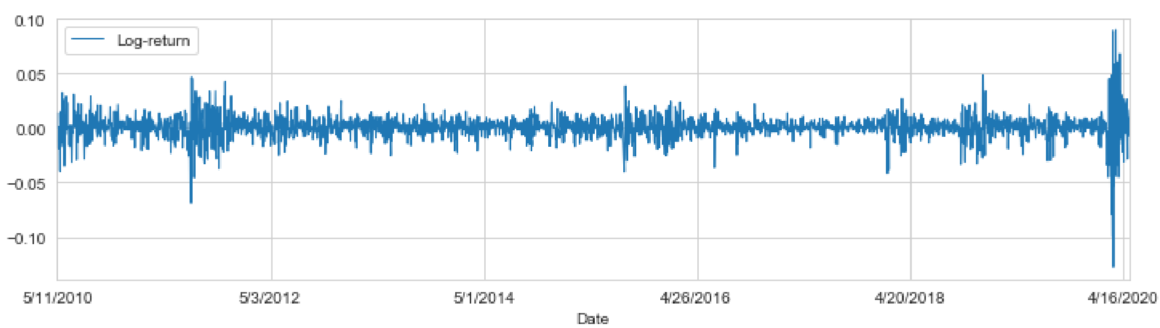

Figure 5.8. S&P 500 Log-Returns from May, 2010 -May, 2020.

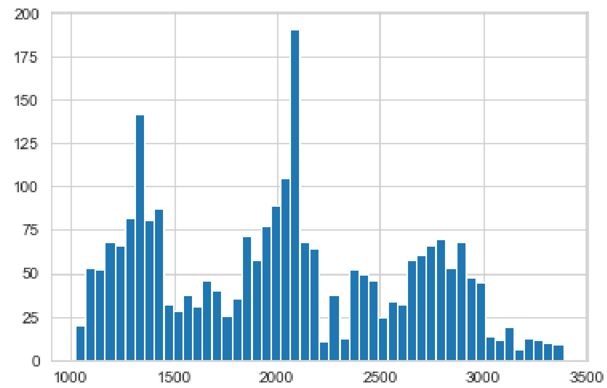

Figure 5.9. Histogram for the S&P 500 Daily Close Price.

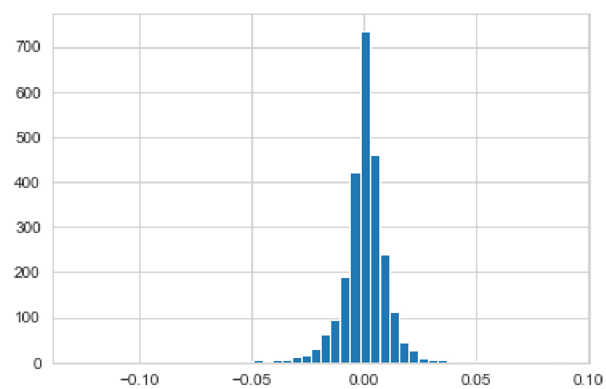

Figure 5.10. Histogram for the Log-Return.

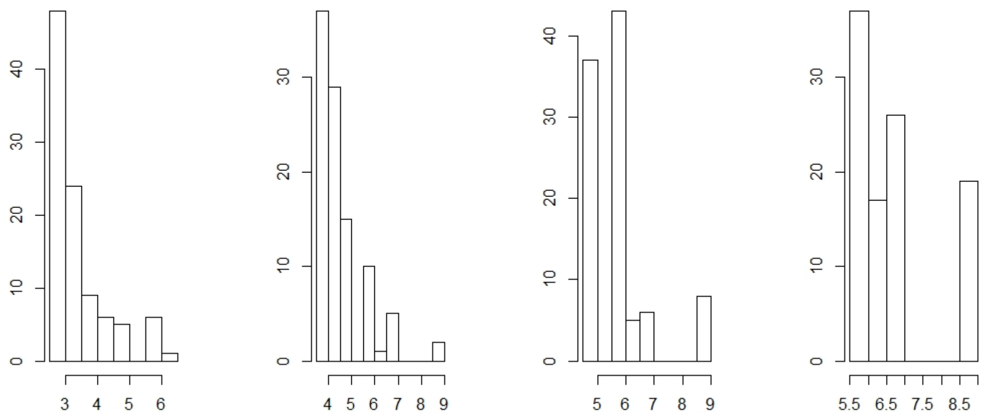

Figure 5.11. Histograms Corresponding to $\inf\{s > 0 : X_s \geq t\}$, for (left to right) $t = 1, 2, 3, 4$.

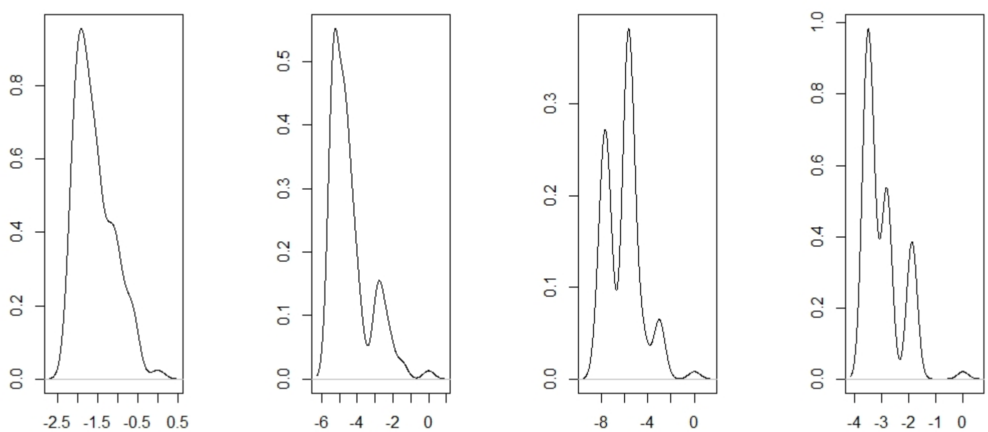

Figure 5.12. Probability Density Functions of $\inf\{s > 0 : W_s = t\}$, for (left to right) $t = 1, 2, 3, 4$.

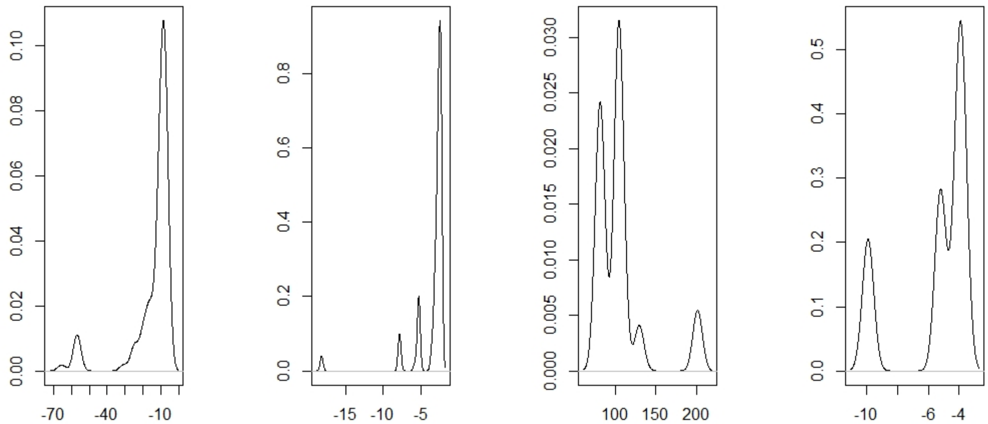

Figure 5.13. Probability Density Functions of $\inf\{s > 0 : Z_s \geq t\}$, for (left to right) $t = 1, 2, 3, 4$.

6. OPTION PRICING AND IMPLIED VOLATILITY

6.1. Option Pricing

Option pricing has a significant influence on quantitative finance. In Black-Scholes model, value of the option depends on future volatility of stock rather than its expected return. One of the biggest drawbacks of Black-Scholes is the mismatch between the model volatility of the underlying option and observed volatility from market. The paper [3] has shown the extension of classical Hull and White formula for option pricing when noise driving the volatility process correlates with the noise driving the stock prices. Price of a European call option, as obtained in [3] is given by,

$$V_t = E^*(C_{BS}(t, X_t; \nu_t)|\mathcal{F}_t) + \frac{1}{2}E^*\left(\int_t^T e^{rt-rs}\left(\frac{\partial^3}{\partial x^3} - \frac{\partial^2}{\partial x^2}\right)C_{BS}(s, X_s, \nu_s)\left(\int_s^T D_s^W \sigma_r^2\right)\sigma_s ds|\mathcal{F}_t\right), \tag{6.1}$$

where, $C_{BS}(t, X_t; \nu_t)$ is the price of European call option, $\sigma = \{\sigma_s, s \in [0, T]\}$ is an adopted and square integrable process and stock price is given by,

$$dS_t = \mu S_t dt + \sigma_t S_t dW_t, t \in [0, T], \tag{6.2}$$

$$X_t = \log(S_t),$$

where E^* denotes the expectation with respect to risk-neutral probalility P^*, $\mathcal{F}_t$ denotes the $\sigma-$ algebra generated by volatility process, r is the risk free interest rate and ν_t^2 is expected average volatility under the risk-neutral probability P^*.

It has been observed that jump in stock prices correlates with jump in volatility. Hence in our approach, we assume that both the stock price and volatility are driven by a correlated jump term.

Assumption 6.1.1. *we assume that jump term in stock price and volatility is bounded.*

We assume that the Stock price X_t^1 is given by,

$$X_t^1 = X_0 + \int_t^T \sigma_s \, dW_s + \frac{1}{2}\int_t^T \sigma_s^2 \, ds + \int_t^T \int_{-n}^{n} c(s,z)\tilde{N}(ds,dz). \tag{6.3}$$

Where $c(s,z)$ be a Skorohod integrable stochastic process. We also assume that volatility process is given by :,

$$Y_t = \int\limits_t^T \int\limits_{-n}^{n} \sigma^2(s,z)\tilde{N}(ds,dz). \tag{6.4}$$

Theorem 6.1.2. *Assume that the stock price and volatility dynamics are given by* (6.3) *and* (6.4) *respectively. Then under* (6.1.1)*, the price of a European call option is given by:*

$$\begin{aligned}
V_t &= E^*(C_{BS}(t, X_t^1; X_t^2)|\mathbb{F}_t) + \frac{1}{2}E^*[\int_t^T x_1 e^{rt-rs} C_{BS}(s, X_{s-})\sigma_s^2 ds] \\
&+ E^*[x_2 e^{rt-rs} C_{BS}(s, X_{s-})\frac{X_s^2 ds}{2(T+\delta-s)}] + E^*[\int_t^T s e^{rt-rs} C_{BS}(s, X_s) ds] \\
&+ E^*[\int_t^T A_{11}\frac{\partial^2 e^{rt-rs} C_{BS}(s-\delta, X_S)}{\partial x_1^2} ds + \int_t^T A_{12}\frac{\partial^2 e^{rt-rs} C_{BS}(s-\delta, X_S)}{\partial x_1 x_2} ds \\
&+ \int_t^T A_{21}\frac{\partial^2 e^{rt-rs} C_{BS}(s-\delta, X_S)}{\partial x_2 x_1} ds + \int_t^T A_{22}\frac{\partial^2 e^{rt-rs} C_{BS}(s-\delta, X_S)}{\partial x_2^2} ds] \\
&+ E^*[\sum_{0\le s\le t}^{\Delta X_s \neq 0} e^{rt-rs} C_{BS}(s, X_s^- + \Delta X_s) - e^{rt-rs} C_{BS}(s, X_s^-) \\
&- \Delta X_s^1 x_1 e^{rt-rs} C_{BS}(s, X_s^-) - \Delta X_s^2 x_2 e^{rt-rs} C_{BS}(s, X_s^-)].
\end{aligned}$$

Here $X_s = (X_s^1, X_s^2)$ are multidimensional Leǘy process and $\Delta X_s = X_s - X_{s-}$.

Proof. With a similar argument as in [3] we observe, $C_{BS}(T, X_T^1; X_T^2) = V_T$, as $e^{-rt}V_t$ is a P^* martiangle. Now we will apply Itô lemma for multidimensional Leǘy process, since derivatives of $C_{BS}(t,x,y)$ are not bounded, we will make use of approximating argument. Consider the process,

$$e^{-rt} C_{BS}(t-\delta, X_t^1; X_t^2), \tag{6.5}$$

where

$$X_t^2 = \sqrt{\frac{Y_t}{2n(T+\delta-t)}}. \tag{6.6}$$

It can be shown that,

$$dX_s^1 = \sigma_s dW_s + \frac{1}{2}\sigma_s^2 + \int_{-n}^{n} c(s,z)\tilde{N}(ds,dz). \tag{6.7}$$

$$dX_s^2 = \frac{X_s^2}{2(T+\delta-s)} + \frac{X_s^2 \int_{-n}^{n} \sigma^2(s,z)\tilde{N}(ds,dz)}{2Y_s}. \tag{6.8}$$

Now using Itô formula for multidimensional Lévy process (see [48]), we obtain

$$\begin{aligned}
e^{-rT}C_{BS}(T-\delta, X_T^1; X_T^2) &= e^{-rt}C_{BS}(t-\delta, X_t^1; X_t^2) + \int_t^T x_1 e^{-rs}C_{BS}(s, X_{s-})dX_s^1 \\
&+ x_2 e^{-rs}C_{BS}(s, X_{s-})dX_s^2) + \int_t^T e^{-rs}sC_{BS}(s, X_s)ds \\
&+ \int_t^T A_{11}\frac{\partial^2 e^{-rs}C_{BS}(s-\delta, X_S)}{\partial x_1^2}ds + \int_t^T A_{12}\frac{\partial^2 e^{-rs}C_{BS}(s-\delta, X_S)}{\partial x_1 x_2}ds \\
&+ \int_t^T A_{21}\frac{\partial^2 e^{-rs}C_{BS}(s-\delta, X_S)}{\partial x_2 x_1}ds + \int_t^T A_{22}\frac{\partial^2 e^{-rs}C_{BS}(s-\delta, X_S)}{\partial x_2^2}ds \\
&+ \sum_{0\le s\le t}^{\Delta X_s \neq 0} [e^{-rs}C_{BS}(s, X_s^- + \Delta X_s) - e^{-rs}C_{BS}(s, X_s^-) \\
&- \Delta X_s^1 x_1 e^{-rs}C_{BS}(s, X_s^-) - \Delta X_s^2 x_2 e^{-rs}C_{BS}(s, X_s^-)],
\end{aligned}$$

where $X_s = (X_s^1, X_s^2)$ is multidimenisonal Leÿy process. Consequently,

$$
\begin{aligned}
e^{-rT}C_{BS}(T-\delta, X_T^1; X_T^2) = e^{-rt}C_{BS}(t-\delta, X_t^1; X_t^2) + \int_t^T x_1 e^{-rs}C_{BS}(s, X_{s-})(\sigma_s dW_s + \frac{1}{2}\sigma_s^2 ds \\
&+ \int_{-n}^{n} c(s,z)\tilde{N}(ds,dz)) \\
&+ x_2 e^{-rs}C_{BS}(s, X_{s-}))(\frac{X_s^2 ds}{2(T+\delta-s)} + \frac{X_s^2\int_{-n}^{n}\sigma(s,z)\tilde{N}(ds,dz)ds}{2Y_s}) \\
&+ \int_t^T s e^{-rs}C_{BS}(s, X_s)ds \\
&+ \int_t^T A_{11}\frac{\partial^2 e^{-rs}C_{BS}(s-\delta, X_S)}{\partial x_1^2}ds + \int_t^T A_{12}\frac{\partial^2 e^{-rs}C_{BS}(s-\delta, X_S)}{\partial x_1 x_2}ds \\
&+ \int_t^T A_{21}\frac{\partial^2 e^{-rs}C_{BS}(s-\delta, X_S)}{\partial x_2 x_1}ds + \int_t^T A_{22}\frac{\partial^2 e^{-rs}C_{BS}(s-\delta, X_S)}{\partial x_2^2}ds \\
&+ \sum_{0\le s\le t}^{\Delta X_s \neq 0}[e^{-rs}C_{BS}(s, X_s^- + \Delta X_s) - e^{-rs}C_{BS}(s, X_s^-) \\
&- \Delta X_s^1 x_1 e^{-rs}C_{BS}(s, X_s^-) - \Delta X_s^2 x_2 e^{-rs}C_{BS}(s, X_s^-)].
\end{aligned}
$$

Now taking conditional expectations and multiplying by e^{rt},we obtain,

$$
\begin{aligned}
E^*\left[C_{BS}(T-\delta, X_T^1; X_T^2)|\mathbb{F}_t\right] = C_{BS}(T-\delta, X_t^1; X_t^2) + E^*[\int_t^T (x_1 e^{rt-rs}C_{BS}(s, X_{s-})(\sigma_s dW_s + \frac{1}{2}\sigma_s^2 \\
&+ \int_{-n}^{n} c(s,z)\tilde{N}(ds,dz))] \\
&+ E^*[x_2 e^{rt-rs}C_{BS}(s, X_{s-}))(\frac{X_s^2 ds}{2(T+\delta-s)} + \frac{X_s^2\int_{-n}^{n}\sigma(s,z)\tilde{N}(ds,dz)ds}{2Y_s})] \\
&+ E^*[\int_t^T s e^{rt-rs}C_{BS}(s, X_s)ds] \\
&+ E^*[\int_t^T A_{11}\frac{\partial^2 e^{rt-rs}C_{BS}(s-\delta, X_S)}{\partial x_1^2}ds + \int_t^T A_{12}\frac{\partial^2 e^{rt-rs}C_{BS}(s-\delta, X_S)}{\partial x_1 x_2}ds \\
&+ \int_t^T A_{21}\frac{\partial^2 e^{rt-rs}C_{BS}(s-\delta, X_S)}{\partial x_2 x_1}ds + \int_t^T A_{22}\frac{\partial^2 e^{rt-rs}C_{BS}(s-\delta, X_S)}{\partial x_2^2}ds] \\
&+ E^*[\sum_{0\le s\le t}^{\Delta X_s \neq 0} e^{rt-rs}C_{BS}(s, X_s^- + \Delta X_s) - e^{rt-rs}C_{BS}(s, X_s^-) \\
&- \Delta X_s^1 x_1 e^{rt-rs}C_{BS}(s, X_s^-) - \Delta X_s^2 x_2 e^{rt-rs}C_{BS}(s, X_s^-)].
\end{aligned}
$$

Now letting δ to 0, we obtain,

$$
\begin{aligned}
V_t &= E^*(C_{BS}(t, X_t^1; X_t^2)|\mathbb{F}_t) + \frac{1}{2}E^*[\int_t^T x_1 e^{rt-rs} C_{BS}(s, X_{s-})\sigma_s^2] \\
&+ E^*[x_2 e^{rt-rs} C_{BS}(s, X_{s-}) \frac{X_s^2 ds}{2(T+\delta-s)}] + E^*[\int_t^T s e^{rt-rs} C_{BS}(s, X_s) ds] \\
&+ E^*[\int_t^T A_{11} \frac{\partial^2 e^{rt-rs} C_{BS}(s-\delta, X_S)}{\partial x_1^2} ds + \int_t^T A_{12} \frac{\partial^2 e^{rt-rs} C_{BS}(s-\delta, X_S)}{\partial x_1 x_2} ds \\
&+ \int_t^T A_{21} \frac{\partial^2 e^{rt-rs} C_{BS}(s-\delta, X_S)}{\partial x_2 x_1} ds + \int_t^T A_{22} \frac{\partial^2 e^{rt-rs} C_{BS}(s-\delta, X_S)}{\partial x_2^2} ds] \\
&+ E^*[\sum_{0 \le s \le t}^{\Delta X_s \neq 0} e^{rt-rs} C_{BS}(s, X_s^- + \Delta X_s) - e^{rt-rs} C_{BS}(s, X_s^-) \\
&- \Delta X_s^1 x_1 e^{rt-rs} C_{BS}(s, X_s^-) - \Delta X_s^2 x_2 e^{rt-rs} C_{BS}(s, X_s^-)].
\end{aligned}
$$

□

6.2. Implied Velocity

Main motivation of this section is to study at-the-money implied volatility of a European call option when stock price is defined by (6.9). We will use Malliavin calculus to derive an exact expression. In this section, we assume log-price of a stock under a risk neutral probability measure P the model is given by,

$$
X_t = X_0 - \frac{1}{2}\int_0^t \sigma_s^2 \, ds + \int_0^t \sigma_s \, dW_s + \int_0^t \int_{R_o} c(s,z)\tilde{N}(ds,dz), \quad t \in [0,T], \tag{6.9}
$$

where X_0 is current log-price, W_t is standard Brownian motions defined on a complete probability space (Ω,$\mathcal{F}$,P) and σ_s is a square integrable and right continuous stochastic process and $\tilde{N}(ds,dz)$ is a compenstaed poisson process. The price of the European call with strike price K is given by the following formula,

$$
V_t = E_t[(e^{X_T} - K)^+], \tag{6.10}
$$

where E_t denotes the $\mathcal{F}_t$ conditional expectation with respect to P. In the following, we used following notation: $v(t,Y_t) = \sqrt{\frac{Y_t}{T-t}}$, where $Y_t = \int_t^T \sigma_u^2 du$. Suppose v represents future average volatility, $BS(t,T,x,k,\sigma)$ denotes the price of a European call option under Black-Scholes model with current log price x, constant volatility σ,time to maturity $T-t$ and strike price $K=\exp(k)$,

$$BS(t,T,x,k,\sigma)=\exp(x)N(d_+(k,\sigma))-\exp(k)N(d_-(k,\sigma)), \tag{6.11}$$

where N is the cumulative distribution function of the standard normal law and

$$d_{\pm}(k,\sigma)=\frac{k_t^*-k}{\sigma\sqrt{T-t}}\pm\frac{\sigma(T-t)}{2}, \tag{6.12}$$

where k_t^* denotes the at-the-money strike,which coincides with x when interest rate is 0. The inverse function $BS^{-1}(t,T,x,k,.)$ of the Black-Scholes formula with respect to the volatility parameter is defined, for all $\lambda>0$, (see [2]):

$$BS(t,T,x,k,BS^{-1}(t,T,x,k,\lambda))=\lambda. \tag{6.13}$$

For any fixed t,T,X_t,k, we define the implied volatility $I(t,T,X_t,k)$ as the quantity such that

$$BS(t,T,X_t,k,I(t,T,X_t,k))=V_t. \tag{6.14}$$

Following notations are used for following section: $\mathbb{D}_W^{1,2}$ is domain of Malliavin derivative operator $\mathbb{D}_W$ with respect to the Brownian motion W_t. We consider the iterated derivative $\mathbb{D}_W^n$ for $n>1$, whose domains are denoted by $\mathbb{D}_W^{n,2}$.We also define $L_W^{n,2}=L^2([0,T]:\mathbb{D}_W^{n,2})$.

It is shown in [2] that at-the-money implied volatility has following form.

$$I(t,T,X_t,k_t^*)=E_t[v_t]-\frac{1}{32(T-t)}E_t\left[\int_t^T\frac{\psi_r}{(N'(d_+(k_t^*,\psi_r)))^2}\left(E_r\left[N'(d_+(k_t^*,v_t))\frac{\int_r^T D_r^W\sigma_s^2ds}{v_t}\right]\right)^2dr\right], \tag{6.15}$$

where, $\Lambda_r=E_r[BS(t,T,X_t,k_t^*,v_t)]$,

and $\psi_r=BS^{-1}(k_t^*,v_t)$.

Consider the following hypotheses:

(1) There exist positive constant a, b such that $a \leq \sigma_t \leq b$ for all $t \in [0, T]$.

(2) $\sigma^2 \in L_W^{1,2}$.

Theorem 6.2.1. *Consider the model* (6.9). *Then, the at-the-money implied volatility is given as,*

$$I(t,T,X_t,k_t^*) = E_t[v_t] - \frac{1}{32(T-t)} E_t\left[\int_t^T \frac{\psi_r}{(N'(d_+(k_t^*,\psi_r)))^2}\left(E_r\left[N'(d_+(k_t^*,v_t))\frac{\int_r^T D_r^W \sigma_s^2 ds}{v_t}\right]\right)^2 dr\right], \tag{6.16}$$

where, $\Lambda_r = E_r[BS(t,T,X_t,k_t^*,v_t)]$

$\psi_r = BS^{-1}(k_t^*, v_t)$.

Proof. We have,

$V_t = E_t[BS(t,T,X_t,k_t^*,v_t)]$.

Then

$$\begin{aligned} I(t,T,X_t,k_t^*) &= BS^{-1}(k_t^*,V_t) \\ &= E_t[BS^{-1}(k_t^*,E_t[BS(t,T,X_t,k_t^*,v_t)])] \\ &= E_t[BS^{-1}(k_t^*,E_t[BS(t,T,X_t,k_t^*,v_t)]) - BS^{-1}(K_t^*,BS(t,T,X_t,k_t^*,v_t))] + E_t[v_t]. \end{aligned} \tag{6.17}$$

where $E_t[BS^{-1}(k_t^*,BS(t,T,X_t,k_t^*,v_t))] = E_t[v_t]$.

From [2], we obtain,

$$BS(t,T,X_t,k_t^*,v_t) = E_t[BS(t,T,X_t,k_t^*,v_t)] + \int_t^T U_S dW_s, \tag{6.18}$$

Where U_s can be computed by Clark-Ocone formula and W is Brownian motion that derives the volatility process.

$$E_t[BS^{-1}(k, E_t[BS(t,T,X_t,k_t^*,v_t)]) - BS^{-1}(k, BS(t,T,X_t,k_t^*,v_t))]$$
$$= E_t[BS^{-1}(k_t^*,\Lambda_t) - BS^{-1}(k_t^*,\Lambda_T)]$$
$$= -E_t[\int_t^T (BS^{-1})'(k_t^*,\Lambda_r)U_r dW_r + \frac{1}{2}\int_t^T (BS^{-1})''(k_t^*,\Lambda_r)U_r^2 dr. \tag{6.19}$$

Where $(BS^{-1})'$,$(BS^{-1})''$ denote first and second derivative of $BS^{-1}$with respect to last variable λ.

$$U_r = E_r\left[\exp(X_t)N'(d_+(k_t^*,v_t))\frac{\int_r^T D_r^W \sigma_s^2}{2\sqrt{T-t}v_t}\right]. \tag{6.20}$$

Clearly (6.20) along with hypothesis (1) imply that

$$E_t\left[\int_t^T \left((BS^{-1})'(k_t^*,\Lambda_r)U_r\right)^2 dr\right] \le C(T,t). \tag{6.21}$$

This gives us expectation of stochastic integral is zero. Then we get,

$$E_t[BS^{-1}(X_t, E_t[BS(t,T,X_t,k_t^*,v_t)]) - BS^{-1}(X_t, BS(t,T,X_t,k_t^*,v_t))]$$
$$= -\frac{1}{2}E_t\left[\int_t^T (BS^{-1})''(k_t^*,\Lambda_r)U_r^2 dr\right]. \tag{6.22}$$

□

Now we estimate at the money implied volatility under specific framework. We assume that volatility follows a mean-reverting OU process, i.e., Stein and Stein model.
Under Stein and Stein Model,volatility process assumes following form,

$$d\sigma_t = -\alpha(m-\sigma_t)dt + c dW_t, \tag{6.23}$$

where α, m and c are positive real constants and W_t is a standard Brownian motion. Analytical soluttion of (6.23) is given as,

$$\sigma_s = m + (\sigma_t - m)e^{-\alpha(s-t)} + c\int_t^s e^{-\alpha(s-u)}dW_u. \tag{6.24}$$

Theorem 6.2.2. *Assuming Model* (6.9) *and volatility process follows Stein and Stein model, the at-the-money implied volatility* $I(t,T,X_t,k_t^*)$ *is given as,*

$$E_t[v_t] - \frac{1}{32(T-t)} E_t \left[\int_t^T \frac{\psi_r}{(N'(d_+(k_t^*,\psi_r)))^2} \left(E_r \left[N^{'}(d_+(k_t^*,v_t)) \frac{2c\rho\sigma_r e^{-\alpha r}[e^{\alpha T} - e^{\alpha r}]}{\alpha v_t} \right] \right)^2 dr \right].$$

Proof. We have already shown in (6.2.1), the explicit expression for implied volatility $I(t,T,X_t,k_t^*)$. Now using (6.24) for volatility process,we compute Malliavin derivative as

$$D_s^W \sigma_r^2 = 2\sigma_r D_s \sigma_r = 2c\rho\sigma_r e^{-\alpha(r-s)}. \tag{6.25}$$

This completes the proof. □

7. CONCLUSION

It is shown in this dissertation that an analytically tractable expression can be obtained for the probability density function of the first-exit time process of an approximate BN-S process. For the financial data, the density function of the first-exit time of the corresponding log-return process provides an important insight. In particular, such density function facilitates the understanding of a "crash-like" future fluctuation of the market. In addition, this analysis has two-fold advantages. Firstly, based on the insight from the probability density function of the first-exit time process, the empirical data analysis for the future market is improved. Secondly, and more importantly, this provides a concrete way to improve existing stochastic models. For example, most of the existing financial models suffer from the lack of long-range dependence problem. An understanding of the density function of the first-exit time of stochastic models driven by a general Lévy process can contribute positively to mitigate this issue.

In the numerical results, we show various plots in support of the theoretical analysis provided in this dissertation. However, the analysis is dependent on the accurate estimation of model parameters for the empirical dataset. At present, we are implementing various machine learning based calibration techniques to improve the estimates of the parameter values for the empirical dataset. In effect, this may significantly improve the numerical results.

In this dissertation, it is also showed that data-science driven models have advantages in providing analysis of competition in the soybean market. Many features of competition in a soybean sector which limits or constrains the ability of traditional "equilibrium" models normally specified in economics. Most important are the randomness of most variables, changes over time in the underlying relationships and functions, as well as periodic interventions. In this dissertation, we develop a data-science driven model of competition in soybean exports to China, the largest and fastest growing market in the world.

The results indicate that export market shares, which are otherwise highly volatile, can be effectively explained (predicted) using deep learning methodologies and a set of logical feature variables. Some of the variables have significant influences, particularly using the Variable Importance.

These factors include “Gulf duenext10D”, “barge mm”, “gulf loaded past 7 days”, in addition to several other variables including monthly binary variables.

We conclude this dissertation with an analysis of option pricing and implied volatility in the case when the market is driven by a jump-stochastic volatility model. We find the price of the European call option in this case. In addition, we implement Malliavin calculus to analyze the implied volatility. This is a novel way generates simple formulas for various stochastic models. We plan to explore this more in our future works.

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