# ABSOLUTE STABILITY OF A CLASS OF SECOND-ORDER FEEDBACK NON-LINEAR, TIME-VARYING SYSTEMS

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**Title**

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The Supervisory Committee certifies that this ***disquisition*** complies with North Dakota State University's regulations and meets the accepted standards for the degree of

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# ABSTRACT

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In this thesis, we consider the problem of absolute stability of continuous time feedback systems with a single, time-varying nonlinearity. Necessary and sufficient conditions for absolute stability of second-order systems in terms of system parameters are developed, which are characterized by eigenvalue locations on the complex plane. More specifically, our results are presented in terms of the associated matrix-pencil $\{\mathbf{A} + b\nu c^*,\ \nu \in [\mu_1, \mu_2]\}$, where $\mu_1, \mu_2 \in \mathbb{R}$, $\mathbf{A}$ is $n \times n$-matrix, $b$ and $c$ are $n$-vectors. The stability conditions require that the eigenvalues of all matrices $\mathbf{A}+b\nu c^*$, $\mu_1 \leq \nu \leq \mu_2$, lie in the interior of a specific region of the complex plane (a cone to be specific). Thus, we have the following reformulation of the problem. Find the maximal cone satisfying the following condition: If all eigenvalues of corresponding linear systems belong to this cone, then system is absolutely stable. Known results show that this cone is not smaller than $\{z \in \mathbb{C} : \frac{3\pi}{4} \leq \arg z \leq \frac{5\pi}{4}\}$ (called the $45^0$-Region). The result is proven using Lyapunov functions of two different types. It is known that usually the approach based on Lyapunov functions provides essentially sufficient conditions for absolute stability. We will use a different technique which provides necessary and sufficient conditions for absolute stability. The problem setting, the approach, and methods to solve the problem will be presented in Chapter 3. The contents of Chapters 1 and 2 include preliminary concepts, definitions, and facts basic to

the theory of feedback control systems. In Sections 3.1 and 3.2, we introduce basic results of the theory of stability for feedback control systems (i.e., for systems of arbitrary order $n \in \mathbb{Z}^+$). In particular, we will introduce the notion of absolute stability for feedback control systems, linear differential inclusions, dual inclusions, and asymptotic stability of linear inclusions. Sections 3.3, 3.4, and 3.5 are devoted to the core of this thesis: the analysis of absolute stability of systems of order two (i.e., $n = 2$). In Section 3.4, we present the proof of a variant of sufficient conditions for absolute stability that was first introduced in [2], and in Section 3.5, we prove the new result that shows the necessity of the condition given in Section 3.4. Chapter 4 is a summary of the results obtained in this thesis and highlight some possible future investigations.

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In memory of Yetunde Anne Adeparusi.

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# CHAPTER 1. INTRODUCTION

In this chapter, we present a collection of basic results in the theory of ordinary differential equations relevant to the subject of feedback control systems. The proofs are omitted for most of the facts, with references given. This chapter includes the following: definition, existence and uniqueness theorems for solutions of ordinary differential equations, definition of stability, and sufficient criteria for different kinds of stability which use the Lyapunov functions.

## 1.1. General Background: Existence and Uniqueness Theorems

Let us consider an ordinary differential equation of the form

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t), \qquad (\mathbf{x}, t) \in E, \tag{1}$$

where $\mathbf{x}$ is an absolutely continuous $n$-vector function; $\dot{\mathbf{x}}$ denotes the ordinary derivative with respect to the independent variable, $t$; $E$ is an open set of $\mathbb{R}^{n+1}$ (where $0 < n \in \mathbb{Z}^+$); and the map $\mathbf{f} : E \to \mathbb{R}^n$ is $t$-measurable and $\mathbf{x}$-continuous. The vector $\mathbf{x}$ is often thought of as space variable and $t$ as time. We will denote the Euclidean norm of a vector $\mathbf{x}$ by $||\mathbf{x}||$ and the corresponding induced norm on $n \times n$ matrices, $\mathbf{A}$, is defined as follows:

$$||\mathbf{A}|| = \sup\{||\mathbf{A}\mathbf{x}|| : \mathbf{x} \in \mathbb{R}^n,\ ||\mathbf{x}|| \leq 1\}$$

**Definition 1.1.1.** *[15] A map $\varphi : I \to \mathbb{R}^n$ is called a solution of equation (1) if $\varphi$ is absolutely continuous and*

$$\dot{\varphi} = \mathbf{f}(\varphi, t) \quad \textit{a.e. on } I.$$

The curve $t \to (\varphi(t), t)$ lying in $E$ is called a *trajectory* of equation (1).
Usually, we look for solutions where the trajectory passes through a specific point, $(t_0, \mathbf{x}^0) \in E$. The problem of finding such solutions is called an initial-value problem. Regarding the existence of a solution for an initial-valued problem, we have the

following general theorem.

**Theorem 1.1.1.** *[15](Carathéodory)*
*Let* $\mathbf{f}(\mathbf{x}, t)$ *be a function defined on* $R = \{(\mathbf{x}, t) : |t - t_0| \leq a, \|\mathbf{x} - \mathbf{x}_0\| \leq b\} \subset E$, *and suppose it is measurable in* $t$ *for each fixed* $\mathbf{x}$ *and continuous in* $\mathbf{x}$ *for each fixed* $t$. *If there exists a Lebesgue-integrable function,* $m(\cdot)$, *on the interval* $|t - t_0| \leq a$ *such that*

$$|f(\mathbf{x}, t)| \leq m(t), \qquad ((\mathbf{x}, t) \in R), \tag{2}$$

*then there exists a solution,* $\varphi$, *of the initial-value problem*

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t) \\ \mathbf{x}(t_0) = \mathbf{x}^0 \end{cases} \tag{3}$$

*on some interval* $|t - t_0| \leq \beta$, $(\beta > 0)$.

Carathéodory's theorem provides a general setting in which the existence of solutions can be taken for granted. In this thesis, we always stay within this context and only consider the initial-value problem, $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$, $\mathbf{x}(t_0) = \mathbf{x}^0$, where $\mathbf{f}$ is a function measurable with respect to $t$ and continuous with respect to $\mathbf{x}$. The following theorem gives a criterion for uniqueness of a solution to an Ordinary Differential Equation (ODE), which is sufficient for many practical cases. We will use the following notation: $C(R)$ is the space of continuous functions on the set $R$ and $C^1(R)$ is the space of continuously differentiable functions on the set $R$.

**Theorem 1.1.2.** *[15] Let* $\psi = \psi(r, t)$ *be a nonnegative function defined on the set*

$$S_a = \{(r, t) : 0 < t < a,\ r \geq 0\} \qquad (a > 0), \tag{4}$$

*such that the function* $\psi$ *is Lebesgue measurable in* $t$ *for fixed* $r$ *and continuous nondecreasing in* $r$ *for fixed* $t$. *Further, for every bounded subset,* $B \subset S_a$, *let there* *exist a function, $\chi_B \in L^1(0,\infty)$, such that*

$$\psi(r,t) \le \chi_B(t) \qquad \forall\ (r,t) \in B.$$

*Suppose that, for each $\alpha \in (0,a)$, the function $\rho$ defined by $\rho(t) = 0$, $0 \le t < \alpha$, is the only absolutely continuous function on $0 \le t < \alpha$ which satisfies*

$$\rho'(t) = \psi(\rho(t),t) \qquad a.e.\ t \in (0,\alpha),$$

*such that $\rho'_+(0)$ exists and $\rho(0) = \rho'_+(0) = 0$. Let $f \in C(R)$ and satisfies*

$$|f(\hat{\mathbf{x}},t) - f(\tilde{\mathbf{x}},t)| \le \psi(|t-t_0|, |\hat{\mathbf{x}} - \tilde{\mathbf{x}}|) \qquad t \neq t_0.$$

*Then, there exists, at most, one solution $\varphi \in C^1(R)$ for the initial-value problem (3).*

In Theorems 1.1.1-1.1.2, we have obtained estimates for the size of the solution domain. However, none of these estimates put a limitation or bound on the domain of the solution. That is, it might be possible that, for a particular equation, the solution is defined for all real $t$ even though the existence theorem only guaranteed the existence of a solution on a small finite interval. Hence, we need a result about how large the domain of the solution may be.

**Theorem 1.1.3.** *(Extension Theorem) [16]. Let $f(\mathbf{x},t)$ be a continuous function on an open $(\mathbf{x},t)$-set $\mathbf{E}$, and let $\mathbf{x}(t)$ be a solution of (1) on some interval. Then, $\mathbf{x}(t)$ can be extended (as a solution) over a maximal interval of existence $(\omega_-,\omega_+)$. Also, if $(\omega_-,\omega_+)$ is a maximal interval of existence, then $\mathbf{x}(t)$ tends to the boundary, $\partial\mathbf{E}$, of $\mathbf{E}$ as $t \to \omega_-$ and $t \to \omega_+$.*

Now, by Carathéodory's theorem, every initial-value problem has at least one solution; by Theorem 1.1.2, there is, at most, one solution; by Theorem 1.1.3, for each initial-value problem, there is a maximal interval to which the solution can be extended. Therefore, let $\mathbf{x}(t,t_0,\mathbf{x}^0)$ denote the maximal solution of system (3) passing through the point $(t_0, \mathbf{x}^0) \in E$. Note that $\mathbf{x}(t, t_0, \mathbf{x}^0)$ is a well-defined function on some subset, $E_f$, of $\mathbb{R}^{n+2}$ with values in $\mathbb{R}^n$. The function $\mathbf{x}(t, t_0, \mathbf{x}^0)$ will play a major role in subsequent results. It provides a tool for investigating the asymptotic behavior of all solutions, in particular stability of solutions to initial-value problems.

## 1.2. Asymptotic Behavior of Systems: Stability

The following definitions are due to A. M. Lyapunov. They are very important in modern control theory.

**Definition 1.2.1.** *[16] A solution $\mathbf{x}(t, t_0, \mathbf{x}^0)$ of system (3) is called Lyapunov stable if, for every $\epsilon > 0$, there exist $\delta > 0$ such that, for every vector $\mathbf{y}^0$ such that $\|\mathbf{x}^0 - \mathbf{y}^0\| < \delta$, it follows that $\|\mathbf{x}(t, t_0, \mathbf{x}^0) - \mathbf{x}(t, t_0, \mathbf{y}^0)\| < \epsilon$ for every $t \geq t_0$.*

Lyapunov stability does not imply that solutions starting at a neighboring point tend to each other. The next definition concerns this issue.

**Definition 1.2.2.** *[16] A solution $\mathbf{x}(t, t_0, \mathbf{x}^0)$ of system (3) is called asymptotically stable if*

*(1)* $\mathbf{x}(t, t_0, \mathbf{x}^0)$ *is Lyapunov stable.*

*(2) There is a positive number, $\gamma$, such that $\|\mathbf{x}(t, t_0, \mathbf{x}^0) - \mathbf{x}(t, t_0, \mathbf{y}^0)\| \to 0$ as $t \to \infty$ for every vector $\mathbf{y}^0$ such that $\|\mathbf{x}^0 - \mathbf{y}^0\| < \gamma$.*

Note that (1) does not imply (2). Now, let us consider solutions starting at arbitrary points, not necessarily in a neighborhood of a given solution.

**Definition 1.2.3.** *[16] A solution $\mathbf{x}(t, t_0, \mathbf{x}^0)$ of system (3) is called globally asymptotically stable if $\mathbf{x}(t, t_0, \mathbf{x}^0)$ is Lyapunov stable and $\|\mathbf{x}(t, t_0, \mathbf{x}^0) - \mathbf{x}(t, t_0, \mathbf{y}^0)\| \to 0$ as $t \to \infty$ for each initial value, $\mathbf{y}^0$.*

Certainly, global asymptotic stability implies asymptotic stability.

Assume $\eta(t)$ is a solution of the initial-value problem (3). Consider the following change of variable; $\mathbf{z}(t) = \mathbf{x}(t) - \eta(t)$. Then, by equation (3), we have

$$\dot{\mathbf{z}} = \mathbf{f}(\mathbf{z}(t) + \eta(t), t) - \mathbf{f}(\eta(t), t). \tag{5}$$

Let us denote the right-hand side of (5) by $\tilde{\mathbf{f}}(\mathbf{z}, t)$. Hence, by this change of variable system (1) becomes

$$\dot{\mathbf{z}} = \tilde{\mathbf{f}}(\mathbf{z}, t). \tag{6}$$

Notice that function $\mathbf{z}(t) \equiv 0$ is a solution of system (6) (known as the zero solution of system (6)). In the stability theory of systems for ordinary differential equations, it is convenient to assume that we are dealing with the stability property of the zero solution for system (1).

**Definition 1.2.4.** *System (6) is called Lyapunov (asymptotically, globally asymptotically, respectively) stable if $\mathbf{z}(t) \equiv 0$ is Lyapunov (asymptotically, globally asymptotically) stable.*

**Definition 1.2.5.** *A solution $\mathbf{x}(t, t_0, \mathbf{x}^0)$ is Lyapunov (asymptotically, globally asymptotically, respectively) stable for system (3) if and only if $\mathbf{z}(t) \equiv 0$ (i.e., zero solution) is Lyapunov (asymptotically, globally asymptotically, respectively) stable for equation (6).*

Based on the above discussion, we have the following revision. Let us consider the system of differential equations

$$\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t), \tag{7}$$

where $\mathbf{x}$ and $\mathbf{f}$ are vector functions with $n$ real coordinates, with the following general assumptions: $\mathbf{f}(\mathbf{x}, t)$ is measurable with respect to $t$ and continuous with respect to $\mathbf{x}$, and satisfies a local Lipschitz condition with respect to $\mathbf{x}$ on

$$R = \{(\mathbf{x}, t) : t \geq t_0, \|\mathbf{x}\| < a < \infty\}.$$

Assume

$$\mathbf{f}(\mathbf{0}, t) = \mathbf{0}, \quad t \geq t_0. \tag{8}$$

Our assumption implies the existence and the uniqueness of the solution passing through any point $(\mathbf{x}^0, t) \in R^{n+1}$ with $\mathbf{x}^0 \in \mathbb{R}$. In addition, from condition (8), it follows that $\mathbf{x} \equiv \mathbf{0}$ is a solution of equation (7), and because uniqueness holds, any solution of equation (7), distinct from $\mathbf{x} = \mathbf{0}$, cannot vanish on its interval of existence.

We have the following definitions. They relate to the concept of stability for the zero solution, $\mathbf{x}(t) \equiv \mathbf{0}$, of equation (7).

**Definition 1.2.6.** *The zero solution,* $\mathbf{x}(t) \equiv 0$, *of equation (7) is called Lyapunov stable, if for every* $\epsilon > 0$, *there exists* $\delta > 0$ *such that* $\|\mathbf{x}(t, t_0, \mathbf{x}^0)\| < \epsilon$ *for every* $t \geq t_0$ *whenever* $\|\mathbf{x}^0\| < \delta$.

**Definition 1.2.7.** *The zero solution,* $\mathbf{x}(t) \equiv 0$, *of equation (7) is called asymptotically stable if it is Lyapunov stable and there exists a positive number,* $\gamma$, *such that*

$$\lim_{t \to \infty} \|\mathbf{x}(t, t_0, \mathbf{x}^0)\| = 0 \tag{9}$$

*for every* $\mathbf{x}(t, t_0, \mathbf{x}^0)$ *with* $\|\mathbf{x}^0\| < \gamma$.

**Definition 1.2.8.** *The zero solution,* $\mathbf{x}(t) \equiv 0$, *of (7) is called globally asymptotically stable if it is Lyapunov stable and* $\lim_{t \to \infty} \|\mathbf{x}(t, t_0, \mathbf{x}^0)\| = 0$ *for every initial value,* $\mathbf{x}^0$.

Now, consider the problem of efficiently checking the stability of the zero solution for a given system. For some particular class of differential equations that we can actually solve, the above definitions of stability can be easily checked. There is a way which may allow us to determine whether the zero solution is stable (Lyapunov stable, asymptotically stable, and globally asymptotically stable, respectively) without knowing the solutions of the differential equation.

Let us introduce the real-valued function $V(\mathbf{x}, t) \subset C^1(E_0 \times \mathbb{R}^+)$, where $E_0$ is given. Suppose $\mathbf{x}(t)$ is a solution of $\dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}, t)$. When $\mathbf{x}(t)$ is in $E_0$, the derivative of $V(\mathbf{x}(t), t)$ with respect to $t$ can be calculated using the chain rule. Thus, it can be determined whether $V$ is increasing or decreasing along the solution through $\mathbf{x}$ by examining

$$\dot{V} = \frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}(t), t) \cdot \mathbf{f}(\mathbf{x}(t), t) + \frac{\partial V}{\partial t}(\mathbf{x}(t), t). \tag{10}$$

The idea of Lyapunov's method and the related results is to impose conditions on the test functions, $V$ and $\dot{V}$, which will imply stability. To formulate the main theorem about Lyapunov functions, we need the following definitions.

**Definition 1.2.9.** *[17] Let $E_0 \in \mathbb{R}^n$ with $\mathbf{0} \in int(E_0)$. Function $V : E_0 \times \mathbb{R}^+ \to \mathbb{R}$ is called positive definite (denoted $V \gg 0$) if*

*(1)* $V(\mathbf{0}, t) = 0$, $V(\mathbf{x}, t) \geq 0$ *for all* $(\mathbf{x}, t) \in E_0 \times \mathbb{R}^+$.

*(2)* *For all $\epsilon > 0$, there exists $\delta > 0$ such that, for all $\mathbf{x} \in E_0$ and $\|\mathbf{x}\| \geq \epsilon$, we have $V(\mathbf{x}, t) \geq \delta$ for all $t \geq t_0$.*

Similarly, function $V(\mathbf{x}, t)$ is called negative definite (*denoted* $V \ll 0$) if $-V(\mathbf{x}, t)$ is positive definite.

**Definition 1.2.10.** *[17] A positive, definite function, $V(\mathbf{x}, t)$, is called a Lyapunov function for system (3) if*

$$\frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}, t) \cdot \mathbf{f}(\mathbf{x}, t) + \frac{\partial V}{\partial t}(\mathbf{x}, t) \leq 0$$

*for each $\mathbf{x} \in E_0$ and $t \geq t_0$.*

The following theorems formulate criteria for different kinds of stability based on the existence of Lyapunov functions for a given system (7).

**Theorem 1.2.1.** *[17] Given equation (7), assume that there exists a Lyapunov function $V(\mathbf{x}, t)$. Then, $\mathbf{x} \equiv 0$ is Lyapunov stable.*

Likewise, for asymptotic stability, we have the following statement.

**Theorem 1.2.2.** *[17] Given equation (7), assume that there exists a Lyapunov function $V(\mathbf{x}, t)$ such that*

$$\frac{\partial V}{\partial \mathbf{x}}(\mathbf{x}, t) \cdot \mathbf{f}(\mathbf{x}, t) + \frac{\partial V}{\partial t}(\mathbf{x}, t) \ll 0$$

*and $V(\mathbf{x}, t) \to 0$ as $\mathbf{x} \to 0$ uniformly with respect to $t \geq t_0$. Then, $\mathbf{x}(t) \equiv 0$ is asypmtotically stable.*

Finally, for global asymptotic stability, we have Krasovsky-LaSalle principle.

**Theorem 1.2.3.** *(Krasovsky-LaSalle principle)[19] If there exists a Lyapunov function $V(\mathbf{x},t)$ such that*

$$\frac{\partial V}{\partial \mathbf{x}}(\mathbf{x},t)\cdot \mathbf{f}(\mathbf{x},t)+\frac{\partial V}{\partial t}(\mathbf{x},t)\gg 0,$$

*$V(\mathbf{x},t)\to 0$ as $\mathbf{x}\to 0$ and $V(\mathbf{x},t)\to\infty$ as $\mathbf{x}\to\infty$ uniformly with respect to $t$, then $\mathbf{x}(t)\equiv 0$ is globally, asymptotically stable.*

Notice that the above Lyapunov theorems provide only sufficient conditions for stability of the solutions for system (3). In particular, these conditions depend on the existence of a certain (Lyapunov) function that satisfies some restrictive properties. The following theorem provides conditions for the converse statement.

**Theorem 1.2.4.** *(Persidsky) Consider $\dot{\mathbf{x}}=\mathbf{f}(\mathbf{x},t)$, $\mathbf{f}(0,t)=\mathbf{0}$. Assume $\mathbf{x}\equiv 0$ is Lyapunov stable. Then, there exists a function $V(\mathbf{x},t)$ such that $V\gg 0$ and for all solutions $\mathbf{x}(\cdot)$ we have $\frac{dV(\mathbf{x}(t),t)}{dt}\leq 0$ for all $t$.*

Lyapunov functions play an important role in stability analysis. In general, it can be difficult to construct a Lyapunov function for an arbitrary system. Hence, stability criteria given by the existence of Lyapunov functions (Lyapunov and Persidsky) are a mere reformulation of the stability problem. However, constructing Lyapunov functions for a certain class of differential equations is quite simple. One such class of differential equations is the class of linear differential equations with a constant coefficients. First, we will introduce linear differential equations, a special but very important class of ordinary differential equations.

### 1.3. Linear Systems

A linear system of differential equations has a form

$$\dot{\mathbf{x}}=\mathbf{A}(t)\mathbf{x}+\mathbf{b}(t), \tag{11}$$

where $\mathbf{A}(t)$ and $\mathbf{b}(t)$ are piecewise continuous $n \times n$ and $n \times 1$ matrices, respectively.

**Theorem 1.3.1.** *[15](Existence Theorem for Linear Systems)*
*For every $\mathbf{x}^0 \in \mathbb{R}^n$, the initial-value problem for system (11) with initial condition*

$$\mathbf{x}(t_0) = \mathbf{x}^0 \tag{12}$$

*and $t_0 \in [a, b]$ has a solution $\mathbf{x} = \mathbf{x}(t)$ on $a \leq t \leq b$, and this solution is unique.*

Notice that the uniqueness of the solutions for equations (11) and (12) imply the following corollary.

**Corollary 1.3.1.** *If $\mathbf{x}(t)$ is a solution of equation (11) with $\mathbf{b}(t) \equiv \mathbf{0}$ and $\mathbf{x}(t_0) = \mathbf{0}$ for some $t_0 \in [a, b]$, then $\mathbf{x}(t) \equiv \mathbf{0}$.*

**Definition 1.3.1.** *If $\mathbf{b}(t)$ in equation (11) is identically $\mathbf{0}$, that is, if equation (11) has the form*

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}, \tag{13}$$

*then the equation is said to be linear homogenous.*

The main property of linear homogenous systems, from which many other properties can be derived, concerns the algebraic structure of the solution set, stated as follows.

**Definition 1.3.2.** *Let $f_1, \cdots, f_m$ be n-vector functions on $(a, b)$. Then, $f_1, \cdots, f_m$ are linearly dependent on $(a, b)$ if there exist constants $c_1, \cdots, c_m$ (not all zero) such that, for all $t \in (a, b)$,*

$$c_1 f_1(t) + \cdots + c_m f_m(t) = 0. \tag{14}$$

*If $f_1, \cdots, f_m$ are not linearly dependent on $(a, b)$, they are linearly independent on $(a, b)$.*

**Theorem 1.3.2.** *[15] The set of all solutions of equation (13) is an n-dimensional linear space.*

**Remark 1.3.1.** *From the preceding theorem, it follows that we can construct the set of all solutions of (13) as soon as we know n linearly independent solutions.*

A basis of the space for all solutions of system (13) is called a fundamental system of solutions. Assume now that we have $n$ solutions of system (13):

$$x^1(t), x^2(t), \cdots, x^n(t).$$

Consider matrix $\mathbf{X}(t)$, the columns of which are vectors

$$x^1(t), x^2(t), \cdots, x^n(t).$$

An important question concerning system (13) is to decide whether $\mathbf{X}(t)$ is the matrix of a fundamental system of solutions. The complete answer to this question is given by the next theorem.

**Theorem 1.3.3.** *[15] The necessary and sufficient conditions that $x^1, x^2, \cdots, x^n$ form a fundamental system for system (13) are the following: $x^1, x^2, \cdots, x^n$ are solutions of system (13), and*

$$\det \mathbf{X}(t) \neq 0, \qquad \textit{for at least one point } t \in (a, b). \tag{15}$$

Given the homogenous linear system,

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}, \tag{16}$$

where $\mathbf{A}(t)$ is a continuous $n \times n$ matrix for $t \in (0, \infty)$. A solution of system (16) with initial condition $\mathbf{x}(t_0) = \mathbf{x}^0$ has the form

$$\mathbf{x}(t; t_0, \mathbf{x}^0) = \mathbf{X}(t)\mathbf{X}^{-1}(t_0)\mathbf{x}^0,$$

where $\mathbf{X}(t)$ is an arbitrary fundamental matrix of system (16).

**Theorem 1.3.4.** *[15] Let* $\mathbf{X}(t)$ *be a fundamental matrix of system* (16)*. System* (16) *is Lyapunov stable if and only if all solutions are bounded on* $[0, \infty)$*. That is, there exists a positive real number,* $M$*, such that*

$$||\mathbf{X}(t)|| \leq M, \quad t \geq 0. \tag{17}$$

*System* (16) *is asymptotically stable if and only if all solutions tend to zero at infinity, that is,*

$$\lim_{t \to \infty} ||\mathbf{X}(t)|| = 0. \tag{18}$$

For the case of homogenous linear systems with constant coefficients,

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}, \tag{19}$$

we have the following results.

**Theorem 1.3.5.** *[17] System* (19) *is Lyapunov stable if and only if, for all eigenvalues* $\lambda_i$ *of* $\mathbf{A}$ *we have* $\Re e(\lambda_i) \leq 0$*, and all the Jordan blocks of matrix* $\mathbf{A}$ *corresponding to eigenvalues* $\lambda_i$ *such that* $\Re e(\lambda_i) = 0$*, have dimension one.*

**Theorem 1.3.6.** *[17] System* (19) *is asymptotically stable if and only if, all eigenvalues of* $\mathbf{A}$ *lie in the half space* $\mathbb{C}^- = \{z \in \mathbb{C} : \Re e(z) < 0\}$*.*

Notice that both criteria in Theorems 1.3.5 and 1.3.6 concern the problem of finding the signs of the real parts of the roots of the characteristic polynomial of matrix $\mathbf{A}$. There exist several conditions that assure that all roots of an algebraic equation,

$$z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0 = 0,$$

have negative real parts. We shall mention here the well-known Hurwitz-Routh criterion.

Let us consider the characteristic polynomial of matrix $A$,

$$f(z) = \det(z\mathbf{I} - \mathbf{A}) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0. \tag{20}$$

**Definition 1.3.3.** *[12] The polynomial $f(z) = \det(z\mathbf{I} - \mathbf{A}) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ is called Hurwitz if all roots of $f(\cdot)$ have a negative real parts.*

We will denote set of all Hurwitz polynomials of degree $n$, by $H_n$. To formulate the Hurwitz-Routh criterion, we need the following $n \times n$ matrix:

$$D = \begin{pmatrix} a_1 & a_3 & a_5 & \ldots & 0 \\ a_0 & a_2 & a_4 & \ldots & 0 \\ 0 & a_1 & a_3 & \ldots & 0 \\ 0 & a_0 & a_2 & \ldots & 0 \\ \vdots & \vdots & \vdots & \ldots & \vdots \\ 0 & 0 & 0 & \ldots & 1 \end{pmatrix},$$

constructed as follows: Define $a_i = 0$ if $i \leq 0$ or $i > n$, $a_n = 1$. Then, $d_{ij} = a_{2j-i}$, $1 \leq i \leq n$, $1 \leq j \leq n$. We will denote the principal minor of order $j$, by $\triangle_j$, $1 \leq j \leq n$.

**Theorem 1.3.7.** *(Hurwitz and Routh)[12]*
*Polynomial $f(z) = z^n + a_{n-1}z^{n-1} + \cdots + a_1 z + a_0$ is Huriwitz if and only if $\triangle_j > 0$ for all $j$, $1 \leq j \leq n$.*

### 1.4. Method of Lyapunov Equations

As mentioned earlier, constructing Lyapunov functions for systems of the linear constant coefficient (ODE) is quite simple. This section is devoted to the basic Lyapunov stability theory of such systems.

Let us consider, again, a linear system with constant coefficients

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}. \tag{21}$$

Recall that system $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ is asymptotically stable if and only if, all the eigenvalues of matrix $\mathbf{A}$ have a negative real part. (i.e., The characteristic polynomial $\det(\lambda\mathbf{I}-\mathbf{A}) = 0$ is Hurwitz.) The following definitions will be useful in the subsequent analysis.

**Definition 1.4.1.** *Matrix* $\mathbf{A}$ *is Hurwitz if* $\det(\lambda\mathbf{I} - \mathbf{A})$ *is Hurwitz.*

Note for any $(n \times m)$-matrix $\mathbf{A} = \{a_{ij}\}_{i=1,j=1}^{n,m}$, we will denote by $\mathbf{A}^*$ an $(m \times n)$-matrix, with entries $\mathbf{A}^* = (\bar{a}_{ji})$ (where $\bar{a}_{ji}$ denotes the complex conjugate of $a_{ji}$).

**Definition 1.4.2.** *Matrix* $\mathbf{A}$ *is positive definite (denoted by* $\mathbf{A} > 0$*) if, for any* $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} \neq \mathbf{0}$, *we have* $\mathbf{x}^*\mathbf{A}\mathbf{x} > 0$. *Similarly, Matrix* $\mathbf{A}$ *is negative definite (denoted by* $\mathbf{A} < 0$*) if, for any* $\mathbf{x} \in \mathbb{R}^n$, $\mathbf{x} \neq \mathbf{0}$, *we have* $\mathbf{x}^*\mathbf{A}\mathbf{x} < 0$.

**Remark 1.4.1.** *If* $\mathbf{A} > 0$, *then* $\mathbf{A}^* > 0$, *and* $\frac{\mathbf{A}^*+\mathbf{A}}{2} > 0$ *(where* $\frac{\mathbf{A}^*+\mathbf{A}}{2}$ *is called the real part of matrix* $\mathbf{A}$ *and denoted by* $\Re e\{\mathbf{A}\}$*).*

Notice that a system $\dot{x} = ax$ is stable if and only if $2\Re e(a) = a + \bar{a} < 0$. The generalization to system (21) is $\mathbf{A}+\mathbf{A}^* < 0$. Clearly, $\mathbf{A}+\mathbf{A}^* < 0$ is satisfied if and only if $2\Re e(\mathbf{A}\mathbf{x}, \mathbf{x}) < 0$ for all nonzero $\mathbf{x}$. If $\mathbf{A} + \mathbf{A}^* < 0$, then system matrix $\mathbf{A}$ is stable. Indeed, consider any eigenvalue, $\lambda$, of $\mathbf{A}$ and a corresponding unit eigenvector, $\mathbf{v}$, that is $\mathbf{A}\mathbf{v} = \lambda\mathbf{v}$ and $||\mathbf{v}|| = 1$. Then, $0 > 2\Re e(\mathbf{A}\mathbf{v}, \mathbf{v}) = 2\Re e(\lambda)$. Since $\lambda$ is arbitrary, $\mathbf{A}$ is stable (Hurwitz). Recall that matrix $\mathbf{A}$ is similar to another matrix $\mathbf{F}$ if there exists a matrix, say $\mathbf{T}$ ($\det \mathbf{T} \neq 0$), such that $\mathbf{F} = \mathbf{T}\mathbf{A}\mathbf{T}^{-1}$. Because similarity transformations preserve eigenvalues, they also preserve stability. We will establish that $\mathbf{A}$ is Hurwitz, if and only if $\mathbf{A}$ is similar to matrix $\mathbf{F}$ that satisfies

$$\mathbf{F} + \mathbf{F}^* < 0. \tag{22}$$

Assume that $\mathbf{A}$ is similar to matrix $\mathbf{F}$, and suppose there exists a matrix, say $\mathbf{T}$, satisfying

$$\mathbf{T}\mathbf{A}\mathbf{T}^{-1} + (\mathbf{T}^*)^{-1}\mathbf{A}^*\mathbf{T}^* = \mathbf{F} + \mathbf{F}^* < 0.$$

Let $\mathbf{P}$ be the positive definite matrix defined by $\mathbf{H} := \mathbf{T}^*\mathbf{T}$. Note that $\mathbf{H} = \mathbf{H}^*$. Pre-multiplying and post-multiplying the previous inequality by $\mathbf{T}^*$ and $\mathbf{T}$, respectively, yields

$$\mathbf{HA} + \mathbf{A}^*\mathbf{H} < \mathbf{0}. \tag{23}$$

If $\mathbf{A}$ is similar to a matrix that satisfies equation (22), then $\mathbf{A}$ is Hurwitz, and there exists a strictly positive matrix $\mathbf{H}$ satisfying (23). The following lemma shows that the existence of a strictly positive matrix $\mathbf{H}$ satisfying (23) guarantees stability.

**Lemma 1.4.1.** *[17] Consider system* 21. *Suppose that there exists a strictly positive matrix* $\mathbf{H}$ *satisfying (*23*). Then,* $\mathbf{A}$ *is Hurwitz.*

A strictly positive matrix $\mathbf{H}$ which satisfies inequality (23) is referred to as a Lyapunov matrix for system (21).

So far, we have shown that, if (23) holds for some matrix $\mathbf{H}$, then system matrix $\mathbf{A}$ is stable. Is the converse true? That is, if $\mathbf{A}$ is stable, does there exist an operator $\mathbf{H}$ such that (23) holds? Moreover, if this is true, how does one find such a matrix $\mathbf{H}$? First, notice that inequality (23) is equivalent to

$$\mathbf{HA} + \mathbf{A}^*\mathbf{H} + \mathbf{C} = \mathbf{0}, \tag{24}$$

where $\mathbf{C}$ is any strictly positive matrix. This linear matrix equation is known as the Lyapunov equation. One approach to check for the existence of a Lyapunov matrix could be to choose a strictly positive matrix, $\mathbf{C}$, and determine whether the Lyapunov equation has a strictly positive solution for matrix $\mathbf{H}$. The following result shows that, if system matrix $\mathbf{A}$ is Hurwitz, then (24) has a solution, $\mathbf{H}$, for every $\mathbf{C}$.

**Lemma 1.4.2.** *(Lyapunov)[17] Assume that system matrix* $\mathbf{A}$ *is Hurwitz and* $\mathbf{C} = \mathbf{C}^* > 0$. *Then, there is a Hermitian matrix,* $\mathbf{H}$ *(i.e.,* $\mathbf{H} = \mathbf{H}^*$*), such that* $\mathbf{HA}+\mathbf{A}^*\mathbf{H} = -\mathbf{C}$. *Matrix* $\mathbf{H}$ *is positive definite and unique. This is given by*

$$\mathbf{H} = \int_0^\infty e^{\mathbf{A}^*t}\mathbf{C}e^{\mathbf{A}t}dt. \tag{25}$$

Using the previous two lemmas, we can state the main theorem of this section as follows.

**Theorem 1.4.1.** *[17] The following statements are equivalent:*

*(1) Linear system* $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ *is Lyapunov stable.*

*(2) There exist strictly positive, definite matrices* $\mathbf{H}$ *and* $\mathbf{C}$ *satisfying the Lyapunov equation,* $\mathbf{HA} + \mathbf{A}^*\mathbf{H} + \mathbf{C} = \mathbf{0}$.

*(3) For any strictly positive matrix* $\mathbf{C}$, *the Lyapunov equation,* $\mathbf{HA} + \mathbf{A}^*\mathbf{H} + \mathbf{C} = \mathbf{0}$, *has a strictly positive, definite, unique solution* $\mathbf{H}$.

*(4) System matrix* $\mathbf{A}$ *is similar to a matrix* $\mathbf{F}$ *that satisfies the inequality* $\mathbf{F} + \mathbf{F}^* < 0$.

**Remark 1.4.2.** $V(\mathbf{x}) = \mathbf{x}^*\mathbf{H}\mathbf{x}$ *is a positive, definite function. It is a Lyapunov function for the linear system with constant coefficients*

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}. \tag{26}$$

Therefore, any stable linear system, $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$, has a Lyapunov function of the form $V(\mathbf{x}) = \mathbf{x}^*\mathbf{H}\mathbf{x}$. Furthermore, the above analysis shows that each strictly positive, definite matrix $\mathbf{C}$ uniquely determines a quadratic Lyapunov function, $V(\mathbf{x}) = \mathbf{x}^*\mathbf{H}\mathbf{x}$, where $\mathbf{H}$ is the unique solution of the Lyapunov equation, $\mathbf{HA} + \mathbf{A}^*\mathbf{H} + \mathbf{C} = \mathbf{0}$.

# CHAPTER 2. FEEDBACK CONTROL SYSTEMS

## 2.1. Introduction

The general theorems of stability find important applications for applied science in various fields. Among many, one of the most interesting problems is the stability of systems in control theory. In this chapter, we give a brief overview of the main concepts in feedback control theory, we introduce the notion of controllability, observability, transfer function, and minimal realization.

We will begin our discussion by considering an initial value problem for autonomous, ordinary differential equations of the form

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t)), & (t > 0) \\ \mathbf{x}(0) = \mathbf{x}^0. \end{cases} \tag{27}$$

where a point $\mathbf{x}^0 \in \mathbb{R}^n$ and the function $\mathbf{f} : \mathbb{R}^n \to \mathbb{R}^n$ are given. The map $\mathbf{x} : [0, \infty) \to \mathbb{R}^n$ is interpreted as the dynamical evolution of the state of some system that equation (27) models. We can generalize this model by assuming that function $\mathbf{f}$ depends upon some parameter belonging to set $\mathbf{U} \subset \mathbb{R}^n$ so that $\mathbf{f} : \mathbb{R}^n \times \mathbf{U} \to \mathbb{R}^n$. Then, if we select some $\mathbf{u} \in \mathbf{U}$ and consider the corresponding dynamics

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t), \mathbf{u}) & (t > 0), \\ \mathbf{x}(0) = \mathbf{x}^0 \end{cases} \tag{28}$$

we can obtain the evolution of our system when the parameter is set to the value $\mathbf{u}$.

We can further generalize this model by allowing the value of the parameter to vary as the system evolves. That is, define a function $u : [0, \infty) \to \mathbf{U}$. Notice that the system behavior is now dependent on the control function, $u(t)$. We call function $u : [0, \infty) \to \mathbf{U}$ a control function. With control function given, we consider the initial-value problem

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{f}(\mathbf{x}(t), \mathbf{u}(t)) & (t > 0). \\ \mathbf{x}(0) = \mathbf{x}^0. \end{cases} \tag{29}$$

The solution $\mathbf{x}(\cdot)$ of equation (29) depends upon control $\mathbf{u}(\cdot)$ and the initial condition. Hence, we will use the following

$$\mathbf{x}(\cdot) = \mathbf{x}(\cdot, \mathbf{u}(\cdot), \mathbf{x}^0).$$

Hereafter, we assume that our ordinary differential equation is linear in both the state $\mathbf{x}(\cdot)$ and the control $\mathbf{u}(\cdot)$. Consequently, it has the form

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x}(t) + \mathbf{B}\mathbf{u}(t) & (t > 0), \\ \mathbf{x}(0) = \mathbf{x}^0 \end{cases} \tag{30}$$

where $\mathbf{A} \in \mathbb{M}^{n \times n}$ and $\mathbf{B} \in \mathbb{M}^{n \times m}$.

### 2.2. Controllability and Observability

In this section, we address the following fundamental questions in mathematical control theory.

**Controllability**: For any given initial point $\mathbf{x}^0$ and a target set $S \subset \mathbb{R}^n$, does there exist a control steering the system to S in finite time?

**Observability**: Given an $n$-vector $\mathbf{c}$ and a function $\mathbf{y}(\cdot) = \mathbf{c}^*\mathbf{x}(\cdot)$, can we, in principle, reconstruct $\mathbf{x}(\cdot)$? In particular, do observations of $\mathbf{y}(\cdot)$ provide enough information for us to deduce the initial value, $\mathbf{x}^0$, in equation (30)?

The following definition, due to R. E. Kalman, plays a crucial role in many control theory problems.

**Definition 2.2.1.** *[13] The pair* $(\mathbf{A}, \mathbf{B})$ *is called controllable if, for any pair of points* $\mathbf{x}_0, \mathbf{x}_1 \in \mathbb{R}^n$ *and any positive number* $T$*, there exists a control function* $u : [0, T] \to \mathbb{R}^n$

*such that the solution of system* $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{B}u$ *with initial data* $\mathbf{x}(0) = \mathbf{x}_0$ *has the property* $\mathbf{x}(T) = \mathbf{x}_1$.

In other words, pair $(\mathbf{A}, \mathbf{B})$ is controllable if, for any initial and terminal points, $\mathbf{x}_0$ and $\mathbf{x}_1$ and any time period, $T$, there exists a control which brings the system from $\mathbf{x}_0$ to $\mathbf{x}_1$ during time period $T$. The following theorem provides several necessary and sufficient conditions of controllability for pair $(\mathbf{A}, \mathbf{B})$.

**Theorem 2.2.1.** *[20] The following statements are equivalent.*

1. *Pair* $(\mathbf{A}, \mathbf{B})$ *is controllable.*
2. *For any* $t > 0$, $\int_0^t e^{\mathbf{A}s}\mathbf{B}\mathbf{B}^* e^{\mathbf{A}^* s}\, ds > 0$.
3. $[\mathbf{A} - \lambda I | \mathbf{B}]$ *has full row rank for all* $\lambda \in \mathbb{C}$.
4. *For any eigenvector,* $z$, *of* $\mathbf{A}^*$, *we have* $\mathbf{B}^* z \neq 0$.
5. *The* $n \times (nm)$ *matrix* $[\mathbf{B}, \mathbf{A}\mathbf{B}, \mathbf{A}^2\mathbf{B}, \cdots, \mathbf{A}^{n-1}\mathbf{B}]$ *has full row rank.*
6. *For any set of complex numbers* $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$, *such that,* $\bar{\Lambda} = \Lambda$, *there exists a matrix* $\mathbf{F}$ *such that,* $\Lambda$ *is the set of eigenvalues of* $\mathbf{A} + \mathbf{B}\mathbf{F}$.

Now we will discuss the observability problem. Let us consider the initial-value problem

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x}(t) & (t > 0) \\ \mathbf{x}(0) = \mathbf{x}^0. \end{cases} \tag{31}$$

We suppose that we can observe or measure the function

$$\mathbf{y}(t) := \mathbf{C}\mathbf{x}(t) \qquad (t \geq 0), \tag{32}$$

where matrix $\mathbf{C} \in \mathbb{M}^{m \times n}$ is given; function $\mathbf{y}$ is usually called measurement.

**Definition 2.2.2.** *Pair* $(\mathbf{A}, \mathbf{C})$ *is called observable if, for any solution of system* $\dot{\mathbf{x}} = \mathbf{A}\mathbf{x}$ *such that* $\mathbf{C}\mathbf{x} \equiv 0$, *we have* $\mathbf{x} \equiv 0$.

The following theorem provides necessary and sufficient conditions of observability for pair $(\mathbf{A}, \mathbf{C})$.

**Theorem 2.2.2.** *[20] The following statements are equivalent.*

1. *Pair* $(\mathbf{A}, \mathbf{C})$ *is observable.*

2. *For any* $t > 0$, $\int_0^t e^{\mathbf{A}^* s}\mathbf{C}^* e^{\mathbf{A}s}\, ds > 0$.

3. *The matrix* $\begin{bmatrix} \mathbf{A} - \lambda I \\ \mathbf{C} \end{bmatrix}$ *has full column rank for all* $\lambda \in \mathbb{C}$.

4. *For any eigenvector,* $z$*, of matrix* $\mathbf{A}^*$, *we have* $\mathbf{C}^* z \neq 0$.

5. *The* $n \times (nm)$ $[\mathbf{C}^*, \mathbf{A}^*\mathbf{C}, \mathbf{A}^{*(2)}\mathbf{C}, \cdots, \mathbf{A}^{*(n-1)}\mathbf{C}^*]$ *has full row rank.*

6. *For any set of complex numbers* $\Lambda = \{\lambda_1, \lambda_2, \cdots, \lambda_n\}$, *such that,* $\bar{\Lambda} = \Lambda$, *there exist a matrix* $\mathbf{F}$ *such that,* $\Lambda$ *is the set of eigenvalues of* $\mathbf{A} + \mathbf{CF}$.

7. *Pair* $(\mathbf{A}^*, \mathbf{C}^*)$ *is controllable.*

Notice that the equivalence of (1) and (7) in Theorem 2.2.2 shows that observability and controllability are dual concepts for linear systems.

### 2.3. Realization Theory: Frobenius Forms

Let us consider system (feedback system) where the dynamics can be described by the following equations:

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x}(t) + \mathbf{b}u(t) & \qquad (t > 0), \\ \mathbf{y} = \mathbf{c}^*\mathbf{x} \end{cases} \tag{33}$$

where $\mathbf{A} \in \mathbb{M}^{n\times n}$, $\mathbf{b}$, $\mathbf{c} \in \mathbb{M}^{n\times 1}$.

**Theorem 2.3.1.** *[18] Let us assume that pair* $(\mathbf{A}, \mathbf{b})$ *is controllable; then, there exists a nonsingular, constant matrix* $\mathbf{K}$ *such that, for change of variable* $\mathbf{z} = \mathbf{K}\mathbf{x} = (z_1, \cdots, z_n)^T$, *we have*

$$\begin{aligned} \dot{z}_1 &= z_2, \\ \dot{z}_2 &= z_3, \\ \vdots & \quad \vdots \\ \dot{z}_n &= -\delta_0 z_1 - \delta_1 z_2 - \cdots - \delta_{n-1} z_n + u, \\ \mathbf{y} &= \mathbf{c}^*\mathbf{K}^{-1}\mathbf{z}, \end{aligned} \tag{34}$$

*where* $\delta_0, \delta_1, \cdots, \delta_{n-1}$ *are coefficients of the characteristic polynomial of matrix* $\mathbf{A}$ :

$$\det(\lambda I - \mathrm{A}) = \lambda^n + \delta_{n-1}\lambda^{n-1} + \cdots + \delta_0.$$

This form is called the first Frobenius form of system (33). On the other hand, if we assume that pair $(\mathbf{A}, \mathbf{c})$ is observable them we have the following result:

**Theorem 2.3.2.** *[18] Let us assume that pair* $(\mathbf{A}, \mathbf{c})$ *is observable, then there exists a nonsingular matrix* $\mathbf{K} \in \mathbb{M}^{n\times n}$ *such that, for the change of variable* $\mathbf{z} = \mathbf{K}\mathbf{x} = (z_1, \cdots, z_n)^T$, *we have*

$$\begin{aligned}
\dot{z}_1 &= -\delta_0 z_n + \beta_0 u, \\
\dot{z}_2 &= z_1 - \delta_1 z_n + \beta_1 u, \\
\dot{z}_3 &= z_2 - \delta_2 z_n + \beta_2 u, \\
\vdots \quad & \quad \vdots \\
\dot{z}_n &= z_{n-1} - \delta_{n-1} z_n + \beta_{n-1} u, \\
\mathbf{y} &= z_1.
\end{aligned} \tag{35}$$

This form is called the second Frobenius form of system (33).

The following function plays an important role in modern mathematical control theory.

**Definition 2.3.1.** *Function* $W(s) = \mathbf{c}^*(\mathbf{A} - sI)^{-1}\mathbf{b}$ *is called the transfer function of system (*33*).*

Note that $W(s)$ is a rational function. If matrix $\mathbf{A}$ is Hurwitz and if $u(t) = e^{i\omega t}$, then $|y(t) - W(i\omega)e^{i\omega t}| \to 0$ as $t \to \infty$.

Now, assume function $W(s)$ is a proper, rational function. Let us pose the following questions. How can we construct system (33) having $W(s)$ as a transfer function? How can we find system (33) of minimal dimension having this property?

Denote the coefficients of the rational function $W$ as follows:

$$W(s) = \frac{\beta_0 + \beta_1 s + \cdots + \beta_{n-1}s^{n-1}}{\delta_0 + \delta_1 s + \cdots + \delta_{n-1}s^{n-1} + s^n} \,. \tag{36}$$

**Theorem 2.3.3.** *[18] System (33) with Frobenius form*

$$\begin{aligned}
\dot{x}_1 &= x_2 + \beta_0 u, \\
\dot{x}_2 &= x_3 + \beta_1 u, \\
\vdots & \quad \vdots \\
\dot{x}_{n-1} &= x_n + \beta_{n-2} u, \\
\dot{x}_n &= -\delta_0 x_1 - \delta_1 x_2 - \cdots - \delta_{n-1} x_n + \beta_{n-1} u, \\
\mathbf{y} &= \beta_0 x_1 + \cdots + \beta_{n-1} x_n
\end{aligned} \tag{37}$$

*has transfer function* $W(s) = \frac{\beta_0 + \beta_1 s + \cdots + \beta_{n-1}s^{n-1}}{\delta_0 + \delta_1 s + \cdots + \delta_{n-1}s^{n-1} + s^n}$.

Notice that pair $(\mathbf{A}, \mathbf{b})$ in system (37) is controllable. Hence, we have the controllable realization of transfer function (36).

Similarly, the following theorem provides the observable realization of transfer function (36).

**Theorem 2.3.4.** *[18] System (33) with Frobenious form*

$$\begin{aligned}
\dot{x}_1 &= -\delta_0 x_n - \beta_0 u, \\
\dot{x}_2 &= x_1 - \delta_1 x_n - \beta_1 u, \\
\vdots & \quad \vdots \\
\dot{x}_n &= x_{n-1} - \delta_{n-1} x_n - \beta_{n-1} u, \\
\mathbf{y} &= x_1,
\end{aligned} \tag{38}$$

*has transfer function* $W(s) = \frac{\beta_0 + \beta_1 s + \cdots + \beta_{n-1}s^{n-1}}{\delta_0 + \delta_1 s + \cdots + \delta_{n-1}s^{n-1} + s^n}$.

The following statement is the complete answer to the question that was posed above.

**Theorem 2.3.5.** *[18] Systems (37) and (38) present the minimal realization of transfer function $W$ if and only if the numerator and denominator of transfer function $W$ have no common roots.*

Henceforth, without any loss of generality, we can consider only minimal realizations of transfer functions.

# CHAPTER 3. STABILITY OF FEEDBACK CONTROL SYSTEMS

## 3.1. Introduction

In what follows, we shall deal with the absolute stability of feedback control systems. The basic concepts of Lyapunov exponents and extremal norms will be introduced and explored to derive necessary and sufficient conditions for the asymptotic stability of differential inclusions and dual inclusions. We will derive an important, particular solution, called the "worst-case solutions." We will use this concept to establish the largest cone on the complex plane such that, if all eigenvalues of corresponding linear systems belong to this region, then the system is absolutely stable.

## 3.2. Absolute Stability of Feedback Control Systems

We shall consider a problem, formulated first by Lur'e and Postnikov, related to the stability of some systems occurring in the theory of feedback control. Let us consider a system (feedback control system with one nonlinearity, also known as Lur'e system) where the dynamics can be described by the following equations:

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}\mathbf{x}(t) + \mathbf{b}\varphi(\sigma, t) \\ \sigma(t) = \mathbf{c}^*\mathbf{x}, \end{cases} \tag{39}$$

where $\mathbf{A}$, $\mathbf{b}$, and $\mathbf{c}$ are constant matrices of dimensions $n \times n$, $n \times 1$, and $n \times 1$, respectively. We assume that matrix $\mathbf{A}$ is Hurwitz and that pairs $(\mathbf{A}, \mathbf{b})$ and $(\mathbf{A}, \mathbf{c})$ are controllable and observable, respectively. Function $\varphi(\sigma, t)$ is a real-valued function defined on the whole plane, and satisfies condition $\varphi(0, t) \equiv 0$, in order for system (39) to have a zero solution. In addition, we also assume that function $\varphi$ is measurable in $t$ and that there exists a number $\mu$ such that $||\varphi(t, \sigma)|| \leq \mu||\sigma||$. This condition guarantees the existence of a solution on $(-\infty, \infty)$ for any initial-value problem that system (39) may have.

Function $\varphi$ is generally nonlinear, and sometimes it is only known approximately. Because, it is not easy to explore the full information about function $\varphi$, rather we will use some reasonable property in order to prove stability. One of the most important properties of $\varphi$ is presented below. If we use only this property, then we will deal with the problem of stability for system (39) with all functions $\varphi$ satisfying this property.

Fix numbers $\mu_1$ and $\mu_2$. For the remainder of this thesis, it will be assumed that $\varphi(\sigma, t)$ is a measurable function with respect to $t$ and continuous with respect to $\sigma$ such that

$$\mu_1 \leq \frac{\varphi(\sigma, t)}{\sigma} \leq \mu_2 \qquad \text{for all } \sigma \neq 0 \text{ and for all } t. \tag{40}$$

The reason for interest in such functions is due to the fact that such information about nonlinear functions, $\varphi$, is readily available in many practical cases (nonlinearities with dead zone, with saturation, with hysteresis, etc.) which give fruitful stability and performance analysis of system (39).

We can simplify condition (40) by making the following change of variables:

$$\varphi_1(\sigma, t) = \varphi(\sigma, t) - \sigma\mu_1$$

and

$$A_1 = A - b\mu_1 c^*.$$

Then, the lower bound of quotient (40) is equal to zero. Hence, without loss of generality, we will assume that $\mu_1 = 0$. Also, we denote the new upper bound by $\mu = \mu_2 - \mu_1$. We use $\mathcal{M}_\mu$ to denote the set of (nonlinear) functions, $\varphi$, that satisfy the above sector condition, i.e.,

$$\mathcal{M}_\mu = \{\varphi : 0 \leq \frac{\varphi(\sigma, t)}{\sigma} \leq \mu \ \forall \sigma \neq 0, t \geq 0\},$$

where $\mu$ is a given positive number.

The property of interest which we studied concerns a special type of stability, an absolute stability.

**Definition 3.2.1.** *We say that system (39) is absolutely stable in class $\mathcal{M}_\mu$ if, for any $\varphi \in \mathcal{M}_\mu$, the zero solution is globally, asymptotically stable and this stability is uniform with respect to $\varphi \in \mathcal{M}_\mu$.*

The subject of this thesis concerns the necessary and sufficient conditions on set $\{\mathbf{A}, \mathbf{b}, \mathbf{c}, \mu\}$ for absolute stability of system (39) in class $\mathcal{M}_\mu$. In particular, we seek criteria that are coordinate independent and can be tested analytically. Before formulating the problem and showing new results based on new approaches, we first look at early results and conjectures. Let us suppose function $\varphi$ is such that $\varphi(\sigma, t) \equiv \nu\sigma$ for all $\nu$. Then, we have the following useful result.

**Theorem 3.2.1.** *[18] System $\dot{\mathbf{x}} = (\mathbf{A} + \mathbf{b}\nu\mathbf{c}^*)\mathbf{x}$ for all $\nu \in [0, \mu]$ is asymptotically stable if and only if matrices $\mathbf{A} + \mathbf{b}\nu\mathbf{c}^*$ are Hurwitz for all $\nu \in [0, \mu]$.*

**Theorem 3.2.2.** *(Mikhailov,Nyquist)[18] Suppose $f(\lambda)$ is a polynomial of degree $n$. If $f(i\omega) \neq 0$ for all $\omega \in \mathbb{R}$, then $f$ is Hurwitz if and only if*

$$\triangle_{-\infty}^{\infty} \arg f(i\omega) = +n\pi.$$

Hence, if we combine Theorems 3.2.1 and 3.2.2, the result is the well known Nyquist criterion.

**Proposition 3.2.1.** *(Nyquist criterion) Matrices $\mathbf{A} + \mathbf{b}\nu\mathbf{c}^*$ are Hurwitz for all $\nu \in [0, \mu]$ if and only if matrix $\mathbf{A}$ is Hurwitz and if for all $\nu \in [0, \mu]$, we have $1 + \nu W(i\omega) \neq 0$ for all $\omega \geq 0$, and*

$$\triangle_{-\infty}^{\infty} \arg(1 + \nu W(i\omega)) = 0,$$

*where $W(s) = \mathbf{c}^*(\mathbf{A} - s\mathbf{I})^{-1}\mathbf{b}$ is the transfer function of the system.*

Now, let us instead assume function $\varphi(\sigma)$ is time-invariant and satisfies sector inequality $0 \leq \frac{\varphi(s)}{s} \leq \mu$. In 1949 Aizerman conjectured that, if system (39) with

$\varphi(s) = \nu s$ is asymptotically stable for all $\nu \in [0, \mu]$, then system (39) with any function $\varphi$ satisfying sector condition $0 \leq \frac{\varphi(s)}{s} \leq \mu$, $\forall s \neq 0$ is globally, asymptotically stable. With a slight abuse of terminology, this conjecture may be formulated as follows: a sector of linear stability implies a sector of nonlinear stability. In 1956, Pliss came up with a counter example [1] to Aizerman's conjecture. In 1957, Kalman conjectured that system

$$\dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}\varphi(\mathbf{c}^*\mathbf{x})$$

is globally, asymptotically stable if $\varphi(0) = 0$ and if $\mathbf{A} + \mathbf{b}\varphi'(\sigma)\mathbf{c}^*$ is Hurwitz for all $\sigma$. Also, a counter example was constructed [7]. If we further generalize and consider systems with time-varying nonlinearities, $\varphi(\sigma, t) \in \mathcal{M}_\mu$. Then, there are criteria that have been derived for the absolute stability of such systems. The most well-known criterion is the circle criterion which is based on the existence of a Lyapunov quadratic form.

Let us consider a positive, definite quadratic form, $V(\mathbf{x}) = \mathbf{x}^*\mathbf{H}\mathbf{x}$. By the Lyapunov Theorem 1.4.2, $\frac{dV(\mathbf{x}(t))}{dt} < 0$ is sufficient for stability. Let us denote $\xi(t) := \varphi(\sigma, t)$. Therefore,

$$\frac{dV(\mathbf{x}(t))}{dt} = \frac{\partial V(\mathbf{x}(t))}{\partial t} \cdot \frac{d\mathbf{x}}{dt} = 2\mathbf{x}^*\mathbf{H}(\mathbf{A}\mathbf{x} + \mathbf{b}\xi).$$

Note that the condition $0 \leq \frac{\varphi(\sigma,t)}{\sigma} \leq \mu$ is equivalent to the inequality $\xi(\mu c^*\mathbf{x} - \xi) \geq 0$. Denote $F(\mathbf{x}, \xi) := \xi(\mu c^*\mathbf{x} - \xi)$. Then, we have the following questions.

**Question 1**: Does there exist a Hermitian positive definite matrix, $\mathbf{H} = \mathbf{H}^*$, such that

$$2\mathbf{x}^*\mathbf{H}(\mathbf{A}\mathbf{x} + \mathbf{b}\xi) < 0 \tag{41}$$

for all $(\mathbf{x}, \xi) \in \mathbb{R}^{n+1}$ such that $\mathbf{x} \neq 0$ and $F(\mathbf{x}, \xi) \geq 0$?

We can pose another question.

**Question 2**: Does there exist a Hermitian matrix, $\mathbf{H} = \mathbf{H}^*$, and a positive number,

$\tau$, such that

$$2\mathbf{x}^*\mathbf{H}(\mathbf{A}\mathbf{x} + \mathbf{b}\xi) + \tau F(\mathbf{x}, \xi) < 0 \tag{42}$$

for all $(\mathbf{x}, \xi) \in \mathbb{R}^{n+1}$? It is obvious that (41) implies (42). The replacement of condition (41) by (42) is called $S$-procedure. Dines theorem [10] states that (41) is equivalent to (42). The necessary and sufficient condition for the existence of such a matrix $\mathbf{H}$ is given by the following famous result.

**Lemma 3.2.1.** *(Kalman and Yakubovich)[14] Assume pair* $(\mathbf{A}, \mathbf{B})$ *is controllable. For the existence of a Hermitian matrix* $\mathbf{H}$ *such that* $2\mathbf{x}^*\mathbf{H}(\mathbf{A}\mathbf{x} + \mathbf{b}\xi) + F(\mathbf{x}, \xi) \leq 0$ *for all* $\mathbf{x}, \xi$*, it is necessary and sufficient that* $\Re e\{F(i\omega\mathbf{I} - \mathbf{A})^{-1}\mathbf{b}\xi, \xi)\} \leq 0$ *(Frequency Domain Inequality) for all* $\xi \in \mathbb{C}$*, and* $\omega \geq 0$ *such that* $\det(\mathbf{A} - i\omega I) \neq 0$.

Then, Frequency Domain Inequality for systems in the class of sector time-varying nonlinearities gives the following inequality known as the circle criterion:

$$\Re e\{W(i\omega) + \frac{1}{\mu}\} \geq 0 \qquad \text{for all } \omega \geq 0. \tag{43}$$

The circle criterion, which is equivalent to the existence of a quadratic Lyapunov function in the majority of cases, only provides an essentially sufficient condition for absolute stability. Hence, we need a different approach to get necessary and sufficient conditions for the absolute stability of system (39) in class $\mathcal{M}_\mu$. These conditions could be formulated in terms of system parameters $\{\mathbf{A}, \mathbf{b}, \mathbf{c}, \mu\}$ or transfer function $W(s)$. In the next section, we will consider a new object which was proven to be useful when studying the absolute stability problem. This object is called linear differential inclusion.

### 3.3. Linear Differential Inclusions: Dual Inclusions and Asymptotic Stability of Inclusions

Assume $\mathcal{A}$ is a set of $n \times n$ real matrices, such that for a constant $c$, we have

$||\mathbf{A}|| \le c$ for all $\mathbf{A} \in \mathcal{A}$. Let us consider the following differential inclusion:

$$\frac{dx}{dt} \in \{\mathbf{A}x \;:\; \mathbf{A} \in \mathcal{A}\}. \tag{44}$$

Solutions of this inclusion are absolutely continuous functions $x(\cdot)$ such that (44) holds for almost all $t$. The following known result shows the relationship between the absolute stability problem and asymptotic stability of inclusion (44).

**Theorem 3.3.1.** *[8] System (39) is absolutely stable in class $\mathcal{M}_\mu$ if and only if inclusion (44), where $\mathcal{A} = \{\mathbf{A} + \mathbf{b}\nu\mathbf{c}^* : 0 \le \nu \le \mu\}$, is asymptotically stable.*

Henceforth, we shall study asymptotic stability of inclusion (44).

One of the pioneering papers about the theory of stability for inclusion (44), was published by Molchanov and Pyatnisky [4]. In this paper, they showed that linear inclusion (44) is asymptotically stable if and only if there exists a Lyapunov function of the form $V(\mathbf{x}) = max\{|\mathbf{P}_i^*\mathbf{x}|,\ i = 1, \cdots, m\}$, where $m$ is an integer number; $\mathbf{P}_1, \cdots, \mathbf{P}_m$ are some constant vectors; and $V(\mathbf{x}(t))$ is decreasing along all nonzero solutions $\mathbf{x}(\cdot)$ of inclusion (44). Unfortunately, there are no available methods to find the number, $m$, and vectors, $\mathbf{P}_1, \cdots, \mathbf{P}_m$. Therefore, this result cannot be considered a solution of the problem for stability of inclusion (44). There were several other papers (one, in particular, an early paper of Mejlakhs [5]) which studied necessary and sufficient conditions for stability of inclusion (44). None led to numerical procedures.

The problem remained open until 1988 when a new approach to stability of inclusion (44) was developed (papers of [7] and [8]), where the concepts of the Lyapunov exponent and the extremal norm for inclusions first appeared.

Let us consider a number

$$\rho(\mathcal{A}) := \sup \overline{\lim}_{t\to\infty} \frac{\ln ||\mathbf{x}(t)||}{t},$$

where the supremum is taken over all solutions of inclusion (44) with $\mathbf{x}(0) \ne 0$.

**Definition 3.3.1.** *The number $\rho(\mathcal{A})$ is called the Lyapunov exponent of inclusion (44).*

It may be shown that inclusion (44) is asymptotically stable if and only if $\rho(\mathcal{A}) < 0$ [8]. Hence, the problem of asymptotic stability for inclusion (44) is reduced to the problem of finding the sign of the Lyapunov exponent, $\rho(\mathcal{A})$. There exist necessary and sufficient conditions for stability of differential inclusions (as illustrated by the theorems below), but these conditions are numerically too complicated.

First, we need to introduce the notion of irreducibility for set $\mathcal{A}$ of $n \times n$ matrices.

**Definition 3.3.2.** [8] *Set $\mathcal{A}$ is irreducible if there does not exist a proper subspace of $\mathbb{R}^n$ invariant with respect to all matrices $\mathbf{A} \in \mathcal{A}$.*

Set $\mathcal{A}$ is called reducible if it is not irreducible.

Set $\mathcal{A}$ is reducible if and only if there exists a number, $k \in \{1, \cdots, n-1\}$, and a basis in $\mathbb{R}^n$ such that each matrix, $\mathbf{A} \in \mathcal{A}$, in this basis is presented in the form

$$\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{A}_{22} \end{pmatrix},$$

where $\mathbf{A}_{11}$ is a $k \times k$-matrix.

The stability problem for inclusion (44) with reducible $\mathcal{A}$ is equivalent to the stability of inclusions with lower order. In particular, if $\mathbf{A} = \begin{pmatrix} \mathbf{A}_{11} & \mathbf{A}_{12} \\ \mathbf{0} & \mathbf{A}_{22} \end{pmatrix}$ for each $A \in \mathcal{A}$, then the asymptotic stability of (44) is equivalent to asymptotic stability of inclusions

$$\dot{\mathbf{x}}_1 \in \{\mathbf{A}_{11}\mathbf{x}_1 : \ \mathbf{A} \in \mathcal{A}\},$$

$$\dot{\mathbf{x}}_2 \in \{\mathbf{A}_{22}\mathbf{x}_2 : \ \mathbf{A} \in \mathcal{A}\} \text{ [8].}$$

Therefore, we will consider only irreducible inclusion (44).

Similar to continuous time inclusions, we can analyze inclusions in discrete time

$$\mathbf{x}_{k+1} \in \{\mathbf{A}\mathbf{x}_k : \ \mathbf{A} \in \mathcal{A}\}, k = 0, 1, 2, \cdots . \tag{45}$$

The Lyapunov exponent for inclusion (45) is the number

$$\rho(\mathcal{A}) := \sup \overline{\lim}_{k\to\infty} \frac{\ln ||\mathbf{x}_k||}{k},$$

where the supremum is taken over all nonzero solutions of inclusion (45). The following algorithm allows us to compute, in finitely many steps, the sign of the Lyapunov exponent, $\rho$, of inclusion (45) for the case $\rho \neq 0$. We will denote the convex hull of a set $Z$ by $conv(Z_k)$. Also, we will denote the set of extremal points of a convex set $Z$ by $ex(Z)$.

**Theorem 3.3.2.** [8] *Assume $Z_0$ is a set of points in $\mathbb{R}^N$ such that $Z_0 = -Z_0$ and that zero is an interior point of $conv(Z_0)$. For any $k = 0, 1, 2, \cdots$, denote $Z'_k = \{\mathbf{A}z : \mathbf{A} \in \mathcal{A}, z \in Z_k\}$ and $Z_{k+1} = ex(conv(Z_k \cup Z'_k)$. The following statements hold:*
*(1) If $\rho(\mathcal{A}) < 0$, then there exists $k$ such that $Z_k = Z_{k+1}$.*
*(2) If $\rho(\mathcal{A}) > 0$, then there exists $k$ such that $Z_k \cap Z_{k+1} = \phi$.*

If $\rho(\mathcal{A}) \neq 0$, then the procedure gives the *sign* of $\rho(\mathcal{A})$ in a finite number of steps. The algorithm could be considered as a solution of the absolute stability problem if the growth rate for the number of polygon $conv(Z_k)$ edges would not be so big as it occurs in many cases even for 4-dimensional inclusion.

Other important results are based on the following concept. Consider function [8]

$$v(\mathbf{y}) = \sup \left(\overline{\lim}_{t\to\infty} e^{-\rho(\mathcal{A})t} ||\mathbf{x}(t)||\right),$$

where the supremum is taken over the set of solutions $\mathbf{x}(t)$ of inclusion (44) with initial data $\mathbf{x}(0) = \mathbf{y}$.

**Definition 3.3.3.** *Function $v$ is called extremal norm of inclusion (44).*

The following statement is a basis for all important results in the theory of stability for linear inclusions.

**Theorem 3.3.3.** [7] *The following statements are true*

*(1) Function $v$ is a norm in $\mathbb{R}^n$.*

*(2) For any solution $\mathbf{x}$ of inclusion (44) and any $t \geq 0$, we have $v(\mathbf{x}(t)) \leq e^{\rho(\mathcal{A})t} v(\mathbf{x}(0))$.*

*(3) For any vector $\mathbf{y} \in \mathbb{R}^n$, there exists a solution $\mathbf{x}(\cdot)$ of inclusion (44) such that $\mathbf{x}(0) = \mathbf{y}$ and $v(\mathbf{x}(t)) = e^{\rho(\mathcal{A})t} v(\mathbf{x}(0))$ for all $t \geq 0$.*

In the case $\rho(\mathcal{A}) < 0$, the function $\nu$ is the Lyapunov function of inclusions (44).

The following proposition is important.

**Proposition 3.3.1.** [7] *Assume $\mathcal{A}$ is irreducible. The following statements are equivalent.*

*(1) Inclusion (44) is asymptotically stable.*

*(2) Inclusion (44) is exponentially stable.*

*(3) Lyapunov index $\rho(\mathcal{A})$ is negative.*

*(4) There exists a positive, definite form of an even degree which is a Lyapunov function of inclusion (44).*

Item (4) in Proposition 3.3.1 above shows that, to prove exponential stability of inclusion (44), it is necessary and sufficient to prove the existence of a Lyapunov function in the class of forms of an even degree. There are no efficient procedures to check the existence of such Lyapunov functions for all forms with a degree bigger than 2.

Another approach to check stability of inclusion (44) concerns methods to find so-called worst-case solutions. In this thesis, a solution is called worst-case solution if all solutions of an inclusion tend to zero provided that this single solution (worst-case solution) tends to zero. The properties of worst-case solutions have been the subject of many papers since the 1970s. One of the first results in this area was obtained by Pyatnitsky [9]. Pyatnitsky used the Pontrygin maximum principle to find these solutions in [9], where the so-called dual system was applied to stability theory for the first time. Next, we will introduce the concept of dual inclusions and subsequent statements that relate certain properties of dual inclusions with the asymptotic stability of inclusion (44) and, therefore, the absolute stability of system (39) in class $\mathcal{M}_\mu$.

Consider inclusion (44) with the set of matrices $\mathcal{A}^* = \{A^* : A \in \mathcal{A}\}$. This inclusion is called the dual to inclusion (44). Denote by $\rho(\mathcal{A}^*)$ its Lyapunov exponent. The Lyapunov exponents of the original and the dual inclusion coincide:

**Theorem 3.3.4.** *[8] $\rho(\mathcal{A}) = \rho(\mathcal{A}^*)$.*

It turns out that the worst-case solutions of inclusion (44) satisfy certain conditions which may be expressed in terms of dual inclusions and which are closely connected to the Pontrygin maximum principle. We will denote the subdifferential of the convex function, $u$, at a point $\mathbf{x}$ as $\partial v(\mathbf{x})$ and denote the inner product of two vectors $\mathbf{x}, \mathbf{y} \in \mathbb{R}^n$ as $(\mathbf{x}, \mathbf{y})$. Assume $\det \mathbf{A} \neq 0 \; \forall \mathbf{A} \in \mathcal{A}$ (otherwise, $\rho(\mathcal{A}) \geq 0$). Denote the unit ball with norm $v$ as $M = \{\mathbf{x} \in R^n : \; v(\mathbf{x}) \leq 1\}$. Set $M$ is a convex, closed, bounded, central symmetric set, such that the origin is its interior point. Denote the polar of $M$ as $M^o = \{\mathbf{l} \in R^n : \; (\mathbf{l}, \mathbf{x}) \leq 1 \; \forall \mathbf{x} \in M\}$. Set $M^o$ has the same properties as $M$, and $v(\mathbf{x}) = \sup\{(\mathbf{l}, \mathbf{x}) : \; \mathbf{l} \in M^o\}$. Denote the boundary of the set $M^o$ as $Q^o = \{\mathbf{l} \in M^o : \; \exists \mathbf{x} \in M,$ such that $(\mathbf{l}, \mathbf{x}) = 1\}$. Let $Q$ be the boundary of $M$. We say that $\mathbf{l} \in M^o$ corresponds to $\mathbf{x} \in M$ if $(\mathbf{l}, \mathbf{x}) = 1$. The following result is an analog of the Pontryagin maximum principle for continuous, time-linear inclusion (44).

**Theorem 3.3.5.** *[8] For all $\mathbf{y} \in Q$ there exists a solution $\mathbf{x}(\cdot)$ of inclusion (44) with initial data $\mathbf{x}(0) = \mathbf{y}$ and such that $v(\mathbf{x}(t)) \equiv 1$ (or $\mathbf{x}(t) \in Q \; \forall t > 0$), a function $\mathbf{l}(t)$ such that $\mathbf{l}(t) = \partial v(\mathbf{x}(t))$ for all $t > 0$ and a matrix function such that $\mathbf{A}(t) \in \mathcal{A}$, $\dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}$, $\dot{\mathbf{l}}(t) = -\mathbf{A}^*(t)\mathbf{l}(t)$, and for all $t \geq$*

*(1)* $(\mathbf{l}(t), \mathbf{x}(t)) = (\mathbf{l}(0), \mathbf{x}(0)) = 1$, $e^{pt}\mathbf{l}(t) \in Q^o$, $e^{-pt}\mathbf{x}(t) \in Q$.

*(2)* $max\{(\mathbf{l}(t), \mathbf{A}\mathbf{x}(t)) : \; \mathbf{A} \in \mathcal{A}_\rho\} = (\mathbf{l}(t), \mathbf{A}(t)\mathbf{x}(t)) = 0$.

Note that, because the function $v$ is a convex function, it is differentiable almost everywhere. Moreover, if $v$ is differentiable at $\mathbf{x}(0)$, $\mathbf{l}(0) = v'(\mathbf{x}(0))$, and if matrix $\mathbf{A}$ is as defined above, then function $v$ is differentiable at $\mathbf{x}(t)$ and $\mathbf{l}(t) = \nu'(\mathbf{x}(t))$ for all $t > 0$. Hence, Theorem 3.3.5 shows that if, for at least one $\mathbf{x}_0$, we are able to find vector $\mathbf{l}_0 \in \partial v(\mathbf{x}_0)$ such that $\mathbf{l}(0) = \mathbf{l}_0$, where $\mathbf{l}(\cdot)$ is a solution pointed out in Theorem 3.3.5, then we are able to construct the solution $\mathbf{x}(\cdot)$ of inclusion (44) such that $\mathbf{x}(0) = \mathbf{y}$ and $v(\mathbf{x}(t)) = e^{\rho(\mathcal{A})t} v(\mathbf{x}(0))$ for all $t \geq 0$. obviously this is the worst-case solution.

An important application of this result for the problem of absolute stability for feedback systems with one time-varying nonlinearity is given below.

Define the set of matrices

$$\mathcal{A}_a = \{\mathbf{A} + \mathbf{b}\nu\mathbf{c}^* : 0 \leq \nu \leq \mu\}, \tag{46}$$

where $\mathbf{A}$ is an $n \times n$-matrix, $\mathbf{b}$ and $\mathbf{c}$ are $n$-vectors, pair $(\mathbf{A}, \mathbf{b})$ is controllable, pair $(\mathbf{A}, \mathbf{c})$ is observable, and $\mu$ is a positive number. Note that, for the case of inclusions arising in the theory of absolute stability, the property of irreducibility is equivalent to the controllability of pair $(\mathbf{A}, \mathbf{b})$ and the observability of pair $(\mathbf{A}, \mathbf{c})$. For the particular set of matrices $\mathcal{A}_a$, we have the following result.

**Theorem 3.3.6.** *The following statements are equivalent*

*(1) The Lyapunov exponent $\rho(\mathcal{A}_a)$ is negative.*

*(2) For all solutions of the system,*

$$\begin{aligned} \frac{d\mathbf{x}}{dt} &= \mathbf{A}\mathbf{x} + \mathbf{b}(\mathbf{c}, \mathbf{x})\mathbf{u} \\ \frac{d\mathbf{l}}{dt} &= -\mathbf{A}^*\mathbf{l} - \mathbf{c}(\mathbf{b}, \mathbf{l})\mathbf{u} \\ \mathbf{u} &= \frac{\mu}{2}\{1 + sign[(\mathbf{c}, \mathbf{x})(\mathbf{b}, \mathbf{l})]\}, \end{aligned}$$

*it follows that* $\mathbf{x}(t) \to 0$ *as* $t \to \infty$.

Notice that for nonzero solutions, functions $\mathbf{c}^*\mathbf{x}(t)$ and $\mathbf{b}^*\mathbf{l}(t)$ have isolated roots [8]. Therefore, function $\mathbf{u}(\cdot)$ is well defined.

To find a worst-case solution, it is sufficient to find for any nonzero vector $\mathbf{x}(0)$ a vector $\mathbf{l}(0)$ from Theorem 3.3.6. Such information is currently unavailable in general case. Therefore, the general problem of finding the necessary and sufficient conditions for stability of inclusion (44) is open. On the other hand, for second-order systems (i.e., if $n = 2$), we can efficiently find such solutions in terms of system parameters $\mathbf{A}, \mathbf{b}, \mathbf{c}$ and $\mu$ as follows.

**Theorem 3.3.7.** *[11] For second-order systems ($n = 2$), the Lyapunov exponent $\rho(\mathcal{A}_a)$ is negative if and only if the following systems is stable:*

$$d\mathbf{x}/dt = \mathbf{A}\mathbf{x} + \mathbf{b}(\mathbf{c},\mathbf{x})\frac{\mu}{2}\{1 + sign[(\mathbf{c},\mathbf{x})(\mathbf{b},\mathbf{x})]\}.$$

Hence, by the previous theorem, the worst-case solution for second-order feedback systems in class $\mathcal{M}_\mu$ satisfies

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} & \text{, if } \mathbf{c}^*\mathbf{x}(t)\cdot\mathbf{b}^*\mathbf{x}(t) < 0, \\ \dot{\mathbf{x}} = (\mathbf{A} + \mathbf{b}\mu\mathbf{c}^*)\mathbf{x} & \text{, if } \mathbf{c}^*\mathbf{x}(t)\cdot\mathbf{b}^*\mathbf{x}(t) \geq 0. \end{cases} \tag{47}$$

## 3.4. Absolute Stability of Second-Order Feedback Systems in Class $\mathcal{M}_\mu$: Preliminaries

The above result provides a useful tool that will be utilized in this thesis to establish an important result in the theory of absolute stability for a particular class of feedback systems. We shall study inclusions of order two with the matrix pencil $\mathcal{A}_a = \{\mathbf{A} + \mathbf{b}\nu\mathbf{c}^* : 0 \leq \nu \leq \mu\}$, where $\mathbf{A} \in \mathbb{M}^{2\times 2}; \mathbf{b} \in \mathbb{M}^{2\times 1}; \mathbf{c} \in \mathbb{M}^{2\times 1}$; and $\mu \in \mathbb{R}^+$, arising from the problem of absolute stability for feedback systems

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} + \mathbf{b}\ \varphi(\sigma, t), \\ \sigma = \mathbf{c}^*\mathbf{x} \end{cases} \tag{48}$$

in class

$$\mathcal{M}_\mu = \{\varphi : 0 \leq \frac{\varphi(\sigma,t)}{\sigma} \leq \mu \ \ \forall\sigma \neq 0, t\}$$

of sector nonlinearities, We will assume that pairs $(\mathbf{A}, \mathbf{b})$ and $(\mathbf{A}, \mathbf{c})$ are controllable and observable, respectively.

The problem of interest is to derive necessary and sufficient conditions in terms of system parameters, $\mathbf{A}, \mathbf{b}, \mathbf{c}$ and $\mu$, that are readily verifiable and coordinate-free for the global asymptotic stability of system (48) for all nonlinearities $\varphi(\sigma, t)$ belonging to class $\mathcal{M}_\mu$. Many stability criteria have been derived for such systems. Duignan and Curran proved [2] the absolute stability of system (48) in class $\mathcal{M}_\mu$ using two types of Lyapunov functions. Wulff et al. [3] characterized the existence of such Lyapunov functions by eigenvalues of the matrices $\mathbf{A} + \mathbf{b}\nu\mathbf{c}^*$, $0 \leq \nu \leq \mu$; a quadratic or unic Lyapunov function exists if all eigenvalues of matrices $\mathbf{A} + \mathbf{b}\nu\mathbf{c}^*$, $0 \leq \nu \leq \mu$ lie in the set

$$S_\gamma = \left\{ z \in \mathbb{C} : \left| \frac{\Im m\{z\}}{\Re e\{z\}} \right| < \gamma, \ \Re e\{z\} \leq 0 \right\}, \ \gamma = 1. \tag{49}$$

Duignan and Curran referred to this region of the complex plane as the $45^0$-Region. In [2], it is proven that, if all eigenvalues of matrices $\mathbf{A} + \mathbf{b}\nu\mathbf{c}^*$, $0 \leq \nu \leq \mu$, lie in the interior of sector $S_1$, then system (48) is absolutely stable in class $\mathcal{M}_\mu$. It is known that, usually, the Lyapunov function approach provides essentially sufficient criteria for absolute stability. Hence, it is reasonable to expect that there exists a number $\gamma > 1$ such that system (48) is absolutely stable in class $\mathcal{M}_\mu$ if all eigenvalues of matrices $\mathbf{A} + \mathbf{b}\nu\mathbf{c}^*$, $0 \leq \nu \leq \mu$, lie in $S_\gamma$.

This thesis contains results showing that, in fact, boundary $\gamma = 1$ is firm. That is, the previous statement, in general, is true only for $\gamma = 1$. Therefore, the problem under consideration may be formulated as follows:

**Main Problem**: Find the maximal value of $\gamma$ such that, if for any second-order system (48), all eigenvalues of matrices $\mathbf{A} + \mathbf{b}\nu\mathbf{c}^*$, $0 \leq \nu \leq \mu$, belong to sector $S_\gamma$, then system (48) is absolutely stable in class $\mathcal{M}_\mu$.

Let us suppose pair $(\mathbf{A}, \mathbf{b})$ is controllable. Then, without loss of generality, we may assume system (48) to be in the controllable, canonical first Frobenius form as below

$$\begin{aligned} \begin{bmatrix} \dot{x}_1 \\ \dot{x}_2 \end{bmatrix} &= \begin{bmatrix} 0 & 1 \\ -a_2 & -a_1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} + \begin{bmatrix} 0 \\ -1 \end{bmatrix} \varphi(\sigma, t), \\ \sigma &= \begin{bmatrix} z & 1 \end{bmatrix} x, \end{aligned} \tag{50}$$

with given transfer function

$$W(s) = \frac{\alpha s + z}{s^2 + a_1 s + a_2}, \tag{51}$$

where $\alpha$, $z$, $a_1$, and $a_2$ are real numbers. We assume matrices $\mathbf{A} + \mathbf{b}\nu\mathbf{c}^*$ are Hurwitz for all $\nu \in [0, \mu]$, that is, $a_1 > 0$, $a_2 > 0$. If $\alpha < 0$, then the change of variables $\xi \mapsto \mu\sigma - \xi$, $\mathbf{A} \mapsto \mathbf{A} + \mathbf{b}\mu\mathbf{c}^*$ results in the transfer function of form (51) with positive $\alpha$. Hence, without loss of generality, we can assume $\alpha \geq 0$.

First, consider the case $\alpha > 0$. The change of the independent time variable, $t \mapsto \frac{t}{\alpha}$, results in the change of variable $s \mapsto \alpha s$, hence getting the transfer function of form (51) with $\alpha = 1$. Hence, in the sequel without loss of generality, we will assume $\alpha = 1$. The system under investigation has a transfer function

$$W(s) = \frac{s + z}{s^2 + a_1 s + a_2}. \tag{52}$$

The following definitions will be useful in stating and proving the main results of this thesis. We begin by defining the region in the complex plane that is of primary importance to our discussion.

**Definition 3.4.1.** *[3] ($45^0$-Region) The $45^0$-Region is the open subset, $S_1$, of the complex plane defined by*

$$S_1 = \left\{ z \in \mathbb{C} : \left| \frac{\Im m\{z\}}{\Re e\{z\}} \right| < 1,\ \Re e\{z\} < 0 \right\}.$$

Now we will introduce the unic function.

**Definition 3.4.2.** *[3] For any nonsingular $n \times n$ matrix $M$, we define a unic function on $\mathbb{R}^n$ by*

$$V(x) = ||Mx||_1 = \sum_{i=1}^{n} |(Mx)_i|$$

*where $(Mx)_i$ denotes the $i$th component of vector $Mx$.*

Note that, a unic function is continuous, bounded, positive, definite and homogenous of degree one, i.e., $V(\alpha x) = |\alpha| V(x)$ for every scalar $\alpha$ and every vector $x$. It satisfies the triangular inequality $V(x_1 + x_2) \leq V(x_1) + V(x_2)$. Therefore, function $V(\cdot)$ is a norm in $\mathbb{R}^n$.

In addition, the following definition is useful to prove subsequent statements about absolute stability.

**Definition 3.4.3.** *We say that inclusion (44) satisfies condition $S_1$ if all roots of matrices $\mathbf{A} + \mathbf{b}\nu\mathbf{c}^*$, $0 \leq \nu \leq \mu$ lie in the set $S_1$.*

A necessary condition for this property, i.e., condition $S_1$, amounts to $a_1^2 - 2a_2 > 0$. Later, it will be shown that this condition is indeed necessary. In addition, we will augment this condition with several others in such a way that the total conditions are necessary and sufficient. These additional conditions are given in the following lemma.

**Lemma 3.4.1.** *Assume $a_1 > 0$, $a_2 > 0$. Then, condition $S_1$ is satisfied if and only if $a_1^2 > 2a_2$, $z > -a_2$ and*
*(1) if $a_1^2 - 2a_2 \in (0, 1)$, then $z < a_1 + \sqrt{a_1^2 - 2a_2}$, as well as*
*(2) if $a_1^2 - 2a_2 \geq 1$, then $z < a_1 + \frac{1}{2}(a_1^2 - 2a_2 + 1)$.*

Before proceeding with the proof of Lemma 3.4.1, we will prove the following auxiliary lemma.

**Lemma 3.4.2.** *Assume $\beta \in [0, 1]$, $\alpha^2 > \beta$ and $\alpha > 0$. Then, $\frac{\beta}{\alpha + \sqrt{\alpha^2 - \beta}} < 1$.*

*Proof.* (Lemma 3.4.2) Because $\alpha^2 > \beta$, $\alpha > 0$, and $\beta \leq 1$, we have $\alpha > \beta$. Therefore, we have $\sqrt{\alpha^2 - \beta} \geq 0 > \beta - \alpha$, which is equivalent to $\beta < \alpha + \sqrt{\alpha^2 - \beta}$, which, in turn, is equivalent to

$$\frac{\beta}{\alpha + \sqrt{\alpha^2 - \beta}} < 1.$$

□

*Proof.* (Lemma 3.4.1) System (50) with $\varphi(\sigma, t) = \nu\sigma$ has characteristic polynomial

$$P(s) = s^2 + (a_1 + \nu)s + a_2 + \nu z.$$

Evidently, condition $S_1$ is satisfied only if $a_2 + \nu z > 0,\ a_1 + \nu > 0,\ \forall \nu \in [0,1]$. Hence, we will assume that this is true. First, the necessity of the condition $S_1$ is given by $a_1^2 - 2a_2 > 0$. Indeed, consider the case $\nu = 0$, and suppose the characteristic polynomial

$$P_{v=0}(s) := s^2 + a_1 s + a_2$$

has all roots in $S_1$. We have $s_{1,2} = \frac{-a_1}{2} \pm \sqrt{(\frac{a_1}{2})^2 - a_2} \in S_1$ if and only if either

$$\left(\frac{a_1}{2}\right)^2 - a_2 > 0$$

or

$$\left(\frac{a_1}{2}\right)^2 - a_2 \leq 0 \quad \text{and} \quad \left(\frac{a_1}{2}\right)^2 > a_2 - \left(\frac{a_1}{2}\right)^2.$$

It is easy to see that the last condition holds if and only if $a_1^2 > 2a_2$.

Condition $S_1$ fails only if the characteristic polynomials have roots on the boundary of $S_1$, i.e., at points $\lambda(-1+i)$, for some $\lambda \geq 0$ and some $\nu \in [0,1]$. We have

$$P(\lambda(-1+i)) = (-1+i)^2\lambda^2 + (a_1+\nu)(-1+i)\lambda + a_2 + \nu z = 0,$$

or

$$-(a_1+\nu)\lambda + a_2 + \nu z + [-2\lambda^2 + (a_1+\nu)\lambda]i = 0,$$

which is equivalent to

$$\begin{cases} -(a_1+\nu)\lambda + a_2 + \nu z = 0, \\ -2\lambda^2 + (a_1+\nu)\lambda = 0. \end{cases} \tag{53}$$

If $\lambda = 0$, then (53) implies $a_2 + \nu z = 0$. According to our assumption, $a_2 + \nu z > 0$. Thus, $\lambda \neq 0$.

If $\lambda > 0$, then (53) implies $\lambda = \frac{a_1+\nu}{2}$, and $\frac{-a_1+\nu}{2}(a_1+\nu) + a_2 + \nu z = 0$. Therefore, $\nu^2 + 2\nu(a_1 - z) + a_1^2 - 2a_2 = 0$, and

$$\nu = -a_1 + z \pm \sqrt{(a_1 - z)^2 - (a_1^2 - 2a_2)}.$$

The goal is to find conditions necessary and sufficient for $\nu \notin [0,1]$. As already established, the necessity of condition $S_1$ is given by $a_1^2 - 2a_2 > 0$. Now, consider the following two separate cases.

*Case I*: Assume $0 < a_1^2 - 2a_2 < 1$. We need to find a condition equivalent to the following property: $\nu = -a_1 + z \pm \sqrt{(a_1 - z)^2 - (a_1^2 - 2a_2)} \notin [0,1]$ or, equivalently, both roots of polynomial $f(\nu) = \nu^2 + 2\nu(a_1 - z) + a_1^2 - 2a_2$ do not belong to $[0,1]$. Notice that, if $a_1 - z \geq 0$, then there are no nonnegative roots, $\nu$, of this polynomial. Hence, $a_1 - z \geq 0$ is a sufficient condition for this property. Notice, our property also holds if $(a_1 - z)^2 - (a_1^2 - 2a_2) < 0$.

Hence, we will consider the subcase $(a_1-z)^2-(a_1^2-2a_2) \geq 0$. We have $(a_1-z)^2 > (a_1-z)^2-(a_1^2-2a_2)$, because $a_1^2-2a_2 > 0$. Therefore, $z-a_1 > \sqrt{(a_1 - z)^2 - (a_1^2 - 2a_2)}$. On the other hand, we have

$$z - a_1 - \sqrt{(a_1 - z)^2 - (a_1^2 - 2a_2)} = \frac{a_1^2 - 2a_2}{(z - a_1) + \sqrt{(z - a_1)^2 - (a_1^2 - 2a_2)}} .$$

Now, denote $\alpha := z - a_1$ and $\beta := a_1^2 - 2a_2$. By assumption, we have $z - a_1 > 0$ and $0 < a_1^2 - 2a_2 < 1$. Because $\alpha^2 - \beta \geq 0$, by Lemma 3.4.2, we have

$$\frac{a_1^2 - 2a_2}{z - a_1 + \sqrt{(z - a_1)^2 - (a_1^2 - 2a_2)}} \in [0,1],$$

and there is an eigenvalue of matrices $\mathbf{A} + \mathbf{b}\nu\mathbf{c}^*$, $0 \leq \nu \leq 1$ which lies outside the set $S_1$. Hence, if condition $S_1$ holds and if $z > a_1$, $0 \leq a_1^2 - 2a_2 \leq 1$, then $(z - a_1)^2 - (a_1^2 - 2a_2) < 0$. This condition is equivalent to

$$-\sqrt{a_1^2 - 2a_2} < z - a_1 < \sqrt{a_1^2 - 2a_2}.$$

Therefore, we have the following statement: If $0 < a_1^2 - 2a_2 \leq 1$, then condition $S_1$ holds if and only if either $-a_2 \leq z \leq a_1$, or $z > a_1$ and $z < a_1 + \sqrt{a_1^2 - 2a_2}$.

*Case II*: Assume $a_1^2 - 2a_2 \geq 1$. If $-a_2 < z \leq a_1$. Then, again as in *Case I* we have condition $S_1$ is satisfied. Now, consider the case $z \geq a_1$. Then, $z - a_1 \pm \sqrt{(a_1 - z)^2 - (a_1^2 - 2a_2)} \notin [0, 1]$ if and only if either

$$(a_1 - z)^2 - (a_1^2 - 2a_2) < 0$$

or

$$(a_1 - z)^2 - (a_1^2 - 2a_2) > 0 \quad \text{and} \quad z - a_1 - \sqrt{(a_1 - z)^2 - (a_1^2 - 2a_2)} > 1.$$

The last inequality is equivalent to $(z - a_1 - 1) > \sqrt{(a_1 - z)^2 - (a_1^2 - 2a_2)}$ which, in turn, is equivalent to the following equations:

$$\Leftrightarrow \begin{cases} z - a_1 - 1 > 0 \\ (z - a_1 - 1)^2 > (a_1 - z)^2 - (a_1^2 - 2a_2) \end{cases}$$

$$\Leftrightarrow \begin{cases} z > a_1 + 1 \\ z < a_1 + \frac{a_1^2 - 2a_2 + 1}{2} \end{cases}$$

$$\Leftrightarrow z \in \left(a_1 + 1,\ a_1 + 1 + \tfrac{a_1^2 - 2a_2 + 1}{2}\right)$$

We have either

$$z - a_1 \leq \sqrt{a_1^2 - 2a_2}$$

or

$$z - a_1 > \sqrt{a_1^2 - 2a_2} \qquad \text{and} \quad z \in \left(a_1 + 1,\ a_1 + 1 + \tfrac{a_1^2 - 2a_2 + 1}{2}\right) \quad .$$

These conditions are equivalent to

$$\begin{cases} z \leq a_1 + \sqrt{a_1^2 - 2a_2} \\ \text{or} \\ a_1 + \sqrt{a_1^2 - 2a_2} < z < a_1 + \frac{a_1^2 - 2a_2 + 1}{2}, \end{cases}$$

which, in turn, is equivalent to $z < a_1 + \frac{a_1^2 - 2a_2 + 1}{2}$. Therefore, condition $S_1$ is satisfied if and only if either

$$0 < a_1^2 - 2a_2 \leq 1 \qquad \text{and} \qquad -a_2 < z < a_1 + \sqrt{a_1^2 - 2a_2}$$

or

$$a_1^2 - 2a_2 \geq 1 \qquad \text{and} \qquad -a_2 < z < a_1 + \tfrac{a_1^2 - 2a_2 + 1}{2}.$$

$\square$

### 3.5. Sufficient Conditions for Absolute Stability in Class $\mathcal{M}_\mu$

In this section, we establish sufficient conditions, in terms of system parameters $\mathbf{A}, \mathbf{b}, \mathbf{c}$ and $\mu$, for the absolute stability of system (48) in class $\mathcal{M}_\mu$. First, we establish conditions necessary and sufficient for the existence of a quadratic and unic Lyapunov function for all $\varphi \in \mathcal{M}_\mu$. It is well known that the existence of a quadratic Lyapunov function is equivalent to the frequency domain inequality, which is a basis of the so called the circle criterion, much of which was discussed in general context earlier in Section 3.1.

In our case, the transfer function has a form (52). We assume the denominator of the transfer function, $s^2 + a_1 s + a_2$, is Hurwitz (i.e., $a_1 > 0$ and $a_2 > 0$). This is a necessary condition for absolute stability in class $\mathcal{M}_\mu$. Hence, the circle criterion for absolute stability acquires the form

$$1 + \Re e\{W(iw)\} > 0 \quad \text{for all } w \geq 0.$$

This inequality is equivalent to the following inequalities:

$$\begin{aligned}
&0 < 1 + \Re e\left\{\frac{iw + z}{-w^2 + iwa_1 + a_2}\right\} \quad \forall w \geq 0,\\
&0 < 1 + \frac{z(a_2 - w^2) + a_1 w^2}{(a_2 - w^2)^2 + a_1^2 w^2} \quad \forall w \geq 0,\\
&0 < w^4 + w^2(-2a_2 + a_1^2 - z + a_1) + a_2 z + a_2^2 \quad \forall w \geq 0.
\end{aligned}$$

The following result gives the conditions on parameters $\mathbf{A}, \mathbf{b}$ and $\mathbf{c}$ if the last inequality above holds for all $w$.

**Lemma 3.5.1.** *There exists a quadratic Lyapunov function if and only if*

(1) $a_2 z + a_2^2 > 0$, *and*

(2) *either* $-2a_2 + a_1^2 - z + a_1 > 0$

*or* $\left( -2a_2 + a_1^2 - z + a_1 < 0 \quad \textit{and} \quad (-2a_2 + a_1^2 - z + a_1)^2 < 4(a_2 z + a_2^2)\right)$.

The proof follows from the inequalities above and the fact that the circle criterion is necessary and sufficient for the existence of the quadratic Lyapunov function.

Taking into account that $a_2 > 0$, we can present condition (1) above in the form $a_2 + z < 0$. Provided that condition (1) is true, condition (2) is equivalent to the following inequality:

$$2a_2 - a_1^2 + z - a_1 < 2\sqrt{a_2(a_2 + z)},$$

which may be rewritten as follows:

$$z < a_1^2 + a_1 - 2a_2 + \sqrt{a_2 z + a_2^2}.$$

Next, we consider conditions for the existence of the unic Lyapunov function of the form

$$V(x_1, x_2) = |x_2| + |\sigma| = |x_2| + |zx_1 + x_2|. \tag{54}$$

**Lemma 3.5.2.** *There exists a unic Lyapunov function (norm) of form (54) if and only if*

$$a_1 z > a_2 \quad \textit{and} \quad a_1 - \sqrt{a_1^2 - 2a_2} < z < a_1 + \sqrt{a_1^2 - 2a_2}.$$

*Proof.* Consider the following change of basis $(x_1, x_2) \to (\sigma, x_2)$. Then, system (50) takes the form

$$\begin{aligned} \dot{\sigma} &= zx_2 - a_2 x_1 - a_1 x_2 - \varphi = -\tfrac{a_2}{z}\sigma - (a_1 - z - \tfrac{a_2}{z})x_2 - \varphi(\sigma, t), \\ \dot{x_2} &= -a_2 x_1 - a_1 x_2 - \varphi = -\tfrac{a_2}{z}\sigma - (a_1 - \tfrac{a_2}{z})x_2 - \varphi(\sigma, t), \end{aligned} \tag{55}$$

(We changed the basis from $(x_1, x_2)$ to $(\sigma, x_2)$ by making the substitution $x_1 =$

$\frac{\sigma - x_2}{z}$, where $z \neq 0$.) Notice that, if $z = 0$, then transfer function (51) takes the form $W(s) = \frac{s}{s^2 + a_1 s + a_2}$, which satisfies the circle criterion trivially. The first (state) equation of system (50) can be presented in a new coordinate system as follows:

$$\begin{aligned}
\frac{d}{dt}\begin{bmatrix} \sigma \\ x_2 \end{bmatrix} &= \begin{bmatrix} -\frac{a_2}{z} & -a_1 + z + \frac{a_2}{z} \\ -\frac{a_2}{z} & -a_1 + \frac{a_2}{z} \end{bmatrix}\begin{bmatrix} \sigma \\ x_2 \end{bmatrix} + \begin{bmatrix} -1 \\ -1 \end{bmatrix}\varphi(\sigma, t) \\
&= \begin{bmatrix} -\frac{a_2}{z} - \frac{\varphi}{\sigma} & -a_1 + z + \frac{a_2}{z} \\ -\frac{a_2}{z} - \frac{\varphi}{\sigma} & -a_1 + \frac{a_2}{z} \end{bmatrix}\begin{bmatrix} \sigma \\ x_2 \end{bmatrix} \\
&:= \mathfrak{D}\begin{bmatrix} \sigma \\ x_2 \end{bmatrix}.
\end{aligned} \tag{56}$$

Taking into account that $0 \leq \frac{\varphi(t)}{\sigma}$, we can deduce that function $V$ is a Lyapunov function if the diagonal elements of matrix $\mathfrak{D}$ are negative, their absolute values are not less than the absolute values of the off-diagonal elements of the second column, and at least one column has a strict inequality. The first column clearly satisfies this property if and only if $z > 0$. We will assume $z > 0$. For the second column, we get the inequality

$$a_1 - \frac{a_2}{z} > \left| a_1 - z - \frac{a_2}{z} \right|,$$

which is equivalent to the following inequalities:

$$\begin{cases} a_1 > \frac{a_2}{z} \\ 0 < z < 2(a_1 - \frac{a_2}{z}) \end{cases} \Leftrightarrow \begin{cases} a_1 z > a_2 \\ z^2 - 2a_1 z + 2a_2 < 0 \end{cases}$$

$$\Leftrightarrow \begin{cases} a_1 z > a_2 \\ a_1 - \sqrt{a_1^2 - 2a_2} < z < a_1 + \sqrt{a_1^2 - 2a_2}. \end{cases}$$

Therefore, system (50) has a unic Lyapunov function if and only if

$$a_1 z > a_2 \text{ and } a_1 - \sqrt{a_1^2 - 2a_2} < z < a_1 + \sqrt{a_1^2 - 2a_2}\ .$$

$\square$

Now, we need to show that, if condition $S_1$ is satisfied, then either there exists a quadratic Lyapunov function (that is, circle criterion satisfied) or there exists a unic Lyapunov function.

**Theorem 3.5.1.** *If condition $S_1$ is satisfied, then there exists either a quadratic Lyapunov function or a unic Lyapunov function.*

*Proof.* Let us suppose condition $S_1$ is satisfied. Assume $a_1^2 - 2a_2 \in (0, 1]$. Then, condition $S_1$ implies $-a_2 < z < a_1 + \sqrt{a_1^2 - 2a_2}$. If $z > a_1 - \sqrt{a_1 - 2a_2}$, then by Lemma 3.5.2, there exists a unic Lyapunov function. On the other hand, if

$$z \leq a_1 - \sqrt{a_1^2 - 2a_2},$$

then the circle criterion is satisfied if

$$a_1 - \sqrt{a_1^2 + 2a_2} < a_1^2 + a_1 - 2a_2 + 2\sqrt{a_2 z + a_2^2}\ .$$

This condition is equivalent to

$$a_1^2 - 2a_2 + 3\sqrt{a_2 z + a_2^2} > 0,$$

which is true since $a_1^2 - 2a_2 > 0$.
Now assume $a_1^2 - 2a_2 > 1$. Then, condition $S_1$ implies $z < a_1 + \frac{1}{2}(a_1^2 - 2a_2 + 1)$. By Lemma 3.5.1 the circle criterion is satisfied if

$$a_1^2 + a_1 - 2a_2 + 2\sqrt{a_2 z + a_2^2} > a_1 + \frac{1}{2}(a_1^2 - 2a_2 + 1),$$

which is equivalent to

$$\frac{1}{2}(a_1^2 - 2a_2 - 1) + 2\sqrt{a_2 z + a_2^2} > 0,$$

which is satisfied because $a_1^2 - 2a_2 - 1 > 0$. Therefore, if condition $S_1$ holds, then there exists either a quadratic or unic Lyapunov function as claimed. □

Lastly, consider the case $\alpha = 0$. The transfer function has the form

$$W(s) = \frac{z}{s^2 + a_1 s + a_2}. \tag{57}$$

Condition $S_1$ is equivalent to $a_2 > 0$, $a_1^2 > 2(a_2 + z) > 0$, $a_1^2 > 2a_2$, $a_1 > 0$, and $a_2 > 0$. The circle criterion acquires the form

$$z(a_2 - w^2) + (a_2 - w^2)^2 + a_1^2 w^2 > 0,$$

which is equivalent to

$$w^4 - w^2(z + 2a_2 - a_1^2) + a_2^2 + a_2 z > 0,$$

$a_1^2 > max\{2a_2, 2a_2 + 2z\}$. Hence, $a_1^2 > \frac{1}{2}(2a_2 + 2a_2 + 2z) = 2a_2 + z$, and this inequality holds.

All in all, condition $S_1$ implies the existence of the Lyapunov function of at least one of the following types: quadratic function and unic function. Therefore, condition $S_1$ implies the absolute stability of system (50) in class $\mathcal{M}_\mu$.

### 3.6. Necessary Conditions for Absolute Stability in Class $\mathcal{M}_\mu$

In this section, we will show that sector $S_1$ is maximal in the following sense. For any $\gamma > 1$ there exists a second-order system (50) that is not absolutely stable in class $\mathcal{M}_\mu$, and all eigenvalues of matrices $\mathbf{A} + \mathbf{b}\nu\mathbf{c}^*$, $0 \le \nu \le \mu$, lie in $S_\gamma$.

To establish this property, we will consider the behavior of the worst-case solutions of second-order linear inclusions. Consider the second-order inclusion of the form

$$\tfrac{d\mathbf{x}}{dt} \in \{(\mathbf{A}+\mathbf{b}\nu\mathbf{c}^*)\mathbf{x} : \ 0 \le \nu \le \mu\}, \tag{58}$$

where

$$\mathbf{A} = \begin{bmatrix} 0 & 1 \\ -\alpha & -\beta \end{bmatrix}, \quad \mathbf{b} = \begin{bmatrix} 0 \\ -1 \end{bmatrix}, \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \end{bmatrix}.$$

Notice that, by Theorem 3.3.7, there exists

$$\mathbf{l}(t) \in \partial v(\mathbf{x}(t)) \qquad \forall t \ge 0 \ [9]$$

such that

$$\max\{\mathbf{l}^*(t)(\mathbf{A}+\mathbf{b}\nu\mathbf{c}^*)\mathbf{x}(t) : \ 0 \le \nu \le \mu\} \ = \ \mathbf{l}^*(t)\mathbf{A}\mathbf{x}(t) + \max_{0\le\nu\le\mu}\{\nu(\mathbf{l}^*(t)\mathbf{b})(\mathbf{c}^*\mathbf{x}(t))\}.$$

Hence, for the worst-case solution we have

$$\begin{cases} \dot{\mathbf{x}} = \mathbf{A}\mathbf{x} & , \text{if } \mathbf{c}^*\mathbf{x}(t)\cdot\mathbf{b}^*\mathbf{x}(t) < 0 \\ \dot{\mathbf{x}} = (\mathbf{A}+\mathbf{b}\mu\mathbf{c}^*)\mathbf{x} & , \text{if } \mathbf{c}^*\mathbf{x}(t)\cdot\mathbf{b}^*\mathbf{x}(t) \ge 0. \end{cases} \tag{59}$$

Recall that the absolute stability of system (50) in class $\mathcal{M}_\mu$ is equivalent to the asymptotic stability of inclusion (58). To find the necessary and sufficient conditions for asymptotic stability of inclusion (58), for the case $c_1 > 0$, it suffices to consider the solution (which is the worst-case solution) defined as follows:

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = -\alpha x_1 - \beta x_2 - (c_1 x_1 + c_2 x_2) \\ x_1(0) = 1 \\ x_2(0) = 0 \end{cases} \tag{60}$$

for $t \in [0, t_0]$, where $t_0$ is the first positive root of function $(c_1 x_1 + c_2 x_2)(t)$, and

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = -\alpha x_1 - \beta x_2 \end{cases} \tag{61}$$

for $t \in [t_0, t_0 + t_1)$, where $t_1$ is the first positive root of function $x_2(t_0 + t)$.

Inclusion (58) is not asymptotically stable if and only if $t_0 < \infty$, $t_1 < \infty$, and $x_1(t_0 + t_1) \leq -1$. In the case when $x_1(t_0 + t_1) \leq -1$, inclusion (58) has a solution such that $x(t + (t_0 + t_1)) = \delta x(t)$ for all $t$, where $\delta = x_1(t_0 + t_1)$. Define $T = t_0 + t_1$; the values $t_0 + kT$, $(k + 1)T$, $k = 0, 1, 2, \cdots$ are called the switching point because the coefficients of the system jump as time traverses through these points. Notice that, if $\delta = 1$, then the worst-case solution is $2T$-periodic.

**Theorem 3.6.1.** *For all $\gamma > 1$, there exist $\epsilon > 0$ and $\beta > 0$ such that all eigenvalues of a system with transfer function*

$$W(s) = \frac{\frac{1-\epsilon}{2\beta} s + 1}{s^2 + \beta s + \beta^4}$$

*lie in*

$$S_\gamma = \{z : \frac{|\Im m(z)|}{|\Re e(z)|} < \gamma, \Re e(z) < 0\},$$

*and the system is not absolutely stable.*

*Proof.* Consider inclusion (58) with $c_1 = 1$, $c_2 > 0$, $\alpha > 0$ and $\beta > 0$. In the sequel, we have $c_2 = \frac{1-\epsilon}{2\beta}$ and $\beta$ sufficiently small.

The idea of the proof is as follows. First, we will show that, for sufficiently small $\beta$, the constructed system is not absolutely stable in class $\mathcal{M}_\mu$. To this end, we will compute the worst-case solution, and we will use the results stated at the beginning of this section. For $t \in [0, t_1]$, the worst-case solution satisfies the following equation:

$$\begin{cases} \dot{x_1} = x_2 \\ \dot{x_2} = -(\alpha+1)x_1 - (\beta + c_2)x_2. \end{cases} \tag{62}$$

Assume the eigenvalues of this system are real, i.e., $(\beta + c_2)^2 > 4(\alpha + 1)$. Denote the corresponding eigenvalues by

$$\lambda_{1,2} = -\frac{\beta + c_2}{2} \pm \sqrt{\left(\frac{\beta + c_2}{2}\right)^2 - \alpha - 1}.$$

Then, there exist numbers $r_1$ and $r_2$ such that, for solution $(x_1, x_2)$ of system (62), we have

$$\begin{aligned} x_1(t) &= r_1 e^{\lambda_1 t} + r_2 e^{\lambda_2 t} \\ x_2(t) &= r_1 \lambda_1 e^{\lambda_1 t} + r_2 \lambda_2 e^{\lambda_2 t}. \end{aligned}$$

Applying initial conditions $x_1(0) = 1$ and $x_2(0) = 0$, we get $r_1 = \frac{-\lambda_2}{\lambda_1 - \lambda_2}$ and $r_2 = \frac{\lambda_1}{\lambda_1 - \lambda_2}$. Hence,

$$\begin{aligned} x_1 &= \tfrac{\lambda_1 e^{\lambda_2 t} - \lambda_2 e^{\lambda_1 t}}{\lambda_1 - \lambda_2} \\ x_2 &= \tfrac{\lambda_1 \lambda_2}{\lambda_1 - \lambda_2}(e^{\lambda_2 t} - e^{\lambda_1 t}). \end{aligned}$$

Then,

$$x_1 + c_2 x_2 = \frac{1}{\lambda_1 - \lambda_2}\left[e^{\lambda_1 t}(-\lambda_2 - c_2\lambda_1\lambda_2) + e^{\lambda_2 t}(\lambda_1 + c_2\lambda_1\lambda_2)\right].$$

The number $t_0$ is the first positive root of function $x_1(t) + c_2 x_2(t)$. Hence,

$$e^{(\lambda_1 - \lambda_2)t_0} = \frac{\lambda_1 + c_2\lambda_1\lambda_2}{\lambda_2 + c_2\lambda_1\lambda_2},$$

and

$$t_0 = \frac{1}{\lambda_1 - \lambda_2} \ln\left(\frac{\lambda_1 + c_2\lambda_1\lambda_2}{\lambda_2 + c_2\lambda_1\lambda_2}\right).$$

We assume that $t_0$ exists, i.e., $\frac{\lambda_1 + c_2\lambda_1\lambda_2}{\lambda_2 + c_2\lambda_1\lambda_2} > 0$. The value of the pair $(x_1(t_0), x_2(t_0))$ is as follows:

$$x_1(t_0) = e^{\lambda_1 t_0} \frac{c_2(\alpha+1}{c_2(\alpha+1)+\lambda_1} \quad \text{and} \quad x_2(t_0) = -e^{\lambda_1 t_0} \frac{\alpha+1}{c_2(\alpha+1)+\lambda_1}.$$

Denote $x_{10} := x_1(t_0)$ and $x_{20} := x_2(t_0)$. This point $(x_{10}, x_{20})$ serves as an initial condition for the next phase after switching, i.e., we consider the following initial value problem

$$\begin{cases} \dot{y_1} = y_2 \\ \dot{y_2} = -\alpha y_1 - \beta y_2 \\ y_1(0) = x_{10} \\ y_2(0) = x_{20} \end{cases} \tag{63}$$

for $t \in [t_0; t_1)$.

Assume the eigenvalues of this linear system are real, i.e., $\beta^2 > 4\alpha$. Denote the eigenvalues as

$$\mu_{1,2} = -\frac{\beta}{2} \pm \sqrt{(\frac{\beta}{2})^2 - \alpha}.$$

Then, the general solution to system (63) is

$$y_1 = p_1 e^{\mu_1 t} + p_2 e^{\mu_2 t}$$
$$y_2 = p_2 \mu_1 e^{\mu_1 t} + p_2 \mu_2 e^{\mu_2 t}.$$

With initial conditions $y_{10} := y_1(0)$ and $y_{20} := y_2(0)$, we have

$$p_1 = \frac{\mu_2 x_{10} - x_{20}}{\mu_2 - \mu_1} \quad \text{and} \quad p_2 = \frac{x_{20} - \mu_2 x_{10}}{\mu_2 - \mu_1}.$$

We use $t_1$ to denote the first positive root of function $y_2(t)$. In addition, we assume that $t_1 > 0$. Then,

$$e^{(\mu_2 - \mu_1)t_1} = -\frac{p_1 \mu_1}{p_2 \mu_2} = \frac{\mu_1 \mu_2 x_{10} - \mu_1 x_{20}}{\mu_1 \mu_2 x_{10} - \mu_2 x_{20}}.$$

Hence, $t_1 = \frac{1}{\mu_2 - \mu_1} \ln \left( \frac{\mu_1 \mu_2 x_{10} - \mu_1 x_{20}}{\mu_1 \mu_2 x_{10} - \mu_2 x_{20}} \right)$.

The assumption regarding $t_1$ means that $(\mu_1 \mu_2 x_{10} - \mu_1 x_{20})(\mu_1 \mu_2 x_{10} - \mu_2 x_{20}) > 0$. In this case,

$$y_1(t_1) = e^{\mu_1 t_1}\left(p_1 - p_1\frac{\mu_1}{\mu_2}\right) = e^{\mu_1 t_1}\left(x_{10} - \frac{1}{\mu_2}x_{20}\right).$$

Taking into account the values of $x_{10}$ and $x_{20}$, we get

$$y_1(t_1) = e^{\mu_1 t_1 + \lambda_1 t_0}\left(\frac{c_2(\alpha+1)}{c_2(\alpha+1)+\lambda_1} + \frac{\alpha+1}{\mu_2(c_2(\alpha+1)+\lambda_1)}\right) = e^{\mu_1 t_1 + \lambda_1 t_0}\left(\frac{c_2\mu_2+1}{c_2\mu_2 + \frac{\lambda_1\mu_2}{\alpha+1}}\right).$$

Recall that inclusion (58) is not absolutely stable if and only if $y_1(t_1) \leq -1$. Now, set $\alpha = \beta^4$, $c_2 = \frac{1-\epsilon}{2\beta}$, $\epsilon \in (0,1)$, and $\beta$ as a positive small number. Then,

$$\lambda_1 t_0 = \frac{\lambda_1}{\lambda_1 - \lambda_2}\ln\left(\frac{c_2(\alpha+1)+\lambda_1}{c_2(\alpha+1)+\lambda_2}\right) = \frac{\frac{\alpha+1}{\lambda_2}}{\frac{\alpha+1}{\lambda_2} - \lambda_2}\ln\left(\frac{c_2(\alpha+1)+\frac{\alpha+1}{\lambda_2}}{c_2(\alpha+1)+\lambda_2}\right)$$

$$= \frac{-1}{\frac{\lambda_2^2}{\alpha+1} - 1}\ln\left(\frac{1+\frac{1}{c_2\lambda_2}}{1+\frac{\lambda_2}{c_2(\alpha+1)}}\right)$$

$$= \frac{-1}{\left(\frac{\beta+c_2}{2} + \sqrt{\left(\frac{\beta+c_2}{2}\right)^2 - \alpha - 1}\right)^2\frac{1}{\alpha+1} - 1}\ln\left(\frac{1 - \frac{1}{c_2\left(\frac{\beta+c_2}{2}+\sqrt{\left(\frac{\beta+c_2}{2}\right)^2-\alpha-1}\right)}}{1 - \frac{1}{\alpha+1}\left(\frac{\beta+c_2}{2c_2} + \sqrt{\left(\frac{\beta+c_2}{2c_2}\right)^2 - \frac{\alpha+1}{c_2^2}}\right)}\right).$$

The last product tends to zero as $\beta \to 0$.

Besides,

$$\mu_1 t_1 = \frac{\alpha}{\mu_2}t_1$$

$$= \left(\frac{-2\alpha}{\beta+\sqrt{\beta^2-4\alpha}}\right)\left(\frac{1}{\sqrt{\beta^2-4\alpha}}\right)\ln\left[\frac{\alpha\frac{c_2(\alpha+1)}{c_2(\alpha+1)+\lambda_1} + \mu_2\frac{\alpha+1}{c_2(\alpha+1)+\lambda_1}}{\alpha\frac{c_2(\alpha+1)}{c_2(\alpha+1)+\lambda_1} + \mu_1\frac{\alpha+1}{c_2(\alpha+1)+\lambda_1}}\right]$$

$$= \frac{-2\alpha}{\beta\sqrt{\beta^2-4\alpha}+\beta^2-4\alpha}\ln\left(\frac{\alpha c_2+\mu_2}{\alpha c_2+\mu_1}\right)$$

$$= \frac{2\alpha}{\beta\sqrt{\beta^2-4\alpha}+\beta^2-4\alpha}\ln\left(\frac{-\frac{\alpha}{\mu_2}-\alpha c_2}{-\mu_2-\alpha c_2}\right)$$

$$= \frac{2\alpha}{\beta\sqrt{\beta^2-4\alpha}+\beta^2-4\alpha}\ln\left(\frac{\frac{2\alpha}{\beta+\sqrt{\beta^2-4\alpha}}-\alpha c_2}{\beta+\sqrt{\beta^2-4\alpha}-\alpha c_2}\right)$$

$$= \frac{2\beta^4}{\beta\sqrt{\beta^2-4\alpha}+\beta^2-4\beta^4}\ln\left(\frac{\frac{2\beta^4}{\beta+\sqrt{\beta^2-4\beta^4}}-\frac{1-\varepsilon}{2}\beta^3}{\beta+\sqrt{\beta^2-4\beta^4}-\frac{1-\varepsilon}{2}\beta^3}\right) \longrightarrow 0 \qquad as\ (\beta\to 0)$$

and

$$\frac{1+c_2\mu_2}{\frac{\lambda_1\mu_2}{\alpha+1}+c_2\mu_2} = \frac{1-\left(\frac{c_2\beta}{2}+\sqrt{\left(\frac{c_2\beta}{2}\right)^2-\alpha c_2^2}\right)}{\frac{\alpha+1}{\frac{\beta+c_2}{2}+\sqrt{\left(\frac{\beta+c_2}{2}\right)^2-\alpha-1}}\left(\frac{\beta}{2}+\sqrt{\frac{\beta^2}{4}-\alpha}\right)\frac{1}{\alpha+1}-\left(\frac{c_2\beta}{2}+\sqrt{\left(\frac{c_2\beta}{2}\right)^2-\alpha c_2^2}\right)}$$

$$= \frac{1-\left(\frac{1-\varepsilon}{4}+\sqrt{\left(\frac{1-\varepsilon}{4}\right)^2-\frac{(1-\varepsilon)^2}{4}\beta^2}\right)}{\frac{\beta+\sqrt{\beta^2-4\beta^4}}{\beta+\frac{1-\varepsilon}{2\beta}+\sqrt{\left(\beta+\frac{1-\varepsilon}{2\beta}\right)^2-4\beta^4-1}}-\left(\frac{1-\varepsilon}{4}+\sqrt{\left(\frac{1-\varepsilon}{4}\right)^2-\frac{(1-\varepsilon)^2}{4}\beta^2}\right)} \quad \underset{\beta\to 0}{\longrightarrow} \quad -\frac{1+\varepsilon}{1-\varepsilon}<-1.$$

Hence, $y_1(t_1) \longrightarrow -\frac{1+\varepsilon}{1-\varepsilon} < -1$ (as $\beta\to 0$), and therefore, inclusion (58) is not absolutely stable in class $\mathcal{M}_\mu$ for sufficiently small positive $\beta$.

To prove the theorem, we only need to check the sector condition for this system, i.e., if all eigenvalues of a system where the transfer function is given by

$$W(s) = \frac{c_2 s+1}{s^2+\beta s+\alpha} \tag{64}$$

with $\varphi = \nu\sigma$, $0\le\nu\le 1$, lie in $S_\alpha$ for some $\alpha>1$. The characteristic polynomial of system (64) with $\varphi(\sigma)=\nu\sigma$ is $P(\lambda,\nu)=\lambda^2+\lambda(\beta+c_2\nu)+\alpha+\nu$. Denote the roots of $P(\lambda,\nu)$ by $\lambda(\nu)$. All roots of this polynomial lie in sector $S_\gamma$ if and only if the following two claims are satisfied:

(1) roots of polynomials

$$P_0(\lambda) := P(\lambda, 0) = \lambda^2 + \lambda\beta + \alpha,$$
$$P_1(\lambda) := P(\lambda, 1) = \lambda^2 + \lambda(\beta + c_2) + \alpha + 1$$

lie in $S_\gamma$.

(2) if there exists $\nu$ such that $\frac{d\lambda(\nu)}{d\nu}$ is parallel to $\lambda(\nu)$, then $\lambda(\nu) \in S_\gamma$.

If $\alpha = \beta^4$, $c_2 = \frac{1-\epsilon}{2\beta}$, and $\beta > 0$, with $\beta$ sufficiently small, then the roots of polynomials $P_0$ and $P_1$ are real and negative, hence belonging to the set $S_1$.

Assume $\nu \in [0,1]$ is such that $\frac{d\lambda(\nu)}{d\nu} = q\lambda(\nu)$, where $q \in (0, \infty)$. If, for all such $\nu$, we have $\Im m\{\lambda(\nu) = 0\}$; then, the roots of $P(\lambda, \nu)$ lie in $S_1$ for all $\nu \in [0,1]$. Assume $\Im m\{\lambda(\nu) \neq 0\}$, then

$$\lambda^2(\nu) + \lambda(\nu) \cdot (\beta + \nu c_2) + \alpha + \nu = 0 \tag{65}$$

for all $\nu$, and therefore,

$$2\frac{d\lambda}{d\nu}\lambda(\nu) + \frac{d\lambda}{d\nu}(\beta + \nu c_2) + 1 + c_2\lambda(\nu) = 0.$$

Taking into account $\frac{d\lambda}{d\nu} = q\lambda$, we get

$$2\lambda^2(\nu)q + q\lambda(\nu)(\beta + c_2\nu) + 1 + c_2\lambda(\nu) = 0. \tag{66}$$

Combining equations (65) and (66) and eliminating $\lambda^2$,

$$q(-\lambda(\nu)(\beta + c_2\nu) - 2(\alpha + \nu)) + \alpha c_2 + 1 = 0$$

or

$$\lambda(\nu)[c_2 - q(\beta + c_2\nu)] + 1 - 2q(\alpha + \nu) = 0.$$

Splitting the real and imaginary parts, we get the following equations:

$$c = q(\beta + cs)$$

$$1 = 2q(\alpha + s).$$

Thus, $q = \frac{1}{2(\alpha+\nu)}$ and $2c_2(\alpha + \nu) = \beta + c_2\nu$ which, in turn, implies $\nu = \frac{\beta - 2c_2\alpha}{c_2}$. For this $\nu$, the roots $\lambda(\nu)$ satisfy the equation

$$\lambda^2(\nu) + (2\beta - c_2\alpha)\lambda(\nu) + \frac{\beta - c_2\alpha}{c_2} = 0.$$

If $\alpha = \beta^4$ and $c_2 = \frac{1-\epsilon}{2\beta}$, then $(\beta - c_2\alpha)^2 < \frac{\beta - c_2\alpha}{c_2}$ for $\beta$ that is sufficiently small. Hence, for sufficiently small $\beta > 0$, we have

$$\lambda(\nu) = c_2\alpha - \beta \pm i\sqrt{\frac{\beta - c_2\alpha}{c_2} - (c_2\alpha)^2}$$

and

$$\left|\frac{\Im m\{\lambda(\nu)\}}{\Re e\{\lambda(\nu)\}}\right| = \frac{\sqrt{\frac{\beta - c_2\alpha}{c_2} - (c_2\alpha - \beta)^2}}{|c_2\alpha - \beta|} = \sqrt{\frac{1}{c_2(\beta - c_2\alpha)} - 1} = \sqrt{-1 + \frac{1}{\frac{1-\epsilon}{2}(1 - \frac{\beta^2(1-\epsilon)}{2})}}.$$

Hence $\left|\frac{\Im m\{\lambda(\nu)\}}{\Re e\{\lambda(\nu)\}}\right|$ converges to $\sqrt{\frac{1+\epsilon}{1-\epsilon}}$ as $\beta \to 0$. Assume $\gamma > 1$. Choose $\epsilon$ such that

$$\sqrt{\frac{1+\epsilon}{1-\epsilon}} < \gamma,$$

$\beta_0$ such that

$$\sqrt{-1 + \frac{1}{\frac{1-\epsilon}{2}(1 - \frac{\beta^2(1-\epsilon)}{2})}} < \gamma$$

for all $\beta \in (0, \beta_0)$, $\beta_1$ such that $y_1(t_1) < -1$ for all $\beta \in (0, \beta_1)$, and $\beta$ such that $0 < \beta < \min\{\beta_0, \beta_1\}$. Then all eigenvalues of $\mathbf{A} + \mathbf{b}\nu\mathbf{c}^*$, $0 \le \nu \le 1$, lie in $S_1$, but inclusion (58) is not absolutely stable in class $\mathcal{M}_\mu$ as claimed. $\square$

# CHAPTER 4. CONCLUSION

We considered the problem of absolute stability for a second-order feedback system in a class of time-varying nonlinearities satisfying the sector condition. It has been shown previously that, if all eigenvalues for matrices of a linear system from this class belong to the cone

$$S_1 = \{z \in \mathbb{C} : -\Re e\{z\} > |\Im m\{z\}|, z \neq 0\},$$

then the system is absolutely stable in class $\mathcal{M}_\mu$.

In this thesis, it was shown that for a bigger cone, this property did not hold in general. More precisely, for any $\gamma > 1$, there existed a system such that all eigenvalues for matrices of linear system with nonlinearities from $\mathcal{M}_\mu$ belong to the cone

$$S_\gamma = \{z \in \mathbb{C} : -\Re e\{z\} > \gamma|\Im m\{z\}|, z \neq 0\}$$

while the system is not absolutely stable.

A similar problem for third-order system may be a subject of future investigation.

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