LOCAL RISK MINIMIZATION UNDER TIME-VARYING TRANSACTION COSTS

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ABSTRACT

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Closely following the results of Lamberton, Pham, and Schweizer [5] we construct a locally risk-minimizing strategy in a general incomplete market including transaction costs. This is done in discrete time under the assumptions of a bounded mean-variance tradeoff and substantial risk. Once we establish all the required integrability conditions, a backward induction argument is implemented to obtain the desired strategy for every square-integrable contingent claim. We model the transaction costs as an adapted stochastic process and provide all necessary proofs in detail.

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CHAPTER 1. INTRODUCTION

In this chapter, we review some of the main historical contributions to Mathematical Finance. This is intended to give this work some context in the larger arena of financial mathematics. We also cover some basic definitions and facts used in this work. In the interest of brevity, proofs are omitted, but references are given.

1.1. Historical Background

The first attempt toward a mathematical description of the evolution of stock prices on the basis of probabilistic concepts was Louis Bachelier in 1900 [1]. He regarded stock prices on the Paris stock exchange $S = (S_t)_{t \geq 0}$ as random stochastic processes. Each process was modeled as a random walk $S_t^{(\Delta)}$ (Δ is a given increment of time) for $t = 0, \Delta, 2\Delta, \ldots$, with

$$S_t^{(\Delta)} = S_0 + \sum_{k \leq [\frac{t}{\Delta}]} \xi_k^{(\Delta)} ,$$

where the $\xi^{(\Delta)}$ are identically distributed random variables taking two possible values $\pm\sigma\sqrt{\Delta}$ each with probability $\frac{1}{2}$. Hence, under this formulation

$$ES_{k\Delta}^{(\Delta)} = S_0 .$$

The major drawback of this model is that it allows negative prices.

Under the assumption of uncertainty in financial markets, the next notable development was made in 1952 by H. Markowitz [6]. This paper established the basis of investment portfolio theory, concentrating on the optimization of investment decisions under uncertainty. This is the origin of mean-variance analysis, which revealed that a key ingredient determining unsystematic risks of a given portfolio is the covariance of prices. This was the first time that diversification was shown to be important in a mathematical context.

The very next year, M. Kendall discovered that it was logarithms of prices $\ln \frac{S_k}{S_{k-1}}$ and not the prices themselves that behave as a random walk [4]. Thus, setting

$h_k = \ln \frac{S_k}{S_{k-1}}$, one obtains

$$S_k = S_0 e^{H_k}, \qquad n \geq 1 \,,$$

where $H_k = h_1 + \cdots + h_k$ and each h_k is an independent random variable. This later lead to P. Samuelson [9] developing the so-called geometric Brownian motion

$$S_t = S_0 e^{W_t}, \qquad t \geq 1$$

where $W = (W_t)_{t \geq 0}$ is standard Brownian motion, the continuous-time generalization of the random walk H_k.

The lack of interest in financial mathematics and the subsequent slow development of the field was due in large part to low market volatility, stable interest rates, and fixed foreign exchange rates. Thus, there was not a consensus among economists that the market could be understood as randomly generated. Simple regression models were sufficient to describe market trends. However, a number of developments in the early 1970's, most notably the Nixon administration's decision to eliminate the gold standard, changed everything. Since the market became much more volatile, investors were more eager to determine ways to hedge against possible loss. This set the stage for the most monumental result in financial mathematics, the Black-Scholes formula for the rational price of a European call option, appearing in 1973.

In that year, two seminal papers were published "The Pricing of Options and Corporate Liabilities" by F. Black and M. Scholes [2] and "The Theory of Rational Option Pricing" by R. Merton [8] that revolutionized option pricing and consequently caused a sharp rise in interest in the theory. An immense number of applications were developed almost overnight. Many consider these results to be the birth of modern mathematical finance.

These results along with other results establishing the theory lead to complete characterization of the fair price of an option in a complete market. That is, pricing theory is well established for a simplified market in which every possible final portfolio value can be exactly attained. However, it can be shown that completeness is readily destroyed by even weak assumptions on the model [11]. Thus, a complete market is not a realistic representation of financial markets. If a market without arbitrage is incomplete, then the situation becomes considerably more complex. In an incomplete market an investor cannot hope to replicate a given claim faithfully. That is, an investor cannot faithfully reach a final portfolio value claimed at the time of initial investment. There is intrinsic error due to the market in any strategy when completeness is destroyed. Hence, we can only hope to choose a strategy that minimizes this error in some suitable sense. Measuring the riskiness of a strategy by a quadratic criterion was first proposed by H. Föllmer and Sondermann in 1986 [3] for the case when the price process is a martingale. This result was extended to the general semimartingale case by Föllmer's student, Martin Schweizer in 1991 [10].

It was Schweizer who first introduced the local quadratic optimality criterion *local risk-minimization.* This approach to risk-minimization has been studied by several authors for frictionless models without transaction costs. In the case of transaction cost models, it has been studied by Mercurio and Vorst in 1997 [7] and, most notably, Lamberton, Pham, and Schweizer in 1998 [5].

However, both of these results assume that transaction costs are fixed over the entire time horizon $[0, N]$. In this thesis, we slightly generalize the 1998 result to consider transaction costs modeled as a random variable. The whole framework of this paper and results essentially follow [5].

1.2. Some Basic Definitions and Facts on Stochastic Processes

In order to describe a financial market consisting of stocks and bonds in discrete time, a (B,S)-market, we introduce a probability space $(\Omega, \mathfrak{F}, P)$ consisting of the sample space Ω, σ-algebra $\mathfrak{F}$ on Ω and a probability measure P on $\mathfrak{F}$. The sample space Ω consists of all elementary events ω, which represent possible market situations.

At the core of any market model are random variables $X : \omega \to \mathbb{R}$, functions of elementary events $\omega \in \Omega$ which are measurable with respect to $\mathfrak{F}$.

Any successful market model should take into account the dynamic nature of random assets, usually modeled as stochastic processes. That is, we need a way to adjust the expected value of assets or other values as new information becomes available to market participants. To model this phenomenon, consider the following construction.

For all Lebesgue sets $A \in \mathbb{R}^k$, define $B_k(A) = A \times \mathbb{R}^{N-k}$. Let $\mathfrak{F}_k = \sigma(B_k(A) : A \in \mathcal{L}(\mathbb{R}^k))$, the σ-algebra generated by such sets and let P be a probability measure on $\mathfrak{F}$. Then, for all sets $C \in \mathfrak{F}_k$ we have that C is of the form $A \times \mathbb{R}^{N-k}$ for $A \in \mathcal{L}(\mathbb{R}^k)$, clearly $P(C) = P(A \times \mathbb{R}^{N-k})$. Next, consider function $f : \mathbb{R}^N \to \mathbb{R}$, $f \in L_1(P)$, and define an $\mathfrak{F}_k$-measurable function g such that for all $C \in \mathfrak{F}_k$, we have that

$$\int_C f dP = \int_C g dP .$$

This function g is called a conditional expectation of f with respect to σ-algebra $\mathfrak{F}_k$. In order to explicitly write $g(x)$, consider the probability density function of future values $x_{k+1}, \ldots, x_N$ given past values $x_1, \ldots, x_k$

$$p_{x_1,\ldots,x_k}(x_{k+1}, \ldots, x_N) .$$

Thus, we define g as:

$$g(x) := \int_{\mathbb{R}^{N-k}} f(x_1, \ldots, x_N) p_{x_1,\ldots,x_k}(x_{k+1}, \ldots, x_N) dx_{k+1} \cdots dx_N .$$

Note that if we consider two vectors whose values are known to be equal for the first k components (up to time k), $x_1 = y_1, \ldots, x_k = y_k$, then $g(x) = g(y)$ P-a.s. since g is $\mathfrak{F}_k$-measurable.

To adjust the expected value to changing information, we repeat this process as new information becomes available. So, we consider a sequence of such sub σ-algebras,

$\mathbb{F} = (\mathfrak{F}_k)_{k\geq 0}$, each more refined than the previous:

$$\mathfrak{F}_0 \subseteq \mathfrak{F}_1 \subseteq \cdots \subseteq \mathfrak{F}_k \subseteq \cdots \subseteq \mathfrak{F}.$$

Such a sequence is called a filtration. In particular, $\mathfrak{F}_0$ may be equal to the trivial σ-algebra $\{\emptyset, \Omega\}$ and $\mathfrak{F}_N$ may be equal to $\mathfrak{F}$. This sets the stage for the definition of conditional expectation.

Definition 1.2.1. *Let $(\Omega, \mathfrak{F}, P)$ be a probability space, $X : \Omega \to \mathbb{R}$ $\mathfrak{F}$-measurable, $E|X| < \infty$, and $\mathfrak{G}$ a σ-algebra on Ω with $\mathfrak{G} \subseteq \mathfrak{F}$. Then, the conditional expectation $E[X|\mathfrak{G}]$ is the $\mathfrak{G}$-measurable random variable such that*

$$\int_B E[X|\mathfrak{G}](\omega)P(d\omega) = \int_B X(\omega)P(d\omega) \ \textit{for all } B \in \mathfrak{G}.$$

The existence of such a variable is a consequence of the Radon-Nikodym theorem.

In order to incorporate time into a model of changing asset prices, we define a stochastic process over a finite time horizon $[0, N]$.

Definition 1.2.2. *A finite stochastic process on a sample space Ω is a sequence of random variables $X_1, X_2, \ldots, X_N$ defined on Ω. This sequence is usually denoted $X = (X_k)_{k\in[0,N]}$.*

A stochastic process X can have some important properties which we now introduce.

Definition 1.2.3. *A predictable process $X = (X_k)$ in discrete time is a process such that each X_k is $\mathfrak{F}_{k-1}$-measurable.*

Definition 1.2.4. *An adapted process $X = (X_k)$ is adapted to the filtration $\mathbb{F}$ if X is $\mathfrak{F}_k$-measurable for all k. We do not mention $\mathfrak{F}$ if it is clear with respect to which filtration the process X is adapted.*

Definition 1.2.5. *A stochastic process X is a martingale with respect to the filtration*

$\mathfrak{F} = (\mathfrak{F}_k)_{k\in[0,N]}$ *if* X *is adapted to* $\mathfrak{F}$, $E|X| < \infty$ *and*

$$E[X_{k+1}|\mathfrak{F}_k] = X_k \qquad P\text{-a.s.} \quad \text{for all } k = 0, 1, \ldots, N-1 .$$

In order to understand how properties of a stochastic sequence depend on the properties of the filtration, it seems natural to decompose the sequence into two subsequences. Assume $(\mathfrak{F}_k)_{k\in[0,N]}$ is a filtration, P is a measure on $\mathfrak{F}_N$, and $(H_k)_{k\in[0,N]}$ is an adapted stochastic sequence $H_k = h_1 + \cdots + h_k$, where $H_0 = 0$ and $\mathfrak{F}_0 = \{\emptyset, \Omega\}$. Assume $E|h_k| < \infty$ for each $k \geq 1$. Then conditional expectations $E[h_k|\mathfrak{F}_{k-1}]$ are well-defined and we can write

$$H_n = \sum_{k\leq n} E[h_k|\mathfrak{F}_{k-1}] + \sum_{k\leq n}(h_k - E[h_k|\mathfrak{F}_{k-1}]) .$$

Define

$$A_n = \sum_{k\leq n} E[h_k|\mathfrak{F}_{k-1}] ,$$

and

$$M_n = \sum_{k\leq n} (h_k - E[h_k|\mathfrak{F}_{k-1}]) .$$

The representation

$$H_n = A_n + M_n \qquad n \geq 1 ,$$

is called the Doob decomposition for stochastic process $H = (H_k)_{k\in[0,N]}$.

The sequence $A = (A_k)_{k\in[0,N]}$ is predictable and the sequence $M = (M_k)_{k\in[0,N]}$ is a martingale.

1.3. Self-Financing Portfolios in a (B,S)-Market

In the framework of the model described in section 1.2, we consider two categories of primary assets: risky assets representing stocks and less-risky assets which represent bonds. We assume an investor has $m + 1$ stocks and a bond available for trade whose value can be described by stochastic processes $S^i = (S^i_k)_{k\in[0,N]}$, for $i = 0, \dots, m$ and $B = (B_k)_{k\in[0,N]}$. We adopt the random walk conjecture of Kendall [4] that the logarithms of the prices $S = (S_k)_{k\in[0,N]}$ behave as a random walk. Thus, setting $h_k = \ln \frac{S_k}{S_{k-1}}$ we obtain a formula for stock prices

$$S_k = S_0 e^{H_k} \qquad k \geq 1$$

where $H_k = h_1 + \dots + h_k$ is the sum of independent random variables $h_1, \dots, h_k$ and $E|h_k| < \infty$ for each $k \geq 1$. Similarly,

$$B_k = B_0 e^{R_k}$$

where $R_k = r_1 + \dots + r_k$ and r_j is the interest rate of the bank account at time j.

There is a distinct difference between these two types of financial assets. B_k is $\mathfrak{F}_{k-1}$-measurable which means that the state of the bank account at time k is already clear at time $k-1$, hence $(B_k)_{k\in[0,N]}$ is predictable. Alternatively, the stocks S^i_k are $\mathfrak{F}_k$-measurable, meaning their actual values are known only after one obtains all information $\mathfrak{F}_k$ arriving at time k. This is why bank accounts (bonds) are considered less risky assets and stocks are considered risky assets.

We normalize the units based on the positive asset B_j, which not only simplifies the model, but also provides a more transparent measure of gains and losses. Thus, in the sequel we work with discounted units $X^i_j = \frac{S^i_j}{B_j}$ for $i = 1, \dots, m$.

A central concept in financial mathematics is the value of an investment portfolio. We define a portfolio ϕ as a stochastic sequence that represents the quantity of stocks and bonds an investor has at discrete time intervals

$$\phi_k = (\theta_{k+1}, \eta_k) \qquad \text{for } k = 0, 1, \dots, N\,,$$

where θ_k are predictable and η_k are adapted random variables. The values $\theta_k(\omega)$ and $\eta_k(\omega)$ can be positive, zero, or negative, which means the investor can borrow from the bank account or sell stock short. Predictability of the process θ is a mathematical formulation of the informational constraint that θ is not allowed to anticipate the movement of risky asset prices. This fact is realized in the notation since at time k, the investor determines θ_{k+1}, the amount of risky asset to be held over the interval $[k, k+1)$.

We make the following canonical assumptions regarding an investor on the (B,S)-market [12]:

1. The investor can:

   (a) deposit money into the bank account and borrow from it.

   (b) buy and sell stock.

2. A transfer of money from one asset into another can be done with no transaction costs (we'll relax this condition later).

3. The assets are infinitely divisible, meaning the investor can buy or sell any portion of stock and withdraw any amount from the bank.

The value of a portfolio is considered a function of the strategy ϕ. It is defined as the sum of assets in the portfolio at time k

$$V_k(\phi) := \theta_{k+1} X_k + \eta_k \,.$$

Thus, the total value of portfolio ϕ up to time k is

$$V_k(\phi) = V_0(\phi) + \sum_{j=1}^{k} (\Delta\eta_j + \Delta\theta_{j+1} X_j) + \sum_{j=1}^{k} \theta_j \Delta X_j \,.$$

The value $\Delta C_j = \Delta\eta_j + \Delta\theta_{j+1} X_j$ represents funds which are invested (if $\Delta C_j > 0$) or withdrawn (if $\Delta C_j < 0$) from an investor's capital at time j. If $\Delta C_j = 0$ for all j, then the portfolio is called self-financing. Therefore, the cumulative value of self-financing portfolios can be written as

$$V_k(\phi) = V_0(\phi) + G_k(\theta) ,$$

where the cumulative capital gains due to the market (and not investment) up to time k can be defined as

$$G_k(\theta) := \sum_{j=0}^{k} \theta_j \triangle X_j .$$

The sequence $C_k = \sum_{j=1}^{k} \Delta C_j$ with $C_0 = 0$ is called the cumulative cost process. This motivates a formal definition of this constraint.

Definition 1.3.1. *A strategy is called self-financing if its cumulative cost process $C = (C_k)_{k\in[0,N]}$ is constant or equivalently if its value process $V = (V_k)_{k\in[0,N]}$ is*

$$V_k = V_0 + G_k(\theta) \qquad \text{for } k \in [0, N] ,$$

where $V_0 = C_0$ is the initial investment.

Thus, after time zero such a strategy is self-supporting, which means no additional investments are made. In fact, any fluctuations in X can be neutralized by rebalancing θ and η in such a way that no further gains or losses result.

Therefore, we can determine the best strategy by systematically adjusting the quantities of assets at each discrete time step. (At each step, the quantity of assets is adjusted to maximize value.)

Along with the martingale property of risky assets, one of the essential properties of a fair financial market is the absence of arbitrage. That is, the opportunity of an investor to make a profit out of zero investment. If an arbitrage situation arises, the market forces of supply and demand would quickly eliminate such opportunities. As an example, suppose gold in Moscow sells for \$2 per ounce and in New York for \$2.10 per ounce. A flood of investors would buy gold in Moscow and sell it in New York, which would raise demand in Moscow and lower it in New York. Thus the price quickly would balance at an equilibrium point. Therefore, a market is "rationally organized" if the investors get no opportunities for riskless profit. This concept is placed into a mathematically rigorous context in the next theorems.

Definition 1.3.2. *A self-financing strategy ϕ brings an opportunity for arbitrage at time N if, for starting capital $V_0(\phi) = 0$, we have that $V_N(\phi) \geq 0$ P-a.s. and $P(V_N(\phi) \geq 0) > 0$.*

Theorem 1.3.1 (First Fundamental Theorem of Asset Pricing). *Assume a (B, S)-market on a filtered probability space is formed by a bank account $B = (B_k)_{k\in[0,N]}$, $B_k > 0$ and finitely many assets $S = (S^1, \ldots, S^m)$, $S^i = (S^i_k)_{k\in[0,N]}$. Assume also that this market operates at instants $k = 0, 1, \ldots, N$, where $\mathfrak{F}_0 = \{\emptyset, \Omega\}$ and $\mathfrak{F}_N = \mathfrak{F}$. Then, this (B,S)-market is arbitrage-free if and only if there exists at least one measure $\widetilde{P}$ equivalent to P such that the discounted price process $X = (X^i_k)_{k\in[0,N]} = (\frac{S^i_k}{B_k})_{k\in[0,N]}$, is a$\widetilde{P}$-martingale. That is, $E_{\widetilde{P}}|X^i_k| < \infty$ for all $i = 1, \ldots, m$ and $k = 0, \ldots, N$ and*

$$E_{\widetilde{P}}\left[X^i_k \middle| \mathfrak{F}_{k-1}\right] = X^i_{k-1} \text{ for all } k = 1, \ldots, N\,.$$

This theorem asserts that in order to guarantee an arbitrage-free market, a measure $\widetilde{P}$ must exist that forces the discounted price process $X = (\frac{S_k}{B_k})_{k\in[0,N]}$ to be a martingale. Thus, as one might suspect, the martingale property is important to model a fair and rational market. These measures $\widetilde{P}$, called martingale measures, are an essential ingredient in a fair and efficient market.

1.4. Upper and Lower Hedges, Put and Call Options

The concept of a hedge plays an important role in financial mathematics. It is an instrument of protection enabling an investor to have guaranteed levels of capital, and to insure transactions on securities markets. This is a key component of the present work.

We will assume that transactions in our (B, S)-market are made only at instants $k = 0, \ldots, N$. Let $f_N = f_N(\omega)$ be a non-negative $\mathfrak{F}_N$-measurable function treated as an "obligation" or terminal payoff. This function is also sometimes called a contingent claim.

Definition 1.4.1. *An investment portfolio* ϕ *is called an upper (lower) hedge if* $V_N(\phi) \geq f_N$ *P-a.s. (respectively* $V_N(\phi) \leq f_N$ *P-a.s.).*

Definition 1.4.2. $H^* = \{\phi : V_N(\phi) \geq f_N \text{ } P\text{-a.s.}\}$ *is the class of upper* f_N*-hedges and* $H_* = \{\phi : V_N(\phi) \leq f_N \text{ } P\text{-a.s.}\}$ *the class of lower* f_N*-hedges.*

A perfect hedge ϕ is a strategy such that $V_N(\phi) = f_N$ P-a.s. In fact, the equality $V_N(\phi) = f_N$ means that the hedge ϕ replicates the contingent claim f_N. When $H_* \bigcap H^* \neq \emptyset$, every contingent claim f_N is exactly replicated, which is essential to a complete market.

Definition 1.4.3. *A* (B, S)*-market is complete with respect to time N if each bounded* $\mathfrak{F}_N$*-measurable payoff is replicable. That is, there exists a perfect hedge* ϕ*, a portfolio such that*

$$V_N(\phi) = f_N \quad P\text{-}a.s.$$

The property of completeness can be formulated in terms of martingale measures $\widetilde{P}$. The next theorem concerns the relationship between completeness and the set $\mathcal{P}(P)$ of equivalent martingale measures $\widetilde{P}$.

Theorem 1.4.1 (Second Fundamental Theorem of Asset Pricing). *An arbitrage-free financial* (B, S)*-market is complete if and only if the set* $\mathcal{P}(P)$ *contains a single unique element.*

Therefore, we can understand the completeness criterion in terms of the set of martingale measures equivalent to P.

Corollary 1.4.1. *A* (B, S)*-market is incomplete if* $|\mathcal{P}| > 1$*. That is, there is more than one unique* $\widetilde{P}$ *equivalent to* P*.*

In general, completeness of a (B, S)-market is a fairly strong condition, which imposes serious constraints on the structure of the market.

This fact invokes natural questions about constraints imposed on the model. That is, one usually considers only self-financing strategies in a complete market, which is a way to measure the true value of a strategy based solely on the initial investment. In this case, we are interested in determining the optimal "rearrangement" of asset quantities, without additional investment. However, if the market is incomplete, we cannot hope to exactly replicate a contingent claim.

In a general incomplete market, it is impossible to impose the self-financing constraint and simultaneously insist on exact replication. Thus, if we force the strategies to be self-financing, we must relax the constraint on exact replication. Alternatively we could fix the value of the contingent claim at time N, and allow continued investment by relaxing the self-financing constraint. In both situations, we need a way to measure the "success" of a given strategy.

One of the most common ways to hedge in a financial market is through options. An option is a contract issued by a financial institution which gives its purchaser the option to buy or sell an underlying asset. Options are used to both hedge against possible loss and speculate on future market trends. The two most common forms of which are American and European options. The difference concerns the investor's flexibility to exercise the contract on or before an agreed upon maturity date. American options allow the investor to exercise the contract at any time on or before the maturity date. On the other hand, European options only allow action at the time when the contract expires. Also, there are two flavors of any option, depending on whether one is buying or selling the contract. A call option is the buyer's option and a put option is the seller's option. For simplicity, we focus only on call options, since the analysis is essentially the same. Thus, the payoff functions for call options are defined as:

$$\begin{aligned} &\text{American Option} \quad f_\tau = (S_\tau - K)^+ \\ &\text{European Option} \quad f_N = (S_N - K)^+ \end{aligned}$$

where τ is a stopping time on $[0, N]$ and $a^+ = \max(a, 0)$. S_N is the price at time N

and K is the strike price.

Definition 1.4.4. *A stopping time $\tau = \tau(\omega)$ is a nonnegative random variable independent of the future such that*

$$\{\omega : \tau(\omega) \leq t\} \in \mathfrak{F}_t$$

for each $t \geq 0$.

By terms of the contract, the buyer has the right to buy shares of stock S at strike price K. Thus, in the case of a European call option if

$S_N > K$	The investor gains
$S_N = K$	The investor breaks even
$S_N < K$	The investor doesn't exercise the option

One of the main problems in financial mathematics is the pricing and hedging of contingent claims using dynamic trading strategies based on the price process S. Contingent claims model financial contracts, the most common example of which is a European call option. Therefore, in order to hedge against future loss, the investor must make sure that $S_N \geq K$. Of course, one cannot forget that in general the issuer of such an option charges a fee, so $S_N \geq K$ does not guarantee a successful hedge by the investor.

In the current paper we consider the problem of hedging in an incomplete market, which includes transaction costs. In particular, we consider the case when the contingent claim f_N is fixed and strategies are not self-financing. We use a certain cost functional which is constant on self-financing strategies. We minimize this cost functional and recursively work backward toward $V_0(\phi)$, determining the best portfolio choice over each local time interval $[k, k+1)$, under appropriate conditions. That is, we minimize the cost of the strategy using a mean-variance criterion over each of these intervals. This can be thought of as a model of a European-type option.

CHAPTER 2. LOCAL RISK-MINIMIZATION

2.1. Formulation of the Problem

In order to properly formulate the problem under investigation, we start with some important definitions. Throughout this paper, we consider only the discrete-time case. That is, we only allow actions at integer time values $k = 0, 1, \ldots, N$ over the finite time horizon $[0, N]$.

Definition 2.1.1. *A trading strategy $\phi = (\theta, \eta)$ is a pair of processes*

$$\theta = (\theta_k)_{k=1,\ldots,N+1} \quad , \eta = (\eta_k)_{k=0,1,\ldots,N}$$

such that (θ_k) is predictable and (η_k) is adapted.

In discrete time, a random variable is predictable if it is measurable with respect to knowledge up through all previous times. That is, θ_k is $\mathfrak{F}_{k-1}$-measurable for each k. Similarly, a random variable is adapted if it is measurable with respect to current information, thus $\mathfrak{F}_k$-measurable.

Remark 2.1.1. *Predictability is a property imposed on the process $\theta = (\theta_k)$ to model the informational constraint that θ is not allowed to anticipate the movement of X.*

Definition 2.1.2. *The value of a portfolio at time k is*

$$V_k(\phi) := \theta_{k+1} X_k + \eta_k ,$$

where $X_k = \frac{S_k}{B_k}$ is the discounted price at time k of the risky asset and the price of each bond is equal to unity.

Remark 2.1.2. *Note that one wishes to formulate the value process so that it is adapted. Hence, at time $t = k$, the number of shares in the risky asset held over the time interval $[k, k+1)$ is determined.*

Definition 2.1.3. *A contingent claim is a pair of random variables* (θ_{N+1}, η_N) *such that both* θ_{N+1} *and* η_N *are* $\mathfrak{F}_N$*-measurable. That is, in this problem setting, a contingent claim is a pair of random amounts of both stocks and bonds known at the end of some finite time interval. The corresponding value of this claim is denoted*

$$H(\omega) := \theta_{N+1} X_N(\omega) + \eta_N(\omega) .$$

Realistically, the amount of capital an investor gains purely due to the market should be bounded during each discrete time interval. For this reason, it is natural to model the gains using functions from a familiar and well-behaved space, namely $L_2(P)$. In this context, we define the set of all admissible processes (ϕ_k).

Definition 2.1.4. *The process* $\phi = (\theta, \eta)$ *is called admissible (with respect to process* X*) if the process* $\theta = (\theta_k)$ *is predictable,* $\theta_k \triangle X_k \in L_2(P)$*, and* $V_k(\phi) \in L_2(P)$*. The set of admissible processes we denote by* $\Theta(X)$*.*

To model transaction costs, we consider an additional term in the cost process mentioned above. Since the cost of making a financial transaction is some fraction of the price of the given asset, this suggests the following mathematical description:

$$\begin{aligned} \text{bid price}: &\quad (1-\lambda_k)X_k \quad \lambda_k(\omega) \in [0, \lambda_0], \; \lambda_0 < 1 \; P-\text{a.s.} \\ \text{ask price}: &\quad (1+\mu_k)X_k \quad \mu_k(\omega) \in [0, \mu_0], \; \mu_0 \neq \infty \; P-\text{a.s.}, \end{aligned}$$

where $\lambda_0 \in \mathbb{R}^+$ and $\mu_0 \in \mathbb{R}^+$. In many realistic market situations, the transaction costs are symmetric which incidently simplifies the model. Hence, we assume $\lambda_k \equiv \mu_k$ and that λ_k is adapted. In this context, we define the cost process.

Definition 2.1.5. *The cost process of a strategy* $\phi = (\theta, \eta)$ *is*

$$C_k(\phi) := V_k(\phi) - \sum_{j=1}^{k} \theta_j \triangle X_j + \sum_{j=1}^{k} X_j \lambda_j \left|\theta_{j+1} - \theta_j\right| \quad k = 0, ..., N . \tag{1}$$

This is the total cost of strategy ϕ up to time k, including transaction costs. This process clearly depends on the adapted transaction cost parameter λ_k and θ_{k+1}

the number of shares of stock traded up through time k. In the next section, we formulate this process more carefully.

The way we measure the success of a given strategy is through the risk process. This is defined as the conditional variance of the remaining cost of implementing strategy ϕ under information available to the investor at the current time k. This quadratic process was first implemented by Hans Follmer and Dieter Sondermann in [3].

Definition 2.1.6. *The risk process of a strategy ϕ is*

$$R_k(\phi) := E\left[(C_T(\phi) - C_k(\phi))^2 | \mathfrak{F}_k\right] \quad k = 0, 1, ..., N\,. \tag{2}$$

Remark 2.1.3. *In this definition we assume the cost process $C_k(\phi)$ is square integrable i.e. $C_k(\phi) \in L_2(P)$.*

<u>Problem:</u> Find an admissible strategy which minimizes the functional $R_k(\phi)$ for all k.

2.2. Local Risk-Minimizing Strategies

At time k an investor following strategy ϕ buys (or sells) $\eta_k - \eta_{k-1}$ bonds and $\theta_{k+1} - \theta_k$ shares. So the total incremental cost due to a transaction during $[k, k+1)$ is

$$\eta_k - \eta_{k-1} + (\theta_{k+1} - \theta_k)X_k(1 + \lambda_k \text{sign}(\theta_{k+1} - \theta_k))\,,$$

where again each λ_k is adapted and $\lambda_k \in [0, \lambda_0]$ P-a.s.

The incremental cost can be presented in terms of the portfolio's value at time k, which will be useful later in this paper. This can be done as in the following series of operations:

$$
\begin{aligned}
&\eta_k - \eta_{k-1} + (\theta_{k+1} - \theta_k)X_k(1 + \lambda_k \text{sign}(\theta_{k+1} - \theta_k)) \\
&= \eta_k - \eta_{k-1} + \theta_{k+1}X_k + \theta_{k+1}X_k\lambda_k \text{sign}(\theta_{k+1} - \theta_k) - \theta_k X_k \\
&\quad -\theta_k X_k \lambda_k \text{sign}(\theta_{k+1} - \theta_k) \\
&= \eta_k + \theta_{k+1}X_k + X_k\lambda_k(\theta_{k+1} - \theta_k)\text{sign}(\theta_{k+1} - \theta_k) - \eta_{k-1} - \theta_k X_k \\
&= \eta_k + \theta_{k+1}X_k - (\eta_{k-1} + \theta_k X_k) + X_k\lambda_k(\theta_{k+1} - \theta_k)\text{sign}(\theta_{k+1} - \theta_k) \\
&= V_k(\phi) - (\eta_{k-1} + \theta_k X_k) + X_k\lambda_k \left|\theta_{k+1} - \theta_k\right| \\
&= V_k(\phi) - (\eta_{k-1} + \theta_k X_k + \theta_k X_{k-1} - \theta_k X_{k-1}) + X_k\lambda_k \left|\theta_{k+1} - \theta_k\right| \\
&= V_k(\phi) - V_{k-1}(\phi) - \theta_k(X_k - X_{k-1}) + X_k\lambda_k \left|\theta_{k+1} - \theta_k\right| \\
&= V_k(\phi) - V_{k-1}(\phi) - \theta\triangle X_k + X_k\lambda_k \left|\theta_{k+1} - \theta_k\right| .
\end{aligned}
$$

Summing all of the incremental costs up to time k yields the total cost of following the strategy ϕ from an initial investment up to time k. This is expressed in equation (1) as the cost process of strategy ϕ.

Ultimately, we're interested in minimizing the risk globally. It turns out that an efficient means of achieving this goal is to minimize risk locally over each time increment $[k, k+1)$. Thus, we're motivated to rewrite the risk process so it can be understood in a local context. That is, instead of remaining risk we should rewrite the risk process to accommodate local changes of cost. This motivates the following definition of a locally risk-minimizing strategy.

Definition 2.2.1. *Let $\phi = (\theta, \eta)$ be a strategy and $k \in \{0, 1, \ldots, N-1\}$. A local perturbation of ϕ at date k is a strategy $\phi' = (\theta', \eta')$ with*

$$
\begin{aligned}
\theta'_j &= \theta_j \quad \text{for } j \neq k+1 \\
\eta'_j &= \eta_j \quad \text{for } j \neq k\,.
\end{aligned}
$$

A strategy ϕ is locally risk-minimizing if and only if

$$R_k(\phi) \leq R_k(\phi')$$

for all admissible strategies ϕ' and for all $k \in [0, N-1]$.

This definition sets the stage for a recursive formula to determine the minimum risk of a given strategy. The first thing to note is that the cost process necessarily is a martingale if the strategy used is locally risk-minimizing. This leads to the following preliminary result.

Proposition 2.2.1. *If the strategy ϕ is locally risk-minimizing, then the cost process $C_k(\phi)$ is a martingale.*

Proof. According to the definition of $C_k(\phi)$, we have

$$\begin{aligned}
\Delta C_k &= C_{k+1} - C_k \\
&= V_{k+1} - \sum_{j=1}^{k+1} \theta_j \Delta X_j + \sum_{j=1}^{k+1} X_j \lambda_j \left|\theta_{j+1} - \theta_j\right| - V_k + \sum_{j=1}^{k} \theta_j \Delta X_j \\
&\quad - \sum_{j=1}^{k} X_j \lambda_j \left|\theta_{j+1} - \theta_j\right| \\
&= V_{k+1} - V_k - \theta_{k+1} \triangle X_{k+1} + X_{k+1} \lambda_{k+1} \left|\theta_{k+2} - \theta_{k+1}\right| \\
&= (V_{k+1} - \theta_{k+1} \triangle X_{k+1}) + (X_{k+1} \lambda_{k+1} \left|\theta_{k+2} - \theta_{k+1}\right| - V_k) \ .
\end{aligned}$$

Note that V_k is the only term in this sum that can be not $\mathfrak{F}_{k+1}$-measurable. The goal is to isolate the only $\mathfrak{F}_k$-measurable term, since it is the only term the investor can control at time k. That is, it's the only term that depends on η_k. So we continue with this goal in mind. In particular, consider $\alpha = V_{k+1} - \theta_{k+1} \triangle X_{k+1} + X_{k+1}\lambda_{k+1} \left|\theta_{k+2} - \theta_{k+1}\right|$ as one term and V_k as the other. Thus,

$$\Delta C_k = (V_{k+1} - \theta_{k+1} \triangle X_{k+1} + X_{k+1} \lambda_{k+1} \left|\theta_{k+2} - \theta_{k+1}\right|) - V_k$$

and so

$$(\Delta C_k)^2 = \alpha^2 - 2\alpha V_k + V_k^2 \ .$$

Therefore,

$$E\left[(\triangle C_k)^2 \left|\mathfrak{F}_k\right.\right] = E[\alpha^2 | \mathfrak{F}_k] - 2 V_k E[\alpha | \mathfrak{F}_k] + V_k^2$$

$$= E[\alpha^2|\mathfrak{F}_k] - 2V_k E[\alpha|\mathfrak{F}_k] + V_k^2 - E[\alpha|\mathfrak{F}_k]^2 + E[\alpha|\mathfrak{F}_k]^2$$

$$= \text{Var}[\alpha|\mathfrak{F}_k] + E[\alpha|\mathfrak{F}_k]^2 - 2V_k E[\alpha|\mathfrak{F}_k] + V_k^2$$

$$= \text{Var}[\alpha|\mathfrak{F}_k] + (E[\alpha|\mathfrak{F}_k] - V_k)^2 .$$

Since the strategy is locally risk-minimizing and V_k is the only term in the last formula which depends on η_k, at time k the value of η_k should be chosen such that

$$V_k = E\left[\alpha|\mathfrak{F}_k\right] .$$

Hence, the optimal choice of η_k implies that
$V_k(\phi) = E\left[V_{k+1} - \theta_{k+1}\triangle X_{k+1} + X_{k+1}\lambda_{k+1}\left|\theta_{k+2} - \theta_{k+1}\right| |\mathfrak{F}_k\right]$ or
$E\left[V_{k+1} - \theta_{k+1}\triangle X_{k+1} + X_{k+1}\lambda_{k+1}\left|\theta_{k+2} - \theta_{k+1}\right| |\mathfrak{F}_k\right] - V_k = 0$ or
$E\left[\triangle V_k - \theta_{k+1}\triangle X_{k+1} + X_{k+1}\lambda_{k+1}\left|\theta_{k+2} - \theta_{k+1}\right| |\mathfrak{F}_k\right] = 0$ or $E\left[\triangle C_{k+1}|\mathfrak{F}_k\right] = 0$, which means that $C_k(\phi)$ is a martingale. □

This result leads directly to the following lemma which allows us to decompose the local risk process.

Lemma 2.2.1. *If the strategy is locally risk-minimizing, then the cost process, $C_k(\phi)$, is a martingale and the corresponding risk process at time k can be written as*

$$R_k(\phi) = E\left[R_{k+1}|\mathfrak{F}_k\right] + Var[\triangle C_{k+1}|\mathfrak{F}_k] , \quad P\text{-a.s. for } k = 0, 1, ..., N-1. \tag{3}$$

Proof. Since the strategy ϕ is risk-minimizing, the process $C_k(\phi)$ is a martingale according to the previous proposition. Formula (3) results from the following transformations, where we define $\triangle C_k^N = C_N(\phi) - C_k(\phi)$.

$$\begin{aligned}
R_k(\phi) &:= E\left[\left(\triangle C_k^N\right)^2 |\mathfrak{F}_k\right] = E\left[\left(\triangle C_k^N + C_{k+1} - C_{k+1}\right)^2 |\mathfrak{F}_k\right] \\
&= E\left[\left(\triangle C_{k+1}^N + \triangle C_{k+1}\right)^2 |\mathfrak{F}_k\right] \\
&= E\left[\left(\triangle C_{k+1}^N\right)^2 + 2\triangle C_{k+1}^N \triangle C_{k+1} + \left(\triangle C_{k+1}\right)^2 |\mathfrak{F}_k\right] \\
&= E\left[\left(\triangle C_{k+1}^N\right)^2 |\mathfrak{F}_k\right] + 2E\left[\triangle C_{k+1}^N \triangle C_{k+1} |\mathfrak{F}_k\right] + E\left[\left(\triangle C_{k+1}\right)^2 |\mathfrak{F}_k\right] \\
&= E\left[\left(\triangle C_{k+1}^N\right)^2 |\mathfrak{F}_k\right] + 2E\left[\triangle C_{k+1}^N |\mathfrak{F}_k\right] E\left[\triangle C_{k+1} |\mathfrak{F}_k\right] + E\left[\left(\triangle C_{k+1}\right)^2 |\mathfrak{F}_k\right] \\
&= E\left[\left(\triangle C_{k+1}^N\right)^2 |\mathfrak{F}_k\right] + E\left[\left(\triangle C_{k+1}\right)^2 |\mathfrak{F}_k\right] \\
&= E\left[E\left[\left(\triangle C_{k+1}^N\right)^2 |\mathfrak{F}_{k+1}\right] |\mathfrak{F}_k\right] + E\left[\left(\triangle C_{k+1}\right)^2 |\mathfrak{F}_k\right] \\
&= E\left[R_{k+1} |\mathfrak{F}_k\right] + \operatorname{Var}\left[\triangle C_{k+1} |\mathfrak{F}_k\right] + E\left[\triangle C_{k+1} |\mathfrak{F}_k\right]^2 \\
&= E\left[R_{k+1} |\mathfrak{F}_k\right] + \operatorname{Var}\left[\triangle C_{k+1} |\mathfrak{F}_k\right] .
\end{aligned}$$

□

So if the cost process is a martingale, the local risk of a strategy is the conditional expectation of the remaining risk under current information plus the conditional variance of the local change in cost. At time k, the investor only has control over θ_{k+1} and η_k. Hence, to minimize the local risk R_k it suffices to consider only minimizing $\operatorname{Var}\left[\triangle C_k|\mathfrak{F}_k\right]$ with respect to θ_{k+1}. Thus, for a martingale cost process minimizing local risk is equivalent to minimizing the local conditional variance.

This leads naturally to the essential criteria of a locally risk-minimizing strategy. The necessary and sufficient conditions for a strategy to be locally risk-minimizing are summarized in the following proposition. Determining such a strategy is the main focus of this paper.

Proposition 2.2.2. *A strategy $\phi = (\theta, \eta)$ is locally risk-minimizing if and only if*

1. *$C(\phi)$ is a martingale.*

2. For each $k \in \{0, ..., N-1\}$, θ_{k+1} satisfies

$$\theta_{k+1} = \underset{\theta'_{k+1} \in \Theta(X)}{\operatorname{argmin}} \ Var[\Delta C_{k+1} | \mathfrak{F}_k] \ , \tag{4}$$

such that $\theta'_{k+1} \Delta X_{k+1} \in L_2(P)$ and $\theta'_{k+1} X_k \in L_2(P)$.

Remark 2.2.1. *The fact that θ'_{k+1} is admissible guarantees the first integrability condition. We prove the integrability of the second condition below which will come from technical constraints imposed on the process X.*

This proposition naturally progresses from the properties of local risk explained above. In order for us to be able to optimize the strategy with the controls available, the cost process must be a martingale. The cost process is a martingale from the optimal choice of η_k shown above. (It is necessarily a martingale from the fact we assume the strategy is locally risk-minimizing.) Once we have this property, we can rewrite the risk process as a sum of two terms only one of which is in our control at time k. Thus, we only concern ourselves with minimizing the variance of the local change of the cost process with information available at time k.

Proof of proposition 2.2.2. By proposition 2.2.1, $C(\phi)$ is a martingale if ϕ is locally risk-minimizing. If $C(\phi)$ is a martingale, then equation (3) and the definition of $C(\phi)$ imply

$$\begin{aligned} R_k(\phi) &= E\left[R_{k+1} \middle| \mathfrak{F}_k\right] + \operatorname{Var}\left[\Delta C_{k+1} \middle| \mathfrak{F}_k\right] \\ &= E\left[R_{k+1} \middle| \mathfrak{F}_k\right] + \operatorname{Var}\left[V_{k+1} - V_k - \theta_{k+1} \Delta X_{k+1} + X_{k+1} \lambda_{k+1} \left|\theta_{k+2} - \theta_{k+1}\right| \middle| \mathfrak{F}_k\right] . \end{aligned}$$

Now, removing all $\mathfrak{F}_k$-measurable terms from the conditional variance, we have

$$R_k(\phi) = \left[R_{k+1} \middle| \mathfrak{F}_k\right] + \operatorname{Var}\left[V_{k+1} - \theta_{k+1} \Delta X_{k+1} + X_{k+1} \lambda_{k+1} \left|\theta_{k+2} - \theta_{k+1}\right| \middle| \mathfrak{F}_k\right] .$$

We're assuming $\theta_k \in \Theta(X)$, so by definition $\theta_{k+1} \Delta X_{k+1} \in L_2(P)$. Fix $k \in \{0, 1, \ldots, N-1\}$ and let ϕ' be a local perturbation of ϕ at time k. Then from the definition of the cost process C_k

$$C_N(\phi') - C_{k+1}(\phi') = V_N - V_{k+1} - \sum_{j=k+1}^{N} \theta'_j \Delta X_j + \sum_{j=k+1}^{N} X_j \lambda_j \left|\theta'_{j+1} - \theta'_j\right|$$
$$= C_N(\phi) - C_{k+1}(\phi) .$$

Since we can assume $C(\phi)$ is a martingale and since by the previous lemma $R_k(\phi) = E[R_{k+1}(\phi)|\mathfrak{F}_k] + \mathrm{Var}[\Delta C_{k+1}(\phi)|\mathfrak{F}_k]$, we obtain that the risk process of a locally perturbed strategy ϕ' is

$$R_k(\phi') = E\left[R_{k+1}(\phi)\middle|\mathfrak{F}_k\right] + E\left[(\Delta C_{k+1}(\phi'))^2\middle|\mathfrak{F}_k\right] . \tag{5}$$

Suppose conditions (1) and (2) hold in the proposition. Since ϕ' is a local perturbation at time k,

$$V_{k+1}(\phi') = V_{k+1}(\phi) \qquad \text{and} \qquad \theta'_{k+2} = \theta_{k+2} . \tag{6}$$

Thus,

$$\Delta C_{k+1}(\phi') = V_{k+1}(\phi) - V_k(\phi') - \theta'_{k+1}\Delta X_{k+1} + \lambda_{k+1} X_{k+1} \left|\theta_{k+2} - \theta'_{k+1}\right| .$$

Using equation (5), we obtain the following inequality:

$$\begin{aligned}
R_k(\phi') &= E\left[R_{k+1}(\phi)\middle|\mathfrak{F}_k\right] + E\left[(\Delta C_{k+1}(\phi'))^2\middle|\mathfrak{F}_k\right] \\
&= E\left[R_{k+1}(\phi)\middle|\mathfrak{F}_k\right] + \mathrm{Var}\left[\Delta C_{k+1}(\phi')\middle|\mathfrak{F}_k\right] + E\left[\Delta C_{k+1}(\phi')|\mathfrak{F}_k\right]^2 \\
&\geq E\left[R_{k+1}(\phi)\middle|\mathfrak{F}_k\right] + \mathrm{Var}\left[\Delta C_{k+1}(\phi')\middle|\mathfrak{F}_k\right] \\
&\geq E\left[R_{k+1}(\phi)\middle|\mathfrak{F}_k\right] + \mathrm{Var}\left[\Delta C_{k+1}(\phi)\middle|\mathfrak{F}_k\right] \\
&= R_k(\phi) ,
\end{aligned}$$

where the third inequality relies on condition (2), equation (5) and the irrelevance of $\mathfrak{F}_k$-measurable terms in the conditional variance. The last equality comes from equation (3). By definition, this means that ϕ is locally risk-minimizing.

Conversely, suppose strategy ϕ is locally risk-minimizing, which means (1) holds by proposition 2.2.1. To show that property (2) follows, recall that

$$R_k(\phi) \leq R_k(\phi')$$

for locally risk-minimizing strategy ϕ and all locally perturbed strategies ϕ'. Since we have assumed that ϕ is locally risk-minimizing and by comparing equations (3) and (5),

$$\begin{aligned} R_k(\phi) &= E\left[R_{k+1}(\phi)\middle|\mathfrak{F}_k\right] + \mathrm{Var}\left[\Delta C_{k+1}(\phi)\middle|\mathfrak{F}_k\right] \\ &\leq E\left[R_{k+1}(\phi)\middle|\mathfrak{F}_k\right] + E\left[(\Delta C_{k+1}(\phi'))^2\middle|\mathfrak{F}_k\right] = R_k(\phi') \,. \end{aligned}$$

This means that we have the following inequality for any $\mathfrak{F}_k$-measurable choice of θ'_{k+1} and η'_k:

$$\mathrm{Var}\left[\Delta C_{k+1}(\phi)\middle|\mathfrak{F}_k\right] \leq E\left[(\Delta C_{k+1}(\phi'))^2\middle|\mathfrak{F}_k\right] \,. \tag{7}$$

In particular, fix θ'_{k+1} and choose η'_k such that $E\left[\Delta C_{k+1}(\phi')\middle|\mathfrak{F}_k\right] = 0$. Hence, putting this together with (7), using the definition of $\Delta C_{k+1}(\phi')$ and property (6) we have that

$$\mathrm{Var}\left[\Delta C_{k+1}(\phi)\middle|\mathfrak{F}\right] \leq \mathrm{Var}\left[\Delta C_{k+1}(\phi')\middle|\mathfrak{F}_k\right]$$

for each $k \in \{0, 1, \ldots, N-1\}$, which means (2) holds. □

We obtain the required measurability properties in the next section after introducing some required technical conditions on the process X.

2.3. Substantial Risk and Mean-Variance Tradeoff

We naively assumed that the units in this new market with friction were equivalent to the usual frictionless market. That is, not only the cost process, but also the value process is different in this new market with transaction costs. Before we can proceed, we must carefully formulate the conditions in the present market model. The main goal of this section is to clearly define the relationship between the nominal frictionless market and the market we consider in this paper. To that end, we introduce the following definition.

Definition 2.3.1. *Denote as* Γ *the class of all adapted processes* $\gamma = (\gamma_k)_{k=0,...,N}$ *such that* $\gamma_k \in [-1, 1]$ *P-a.s. Similarly, denote* Λ *the class of all adapted processes* $\lambda = (\lambda_k)_{k\in[0,N]}$ *such that* $\lambda_k \in [0, \lambda_0]$ *P-a.s. for* $\lambda_0 < 1$. *So, for* $\gamma \in \Gamma$ *and* $\lambda \in \Lambda$,

$$X_k^{\lambda\gamma} := X_k (1 + \lambda_k \gamma_k) \quad \text{for } k = 0, 1, ..., N\,,$$

and the corresponding value process in the new units is defined as

$$V_k^{\lambda\gamma} := \theta_{k+1} X_k^{\lambda\gamma} + \eta_k \quad \text{for } k = 0, 1, ..., N\,.$$

Note that this generalizes the value of a risky asset with transaction costs since we allow γ_k to take any value in $[-1, 1]$. In fact, $\gamma_k(\omega)$ is usually -1 or 1 and, as we'll see later, all other possible values only concern the case when no transactions occur during the given time increment (i.e. when $\theta_{k+1} = \theta_k$).

The point of this section is to carefully show that the choice of units does not effect the strategies. That is, we wish to show that under additional technical assumptions, $\Theta(X) = \Theta(X^{\lambda\gamma})$.

In order to obtain all the necessary integrability conditions required in this model, a constraint on the process X must be imposed. It turns out that this constraint not only results in the desired integrability conditions, but also has a quite natural physical interpretation in the context of our model. Therefore, we do not consider it as an unnatural restriction on the subsequent results.

Definition 2.3.2. X *has substantial risk if there exists a constant* $c < \infty$ *such that*

$$\frac{X_{k-1}^2}{E\left[\triangle X_k^2 | \mathfrak{F}_{k-1}\right]} \le c \quad P\text{-a.s. for } k = 1, ..., N\,. \tag{8}$$

The smallest such constant is denoted c_{sr}.

This means that the increments of the value of each risky asset are sufficiently "spread out" away from the mean (sufficiently risky). That is, substantial risk places a lower bound on the conditional variance of increments of X. Substantial risk also has a very intuitive interpretation. If we define the return process $\rho = (\rho_k)_{k\in[0,N]}$ of X as

$$\rho_k = \frac{\Delta X_k}{X_{k-1}} \qquad \text{for each } k = 1, \ldots, N,$$

then (8) can be written in terms of ρ_k as

$$E\left[\rho_k \middle| \mathfrak{F}_{k-1}\right] = \frac{E\left[\Delta X_k^2 \middle| \mathfrak{F}_{k-1}\right]}{X_{k-1}^2} \geq \frac{1}{c} > 0 \quad P-\text{a.s. for } k = 1, \ldots, N.$$

Therefore, X has substantial risk if and only if there is a lower bound on the returns of X.

Insisting that X satisfies substantial risk has some very fruitful consequences summarized in the next lemma.

Lemma 2.3.1. *Suppose X has substantial risk. Then the following are true:*

1. $\Theta\left(X^{\lambda\gamma}\right) \supseteq \Theta(X)$.
2. $V_k^{\lambda\gamma}(\phi) \in L_2(P)$ *for* $k = 0, 1, ..., N$ *for all* $\gamma \in \Gamma$, $\lambda \in \Lambda$ *and for all* ϕ.
3. $\theta_{k+1}X_k \in L_2(P)$ *for* $k = 0, 1, ..., N$ *and for all* $\theta \in \Theta(X)$.
4. $C_k(\phi) \in L_2(P)$ *for* $k = 0, ..., N$ *and for all* ϕ.

Proof. First we prove (3) and the other results easily follow. Suppose X has substantial risk. To show the product $\theta_{k+1}X_k \in L_2(P)$, we must show $E\left[(\theta_{k+1}X_k)^2\right] < \infty$. So consider

$$
\begin{aligned}
E\left[(\theta_{k+1}X_k)^2\right] &= E\left[(\theta_{k+1}X_k)^2 \frac{E\left[(\Delta X_{k+1})^2|\mathfrak{F}_k\right]}{E\left[(\Delta X_{k+1})^2|\mathfrak{F}_k\right]}\right] \\
&= E\left[\theta_{k+1}^2 E\left[(\Delta X_{k+1})^2|\mathfrak{F}_k\right] \frac{X_k^2}{E\left[(\Delta X_{k+1})^2|\mathfrak{F}_k\right]}\right] \\
&= E\left[E\left[(\theta_{k+1}\Delta X_{k+1})^2|\mathfrak{F}_k\right] \frac{X_k^2}{E\left[(\Delta X_{k+1})^2|\mathfrak{F}_k\right]}\right] \\
&\le c_{sr}E\left[E\left[(\theta_{k+1}\Delta X_{k+1})^2|\mathfrak{F}_k\right]\right] \\
&= c_{sr}E\left[(\theta_{k+1}\Delta X_{k+1})^2\right] < \infty\,.
\end{aligned}
$$

The third equality comes from the fact that θ_{k+1} is $\mathfrak{F}_k$-measurable and the inequality from equation (8). Hence, (3) is proved. Next we prove (1), so consider θ_k an arbitrary element of $\Theta(X)$.

$$
\begin{aligned}
\theta_k\triangle X_k^{\lambda\gamma} &= \theta_k\left[X_k\left(1+\lambda_k\gamma_k\right) - X_{k-1}\left(1+\lambda_{k-1}\gamma_{k-1}\right)\right] \\
&= \theta_k\left[X_k - X_{k-1} + X_k\lambda_k\gamma_k - X_{k-1}\lambda_{k-1}\gamma_{k-1} - X_{k-1}\lambda_k\gamma_k + X_{k-1}\lambda_k\gamma_k\right] \\
&= \theta_k\triangle X_k + \theta_k\triangle X_k\lambda_k\gamma_k + \theta_k X_{k-1}\triangle\lambda_k\triangle\gamma_k\,.
\end{aligned}
$$

Since λ_k and γ_k are uniformly bounded for each k as functions of ω, $\theta_k \in \Theta(X)$, and by property (3) all of the terms in the above sum are square integrable. Therefore, $\theta_k\triangle X_k^{\lambda\gamma} \in L_2(P)$ which implies that $\theta_k \in \Theta(X^{\lambda\gamma})$. Thus, we have the desired containment in property (1).

Similarly, we prove property (2), so consider the value of ϕ at time k

$$
\begin{aligned}
V_k^{\lambda\gamma}(\phi) &= \theta_{k+1}X_k^{\lambda\gamma} + \eta_k \\
&= \theta_{k+1}\left(X_k\left(1+\lambda_k\gamma_k\right)\right) + \eta_k \\
&= \theta_{k+1}X_k + \theta_{k+1}X_k\lambda_k\gamma_k + \eta_k\,.
\end{aligned}
$$

Thus, as before since every $\lambda_k \in \Lambda$ and $\gamma_k \in \Gamma$, both λ_k and γ_k that appear above are uniformly bounded. From this fact and property (3) it is clear that $V_k^{\lambda\gamma}(\phi) \in L_2(P)$.

Finally, we consider the cost process

$$\begin{aligned} C_k(\phi) &= V_k(\phi) - \sum_{j=1}^{k} \theta_{j+1} \triangle X_j + \sum_{j=1}^{k} X_j \lambda_j \left| \theta_{j+1} - \theta_j \right| \\ &= V_k(\phi) - \sum_{j=1}^{k} \theta_{j+1} \triangle X_j + \sum_{j=1}^{k} X_j \lambda_j \gamma_j \,. \end{aligned}$$

Again, by definition $V_k(\phi) \in L_2(P)$, $\theta_{j+1} \triangle X_j \in L_2(P)$, and $X_j \lambda_j \gamma_j \in L_2(P)$. So we've established the final property (4) of the lemma and hence it is proved. □

Another technical condition is needed in order to prove the reverse inclusion $\Theta(X^{\lambda\gamma}) \subseteq \Theta(X)$. To measure the relative diffusion of the random amount X with respect to its drift in each time interval, we introduce the following process:

Definition 2.3.3. *The mean-variance tradeoff process of X for $\gamma \in \Gamma$ and $\lambda \in \Lambda$ up to time ℓ is defined as*

$$K_\ell^{\lambda\gamma} := \sum_{j=1}^{\ell} \frac{\left(E\left[\triangle X_j^{\lambda\gamma} | \mathfrak{F}_{j-1} \right] \right)^2}{Var\left[\triangle X_j^{\lambda\gamma} | \mathfrak{F}_{j-1} \right]} \quad for\ \ell = 0, \dots, N \,. \tag{9}$$

The "boundedness" of the above process is usually included in the canonical structure conditions of a random market model. We follow this convention in the model under consideration. Denote by $c_{mvt}(\lambda\gamma)$ the smallest of $K_k^{\lambda\gamma} - K_{k-1}^{\lambda\gamma}$ for all $k \in [0, N]$, then

$$\left(E\left[\triangle X_k^{\lambda\gamma} | \mathfrak{F}_{k-1} \right] \right)^2 \le c_{mvt} \mathrm{Var}\left[\triangle X_k^{\lambda\gamma} | \mathfrak{F}_{k-1} \right] \,. \tag{10}$$

Intuitively, this means that for each k, the conditional distribution of X_k given $\mathfrak{F}_{k-1}$ is P-a.s. not concentrated at one point. This condition is similar to that of substantial risk in that it assumes a sufficiently random market. The structure condition (10) is needed along with an additional constraint on the variance to obtain our immediate goal of set containment $\Theta(X^{\lambda\gamma}) \subseteq \Theta(X)$. This, along with the reverse inclusion shown above, due to condition (8), is used in the next proposition to obtain set equivalence.

Proposition 2.3.1. *Assume X has bounded mean-variance tradeoff and substantial* *risk. Fix $\gamma \in \Gamma$, $\lambda \in \Lambda$ and assume there is a constant $c > 0$ such that*

$$Var\left[\triangle X_k^{\lambda\gamma}|\mathfrak{F}_{k-1}\right] \geq c \; Var[\triangle X_k|\mathfrak{F}_{k-1}] \; . \tag{11}$$

Then $X^{\lambda\gamma}$ has a bounded mean-variance tradeoff and moreover $\Theta(X^{\lambda\gamma}) = \Theta(X)$.

Proof. First, we show $X^{\lambda\gamma}$ has bounded mean-variance tradeoff (i.e. $(E[\triangle X_k^{\lambda\gamma}|\mathfrak{F}_{k-1}])^2 \leq c_{mvt}\mathrm{Var}\left[\triangle X_k^{\lambda\gamma}|\mathfrak{F}_k\right]$) To this end, consider

$$\begin{aligned}
&\triangle X_k^{\lambda\gamma} = X_k\left(1 + \lambda_k\gamma_k\right) - X_{k-1}\left(1 + \lambda_{k-1}\gamma_{k-1}\right) \\
&= X_k - X_{k-1} + X_k\lambda_k\gamma_k - X_{k-1}\lambda_{k-1}\gamma_{k-1} + X_{k-1}\lambda_k\gamma_k - X_{k-1}\lambda_k\gamma_k \\
&= \triangle X_k + \triangle X_k\lambda_k\gamma_k + X_{k-1}\triangle\lambda_k\triangle\gamma_k \, .
\end{aligned}$$

By definition $0 \leq \lambda_k \leq \lambda_0 < 1$ P-a.s. for each k, $-1 \leq \gamma_k \leq 1$ P-a.s. for each k and X_{k-1} is $\mathfrak{F}_{k-1}$-measurable, so

$$\begin{aligned}
&\triangle X_k + \triangle X_k\lambda_k\gamma_k + X_{k-1}\triangle\lambda_k\triangle\gamma_k \\
&\leq \left(1 + \lambda_0\right)\triangle X_k + 2\lambda_0 X_{k-1} \, .
\end{aligned}$$

Thus, since we assumed X has substantial risk and bounded mean-variance tradeoff, we have the following series of equations:

$$\begin{aligned}
&E\left[\left(\triangle X_k^{\lambda\gamma}\right)^2|\mathfrak{F}_{k-1}\right] \leq (1+\lambda_0)^2 E\left[\triangle X_k^2|\mathfrak{F}_{k-1}\right] + 4\lambda_0^2 X_{k-1}^2 \\
&\leq (1+\lambda_0)^2 E\left[\triangle X_k^2|\mathfrak{F}_{k-1}\right] + 4\lambda_0^2 c_{sr} E\left[\triangle X_k^2|\mathfrak{F}_{k-1}\right] \\
&= \left((1+\lambda_0)^2 + 4\lambda_0^2 c_{sr}\right) E\left[\triangle X_k^2|\mathfrak{F}_{k-1}\right] \\
&= m\left(\mathrm{Var}\left[\triangle X_k|\mathfrak{F}_{k-1}\right] + E\left[\triangle X_k|\mathfrak{F}_{k-1}\right]^2\right) \\
&\leq m\mathrm{Var}\left[\triangle X_k|\mathfrak{F}_{k-1}\right] + c_{mvt}\mathrm{Var}\left[\triangle X_k|\mathfrak{F}_{k-1}\right] \\
&= m(1 + c_{mvt})\mathrm{Var}\left[\triangle X_k|\mathfrak{F}_{k-1}\right] \, .
\end{aligned}$$

The second inequality follows from the assumption of substantial risk and inequality (8) using the constant c_{sr}. After the constant terms are collected ($m = (1+\lambda_0)^2 + 4\lambda_0^2 c_{sr}$) the final inequality follows from relation (10) and bounded mean-variance tradeoff with corresponding constant bound $c_{mvt}(0)$. Therefore, $X^{\lambda\gamma}$ has bounded mean-variance tradeoff.

Finally, we show that $\Theta(X^{\lambda\gamma}) \subseteq \Theta(X)$ which will lead to the desired result since the reverse inclusion was previously shown. With this in mind, let $\theta_k \in \Theta(X^{\lambda\gamma})$ which implies that $\theta_k \triangle X^{\lambda\gamma} \in L_2(P)$. Next, consider the Doob decomposition of this product $(X_k = X_0 + M_k + A_k)$, where $\Delta X_0 = 0$.

$$\theta_k \triangle X_k^{\lambda\gamma} = \theta_k \triangle M_k^{\lambda\gamma} + \theta_k \triangle A_k^{\lambda\gamma} .$$

Since $M_k^{\gamma\lambda}$ is a martingale and $A_k^{\gamma\lambda}$ is predictable, we have that

$$E\left[\triangle X_k^{\lambda\gamma} | \mathfrak{F}_{k-1}\right] = E\left[\triangle M_k^{\lambda\gamma} | \mathfrak{F}_{k-1}\right] + E\left[\triangle A_k^{\lambda\gamma} | \mathfrak{F}_{k-1}\right] = \triangle A_k^{\lambda\gamma} .$$

Thus,

$$\theta_k \triangle X_k^{\lambda\gamma} = \theta_k \triangle M_k^{\lambda\gamma} + \theta_k E\left[\triangle X_k^{\lambda\gamma} | \mathfrak{F}_{k-1}\right] .$$

Also, applying the Doob decomposition to the variance gives

$$\begin{aligned}
&\operatorname{Var}\left[\triangle X_k^{\lambda\gamma} \middle| \mathfrak{F}_{k-1}\right] = E\left[\left(\triangle X_k^{\lambda\gamma} - E\left[\triangle X_k^{\lambda\gamma} \middle| \mathfrak{F}_{k-1}\right]\right)^2 \middle| \mathfrak{F}_{k-1}\right] \\
&= E\left[\left(\triangle M_k^{\lambda\gamma} + \triangle A_k^{\lambda\gamma} - \triangle A_k^{\lambda\gamma}\right)^2 \middle| \mathfrak{F}_{k-1}\right] = E\left[\left(\triangle M_k^{\lambda\gamma}\right)^2 \middle| \mathfrak{F}_{k-1}\right] .
\end{aligned}$$

Also, since we assumed X has bounded mean-variance tradeoff, (10) implies

$$\left(E\left[\Delta X_k^{\lambda\gamma} | \mathfrak{F}_{k-1}\right]\right)^2 \leq c_{mvt}\, E\left[\left(\Delta M_k^{\lambda\gamma}\right)^2 \middle| \mathfrak{F}_{k-1}\right] \quad \text{for each } k = 1, \ldots, N .$$

Hence, $\theta_k \Delta X_k^{\lambda\gamma} \in L_2(P)$ if and only if $\theta_k \Delta M_k^{\lambda\gamma} \in L_2(P)$. Thus, we have the reverse implication for predictable θ_k i.e. $\theta_k \Delta M_k^{\lambda\gamma} \in L_2(P)$ implies $\theta_k \Delta X_k^{\lambda\gamma} \in L_2(P)$. So, $\theta_k \in \Theta(X^{\lambda\gamma})$ if and only if $\theta_k \Delta M_k^{\lambda\gamma} \in L_2(P)$. Now, θ_k predictability and inequality

(10) together imply that

$$
\begin{aligned}
&E\left[(\theta_k \Delta M_k)^2 \,|\mathfrak{F}_{k-1}\right] = \theta_k^2 E\left[(\Delta M_k)^2 \,|\mathfrak{F}_{k-1}\right] = \theta_k^2 \mathrm{Var}\left[\Delta X_k |\mathfrak{F}_{k-1}\right] \\
&\leq \tfrac{1}{c_{mvt}} \theta_k^2 \mathrm{Var}\left[\Delta X_k^{\lambda\gamma} |\mathfrak{F}_{k-1}\right] = \tfrac{1}{c_{mvt}} \theta_k^2 E\left[\left(\Delta M_k^{\lambda\gamma}\right)^2 |\mathfrak{F}_{k-1}\right] \\
&= \tfrac{1}{c_{mvt}} E\left[\left(\theta_k \Delta M_k^{\lambda\gamma}\right)^2 |\mathfrak{F}_{k-1}\right].
\end{aligned}
$$

Hence, $\theta_k \Delta M_k^{\lambda\gamma} \in L_2(P)$ implies $\theta_k \Delta M_k \in L_2(P)$, which means that $\theta_k \in \Theta(X^{\lambda\gamma})$ implies $\theta_k \in \Theta(X)$. Therefore, we have the desired inclusion $\Theta(X^{\lambda\gamma}) \subseteq \Theta(X)$ and the proposition is proved. □

Thus, with some mild technical assumptions on the process X, we have that the sets $\Theta(X)$ and $\Theta(X^{\lambda\gamma})$ are equal. This means that the same strategies can be used regardless of the specific units. Hence, there is ultimately no structural difference between the techniques used in a market model with and without transaction costs. Note that both technical assumptions, bounded mean-variance tradeoff and substantial risk, regard the underlying randomness of the market. These are both very natural and non-restrictive conditions.

To be sure that the previous lemma holds, the condition (11) must hold. In order to check this condition on the variance, γ_k and λ_k must be sufficiently known. However, since both quantities are random variables this is not very likely. Therefore, it's useful to impose an additional auxiliary condition independent of both γ and λ that assures condition (11) holds.

Proposition 2.3.2. *If there is a constant $\delta < 1$ such that*

$$
2\sqrt{\frac{E\left[X_k^2|\mathfrak{F}_{k-1}\right]}{Var\left[\triangle X_k|\mathfrak{F}_{k-1}\right]}} \leq \delta \quad P\text{-}a.s. \text{ for } k = 1, ..., N \tag{12}
$$

then inequality (11) holds simultaneously for all $\gamma \in \Gamma$, with $c = 1 - \delta$.

Proof. Since $X_k^{\lambda\gamma} = X_k(1+\lambda_k\gamma_k)$, we can write $\triangle X_k^{\lambda\gamma} = \triangle X_k + X_k\lambda_k\gamma_k - X_{k-1}\lambda_{k-1}\gamma_{k-1}$. Also, since we're only concerned with the conditional variance, we drop the $\mathfrak{F}_{k-1}$-

measurable terms. Doing this and rewriting the left-hand side of (11) using definition of variance, we get the following series of equations:

$$
\begin{aligned}
&\operatorname{Var}\left[\triangle X_k^{\lambda\gamma}|\mathfrak{F}_{k-1}\right] = \operatorname{Var}\left[\triangle X_k + X_k\lambda_k\gamma_k|\mathfrak{F}_{k-1}\right] \\
&= E\left[\left((\triangle X_k + X_k\lambda_k\gamma_k) - E\left[\triangle X_k + X_k\lambda_k\gamma_k\right]\right)^2|\mathfrak{F}_{k-1}\right] \\
&= E\left[\left(\triangle X_k - E\left[\triangle X_k|\mathfrak{F}_{k-1}\right] + X_k\lambda_k\gamma_k - E\left[X_k\lambda_k\gamma_k|\mathfrak{F}_{k-1}\right]\right)^2\right] \\
&= E[\left(\triangle X_k - E\left[\triangle X_k|\mathfrak{F}_{k-1}\right]\right)^2 + 2\left(\triangle X_k - E\left[\triangle X_k|\mathfrak{F}_{k-1}\right]\right)\left(X_k\lambda_k\gamma_k - E\left[X_k\lambda_k\gamma_k|\mathfrak{F}_{k-1}\right]\right) \\
&\qquad + \left(X_k\lambda_k\gamma_k - E\left[X_k\lambda_k\gamma_k|\mathfrak{F}_{k-1}\right]\right)^2|\mathfrak{F}_{k-1}] \\
&\geq \operatorname{Var}\left[\triangle X_k|\mathfrak{F}_{k-1}\right] - 2E\left[\left(\triangle X_k - E\left[\triangle X_k|\mathfrak{F}_{k-1}\right]\right)\left(X_k\lambda_k\gamma_k - E\left[X_k\lambda_k\gamma_k|\mathfrak{F}_{k-1}\right]\right)|\mathfrak{F}_{k-1}\right] \\
&= \operatorname{Var}\left[\triangle X_k|\mathfrak{F}_{k-1}\right] - 2\int\left(\triangle X_k - E\left[\triangle X_k|\mathfrak{F}_{k-1}\right]\right)\left(X_k\lambda_k\gamma_k - E\left[X_k\lambda_k\gamma_k|\mathfrak{F}_{k-1}\right]\right)dP \\
&\geq \operatorname{Var}\left[\triangle X_k|\mathfrak{F}_{k-1}\right] - 2\sqrt{\int\left(\triangle X_k - E\left[\triangle X_k|\mathfrak{F}_{k-1}\right]\right)^2 dP\int\left(X_k\lambda_k\gamma_k - E\left[X_k\lambda_k\gamma_k|\mathfrak{F}_{k-1}\right]\right)^2 dP} \\
&= \operatorname{Var}\left[\triangle X_k|\mathfrak{F}_{k-1}\right] - 2\sqrt{\operatorname{Var}\left[\triangle X_k|\mathfrak{F}_{k-1}\right]\operatorname{Var}\left[X_k\lambda_k\gamma_k|\mathfrak{F}_{k-1}\right]}\,.
\end{aligned}
$$

After rewriting variance in terms of its underlying expectations, we get a lower bound by excluding positive terms. The last estimate is obtained by using the Cauchy-Schwartz inequality. Therefore, we obtain a lower bound on the conditional variance. Now, since $-1 < -\lambda_0 \leq \lambda_k\gamma_k \leq \lambda_0 < 1$, P-a.s for all k, we have an additional relation between conditional variances

$$
\operatorname{Var}\left[X_k\lambda_k\gamma_k|\mathfrak{F}_{k-1}\right] \leq E\left[(X_k\lambda_k\gamma_k)^2|\mathfrak{F}_{k-1}\right] \leq E\left[X_k^2|\mathfrak{F}_{k-1}\right] \leq \frac{\delta^2}{4}\operatorname{Var}\left[\triangle X_k|\mathfrak{F}_{k-1}\right]\,.
$$

Therefore, using this estimate we get the desired lower bound.

$$
\begin{aligned}
&\operatorname{Var}\left[\triangle X_k^{\lambda\gamma}\middle|\mathfrak{F}_{k-1}\right] \geq \\
&\geq \operatorname{Var}\left[\triangle X_k|\mathfrak{F}_{k-1}\right] - 2\sqrt{\operatorname{Var}\left[\triangle X_k|\mathfrak{F}_{k-1}\right]\operatorname{Var}\left[X_k\lambda_k\gamma_k|\mathfrak{F}_{k-1}\right]} \\
&\geq \operatorname{Var}\left[\triangle X_k|\mathfrak{F}_{k-1}\right] - 2\sqrt{\operatorname{Var}\left[\triangle X_k|\mathfrak{F}_{k-1}\right]\tfrac{\delta^2}{4}\operatorname{Var}\left[\triangle X_k|\mathfrak{F}_k\right]} \\
&= \operatorname{Var}\left[\triangle X_k|\mathfrak{F}_{k-1}\right] - \delta\operatorname{Var}\left[\triangle X_k|\mathfrak{F}_{k-1}\right] \\
&= \operatorname{Var}\left[\triangle X_k|\mathfrak{F}_{k-1}\right](1-\delta)\,.
\end{aligned}
$$

Hence, $\operatorname{Var}\left[\triangle X_k^{\lambda\gamma}|\mathfrak{F}_{k-1}\right] \geq c\operatorname{Var}\left[\triangle X_k|\mathfrak{F}_{k-1}\right]$ for positive constant $c = 1-\delta$. Therefore, imposing condition (12), which does not depend on λ or γ, on the conditional variance of $\Delta X_k^{\gamma\lambda}$ results in the desired lower bound (11). $\square$

2.4. Existence and Structure of an Optimal Strategy

Now that we've established the technical assumptions necessary to properly implement the model, we address more general questions regarding existence and uniqueness of minimizers. That is, we're ultimately interested in determining a locally risk-minimizing strategy which must satisfy proposition 2.2.2 above. So, going backward from a contingent claim (θ_{N+1}, η_N), we determine θ_{k+1} over each finite time interval such that

$$\begin{aligned}\theta_{k+1} &= \underset{\theta'_{k+1}\in\Theta(X)}{\operatorname{argmin}} \operatorname{Var}\left[\triangle C_{k+1}|\mathfrak{F}_k\right] \\ &= \underset{\theta'_{k+1}\in\Theta(X)}{\operatorname{argmin}} \operatorname{Var}\left[V_{k+1} - \theta'_{k+1}\triangle X_{k+1} + X_{k+1}\lambda_{k+1}\left|\theta_{k+2} - \theta'_{k+1}\right|\left|\mathfrak{F}_k\right.\right].\end{aligned}$$

Therefore, it behooves us to be sure that each local minimizer exists and is unique. To this end, we consider the general conditional variance function defined as:

$$f(c,\omega) := \operatorname{Var}\left[U - cZ + \lambda Z\left|Y - c\right|\left|\mathfrak{G}\right.\right](\omega),$$

where we assume $\mathfrak{G} \subseteq \mathfrak{F}$ is a sub sigma algebra of $\mathfrak{F}$. Also, let U, Y, and Z be $\mathfrak{F}$-measurable, real-valued random variables such that $U \in L_2(P)$, $Z \in L_2(P)$ and $YZ \in L_2(P)$. Additionally λ is a uniformly bounded random variable such that $\lambda_k \in [0,1)$ P-a.s. for each k. First, we construct the left and right-hand derivatives of this general conditional variance function, which will be used to prove subsequent existence results.

Lemma 2.4.1. *For P-almost every ω, $c \mapsto f(c,\omega)$ is a continuous function with left and right-hand derivatives $f'_-(c,\omega)$ and $f'_+(c,\omega)$ given by*

$$f'_-(c,\omega) = -2Cov(U - cZ + \lambda Z|Y - c|, Z(1 + \lambda\overline{sign}(Y - c))\big|\mathfrak{G})(\omega)$$

and

$$f'_+(c,\omega) = -2Cov(U - cZ + \lambda Z|Y - c|, Z(1 + \lambda \underline{sign}(Y - c))|\mathfrak{G})(\omega)\,,$$

where we define

$$\begin{aligned} \overline{sign}(x) &:= sign(x) + I_{\{x=0\}} := \begin{cases} +1 & x \geq 0 \\ -1 & x < 0 \end{cases}, \\ \underline{sign}(x) &:= sign(x) - I_{\{x=0\}} := \begin{cases} +1 & x > 0 \\ -1 & x \leq 0 \end{cases}. \end{aligned} \tag{13}$$

Proof. The continuity in c of $f(c,\omega)$ is evident by the definition of conditional variance. Also, by symmetry it suffices to determine the right-hand derivative $f'_+(c,\omega)$. We do this by computing the corresponding difference quotient. In the following list of equalities, we write the definition of f in terms of conditional variance and decompose the result in terms of the underlying expectations. Collecting the appropriate terms leads to the conditional expectations shown:

$$\begin{aligned} & f(c+h,\omega) - f(c,\omega) \\ &= \mathrm{Var}\left[U - (c+h)Z + \lambda Z|Y - (c+h)||\mathfrak{G}\right](\omega) - \mathrm{Var}\left[U - cZ + \lambda Z|Y - c||\mathfrak{G}\right](\omega) \\ &= E\left[(U - (c+h)Z + \lambda Z|Y - (c+h)|)^2 |\mathfrak{G}\right](\omega) \\ &\qquad - \left(E\left[U - (c+h)Z + \lambda Z|Y - (c+h)||\mathfrak{G}\right](\omega)\right)^2 \\ & -E\left[(U - cZ + \lambda Z|Y - c|)^2 |\mathfrak{G}\right](\omega) \\ &\qquad + \left(E\left[U - cZ + \lambda Z|Y - c||\mathfrak{G}\right](\omega)\right)^2 \\ &= E\left[(U - (c+h)Z + \lambda Z|Y - (c+h)|)^2 - (U - cZ + \lambda Z|Y - c|)^2 \middle|\mathfrak{G}\right] \\ &\qquad - \left(E\left[U - (c+h)Z + \lambda Z|Y - (c+h)||\mathfrak{G}\right]^2 - E\left[U - cZ + \lambda Z|Y - c||\mathfrak{G}\right]^2\right). \end{aligned}$$

Next, to achieve the desired result we cleverly rearrange the previous two terms involving conditional expectation. In the interest of simplicity, we make the following substitutions:

$$\kappa = U - (c+h)Z + \lambda Z\,|Y - (c+h)|$$
$$\tau = U - cZ + \lambda Z\,|Y - c|\ .$$

With these substitutions, the expression obtained above simplifies to

$$f(c+h,\omega) - f(c,\omega) = E[\kappa^2 - \tau^2|\mathfrak{G}] - \left(E[\kappa|\mathfrak{G}]^2 - E[\tau|\mathfrak{G}]^2\right)\ . \tag{14}$$

Note that the terms inside of the first expectation of (14) can be rewritten as the following product of terms:

$$\begin{aligned}
&\kappa^2 - \tau^2 = (\kappa - \tau)(\kappa + \tau)\\
&= (-hZ + \lambda Z(|Y - (c+h)| - |Y - c|))\\
&\qquad (2U - 2cZ - hZ + \lambda Z(|Y - (c+h)| + |Y - c|))\ .
\end{aligned}$$

Similarly, the second component of equation (14) can be written as:

$$\begin{aligned}
&(E[\tau|\mathfrak{G}])^2 - (E[\kappa|\mathfrak{G}])^2\\
&= \left(E\left[U - cZ + \lambda Z\,|Y - c|\,|\mathfrak{G}\right] - E\left[U - (c+h)Z + \lambda\,|Y - (c+h)|\,|\mathfrak{G}\right]\right)\\
&\qquad \left(E\left[U - cZ + \lambda Z\,|Y - c|\,|\mathfrak{G}\right] + E\left[U - (c+h)Z + \lambda Z\,|Y - (c+h)|\,|\mathfrak{G}\right]\right)\\
&= E\left[-hZ + \lambda Z\left(|Y - c| - |Y - (c+h)|\right)|\mathfrak{G}\right]\\
&\qquad E\left[2U - 2cZ - hZ + \lambda Z\left(|Y - c| + |Y - (c+h)|\right)|\mathfrak{G}\right]\ .
\end{aligned}$$

Thus, putting all of this together we obtain the following expression:

$$\begin{aligned}
&f(c+h,\omega) - f(c,\omega) =\\
&= E[(-hZ + \lambda Z\left(|Y - (c+h)| - |Y - c|\right))\\
&\qquad \left(2U - 2cZ - hZ + \lambda Z\left(|Y - (c+h)| + |Y - c|\right)|\mathfrak{G}\right]\\
&-E\left[-hZ + \lambda Z\left(|Y - c| - |Y - (c+h)|\right)|\mathfrak{G}\right]\\
&\qquad \cdot E\left[2U - 2cZ - hZ + \lambda Z\left(|Y - c| + |Y - (c+h)|\right)|\mathfrak{G}\right]\\
&= \mathrm{Cov}[-hZ + \lambda Z(|Y - c| - |Y - (c+h)|),\\
&\qquad 2U - 2cZ - hZ + \lambda Z(|Y - c| + |Y - (c+h)|)|\mathfrak{G}]\ .
\end{aligned}$$

This allows one to construct the desired difference quotient, which is expressed as the following covariance function:

$$\frac{f(c+h,\omega)-f(c,\omega)}{h} =$$
$$= \operatorname{Cov}\left[-Z + \lambda Z \left(\frac{|Y-c-h|-|Y-c|}{h}\right),\right.$$
$$\left. 2U - 2cZ - hZ + \lambda Z \left(|Y-c-h| + |Y-c|\right) | \mathfrak{G}\right].$$

Note that $\lim_{h\to 0^+} \operatorname{sign}(Y-c-h) = \operatorname{sign}(Y-c)$, and we have that

$$\lim_{h\to 0^+} \frac{|Y-c-h|-|Y-c|}{h} = -\underline{\operatorname{sign}}\,(Y-c) \quad \text{P-a.s.}$$

Also, by the triangle inequality we have that $|Y-c-h| \le |Y-c| + |-h|$, which means that $|Y-c-h| - |Y-c| \le |-h|$. Thus we have the following important bound:

$$||Y-c-h| - |Y-c|| \le h \,\text{for a.e.}\,\omega \text{ and for all } h.$$

Now, since we've established the limit

$$\lim_{h\to 0^+} \frac{|Y-c-h|-|Y-c|}{h} = -\underline{\operatorname{sign}}(Y-c)$$

and since $\left|\frac{|Y-c-h|-|Y-c|}{h}\right| \le 1$, we can apply the dominated convergence theorem yielding the following series of equalities:

$$\lim_{h\to 0^+} \frac{f(c+h,\omega)-f(c,\omega)}{h} =$$
$$= \lim_{h\to 0^+} \operatorname{Cov}\left[-Z + \lambda Z \frac{|Y-c-h|-|Y-c|}{h},\right.$$
$$\left. 2U - 2cZ - hZ + \lambda Z \left(|Y-c-h| + |Y-c|\right) | \mathfrak{G}\right]$$
$$= \operatorname{Cov}\left[-Z + \lim_{h\to 0^+} \lambda Z \frac{|Y-c-h|-|Y-c|}{h},\right.$$
$$\left. 2U - 2cZ + \lim_{h\to 0^+} \left(-hZ + \lambda Z \left(|Y-c-h| + |Y-c|\right)\right) | \mathfrak{G}\right]$$
$$= \operatorname{Cov}\left[-Z - \lambda Z \underline{\operatorname{sign}}\,(Y-c), 2U - 2cZ + 2\lambda Z\,|Y-c|\, | \mathfrak{G}\right].$$

Therefore, we have successfully constructed the right-hand derivative of the general conditional variance function $f(c,\omega)$.

$$\begin{aligned} f'_+(c,\omega) &= \lim_{h\to 0+} \frac{f(c+h,\omega)-f(c,\omega)}{h} \\ &= \mathrm{Cov}\left[-Z-\lambda Z\underline{\mathrm{sign}}\,(Y-c)\,, 2U-2Z+2\lambda Z\,|Y-c|\,\middle|\mathfrak{G}\right](\omega)\,. \quad (15) \end{aligned}$$

By a symmetrical argument, the left-hand derivative is computed. □

Next, we get at the essential ingredient to ensure the existence of a minimizer for the problem at hand.

Proposition 2.4.1. *Suppose $Var[Z|\mathfrak{G}] > 0$ P-a.s., then there exists a $\mathfrak{G}$-measurable random variable $c^*(\omega)$ such that for P - almost every ω,*

$$f(c^*(\omega),\omega) \leq f(c,\omega) \quad \textit{for all } c \in \mathbb{R}\,.$$

Proof. To prove this statement, we first show that $f(c,\omega)$ has a minimum in c then we construct such a minimizer. With the first goal in mind, we decompose $f(c,\omega)$ into its variance and covariance components. For notational simplicity, let $s = \mathrm{sign}\,(Y-c)$.

Since $|Y-c| = (Y-c)\,\mathrm{sign}\,(Y-c)$, we can decompose $f(c,\omega)$ as follows:

$$f(c(\omega),\omega) = \mathrm{Var}\left[U - cZ + \lambda Z\,|Y-c|\,|\mathfrak{G}\right](\omega)$$

$$= \mathrm{Var}\left[U - cZ + \lambda Z\,(Y-c)\,s|\mathfrak{G}\right](\omega)$$

$$= \mathrm{Var}\left[(U + \lambda ZYs) - c\,(Z(1+\lambda s))\,|\mathfrak{G}\right](\omega)$$

$$= E\left[((U+\lambda ZYs) - c\,(Z(1+\lambda s)))^2\,|\mathfrak{G}\right](\omega) - E\left[(U+\lambda ZYs) - c\,(Z(1+\lambda s))\,|\mathfrak{G}\right]^2(\omega)$$

$$= E\left[(U+\lambda ZYs)^2 - 2c\,(U+\lambda ZYs)\,(Z(1+\lambda s)) + c^2\,(Z(1+\lambda s))^2\,|\mathfrak{G}\right](\omega)$$

$$= E\left[(U+\lambda ZYs)^2\,|\mathfrak{G}\right] - 2cE\left[(U+\lambda ZYs)\,(Z(1+\lambda s))\,|\mathfrak{G}\right]$$

$$+c^2E\left[(Z(1+\lambda s))^2\,|\mathfrak{G}\right](\omega) - E\left[U+\lambda ZYs|\mathfrak{G}\right]^2(\omega)$$

$$+2cE\left[U+\lambda ZYs|\mathfrak{G}\right]E\left[Z(1+\lambda s)|\mathfrak{G}\right](\omega) - c^2E\left[Z(1+\lambda s)|\mathfrak{G}\right]^2(\omega)$$

$$= \mathrm{Var}\left[U+\lambda ZYs|\mathfrak{G}\right](\omega) - 2c\,\mathrm{Cov}\left[(U+\lambda ZYs)\,,Z\,(1+\lambda s)\,|\mathfrak{G}\right](\omega)$$

$$+c^2\,\mathrm{Var}\left[Z(1+\lambda s)|\mathfrak{G}\right](\omega)\,.$$

Now let

$$h_1(c,\omega) := \mathrm{Var}\left[Z(1+\lambda s)|\mathfrak{G}\right](\omega)\,,$$
$$h_2(c,\omega) := \mathrm{Cov}\left[(U+\lambda ZYs)\,,Z(1+\lambda s)|\mathfrak{G}\right](\omega)\,,$$
$$h_3(c,\omega) := \mathrm{Var}\left[U+\lambda ZYs|\mathfrak{G}\right](\omega)\,.$$

This, along with the formula for $f(c,\omega)$ formulated above results in the following representation of function $f(c,\omega)$:

$$f(c,\omega) = c^2h_1(c,\omega) - 2ch_2(c,\omega) + h_3(c,\omega)\,.$$

Since, by assumption $U \in L_2(P)$, $Z \in L_2(P)$, $YZ \in L_2(P)$, and $\lambda \in [0,1)$ P-a.s., we have that $h_1(c,\omega)$, $h_2(c,\omega)$, and $h_3(c.\omega)$ are all bounded as functions of c. We need only check that $h_1(c,\omega)$ doesn't go negative almost surely since it is clearly the leading coefficient. To that end, consider the limit of this function evaluated as follows:

$$\begin{aligned}&\lim_{|c|\to\infty} \operatorname{Var}\left[Z(1+\lambda\operatorname{sign}(Y-c))|\mathfrak{G}\right](\omega)\\&= \operatorname{Var}\left[Z(1\mp\lambda)|\mathfrak{G}\right](\omega)\\&= (1\mp\lambda)\operatorname{Var}\left[Z|\mathfrak{G}\right](\omega)\,.\end{aligned}$$

Since $0 < \lambda \leq \lambda_0 < 1$ P-a.s. and $\operatorname{Var}\left[Z|\mathfrak{G}\right](\omega)$ is assumed positive, this implies that

$$\lim_{|c|\to\infty} \operatorname{Var}\left[Z(1+\lambda s)|\mathfrak{G}\right](\omega) > 0\,.$$

Therefore,

$$\lim_{|c|\to\infty} c^2 h_1(c,\omega) - 2ch_2(c,\omega) + h_3(c,\omega) = +\infty$$

for P-almost every ω, and since $h_1(c,\omega)$, $h_2(c,\omega)$, and $h_3(c,\omega)$ are continuous in c, $f(c,\omega)$ must have a minimum value P-almost everywhere. Next, we construct such a $\mathfrak{G}$-measurable minimizer. Thus, consider the dyadic integers of order n, $D_n = \{\frac{j}{2^n}|j \in \mathbb{Z}\}$. Then, define he sequence

$$c_n(\omega) := \inf\{c \in D_n : f(c,\omega) \leq f(c',\omega) \text{ for all } c' \in D_n\}\,.$$

Since, by definition the mapping $\omega \mapsto f(c,\omega)$ is $\mathfrak{G}$-measurable for fixed c, the random variable $c_n(\omega)$ is $\mathfrak{G}$-measurable for every n. That is, for arbitrary $x \in \mathbb{R}$ we have

$$\{c_n(\omega) \leq x\} = \bigcup_{c\leq x, c\in D_n} \bigcap_{c'\in D_n} \{\omega : f(c',\omega) \geq f(c,\omega)\}\,.$$

Each f is $\mathfrak{G}$-measurable, therefore each set $\{\omega : f(c',\omega) \geq f(c,\omega)\}$ is $\mathfrak{G}$-measurable. Hence, $\{c_n(\omega) \leq x\}$ is $\mathfrak{G}$-measurable. Since $\lim_{|c|\to\infty} f(c,\omega) = \infty$, the sequence $(c_n)_{n\in\mathbb{N}}$ is bounded in n for P-almost every ω. Also, for any finite subset of $\mathbb{R}$, D_n is finite for any $n \in \mathbb{N}$. Thus, over each finite n, the sequence $c_n(\omega)$ attains a minimum. Finally, the set of dyadic integers is dense in $\mathbb{R}$. Therefore, along with continuity of $f(c,\omega)$ in

c, the above properties imply the minimum c is attained defined by c^* as:

$$c^*(\omega) := \liminf_{n\to\infty} c_n(\omega) \qquad \text{P-a.s.}$$

□

Proposition 2.4.1 guarantees the existence of a minimizer $c^*(\omega)$ for the general conditional variance function $f(c,\omega)$. However, this result only concerns the existence of a minimizer and not the properties of $c^*(\omega)$. Therefore, with this in mind, we introduce the function $g(c,\beta,\omega)$ (with $0 \le \beta \le 1$) which transitions between the left and right-hand derivatives of $f(c,\omega)$. Thus,

$$g(c(\omega),\beta(\omega),\omega) := \text{Cov}\left[U - cZ + \lambda_k Z\,|Y-c|\,, Z\left(1+\lambda_k S(\beta,c)\right)|\mathfrak{G}\right](\omega)\,, \tag{16}$$

where

$$S(\beta,c) := \beta\overline{\text{sign}}\,(Y-c) + (1-\beta)\,\underline{\text{sign}}\,(Y-c)\ . \tag{17}$$

Note that $S(\beta,c)$ has been defined so that $g(c,\beta,\omega)$ transitions between $f'_+(c,\omega)$ and $f'_-(c,\omega)$ as $\beta(\omega)$ takes on values from $[0,1]$. In particular,

$$\begin{aligned} g(c,0,\omega) &= \text{Cov}\left[U - cZ + \lambda Z\,|Y-c|\,, Z\left(1+\lambda\underline{\text{sign}}(Y-c)\right)\Big|\mathfrak{G}\right](\omega) \\ &= -\tfrac{1}{2}f'_+(c,\omega)\,. \end{aligned}$$

$$\begin{aligned} g(c,1,\omega) &= \text{Cov}\left[U - cZ + \lambda Z\,|Y-c|\,, Z\left(1+\lambda\overline{\text{sign}}(Y-c)\right)\Big|\mathfrak{G}\right](\omega) \\ &= -\tfrac{1}{2}f'_-(c,\omega)\,. \end{aligned}$$

Next, we prove the existence of an optimal random variable $\beta^*(\omega)$ that is exactly between $f'_+(c,\omega)$ and $f'_-(c,\omega)$. That is, for $c^*(\omega)$ given, there is a $\beta^*(\omega) \in [0,1]$ such that $g(c^*,\beta^*,\omega) = 0$. This equation will be used to determine an analytic expression for the optimal c.

Proposition 2.4.2. *Assume that $Var[Z|\mathfrak{G}] > 0$ P-a.s. and let $c^*(\omega)$ be given as in Proposition 2.4.1. Then there exists a $\mathfrak{G}$-measurable random variable $\beta^*(\omega)$ with* *values in* $[0,1]$ *such that*

$$g(c^*(\omega), \beta^*(\omega), \omega) = 0 \text{ for } P\text{-almost every } \omega\,.$$

Proof. Since c^* minimizes $c \mapsto f(c,\omega)$ for P-almost every ω,

$$f'_-(c^*(\omega), \omega) \le 0 \le f'_+(c^*(\omega), \omega) \qquad P - a.s.$$

Also, by the above discussion $0 \ge g(c^*(\omega), 0, \omega) = -\frac{1}{2} f'_+(c^*(\omega), \omega)$, $g(c^*(\omega), 1, \omega) = -\frac{1}{2} f'_-(c^*(\omega), \omega) \ge 0$ so

$$g(c^*, 0, \omega) \le 0 \le g(c^*, 1, \omega) \qquad \text{P-a.s.}$$

Now, let

$$\beta_n(\omega) = \text{argmin}\{|g(c_n(\omega), \beta, \omega)| \,:\, \beta \in D_n\}\,.$$

By properties already established for $f(c,\omega)$, it follows that $\omega \mapsto g(c^*(\omega), \beta, \omega)$ is $\mathfrak{G}$-measurable for every fixed β and $\beta \mapsto g(c^*(\omega), \beta, \omega)$ is P-almost surely continuous on $[0,1]$. Thus, for P-almost every ω, $\beta \mapsto g(c^*(\omega), \beta, \omega)$ has a zero in $[0,1]$, so that

$$\min_{\beta \in [0,1]} |g(c^*(\omega), \beta, \omega)| = 0 \qquad \text{P-a.s.}$$

□

Now that we've established the existence of a random variable $\beta^*(\omega)$ such that $g(c^*, \beta^*, \omega) = 0$ for P-almost every ω, we can use this equation to get an expression for the minimum $c^*(\omega)$ in the context of the current problem. Therefore, we rearrange g

to solve for the optimal c, using the fact that $\mathrm{Cov}(A, A+B) = \mathrm{Cov}(A,B) + \mathrm{Var}(A)$.

$$
\begin{aligned}
g(c^*, \beta^*, \omega) &= \mathrm{Cov}\left(U - c^* Z + \lambda Z \left|Y - c^*\right|, Z((1+\lambda S(\beta, c^*)) \,|\mathfrak{G}\right)(\omega) \\
&= \mathrm{Cov}\left(U - c^* Z + \lambda Z\left(Y - c^*\right) S(\beta, c^*), Z\left(1+\lambda S(\beta, c^*)\right) |\mathfrak{G}\right)(\omega) \\
&= \mathrm{Cov}\left(U + \lambda Z Y S(\beta, c^*) - c^*\left(Z\left(1+\lambda S(\beta, c^*)\right)\right), Z\left(1+\lambda S(\beta, c^*)\right)\right)(\omega) \\
&= \mathrm{Cov}\left(U + \lambda Z Y S(\beta, c^*), Z\left(1+\lambda S(\beta, c^*)\right) |\mathfrak{G}\right)(\omega) \\
&\quad - c^* \mathrm{Var}\left[Z\left(1+\lambda S(\beta, c^*)\right) |\mathfrak{G}\right](\omega) = 0\,.
\end{aligned}
$$

Hence, we obtain a relation that c^* should satisfy for an analytically tractable formula of the minimizer of $f(c, \omega)$.

$$
c^*(\omega) = \frac{\mathrm{Cov}\left(U + \lambda Z Y S(\beta, c^*), Z\left(1+\lambda S(\beta, c^*)\right) |\mathfrak{G}\right)(\omega)}{\mathrm{Var}\left[Z\left(1+\lambda S(\beta, c^*)\right) |\mathfrak{G}\right](\omega)}\,. \tag{18}
$$

2.5. The Main Result

Now that we have established the general framework of our model and carefully checked that local minimizers exist, we are ready to state the main theorem. The statement and subsequent proof of the following theorem follow directly from all of the previous results. The main point is to be sure all the proper integrability conditions are satisfied and to concisely record our main result. In fact, the main theorem below formalizes the result of proposition 2.2.2 in terms of transaction costs, and uses generalized existence results concerning the functions $f(c, \omega)$ and $g(c, \beta, \omega)$ to determine an optimal strategy.

Theorem 2.5.1. *Assume that X has a bounded mean-variance tradeoff for $\lambda = 0$, substantial risk, satisfies (12) and*

$$
Var\left[\triangle X_k \middle| \mathfrak{F}_{k-1}\right] > 0 \;\; P\text{-almost surely for } k = 1, \ldots, N\,.
$$

Then for any contingent claim $(\bar{\theta}_{N+1}, \bar{\eta}_N)$, there exists a locally risk-minimizing strat-

egy $\phi^* = (\theta^*, \eta^*)$ *with* $\theta^*_{N+1} = \bar{\theta}_{N+1}$ *and* $\eta^*_N = \bar{\eta}_N$. *The first component of this optimal strategy is constructed as follows. There exists a process* $\delta^* \in \Gamma$ *such that if we define* $\xi \in \Gamma$ *by*

$$\xi_k := \begin{cases} sign(\theta^*_{k+1} - \theta^*_k) + \delta^* I_{\{\theta^*_{k+1} = \theta^*_k\}} & for\ k = 1, ..., N \\ \\ 0 & for\ k = 0 \end{cases} \tag{19}$$

then we have that the first component in question is

$$\theta^*_k = \frac{Cov\left[\triangle V^{\lambda\xi}_k(\phi^*), \triangle X^{\lambda\xi}_k \middle| \mathfrak{F}_{k-1}\right]}{Var\left[\triangle X^{\lambda\xi}_k \middle| \mathfrak{F}_{k-1}\right]} \quad P\text{-}a.s.\ for\ k = 1, ..., N\,. \tag{20}$$

Proof. We prove the required integrability and existence criteria by a backward induction argument and then construct the optimal strategy ϕ^*. To begin, we define $\theta^*_{N+1} := \bar{\theta}_{N+1}$, which by definition 2.1.3 implies that the following hold for $k = N$

$$\theta^*_{k+1} X_k \in L_2(P)\,,$$
$$W^*_k := H - \sum_{j=k+1}^N \theta^*_j \triangle X_j + \sum_{j=k+1}^N \lambda_j X_j \left|\triangle \theta^*_{j+1}\right| \in L_2(P)\,.$$

Now, assume the above conditions hold for $k = 1, \ldots, N$. We're interested in determining θ^*_k assuming θ^*_{k+1} is given for each k, so we define the following (where the general random variable Y is replaced by θ^*_{k+1}):

$$f_k(c, \omega) := \mathrm{Var}\left[E\left[W^*_k|\mathfrak{F}_k\right] - cX_k + \lambda_k X_k \left|\theta^*_{k+1} - c\right| \middle| \mathfrak{F}_{k-1}\right](\omega)\,,$$

and

$$g_k(c, \beta, \omega) := \mathrm{Cov}\left[E\left[W^*_k|\mathfrak{F}_k\right] - cX_k + \lambda_k X_k \left|\theta^*_{k+1} - c\right|, X_k\left(1 + \lambda_k S_k(\beta, c)\right) \middle| \mathfrak{F}_{k-1}\right](\omega)\,.$$

$S_k(\beta, c)$ is defined so that $\overline{\text{sign}}$ and $\underline{\text{sign}}$ do not appear separately:

$$\begin{aligned} S_k(\beta, c) &:= \beta\overline{\text{sign}}\left(\theta^*_{k+1} - c\right) + (1-\beta)\,\underline{\text{sign}}\left(\theta^*_{k+1} - c\right) \\ &= \text{sign}\left(\theta^*_{k+1} - c\right) + (2\beta(\omega) - 1)\, I_{\{\theta^*_{k+1} = c\}}\,. \end{aligned}$$

In propositions 2.4.1 and 2.4.2 we established the existence of $\mathfrak{F}_{k-1}$-measurable random variables θ^*_k and β^*_k such that

$$f_k(\theta^*_k(\omega), \omega) \leq f_k(c, \omega) \quad \text{P-a.s. for all } c \in \mathbb{R} \text{ and for each } k$$

and

$$g_k(\theta^*_k(\omega), \beta^*_k(\omega), \omega) = 0 \quad \text{P-a.s. for each } k\,.$$

For notational simplicity, we define the process $\delta^*_k := 2\beta^*_k - 1$, so we have

$$S_k(\beta^*(\omega), \theta^*_k(\omega), \omega) = \text{sign}\left(\theta^*_{k+1} - \theta^*_k\right) + \delta^*_k I_{\{\theta^*_{k+1} = \theta^*_k\}} = \xi_k\,.$$

Therefore, we reformulate the general equation (18) into the context of the present theorem as follows:

$$\theta^*_k(\omega) = \frac{\text{Cov}\left[E\left[W^*_k \middle| \mathfrak{F}_k\right] + \lambda_k X_k \xi_k \theta^*_{k+1}, X_k\left(1 + \lambda_k \xi_k\right) \middle| \mathfrak{F}_{k-1}\right](\omega)}{\text{Var}\left[X_k\left(1 + \lambda_k \xi_k\right) \middle| \mathfrak{F}_{k-1}\right](\omega)}\,, \tag{21}$$

P-a.s. for each k. By assumption $X_k \theta^*_{k+1}$ and W^*_k are both square integrable, both λ_k and ξ_k are uniformly bounded for each k, thus (21) is well-defined. Therefore we've established the existence of a process $\delta^* \in \Gamma$ such that if we define ξ_k as in (19), equation (21) determines the first component of risk-minimizing strategy ϕ^*. Now that we have proved the main existence question, we focus on verifying that the proper integrability conditions hold. With this in mind, we first verify that the product $\theta^*_k \triangle X_k$ is an element of $L_2(P)$. For simplicity, let γ be an arbitrary process in Γ such that $\gamma_k = \xi_k$ for all k, and define

$$W_k^{\lambda\gamma} := E[W_k^* | \mathfrak{F}_k] + \lambda_k \xi_k X_k \theta_{k+1}^* ,$$

which is in $L_2(P)$ by our assumptions on $\theta_{k+1}^* X_k$ and W_k^*. This definition allows one to write (21) in the more compact form:

$$\theta_k^* = \frac{\text{Cov}\left[W_k^{\lambda\gamma}, \triangle X_k^{\lambda\gamma} | \mathfrak{F}_{k-1}\right]}{\text{Var}\left[\triangle X_k^{\lambda\gamma} | \mathfrak{F}_{k-1}\right]} . \tag{22}$$

The Cauchy-Schwartz inequality, the assumption that X satisfies (10) along with proposition 2.3.2 imply the following series of relations:

$$\begin{aligned}
&E\left[(\theta_k^* \triangle X_k)^2\right] \\
&= E\left[\left(\frac{E[(W_k^{\lambda\gamma} - E[W_k^{\lambda\gamma}|\mathfrak{F}_{k-1}])(\triangle X_k^{\lambda\gamma} - E[\triangle X_k^{\lambda\gamma}|\mathfrak{F}_{k-1}]) | \mathfrak{F}_{k-1}]}{E[(\triangle X_k^{\lambda\gamma} - E[\triangle X_k^{\lambda\gamma}|\mathfrak{F}_{k-1}])^2 | \mathfrak{F}_{k-1}]} \triangle X_k\right)^2\right] \\
&= E\left[E\left[\frac{W_k^{\lambda\gamma} - E[W_k^{\lambda\gamma}|\mathfrak{F}_{k-1}]}{\triangle X_k^{\lambda\gamma} - E[\triangle X_k^{\lambda\gamma}|\mathfrak{F}_{k-1}]} \Big| \mathfrak{F}_{k-1}\right]^2 \triangle X_k^2\right] \\
&\leq E\left[\frac{E[(W_k^{\lambda\gamma} - E[W_k^{\lambda\gamma}|\mathfrak{F}_{k-1}])^2 | \mathfrak{F}_{k-1}]}{E[(\triangle X_k^{\lambda\gamma} - E[\triangle X_k^{\lambda\gamma}|\mathfrak{F}_{k-1}])^2 | \mathfrak{F}_{k-1}]} \triangle X_k^2\right] \\
&\leq E\left[\frac{\text{Var}[W_k^{\lambda\gamma}|\mathfrak{F}_{k-1}]}{c\text{Var}[\triangle X_k|\mathfrak{F}_{k-1}]} E[\triangle X_k^2 | \mathfrak{F}_{k-1}]\right] \\
&\leq E\left[\frac{\text{Var}[W_k^{\lambda\gamma}|\mathfrak{F}_{k-1}]}{c\text{Var}[\triangle X_k|\mathfrak{F}_{k-1}]} \text{Var}\left[\triangle X_k | \mathfrak{F}_{k-1}\right] + c_{mvt}(0)\text{Var}\left[\triangle X_k | \mathfrak{F}_{k-1}\right]\right] \\
&= E\left[\frac{\text{Var}[W_k^{\lambda\gamma}|\mathfrak{F}_{k-1}]}{c\text{Var}[\triangle X_k|\mathfrak{F}_{k-1}]} (1 + c_{mvt}(0))\text{Var}[\triangle X_k | \mathfrak{F}_{k-1}]\right] \\
&\leq E\left[\frac{E[(W_k^{\lambda\gamma})^2 | \mathfrak{F}_{k-1}]}{c\text{Var}[\triangle X_k|\mathfrak{F}_{k-1}]} (1 + c_{mvt}(0))\text{Var}[\triangle X_k | \mathfrak{F}_{k-1}]\right] \\
&= \tfrac{1}{c}(1 + c_{mvt}(0)) E\left[\left(W_k^{\lambda\gamma}\right)^2\right] < \infty .
\end{aligned}$$

Note that the first estimate is due to the Cauchy-Schwartz inequality, the second is from inequality (11), and the final two estimates come from the assumption of bounded mean-variance tradeoff with $\lambda = 0$ and the definition of variance. Thus, since we assumed $W_k^* \in L_2(P)$, we have verified that indeed $\theta_k^* \triangle X_k \in L_2(P)$. Also, by our assumption that X has substantial risk as in the proof of lemma 2.3.1 we conclude,

$$E\left[(\theta_k^* X_{k-1})^2\right] = E\left[E\left[(\theta_k^* \triangle X_k)^2 \,|\mathfrak{F}_{k-1}\right] \frac{X_{k-1}^2}{E[\triangle X_k^2|\mathfrak{F}_{k-1}]}\right]$$
$$\leq c_{sr} E\left[(\theta_k^* \triangle X_k)^2\right] < \infty .$$

So, $\theta_k^* X_{k-1} \in L_2(P)$ which means our assumption holds for $k-1$. This puts us into position to prove that the θ_k^* obtained previously in fact satisfies the desired relation

$$\theta_k^* = \underset{\theta_k \in \Theta(X)}{\operatorname{argmin}} \operatorname{Var}\left[E[W_k^*|\mathfrak{F}_k] - \theta_k \triangle X_k + \lambda_k X_k \left|\theta_{k+1}^* - \theta_k\right| |\mathfrak{F}_{k-1}\right] . \qquad (23)$$

Since we've shown that both $\theta_k^* \triangle X_k$ and $\theta_k^* X_{k-1}$ are elements of $L_2(P)$, we have that the sum is integrable, hence $\theta_k^* \triangle X_k + \theta_k^* X_{k-1} = \theta_k^* X_k \in L_2(P)$. This means that if θ_k is $\mathfrak{F}_{k-1}$-measurable and satisfies $\theta_k \triangle X_k \in L_2(P)$ and $\theta_k X_k \in L_2(P)$ then we can write

$$f_k(\theta_k(\omega), \omega) = \operatorname{Var}\left[E[W_k^*|\mathfrak{F}_k] - \theta_k \triangle X_k + \lambda_k X_k \left|\theta_{k+1}^* - \theta_k\right| |\mathfrak{F}_{k-1}\right](\omega)\; P\text{-a.s.} \qquad (24)$$

Thus (23) is satisfied for k by (24) and optimality conditions previously obtained. From the assumption on W_k^* and $\theta_{k+1}^* X_k$ for k and square integrability of $\theta_k^* X_k$ established above,

$$W_{k-1}^* = W_k^* - \theta_k^* \triangle X_k + \lambda_k X_k \left|\triangle \theta_{k+1}^*\right| \in L_2(P) .$$

Therefore, the assumption that $W_k^* \in L_2(P)$ for all k also holds for $k-1$, and induction is complete.

Now that we have the required integrability and optimality properties, we obtain the optimal strategy ϕ^*. Therefore, we define the second component η^* of ϕ^* as

$$\eta_k^* := E\left[W_k^* |\mathfrak{F}_k\right] - \theta_{k+1}^* X_k \quad \text{for } k = 0, 1, ..., N . \qquad (25)$$

Since we've established the square integrability of both W_k^* and $\theta_{k+1}^* X_k$ for $k = 0, \ldots, N$, η_k^* is adapted and the sum $\theta_{k+1}^* X_k + \eta_k^* \in L_2(P)$. Also, since we've shown that $\theta_k^* \Delta X_k \in L_2(P)$, by definition 2.1.4 $\theta_k^* \in \Theta(X) = \Theta(X^{\lambda\gamma})$. Thus, $\phi^* = (\theta^*, \eta^*)$ is a strategy such that

$$\theta_{N+1}^* X_N + \eta_N^* = E[W_N^* | \mathfrak{F}_N] = E[H | \mathfrak{F}_N] = \bar{\theta}_{N+1} X_N + \bar{\eta}_N \, .$$

Hence, we've obtained the contingent claim exactly

$$\theta_{N+1}^* = \bar{\theta}_{N+1} \, , \qquad \eta_N^* = \bar{\eta}_N \, .$$

Therefore, by proposition 2.2.2 and equation (23) we conclude that ϕ^* is locally risk-minimizing. Since the optimal second component of ϕ^* is given by (25), we have that

$$V_k^{\lambda\xi}(\phi) = \theta_{k+1}^* X_k \lambda_k \xi_k + E\left[W_k^* | \mathfrak{F}_k\right] = \eta^* \, .$$

Thus, equation (22) can be written in the more compact form (20) stated in the theorem. □

Note that if $\mathrm{Var}[\Delta X_k | \mathfrak{F}_{k-1}] = 0$, ΔX_k is $\mathfrak{F}_k$-measurable. In this case, the actions at times k and $k+1$ can be combined and one time step can be eliminated without loss of generality. Thus, the assumption that $\mathrm{Var}[\Delta X_k | \mathfrak{F}_{k-1}] > 0$ is natural and not restrictive on the results.

CONCLUSION

We have obtained a locally risk-minimizing strategy ϕ^* for every square-integrable contingent claim in a general incomplete market. This was done in discrete time with transaction costs modeled as adapted random variables. Closely following the approach of Lamberton, Pham, and Schweizer [5], we prove the existence of such a strategy under the assumptions of a bounded mean-variance tradeoff, substantial risk, and a non-degeneracy condition on the conditional variances of asset returns. This was done in parallel with usual dynamic programming algorithms.

An immediate possible extension to this work is the formation of a carefully constructed example, illustrating the effect of time-varying transaction costs. A more long-term prospect is the possibility of extending these results to continuous time models.

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